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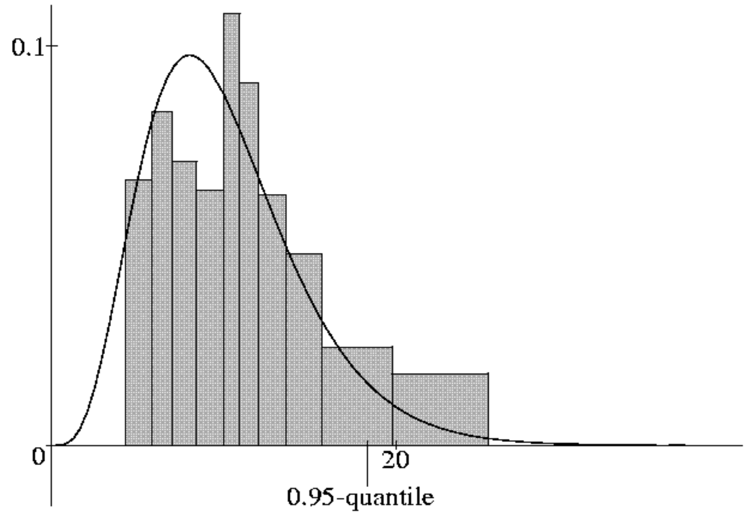

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MARTIN-DYNKIN BOUNDARIES OF THE BOSE GAS

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Abstract

The Ginibre gas is a Poisson point process defined on a space of loops related to the Feynman-Kac representation of the ideal Bose gas. Here we study thermodynamic limits of different ensembles via Martin-Dynkin boundary technique and show, in which way infinitely long loops occur. This effect is the so-called Bose-Einstein condensation.

Keywords: Martin-Dynkin boundary; Bose-Einstein condensation; Point process; Loop space; Gibbs state

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1. Intoduction

Recently, the interest in research on gases of Bosons has increased and models of different kinds have been studied, in particular their connection to cycle percolations. Sütő [17, 18] and Benfatto et all [1] considered limiting models of random permutations without interaction and in the mean field, respectively. Ueltschi [19, 20] examined lattice models on the basis of Sütő's work. In particular both, Sütő and Ueltschi, indicated in which way cycle percolation and Bose Einstein condensation are connected. Very recently, Ueltschi and Betz [2, 21] generalised the lattice model to models of random point configurations in space.

From the point of view of point processes Fichtner [8] started the investigation of the position distribution of the Bose gas and gave a characterisation in terms of its moment measures of a point process on $\mathbb{R}^d$.

The initial point of our investigations is the work of Ginibre [10] who investigated quantum gases and derived a representation of the reduced density matrices of the Bose gas in terms of Wiener measures. His results are a valuable starting point for defining a point process on a certain loop space, which will be called Ginibre gas. Our aim is to characterise limits corresponding to various specifications in the sense of Preston [15].

The paper is organised as follows: In section 2, firstly the Ginibre gas model is introduced. We then recall the notion of specifications and Martin-Dynkin boundary due to Föllmer [7] and Dynkin [5, 6] and define the filtrations which this paper is concerned with. After that, we study different examples of specifications. These are in sections 3– 5 the microcanonical, canonical and grand canonical loop ensemble. Finally, we turn second canonical ensemble in section 6, which could be more convenient from a physical point of view.

2. The setting

2.1. The Ginibre Gas

For an arbitrary integer $j \geqslant 1$ consider a continuous function $x : [0, j\beta] \mapsto \mathbb{R}^d$ with $x(0) = x(j\beta)$. The image of x in $\mathbb{R}^d$ is a *j-loop* at *inverse temperature* β. It represents j simultaneously moving particles starting at $x(k\beta)$, $k = 0, \ldots, j-1$ and changing its positions during a time intervall of length β. Hence, $x\big([k\beta, (k-1)\beta]\big)$ is the trace of a single particle or *elementary component*. Let X_j denote the space of j-loops

$$X_j := \left\{x \in C\left([0, j\beta], \mathbb{R}^d\right) : x(0) = x(j\beta)\right\}, \qquad X := \bigcup_{j \geqslant 1} X_j$$

the space of loops at the inverse temperature β. X contains continuous trajectories of multiple length of β. Each of the spaces of j-loops is endowed with the Borel topology, and we equip X with the corresponding disjoint union topology, that is the finest topology such that the canonical injections $X_j \to X$ are continuous.

Let $\mathcal{B}_0(\mathbb{R}^d)$ be the algebra of bounded Borel sets of $\mathbb{R}^d$, which is a partially ordered set when endowed with the inclusion $\big(\mathcal{B}_0(\mathbb{R}^d), \subseteq\big)$. For $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$ define the set of bounded sets of X to be

$$\mathcal{B}_0 := \mathcal{B}_0(X) = \left\{B \in \mathcal{B}(X) : B \subseteq X_\Lambda \text{ for some } \Lambda \in \mathcal{B}_0(\mathbb{R}^d)\right\}$$

where X_Λ is the set of all the loops contained in Λ,

$$X_\Lambda = \{x \in X : \operatorname{range} x \subseteq \Lambda\}.$$

In that way we speak of a loop x *contained in* some region Λ, whenever the image of the loop is fully contained in Λ, for which we write $x \subset \Lambda$; a set of loops is bounded, whenever there exists some bounded region Λ, which contains these loops. Note that if $\Lambda_1, \Lambda_2 \in \mathcal{B}_0(\mathbb{R}^d)$ are two disjoint bounded regions, then $X_{\Lambda_1} \cup X_{\Lambda_2} \subseteq X_{\Lambda_1 \cup \Lambda_2}$ without equality in general, since loops may start in one region and cross the other one.

Let ψ_β be the density of the centered normal distribution on $\mathbb{R}^d$ with covariance matrix βI and consider on $(\mathbb{R}^d)^j$ the measure

$$\bar{\rho}_j(\mathrm{d}a) = \psi_\beta(a_1 - a_0) \cdot \ldots \cdot \psi_\beta(a_{j-1} - a_{j-2}) \psi_\beta(a_0 - a_{j-1}) \mathrm{d}a_0 \ldots \mathrm{d}a_{j-1}.$$

Furthermore, let $p : X_j \to (\mathbb{R}^d)^j$ be the projection $x \mapsto \big(x(0), x(\beta), \dots, x((j-1)\beta)\big)$ and ρ_j be a measure on X_j such that $\rho_j \circ p^{-1} = \bar{\rho}_j$. Of particular interest are Brownian bridge measures or measures of a random walk bridge with normally distributed steps.

Now, sum up the measures ρ_j to get a family of measures ρ_z on X for $z \in (0,1]$,

$$\rho_z := \sum_{j \geqslant 1} \frac{z^j}{j} \rho_j. \tag{2.1}$$

The parameter z is the *fugacity*.

Lemma 1. *For any $z \in (0,1]$ and any $d \geqslant 1$, ρ_z is a σ-finite but infinite measure on X.*

Proof. Consider the projection $s : X \to \mathbb{R}^d$ on the initial point of a loop, $s : x \mapsto x(0)$, then

$$\rho_j \circ s^{-1} = \frac{1}{(2\pi\beta j)^{d/2}} \lambda$$

with λ denoting the Lebesgue measure on $\mathbb{R}^d$. This can be seen from

$$\begin{aligned}\int f(a_0) \rho_j \circ s^{-1}(\mathrm{d}a_0) &= \int f(a_0) \psi_\beta(a_1 - a_0) \cdot \ldots \cdot \psi_\beta(a_{j-1} - a_{j-2}) \\ &\qquad \times \psi_\beta(a_0 - a_{j-1}) \mathrm{d}a_0 \dots \mathrm{d}a_{j-1} \\ &= \int f(a_0) \psi_{j\beta}(a_0 - a_0) \mathrm{d}a_0 = (2\pi\beta j)^{-d/2} \int f(a_0) \mathrm{d}a_0,\end{aligned}$$

since the rhs of the first line is a convolution of normal distributions. That way we get

$$\rho_z \circ s^{-1} = (2\pi\beta)^{-d/2} g_{1+\frac{d}{2}}(z) \lambda, \tag{2.2}$$

where $g_\alpha : [0,1] \to \mathbb{R}_+ \cup \{\infty\}$ is for any $\alpha > 0$ defined as

$$g_\alpha(z) = \sum_{j \geqslant 1} \frac{z^j}{j^\alpha}. \tag{2.3}$$

Since for any $\alpha > 1$ this g_α is finite on $[0,1]$, we get the claim.

Observe that for $0 < \alpha \leqslant 1$ the series g_α is only finite on $[0,1)$ without the right boundary. Furthermore g_α is strictly increasing and continuous whenever it is finite.

A *configuration* μ is an element of the phase space $\mathcal{M}^{\cdot\cdot}(X)$ of *locally finite point measures* on X. Particularly, μ is a collection of loops such that any bounded region Λ

contains only a finite number of loops. For simplicity write $x \in \mu$ whenever $\mu = \delta_x + \nu$ for some $\nu \in \mathcal{M}^{\cdot\cdot}(X)$. Every configuration μ has a representation $\mu = \sum_{x \in \mu} \delta_x$. The following restrictions of a configuration μ will be used throughout this article:

$$\mu_j := \sum_{x \in \mu : x \in X_j} \delta_x \quad \text{is the configuration of } j\text{-loops of } \mu,$$

$$\mu_{(\Lambda)} := \sum_{x \in \mu : x \subset \Lambda} \delta_x \quad \text{the restriction of } \mu \text{ to loops which stay in } \Lambda.$$

A configuration μ is said to be simple, if for all $x \in X$ the relation $\mu(\{x\}) \leqslant 1$ is satisfied, i.e. at each site x there is at most one loop in the configuration μ. The set of *locally finite simple point measures* is denoted by $\mathcal{M}^{\cdot}(X)$.

A probability measure on $\mathcal{M}^{\cdot\cdot}(X)$ and on $\mathcal{M}^{\cdot}(X)$ is a point process and a simple point process, respectively. Of special interest are the Poisson point processes $\mathbf{P}_{\rho_z}$, which are even concentrated on simple configurations of loops $\mathcal{M}^{\cdot}(X)$, since ρ_z has no fixed atoms. If $\mathbf{P}_{\rho_z,\Lambda}$ denotes the restriction of $\mathbf{P}_{\rho_z}$ to $\mathcal{M}^{\cdot}(X_\Lambda)$ and $\rho_{z,\Lambda}$ the restriction of ρ_z to X_Λ, respectively, then $\mathbf{P}_{\rho_z,\Lambda} = \mathbf{P}_{\rho_{z,\Lambda}}$.

Definition 1. The *Ginibre gas* with fugacity $z \in (0,1]$ is the Poisson process $\mathbf{P}_{\rho_z}$ on $\mathcal{M}^{\cdot\cdot}(X)$.

A *composition* η is a finite point measure on $\mathbb{N}^*$, i.e. an element of $\mathcal{M}^{\cdot\cdot}_f(\mathbb{N}^*)$. Observe that there is a canonical partial order on $\mathcal{M}^{\cdot\cdot}_f(\mathbb{N}^*)$,

$$\gamma \leqslant \eta :\Leftrightarrow \gamma(j) \leqslant \eta(j)$$

for all $j \in \mathbb{N}^*$. For $\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}^*)$ let denote η^* the element in $\mathcal{M}^{\cdot\cdot}_f(\mathbb{N}^*)$ which represents the support of η, namely

$$\eta^* = \sum_{j \in \mathbb{N}^* : \eta(j) \geqslant 1} \delta_j \quad \text{for } \eta = \sum_{j \geqslant 1} \eta(j) \delta_j.$$

2.2. Specifications and Martin-Dynkin boundary

Specifications were studied intensely by Preston [15], who contributed the notion of microcanonical and canonical ensemble in this context. A further tessera is the work of Föllmer [7], who extended the Martin boundary technique to specifications. As a consequence, a characterisation of Poisson processes by their local specifications was given by Nguyen and Zessin [13].

Consider the measurable space $(\mathcal{M}^{\cdot}(X), \mathcal{E})$ of simple point measures on X and fix a with respect to $(\mathcal{B}_0(\mathbb{R}^d), \subseteq)$ decresing family of sub-σ-fields $\mathbb{E} = \{\mathcal{E}_\Lambda\}_\Lambda$ of $\mathcal{E}$. A probability kernel π' is a mapping $\mathcal{M}^{\cdot}(X) \times \mathcal{E} \to \mathbb{R}_{\geqslant 0}$ with the properties

1. $\forall \mu \in \mathcal{M}^{\cdot}(X) : \pi'(\mu, \cdot\,)$ is a measure,

2. $\forall E \in \mathcal{E} : \pi'(\,\cdot\,, E)$ is $\mathcal{E}$-measurable.

An $\mathbb{E}$-*specification* $\pi = \{\pi_\Lambda\}_\Lambda$ is a collection of probability kernels on $\mathcal{M}^{\cdot}(X) \times \mathcal{E}$ such that

1. $\forall A \in \mathcal{E} : \pi_\Lambda(\,\cdot\,, A)$ is $\mathcal{E}_\Lambda$-measurable,

2. $\forall A \in \mathcal{E}_\Lambda : \pi_\Lambda(\,\cdot\,, A) = 1_A$,

3. $\forall \mu \in \mathcal{M}^{\cdot}(X) : \pi_\Lambda(\mu, \mathcal{M}^{\cdot}(X)) \in \{0, 1\}$,

4. $\forall \Lambda \subseteq \Lambda' : \pi_{\Lambda'} = \pi_{\Lambda'}\pi_\Lambda$.

A *Gibbs state* or a *phase* with respect to the $\mathbb{E}$-specification is a probability measure $\mathbb{P}$ on $\mathcal{M}^{\cdot}(X)$ such that its conditional expectations given the σ-algebras in $\mathbb{E}$ is given by the corresponding kernel, $\mathbf{E}_{\mathbb{P}}\big(\,\cdot\,|\mathcal{E}_\Lambda\big)(\mu) = \pi_\Lambda(\mu, \cdot\,)$. Let $C = C(\pi)$ denote the set of those phases. If C contains more than one phase, a *phase transition* occurs.

To define the Martin-Dynkin boundary fix a countable base $(\Lambda_k)_k$ of $\mathcal{B}_0(\mathbb{R}^d)$ and a polish topology on $\mathcal{M}^{\cdot}(X)$ compatible with $\{\mathcal{E}_{\Lambda_k}\}_k$. One obtains a polish topology on the set of probability measures on $\mathcal{M}^{\cdot}(X)$. Furthermore, let $C_\infty = C_\infty(\pi)$ be the set of all limits

$$\lim_{k\to\infty} \pi_{\Lambda_k}(\mu_k, \cdot\,) \tag{2.4}$$

for sequences $(\mu_k)_k \subset \mathcal{M}^{\cdot}(X)$. Now C_∞ does not depend on the choice of the family $(\Lambda_k)_k$ and is complete in the set of probability measures on $\mathcal{M}^{\cdot}(X)$, hence polish with the induced Borel field $\mathcal{C}_\infty$. The *Martin-Dynkin boundary* associated to π is the measurable space $(C_\infty, \mathcal{C}_\infty)$.

Finally, let Q_μ for $\mu \in \mathcal{M}^{\cdot}(X)$ be the limit

$$Q_\mu := \lim_{k\to\infty} \pi_{\Lambda_k}(\mu, \cdot\,), \tag{2.5}$$

then the *essential part* Δ of the Martin-Dynkin boundary is the set of those $\mathbb{P} \in C_\infty \cap C$, for which $Q_\cdot = \mathbb{P}$ holds $\mathbb{P}$-a.s, i.e.

$$Q_\mu(A) = \mathbb{P}(A) \qquad \text{for } \mathbb{P}\text{-a.e. } \mu. \tag{2.6}$$

2.3. Counting Loops

The purpose of this subsection is to define filtrations on $\mathcal{M}^{\cdot}(X)$, that is to define how to count loops. As already pointed out, $x \subset \Lambda$ for some bounded region Λ and some loop $x \in X$ iff x is completely contained in Λ. Consequently, the loop x is outside Λ iff x leaves Λ at least once. Define a collection of counting variables $\{n_\Lambda\}_{\Lambda \in \mathcal{B}_0(\mathbb{R}^d)}$, each n_Λ counting the number of loops of each kind in some region Λ

$$n_\Lambda : \mathcal{M}^{\cdot}(X) \to \mathcal{M}_f^{\cdot\cdot}(\mathbb{N}^*), \qquad n_\Lambda\mu := \sum_{j \geqslant 1} |\mu_{(\Lambda),j}|\delta_j, \tag{2.7}$$

where $|\nu| := \nu(X)$ is the total mass of ν of the point measure ν. $n_\Lambda\mu$ is indeed an almost surely finite measure under $\mathbf{P}_{\rho_z}$, since $\mathbf{P}_{\rho_z}$ is locally finite and hence $\mu_{(\Lambda)}(X) < \infty$ almost surely for any bounded region Λ. From the definition immediatly follows that $n_\Lambda\mu \leqslant n_{\Lambda'}\mu$ for each configuration μ and bounded regions $\Lambda \subseteq \Lambda'$. Therfore, we can define spatial increments, that is for $\Lambda, \Lambda' \in \mathcal{B}_0(\mathbb{R}^d)$ with $\Lambda \subseteq \Lambda'$

$$n_{\Lambda',\Lambda} : \mathcal{M}^{\cdot}(X) \to \mathcal{M}_f^{\cdot\cdot}(\mathbb{N}^*), \qquad n_{\Lambda',\Lambda} := n_{\Lambda'} - n_\Lambda.$$

Now we are able to give the definition of the filtration of the outside events $\mathbb{E} = \{\mathcal{E}_\Lambda\}_\Lambda$,

$$\mathcal{E}_\Lambda = \sigma\Big(\big\{n_{\Lambda',\Lambda} = \eta | \Lambda \supseteq \Lambda' \in \mathcal{B}_0(\mathbb{R}^d), \eta \in \mathcal{M}_f^{\cdot\cdot}(\mathbb{N}^*)\big\}\Big), \tag{2.8}$$

that is the smallest σ-algebra such that the increments of the region Λ are measurable. In keeping the terminology of Preston, the phases corresponding to $\mathbb{E}$ will form the *grand canonical loop ensemble*.

Adding more detailled information about the interior leads to the filtration $\mathbb{F} = \{\mathcal{F}_\Lambda\}_\Lambda$,

$$\mathcal{F}_\Lambda = \mathcal{E}_\Lambda \vee \sigma\Big(\big\{n_\Lambda = \eta | \eta \in \mathcal{M}_f^{\cdot\cdot}(\mathbb{N}^*)\big\}\Big), \tag{2.9}$$

which is associated to the *microcanonical loop ensemble.*

For a configuration $\mu \in \mathcal{M}^{\cdot}(X)$ let $c_\Lambda\mu = \sum_{j \geqslant 1} n_\Lambda\mu(j)$ be the total number of loops inside Λ and

$$\mathcal{G}_\Lambda = \mathcal{E}_\Lambda \vee \sigma\Big(\big\{c_\Lambda = k | k \in \mathbb{N}^*\big\}\Big), \tag{2.10}$$

the $\mathbb{G} = \{\mathcal{G}_\Lambda\}_\Lambda$ defines the *canonical loop ensemble*. Similiar to n_Λ, we have for $\Lambda \subseteq \Lambda'$, $c_\Lambda \leqslant c_{\Lambda'}$ and $c_{\Lambda',\Lambda} = c_{\Lambda'} - c_\Lambda$ is $\mathcal{E}_\Lambda$-measurable.

Finally, we are interested in what happens if we give weights to loops of different lengths, in particular we consider the counting variable

$$N_\Lambda : \mathcal{M}^\cdot(X) \to \mathbb{N}^*, \quad N_\Lambda\mu = \sum_{j \geqslant 1} j n_\Lambda \mu(j), \tag{2.11}$$

which counts the number of elementary components of the loops inside Λ. It is clear that N_Λ fulfills the same monotonicity and measurablility properties of the increments as c_Λ. Let

$$\mathcal{H}_\Lambda = \mathcal{E}_\Lambda \vee \sigma\Big(\big\{N_\Lambda = k | k \in \mathbb{N}^*\big\}\Big) \tag{2.12}$$

and call the corresponding ensemble $\mathbb{H} = \{\mathcal{H}_\Lambda\}_\Lambda$ *canonical ensemble.*

In the following sections specifications with respect to these Filtrations are going to be discussed: In section 3 the microcanonical loop ensemble $\mathbb{F}$, in section 4 the canonical loop ensemble $\mathbb{G}$, in section 5 the grand canonical loop ensemble $\mathbb{E}$ and finally in section 6 the canonical ensemble $\mathbb{H}$.

3. The microcanonical loop ensemble

In this section the specification for the filtration $\mathbb{F} = \{\mathcal{F}_\Lambda\}_\Lambda$ is discussed. As an intermediate step, a representation of the Poisson process $\mathbf{P}_{\rho_z}$ conditioned on certain events derived. Afterwards we turn to the Martin boundary technique.

Fix a fugacity $z \in (0,1]$. For each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, n_Λ maps the simple Poisson process $\mathbf{P}_{\rho_z}$ on $\mathcal{M}^\cdot(X)$ into a finite Poisson process $\mathbf{P}_{\tau_{z,\Lambda}}$ on $\mathcal{M}^{\cdot\cdot}_f(\mathbb{N}^*)$ with intensity measure $\tau_{z,\Lambda}$ given by

$$\tau_{z,\Lambda}(j) = \frac{z^j}{j}\rho_j(X_\Lambda). \tag{3.1}$$

Indeed,

$$\begin{aligned}\mathbf{P}_{\tau_{z,\Lambda}}(\eta) = \mathbf{P}_{\rho_z}(n_\Lambda = \eta) = \exp(-\rho_z(X_\Lambda)) \prod_{j \in \eta^*} \frac{z^{j\eta(j)}\rho_j(X_\Lambda)^{\eta(j)}}{j^{\eta(j)}\eta(j)!} \\ = \exp(-\tau_{z,\Lambda}(\mathbb{N}^*)) \prod_{j \in \eta^*} \frac{\tau_{z,\Lambda}(j)^{\eta(j)}}{\eta(j)!},\end{aligned}$$

since $\rho_z(X_\Lambda) = \tau_{z,\Lambda}(\mathbb{N}^*)$. Let $\tilde{\rho}_{z,\Lambda}$ denote the normalisation of the finite measure $\rho_{z,\Lambda}$; then we define the η-convolution $P^\eta_{\rho_z,\Lambda}$ of the probability measures $\tilde{\rho}_{j,\Lambda}, j \geqslant 1$ for some $\eta \in \ddot{\mathcal{M}}_f(\mathbb{N}^*)$ as

$$P^\eta_{\rho,\Lambda} := \tilde{\rho}^\eta_\Lambda = \mathop{\circledast}_{j\in\eta^*} \tilde{\rho}^{*\eta(j)}_{j,\Lambda}, \tag{3.2}$$

which represents the superposition of loops of a given length j according to the number $\eta(j)$. The $\mathbf{P}_{\tau_{z,\Lambda}}$-combination of that convolution is

$$\tilde{P}_{\rho_z,\Lambda} = \sum_{\eta\in\ddot{\mathcal{M}}_f(\mathbb{N}^*)} \mathbf{P}_{\tau_{z,\Lambda}}(\eta) P^\eta_{\rho,\Lambda}. \tag{3.3}$$

Thus $\tilde{P}_{\rho_z,\Lambda}$ is given by a two step mechanism: At first choose a composition $\eta \in \ddot{\mathcal{M}}_f(\mathbb{N}^*)$ defining the number of loops in some bounded region Λ and then realise a configuration according to this composition. An effect is that the fugacity z does only affect the choice of the composition and not $P^\eta_{\rho,\Lambda}$.

These probability measures are closely related to the Ginibre gas restricted to bounded sets Λ, $\mathbf{P}_{\rho_z,\Lambda}$.

Lemma 2. $\mathbf{P}_{\rho_z,\Lambda}(A|n_\Lambda = \eta) = P^\eta_{\rho,\Lambda}(A)$.

Proof. Since we have exactly $K = \sum_j \eta(j)$ loops in Λ and if we order them in increasing length, we obtain

$$\begin{aligned}
\mathbf{P}_{\rho_z,\Lambda}(A \cap n_\Lambda = \eta) &= \exp(-\rho_z(X_\Lambda)) \sum_{n\geqslant 0} \frac{1}{n!} \times \\
&\quad \times \int \cdots \int 1_A 1_{n_\Lambda=\eta}(\delta_{x_1} + \ldots + \delta_{x_n}) \rho_{z,\Lambda}(dx_1) \cdots \rho_{z,\Lambda}(dx_n) \\
&= \exp(-\rho_z(X_\Lambda)) \frac{1}{K!} \times \\
&\quad \times \int \cdots \int 1_A 1_{n_\Lambda=\eta}(\delta_{x_1} + \ldots + \delta_{x_N}) \rho_{z,\Lambda}(dx_1) \cdots \rho_{z,\Lambda}(dx_N) \\
&= \exp(-\rho_z(X_\Lambda)) \prod_{j\in\eta^*} \frac{z^{j\eta(j)} \rho_j(X_\Lambda)^{\eta(j)}}{j^{\eta(j)} \eta(j)!} \times \\
&\quad \times \int \cdots \int 1_A 1_{n_\Lambda=\eta}(\delta_{x_1} + \ldots + \delta_{x_N}) \tilde{\rho}^\eta_{z,\Lambda}(dx_1, \ldots, dx_N) \\
&= \exp(-\rho_z(X_\Lambda)) \prod_{j\in\eta^*} \frac{z^{j\eta(j)} \rho_j(X_\Lambda)^{\eta(j)}}{j^{\eta(j)} \eta(j)!} P^\eta_{\rho,\Lambda}(A \cap \{n_\Lambda = \eta\}),
\end{aligned}$$

Finally, setting $A = \mathcal{M}^{\cdot}(X)$ we get the normalisation constant and using the fact that $P^{\eta}_{\rho,\Lambda}(n_\Lambda = \eta) = 1$ we get the assertion.

Corollary 1. $\tilde{P}_{\rho_z,\Lambda} = \mathbf{P}_{\rho_z,\Lambda}$.

Proof. This follows immediatly since

$$\begin{aligned}\tilde{P}_{\rho_z,\Lambda}(\varphi) &= \sum_{\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}^*)} \mathbf{P}_{\tau_z,\Lambda}(\eta) P^{\eta}_{\rho,\Lambda}(\varphi) \\ &= \sum_{\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}^*)} \mathbf{P}_{\tau_z,\Lambda}(\eta) \mathbf{P}_{\rho,\Lambda}(\varphi | n_\Lambda = \eta) = \mathbf{P}_{\rho_z,\Lambda}(\varphi)\end{aligned}$$

for any measurable, nonnegative function φ.

That way we found a new representation of $\mathbf{P}_{\rho_z}$. Define on $X \times \mathcal{M}^{\cdot}(X)$

$$\begin{aligned}\pi^{\mathbb{F}}_{\Lambda}(\mu,\varphi) &= \mathbf{P}_{\rho_z,\Lambda}\left(\varphi\left(\,\cdot\, + \mu^{(\Lambda^c)}\right)\Big| n_\Lambda = n_\Lambda \mu\right) \\ &= P^{n_\Lambda \mu}_{\rho_z,\Lambda}\left(\varphi\left(\,\cdot\, + \mu^{(\Lambda^c)}\right)\right)\end{aligned}$$

and observe that $\pi^{\mathbb{F}}_{\Lambda}$ is a probability kernel. $\pi^{\mathbb{F}} = \{\pi^{\mathbb{F}}_{\Lambda}\}_{\Lambda}$ is indeed an $\mathbb{F}$-specification, which follows from the conditioning procedure of the Poisson process. By definition, $\mathbf{P}_{\rho_z} \in C(\pi^{\mathbb{F}})$, hence the set of phases $C(\pi^{\mathbb{F}})$ associated to $\pi^{\mathbb{F}}$ is not empty.

Fix an expanding sequence $(\Lambda_k)_k \subset \mathcal{B}_0(\mathbb{R}^d)$ with $\bigcup_{k \geqslant 1} \Lambda_k = \mathbb{R}^d$, let $\mathcal{F}_\infty = \bigcap_k \mathcal{F}_{\Lambda_k}$ be the tail-field and $\mathbb{P} \in C_\infty(\pi^{\mathbb{F}})$, then for $\varphi \in L^1(\mathbb{P})$,

$$\mathbb{P}(\varphi | \mathcal{F}_\infty) = \lim_{k \to \infty} \pi^{\mathbb{F}}_{\Lambda_k}(\,\cdot\,,\varphi) \qquad \mathbb{P}\text{-a.s.}. \tag{3.4}$$

Therefore, the limits $Q_\mu = \lim_k \pi^{\mathbb{F}}_{\Lambda_k}(\mu,\,\cdot\,)$ exist $\mathbb{P}$-a.s. in μ and are by construction contained in the Martin-Dynkin boundary $C_\infty(\pi^{\mathbb{F}})$.

Define the j-loop density of some configuration μ in Λ_k as

$$Y_{j,k}(\mu) = \frac{n_{\Lambda_k}\mu(j)}{\rho_j(X_{\Lambda_k})}, \tag{3.5}$$

let Y_j be its limit as $k \to \infty$ provided that the limit exists and write $Y = (Y_j)_j$. Let M be the set of all those $\mu \in \mathcal{M}^{\cdot}(X)$, such that Y_j exists for each $j \in \mathbb{N}^*$ and is finite. Note that instead of the volume of Λ_k the volume of X_{Λ_k} is used to define the density. However, it has been shown in lemma 1 that, asymptotically, their volume is the same up to a constant given by $(2\pi\beta j)^{-d/2}$. For notationally purpose we write for the convex y-combination

$$y \bullet \rho := \sum_{j\geqslant 1} y_j\rho_j \tag{3.6}$$

for any sequence $y = (y_j)_j$ of nonnegative real numbers.

Proposition 1. *Let $f : X \to \mathbb{R}$ be nonnegative and measurable with bounded support, $\mu \in M$ and $Y(\mu) \bullet \rho\Big(\exp(-f) - 1\Big)$ convergent. Then for any $\varphi \in L^1(\mathbb{P})$*

$$\mathbb{P}(\varphi|\mathcal{F}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{F}}_{\Lambda_k}(\,\cdot\,, \varphi) = \mathbf{P}_{Y\bullet\rho}(\varphi) \qquad \mathbb{P}\text{-}a.s. \tag{3.7}$$

Proof. At first we show that the following limit exists and equals

$$\lim_{k\to\infty} \widehat{\pi^{\mathbb{F}}_{\Lambda_k}(\mu, \cdot\,)}(if) = \widehat{Q_\mu}(if) = \exp\Big(-\sum_{j\geqslant 1} Y_j(\mu)\rho_j\Big(1 - \exp(-f)\Big)\Big). \tag{3.8}$$

Let $\mathcal{N}$ be the set of "good configurations",

$$\mathcal{N} = \{\mu \in \mathcal{M}^{\cdot}(X) : \lim_{k\to\infty} \pi^{\mathbb{F}}_{\Lambda_k}(\mu, \cdot\,) \text{ exists}\}.$$

Let $f : X \to \mathbb{R}$ be nonnegative and measurable with bounded support and such that $\int\Big(\exp(-f) - 1\Big)d\rho \neq 0$, then there exists k_0 such that $\operatorname{supp} f \subset \Lambda_k$ for $k \geqslant k_0$. Provided $\mu \in \mathcal{N}$, we get

$$\begin{aligned}
\widehat{\pi^{\mathbb{F}}_{\Lambda}(\mu, \cdot\,)}(if) &= \int \exp(-\int f \mathrm{d}\nu)\pi^{\mathbb{F}}_{\Lambda_k}(\mu, \mathrm{d}\nu) = \int \exp(-\int f \mathrm{d}\nu) P^{n_{\Lambda_k}\mu}_{\rho_{\Lambda_k}}(\mathrm{d}\nu)\\
&= \int \exp\Big(-(\delta_{x_1} + \ldots + \delta_{x_{n_{\Lambda_k}\mu(\mathbb{N}*)}})f\Big)\tilde{\rho}^{n_{\Lambda_k}\mu}_{\Lambda_k}(\mathrm{d}x_1, \ldots, \mathrm{d}x_{n_{\Lambda_k}\mu(\mathbb{N}*)})\\
&= \prod_{j\in(n_{\Lambda_k}\mu)*} \Big[\int \exp\big(-f(x)\big)\tilde{\rho}_{j,\Lambda_k}(\mathrm{d}x)\Big]^{n_{\Lambda_k}\mu(j)}\\
&= \prod_{j\in(n_{\Lambda_k}\mu)*} \Big[1 + \tilde{\rho}_{j,\Lambda_k}\Big(\exp\big(-f(x)\big) - 1\Big)\Big]^{n_{\Lambda_k}\mu(j)}\\
&= \prod_{j\in(n_{\Lambda_k}\mu)*} \left\{\left[1 + \frac{\rho_j\Big(\exp\big(-f(x)\big) - 1\Big)}{\rho_j(X_{\Lambda_k})}\right]^{\rho_j(X_{\Lambda_k})}\right\}^{\frac{n_{\Lambda_k}\mu(j)}{\rho_j(X_{\Lambda_k})}}.
\end{aligned}$$

Use $\operatorname{supp} f = \operatorname{supp}\Big(\exp(-f) - 1\Big)$ to obtain the last line. Since the lhs converges by assumption, so the rhs does. Therefore $\mathcal{N} \subseteq M$. Vice versa, if $\mu \in M$, the rhs converges and so the lhs does, hence $M \subseteq \mathcal{N}$ and (3.8) is shown.

Immediatly we get that Q_μ is a Poisson process with intensity measure $Y(\mu) \bullet \rho$, which is the claim.

In case of divergence of the series, $\widehat{Q_\mu}(if) = 0$ whenever $f \neq 0$, and there is no suitable limit for Q_μ. Thus we have shown that the only possible limits for Q_μ are Poisson processes.

For $\mathcal{F}_\infty$-measurable φ the proposition 1 implies $\mathbb{P}\big(\varphi\, f(Q.)\big) = \mathbb{P}\big(\varphi\, \mathbf{P}_{Y\bullet\rho}\big(f(Q.)\big)\big)$ and we get

$$\mathbf{P}_{Y(\mu)\bullet\rho}(Q. = Q_\mu) = 1 \qquad \mathbb{P}\text{-a.s.}$$

Particularly, $Y_j = Y_j(\mu)$ $\mathbb{P}$-a.s. for each j.

Let $\Delta^{\mathbb{F}} = \{P \in C_\infty \cap C | Q. = P \quad P\text{-a.s.}\}$ be the essential part of the Martin-Dynkin boundary associated to $\mathbb{F}$. Since $\Delta^{\mathbb{F}}$ is a Borel set in C_∞, $\Delta^{\mathbb{F}}$ is a Borel space itsself with the induced field. For a phase $\mathbb{P} \in C$ define a probability measure $V^{\mathbb{P}}$ on $\Delta^{\mathbb{F}}$ as

$$V^{\mathbb{P}}(A) = \mathbb{P}(Q. \in A),$$

hence by conditioning

$$\mathbb{P}(\varphi) = \mathbb{P}(Q.(\varphi)) = \int_{\Delta^{\mathbb{F}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P)$$

can be written as a Cox process. Vice versa, any probability measure V on $\Delta^{\mathbb{F}}$ induces a phase $P \in C$.

Theorem 1. *The essential part of the Martin-Dynkin boundary of $\pi^{\mathbb{F}}$ consists of all Poisson processes with intensity measure $y \bullet \rho$ for nonnegative sequences $y = (y_j)_j$ such that $y \bullet \rho$ is a σ-finite measure on X,*

$$\Delta^{\mathbb{F}} = \{\mathbf{P}_{y\bullet\rho} | y \bullet \rho \ \sigma\text{-}\mathit{finite}\}.$$

Proof. Let $y \bullet \rho$ be σ-finite. As already seen, $\mathbf{P}_{y\bullet\rho} \in C(\pi^{\mathbb{F}})$, and clearly

$$Q_\mu = \mathbf{P}_{y\bullet\rho} \quad \mathbf{P}_{y\bullet\rho}\text{-a.s.}$$

For arbitrary $\mathbb{P} \in \Delta^{\mathbb{F}}$ we have

$$\int_{\Delta^{\mathbb{F}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P) = \mathbb{P}(\varphi) = Q.(\varphi) \qquad \mathbb{P}\text{-a.s.}$$

This implies $V^{\mathbb{P}} = \delta_{\mathbf{P}_{y\bullet\rho}}$ for some σ-finite intensity measure $y \bullet \rho$.

Remark 1. Observe that the fugacity z played only a minor role in the analysis of the microcanonical ensemble.

4. The canonical loop ensemble

In the previous section we conditioned on the different types of loops; now we drop this distinguishing feature and consider the total number of loops. This means to count like c_Λ. Intuitively, this means to forget the superposition of the different Poisson processes on each space of j-loops. Since the reasoning generally is the same as in the previous section, some details are left out, whenever already given. Troughout this section the fugacity z remains fixed.

Lemma 3. *Let* $B_k = \{\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}^*) : \sum \eta(j) = k\}$ *the set of compositions of mass* k, *then*

$$\mathbf{P}_{\tau_z,\Lambda}(B_k) = \sum_{\eta \in B_k} \mathbf{P}_{\tau_z,\Lambda}(\eta) = \frac{\rho_z(X_\Lambda)^k}{k!} \exp\big(\rho_z(X_\Lambda)\big)$$

Proof.

$$\mathbf{P}_{\tau_z,\Lambda}(B_k) = \mathbf{P}_{\rho_z,\Lambda}(c_\Lambda = k).$$

Since c_Λ is the sum of independently Poisson distributed random variables, c_Λ is Poisson distributed itself with the given intensity.

From the decomposition of $\mathbf{P}_{\rho_z,\Lambda}$ in Corollary 1 immediatly follows

$$\mathbf{P}_{\rho_z,\Lambda}(\varphi | c_\Lambda = c_\Lambda \mu) = \Big(\sum_{\eta \in B_{c_\Lambda \mu}} \mathbf{P}_{\tau_z,\Lambda}(\eta)\Big)^{-1} \sum_{\eta \in B_{c_\Lambda \mu}} \mathbf{P}_{\tau_z,\Lambda}(\eta) P^\eta_{\rho,\Lambda}(\varphi) \tag{4.1}$$

for any measurable function φ on X_Λ, which again emphasises the two step mechanism: At first choose a composition according to some law and then realise the loops according to the given composition.

Let the kernel be

$$\pi^{\mathbb{G}}_\Lambda(\mu, \varphi) = \mathbf{P}_{\rho_z,\Lambda}\bigg(\varphi\Big(\,\cdot\, + \mu^{(\Lambda^c)}\Big)\Big| c_\Lambda = c_\Lambda \mu\bigg). \tag{4.2}$$

It follows that $\pi^{\mathbb{G}} = \{\pi^{\mathbb{G}}_\Lambda\}_\Lambda$ is an $\mathbb{G}$-specification. Again, $\mathbf{P}_{\rho_z} \in C(\pi^{\mathbb{G}})$, hence $C(\pi^{\mathbb{G}})$ is not empty. Fix an expanding sequence $(\Lambda_k)_k \subset \mathcal{B}_0(\mathbb{R}^d)$ with $\bigcup_{k\geqslant 1} \Lambda_k = \mathbb{R}^d$, let $\mathcal{G}_\infty = \bigcap_k \mathcal{G}_{\Lambda_k}$ be the tail-field and $\mathbb{P} \in C_\infty(\pi^{\mathbb{G}})$, then for $\varphi \in L^1(\mathbb{P})$,

$$\mathbb{P}(\varphi | \mathcal{G}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{G}}_{\Lambda_k}(\,\cdot\,, \varphi) \qquad \mathbb{P}\text{-a.s.}. \tag{4.3}$$

Therefore the limits

$$Q_\mu = \lim_{k\to\infty} \pi^{\mathbb{G}}_{\Lambda_k}(\mu, \cdot\,) \tag{4.4}$$

exist $\mathbb{P}$-a.s. in μ and are by construction contained in the Martin-Dynkin boundary $C_\infty(\pi^{\mathbb{G}})$ in case of existence.

Let the loop density of a configuration μ in Λ_k be

$$W_k(\mu) = \frac{c_{\Lambda_k}\mu}{\rho_z(X_{\Lambda_k})}, \tag{4.5}$$

and let W be its limit as $k \to \infty$ provided that the limit exists. Let M be the set of all those $\mu \in \mathcal{M}^\cdot(X)$, such that W exists.

Proposition 2. *Let $f : X \to \mathbb{R}$ be nonnegative and measurable with bounded support and $W(\mu) < \infty$. Then for any $\varphi \in L^1(\mathbb{P})$*

$$\mathbb{P}(\varphi|\mathcal{G}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{G}}_{\Lambda_k}(\,\cdot\,, \varphi) = \mathbf{P}_{W\rho_z}(\varphi) \qquad \mathbb{P}\text{-}a.s.. \tag{4.6}$$

Proof. Essentially the arguments as in the previous section apply,

$$\widehat{\pi^{\mathbb{G}}_{\Lambda_k}(\mu, \cdot\,)}(if) = \frac{\rho_z(\exp(-f))^{c_{\Lambda_k}\mu}}{\rho_z(X_\Lambda)^{c_{\Lambda_k}\mu}} = \left\{\left[1 + \frac{\rho_z\big(\exp(-f(x)) - 1\big)}{\rho_z(X_{\Lambda_k})}\right]^{\rho_z(X_{\Lambda_k})}\right\}^{\frac{c_{\Lambda_k}\mu}{\rho_z(X_{\Lambda_k})}}$$
$$\to \exp\Big(-W(\mu)\rho_z\big(1 - \exp(-f)\big)\Big).$$

Hence we get

$$\widehat{Q_\mu}(if) = \exp\Bigg(-W(\mu)\rho_z\Big(1 - \exp(-f)\Big)\Bigg),$$

that is that Q_μ is a Poisson process with intensity measure $W(\mu)\rho_z$.

Similiar to the microcanonical case, if $W(\mu)$ is not finite, $\widehat{Q_\mu}(if) = 0$ whenever $f \neq 0$, and there is no suitable limit for Q_μ. Furthermore, the possible limits Q_μ are Poisson processes.

Since this implies for $\mathcal{G}_\infty$-measurable φ, $\mathbb{P}(\varphi\, f(Q.)) = \mathbb{P}(\varphi\, \mathbf{P}_{W\rho_z}(f(Q.)))$ one gets

$$\mathbf{P}_{W(\mu)\rho_z}(Q. = Q_\mu) = 1 \qquad \mathbb{P}\text{-a.s.}$$

Particularly $W = W(\mu)$ $\mathbb{P}$-a.s. holds.

Let $\Delta^{\mathbb{G}} = \{P \in C_\infty \cap C | Q_. = P \quad P\text{-a.s.}\}$ be the essential part of the Martin-Dynkin boundary. For $\mathbb{P} \in C$ define a probability measure $V^{\mathbb{P}}$ on $\Delta^{\mathbb{G}}$ as

$$V^{\mathbb{P}}(A) = \mathbb{P}(Q_. \in A),$$

hence

$$\mathbb{P}(\varphi) = \mathbb{P}(Q_.(\varphi)) = \int_{\Delta^{\mathbb{G}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P)$$

can be written as a mixed Poisson process. Vice versa, any probability measure V on $\Delta^{\mathbb{G}}$ induces a phase $P \in C$.

Theorem 2. *The essential part of the Martin-Dynkin boundary of $\pi^{\mathbb{G}}$ consists of all Poisson processes with intensity measure $w\rho_z$ for any positive real number w,*

$$\Delta^{\mathbb{G}} = \{\mathbf{P}_{w\rho_z} | w > 0\}.$$

Proof. If w is a positive real number, $w\rho_z$ is a σ-finite measure on X. As already seen, $\mathbf{P}_{w\rho_z} \in C(\pi^{\mathbb{G}})$, and clearly $Q_\mu = \mathbf{P}_{w\rho_z} \quad \mathbf{P}_{w\rho_z}$-a.s. For arbitrary $\mathbb{P} \in \Delta^{\mathbb{F}}$ we have

$$\int_{\Delta^{\mathbb{G}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P) = \mathbb{P}(\varphi) = Q_.(\varphi) \qquad \mathbb{P}\text{-a.s.}$$

This implies $V^{\mathbb{P}} = \delta_{\mathbf{P}_{w\rho_z}}$ for some σ-finite intensity measure $w\rho_z$.

Remark 2. It is remarkable that in the microcanonical case any fugacity z leads to the same set of Gibbs states, where in the canonical loop case these Gibbs states depend on this parameter. Essentially a similiar result for Poisson processes on $\mathbb{R}^d$ can already be found in [13].

5. The grand canonical loop ensemble

This last ensemble completes the considerations about loop ensembles to the last case, when we do not condition on a number of loops of a given configuration. For that, define the kernel as follows

$$\pi_\Lambda^{\mathbb{E}}(\mu, \varphi) = \mathbf{P}_{\rho_z, \Lambda}\Big(\varphi\Big(\cdot + \mu^{(\Lambda^c)}\Big)\Big). \tag{5.1}$$

Similiar to the previous sections $\pi^{\mathbb{E}} = \{\pi_\Lambda^{\mathbb{E}}\}_\Lambda$ is an $\mathbb{E}$-specification.
Fix an expanding sequence $(\Lambda_k)_k \subset \mathcal{B}_0(\mathbb{R}^d)$ with $\bigcup_{k \geqslant 1} \Lambda_k = \mathbb{R}^d$, let $\mathcal{E}_\infty = \bigcap_k \mathcal{E}_{\Lambda_k}$ be

the tail-field and $\mathbb{P} \in C_\infty(\pi^{\mathbb{E}})$, then for $\varphi \in L^1(\mathbb{P})$,

$$\mathbb{P}(\varphi|\mathcal{E}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{E}}_{\Lambda_k}(\,\cdot\,,\varphi) \qquad \mathbb{P}\text{-a.s.}. \tag{5.2}$$

Therefore the limits

$$Q_\mu = \lim_{k\to\infty} \pi^{\mathbb{E}}_{\Lambda_k}(\mu,\,\cdot\,) \tag{5.3}$$

exist $\mathbb{P}$-a.s. in μ and are by construction contained in the Martin-Dynkin boundary $C_\infty(\pi^{\mathbb{E}})$.

Proposition 3. *Let f be nonnegative and measureable with bounded support. Then $\widehat{Q_\mu}(if) = \lim \widehat{\pi^{\mathbb{E}}_{\Lambda_k}(\mu,\,\cdot\,)}(if)$ exists, is non-degenerate and*

$$\widehat{Q_\mu}(if) = \exp\Big(-\rho_z\big(1-\exp(-f)\big)\Big). \tag{5.4}$$

Proof. The proof of the corresponding microcanonical loop ensemble applies with $Y_j = \frac{z^j}{j}$.

This means that the Poisson process with intensity measure ρ_z is the only limit, hence there is no phase transition. Clearly,

Theorem 3. *The essential part of the Martin-Dynkin boundary of $\pi^{\mathbb{E}}$ consists of the Poisson process with intensity measure ρ_z.*

6. The canonical ensemble of elementary components

From a physical point of view it could be more convenient to work on the level of elementary constituents instead of the composite loops, since they represent the elementary particles, the bosons; and it is more interesting to find statements about the number of particles in some bounded region Λ rather than the number of families they align with. Recall from eq. (2.11) that the number of elementary components in a bounded region Λ is

$$N_\Lambda \mu = \sum_{j\geqslant 1} j n_\Lambda \mu(j).$$

Hence, under $\mathbf{P}_{\rho_z}$, N_Λ has a compound Poisson distribution whenever $z \leqslant 1$ for $d \geqslant 3$ and $z < 1$ for $d = 1, 2$. However, the conditions

$$\mathcal{H}_\Lambda = \mathcal{E}_\Lambda \vee \sigma\Big(\big\{N_\Lambda = k : k \in \mathbb{N}^*\big\}\Big).$$

do not allow a direct computation as in the previous sections. Instead we are going to define $\pi_\Lambda^{\mathbb{H}}$ in a similiar way as before as a conditioned Poisson process, to represent it as a convex combination of $P^\eta_{\rho_\Lambda}$ and to show a large deviation principle for the mixing measure. If the latter measure converges to a suitable limiting probability measure, then, since the microcanonical weak limits are known, $\pi_\Lambda^{\mathbb{H}}$ will converge as well.

From now on fix $d \geqslant 3$, $z = 1$ and write ρ instead of ρ_1, etc. At first we derive the representation in terms of $P^\eta_{\rho,\Lambda}$.

Lemma 4. *With $C_M = \{\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}^*) : \sum j\eta(j) = M\}$ being the set of compositions with first moment M and $\mu \in \mathcal{M}^{\cdot}(X)$ a fixed configuration with $N_\Lambda\mu = M$, it follows*

$$\int \varphi\Big(\nu + \mu^{(\Lambda^c)}\Big) 1_{C_M}(\nu) \mathbf{P}_{\rho_\Lambda}(\mathrm{d}\nu) = \sum_{\eta \in C_M} \mathbf{P}_{\tau_\Lambda}(\eta) P^\eta_{\rho,\Lambda}\bigg(\varphi\Big(\,\cdot\, + \mu^{(\Lambda^c)}\Big)\bigg). \tag{6.1}$$

Proof. This can be seen from disintegration of conditional expectations as in the beginning of section 3.

If we now condition $\mathbf{P}_{\rho_\Lambda}$ on the event $\{N_\Lambda = M\}$ on the lhs. of eq. (6.1), this turns into $\mathbf{P}_{\tau_\Lambda}$ conditioned on C_M on the rhs. Though define

$$\pi_\Lambda^{\mathbb{H}}(\mu, \varphi) = \mathbf{P}_{\rho,\Lambda}\bigg(\varphi\Big(\,\cdot\, + \mu^{(\Lambda^c)}\Big)\Big| N_\Lambda = N_\Lambda\mu\bigg) \tag{6.2}$$

$$= \int P^\eta_{\rho,\Lambda}\bigg(\varphi\Big(\,\cdot\, + \mu^{(\Lambda^c)}\Big)\bigg) \mathbf{P}_{\tau_\Lambda}(\mathrm{d}\eta | C_{N_\Lambda\mu}), \tag{6.3}$$

which is indeed a probability kernel on $X \times \mathcal{M}^{\cdot}(X)$.
It even follows that $\pi^{\mathbb{H}} = \{\pi_\Lambda^{\mathbb{H}}\}_\Lambda$ is an $\mathbb{H}$-specification. As in the previous sections, fix an expanding sequence $(\Lambda_k)_k$ of bounded regions.
Before we turn to the analysis of the Martin-Dynkin boundary of $\pi^{\mathbb{H}}$, we derive a large deviation principle for $\mathbf{P}_{\tau_\Lambda}(\,\cdot\,|C_{N_\Lambda\mu})$. This one can be shown in using a large deviation principle for $\mathbf{P}_{\tau_\Lambda}(\,\cdot\,)$. Since the deviation is done for fixed μ, we write M_k instead of $N_{\Lambda_k}\mu$ and think of it as an increasing parameter in k such that $\frac{M_k}{|\Lambda_k|}$ converges to some finite limit as $k \to \infty$.

Large deviation principle for $\mathbf{P}_{\tau_\Lambda}$. The intensity measure τ_{Λ_k} grows asymptotically like the volume of Λ_k, already seen in Lemma 1,

$$\tau := \lim_{k\to\infty} \frac{\tau_{\Lambda_k}}{|\Lambda_k|} = (2\pi\beta)^{-d/2} \sum_{j\geqslant 1} \frac{1}{j^{1+\frac{d}{2}}} \delta_j. \tag{6.4}$$

τ in some sense represents the *critical limiting loop densities.* From that already follows that there is a law of large numbers,

$$\forall A \subset \mathbb{N}^* \forall \delta > 0 : \lim_{k\to\infty} \mathbf{P}_{\tau_{\Lambda_k}} \left(\left\{ \eta : \left| \frac{\eta(A)}{|\Lambda_k|} - \tau(A) \right| > \delta \right\} \right) = 0,$$

which says that the mean density of loops of any kind tends to τ.

The large deviation principle for $\mathbf{P}_{\tau_\Lambda}$ is established in [11] using Cramer's method: On $\mathcal{M}(\mathbb{N}^*)$, $\mathbf{P}_{\tau_{\Lambda_k}}\left(\frac{\eta}{|\Lambda_k|} \in \cdot\right)$ satisfies a LDP with speed $|\Lambda_k|$ and good rate function $I : \mathcal{M}(\mathbb{N}^*) \to [0, \infty]$ given by

$$I(\kappa;\tau) = \begin{cases} \tau(f \log f - f + 1) & \text{if } \kappa \ll \tau, f := \frac{\mathrm{d}\kappa}{\mathrm{d}\tau}, f \log f - f + 1 \in L^1(\tau) \\ \infty & \text{otherwise} \end{cases},$$

which means that $\{I \leqslant c\}$ is compact for any $c \geqslant 0$ and for any $G \subseteq \mathcal{M}(\mathbb{N}^*)$ vaguely open

$$\liminf_{k\to\infty} \frac{1}{|\Lambda_k|} \log \mathbf{P}_{\tau_{\Lambda_k}} \left(\left\{ \eta : \frac{\eta}{|\Lambda_k|} \in G \right\} \right) \geqslant - \inf_{\kappa \in G} I(\kappa;\tau) \tag{6.5}$$

and for any $F \subseteq \mathcal{M}(\mathbb{N}^*)$ vaguely closed

$$\limsup_{k\to\infty} \frac{1}{|\Lambda_k|} \log \mathbf{P}_{\tau_{\Lambda_k}} \left(\left\{ \eta : \frac{\eta}{|\Lambda_k|} \in F \right\} \right) \leqslant - \inf_{\kappa \in F} I(\kappa;\tau). \tag{6.6}$$

Large deviation principle for $\mathbf{P}_{\tau_{\Lambda_k}}(\,\cdot\,|C_{M_k})$. The conditioned Poisson process can be interpreted as being absolutely continuous with respect to the unconditioned process, where the density is roughly an indicator function times a normalisation constant. That way the LDP for $\mathbf{P}_{\tau_{\Lambda_k}}$ transforms into some LDP for $\mathbf{P}_{\tau_{\Lambda_k}}(\,\cdot\,|C_{M_k})$.

$$\mathbf{P}_{\tau_{\Lambda_k}}(\eta|C_{M_k}) = \Big(\mathbf{P}_{\tau_{\Lambda_k}}\big(\exp(-\chi_{C_{M_k}})\big)\Big)^{-1} \exp\big(-\chi_{C_{M_k}}(\eta)\big)\mathbf{P}_{\tau_{\Lambda_k}}(\eta),$$

where the functional χ_A for some set $A \subseteq \mathcal{M}(\mathbb{N}^*)$ is defined to be $\chi_A = \infty 1_{A^c}$. As known in large deviation theory, the rate function for $\mathbf{P}_{\tau_{\Lambda_k}}(\,\cdot\,|C_{M_k})$ will be the rate function for $\mathbf{P}_{\tau_{\Lambda_k}}$ plus a functional of the form χ_A for a suitable set A, see i.e. [4]. However, because of poor continuity properties of these functionals χ_A additional care has to be taken. Let

$$D_u := \big\{\kappa \in \mathcal{M}(\mathbb{N}^*) | \sum j\kappa(j) = u\big\}$$

be the set of measures on $\mathbb{N}^*$ with first moment u representing the densities of the loops of the different kinds. Observe that in the vague topology χ_{D_u} is neither upper nor lower semicontinuous. But if its upper or lower semicontinuous regularisations are not infinite for any $\kappa \in \mathcal{M}(\mathbb{N}^*)$, one may deduce the lower and upper large deviation bound, respectively, as will do in the sequel.

Lemma 5. *The upper and lower semicontinuous regularisations $\chi_{D_u}^{usc}$ and $\chi_{D_u}^{lsc}$ of χ_{D_u} with respect to the vague topology are*

$$\chi_{D_u}^{usc}(\kappa) = \infty, \qquad \chi_{D_u}^{lsc}(\kappa) = \begin{cases} \infty & \text{if } \sum j\kappa(j) > u \\ 0 & \text{otherwise} \end{cases}. \tag{6.7}$$

Proof. First note that $\chi_A^{usc} = \chi_{\operatorname{int} A}$ and $\chi_A^{lsc} = \chi_{\operatorname{cl} A}$, where $\operatorname{int} A$ and $\operatorname{cl} A$ denote the interior and the closure of A, respectively. But $\operatorname{cl} D_u = \{\kappa \in \mathcal{M}(\mathbb{N}^*) : \sum j\kappa(j) \leqslant u\}$, hence we get the lower semicontinuous regularisation of χ_{D_u}. By the same argument we get $\operatorname{int} D_u = (\operatorname{cl} D_u^c)^c = \varnothing$ and the upper semicontinuous regularisation.

Upper bound of the partition function. In applying [4, Lemma 2.1.7] we get the upper bound as

$$\limsup_{k\to\infty} \frac{1}{|\Lambda_k|} \log \mathbf{P}_{\tau_{\Lambda_k}} \big(\exp(-\chi_{C_{M_k}})\big) \leqslant - \inf_{\mathcal{M}(\mathbb{N}^*)} \Big[I + \chi_{D_u}^{lsc} \Big]. \tag{6.8}$$

Since χ_{D_u} is not lower semicontinuous, it is replaced by its lower semicontinuous regularisation on the rhs. We solve the variational problem on the rhs. of eq. (6.8), which is a minimisation problem with a constraint.

Proposition 4. *Let z_u be the solution of*

$$(2\pi\beta)^{-d/2} g_{d/2}(z) = u \wedge u^*, \tag{6.9}$$

where $u^* := (2\pi\beta)^{-d/2} g_{d/2}(1)$ and $g_{d/2}$ is given in eq. (2.3). Then

$$\inf_{\mathcal{M}(\mathbb{N}^*)} \Big[I + \chi_{D_u}^{lsc} \Big] = \sum_{j\geqslant 1} (1 - z_u^j)\tau(j). \tag{6.10}$$

Proof. The minimisation of $I + \chi_{D_u}^{lsc}$ is equivalent to the minimisation of I under the constraint $\sum j\kappa(j) \leqslant u$. For the moment, assume $u \leqslant u^*$ and minimise I

given $\sum j\kappa(j) = \upsilon$ for any $\upsilon \leqslant u$. By the Euler-Lagrange method of conditional minimisation,

$$\begin{aligned} I(\kappa) - \sum_{j\geqslant 1} j\kappa(j)\log z &= \sum_{j\geqslant 1} \kappa_j \Big(\log\frac{\kappa(j)}{\tau(j)} - 1\Big) + \tau(\mathbb{N}^*) - \sum_{j\geqslant 1} \log z^j \kappa(j) \\ &= \sum_{j\geqslant 1} \kappa_j \Big(\log\frac{\kappa(j)}{z^j\tau(j)} - 1\Big) + \tau(\mathbb{N}^*), \end{aligned}$$

which has a unique minimiser on $\mathcal{M}(\mathbb{N}^*)$, $\bar\kappa = \sum_{j\geqslant 1} z_\upsilon^j \tau(j)\delta_j$ with z_υ being the solution of eq. (6.9) with u replaced by υ. Immediatly

$$I(\bar\kappa) = -\sum_{j\geqslant 1} z_\upsilon^j \tau(j) + \tau(\mathbb{N}^*) = \sum_{j\geqslant 1} (1 - z_\upsilon^j)\tau(j)$$

follows. Since necessarily $z_\upsilon \leqslant 1$ and z_υ is an increasing function of υ, eq. (6.10) holds.

Now let $u > u^*$, so there is no solution of eq. (6.9). Let $u_0 = u^* - (2\pi\beta)^{-d/2} g_{d/2}(1)$ be the excess mass. Define $\bar\kappa = \tau$ and $\bar\kappa^{(n)} = \bar\kappa + \frac{u_0}{n}\delta_n$, then clearly for all n

$$\sum_{j\geqslant 1} j\bar\kappa^{(n)}(j) = \sum_{j\geqslant 1} j\bar\kappa(j) + u_0 = u$$

while $\bar\kappa^{(n)} \to \bar\kappa$ vaguely. Furthermore

$$\begin{aligned} I\big(\bar\kappa^{(n)}\big) &= \sum_{j\neq n} \bar\kappa(j)\Big(\log\frac{\bar\kappa(j)}{\tau(j)} - 1\Big) + \Big(\bar\kappa(n) + \frac{u_0}{n}\Big)\Big(\log\frac{\bar\kappa(n) + \frac{u_0}{n}}{\tau(n)} - 1\Big) + \tau(\mathbb{N}^*) \\ &= -\tau(\mathbb{N}^*\backslash\{n\}) + \Big(\tau(n) + \frac{u_0}{n}\Big)\Big(\log\Big(1 + \frac{u_0}{n\tau(n)}\Big) - 1\Big) + \tau(\mathbb{N}^*) \\ &\to I(\bar\kappa) = I(\tau) = 0 \qquad \text{as } n \to \infty. \end{aligned}$$

Lower bound of the partition function. As we have seen in lemma 5, the upper semicontinuous regularisation $\chi_{D_u}^{usc}$ of χ_{D_u} is not finite and the analogue argument for the lower bound does not apply. The reason is the sparseness of D_u in the vague topology which even holds for the blow ups D_u^ε of D_u of the form $D_u^\varepsilon = \{\kappa \in \mathcal{M}(\mathbb{N}^*) : |\sum j\kappa(j) - u| \leqslant \varepsilon\}$ for any $\varepsilon > 0$. Otherwise this could have been used for some kind of Boltzmann principle, see e.g. [16].

However, the 2-parameter sets

$$D_{m,s} := \Big\{\kappa \in \mathcal{M}(\mathbb{N}^*) : \sum_{j\leqslant m} j\kappa(j) < s\Big\}, \tag{6.11}$$

of those measures in $\mathcal{M}(\mathbb{N}^*)$, whose first moment restricted to $\{1,\dots,m\}$ does not reach s are vaguely open. Furthermore

$$\bigcap_{\varepsilon>0}\bigcap_{m\geqslant 1} D_{m,s+\varepsilon} = \mathrm{cl}\, D_s,$$

i.e. if the first moment of some measure on $\mathbb{N}^*$ restricted to $\{1,\dots,m\}$ is bounded by $s+\varepsilon$ for any $m\in\mathbb{N}^*$ and $\varepsilon>0$, then the first moment of the whole measure is bounded by s. Since now $\chi_{D_{m,s+\varepsilon}}$ is upper semicontinuous for any $m\in\mathbb{N}^*$ and

$$\lim_{L\to\infty}\limsup_{k\to\infty}\frac{1}{|\Lambda_k|}\log \mathbf{P}_{\tau_{\Lambda_k}}\Big(\exp\big(-\chi_{D_{m,s+\varepsilon}}\big)1_{\{\chi_{D_{m,s+\varepsilon}}\leqslant -L\}}\Big) = -\infty, \tag{6.12}$$

we get for any m and ε by [4, Lemma 2.1.8] a lower bound

$$\liminf_{k\to\infty}\frac{1}{|\Lambda_k|}\log \mathbf{P}_{\tau_{\Lambda_k}}\big(\exp(-\chi_{D_{m,s+\varepsilon}})\big) \geqslant -\inf_{\mathcal{M}(\mathbb{N}^*)}\Big[I+\chi_{D_{m,s+\varepsilon}}\Big] \tag{6.13}$$

for the system restricted to the first m components. Therefore, we get the lower bound as $m\to\infty$ and $\varepsilon\to 0$.

Consider now the family of minimisation problems on the rhs of eq. (6.13). Here we have to link the two parameters m and s. Since $\sum_{j>m}\frac{1}{j^{d/2}}$ is strictly decreasing to 0, there exists $m_0\in\mathbb{N}^*$ such that for any $m\geqslant m_0$, $u-(2\pi\beta)^{-d/2}\sum_{j>m}\frac{1}{j^{d/2}}\geqslant 0$.

Proposition 5. *Let $\varepsilon>0$ and $m\in\mathbb{N}^*$ be such that*

$$s_{m,\varepsilon} := u+\varepsilon-(2\pi\beta)^{-d/2}\sum_{j>m}\frac{1}{j^{d/2}}\geqslant 0$$

and $z_{(m,\varepsilon)}$ be the solution of $(2\pi\beta)^{-d/2}\sum_{j\leqslant m}\frac{z^j}{j^{d/2}} = s_{m,\varepsilon}$. Then $\inf_{\mathcal{M}(\mathbb{N}^*)}\Big[I+\chi_{D_{m,s_m+\varepsilon}}\Big] = \sum_{j\leqslant m}\big(1-z^j_{(m,\varepsilon)}\big)\tau(j)$ and as firstly $m\to\infty$ and then $\varepsilon\to 0$, $z_{(m,\varepsilon)}\to z_u$, where z_u is given in proposition 4.

Proof. The first part is similar to the previous proof where the minimiser in this case is

$$\bar\kappa(j) = \frac{1}{(2\pi\beta)^{d/2}}\begin{cases}\frac{1}{j^{1+d/2}} & j>m\\ \frac{z_{(m,\varepsilon)}}{j^{1+d/2}} & j\leqslant m\end{cases}. \tag{6.14}$$

To see the second part, assume for the moment $u=u^*$, then $s_{m,\varepsilon}$ is not exactly the m-th partial sum of the series of $(2\pi\beta)^{-d/2}g_{d/2}(1)$, but close to it. Observe that $z_{(m,\varepsilon)}>1$

for each $m \geqslant m_0$ and $(z_{(m,\varepsilon)})_{m\geqslant m_0}$ is an decreasing sequence for any $\varepsilon > 0$. Indeed, from

$$(2\pi\beta)^{-d/2}\sum_{j\leqslant m}\frac{1}{j^{d/2}} + \varepsilon = s_{m,\varepsilon} = (2\pi\beta)^{-d/2}\sum_{j\leqslant m}\frac{z^j_{(m,\varepsilon)}}{j^{d/2}}$$

immediatly follows $z_{(m,\varepsilon)} > 1$ and

$$s_{m+1,\varepsilon} - s_{m,\varepsilon} = (2\pi\beta)^{-d/2}\frac{1}{(m+1)^{d/2}} < (2\pi\beta)^{-d/2}\frac{z^{m+1}_{(m,\varepsilon)}}{(m+1)^{d/2}}$$

yields the decrease. Finally the sequence $(z_{(m,\varepsilon)})_m$ can not be bounded away from 1 for any $\varepsilon > 0$ since otherwise the sequence of sums $\left(\sum_{j\leqslant m}\frac{z^j_{(m,\varepsilon)}}{j^{d/2}}\right)_{m\geqslant m_0}$ would diverge. Hence $z_{(m,\varepsilon)} \to 1$ for any $\varepsilon > 0$ as $m \to \infty$.

For $u > u^*$ these arguments apply as well.

Let now $u < u^*$, fix $\varepsilon > 0$ such that $u + \varepsilon < u^*$ and m_0 be even large enough, such that $s_{m,\varepsilon} > 0$. Then firstly $z_{(m,\varepsilon)} < 1$ for each $m \geqslant m_0$ follows since

$$(2\pi\beta)^{-d/2}\sum_{j\leqslant m}\frac{z^j_{(m,\varepsilon)}}{j^{d/2}} = s_{m,\varepsilon} < u^* - (2\pi\beta)^{-d/2}\sum_{j>m}\frac{1}{j^{d/2}} = (2\pi\beta)^{-d/2}\sum_{j\leqslant m}\frac{1}{j^{d/2}}.$$

Next we show that $(z_{(m,\varepsilon)})_{m\geqslant m_0}$ is an increasing sequence in m and tends to $z_{u+\varepsilon}$. Since

$$s_{m+1,\varepsilon} - s_{m,\varepsilon} = (2\pi\beta)^{-d/2}\frac{1}{(m+1)^{d/2}} > (2\pi\beta)^{-d/2}\frac{z^{m+1}_{(m,\varepsilon)}}{(m+1)^{d/2}},$$

$z_{(m+1,\varepsilon)}$ needs to be bigger than $z_{(m,\varepsilon)}$. Since necessarily $(z_{(m,\varepsilon)})_m$ is bounded from above by 1, the sequence converges and the only limit can be $z_{u+\varepsilon}$ since $s_{m,\varepsilon}$ tends to $u + \varepsilon$ as $m \to \infty$. By the continuity of $g_{d/2}$ the claim follows as $\varepsilon \to 0$.

Thus we have shown that the following limit exits and equals

$$\lim_{k\to\infty}\frac{1}{|\Lambda_k|}\log \mathbf{P}_{\tau_{\Lambda_k}}\left(\exp(-\chi_{C_{M_k}})\right) = -\tau(G_{z_u}),$$

where $u = \lim_k \frac{M_k}{|\Lambda_k|}$ is the limiting particle density, z_u is given in proposition 4 and

$$G_z : \mathbb{N}^* \to \mathbb{R}, \qquad j \mapsto 1 - z^j.$$

Since the minimiser of the minimisation problem was unique, the conditioned Poisson process is asymptotically degenerate and

$$\lim_{k\to\infty}\mathbf{P}_{\tau_{\Lambda_k}}\left(\left\{\eta : \frac{\eta}{|\Lambda_k|} \in \cdot\right\}\middle| C_{M_k}\right) = \delta_{\tau_{z_u}}. \tag{6.15}$$

Martin-Dynkin boundary. Back to Martin-Dynkin boundary technique, we interpret the boundary condition $\mu \in \mathcal{M}^{\cdot}(X)$ as a random element and write capital letters instead of small ones to emphasize the dependence on μ. Let U be the limiting particle density, $U(\mu) = \lim_{k\to\infty} \frac{N_{\Lambda_k}\mu}{|\Lambda_k|}$, in case of existence of the limit and put $U(\mu) = \infty$ if the limit does not exist. For each configuration μ with $U(\mu) < \infty$ there exists $Z = Z(\mu)$ such that

$$(2\pi\beta)^{-d/2} g_{d/2}(Z) = U \wedge u^*. \tag{6.16}$$

Proposition 6. *Let $f : X \to \mathbb{R}$ be nonnegative and measurable with bounded support and $\mu \in M$. Then for any $\varphi \in L^1(\mathbb{P})$*

$$\mathbb{P}(\varphi|\mathcal{H}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{H}}_{\Lambda_k}(\,\cdot\,, \varphi) = \mathbf{P}_{\rho Z}(\varphi) \qquad \mathbb{P}\text{-}a.s.$$

Proof. From eq. (6.15) we get

$$\mathbf{P}_{\tau_{\Lambda_k}}\left(\left\{\eta : \frac{\eta}{|\Lambda_k|} \in \cdot\right\}\Big| C_{N_{\Lambda_k}\mu}\right) \to \delta_{\tau_{Z(\mu)}}$$

as $k \to \infty$. Now we can use the results of section 3 to deduce that the measures converge

$$\lim_{k\to\infty} \pi^{\mathbb{H}}_{\Lambda_k}(\mu, \,\cdot\,) = \mathbf{P}_{\rho_{Z(\mu)}}.$$

Again the reasoning of the preceeding sections applies. Since for $\mathcal{H}_\infty$-measurable φ, $\mathbb{P}(\varphi\, f(Q.)) = \mathbb{P}(\varphi\, \mathbf{P}_{\rho_Z}(f(Q.)))$ holds, we get

$$\mathbf{P}_{\rho_{Z(\mu)}}(Q. = Q_\mu) = 1 \qquad \mathbb{P}\text{-a.s.}$$

In particular $Z = Z(\mu)$ $\mathbb{P}$-a.s.
Let $\Delta^{\mathbb{H}} = \{P \in C_\infty \cap C | Q. = P \quad P\text{-a.s.}\}$ be the essential part of the Martin-Dynkin boundary associated to $\mathbb{H}$.

Theorem 4. *The essential part of the Martin-Dynkin boundary of $\pi^{\mathbb{H}}$ consists of all Poisson processes with intensity measure ρ_z for $z \in [0, 1]$ and $d \geqslant 3$,*

$$\Delta^{\mathbb{H}} = \{\mathbf{P}_{\rho_z} | 0 \leqslant z \leqslant 1\}.$$

Proof. As already seen, $\mathbf{P}_{\rho_z} \in C(\pi^{\mathbb{H}})$, and clearly $Q_\mu = \mathbf{P}_{\rho_z}$ $\mathbf{P}_{\rho_z}$-a.s.For arbitrary $\mathbb{P} \in \Delta^{\mathbb{H}}$ we have

$$\int_{\Delta^{\mathbb{H}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P) = \mathbb{P}(\varphi) = Q.(\varphi) \qquad \mathbb{P}\text{-a.s.}$$

This implies $V^{\mathbb{P}} = \delta_{\mathbf{P}_{\rho_z}}$.

Remark 3. Finally note that, starting with the intensity measure $\rho_{z'}$ for $z' < 1$, the calculations stay the same in principle. The difference is that the Lagrange multiplier z, which occures during the minimisation procedure using ρ, will be, given $\rho_{z'}$, some $\tilde{z}$ related to z via $z = z'\tilde{z}$. In particular, the analysis is not restricted to $d \geqslant 3$ and applies in this fashion, using the intensity measure $\rho_{z'}$ for some $z' < 1$ instead of ρ and $u^* = \infty$, to the one- and two-dimensional Ginibre gas. In contrast to the cases $d \geqslant 3$, $d = 1, 2$ do not show the critical behaviour.

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References

[1] Benfatto, G., Cassandro, M., Merola, I. and Presutti, E. (2005). Limit theorems for statistics of combinatorial partitions with applications to mean field Bose gas. *J. of Math. Phys.* **46** 033303–033341.

[2] Betz, V. and Ueltschi, D. (2008) Spatial random permutations and infinite cycles, arXiv: 0711.1188v2

[3] Dembo, A. and Zeitouni, O. (1998). *Large Deviations Techniques and Applications*, 2nd edn. Springer.

[4] Deuschel, J.-D. and Stroock, D. W. (2000). *Large Deviations*, 2nd edn. AMS Chelsea Publishing.

[5] Dynkin, E. B. (1971). Entrance and exit spaces for a Markov process. *Actes Congres Intern. Math. 1970* **Tome 2** 507–12.

[6] Dynkin, E. B. (1971). The initial and final behaviour of trajectories of a Markov process. *Russian Math. Surveys* **26, no. 4** 165–85.

[7] Föllmer, H. (1975). Phase transition and Martin Boundary. *Seminaire de probabilites (Strasbourg)* **9** 305–17.

[8] Fichtner, K. H. (1991). On the Position Distribution of the Ideal Bose Gas. *Math. Nachr.* **151** 59–67.

[9] Georgii, H. O. and Zessin, H. (1993). Large deviations and the maximum entropy principle for marked point random fields. *Probab. Theory Relat. Fields* **96** 177–204.

[10] Ginibre, J. (1971). Some Applications of functional Integration in Statistical Mechanics. Statist. Mech. and Quantum Field Theory, Les Houches Summer School Theoret. Phys, (Gordon and Breach), 327-427

[11] Guo, M. Z. and Wu, L. M. (1995). Several large deviation estimations for the Poisson point processes. *Adv. in Math.(China)* **24, no. 4** 313–319.

[12] Kallenberg, O. (2002). *Foundations of Modern probability*, 2nd edn. Springer.

[13] Nguyen, X. X. and Zessin, H. (1976/77). Martin-Dynkin boundary of mixed Poisson processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **37, no. 3** 191–200.

[14] Nguyen, X. X. and Zessin, H. (1979). Ergodic Theorems for Spatial Processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **48** 133–58.

[15] Preston, C. (1979). Canonical and Microcanonical Gibbs States. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **46** 125–58.

[16] Rœlly, S. and Zessin, H. (1993). The Equivalence of Equilibrium Principles in Statistical Mechanics and some Applications to Large Particle Systems. *Expo. Math.* **11** 385–405.

[17] Sütő, A. (1993). Percolation transition in the Bose gas. *J. Phys. A: Math. Gen.* **26** 4689–710.

[18] Sütő, A. (2002). Percolation transition in the Bose gas: II. *J. Phys. A: Math. Gen.* **35** 6995–7002.

[19] Ueltschi, D. (2006). Feynman cycles in the Bose gas. *J. Math. Phys.* **47, no. 12** 123303, 15 pp.

[20] Ueltschi, D. (2006). Relation between Feynman cycles and off-diagonal long-range order. *Phys. Rev. Lett.* **97, no. 17**, 170601, 4 pp.

[21] Ueltschi, D. (2007) The Model of interacting spatial permutations and its relation to the Bose gas, arXiv: 0712.2443v3