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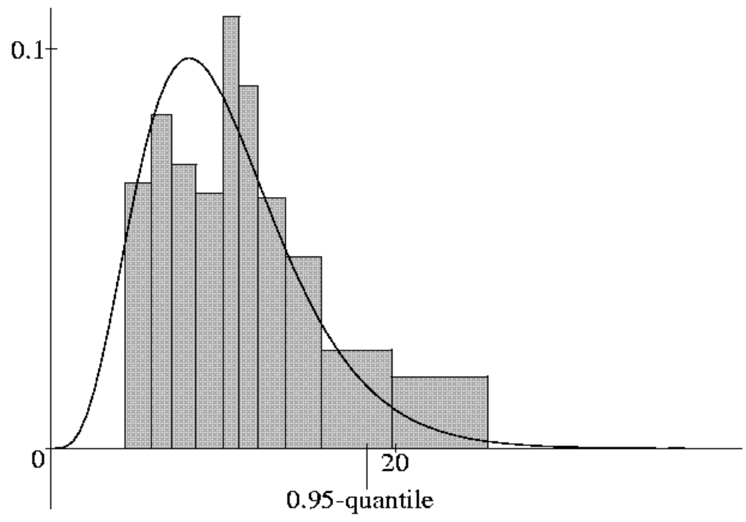

Mathematische Statistik und Wahrscheinlichkeitstheorie

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A path integral representation of the moment measures of the general ideal Bose gas

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Abstract

We reconsider the fundamental work of Fichtner ([2]) and exhibit the permanental structure of the ideal Bose gas again, using another approach which combines a characterization of infinitely divisible random measures (due to Kerstan,Kummer and Matthes [5, 6] and Mecke [8, 9]) with a decomposition of the moment measures into its factorial measures due to Krickeberg [4]. To be more precise, we exhibit the moment measures of all orders of the *general ideal Bose gas* in terms of certain path integrals. This representation can be considered as a point process analogue of the old idea of Symanzik [11] that local times and self-crossings of the Brownian motion can be used as a tool in quantum field theory.

Behind the notion of a general ideal Bose gas there is a class of infinitely divisible point processes of all orders with a *Lévy − measure* belonging to some large class of measures containing the one of the classical ideal Bose gas considered by Fichtner.

It is well known that the calculation of moments of higher order of point processes are notoriously complicated. See for instance Krickeberg's calculations for the Poisson or the Cox process in [4].

1 An integration by parts formula for infinitely divisible random measures

The aim here is to characterize infinitely divisible random measures on a general state space from the point of view of its Campbell measure. It is shown that such random measures are characterized by some integration by parts formula. Our proof seems to be a more direct approach if compared to the one of Kallenberg [3], Kerstan et al.[5, 6, 7] and Wegmann [12], where in principle such results can be found already. The main ideas of the following reasoning can already be found in the seminal work of Mecke [8, 9].

The integration by parts formula

In the sequel we use freely the notions of random measure theory and refer for their definitions to the monographies of Kallenberg [3] and Matthes et al. [7].

X denotes a Polish state space, $\mathcal{B}(X)$ resp. $\mathcal{B}_0(X)$ its Borel resp. bounded Borel sets. $\mathcal{M}(X)$ is the vaguely Polish space of locally finite measures on X (i.e. of Radon measures on X).

We are given a measure $\alpha \in \mathcal{M}(X)$ and a measure L on $Y = \mathcal{M}(X) \setminus \{0\}$ of first order. This means that ν_L^1, the first moment measure of L, is a Radon measure, i.e.

$$\nu_L^1 \in \mathcal{M}(X). \tag{1.1}$$

We consider random measures P on X, i.e. laws P on $\mathcal{M}(X)$, for which we write $P \in \mathcal{PM}(X)$. We are interested in such P which solve the following integration by parts formula

$$(\mathbf{\Sigma}_{\mathbf{L},\alpha}) \qquad C_P = C_L \star P + \alpha \otimes P.$$

Here C_P resp. C_L denote the Campbell measure of P resp. L . $\otimes$ is the usual product of measures, and the operation $\star$ is a version of the convolution of

C_L and P, defined by

$$(1.2) \qquad C_L \star P(h) = \int h(x, \kappa + \nu) C_L(dx, d\nu) P(d\kappa), h \in F_+(X \times \mathcal{M}(X)).$$

(F_+ denotes the collection of measurable, non-negative numerical functions. We often do not write the underlying measurable space.)

Lemma 1.1 *If P is a solution of $(\Sigma_{L,\alpha})$, then its Laplace transform is*

$$(1.3) \qquad \mathcal{L}_P(f) = \exp\big(-\big[\alpha(f) + L(1 - \exp(-\zeta_f))\big]\big), f \in F_+(X).$$

Proof. We establish a differential equation for the function $t \longmapsto L(t \cdot f)$. Let $t > 0$ and $f \in F_+(X)$. Since P is of first order, one can interchange differentiation and integration to obtain by means of the partial integration formula

$$(1.4) \qquad \frac{d}{dt} \mathcal{L}_P(tf) = -C_P(f \otimes \exp(-t\zeta_f)).$$

Using the integration by parts formula this equals

$$= \big[-\big(\alpha(f) + \int \nu(f) \exp(-t\nu(f)) L(d\nu)\big] \cdot \mathcal{L}_P(tf).$$

The initial condition is $\mathcal{L}_P(0 \cdot f) = 1$. Thus

$$\mathcal{L}_P(tf) = \exp(-[t\alpha(f) + L(1 - \exp(-t\zeta_f)]), t \leq 0, f \in F_+(X). \quad \textbf{qed}$$

As a consequence, if $(\Sigma_{L,\alpha})$ has a solution P at all, it must be unique.

Remark 1.2 *It is well known (see [3], thm 6.1, or [12]) that under the conditions above the relation*

$$(1.5) \qquad \mathcal{L}_P(f) = \exp\big(-\big[\alpha(f) + L(1 - \exp(-\zeta_f))\big]\big), f \in F_+(X)$$

defines a 1-1 correspondence between the distributions P of all infinitely divisible random measures ξ and the class of all pairs (α, L).

Construction of a solution of $(\Sigma_{L,\alpha})$ and characterizations

Consider P_L, the Poisson process with intensity L. Furthermore, let $\Im := \Im_{L,\alpha}$ denote the image of P_L under the measurable transformation

$$\xi : \mathcal{M}^{\cdot\cdot}(Y) \longrightarrow \mathcal{M}(X), \mu \longmapsto \alpha + \int_Y \nu\, \mu(d\nu).$$

Set $\chi(\mu) = \int_Y \nu\, \mu(d\nu)$ for short. Note that this is the intensity of μ, which, by assumption on L, is $P_L - a.s.$ locally finite, so that ξ is a random element in $\mathcal{M}(X)$ of first order. We call ξ resp. its distribution $\Im$ the *random KMM-measure in* X *for* (L, α). (These are the initials of J. Kerstan, K. Matthes and J. Mecke. $\Im$ is the initial of Jena.) It seems that the idea of the above construction appeared for the first time in Jena in the works [5, 6] respectively in [8, 9].

Given L of first order, it remains to show that ξ, the random measure constructed above, solves the integration by parts formula. By definition one has for any $h \in F_+$

$$C_\xi(h) = P_L(\int h(x, \xi)\, \xi(dx)) = \int h(x, \chi(\mu) + \alpha)\, (\chi(\mu) + \alpha)(dx) P_L(d\mu)$$

This is the sum of two integrals I_1 and I_2. Using Mecke's characterization of the Poisson process in [9] yields

$$\begin{aligned} I_1 &= \int h(x, \chi(\mu) + \alpha)\, \nu(dx)\mu(d\nu)P_L(d\mu) \\ &= \int h(x, \chi(\mu + \delta_\nu) + \alpha)\, \nu(dx)L(d\nu)P_L(d\mu). \end{aligned}$$

Observe then that $\chi(\mu + \delta_\nu) + \alpha = (\chi(\mu) + \alpha) + \nu$, so that this is equal to

$$\int h(x, \kappa + \nu)\, \nu(dx)L(d\nu)\Im(d\kappa) = C_L \star \Im(h).$$

On the other hand

$$I_2 = \int h(x, \kappa)\, \alpha(dx)\Im(d\kappa).$$

This shows that $\Im$ solves $(\Sigma_{L,\alpha})$:

To summarize, we have shown the following theorem.

Theorem 1.3 *Let $\alpha \in \mathcal{M}(X)$ and L a measure on $Y = \mathcal{M}(X) \setminus \{0\}$ of first order. Then the following conditions are equivalent for a given random measure $\tilde{\xi}$:*

(1) $\tilde{\xi}$ is infinitely divisible for (α, L);

(2) the Laplace transform of $\tilde{\xi}$ is given by (1.3);

(3) $\tilde{\xi}$ is a solution of $(\Sigma_{\alpha,L})$;

(4) $\tilde{\xi}$ is the random KMM-measure ξ in X for (α, L).

In particular $(\Sigma_{\alpha,L})$ has $\Im$ as a unique solution, which is of first order.

This basic result can be stated also in the following form: The transformation ξ gives a one-to-one correspondence between Poisson processes on Y with an intensity measure L of first order and the set of infinitely divisible random measures on X with so called *Lévy measure* L.

In the sequel we consider always the case $\alpha = 0$.

2 The general ideal Bose gas

By specifying a class of Lévy measures L we now consider a large class of infinitely divisible point processes which we call general ideal Bose gases. Our aim is to analyse the structure of their moment measures. All this is motivated by the work of Fichtner and Freudenberg, in particular by [2]. Some of their results are reestablished in a generalized form in the present context.

As above X denotes a Polish space. Let ϱ be a given Radon measure on X. Furthermore, we are given a measurable family of *finite* measures

(2.1) $$\mathcal{B}_m^x(dx_1 \dots dx_{m-1}) \text{ on } X^{m-1}, m \geq 1.$$

Here $X^0 = \{\emptyset\}$ and $\mathcal{B}_1^x$ is the Dirac measure on the empty tupel $\emptyset$. *Measurability* means that $x \longmapsto \mathcal{B}_m^x(f)$ is measurable for any $f \in F_+(X^{m-1})$.

We assume that

(2.2)

$\varrho(dx)\mathcal{B}_m^x(dx_1 \dots dx_{m-1})$ is a cyclic invariant Radon measure on X^m for $m \geq 1$.

Example 2.1 *In the simplest case $\mathcal{B}_m^x$ is the $0-$measure for any $m \geq 2$.*

Example 2.2 *(Fichtner [2])*

Now (X, ϱ) is the Euclidean space $\mathbb{R}^d$ with the the Lebesgue measure λ. The measure $\mathcal{B}_m^x$ is defined by

$$\mathcal{B}_m^x(dx_1 \dots dx_{m-1}) = g(x - x_{m-1}) g(x_1 - x) \prod_{j=0}^{m-2} g(x_{j+1} - x_j)\, dx_1 \dots dx_{m-1}.$$

Here g the centered Gaussian density with covariance matrix βI, I denoting the identity and $\beta > 0$ is a given parameter:

$$g(x) = \frac{1}{(2\pi\beta)^{d/2}} \exp(-\frac{\|x\|^2}{2\beta}), x \in X.$$

Thus $\mathcal{B}_m^x$ is a non-normalized random walk bridge of length m starting at x, which has normally distributed increments.

Finally we consider the following *cluster measure* on the space $\mathcal{M}_f^{\cdot\cdot}$ of finite configurations in X.

$$L(\varphi) = \sum_{m \geq 1} \frac{z^m}{m} \int \varphi(\delta_{x_1} + \dots + \delta_{x_{m-1}} + \delta_x)\ \mathcal{B}_m^x(dx_1 \dots dx_{m-1})\ \varrho(dx), \varphi \in F_+.$$

Here $0 < z \leq 1$ is a given parameter.

In the first example above the measure L is given by $z\varrho$; and in the second one obtains a measure $\mathcal{F}$ which we call *Fichtner's measure.*

We next compute the Campbell measure of L. To do this we introduce the measure

$$\mathcal{B}^x = \sum_{m \geq 1} z^m\ \mathcal{B}_m^x \tag{2.3}$$

on the sum space $\mathbf{X} = \sum_{n \geq 0} X^n$. Define also

$$M : \mathbf{X} \to \mathcal{M}_f^{\cdot\cdot}, y = (x_1, \dots, x_m) \mapsto \mu_y = \delta_{x_1} + \dots + \delta_{x_m}$$

Lemma 2.1 *The Campbell measure of L is given by*

$$C_L(h) = \int_X \int_{\mathbf{X}} h(x, \mu_y + \delta_x)\ \mathcal{B}^x(dy)\ \varrho(dx), h \in F_+.$$

The proof uses only the cyclic invariance of $\mathcal{B}_m^{x_1}(dx_2...dx_m)\ \rho(dx_1)$. This lemma represents the Campbell measure $\mathcal{C}_L$ as the image of the measure $\mathcal{B}^x(dy)\ \varrho(dx)$ on $X \times \mathbf{X}$ under the mapping $(y, x) \longmapsto \mu_y + \delta_x$.

Example 2.3 *(Example 2.2 continued)*

A more probabilistic formulation of Fichtner's example is as follows: The total measure of $\mathcal{B}_m^x$ is $(2\pi m\beta)^{-d/2}$. The corresponding random walk bridge law thus is given by

$$B_m^x = (2\pi m\beta)^{d/2} \cdot \mathcal{B}_m^x.$$

Consider now the mixture of these laws with respect to the law

$$p_z(m) = \frac{1}{g_{d/2}(z)} \cdot \frac{z^m}{m^{d/2}}, m \in \mathbb{N},$$

defined by

$$B^x = \sum_{m \geq 1} p_z(m) \cdot B_m^x.$$

$g_{d/2}(z)$ is the normalizing constant. Note that p_1 is only well defined if $d \geq 3$. If we finally set

$$\varrho_z(dx) = \frac{g_{d/2}(z)}{(2\pi\beta)^{d/2}},$$

then the Campbell measure of Fichtner's measure $\mathcal{F}$ is given by

$$\mathcal{C}_F(h) = \int_X \int_{\mathbf{X}} h(x, \mu_y + \delta_x)\, B^x(dy) \varrho_z(dx),\ h \in F_+.$$

As an immediate consequence of the above lemma we obtain the first moment measure of L as a multiple of ϱ:

$$\nu_L^1 = \mathcal{B}^{(\cdot)}(\mathbf{X}) \cdot \varrho. \tag{2.4}$$

Recall that L should be of first order. Thus we assume from now on

$$L \text{ is of first order .} \tag{2.5}$$

This assumption implies that $\mathcal{B}^x$ is a finite measure for $\varrho-$almost all x. Without restricting the generality we assume that *all* $\mathcal{B}^x$ are finite. (Thus the assumption made above in (2.1) is a consequence of (2.5).)

Under the additional condition (2.5) we are in the situation of theorem 1.1 above. Thus ξ is infinitely divisible with *Lévy* measure L and thereby characterized by an integration by parts formula. Note that ξ, the random KMM-measure, is now a point process, which we call *general ideal Bose gas for L*.

We remark as an aside: If ϱ is diffuse then ν_L^1 has this property too, which then implies that the point process ξ is simple, i.e. realizes only simple point measures. In this case the clusters, which are generated by L, do not meet one another. (For a proof see [10].)

To summarize we have the

Theorem 2.2 *Let ϱ be a Radon measure on X and $\mathcal{B}_m^x, x \in X, m \geq 1$, a family of kernels satisfying the assumptions (2.2) and (2.5). Then the distribution $\Im_L$ of the general ideal Bose gas ξ is the unique solution of the following integration by parts formula:*

$$\mathcal{C}_{\Im_L}(h) = \int h(x, \kappa + \mu_y + \delta_x)\, \mathcal{B}^x(dy)\varrho(dx)\Im_L(d\kappa), h \in F_+. \tag{2.6}$$

This is our main result concerning the *general ideal Bose gas* $\Im_L = \xi P_L$. Equation (2.6) should be seen as follows: Given the measure $\varrho(dx)\mathcal{B}^x(dy)$ on $X \times \mathbf{X}$, the ideal Bose gas is characterized as the unique solution of the integration by parts formula (2.6).

This theorem is a far reaching generalization of Mecke's characterization of the Poisson process ([9]): Mecke's equation appears if $\mathcal{B}^x$ is the empty random walk bridge, i.e. if $M\mathcal{B}^x = \Delta_0$. And this is the case in our first example above. We illustrate its power in deducing some of Fichtner's fundamental results in [2].

We start to demonstrate the power of the theorem by stating first some general observations: An immediate consequence of (2.6) is that the intensity of the general ideal Bose gas ξ is given by

$$\nu_{\Im_L}^1 = \nu_L^1 = \mathcal{B}^{(\cdot)}(\mathbf{X}) \cdot \varrho. \tag{2.7}$$

Furthermore, the *Palm law of the general ideal Bose gas* $\Im_L$ can be read off directly as the convolution

$$\Im_L{}^x = \mathcal{B}^x(\mathbf{X})^{-1} \cdot (M\mathcal{B}^x_\beta) * \Im_L * \Delta_x. \tag{2.8}$$

Here $M\mathcal{B}^x_\beta$ denotes the image of $\mathcal{B}^x_\beta$ under M. Thus the Palm law $\Im_L{}^x$ realizes

configurations of particles in $\mathbb{R}^d$ consisting of x, augmented by independent realizations of the process $\Im_L$ and the cluster measure $M\mathcal{B}^x_\beta$.

3 A loop expansion

We now use the last theorem in connection with the general decomposition of the moment measures of a point process into its factorial measures to represent the $k-th$ moment measure of the general ideal Bose gas $\Im = \Im_L$, L given as above, in terms of certain loop measures. Such a representation has its origin in the work of Symanzik [11]. (We refer also to Dynkin [1].)

Until now we assumed that L is of first order. In the sequel we even assume that

$$L \text{ is of any order.} \tag{3.1}$$

Thus all $\nu^k_L, k \geq 1$, are Radon measures. This is the case in the above examples.

Let $f_1, \ldots, f_k$ be elements of $\mathcal{K}(\mathbb{R}^d)$, the space of all continuous functions with compact support. Theorem 2.2 immediately implies

$$\begin{aligned} &\nu^k_\Im(f_1 \otimes \cdots \otimes f_k) := \mathcal{C}_\Im(f_1 \otimes (\zeta_{f_2} \cdots \zeta_{f_k})) \\ (\mathcal{R}) \qquad &= \sum_{B \subset \{2,\ldots,k\}} \nu^{|B^c|}_\Im(\otimes_{j \in B^c} f_j) \cdot \nu^{|B|+1}_L(\otimes_{j \in B \cup \{1\}} f_j) \end{aligned}$$

This shows that the moment measures of $\Im$ are recursively determined by the moment measures of L. The next lemma shows that the factorial measures $\check{\nu}^k_\Im$ of $\Im$ (defined and investigated in the scholion at the end) satisfy the same kind of recursion.

Lemma 3.1

$$\check{\nu}^k_{\Im}(\otimes_{j=1}^k f_j) = \sum_{B\in\mathcal{P}\{2,\ldots,k\}} \check{\nu}^{|B^c|}_{\Im}(\otimes_{j\in B^c} f_j) \cdot \check{\nu}^{|B|+1}_L(\otimes_{j\in B\cup\{1\}} f_j).$$

Here $\mathcal{P}(.)$ denotes power set building.

Proof. Equation $(\mathcal{R})$ combined with theorem 4.1, the decomposition of the moment measures into its factorial measures, yields with $f_J = \prod_{j\in J} f_j$

$$\nu^k_{\Im}(f_1 \otimes \cdots \otimes f_k) = \sum_{B\in\mathcal{P}\{2,\ldots,k\}} \sum_{\mathcal{J}\in\pi(B^c)} \sum_{\mathcal{K}\in\pi(B\cup\{1\})} \check{\nu}^{|\mathcal{J}|}_{\Im}(\otimes_{\mathcal{J}} f_J) \cdot \check{\nu}^{|\mathcal{K}|}_L(\otimes_{\mathcal{K}} f_K).$$

Here the inner sums are taken over all partitions π of B^c respectively $B\cup\{1\}$. Now reorder the sum on the right hand side to obtain

$$= \sum_{\mathcal{N}\in\pi[k]} \sum_{Q\in\mathcal{P}\{2,\ldots,|\mathcal{N}|\}} \check{\nu}^{|Q^c|}_{\Im}(\otimes_{j\in Q^c} f_{N_j}) \cdot \check{\nu}^{|Q|+1}_L(\otimes_{j\in Q\cup\{1\}} f_{N_j}).$$

Here $[k] = \{1,\ldots,k\}$, and the partition $\mathcal{N} = \{N_1,\ldots,N_{|\mathcal{N}|}\}$ is always chosen in such a way that $1 \in N_1$.

The uniqueness of the decomposition of moment measures into its factorial measures then shows the result. **qed**

The next lemma gives an explicit representation of the measures satisfying the recursion of the last lemma. An induction with respect to $k \geq 1$ immediately gives a proof.

Lemma 3.2 *Let $(\eta_k)_{k\geq 1}$ and $(\lambda_k)_{k\geq 1}$ be two families of symmetric Radon measures on X^k which are related to one another by the following recursion:*

$$\begin{aligned} \eta_0 &\equiv 1; \text{ and if } k \geq 1 \\ \eta_k(dx_1 \ldots dx_k) &= \sum_{B\in\mathcal{P}(\{2,\ldots,k\})} \eta_{|B^c|}((dx_j)_{j\in B^c})\lambda_{|B|+1}((dx_j)_{j\in B\cup\{1\}}). \end{aligned}$$

Then

$$\eta_k(dx_1 \ldots dx_k) = \sum_{\mathcal{J}\in\pi([k])} \lambda_{|J_1|}((dx_j)_{j\in J_1}) \ldots \lambda_{|J_{|\mathcal{J}|}|}((dx_j)_{j\in J_{|\mathcal{J}|}}) \tag{3.2}$$

where each $\mathcal{J} \in \pi([k])$ is numbered here in an arbitrary way, say $\mathcal{J} = \{J_1,\ldots,J_{|\mathcal{J}|}\}$.

In order to determine the factorial measures of $\Im_L$ we have to compute now the factorial measures of L. This has been done by Fichtner [2] in the special case of Gaussian bridges.

Lemma 3.3 *The factorial moment measures of L, as defined above, are given by*

$$\check{\nu}_L^k(dx_1 \dots dx_k) = \tilde{\mathcal{B}}_{k-1}^{x_1}(dx_2 \dots dx_k)\varrho(dx_1), \tag{3.3}$$

where

$$\tilde{\mathcal{B}}_k^x = \sum_{\sigma \in \mathcal{S}_k} \sigma\Big(\sum_{n \geq 1} z^n \cdot \mathcal{B}_{k,n}^x\Big),$$

with

$$\mathcal{B}_{k,n}^x = \sum_{R \in \mathcal{P}_k([n-1])} \mathcal{B}_{R,n}^x,$$

$\mathcal{B}_{R,n}^x$ being the image of $\mathcal{B}_n^x$ under the projection $(x_1, \dots, x_{n-1}) \mapsto (x_j)_{j \in R}$. Furthermore, $\mathcal{P}_k$ means building subsets of cardinality k.

Proof. By lemma 2.1 one has

$$\nu_L^k(f_1 \otimes \cdots \otimes f_k) = \sum_{C \in \mathcal{P}\{2,\dots,k\}} \int_X f_{C \cup \{1\}}(x) \int_{\mathbf{X}} \mu_y^{|C^c|}(\otimes_{j \in C^c} f_j) \mathcal{B}^x(dy) \varrho(dx).$$

We now apply the main lemma (see lemma 4.2 from the scholion at the end) which in the present context implies

$$\int_{\mathbf{X}} \mu_y^k(\otimes_{j=1}^k f_j)\, \mathcal{B}^x(dy) = \sum_{\mathcal{J} \in \pi([k])} \tilde{\mathcal{B}}_{|\mathcal{J}|}^x(\otimes_{\mathcal{J}} f_J)$$

Thus

$$\begin{aligned} \nu_L^k(f_1 \otimes \cdots \otimes f_k) &= \sum_{C \in \mathcal{P}\{2,\dots,k\}} \sum_{\mathcal{J} \in \pi(C^c)} \int_X f_{C \cup \{1\}}(x) \tilde{\mathcal{B}}_{|\mathcal{J}|}^x(\otimes_{\mathcal{J}} f_J) \varrho(dx) \\ &= \sum_{\mathcal{J} \in \pi([k])} \int \int \prod_{j=1}^{|\mathcal{J}|} f_{J_j}(x_j)\, \tilde{\mathcal{B}}_{|\mathcal{J}|-1}^{x_1}(dx_2 \dots dx_{|\mathcal{J}|}) \varrho(dx_1). \end{aligned}$$

(Here as above $1 \in J_1$.) Now use the uniqueness of the decomposition to obtain the result. **qed**

Remark 3.4 *Since L is assumed to be of any order, lemma 3.3 implies that all measures $\tilde{\mathcal{B}}_k^x$ and thereby all measures $\mathcal{B}_{R,n}^x$ can be assumed to be Radon measures.*

Now that we know the factorial measures of L we obtain from lemma 3.2 the

Corollary 3.5

$$\check{\nu}_{\Im}^k(dx_1 \dots dx_k) = \sum_{\mathcal{J} \in \pi([k])} \tilde{\mathcal{B}}_{|J_1|-1}^{x_{i_1}}((dx_j)_{j \in J_1 \setminus \{i_1\}}) \varrho(dx_{i_1}) \dots \tilde{\mathcal{B}}_{|J_{|\mathcal{J}|}|-1}^{x_{i_{|\mathcal{J}|}}}((dx_j)_{j \in J_{|\mathcal{J}|} \setminus \{i_{|\mathcal{J}|}\}}) \varrho(dx_{i_{|\mathcal{J}|}}).$$

Here $\mathcal{J} = \{J_1, \dots, J_{|\mathcal{J}|}\}$ is any numbering of the partition $\mathcal{J}$ and $i_j \in J_j$ are arbitrarily chosen elements.

As in [2] we now use that $\mathcal{S}_k^{cy}$, the set of all cyclic permutations of the set $[k]$, is isomorphic to $\mathcal{S}_{k-1}$, and obtain

$$\tilde{\mathcal{B}}_{k-1}^x(dx_1 \dots dx_{k-1}) = \sum_{\omega \in \mathcal{S}_k^{cy}} \mathcal{B}_{k-1}^x(dx_{\omega(k)} \dots dx_{\omega^{k-1}(k)}).$$

To summarize, we have

Theorem 3.6 *Under the assumptions (2.2) and (3.1) made above on ϱ, $(\mathcal{B}_m^x)_m$ and L, the factorial moment measures of the associated general ideal Bose gas $\Im_L$ can be represented as*

$$\check{\nu}_{\Im_L}^k(dx_1 \dots dx_k) = \sum_{\sigma \in \mathcal{S}_k} \mathcal{B}_{|\omega_1|-1}^{x_{i_1}}(dx_{\omega_1(i_1)} \dots dx_{\omega_1^{|\omega_1|-1}(i_1)}) \varrho(dx_{i_1}) \dots \mathcal{B}_{|\omega_n|-1}^{x_{i_n}}(dx_{\omega_n(i_n)} \dots dx_{\omega_n^{|\omega_n|-1}(i_n)}) \varrho(dx_{i_n})$$

Here the permutation σ has been decomposed uniquely into its cycles: $\sigma = \omega_1 \cdots \omega_n$; $i_j \in \omega_j$, and $|\omega_j|$ denotes the cycle length.

Combined with theorem 4.1, the general decomposition of moment measures into its factorial measures, we thereby obtain finally a representation of all moment measurres of the general ideal Bose gas $\Im_L$.

Remark 3.7 *Theorem 3.6 is the announced loop expansion in the spirit of Symanzik [11]: Note that the measures $\mathcal{B}_{k-1}^x(dx_{\omega(k)} \dots dx_{\omega^{k-1}(k)})$ are supported by loops. Thus the right hand side of the representation of the factorial moment of $\Im_L$ is a symmetric measure on a 'gas of such loops'.*

4 Scholion: A decomposition of moment measures into its factorial measures

Following a device of Krickeberg in [4], we present here for the convenience of the reader a decomposition of the moment measures of point processes into its factorial measures as well as the main lemma.

4.1 The general decomposition into factorial measures

Let P be a point process in X. In a first step we represent P by means of a simple point processes Q in the product space $X \times \mathbb{N}$. The idea is to dissolve the particles of a realization μ seperately as elements of this product space. This is done by transforming P by means of

$$\chi : \mathcal{M}^{\cdot\cdot}(X) \longrightarrow \mathcal{M}^{\cdot}(X \times \mathbb{N}), \mu \longmapsto \kappa = \sum_{x \in supp\, \mu} \sum_{i=1}^{\mu(x)} \delta_{(x,i)}.$$

It is obvious that χ is measurable. Denote by Q the image of P under χ. By construction, χ resp. Q is a simple point process in $X \times \mathbb{N}$.

Assume in addition that P is of $k-$th order. Observe that for $f_1, \ldots, f_k \in \mathcal{K}(X)$

$$\nu_P^k(f_1 \otimes \cdots \otimes f_k) = \nu_Q^k(f_1 \circ pr_1 \otimes \cdots \otimes f_k \circ pr_k).$$

This equation combined with Krickeberg's decomposition for simple point processes will enable us to derive its generalization for non-simple processes. Let $g_j = f_j \circ pr_j$. Now theorem 4 and theorem 3, corollary 2 of [4], which we call *Krickeberg's decomposition* here, implies that

$$\nu_Q^k(g_1 \otimes \cdots \otimes g_k) = \sum_{\mathcal{J}} \grave{\nu}_Q^{|\mathcal{J}|}(\otimes_{\mathcal{J}} g_J).$$

Here the sum is taken over all partitions of $[k] = \{1, \ldots, k\}$ and $g_J = \prod_J g_j$. Moreover this decomposition is unique in the sense that the measures $\grave{\nu}_Q^m, 1 \leq m \leq k$, are uniquely determined by ν_Q^k and thereby by ν_P^k. The measures $\grave{\nu}_Q^m$ are the restrictions of ν_Q^m to the space $\grave{X}^m$ of all $m-$tupel with distinct components.

On the other hand we observe that

$$\grave{\nu}_Q^{|\mathcal{J}|}(\otimes_{\mathcal{J}} g_J) = \check{\nu}_P^{|\mathcal{J}|}(\otimes_{\mathcal{J}} f_J),$$

where the right hand side is defined for a partition $\mathcal{J} = \{J_1, \dots, J_m\}$ by

$$\int P(d\mu)\mu(dx_1)(\mu-\delta_{x_1})(dx_2)\dots(\mu-\delta_{x_1}-\dots-\delta_{x_{m-1}})(dx_m)f_{J_1}(x_1)\cdots f_{J_m}(x_m).$$

$\grave{\nu}_Q^{|\mathcal{J}|}$ resp. $\check{\nu}_P^{|\mathcal{J}|}$ are called *factorial measures of* P *of order* $|\mathcal{J}|$.

To summarize we have the following generalization of Krickeberg's decomposition:

Theorem 4.1 *If* P *is a point process in* X *of order* k*, then all factorial measures of* ν_P^k *are Radon measures, and* ν_P^k *can be uniquely decomposed into its factorial measures in the following way:*

$$\nu_P^k(f_1 \otimes \dots \otimes f_k) = \sum_{\mathcal{J}} \check{\nu}_P^{|\mathcal{J}|}(\otimes_{\mathcal{J}} f_J), f1, \dots, f_k \in \mathcal{K}. \tag{4.1}$$

We remark that this reduces to Krickeberg's decomposition if P is simple.

4.2 The main lemma

Let K be a law on $X^n, n \geq 1$. Consider the associated point process $P = MK$, the image of K under M, defined by

$$P(\varphi) = \int \varphi(\delta_{x_1} + \dots + \delta_{x_n})K(dx_1, \dots, dx_n), \varphi \in F_+.$$

P realizes configurations of n points in X. P is simple iff $K(\grave{X}^n) = 1$. Finally it is also obvious that P *is of any order* k, i.e. for any $k \geq 1$

$$\nu_P^k(f_1 \otimes \dots \otimes f_k) = \int \mu(f_1)\dots\mu(f_k)P(d\mu), f_1, \dots, f_k \in \mathcal{K},$$

defines a Radon measure on X^k, called the *$k-th$ moment measure of* P.

We now derive a useful decomposition of ν_P^k into its factorial measures: Denoting $\tilde{f}_j = f_{J_j}$ one has

$$\begin{aligned}\nu_P^k(f_1 \otimes \cdots \otimes f_k) &= \int \sum_{j_1,\ldots,j_k=1}^{n} f_1(x_{j_1}) \ldots f_k(x_{j_k}) K(dx_1 \ldots dx_n) \\ &= \sum_{m=1}^{k} \sum_{\mathcal{J} \in \pi_m([k])} \int_{X^n} \sum_{i_1 \neq \cdots \neq i_m}^{n} \tilde{f}_1(x_{i_1}) \ldots \tilde{f}_m(x_{i_m}) K(dx_1 \ldots dx_n).\end{aligned}$$

This is already a decomposition of ν_P^k into its factorial measures.

We continue the analysis of these measures: Let $\sigma \in \mathcal{S}_m$ be a permutation which orders a given tupel $(i_1, \ldots, i_m)$ in such a way that $i_{\sigma(1)} < \cdots < i_{\sigma(m)}$. Then general term of the last double sum equals

$$\int_{X^n} \sum_{i_1 \neq \cdots \neq i_m}^{n} \tilde{f}_{\sigma(1)}(x_{i_{\sigma(1)}}) \ldots \tilde{f}_{\sigma(m)}(x_{i_{\sigma(m)}}) K(dx_1 \ldots dx_n).$$

The right hand side can be written as

$$\sum_{\sigma \in \mathcal{L}_m} \int_{X^m} \tilde{f}_1(y_1) \ldots \tilde{f}_m(y_m)\, \sigma\big(\sum_{R \in \mathcal{P}_m([n])} K_R(dy_1 \ldots dy_m)\big).$$

Here K_R denotes the image of K under the projection $pr_R : (x_1, \ldots, x_n) \mapsto (x_j)_{j \in R}$, and $\sigma K_R(\otimes_{j=1}^m \tilde{f}_j) = K_R(\tilde{f}_{\sigma(1)} \otimes \cdots \otimes \tilde{f}_{\sigma(m)})$.

To summarize we proved the *main lemma* used above in the proof of lemma 3.3 .

Lemma 4.2 *If K is a probability on X^n and $k \geq 1$, then the point process $P = MK$ is of $k-$th order, and its $k-$th moment measure can be represented as*

$$\nu_P^k = \sum_{m=1}^{k} \sum_{J \in \pi_m([k])} (\sum_{\sigma \in \mathcal{S}_m} \sigma K_{(m)}) \circ \chi_{\mathcal{J}}^{-1}, \tag{4.2}$$

where $K_{(m)} = \sum_{R \in \mathcal{P}_m([n])} K_R$. Here $\sum_{\sigma \in \mathcal{S}_m} \sigma K_{(m)}$ is a representation of the $m-$th factorial measure of P. And, if $\mathcal{J} = \{J_1, \ldots, J_m\}$, the mapping $\chi_{\mathcal{J}}$ is given by $(z_1, \ldots, z_m) \longmapsto ((z_j^1)_{j \in J_1}, \ldots, (z_j^m)_{j \in J_m})$ with $z_j^1 = z_1$ for all $j \in J_1$ etc..

Historical note

The present work has its origins in the fundamental work of the Jena-school of probability in the sixties. Its *spiritus rector* was Johannes Kerstan who stimulated Klaus Matthes, Joseph Mecke and Karl-Heinz Fichtner. The great impact of their ideas in stochastic geometry and quantum statistical mechanics has now become manifest.

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