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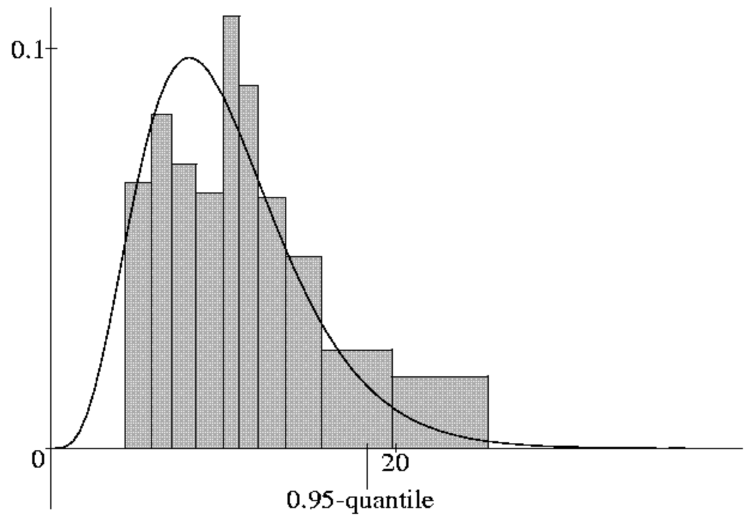

Mathematische Statistik und Wahrscheinlichkeitstheorie

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CHARACTERIZATION OF LÉVY PROCESSES BY A DUALITY FORMULA AND RELATED RESULTS

RÜDIGER MURR

Abstract. Processes with independent increments are characterized via a duality formula, including Malliavin derivative and difference operators. This result is based on a characterization of infinitely divisible random vectors by a functional equation. A construction of the difference operator by a variational method is introduced and compared to approaches used by other authors for Lévy processes involving the chaos decomposition. Finally we extend our method to characterize infinitely divisible random measures.

Introduction

The term duality formula mentioned in the title refers to the duality relation between annihilation and creation operator on the Fock space. If X is a process with independent increments and $\mathcal{H}_X$ the space of square integrable functionals that are $\sigma(X)$-measurable, then $\mathcal{H}_X$ has a chaos decomposition that is isomorphic to the Fock space, see Itô [11]. Thus the abstract duality relation on the Fock space has a probabilistic interpretation. For smooth, cylindrical test functionals $F(X) = f(X_{t_1}, \dots, X_{t_n})$ and step functions $\beta_t = \sum_{j=1}^k \beta_j 1_{(s_j, s_{j+1}]}(t)$ this means that the operators of

$$\text{annihilation} \quad F(X) \mapsto D_t F(X) + \Psi_{t,q} F(X) := \sum_{j=1}^n \partial_j f(X_{t_1}, \dots, X_{t_n}) 1_{[0,t_j]}(t) + F(X + q1_{[t,\infty)}) - F(X),$$

$$\text{and creation} \quad \beta \mapsto \int_{\mathbb{R}_+} \beta_t dX_t = \sum_{j=1}^{k-1} \beta_j \left(X_{s_{j+1}} - X_{s_j} \right),$$

are in duality. The main result of this article is Theorem 2.4: We prove that an integrable process X is a process with independent increments if and only if our duality formula holds for a small class of test functionals and step functions.

This result is based on a characterization of infinitely divisible random vectors presented in Section 1. A random vector Z is infinitely divisible if and only if for

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every $f \in C_b^\infty(\mathbb{R}^d)$ the functional equation

$$\mathbb{E}\left(f(Z)(Z-b)\right) = \mathbb{E}\left(A\nabla f(Z)\right) + \mathbb{E}\left(\int_{\mathbb{R}^d_*}(f(Z+q)-f(Z))qL(dq)\right)$$

holds for some $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ a non-negative definite matrix and L a Lévy measure, where ∇f is the gradient of f. This is an extension of Stein's lemma for standard Gaussian random variables as well as Chen's result for Poisson random variables, see [28] and [5].

In Section 2 we apply the techniques used for random vectors to processes with independent increments. This allows us to prove the aforementioned duality relation as well as the related characterization. The approach of proving the duality formula for Lévy processes seems to be new for the jump case. Rœlly and Zessin proved a characterization result for Brownian diffusions, Hsu extended this to Wiener processes on manifolds, see [25] and [10]. A related characterization of Poisson point processes was given by Mecke [19], see Corollary 4.3.

Section 3 introduces a variational definition of the annihilation operator for Lévy processes, which splits into a derivative and a difference operator. Our perturbation is the addition of a few independent jumps in the direction of the jumps of the reference process. We provide another proof of the duality formula for Lévy processes using our variational definition, then compare our definition of the operators to the annihilation operator from the chaos decomposition. We can show that the definitions coincide by using related works of Solé, Utzet, Vives and Geiss, Laukkarinen on the interpretation of the annihilation operator on $\mathcal{H}_X$. see [27] and [9]. First results in this direction for Lévy processes with jumps where given by Løkka [18]. The case of Poisson processes was treated earlier by Y. Ito and Nualart, Vives, see [12] and [22].

There are mainly two other variational approaches on processes with jumps in the literature. Bichteler and Jacod introduced a perturbation of the jump-size, Carlen and Pardoux respectively Elliott and Tsoi perturbated the jump-times of a Poisson process, see [1], [4] and [7]. These approaches lead to different derivative operators, and restrict the jump-types of the processes, see also Privault [24] for a generalization of the jump-time perturbation and the connection to a chaos decomposition.

In the last Section we extend the results of Section 1 to infinitely divisible random measures. Our method indicates a direct correspondence between infinitely divisible random objects and duality formulae. A characterization of infinitely divisible random measures was first presented by Kummer and Matthes, see [15] and [16].

1. A characterization of infinitely divisible random vectors

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. A measurable application $Z : \Omega \to \mathbb{R}^d$ is called a random vector. With $\langle q_1, q_2\rangle$ we denote the Euclidean scalar-product for $q_1, q_2 \in \mathbb{R}^d$. For $q \in \mathbb{R}^d$ let $\|q\| := \sqrt{\langle q, q\rangle}$ be the Euclidean norm and $|q|$ the sum of the absolute values of the components of q. All vectors are column vectors, with q' we mean the transposed row vector.

We call $\chi : \mathbb{R}^d \to \mathbb{R}^d$ a cutoff function if

$$\chi(q) = q + o(\|q\|^2) \text{ in a neighbourhood of zero and } \chi \text{ is bounded.} \tag{1.1}$$

Let $\mathbb{R}^d_* := \mathbb{R}^d \backslash \{0\}$, later we will also need $\mathbb{R}_+ := [0, \infty)$. A Lévy measure L on $(\mathbb{R}^d_*, \mathcal{B}(\mathbb{R}^d_*))$ is a σ-finite measure such that

$$\int_{\mathbb{R}^d_*} (\|q\|^2 \wedge 1) L(dq) < \infty.$$

By definition every cutoff function χ is in $\mathbb{L}^2(L)$.

We are interested in the class of infinitely divisible random vectors. A random vector Z is called infinitely divisible if for every $k \in \mathbb{N}$ there exist independent and identically distributed random vectors $Z^{(1)}, \dots, Z^{(k)}$ such that

$$Z \overset{\mathcal{L}}{=} Z^{(1)} + \cdots + Z^{(k)},$$

where $\mathcal{L}$ stands for equality in law. The classical Lévy-Khintchine formula gives a representation for the characteristic function of Z. For $\gamma \in \mathbb{R}^d$ we have

$$\log \mathbb{E}\left(e^{i\langle \gamma, Z\rangle}\right) = i\langle \gamma, b\rangle - \frac{1}{2}\langle \gamma, A\gamma\rangle + \int_{\mathbb{R}^d_*} \left(e^{i\langle \gamma, q\rangle} - 1 - i\langle \gamma, \chi(q)\rangle\right) L(dq), \tag{1.2}$$

where $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ is a symmetric non-negative definite matrix and L a Lévy measure (see e.g. [26] Theorem 8.1). The triple $(b, A, L)_\chi$ is called Fourier characteristics of Z with respect to the cutoff function χ. It is well known that only b depends on the choice of χ.

Example 1.1. A Gaussian random vector with mean b and covariance matrix A is infinitely divisible with characteristics $(b, A, 0)_\chi$ for any cutoff function χ.

A Poisson random variable with mean $\lambda \geq 0$ corresponds to an infinitely divisible random variable with characteristics $(\lambda\chi(1), 0, \lambda\delta_{\{1\}})_\chi$.

Remark 1.2. Note that for an infinitely divisible random vector Z we have equivalence of

$$Z \text{ is integrable } \Leftrightarrow \int_{\mathbb{R}^d_*} \left(\|q\|^2 \wedge |q|\right) L(dq) < \infty.$$

In this case we don't need a cutoff function in (1.2) and take $\chi = \mathrm{Id}$. The associated characteristics will be denoted simply by (b, A, L).

The following Theorem presents an equation that characterizes the law of an integrable infinitely divisible random vector. Denote by $C_b^\infty(\mathbb{R}^d)$ the space of smooth and bounded functions from $\mathbb{R}^d$ into $\mathbb{R}$ with all derivatives bounded.

Theorem 1.3. *Let Z be an integrable random vector. Then Z is infinitely divisible with characteristics (b, A, L) if and only if for every $f \in C_b^\infty(\mathbb{R}^d)$ we have*

$$\mathbb{E}\left(f(Z)(Z - b)\right) = \mathbb{E}\left(A\nabla f(Z)\right) + \mathbb{E}\left(\int_{\mathbb{R}^d_*} \left(f(Z + q) - f(Z)\right) q L(dq)\right). \tag{1.3}$$

Proof. Assume that Z is an integrable infinitely divisible random vector with characteristics (b, A, L). We are going to prove the equality (1.3) separately for each component. Fix any $1 \leq j \leq d$. Define

$$f_\gamma(q) := e^{i\langle \gamma, q\rangle}, \text{ for } \gamma, q \in \mathbb{R}^d.$$

We can permute differentiation and integration to obtain

$$\partial_{\gamma_j}\mathbb{E}\left(f_\gamma(Z)\right) = i\,\mathbb{E}\left(f_\gamma(Z)Z_j\right). \tag{1.4}$$

On the other hand the Lévy-Khintchine formula (1.2) gives

$$\partial_{\gamma_j}\mathbb{E}\left(f_\gamma(Z)\right) = \left(i\,b_j - (A\gamma)_j + i\int_{\mathbb{R}^d_*} q_j\left(e^{i\langle\gamma,q\rangle} - 1\right)L(dq)\right)\mathbb{E}\left(f_\gamma(Z)\right). \tag{1.5}$$

The second term on the right reduces to

$$-(A\gamma)_j\mathbb{E}\left(f_\gamma(Z)\right) = i\,\mathbb{E}\left((A\nabla f_\gamma(Z))_j\right).$$

The last term on the right hand side can be reformulated as

$$i\int_{\mathbb{R}^d_*} q_j\left(e^{i\langle\gamma,q\rangle} - 1\right)L(dq)\mathbb{E}\left(f_\gamma(Z)\right) = i\,\mathbb{E}\left(\int_{\mathbb{R}^d_*}\left(f_\gamma(Z+q) - f_\gamma(Z)\right)q_jL(dq)\right).$$

Comparing (1.4) and (1.5) and using the above reformulations we get (1.3) for f_γ. By linearity the equation holds for all real valued trigonometric functions. A Fourier approximation on compact sets and tightness of measure argument then shows that (1.3) holds for all $f \in C_b^\infty(\mathbb{R}^d)$.

To prove sufficiency we only need that (1.3) holds for trigonometric functions. The proof we give is suggested by Mecke ([19] Satz 3.1) and Rœlly, Zessin ([25] Théorème 2). For $\lambda \in \mathbb{R}$ define

$$\Phi(\lambda) = \mathbb{E}\left(e^{i\lambda\langle\gamma,Z\rangle}\right).$$

Then

$$\frac{d}{d\lambda}\Phi(\lambda) = i\,\mathbb{E}\left(e^{i\lambda\langle\gamma,Z\rangle}\langle\gamma, Z\rangle\right),$$

and since the real and complex component of $e^{i\lambda\langle\gamma,\cdot\rangle}$ are in $C_b^\infty(\mathbb{R}^d)$ we can use equation (1.3) to get

$$\frac{d}{d\lambda}\Phi(\lambda) = i\left(\langle\gamma, b\rangle + i\lambda\langle\gamma, A\gamma\rangle + \int_{\mathbb{R}^d_*}\left(e^{i\lambda\langle\gamma,q\rangle} - 1\right)\langle\gamma, q\rangle L(dq)\right)\Phi(\lambda).$$

This is an ordinary differential equation in λ with boundary condition $\Phi(0) = 1$ which admits the unique solution

$$\Phi(\lambda) = \exp\left(i\lambda\langle\gamma, b\rangle - \lambda^2\frac{1}{2}\langle\gamma, A\gamma\rangle + \int_{\mathbb{R}^d_*}\left(e^{i\lambda\langle\gamma,q\rangle} - 1 - i\lambda\langle\gamma, q\rangle\right)L(dq)\right).$$

For $\lambda = 1$ we recognize the characteristic function of an integrable infinitely divisible random vector with characteristics (b, A, L). □

Special cases of this characterization are known and used in Stein's calculus, see Examples 1.5, 1.6 and 1.7 below. In [29] Steutel gave a similar characterization theorem for infinitely divisible distributions on $\mathbb{R}_+$, see also [26] Section 51. Steutels result suggests, that we can avoid the integrability condition for random vectors which are *a.s.* positive.

Let Z be an infinitely divisible random vector that is non-negative, $\mathbb{P}(Z \in \mathbb{R}^d_+) = 1$. It is known that the Laplace transform of Z is such that for all $\gamma \in \mathbb{R}^d_+$ we have

$$-\log \mathbb{E}\left(e^{-\langle\gamma, Z\rangle}\right) = \langle\alpha, \gamma\rangle + \int_{\mathbb{R}^d_+\setminus\{0\}} \left(1 - e^{-\langle\gamma, q\rangle}\right) L^+(dq), \tag{1.6}$$

where $\alpha \in \mathbb{R}^d_+$ and L^+ is a Lévy measure on $\mathbb{R}^d_*$ with

$$\int_{\mathbb{R}^d_*} (|q| \wedge 1) L^+(dq) < \infty, \quad L^+(\mathbb{R}^d_* \setminus \mathbb{R}^d_+) = 0.$$

The relation between the Laplace characteristics (α, L^+) and the Fourier characteristics $(b, A, L)_\chi$ of Z is given by

$$b = \alpha + \int_{\mathbb{R}^d_*} \chi(q) L^+(dq), \quad A = 0, \quad L = L^+. \tag{1.7}$$

Denote by $C^\infty_c(\mathbb{R}^d_+)$ the space of functions from $\mathbb{R}^d_+$ into $\mathbb{R}_+$ that are smooth and have compact support.

Corollary 1.4. *Let Z be a non-negative random vector. Then Z is infinitely divisible with Laplace characteristics (α, L^+) if and only if for every $f \in C^\infty_c(\mathbb{R}^d_+)$ we have*

$$\mathbb{E}\left(f(Z)Z\right) = \mathbb{E}\left(f(Z)\alpha\right) + \mathbb{E}\left(\int_{\mathbb{R}^d_*} f(Z+q) q L^+(dq)\right). \tag{1.8}$$

Proof. Suppose first that Z is infinitely divisible. We repeat the proof of identity (1.3), but replace the function f_γ by $q \mapsto e^{-\langle\gamma, q\rangle}$ and use the Laplace transform instead of the Fourier transform.

For the converse we extend equation (1.8) to functions of the type $q \mapsto e^{-\langle\gamma, q\rangle}$ by monotone convergence and then work in the same lines as in Theorem 1.3. □

Example 1.5. If Z is a standard Gaussian random variable equation (1.3) reduces to

$$\mathbb{E}\left(f(Z)Z\right) = \mathbb{E}\left(f'(Z)\right).$$

This characterization result is also known as Stein's lemma, see [28].

Example 1.6. For a Poisson random variable Z with mean $\lambda \in \mathbb{R}_+$ identities (1.3) respectively (1.8) are

$$\mathbb{E}\left(f(Z)(Z-\lambda)\right) = \mathbb{E}\left((f(Z+1) - f(Z))\lambda\right) \text{ and } \mathbb{E}\left(f(Z)Z\right) = \mathbb{E}\left(f(Z+1)\lambda\right).$$

Chen introduced the corresponding characterization result in [5] as an analogue to Stein's lemma for the Poisson distribution.

Example 1.7. Let Z have a Gamma distribution with density $1_{(0,\infty)}(q)e^{-q}q^{\alpha-1}/\Gamma(\alpha)$ for some $\alpha > 0$. Then Z is infinitely divisible with Fourier characteristics given by $(\alpha, 0, \alpha 1_{(0,\infty)}(q)q^{-1}e^{-q}dq)$. Theorem 1.3 leads to the following formula:

$$\mathbb{E}\left(f(Z)(Z-\alpha)\right) = \mathbb{E}\left(\int_{\mathbb{R}_*} \left(f(Z+q) - f(Z)\right) \alpha e^{-q} dq\right).$$

Diaconis and Zabell proposed a different characterizing formula based on an integration by parts on the density function. According to [6] the random variable Z has a Gamma distribution with parameter α if and only if

$$\mathbb{E}\left(f(Z)\,(Z-\alpha)\right) = \mathbb{E}\left(f'(Z)Z\right)$$

for every smooth function f with compact support.

2. A characterization of processes with independent increments

Whenever $\mathbb{R}_+$ is used as a time index of processes we use the notation $\mathcal{I} = \mathbb{R}_+$. Define

$$\Delta_{\mathcal{I}} := \{\tau_n = \{t_1, \ldots, t_n\} : \ 0 \le t_1 \le \cdots \le t_n < \infty, \ n \in \mathbb{N}\}.$$

On the space $\mathbb{D}$ of càdlàg (right continuous with left limits) functions on $\mathcal{I}$ with values in $\mathbb{R}^d$ we define the usual σ-field $\mathcal{D} := \sigma\left(\pi_\tau, \ \tau \in \Delta_{\mathcal{I}}\right)$, where π_{τ_n} is the finite dimensional projection defined by $x \mapsto (x_{t_1}, \ldots, x_{t_n}) =: x_{\tau_n} \in \mathbb{R}^{nd}$ for $x \in \mathbb{D}$, $n \in \mathbb{N}$. We will work with the canonical setup: Our probability space is $(\mathbb{D}, \mathcal{D}, \mathbb{P})$ and X is the canonical process (the identity). The natural filtration is denoted by $\mathcal{F}_t := \sigma(X_s, \ s \le t)$. To avoid problems with the initial value X_0 we assume throughout the following condition which is standard.

$$\text{The processes we consider are starting from zero } a.s.: \ \mathbb{P}(X_0 = 0) = 1. \tag{2.1}$$

The vector space of elementary functions over $\mathcal{I}$ is

$$\mathcal{E} := \left\{\beta : \mathcal{I} \to \mathbb{R}^d, \ \beta = \sum_{i=1}^{n-1} \beta_i 1_{(t_i, t_{i+1}]} \text{ for some } \beta_i \in \mathbb{R}^d, \ \tau_n \in \Delta_{\mathcal{I}}, \ n \in \mathbb{N}\right\}.$$

We define a real-valued integral of functions β in $\mathcal{E}$ by

$$\int_{\mathcal{I}} \beta_s dx_s := \sum_{i=1}^{n-1} \langle \beta_i, x_{t_{i+1}} - x_{t_i} \rangle = -\sum_{i=1}^{n} \langle \beta_i - \beta_{i-1}, x_{t_i} \rangle, \quad x \in \mathbb{D}, \tag{2.2}$$

with $\beta_0 = \beta_n = 0$.

2.1. PII - Processes with independent increments. We want to characterize the following class of processes.

Definition 2.1. We say that X is a PII if it has independent increments, is starting from zero and is stochastically continuous.

If X is a PII then X_t is an infinitely divisible random vector for each $t \in \mathcal{I}$ (see e.g. [26] Theorem 9.1).

Remark 2.2. Fix some cutoff function χ, denote by $(b_t, A_t, L_t)_\chi$ the Fourier characteristics of X_t. Take some $t \in \mathcal{I}$, any $s \le t$. Theorem 9.8 in [26] states that as functions on $\mathcal{I}$

(1) $b_t \in \mathbb{R}^d$ with $b_0 = 0$ and $t \mapsto b_t$ is continuous;
(2) $A_t \in \mathbb{R}^{d\times d}$ is a non-negative definite matrix with $A_0 = 0$, $\langle q, A_s q\rangle \le \langle q, A_t q\rangle$ and $t \mapsto \langle q, A_t q\rangle$ is continuous for any $q \in \mathbb{R}^d$;
(3) L_t is a Lévy measure on $\mathbb{R}^d_*$ with $L_0(\mathbb{R}^d_*) = 0$, $L_s(Q) \le L_t(Q)$ and $t \mapsto L_t(Q)$ is continuous for any compact $Q \subset \mathbb{R}^d_*$.

If X is integrable we can add the property

(4) $\int_{\mathbb{R}^d_*}(\|q\|^2 \wedge |q|)L_t(dq) < \infty$.

If conversely there are $(b_t)_{t\in\mathcal{I}}$, $(A_t)_{t\in\mathcal{I}}$ and $(L_t)_{t\in\mathcal{I}}$ such that such that conditions (1)-(3) hold, then there exists a unique law $\mathbb{P}$ such that X is a PII and $(b_t, A_t, L_t)_\chi$ are the Fourier characteristics of X_t. Adding condition (4) we get the existence of a unique integrable PII.

Example 2.3. Assume $d = 1$. If X is a PII such that X_t has characteristics $(b_t, A_t, 0)_\chi$, then it is equal in law to $b + \sqrt{A}W$, where W is a real-valued Wiener process.

Take a finite Lévy measure L and let X be a PII such that X_t has characteristics $(t\int_{\mathbb{R}_*}\chi(q)L(dq), 0, tL(dq))_\chi$. Then X is a compound Poisson process: if N is a Poisson process with intensity $L(\mathbb{R}_*)$ and $(Y_j)_{j\ge1}$ a sequence of iid random variables with distribution $L(\mathbb{R}_*)^{-1}L(dq)$ then $\sum_{j=1}^{N_.} Y_j$ has the same law as X.

Using (2.2) we can give a convenient form to the characteristic function of the increments of a PII. Define a measure on $\mathcal{I}\times\mathbb{R}^d_*$ by $\bar{L}([0,t]\times Q) := L_t(Q)$ for $t \in \mathcal{I}$, $Q \in \mathcal{B}(\mathbb{R}^d_*)$. Using independence of increments we see that the Fourier characteristics of $X_t - X_s$ for any $s \le t$ are $(b_t - b_s, A_t - A_s, \bar{L}((s,t]\times dq))_\chi$. Thus for any $\beta \in \mathcal{E}$

$$\mathbb{E}\left(\exp\left(i\int_{\mathcal{I}}\beta_s dX_s\right)\right) = \prod_{k=1}^{n-1}\exp\Big(i\langle\beta_k, b_{t_{k+1}} - b_{t_k}\rangle - \frac{1}{2}\langle\beta_k, (A_{t_{k+1}} - A_{t_k})\beta_k\rangle + \int_{\mathbb{R}^d_*}\left(e^{i\langle\beta_k, q\rangle} - 1 - i\langle\beta_k, \chi(q)\rangle\right)\bar{L}((t_k, t_{k+1}]\times dq)\Big).$$

Define the integral

$$\int_{\mathcal{I}}\beta_s\cdot dA_s\beta_s := \sum_{k=1}^{n-1}\langle\beta_k, (A_{t_{k+1}} - A_{t_k})\beta_k\rangle.$$

We deduce the identity

$$\mathbb{E}\left(\exp\left(i\int_{\mathcal{I}}\beta_s dX_s\right)\right) = \exp\Big(i\int_{\mathcal{I}}\beta_s db_s - \frac{1}{2}\int_{\mathcal{I}}\beta_s\cdot dA_s\beta_s + \int_{\mathcal{I}\times\mathbb{R}^d_*}\left(e^{i\langle\beta_s, q\rangle} - 1 - i\langle\beta_s, \chi(q)\rangle\right)\bar{L}(dsdq)\Big). \quad (2.3)$$

This equation characterizes the increments of X and therefore, by assumption (2.1), the law of X.

2.2. Characterization of PII by a duality formula. In this Section we extend Theorem 1.3 to PII. Instead of test functions $f \in C_b^\infty(\mathbb{R}^d)$ we need smooth, bounded and cylindrical functionals

$$\mathcal{S} := \{F : \mathbb{D} \to \mathbb{R} : F(x) = f(x_{t_1}, \dots, x_{t_n}),\ f \in C_b^\infty(\mathbb{R}^{nd}),\ \tau_n \in \Delta_{\mathcal{I}},\ n \in \mathbb{N}\}.$$

Accordingly we need to define an extension of the gradient and difference operator appearing in equation (1.3). For $F \in \mathcal{S}$ we define the

$$\begin{aligned} \text{derivative operator} \quad D_{s,j}F(x) &:= \sum_{k=0}^{n-1} \partial_{kd+j} f(x_{t_1}, \dots, x_{t_n}) 1_{(0,t_k]}(s), \quad j \in \{1, \dots, d\}, \\ & \quad \text{and } D_s F(x) := (\dots, D_{s,j}F(x), \dots)' \text{ for } s \in \mathcal{I} \\ \text{difference operator} \quad \Psi_{s,q}F(x) &:= f(x_{t_1} + q1_{[0,t_1]}(s), \dots, x_{t_n} + q1_{[0,t_n]}(s)) - f(x_{\tau_n}) \\ &= F(x + q1_{[s,\infty)}) - F(x), \quad \text{for } s \in \mathcal{I},\ q \in \mathbb{R}_*^d. \end{aligned} \tag{2.4}$$

Theorem 2.4. *Let X be an integrable process and $(b_t)_{t\in\mathcal{I}}$, $(A_t)_{t\in\mathcal{I}}$ and $(L_t)_{t\in\mathcal{I}}$ be such that (1)-(4) of remark 2.2 hold. Then X is a PII and X_t has characteristics (b_t, A_t, L_t) if and only if for each $\beta \in \mathcal{E}$, $F \in \mathcal{S}$ the duality formula*

$$\begin{aligned} \mathbb{E}\left(F(X)\left(\int_{\mathcal{I}} \beta_s d(X-b)_s\right)\right) &= \mathbb{E}\left(\int_{\mathcal{I}} D_s F(X) \cdot dA_s \beta_s\right) \\ &\quad + \mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}_*^d} \Psi_{s,q}F(X)\langle \beta_s, q\rangle \bar{L}(dsdq)\right) \end{aligned} \tag{2.5}$$

holds.

Proof. The proof is an application of Theorem 1.3 to the random vector $(X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ for any $\tau_n \in \Delta_{\mathcal{I}}$, $n \in \mathbb{N}$. It will be more convenient to use the form of the characteristic function given in (2.3).

Assume that X is an integrable PII such that X_t has characteristics (b_t, A_t, L_t). Take any $\beta, \tilde\beta \in \mathcal{E}$ and let $F(X) = \exp\left(i\int_{\mathcal{I}} \tilde\beta_s dX_s\right)$ be a trigonometric path functional. Without loss of generality we assume that

$$\beta = \sum_{j=1}^{n-1} \beta_j 1_{(t_j,t_{j+1}]}, \text{ and } \tilde\beta = \sum_{j=1}^{n-1} \tilde\beta_j 1_{(t_j,t_{j+1}]}.$$

Differentiating $\mathbb{E}\left(\exp\left(i\int_{\mathcal{I}} \tilde\beta_s dX_s\right)\right)$ in each of the d components of $\tilde\beta_k$ and using (2.3),

$$\begin{aligned} & i\,\mathbb{E}\left(e^{i\int_{\mathcal{I}}\tilde\beta_s dX_s}(X_{t_{k+1}} - X_{t_k})\right) \\ =\ & \mathbb{E}\Big(i(b_{t_{k+1}} - b_{t_k}) - (A_{t_{k+1}} - A_{t_k})\tilde\beta_k \\ & \quad + i\int_{\mathbb{R}_*^d}\left(e^{i\langle\tilde\beta_k, q\rangle} - 1\right) q\bar{L}((t_k, t_{k+1}] \times dq)\Big)\,\mathbb{E}\left(e^{i\int_{\mathcal{I}}\tilde\beta_s dX_s}\right). \end{aligned}$$

Next we take the scalar product with β_k, sum over $1 \le k \le n-1$ and use the definition of the derivative and difference operator (2.4) to get identity (2.5) for $F(X) = \exp\left(i\int_{\mathcal{I}} \tilde\beta_s dX_s\right)$ and $\beta \in \mathcal{E}$. The extension to $F \in \mathcal{S}$ works in the same lines as in the proof of Theorem 1.3.

Now assume that X is an integrable process that satisfies (2.5). Fix any $\beta \in \mathcal{E}$ and define

$$\Phi(\lambda) := \mathbb{E}\left(\exp\left(i\lambda \int_{\mathcal{I}} \beta_s dX_s\right)\right) \text{ for } \lambda \in \mathbb{R}.$$

We see that

$$\begin{aligned}\frac{d}{d\lambda}\Phi(\lambda) &= i\,\mathbb{E}\left(e^{i\lambda \int_{\mathcal{I}} \beta_s dX_s} \int_{\mathcal{I}} \beta_s dX_s\right) \\ &= i\left[\int_{\mathcal{I}} \beta_s db_s + i\lambda \int_{\mathcal{I}} \beta_s \cdot dA_s \beta_s + \int_{\mathcal{I}\times\mathbb{R}^d_*} (e^{i\lambda\langle \beta_s, q\rangle} - 1)\langle \beta_s, q\rangle \bar{L}(dsdq)\right]\Phi(\lambda),\end{aligned}$$

where we used the duality formula (2.5) to get the second line. This is an ordinary differential equation with initial condition $\Phi(0) = 1$. It admits the unique solution

$$\Phi(\lambda) = \exp\left(i\lambda \int_{\mathcal{I}} \beta_s db_s - \frac{\lambda^2}{2}\int_{\mathcal{I}} \beta_s \cdot dA_s \beta_s + \int_{\mathcal{I}\times\mathbb{R}^d_*} \left(e^{i\lambda\langle \beta_s, q\rangle} - 1 - i\lambda\langle \beta_s, q\rangle\right)\bar{L}(dsdq)\right).$$

For $\lambda = 1$ we recognize (2.3) and thus identify the law of X. □

For Lévy processes without Gaussian part the duality formula (2.5) was first mentioned by Picard, see [23]. For Wiener processes it is a standard result. The duality formula is due to Cameron, and later to Bismut in the realm of Malliavin's calculus, see [3] and [2]. Gaveau and Trauber furnished in [8] the interpretation as duality of operators on the isomorphic Fock space. The characterizing property is known for Wiener processes:

Example 2.5. Take a one-dimensional integrable process X. By Theorem 2.4 X is a Wiener process if and only if

$$\mathbb{E}\left(F(X)\int_{\mathcal{I}} \beta_s dX_s\right) = \mathbb{E}\left(\int_{\mathcal{I}} D_s F(X)\beta_s ds\right)$$

holds for all $F \in \mathcal{S}$ and $\beta \in \mathcal{E}$. This result, using slightly larger classes of test functions, was first obtained by Rœlly and Zessin in [25]. Hsu generalized it to Wiener processes on manifolds in [10].

Example 2.6. Let $\lambda : \mathcal{I} \to \mathbb{R}_+$ be integrable with respect to Lebesgue measure. Then X is a Poisson process with time dependent intensity λ_t if and only if

$$\mathbb{E}\left(F(X)\int_{\mathcal{I}} \beta_s dX_s\right) = \mathbb{E}\left(\int_{\mathcal{I}} F(X + 1_{[t,\infty)})\beta_t \lambda_t dt\right)$$

holds for all $F \in \mathcal{S}$ and $\beta \in \mathcal{E}$. A related characterization of Poisson point processes was proved by Mecke in [19].

Example 2.7. A PII is called isotropic α-stable if its characteristics for a given $t \in \mathcal{I}$ are $(0, 0, (tC/|q|^{1+\alpha})dq)$ for some $\alpha \in (0, 2)$ and any $C \in \mathbb{R}$. It is well known that an α-stable process is integrable only if $\alpha \in (1, 2)$. By Theorem 2.4 X is α-stable for $\alpha \in (1, 2)$ if and only if

$$\mathbb{E}\left(F(X)\int_{\mathcal{I}} \beta_s d(X-b)_s\right) = \mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}^d_*} \left(F(X + q1_{[t,\infty)}) - F(X)\right)\langle \beta_t, q\rangle \frac{C}{|q|^{1+\alpha}} dtdq\right)$$

holds for all $F \in \mathcal{S}$ and $\beta \in \mathcal{E}$.

One method to extend Theorem 2.4 to non-integrable PII is to cut off the large jumps of the process. For any càdlàg trajectory $x \in \mathbb{D}$ the jump at time $t \in \mathcal{I}$, $\Delta x_t := x_t - x_{t-}$, is well defined. Moreover for any $K > 0$, $t \in \mathcal{I}$ the sum $\sum_{s \le t} \Delta x_s 1_{|\Delta x_s| > K}$ is finite. This allows us to define the measurable application

$$x \mapsto x^K := x - \sum_{s \le .} \Delta x_s 1_{|\Delta x_s| > K} .$$

If X is a PII the process X^K is an integrable PII.

Corollary 2.8. *Let X be a process such that X^K is integrable for all $K \in (K_0, \infty)$, where $K_0 > 0$. Let $(b_t)_{t \in \mathcal{I}}$, $(A_t)_{t \in \mathcal{I}}$ and $(L_t)_{t \in \mathcal{I}}$ satisfy conditions (1)-(3) of remark 2.2. If for every $\beta \in \mathcal{E}$, $F \in \mathcal{S}$ and $K \in (K_0, \infty)$*

$$\begin{aligned} \mathbb{E}\left(F(X^K) \left(\int_{\mathcal{I}} \beta_s d(X^K - b^K)_s \right) \right) &= \mathbb{E}\left(\int_{\mathcal{I}} D_s F(X^K) \cdot dA_s \beta_s \right) \\ &\quad + \mathbb{E}\left(\int_{\mathcal{I} \times \mathbb{R}^d_*} \Psi_{s,q} F(X^K) \langle \beta_s, q \rangle 1_{|q| \le K} \bar{L}(dsdq) \right), \end{aligned} \tag{2.6}$$

where $b^K_t := b_t - \int_{\mathbb{R}^d_*} \left(\chi(q) - q 1_{|q| \le K} \right) L_t(dq) \in \mathbb{R}^d$, then X is a PII and X_t has characteristics $(b_t, A_t, L_t)_\chi$.

Proof. By Theorem 2.4 we see that X^K is an integrable PII and X^K_t has Fourier characteristics $(b^K_t, A_t, 1_{|q| \le K} L_t)$. The result follows by letting K tend to infinity. □

Example 2.9. Take some $\alpha \in (0, 2)$. By Theorem 2.4 X is isotropic α-stable if and only if

$$\begin{aligned} &\mathbb{E}\left(F(X) \int_{\mathcal{I}} \beta_s d(X^K - b^K)_s \right) \\ = \; &\mathbb{E}\left(\int_{\mathcal{I} \times \mathbb{R}^d_*} \left(F(X^K + q 1_{[t,\infty)}) - F(X^K) \right) \langle \beta_t, q \rangle 1_{|q| \le K} \frac{C}{|q|^{1+\alpha}} dt dq \right) \end{aligned}$$

holds for all $F \in \mathcal{S}$, $\beta \in \mathcal{E}$, $K > K_0$ and some $C \in \mathbb{R}$, where

$$b^K_t = t \int_{\mathbb{R}^d_*} \left(q 1_{|q| \le K} - \chi(q) \right) \frac{C}{|q|^{1+\alpha}} dq.$$

3. Variational approach to Malliavin calculus for Lévy processes

The reader familiar to Malliavin calculus for Lévy processes already recognized that equation (2.5) seems to be a special case of the duality formula on the chaos decomposition of Lévy processes. We make this statement rigorous by introducing a variational way to define the derivative and difference operators appearing in the duality formula, and comparing this approach to results obtained through the chaos decomposition.

Our variational definition of derivative and difference operators is given in Section 3.1. We use this definition in Section 3.2 to prove an extension of the duality formula (2.5) in terms of the classes of test functions. This is inspired by the method used by Bismut to prove a duality formula for the Wiener process in [2].

Recall that a Lévy process is a stationary PII, in particular X_t has characteristics

$$b_t = tb, \quad A_t = tA, \quad L_t(dq) = tL(dq)$$

for some $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ symmetric non-negative definite and L a Lévy measure on $\mathbb{R}^d_*$. This implies $\bar{L}(dsdq) = dsL(dq)$.

Since the law of a Lévy process is determined by the Fourier characteristics of X_1, we just say from now on that X has characteristics $(b, A, L)_\chi$.

3.1. Variational definition of derivative and difference operator. In this Section we give variational definitions of the derivative and difference operator on $\mathcal{S}$ defined in (2.4). This variational approach extends to larger classes of functionals.

Let $\mathbb{P}$ be any measure on $\mathbb{D}$. The following interpretation of the derivative operator on $\mathcal{S}$ as Gâteaux-derivative is well known. Take some $\beta \in \mathcal{E}$, $F \in \mathcal{S}$, then for any symmetric non-negative definite matrix A

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(F(X+\varepsilon\int_{[0,.]}A\beta_s ds) - F(X)\right) = \int_{\mathcal{I}}\langle D_sF(X), A\beta_s\rangle ds, \ \mathbb{P}\text{-}a.s.. \tag{3.1}$$

Using the Lipschitz regularity of $F \in \mathcal{S}$ and the dominated convergence theorem we can show that this convergence also holds in $\mathbb{L}^2(\mathbb{P})$. Denote by $\mathbb{L}^2(Adt)$ the space of functions $\beta : \mathcal{I} \to \mathbb{R}^d$ with $\int_{\mathcal{I}}\langle\beta_s, A\beta_s\rangle ds < \infty$.

Definition 3.1. For $F \in \mathbb{L}^2(\mathbb{P})$ we say that F is $\mathcal{E}_A$-differentiable if there exists a process $D_sF(X) \in \mathbb{L}^2(Adt \otimes \mathbb{P})$ such that for every $\beta \in \mathcal{E}$ the equality (3.1) holds in $\mathbb{L}^2(\mathbb{P})$.

Remark 3.2. The definition of $\mathcal{E}_A$-differentiability depends on the matrix A and on $\mathbb{P}$. Take for example $d = 2$ and $\mathbb{P}$ such that for $X = (X_1, X_2)$ the first component X_1 is a real-valued Wiener process and $X_2 \equiv 0$. Then X is a Lévy process in $\mathbb{R}^2$ with characteristics

$$b = 0,\ L = 0 \text{ and } A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let us consider the simple functional $F(X) = f_1(X_{1,t})f_2(X_{2,t})$ for some $t \in \mathcal{I}$. Then the fact that F is $\mathcal{E}_A$-differentiable is independent of f_2, but F is $\mathcal{E}_{Id}$-differentiable only if f_2 is differentiable in zero. In both cases f_1 has to be differentiable in a weak sense such that its weak derivative is square integrable with respect to the Gaussian distribution with mean 0 and variance t.

To introduce the variational definition of the difference operator we need the following space of elementary test functions:

$$\bar{\mathcal{E}} := \left\{\xi : \mathcal{I}\times\mathbb{R}^d_* \to \mathbb{R}_+,\ \xi = \sum_{j=1}^{n-1}\xi_j \mathbf{1}_{(t_j,t_{j+1}]\times Q_j} \right.$$
$$\left. \text{for } \xi_j \in \mathbb{R}_+, Q_j \subset \mathbb{R}^d_* \text{ compact},\ \tau_n \in \Delta_{\mathcal{I}}, n \in \mathbb{N}\right\}.$$

The analogous to the deterministic perturbation process $\left(\varepsilon\int_{[0,t]}A\beta_s ds\right)_{t\in\mathcal{I}}$ will be a sequence of random processes $(Y^{\varepsilon\xi})_{\varepsilon\in[0,1]}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ as follows.

Let $\xi = \xi_0 1_{(u,v]\times Q} \in \bar{\mathcal{E}}$ and L any Lévy measure be fixed. We consider a family $(N^{\varepsilon\xi})_{\varepsilon\in[0,1]}$ of Poisson point processes on $\mathcal{I} \times \mathbb{R}^d_*$ with intensity $\varepsilon\xi(s,q)dsL(dq)$. Define the $\mathbb{R}^d$-valued process $Y^{\varepsilon\xi}_t = \int_{[0,t]\times\mathbb{R}^d_*} qN^{\varepsilon\xi}(dsdq)$. Then the marginals of $Y^{\varepsilon\xi}$ satisfy

$$P(Y^{\varepsilon\xi}_t \in dq) = e^{-\varepsilon((t\vee u)\wedge v)\xi_0 L(Q)} \left[\delta_{\{0\}}(dq) + \varepsilon((t\vee u)\wedge v)\xi_0 1_Q(q)L(dq) + O(\varepsilon^2)\right]. \quad (3.2)$$

For $F \in \mathcal{S}$ this implies

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} E\left(F(X + Y^{\varepsilon\xi}) - F(X)\right)\\
=\ & \sum_{j=1}^{n} \int_{\mathbb{R}^d_*} \left(f(X_{t_1} + q_j, \dots, X_{t_j} + q_j, X_{t_{j+1}}, \dots X_{t_n}) - f(X_{\tau_n})\right)\cdot\\
&\quad \cdot((t_j \vee u)\wedge v - (t_{j-1}\vee u)\wedge v)1_Q(q_j)\xi_0 L(dq_j)\\
=\ & \int_{\mathcal{I}\times\mathbb{R}^d_*} \left(F(X + q1_{[s,\infty)}) - F(X)\right)\xi(s,q)dsL(dq) = \int_{\mathcal{I}\times\mathbb{R}^d_*} \Psi_{s,q}F(X)\xi(s,q)dsL(dq)
\end{aligned}$$

holds $\mathbb{P}$-*a.s.*, where E (.) denotes the integral with respect to P. By linearity the same limit exists for all $\xi \in \bar{\mathcal{E}}$, and the convergence holds in $\mathbb{L}^2(\mathbb{P})$ too.

Definition 3.3. For $F \in \mathbb{L}^2(\mathbb{P})$ we say that F is $\bar{\mathcal{E}}_L$-differentiable if there exists $\Psi_{s,q}F(X) \in \mathbb{L}^2(dt \otimes L \otimes \mathbb{P})$ such that for every $\xi \in \bar{\mathcal{E}}$ the equality

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} E\left(F(X + Y^{\varepsilon\xi}) - F(X)\right) = \int_{\mathcal{I}\times\mathbb{R}^d_*} \Psi_{t,q}F(X)\xi(s,q)dsL(dq)$$

holds in $\mathbb{L}^2(\mathbb{P})$.

The notion of $\bar{\mathcal{E}}_L$-differentiability depends on $\mathbb{P}$ and on the Lévy measure L in the same way as $\mathcal{E}_A$-differentiability is dependent on A and $\mathbb{P}$. Note that D can be interpreted as operator from $\mathbb{L}^2(\mathbb{P})$ into $\mathbb{L}^2(Adt \otimes \mathbb{P})$ and Ψ as operator from $\mathbb{L}^2(\mathbb{P})$ into $\mathbb{L}^2(dt \otimes L \otimes \mathbb{P})$

Other variational definitions of derivative operators for jumping processes exist. In [1] Bichteler and Jacod perturbed the jumps heights of the reference process. The drawback is that the Lévy measure has to be absolutely continuous $L(dq) \ll dq$. Carlen and Pardoux as well as Elliott and Tsoi perturbed the jump-times of a Poisson process, i.e. they restricted their approach to $L = \lambda\delta_{\{1\}}$ for some $\lambda > 0$, see [4] and [7]. Privault extended this approach in [24] to Lévy processes with Lévy measure $L = \sum_{j=1}^{k} \lambda_j \delta_{\{q_j\}}$ with $\lambda_j > 0$, $q_j \in \mathbb{R}^d_*$ and $k \in \mathbb{N}$. Both approaches lead to a derivative instead of a difference operator.

3.2. Application: An alternative proof of the duality formula. Let X be a Lévy process with characteristics $(b, A, L)_\chi$. As an application of the variational definition of derivative and difference operators we give another proof of the duality formula (2.5).

It is well known (see e.g. Sato [26] Paragraph 19) that X $\mathbb{P}$-*a.s.* admits the Lévy-Itô decomposition

$$X_t = tb + M^X_t + \int_{[0,t]\times\mathbb{R}^d_*} \chi(q)\tilde{N}^X(dsdq) + \int_{[0,t]\times\mathbb{R}^d_*} (q - \chi(q))N^X(dsdq), \quad (3.3)$$

where M^X is a continuous martingale with quadratic variation process $(tA)_{t\in\mathbb{R}_+}$, N^X is a Poisson measure with intensity $dsL(dq)$ on $\mathcal{I}\times\mathbb{R}^d_*$ and $\tilde{N}^X$ is the compensated Poisson measure.

Proposition 3.4. *Let X be a Lévy process with characteristics $(b, A, L)_\chi$. For every functional F that is $\mathcal{E}_A$- and $\bar{\mathcal{E}}_L$-differentiable and every $\beta\in\mathbb{L}^2(Adt)$, $\xi\in\mathbb{L}^2(dt\otimes L)$ the following duality formula holds*

$$\begin{aligned}&\mathbb{E}\left(F(X)\left(\int_{\mathcal{I}}\beta_s dM^X_s+\int_{\mathcal{I}\times\mathbb{R}^d_*}\xi(s,q)\tilde{N}^X(dsdq)\right)\right)\\ =\ &\mathbb{E}\left(\int_{\mathcal{I}}\langle D_sF(X),A\beta_s\rangle ds\right)+\mathbb{E}\left(\int_{\mathcal{I}\times\mathbb{R}^d_*}\Psi_{s,q}F(X)\xi(s,q)dsL(dq)\right).\end{aligned}\tag{3.4}$$

Proof. First we explain how we use the Girsanov theorem in our proof. Take any $\beta\in\mathcal{E}$, $\xi\in\bar{\mathcal{E}}$, then there exists some $T\in\mathcal{I}$ such that β and ξ have their supports in $[0,T]$ and $[0,T]\times\mathbb{R}^d_*$ respectively. The process defined by

$$G_t=\int_{[0,t]}\beta_s dM^X_s+\int_{[0,t]\times\mathbb{R}^d_*}\xi(s,q)\tilde{N}^X(dsdq),\ t\in\mathcal{I},\tag{3.5}$$

is a martingale and $G_t=G_T$, $\forall t\geq T$. Therefore we can define its Doléans-Dade exponential as the solution of the stochastic integral equation

$$\mathcal{E}(G)=1+\int_{[0,.]}\mathcal{E}(G)_{s-}dG_s.\tag{3.6}$$

A solution to this equation exists in a pathwise sense and is a uniformly integrable martingale (see Theorem IV.3 of Lepingle, Mémin [17]). Therefore we can define a new measure on $(\mathbb{D},\mathcal{D})$ by $\tilde{\mathbb{P}}:=\mathcal{E}(G)_T\mathbb{P}$. An application of the Girsanov theorem for semimartingales (see e.g. [13] Theorem III.3.24) shows that X is a PII under $\tilde{\mathbb{P}}$ and X_t has characteristics

$$tb+\int_{[0,t]}A\beta_s ds+\int_{[0,t]\times\mathbb{R}^d_*}\chi(q)\xi(s,q)dsL(dq),\qquad tA,\qquad\left(\int_{[0,t]}(1+\xi(s,q))ds\right)L(dq)$$

with respect to χ. Note that using Burkhölder-Davis-Gundy inequalities and Gronwall's lemma we can show that

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left(\mathcal{E}(\varepsilon G)_T-1\right)=G_T\qquad\text{in }\mathbb{L}^2(\mathbb{P}).$$

We first prove (3.4) for $\xi\equiv0$, which is the first half of that identity. Given $\varepsilon>0$ define the perturbed process

$$X^{\varepsilon\beta}:=X+\varepsilon\int_{[0,.]}A\beta_s ds.$$

Then $X^{\varepsilon\beta}$ clearly is a PII and $X^{\varepsilon\beta}_t$ has characteristics

$$tb+\varepsilon\int_{[0,t]}A\beta_s ds,\qquad tA,\qquad tL(dq).$$

Now we observe by the Girsanov theorem recalled above that $X^{\varepsilon\beta}$ under $\mathbb{P}$ has the same characteristics as X under $\mathbb{P}^{\varepsilon\beta} := \mathcal{E}(\varepsilon \int_{[0,.]} \beta_s dM_s^X)_T \mathbb{P}$. By equality of the characteristic functions we get that $\mathbb{P}^{\varepsilon\beta} \circ X^{-1} = \mathbb{P} \circ (X^{\varepsilon\beta})^{-1}$ and therefore

$$\mathbb{E}\left(F(X)\frac{1}{\varepsilon}\left(\mathcal{E}(\varepsilon \int_{[0,.]} \beta_s dM_s^X)_T - 1\right)\right) = \frac{1}{\varepsilon}\mathbb{E}\left(F(X^{\varepsilon\beta}) - F(X)\right)$$

for arbitrary $\mathcal{E}_A$-differentiable F. We use the definition of the derivative operator to take limits, thus

$$\mathbb{E}\left(F(X)\int_I \beta_t dM_t^X\right) = \mathbb{E}\left(\int_I \langle D_t F(X), A\beta_t\rangle dt\right). \tag{3.7}$$

We now prove (3.4) for $\beta \equiv 0$. Define $Y^{\varepsilon\xi}$ as in Section 3.1, then the perturbed process $X^{\varepsilon\xi} := X + Y^{\varepsilon\xi}$ is a PII under $\mathbb{P} \otimes P$ and $X_t^{\varepsilon\xi}$ has characteristics

$$tb + \int_{[0,t]\times\mathbb{R}^d_*} \chi(q)\xi(s,q)ds L(dq), \qquad tA, \qquad \left(\int_{[0,t]} (1 + \varepsilon\xi(s,q))ds\right) L(dq).$$

This is an easy consequence of the fact that the sum of two independent Poisson measures is still a Poisson random measure whose intensity is the sum of the intensities. By Girsanov theory X has the same characteristics under the measure $\mathbb{P}^{\varepsilon\xi} := \mathcal{E}\left(\varepsilon \int_{[0,.]\times\mathbb{R}^d_*} \xi(s,q)\tilde{N}^X(dsdq)\right)_T \mathbb{P}$, which implies $\mathbb{P}^{\varepsilon\xi} \circ X^{-1} = (\mathbb{P} \otimes P) \circ (X^{\varepsilon\xi})^{-1}$. For an arbitrary $\bar{\mathcal{E}}_L$-differentiable F this means

$$\mathbb{E}\left(F(X)\frac{1}{\varepsilon}\left(\mathcal{E}\left(\varepsilon \int_{[0,.]\times\mathbb{R}^d_*} \xi(s,q)\tilde{N}^X(dsdq)\right)_T - 1\right)\right) = \frac{1}{\varepsilon}\mathbb{E}\left(E\left(F(X^{\varepsilon\xi}) - F(X)\right)\right).$$

We can apply the definition of the difference operator to get

$$\mathbb{E}\left(F(X)\int_{I\times\mathbb{R}^d_*} \xi(s,q)\tilde{N}^X(dsdq)\right) = \mathbb{E}\left(\int_{I\times\mathbb{R}^d_*} \Psi_{s,q}F(X)\xi(s,q)ds L(dq)\right). \tag{3.8}$$

Adding (3.7) and (3.8) we get (3.4) for $\beta \in \mathcal{E}$ and all $\xi = \xi_1 - \xi_2$ with $\xi_1, \xi_2 \in \bar{\mathcal{E}}$. By density of step functions in $\mathbb{L}^2(Adt)$ and $\mathbb{L}^2(dt \otimes L)$ we get the result. □

Note that (2.5) is a special case of (3.4). The result is easily extended to predictable $\beta \in \mathbb{L}^2(Adt \otimes \mathbb{P})$ and $\xi \in \mathbb{L}^2(dt \otimes L \otimes \mathbb{P})$.

3.3. $\mathbb{L}^2$-extension of derivative and difference operator. In this Section we investigate the connection between the variational definition of the derivative and difference operators and the annihilation operator defined via the chaos decomposition of Lévy processes.

Proposition 3.5. *Let X be a Lévy process with characteristics $(b, A, L)_\chi$. Define the derivative operator and the difference operator as in Definitions 3.1 and 3.3. Then*

- *the derivative operator D is closable as operator from $\mathbb{L}^2(\mathbb{P})$ into $\mathbb{L}^2(Adt \otimes \mathbb{P})$;*
- *the difference operator Ψ is closable as operator from $\mathbb{L}^2(\mathbb{P})$ into $\mathbb{L}^2(dt \otimes L \otimes \mathbb{P})$.*

Moreover for each F in the closure of the domain of Ψ the difference representation

$$\Psi_{t,q}F(X) = F(X + q1_{[t,\infty)}) - F(X), \ dt \otimes L \otimes \mathbb{P}\text{-a.e.} \tag{3.9}$$

holds.

Proof. There is a standard proof for the closability of the derivative operator D using the duality formula (3.4) for $\xi \equiv 0$ (see e.g. [21] Section 1.2).

To show that Ψ is closable, we first prove the representation (3.9) for every $\bar{\mathcal{E}}_L$-differentiable functional F. We have already seen that this representation holds for $F \in \mathcal{S}$. Take $\xi = \xi_0 1_{(u,v]\times Q} \in \bar{\mathcal{E}}$ and let $Y^{\varepsilon\xi}$ be defined as in Section 3.1. Instead of (3.2) we use the density expansion

$$E\left(F(X+Y^{\varepsilon\xi})\right) = \sum_{j=0}^{\infty} \int_{(\mathcal{I}\times\mathbb{R}^d_*)^j} F(X+q_1 1_{[t_1,\infty)}+\cdots+q_j 1_{[t_j,\infty)}) e^{-\varepsilon\xi_0(u-v)L(Q)} (\varepsilon\xi_0)^j 1_Q(q_1)L(dq_1)\cdots 1_Q(q_j)L(dq_j) 1_{\{u<t_1\le t_j\le v\}} dt_1\cdots dt_j.$$

The fact that $E\left(\left|F(X+Y^{\xi})+F(X)\right|\right)<\infty$ allows us to use the dominated convergence theorem to show that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E\left(F(X+Y^{\varepsilon\xi})-F(X)\right) = \int_{\mathcal{I}\times\mathbb{R}^d_*}\left(F(X+q1_{[t,\infty)})-F(X)\right)\xi(t,q)dtL(dq),\ \mathbb{P}\text{-}a.s..$$

By the variational definition of the difference operator there exist subsequences $(\varepsilon_j)_{j\ge1}$, $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty}\frac{1}{\varepsilon_j}E\left(F(X+Y^{\varepsilon_j\xi})-F(X)\right) = \int_{\mathcal{I}\times\mathbb{R}^d_*}\Psi_{t,q}F(X)\xi(t,q)dtL(dq),\ \mathbb{P}\text{-}a.s.,$$

which implies $\Psi_{t,q}F(X) = F(X+q1_{[t,\infty)})-F(X)$ holds $dt\otimes L\otimes\mathbb{P}$-*a.e.*.

The proof of closability of Ψ is suggested by Nualart, Vives ([22] Theorem 6.2). Let $(F_j)_{j\ge1}$ be a sequence of $\bar{\mathcal{E}}_L$-differentiable functionals such that $F_j\to 0$ in $\mathbb{L}^2(\mathbb{P})$ and $\Psi F(X)\to\eta$ in $\mathbb{L}^2(dt\otimes L\otimes\mathbb{P})$. Ψ is closable if $\eta = 0$.

We can find a subsequence $(j_k)_{k\ge1}$ such that $F_{j_k}\to 0$ $\mathbb{P}$-*a.s.* and $\Psi F(X)_{j_k}\to\eta$ holds $dt\otimes L\otimes\mathbb{P}$-*a.e.* By the above representation of ΨF_j we also get

$$\lim_{k\to\infty}\Psi F_{j_k} = \lim_{k\to\infty}\left(F_{j_k}(X+q1_{[t,\infty)})-F_{j_k}(X)\right) = 0,\ dt\otimes L\otimes\mathbb{P}\text{-}a.e.,$$

which implies $\eta = 0$. In the same way we can prove the difference representation on the closed domain of Ψ. □

Define Dom$D$ and DomΨ to be the domains of the closed operators. We observe from the representation (3.9) that our definition of DomΨ is equivalent to the following one:

$$\mathrm{Dom}\Psi := \left\{F\in\mathbb{L}^2(\mathbb{P}):\ (t,q,x)\mapsto F(x+q1_{[t,\infty)})-F(x)\in\mathbb{L}^2(dt\otimes L\otimes\mathbb{P})\right\}. \quad (3.10)$$

In the literature there are two other approaches to define similar derivative and difference operators.

(1) Starting from definition (2.4) on $\mathcal{S}$ extend these as operators from $\mathbb{L}^2(\mathbb{P})$ into $\mathbb{L}^2(Adt\otimes\mathbb{P})$ respectively $\mathbb{L}^2(dt\otimes L\otimes\mathbb{P})$, see recent work by Geiss and Laukkarinen [9].
(2) Introducing a chaos decomposition of $\mathbb{L}^2(\mathbb{P})$ (see Itô [11]) and defining the operators as annihilation operators on the chaos, see [18], [27] and [9].

Geiss and Laukkarinen prove that these two approaches coincide. Solé, Utzet and Vives show that (3.10) and the definition of the difference operator on the chaos are equivalent for Lévy processes without Gaussian part. Using Proposition 5.5 of [27] and the closability of Ψ proved in Proposition 3.5 we see that the variational definition we give is equivalent to the other approaches.

All three ways of defining the derivative and difference operator provide for a proof of the duality formula. We presented two proofs in Theorem 2.4 and Proposition 3.4. An abstract proof of the duality formula on the isomorphic Fock space can be found in Proposition 4.2 of [22].

4. A characterization of infinitely divisible random measures

In this last Section we show an extension of the characterization of infinitely divisible random vectors given in Section 1 to infinitely divisible random measures.

Let $\mathbb{A}$ be a polish space, $\mathcal{A}_0$ the ring of bounded Borel sets and $\mathcal{A}$ the σ-field generated by $\mathcal{A}_0$. Define the space of all σ-finite measures on $(\mathbb{A},\mathcal{A})$ by

$$\mathbb{M} := \{\mu \ \sigma\text{-finite measure on } (\mathbb{A},\mathcal{A})\}. \tag{4.1}$$

Define the finite dimensional projections $\pi_{(A_1,\dots,A_n)} : \mathbb{M} \to \mathbb{R}_+^n$ by $\pi_{(A_1,\dots,A_n)}(\mu) = (\mu(A_1),\dots,\mu(A_n))'$, $A_i \in \mathcal{A}_0$. We equip the space $\mathbb{M}$ with the σ-field generated by the finite dimensional projections: $\mathcal{M} := \sigma(\pi_A,\ A \in \mathcal{A}_0)$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be some probability space. A random measure N on $(\mathbb{A},\mathcal{A})$ is a measurable mapping from Ω into $\mathbb{M}$.

The notion of infinite divisibility is similar to the one for random vectors. A random measure N on $(\mathbb{A},\mathcal{A})$ will be called infinitely divisible if for every $k \in \mathbb{N}$ there exist independent and identically distributed random measures $N^{(1)},\dots,N^{(k)}$ on $(\mathbb{A},\mathcal{A})$ such that

$$N \stackrel{\mathcal{L}}{=} N^{(1)} + \cdots + N^{(k)}. \tag{4.2}$$

Define $\mathbb{M}_* := \mathbb{M}\backslash\{0\}$ and $\mathcal{M}_* = \mathcal{M} \cap \mathbb{M}_*$ (here 0 means the measure that has $\mu(\mathbb{A}) = 0$). Remark that $\{0\} \in \mathcal{M}$. The Laplace transform of an infinitely divisible random measure is of the following form.

Proposition 4.1. *If N is infinitely divisible, there exist $\alpha \in \mathbb{M}$ and a σ-finite measure Γ over $(\mathbb{M}_*,\mathcal{M}_*)$ with $\int_{\mathbb{M}_*}(\mu(A)\wedge 1)\Gamma(d\mu) < \infty$ for every $A \in \mathcal{A}$ such that for all $\xi : \mathbb{A} \to \mathbb{R}_+$ we have*

$$-\log \mathbb{E}\left(e^{-\int_{\mathbb{A}}\xi(a)N(da)}\right) = \int_{\mathbb{A}} \xi(a)\alpha(da) + \int_{\mathbb{M}_*}\left(1 - e^{-\int_{\mathbb{A}}\xi(a)\mu(da)}\right)\Gamma(d\mu), \tag{4.3}$$

where $\log(0) := -\infty$.

Proof. See e.g. Theorem 6.1 in Kallenberg [14]. The integrability condition on Γ given there is actually $\int_{\mathbb{M}_*}(1-e^{-\mu(A)})\Gamma(d\mu) < \infty$. This is equivalent to our condition because

$$\frac{1}{2}q \le (1-e^{-q}) \le q,\ q \in [0,1] \quad \text{and} \quad \frac{1}{2} \le (1-e^{-q}) \le 1,\ q \in (1,\infty).$$

The existence of α and Γ can be proven by projection of the Laplace characteristics of the infinitely divisible random vectors $\pi_{(A_1,\dots,A_n)}(N)$. ◻

(α, Γ) are called characteristics of the infinitely divisible random measure N. To state the characterization theorem we are going to use the following sets of test functions. The set of elementary functions is defined by

$$\mathcal{E}_{\mathbb{A}} := \left\{ \xi : \mathbb{A} \to \mathbb{R}_+,\ \xi = \sum_{i=1}^n \xi_i 1_{A_i},\ \xi_i \in \mathbb{R}_+, A_i \in \mathcal{A}_0, n \in \mathbb{N} \right\}.$$

And the set of smooth and cylindrical functionals with compact support is

$$\mathcal{S}_{\mathbb{M}} := \{F : \mathbb{M} \to \mathbb{R}_+,\ F(\mu) = f(\mu(A_1), \dots, \mu(A_n)), f \in C_c^\infty(\mathbb{R}_+^n), A_i \in \mathcal{A}_0, n \in \mathbb{N}\}.$$

Theorem 4.2. *A random measure N is infinitely divisible with characteristics (α, Γ) if and only if for all $F \in \mathcal{S}_{\mathbb{M}}$, $\xi \in \mathcal{E}_{\mathbb{A}}$ the duality formula*

$$\begin{aligned} \mathbb{E}\left(F(N)\int_{\mathbb{A}} \xi(a)N(da)\right) &= \mathbb{E}\left(F(N)\int_{\mathbb{A}} \xi(a)\alpha(da)\right) \\ &\quad + \mathbb{E}\left(\int_{\mathbb{M}_*} F(N+\mu)\left(\int_{\mathbb{A}} \xi(a)\mu(da)\right)\Gamma(d\mu)\right). \end{aligned} \tag{4.4}$$

holds.

Proof. By Lemma 6.3 in Kallenberg [14], N is infinitely divisible if and only if $(N(A_1), \dots, N(A_n))'$ is infinitely divisible as a random vector in $\mathbb{R}_+^n$ for any $A_1, \dots, A_n \in \mathcal{A}_0$. Remark that for $\gamma \in \mathbb{R}_+^n$ we have $\langle \gamma, (N(A_1), \dots, N(A_n))' \rangle = \int_{\mathbb{A}} \xi(a)N(da)$ if $\xi = \sum_{i=1}^n \gamma_i 1_{A_i} \in \mathcal{E}_{\mathbb{A}}$. By (4.3) the Lévy measure corresponding to $(N(A_1), \dots, N(A_n))'$ is given by $\Gamma \circ \pi^{-1}_{(A_1,\dots,A_n)}$. Then we conclude using Corollary 1.4 and the linearity of (4.4) with respect to ξ. □

This result underlines once more a direct correspondence between infinitely divisible random objects and duality formulae of the type (1.3) or (2.5).

The above theorem was first proven by Kummer and Matthes [16], applying a characterization of infinitely divisible point processes proved in [15]. Nehring and Zessin [20] simplified the proof using a Poissonian representation of infinitely divisible random measures.

We now show that Theorem 4.2 implies the corresponding characterization of Poisson point processes originally given by Mecke [19] in Satz 3.1.

Corollary 4.3. *Let Λ be some σ-finite measure on $(\mathbb{A}, \mathcal{A})$ with no atoms. The random measure N is a Poisson measure with intensity Λ if and only if for all $F \in \mathcal{S}_{\mathbb{M}}$, $\xi \in \mathcal{E}_{\mathbb{A}}$*

$$\mathbb{E}\left(F(N)\int_{\mathbb{A}} \xi(a)N(da)\right) = \mathbb{E}\left(\int_{\mathbb{A}} F(N+\delta_{\{a\}})\xi(a)\Lambda(da)\right). \tag{4.5}$$

Proof. Suppose N is a Poisson measure with intensity Λ. Then N is infinitely divisible with independent increments. Denote by (α, Γ) its characteristics. Since N is a point process we have $\alpha \equiv 0$. By independence of increments the support of Γ is included in the set of degenerate integer-valued measures

$$\{\mu = n\delta_{\{a\}},\ a \in \mathbb{A},\ n \in \mathbb{N}\} = \bigcup_{n \geq 1} M_n \text{ where } M_n := \{\mu = n\delta_{\{a\}},\ a \in \mathbb{A}\},\ n \in \mathbb{N}$$

(see [14], Theorem 7.2 and Lemma 7.3). Thus Γ can be projected onto some measure $\bar{\Lambda}$ on $\mathbb{N} \times \mathbb{A}$ in the sense that for every $g : \mathbb{M} \times \mathbb{A} \to \mathbb{R}_+$ we have

$$\int_{\mathbb{M}_* \times \mathbb{A}} g(\mu, a)\mu(da)\Gamma(d\mu) = \int_{\mathbb{N} \times \mathbb{A}} g(n\delta_{\{a\}}, a)\bar{\Lambda}(dnda). \tag{4.6}$$

Putting $F \equiv 1$ in (4.4) we obtain

$$\Lambda(A) = \mathbb{E}\,(N(A)) = \int_{\mathbb{M}_*} \mu(A)\Gamma(d\mu) = \bar{\Lambda}(\mathbb{N} \times A),\ \forall A \in \mathcal{A}_0.$$

Extending (4.4) to $F = 1_{M_n}$ and $\xi = 1_A$ for $A \in \mathcal{A}_0$ and applying (4.6) to $g = 1_{M_n} 1_A$ we see that $\bar{\Lambda}(\mathbb{N} \times A) = \bar{\Lambda}(\{1\} \times A)$. Therefore

$$\int_{\mathbb{M}_* \times \mathbb{A}} g(\mu, a)\mu(da)\Gamma(d\mu) = \int_{\mathbb{A}} g(\delta_{\{a\}}, a)\Lambda(da),\ g : \mathbb{M} \times \mathbb{A} \to \mathbb{R}_+\ ,$$

which implies (4.5).

The sufficiency of the duality is due to the identification of the Laplace transform of a Poisson random measure, similar to the proof of Theorem 4.2. □

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