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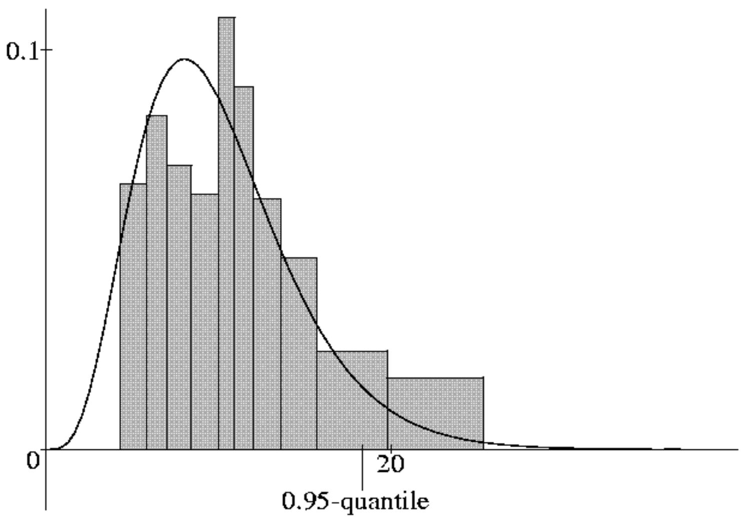
Classical Symmetric Point Processes

Lectures held at ICIMAF, La Habana, Cuba, 2010

Hans Zessin



Mathematische Statistik und
Wahrscheinlichkeitstheorie

Universität Potsdam – Institut für Mathematik

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Classical Symmetric Point Processes

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Abstract

The aim of these lectures is a reformulation and generalization of the fundamental investigations of Alexander Bach [2, 3] on the concept of probability in the work of Boltzmann [6] in the language of modern point process theory. The dominating point of view here is its subordination under the disintegration theory of Krickeberg [14]. This enables us to make Bach's consideration much more transparent. Moreover the point process formulation turns out to be the natural framework for the applications to quantum mechanical models.

Chapter 1

Disintegration of invariant probabilities

We introduce here in a discrete form Krickeberg's theory of disintegration of probability measures which are invariant under the action of a group. Thus the point of view of Felix Klein is adopted, which starts with a group acting on some given space and then asks for the collection of probabilities, which remain invariant under this action.

The character of this chapter is theoretical. It will serve as the base for all later applications.

1.1 Disintegration of probabilities

Here we start with the disintegration of probabilities with respect to a given equivalence relation given on the underlying space.

Let (Y, ν) be a discrete probability space, i.e. Y is a non-empty countable set and $\nu : Y \to [0, 1]$ a function summing up to 1.

Now consider an equivalence relation $\sim$ in Y. Then there exists a countable set Γ and a mapping r from Y onto Γ, having the following properties:

$$(x \sim y \iff r(x) = r(y)). \tag{1.1}$$

Exercise 1 *Show the existence of such a pair* (Γ, r)*.*

[Hint: One can choose Γ *as the set of all equivalence classes and* r *as the mapping that associates to an element* y *the equivalence class to which it belongs.]*

For a given $\gamma \in \Gamma$ let $Y_\gamma = \{r = \gamma\}$. And by κ we denote the distribution of r, thus $\kappa = r\nu$.

In this situation one can decompose the probability ν with respect to r as follows:

Lemma 1 *There exists a family of probabilities $\nu_\gamma, \gamma \in \Gamma$, on Y_γ satisfying*

$$\nu = \sum_{\gamma \in \Gamma} \nu_\gamma \cdot \kappa(\gamma). \tag{1.2}$$

This decomposition is unique in the sense that the family $(\nu_\gamma)_\gamma$ is $\kappa-$almost surely uniquely determined.

[In the sequel an exercise with a star $*$ is essential for the understandig of the main ideas.]

Exercise 2 * *Give a proof of this lemma.*

[Hint: For $\kappa-$almost all γ, i.e. for all γ with $\kappa(\gamma) > 0$, the probabilities ν_γ are the conditional probabilities $\nu(.|r = \gamma)$.]

Consider a function $\varphi : Y \longrightarrow Y$. We say that ν is *invariant under* φ if $\varphi\nu = \nu$. Thus the image of ν under φ is the same as ν itself. We now ask what does it mean for the ν_γ that ν is invariant under φ?

Lemma 2 *Let $\varphi : Y \longrightarrow Y$ be a function which preserves Y_γ for $\kappa-$almost all γ. Then ν is invariant under φ iff ν_γ is $\varphi-$invariant for $\kappa-$almost all γ.*

Proof. By (1.2) the $\varphi-$invariance of ν is equivalent to

$$\sum_{\gamma \in \Gamma} \varphi\nu_\gamma(x) \cdot \kappa(\gamma) = \sum_{\gamma \in \Gamma} \nu_\gamma(x) \cdot \kappa(\gamma) \text{ for all } x \in X. \tag{1.3}$$

It is obvious from (1.3) that the invariance of ν_γ implies for all γ with $\kappa(\gamma) > 0$ the invariance of ν.

It remains to prove the converse. Let $\gamma \in \Gamma$ with $\kappa(\gamma) > 0$. Both, ν_γ as well as $\varphi\nu_\gamma$, are probabilities on Y_γ; and they conincide there because of the uniqueness of the representation (1.2). **qed**

1.2 Invariance under a group

We now consider the special situation when the equivalence relation is induced by some group acting or operating in the underlying probability space.

1.2.1 Some facts about groups

We now consider a finite group $\mathcal{G}$ which *operates* or *acts* on Y. By this important notion we mean the following:

Definition 1 *An action of $\mathcal{G}$ on Y is an application*

$$\phi : \mathcal{G} \times Y \longrightarrow Y, (g, x) \longmapsto g.x, \tag{1.4}$$

satisfying the following conditions:

(1) $\forall g, g' \in \mathcal{G}, \forall x \in Y, g.(g'.x) = (gg').x;$

(2) $\forall x \in Y, 1.x = x.$ *(*1 *denotes the identy of* $\mathcal{G}$*.)*

We say that $\mathcal{G}$ operates or acts on Y if there exists such an action.

Exercise 3 * *Show that $\mathcal{G}$ operates on Y iff*

$$\chi : \mathcal{G} \longrightarrow \mathcal{S}(Y), g \longmapsto g := (x \mapsto g.x), \tag{1.5}$$

defines a homomorphism between $\mathcal{G}$ and the group $\mathcal{S}(Y)$ of all bijections of Y.

[Note that we use the same symbol g for an element of the group $\mathcal{G}$ and the mapping $g : x \mapsto g.x$.]

Exercise 4 * *Assume that ϕ is an action of $\mathcal{G}$ on Y. Show that then the mapping*

$$\tilde{\phi} : (g, x) \longmapsto \phi(g^{-1}, x) = g^{-1}.x$$

does not define an action of $\mathcal{G}$ on Y, unless the group $\mathcal{G}$ is abelian.

Comment 1 *The notion of an action is of fundamental importance. It describes situations which one meets typically in geometry (Euclidean with the group of isometries, hyperbolic with the Lorentz group etc.), but also, as we'll see in the sequel, in classical and quantum statistical mechanics.*

We are now interested in situations, where $\mathcal{G}$ operates in a special way.

Definition 2 *$\mathcal{G}$ operates transitively on Y if*

$$\forall x \in Y, \forall y \in Y, \exists g \in \mathcal{G}, g.x = y. \tag{1.6}$$

If $\mathcal{G}$ does not operate in a transitive way on Y, and in such situations we shall be interested in, we introduce the following equivalence relation:

$$x \sim y \Longleftrightarrow \exists g \in \mathcal{G}, y = g.x. \tag{1.7}$$

This relation describes the deviation from transitivity. The equivalence classes are called *orbits* of Y under $\mathcal{G}$.

Exercise 5 *Show that $\mathcal{G}$ operates transitively on each equivalence class.*

Exercise 6 *(Decomposition of a permutation into a product of cycles) Let $Y = \{1, \ldots, n\}$ and $\mathcal{S}_n = \mathcal{S}(Y)$. $\mathcal{S}_n$ operates on Y in a natural way. If σ is an element of $\mathcal{S}_n$ denote by $< \sigma >$ the cyclic group generated by σ in the following sense: $< \sigma >$ is the smallest subgroup of $\mathcal{S}_n$ containing σ.*

(1) Show that

$$< \sigma >= \{id, \sigma, \sigma^2, \ldots, \sigma^k\}. \tag{1.8}$$

Here k is the order of k, i.e. the smallest integer satisfying $\sigma^k = 1$. The cyclic group $< \sigma >$ operates also on Y. Denote by $F_1, \ldots, F_r$ the orbits of Y under $< \sigma >$.

(2) Show that the permutations σ_i, defined by

$$\sigma_i(x) = \begin{cases} \sigma(x), & \text{if } x \in F_i \\ = id, & \text{else ,} \end{cases}$$

are cycles of cardinality $|F_i|$, which commute and satisfy $\sigma = \sigma_1 \ldots \sigma_r$.

(3) Consider for Y={1,...,8} the permutation

$$\sigma = \begin{pmatrix} 12345678 \\ 36451872 \end{pmatrix}$$

Show that σ has the following cycle decomposition: $\sigma = (1345)(268)(7)$.

1.2.2 Krickeberg's disintegration [14]

Here we present Krickeberg's disintegration of probabilities which are invariant under certain groups. It is a special version of a much more general theorem.

Consider now the set $\mathcal{P}_0 Y$ of all probabilities ν on Y which are invariant under a given finite group $\mathcal{G}$. Thus $\nu \in \mathcal{P}_0 Y$ iff $g\nu = \nu$ for any $g \in \mathcal{G}$. (Recall that $g\nu$ denotes the image of ν under $g : x \mapsto g.x$.) As explained above $\mathcal{G}$ induced an equivalence $\sim$ relation in Y, to which belongs a set Γ and a surjective mapping $r : Y \longrightarrow \Gamma$ such that $(x \sim y \Longleftrightarrow r(x) = r(y))$. Its equivalence classes are the orbits of Y under $\mathcal{G}$.

Applying lemma 1 and lemma 2 we obtain that any $\nu \in \mathcal{P}_0 Y$, i.e. any $\mathcal{G}-$invariant probability ν, can be decomposed with respect to r in such a way that (1.2) holds true, where $\kappa-$almost all probabilities ν_γ are also $\mathcal{G}-$invariant.

But the *converse* is also true: If κ is a probability on Γ and $(\nu_\gamma)_{\gamma\in\Gamma}$ a family of probabilities on Y_γ which are $\mathcal{G}-$invariant for $\kappa-$almost all γ, then the probability ν, defined on Y by (1.2), is $\mathcal{G}-$invariant.

In this way we obtained a survey over all $\mathcal{G}-$invarant probabilties on Y. This result, if considered in the special case that all Y_γ are non-empty and finite, will be our main tool in the sequel.

Theorem 1 *(Krickeberg's disintegration) Consider the above situation where we assume in addition that*

$$\forall \gamma \in \Gamma, 0 < |Y_\gamma| < +\infty, \tag{1.9}$$

and denote by λ_γ the uniform distribution on Y_γ, then the equation

$$\nu = \sum_{\gamma\in\Gamma} \lambda_\gamma \cdot \kappa(\gamma) \tag{1.10}$$

induces a one-to-one correspondence between $\mathcal{G}-$invariant probabilities ν on Y and probabilities κ on Γ. In this case one has

$$\nu(y) = \frac{1}{|Y_{r(y)}|} \cdot \kappa(r(y)), y \in Y. \tag{1.11}$$

For a proof note that $\mathcal{G}$ operates transitively on each orbit and that by assumption (1.9) the only $\mathcal{G}-$invariant laws on the orbits Y_γ are the uniform laws.

Comment 2 *In passing we remark that the importance of this theorem for statistics is connected with the following interpretation: The transformation r is a so called sufficient statistic (basic notion in statistics) for the collection of all $\mathcal{G}-$invariant probabilities ν on Y.*

Chapter 2

Distributions of indistinguishable classical particles

Following the ideas of Bach [2, 3] we deduce by means of the above theorem the statistics of Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac. (This terminology is used in statistical mechanics.)

We here consider the following situation, which can be considered as the *microscopic level*: Let X be a finite set, having $d \geq 1$ elements, and Y the cartesian product X^n for some given n. Let $\mathcal{G}$ be the symmetric group $\mathcal{S}_n$ of all permutations σ of the set $[n] = \{1, \ldots, n\}$. This group operates on Y by means of the action

$$\phi : (\sigma, (x_1, \ldots, x_n)) \longmapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}); \text{ or equivalently} \tag{2.1}$$

$$\chi : \sigma \longmapsto ((x_1, \ldots, x_n) \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})). \tag{2.2}$$

This can be interpreted as follows: Given $(\sigma, (x_1, \ldots, x_n))$, by means of the action ϕ particle i is replaced by particle $\sigma(i)$ in the sense that *if particle i is in state x_i before the transformation, then after the transformation particle $\sigma(i)$ is in state x_i and particle i in state $x_{\sigma^{-1}(i)}$*. (For a more careful interpretation we refer to Hermann Weyl [23], p.119.)

Exercise 7 *(1∗) Show that the mapping defined in (2.1) is an action.*

(2) Deduce that this is not the case for the mapping

$$\tilde{\phi} : (\sigma, (x_1, \ldots, x_n)) \longmapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \tag{2.3}$$

As we saw above the group $\mathcal{G}$ induces in X^n the following equivalence relation: $x \sim y$ iff $\sigma x = y$ for some $\sigma \in \mathcal{S}_n$.

Exercise 8 * *Show that this equivalence relation can be represented by the following pair* (Γ, r)*:*

(2.4) $\Gamma = \mathcal{M}_n^{\cdot\cdot}(X) =$ *the collection of all point measures* γ *of total mass n;*
(2.5) $r : y = (x_1, \ldots, x_n) \longmapsto \gamma = \delta_{x_1} + \cdots + \delta_{x_n}$.

Note that both, Γ and all Y_γ, are finite sets. More precisely one has:

Exercise 9 * *Show that*

(1) $|\Gamma| = \binom{d+n-1}{n}$; *(recall that* $d = |X|$*.)*

[Indication (cf. Feller I [12]): Place n numbered particles into d cells. The particles are considered to be indistinguishable. We focus our attention on events that are independent of numbering. Such events depend only on configurations $\gamma = r(y) = \delta_{x_1} + \cdots + \delta_{x_n}$ *and not on* $y = (x_1, \ldots, x_n)$*. To understand formula* (1) *represent particles by stars and indicate the d cells by the d spaces between* $d + 1$ *bars. Thus*

$$| * * * | * |||| * * * * |$$

illustrates a configuration μ *of* $n = 8$ *balls in* $d = 6$ *cells* $\{1, \ldots, 6\}$*, where* $\gamma = 3\delta_1 + \delta_2 + 4\delta_6$ *and the cells are numbered from left to right. The question now is how many subsets of cardinality* $n = 8$ *exist in a set of cardinality* $n + d - 1 = 13$*?]*

(2) $|Y_\gamma| = \binom{n}{\gamma}$.

Here the right hand side of (2) is defined by

$$\left(\frac{n}{\prod_{a \in X} \gamma(a)!} \right).$$

[Indication: An element $y = (x_1, \ldots, x_n) \in Y_\gamma$ *can be identified with a partition of* $[n]$ *into the sets* $\{j | x_j = a\}, a \in X,$ *of cardinality* $\gamma(a)$*.]*

To summarize: We are in the above setting of Krickeberg's disintegration theorem. Denoting by Λ_γ the uniform distribution on Y_γ we have the

Theorem 2 *The equation*

$$P = \sum_{\gamma \in \mathcal{M}_n^{\cdot\cdot}} \Lambda_\gamma \cdot R(\gamma) \tag{2.6}$$

induces a one-to-one correspondence between $\mathcal{G}-$invariant probabilities P on Y and probabilities R on $\mathcal{M}_n^{\cdot\cdot}$. In this case one has

$$P(y) = \frac{1}{|Y_{r(y)}|} \cdot R(r(y)), y \in Y. \tag{2.7}$$

(From now on we often write $\mathcal{M}_n^{\cdot\cdot}$ if the underlying space is obvious.) This result allows to describe in principle all $\mathcal{G}-$invariant probabilities P on Y.

Comment 3 *In the sequel we call such $\mathcal{G}-$invariant probabilities P on Y symmetric; and elementary events $x = (x_1, \ldots, x_n)$ of such a probability space describe indistinguishable particles in the sense of Weyl's interpretation. Thus symmetry or indistinguishablity means that the law or state of the particle system is invariant under the action of the group $\mathcal{G}$. Probabilities R on $\mathcal{M}_n^{\cdot\cdot}(X)$ are called point processes in X realizing n particles. We then write $R \in \mathcal{PM}_n^{\cdot\cdot}(X)$. Such processes describe laws which realize n points or particles resp. data in the state space X. These processes are of fundamental importance for describing the chance mechanisms behind the data in numerous real situations, in particular in quantum mechanical models.*

Example 1 * **(The Maxwell-Boltzmann process and Maxwell-Boltzmann statistics)** *Let $n \in \mathbb{N}$ and ϱ denote a probability on X; thus $\varrho \in \mathcal{P}(X)$. The $n-$times product ϱ^n of ϱ is a probability on $Y = X^n$ and obviously symmetric. Using (2.7) the corresponding point process is given by the Maxwell-Boltzmann process for (n, ϱ):*

$$\mathcal{B}_\varrho^n(\gamma) = \begin{pmatrix} n \\ \gamma \end{pmatrix} \cdot \prod_{a \in X} \varrho(a)^{\gamma(a)}, \gamma \in \mathcal{M}_n^{\cdot\cdot}. \tag{2.8}$$

In case of $\varrho = \lambda$, the uniform law on X, one obtains the Boltzmann process for (n, d), denoted also by $\mathcal{B}_d^n$.

Example 2 * **(The Bose-Einstein statistics)** *Consider now the following point process in* X

$$\mathcal{E}_d^n(\gamma) = \frac{1}{\begin{pmatrix} d+n-1 \\ n \end{pmatrix}}, \gamma \in \mathcal{M}_n^{\cdot\cdot}. \tag{2.9}$$

[Convince yourself that this defines a probability on $\mathcal{M}_n^{\cdot\cdot}$*.]*

This is the uniform distribution on $\mathcal{M}_n^{\cdot\cdot}$ *and is called in quantum statistical mechanics the Bose-Einstein process for the parameters* (n,d)*. The corresponding symmetric distribution on* X^n*, called Bose-Einstein statistics, is given by*

$$E_d^n(y) = \frac{1}{\begin{pmatrix} n \\ r(y) \end{pmatrix}} \cdot \frac{1}{\begin{pmatrix} d+n-1 \\ n \end{pmatrix}}, y \in X^n. \tag{2.10}$$

Example 3 * **(The Fermi-Dirac statistics)** *Finally we consider the point process in* X *defined by*

$$\mathcal{D}_d^n(\gamma) = \frac{1}{\begin{pmatrix} d \\ n \end{pmatrix}}, \gamma \in \mathcal{M}_n^{\cdot}(X), 0 \leq n \leq d. \tag{2.11}$$

We set $\mathcal{D}_d^n \equiv 0$ *otherwise. Thus* $\mathcal{D}_d^n$ *is the uniform distribution on the space*

$$\mathcal{M}_n^{\cdot}(X) = \{\gamma \in \mathcal{M}_n^{\cdot\cdot}(X) | \gamma(a) \leq 1 \text{ for any } a \in X\} \tag{2.12}$$

of all subsets of X *of cardinality* n*. In classical statistical mechanics its name is Fermi-Dirac process for the parameters* (n,d)*.*

The corresponding symmetric distribution on X^n*, called the Fermi-Dirac statistics is*

$$D_d^n(y) = \frac{1}{n!} \cdot \frac{1}{\begin{pmatrix} d \\ n \end{pmatrix}}, y \in \grave{X}^n = \{r \in \mathcal{M}_n^{\cdot}(X)\}, n \leq n. \tag{2.13}$$

$D_d^n(y) = 0$ *else.* $\grave{X}^n$ *is the space of* $n-$*tupel with pairwise distinct components.*

Comment 4 $\mathcal{D}_d^n$ *is our first example of a simple point process in* X *realizing* n *particles, i.e. a law which is concentrated on* $\mathcal{M}_n^{\cdot}(X)$*. Physically simplicity is a formalization of Pauli's exclusion principle: It is not allowed that more than one particle occupies the same state.*

Chapter 3

Symmetric point processes

On the next higher level we now consider the action of the symmetric group on $\mathcal{M}^{\cdot\cdot}(X)$.

We now replace the space X^n by the spaces of configurations $\mathcal{M}^{\cdot\cdot}_n(X)$ resp.

$$\mathcal{M}^{\cdot\cdot}(X) = \text{ collection of all mappings } \mu : X \longrightarrow \mathbb{N}_0. \tag{3.1}$$

We proceed exactly as we did before.

As before, X is a finite set of cardinality d. And Y now denotes the collection $\mathcal{M}^{\cdot\cdot}(X)$ of all point measures on X, whereas $\mathcal{G}$ is the finite symmetric group $\mathcal{S}(X)$ of all permutations of X.

Exercise 10 * *Show that $\mathcal{M}^{\cdot\cdot}(X)$ is a countably infinite set.*

Thus all integrals taken over $\mathcal{M}^{\cdot\cdot}(X)$ are series!

The action of $\mathcal{S}(X)$ on $\mathcal{M}^{\cdot\cdot}(X)$ is given by

$$\chi : \sigma \longmapsto \sigma := (\mu \mapsto \sum_{a \in X} \mu(a) \cdot \delta_{\sigma(a)}). \tag{3.2}$$

[Verify that this is an action. Note also that $\sigma\mu$ is the image of the point measure μ under the transformation σ.]

This action can also be considered as a *time evolution* of particle configurations μ.

The following exercise shows that also now we are in the setting of Kricke-berg's theorem. (Recall that $d = |X|$.)

Exercise 11 * *Show that the equivalence relation induced by* $\mathcal{S}(X)$ *in* $\mathcal{M}^{\cdot\cdot}(X)$ *can be represented by the following pair* (Γ, r)*:*

(1) $\Gamma = \mathcal{M}^{\cdot\cdot}_{d}(\mathbb{N}_0)$; Γ *is countably infinite;*

(2) $r : \mu \longmapsto \gamma,$ *where* $\gamma(j) = cd\{\mu = j\}, j \geq 0$;

(3) $|Y_\gamma| = \begin{pmatrix} d \\ \gamma \end{pmatrix}$.

[Hint: One can identify a point configuration $\mu \in Y_\gamma$ *with a partition of the set* X *whose elements* $M_j = \{\mu = j\}$ *have the cardinality* $\gamma(j)$*. Furthermore we use that a permutation preserves the particle numbers.]*

Denoting again by Λ_γ the uniform distribution on Y_γ we therefore obtain

Theorem 3 *The equation*

$$P = \sum_{\gamma \in \mathcal{M}^{\cdot\cdot}_d(\mathbb{N}_0)} \Lambda_\gamma \cdot R(\gamma) \tag{3.3}$$

induces a one-to-one correspondence between point processes $R \in \mathcal{PM}^{\cdot\cdot}_d(\mathbb{N}_0)$ *and* $\mathcal{G}-$*invariant point processes* P *in* X. *In this case one has*

$$P(\mu) = \frac{1}{|Y_{r(\mu)}|} \cdot R(r(\mu)), \mu \in \mathcal{M}^{\cdot\cdot}(X). \tag{3.4}$$

Again such $\mathcal{G}-$invariant point processes will be called *symmetric*. If the action is interpreted as a time evolution then $\mathcal{G}-$invariant point processes are also called *equilibrium states*.

We now give some applications of this result, which go back to Ludwig Boltzmann [6] and which have been reconstructed by Alexander Bach [2, 3]. The mapping r acts as intermediary between the levels $\mathcal{M}^{\cdot\cdot}(X)$ and $\mathcal{M}^{\cdot\cdot}_d(\mathbb{N}_0)$.

Exercise 12 * *Show that the Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac processes* $(\mathcal{B}^n_d)_{n\geq 0}, (\mathcal{E}^n_d)_{n\geq 0}$ *as well as* $(\mathcal{D}^n_d)_{n\geq 0}$ *are symmetric with respect to the action (3.1).*

[Hint: Under a permutation σ *of* X *the particle number and the simplicity of a configuration is a constant of the motion; and thus* $\mathcal{M}^{\cdot\cdot}_n$ *as well as* $\mathcal{M}^{\cdot\cdot}_n$ *and* $\mathcal{M}^{\cdot}$ *remain unchanged under* σ*.]*

We'll now discover by means of equation (3.3) the point processes R which correspond to these classical processes.

Example 4 (Fermi-Dirac) *The image of* $\mathcal{D}^n_d$ *under the transformation* r *obviously is the following point process in* $\mathbb{N}_0$*:*

$$\mathbb{D}^n_d = \Delta_{\gamma_0}, \tag{3.5}$$

where $\gamma_0 \in \mathcal{M}^{\cdot\cdot}_d(\mathbb{N}_0)$ *is the following point measure:* $\gamma_0(0) = d - n, \gamma_0(1) = n$, *and* $\gamma_0(j) = 0$ *for* $j \geq 2$.

Example 5 (Bose-Einstein) *The image of* $\mathcal{E}^n_d$ *under* r *is the point process in* $\{0, \ldots, n\}$, *defined by*

$$\mathbb{E}^n_d(\gamma) = \begin{pmatrix} d \\ \gamma \end{pmatrix} \cdot \frac{1}{\begin{pmatrix} d+n-1 \\ n \end{pmatrix}}, \gamma \in \mathcal{M}^{\cdot\cdot}_d(\{0, \ldots, n\}). \tag{3.6}$$

As an aside we obtain from this result the following combinatorial identity:

$$\sum_{\gamma \in \mathcal{M}^{\cdot\cdot}_d(\{0,\ldots,n\})} \binom{d}{\gamma} = \binom{d+n-1}{n}. \tag{3.7}$$

Example 6 (Maxwell-Boltzmann) *Finally it is easy to see that the image of $\mathcal{B}^n_\varrho$ under r is given by*

$$\mathbb{B}^n_\varrho(\gamma) = \binom{d}{\gamma} \cdot n! \cdot \prod_{j \geq 0} \frac{1}{(j!)^{\gamma(j)}} \cdot \prod_j \prod_{a:\mu(\gamma)(a)=j} \varrho(a)^j. \tag{3.8}$$

Here $\mu(\gamma)$ denotes an arbitrary element from $Y_\gamma = \{r = \gamma\}$. If ϱ is the uniform distribution on X one obtains

$$\mathbb{B}^n_d(\gamma) = \binom{d}{\gamma} \cdot n! \cdot \prod_{j \geq 0} \frac{1}{(j!)^{\gamma(j)}} \cdot \frac{1}{d^n}. \tag{3.9}$$

As Bach [2, 3] observed the formulas (3.7) and (3.8) are due to Boltzmann [6].

Recall that any point process $R \in \mathcal{PM}^{\cdot\cdot}_d(\mathbb{N}_0)$ defines via (3.2) resp. (3.3) a symmetric process in X. To stress this we give a non-classical example.

Example 7 *Consider the following law on $\mathbb{N}$: Given $0 < z < 1$, let*

$$\tau(j) = \frac{1}{\kappa(z)} \cdot \frac{z^j}{j}, j \in \mathbb{N}.$$

Then consider the finite point process $\mathcal{B}^d_\tau$ in $\mathbb{N}$. It is easily shown that the corresponding symmetric process is given by

$$P^{(n)}_\tau(\mu) = \frac{1}{\kappa(z)^d} \cdot z^{N(r(\mu))} \cdot \frac{1}{\prod_{j \geq 1}} j^{r(\mu)(j)}, \mu \in \mathcal{M}^{\cdot\cdot}(X). \tag{3.10}$$

We finally consider a mixing of the families $(\mathcal{B}^n_d)_{n \geq 0}, (\mathcal{E}^n_d)_{n \geq 0}$ resp. $(\mathcal{D}^n_d)_n$ with respect to n. We saw in exercise 13 that they define symmetric point processes in X. To be more precise, we consider two examples.

Example 8 (Mixing of Fermi-Dirac processes) *We here consider the mixture of the family $(\mathcal{D}^n_d)_{n \geq 0}$ with respect to the Binomial distribution for the parameters $(d, p), 0 < p < 1$. This is the point process*

$$\mathcal{F}^d_p = \sum_{n=0}^{d} \binom{d}{n} \cdot p^n (1-p)^{d-n} \cdot \mathcal{D}^n_d. \tag{3.11}$$

Thus in a first step the number n is realized according to the Binomial distribution for (d,p), and the in a second step a configuration consisting of n particles is realized in X according to the process $\mathbb{D}_d^n$. We call $\mathcal{F}_p^d$ a mixed Fermi-Dirac process for the parameters (d,p).

It is obvious that

$$\mathcal{F}_p^d(\mu) = p^{|\mu|} \cdot (1-p)^{d-|\mu|}, \mu \in \mathcal{M}^{\cdot}(X). \tag{3.12}$$

Here $\mathcal{M}^{\cdot}(X)$ is the union of all $\mathcal{M}_n^{\cdot}(X), n \geq 0$. Thus this point process in X describes nothing else than coin tossing in X.

Exercise 13 *Show that in the last example the image under r is the Binomial distribution for (d,p).*

An important step for the developement in the sequel is the

Example 9 (Mixing of Maxwell-Boltzmann processes yields the Poisson process) *We consider an arbitrary non-trivial finite measure ϱ on X together with its normalization $\hat{\varrho}$ to a probability law. We then mix the family $\mathcal{B}_{\hat{\varrho}}^n, n \geq 0$, by means of the Poisson distribution on $\mathbb{N}_0$ for the parameter $\varrho(X)$. Thus we obtain the point process P_ϱ defined by*

$$P_\varrho = e^{-\varrho(X)} \sum_{n \geq 0} \frac{\varrho(X)^n}{n!} \mathcal{B}_{\hat{\varrho}}^n. \tag{3.13}$$

This is another famous process called the Poisson point process in X with the intensity measure ϱ. (The following scholion will justify this terminology.) If we now choose $\varrho = z \cdot \lambda, z > 0$, where λ denotes the uniform distribution on X, we obtain the Poisson processes $P_{z\lambda}, z > 0$. Its symmetry is a famous result of Doob [8]. It says that the laws $P_{z\lambda}, z > 0$, are equilibrium states for the time evolutions $\mathcal{G}$. (This shows in particular that a time evolution can have a continuity of equilibrium states.)

Scholion: The Poisson process. The method of the Campbell measure

The aim here is to study in detail the Poisson processes constructed above. This is done by means of the socalled Campbell measure.

The method of the Campbell measure

Let X be finite and non-empty and P a point process in X. The *Campbell measure* of P, denoted by $\mathcal{C}_P$, is a measure on the product space $X \times \mathcal{M}^{\cdot\cdot}_f(X)$ defined by

$$\mathcal{C}_P(h) = \sum_{\mu \in \mathcal{M}^{\cdot\cdot}_f(X)} \sum_{a \in X} h(a, \mu)\mu(a)P(\mu), h \in F_+. \tag{3.14}$$

This measure contains all informations about the process P and plays a similar role as the Fourier or the Laplace transform of P. It determines completely P by the following argument: By definition

$$\mathcal{C}_P((a, \mu)) = \mu(a)P(\mu), (a, \mu) \in X \times \mathcal{M}^{\cdot\cdot}_f(X). \tag{3.15}$$

Thus P is determined by $\mathcal{C}_P$ on $\mathcal{M}^{\cdot\cdot}_f(X) \setminus \{0\}$, and thereby on the whole space because it is a probability.

Furthermore, if the Campbell measure is explicitly known, one often can calculate the *moment measures* of P. The first moment measure, called *intencity of* P, is given by

$$\nu^1_P(B) = \mathcal{C}_P(B \times \mathcal{M}^{\cdot\cdot}_f(X)), B \subset X. \tag{3.16}$$

The second *moment measure* of P is

$$\nu^2_P(B \times C) = \mathcal{C}_P(1_B \cdot \zeta_C), B, C \subset X. \tag{3.17}$$

Here ζ_C is defined by

$$\zeta_C : \mathcal{M}^{\cdot\cdot}_n(X) \longrightarrow \mathbb{N}_0, \mu \longmapsto \mu(C). \tag{3.18}$$

Exercise 14 * *Show that*

$$\nu^1_P(B) = P(\zeta_B); \tag{3.19}$$

$$\nu^2_P(B \times C) = P(\zeta_B \cdot \zeta_C). \tag{3.20}$$

The Poisson process

Let X be finite, non-empty and ϱ a finite measure on X. The associated Poissonprocess P_ϱ has been constructed above.

Exercise 15 ** **(The Poisson process)** *Let $\varrho \in \mathcal{M}(X) \setminus \{0\}$. Show that*

(1)

$$P_\varrho(\mu) = e^{-\varrho(X)} \frac{1}{\prod_{a\in X} \mu(a)!} \prod_{a\in X} \varrho(a)^{\mu(a)}, \mu \in \mathcal{M}_n^{\cdot\cdot}(X).$$

Consider now on the discrete probability space $(\mathcal{M}_n^{\cdot\cdot}(X), P_\varrho)$ the random variables $\zeta_x, x \in X$. We call the tripel

$$\mathcal{P}_\varrho = (\mathcal{M}_n^{\cdot\cdot}(X), P_\varrho, (\zeta_x)_{x\in X}) \tag{3.21}$$

the Poisson random point field in X with intensity measure ϱ. This is justified as follows:

(2) Deduce from (1) that $\mathcal{P}_\varrho$ is independent, i.e. the field variables $\zeta_x, x \in X$, are independent.

(3) Deduce that if B, C are non-empty disjoint subsets of X then ζ_B and ζ_C are independent random variables.

(4) Deduce from (1) that ζ_x has a Poisson distribution with parameter $\varrho(x)$. This implies that $\zeta_B, B \subset X$, has a Poisson distribution with parameter $\varrho(B)$. (Why?)

Part (1) of the last exercise immediately implies the fundamental

Theorem 4 **(Mecke's formula)** *The Campbell measure of the Poisson process is given by*

$$\mathcal{C}_{P_\varrho}(h) = \sum_\mu \sum_a h(a, \mu + \delta_a)\varrho(a) P_\varrho(\mu), h \in F_+. \tag{3.22}$$

Thus the Campbell measure is given by

$$\mathcal{C}_{P_\varrho}((a,\mu)) = \varrho(a) \cdot P_\varrho(\mu - \delta_a), (a,\mu) \in X \times \mathcal{M}_f^{\cdot\cdot}(X). \tag{3.23}$$

(Here the right hand side is understood to be zero in the case that $\mu(a) = 0$.)

Exercise 16 * *Deduce from Mecke's equation the first and second moment measure of the Poisson process. To be more precise show that*

(1) $\nu_P^1 = \varrho$. This means that the measure ϱ is the socalled intensity measure $\nu_{P_\varrho}^1$ of P_ϱ, which measures the mean number of particles in B.

(2) *For any subsets* $B, C \subset X$

$$\nu^2_{P_\varrho}(B \times C) = P_\varrho(\zeta_B \cdot \zeta_C) = \varrho(B) \cdot \varrho(C) + \varrho(B \cap C). \tag{3.24}$$

Example 10 (Einstein's fluctuation formula, as derived 1927 by W. Bothe [7]) *Consider the family* $(P_{z\lambda})_{z>0}$ *of symmetric Poisson processes in* X. *We can mix this process with respect to the exponential law for a given parameter* $r > 0$ *to obtain another symmetric point process in* X:

$$Q^{(r)}_\lambda = \int_0^\infty P_{z\lambda} \frac{1}{r} \cdot \exp(-\frac{z}{r}) dz. \tag{3.25}$$

Such a process is called a mixed Poisson process for the parameters (r, λ).

(1) *Show that the intensity measure of* $Q^{(r)}_\lambda$ *is given by* $r \cdot \lambda$.

(2) *Show that the covariance measure of* $Q^{(r)}_\lambda$, *defined by* $\gamma_{Q^{(r)}_\lambda}(B \times C) = cov_{Q^{(r)}_\lambda}(\zeta_B, \zeta_C), B, C \subset X$, *is given by the signed measure*

$$\gamma_{Q^{(r)}_\lambda} = \nu^2_{Q^{(r)}_\lambda} - (\nu^1_{Q^{(r)}_\lambda})^2. \tag{3.26}$$

(3) *Show that*

$$\nu^2_{Q^{(r)}_\lambda}(B \times C) = 2 \cdot r^2 \cdot \lambda^2(B \times C) + r\lambda(B \cap C).$$

Deduce that

$$\gamma_{Q^{(r)}_\lambda}(B \times C) = r^2 \cdot \lambda^2(B \times C) + r\lambda(B \cap C). \tag{3.27}$$

In particular one obtains from this formula the variance of the counting variable ζ_B:

$$V_{Q^{(r)}_\lambda}(\zeta_B) = (r \cdot \lambda(B))^2 + r \cdot \lambda(B) = Q^{(r)}_\lambda(\zeta_B)^2 + Q^{(r)}_\lambda(\zeta_B). \tag{3.28}$$

Formula (3.27) is called *Einstein's fluctuation formula.* ([10])

Exercise 17 (Einstein's fluctuation formula as derived 1915 by Max von Laue [16]) *Show in the situation of the last exercise that the counting variables* $\zeta_x, x \in X$, *are identically distributed according to the geometric distribution for the parameter* $\frac{1}{1+r}$, *i.e.*

$$Q^{(r)}_\lambda\{\zeta_x = k\} = \frac{1}{1+r} \cdot (\frac{r}{1+r})^k, k \geq 0, x \in X.$$

Deduce from this fact again that

$$(3.29) \qquad \nu^1_{Q^{(r)}_\lambda}(B) = Q^{(r)}_\lambda(\zeta_B) = r \cdot \lambda(B), B \subset X,$$

and the fluctuation formula

$$(3.30) \qquad V_{Q^{(r)}_\lambda}(\zeta_x) = Q^{(r)}_\lambda(\zeta_x)^2 + Q^{(r)}_\lambda(\zeta_x), x \in X.$$

Comment 5 *We indicate shortly a physical interpretation of the point process $Q^{(r)}_\lambda$. Recall that this process is a mixture of the Poisson processes $P_{Z\lambda}$, where Z is a random variable with values in $]0, +\infty[$, distributed according to an exponential distribution with a given parameter $r > 0$. Consider the random variable ζ_B. It counts the number of 'quanta' in the region B. The mean number of quanta in B thus is given by the random variable $Z \cdot \lambda(B)$. In this model ζ_x, the number of quanta in the 'cell' x, is geometrically distributed for the parameter $\frac{1}{1+r}$. Thus its mean number there is r and its variance $r^2 + r = r(1 - r)$.*

Comment 6 *In example 9 we mixed the Poisson family $P_{z\lambda}, z > 0$, with respect to an exponential distribution. More generally one can mix with respect to other laws on the positive half axis. An important example is, if we mix with respect to a $\Gamma-$distribution $\Gamma(\kappa, \nu), \kappa, \nu > 0$. The counting variable in question then has a negative binomial distribution. We'll come to this shortly. Even more generally one could consider the whole intensity measure of the Poisson process as random. (This is the concept of a socalled Cox process.) The question is what kind of random intensity measures appear in nature (i.e. in quantum mechanical, ecological, epidemiological etc. models)? We can only pose this fundamental question here. (See [1] for more details.)*

Chapter 4

The finite Papangelou process

We now generalize the class of finite point processes and present a general construction of socalled Papangelou processes. The Poisson process is one example but another fundamental one is the Polya sum process.

Again X is non-empty and finite. We now consider kernels of the form

$$\pi : \mathcal{M}^{\cdot\cdot}(X) \times X \longrightarrow [0, +\infty[, (\eta, a) \longmapsto \pi(\eta, a). \tag{4.1}$$

Thus $\pi(\eta, .)$ is a measure on X. Such a kernel is the point of departure for the construction.

Example 11 *Let ϱ be some finite measure on X and $0 < z < 1$.*

(1) $\pi(\eta, .) \equiv \varrho$ *;*

(2) $\pi(\eta, .) = z \cdot (\varrho + \eta)$.

(3) $\pi(\eta, .) = V(., \eta) \cdot \varrho$. *Here V is a function with values in $]0, +\infty[$. In classical statistical mechanics V has the form $\exp(-E(a, \eta))$ where $E(a, \eta)$ is the energy of particle a in the environment η.*

π defines a sequence of kernels on the cartesian products of X: Given $\eta, m \geq 1$ and $a_1, \ldots, a_m \in X$ let

$$\pi^{(m)}(\eta, (a_1, \ldots, a_m)) = \pi(\eta, a_1) \cdot \pi(\eta + \delta_{a_1}, a_2) \cdots \pi(\eta + \delta_{a_1} + \cdots + \delta_{a_{m-1}}, a_m).$$

The $\pi^{(m)}(\eta, .)$ are finite measures on X^m. Let

$$Z^{(m)}(\eta) = \pi^{(m)}(\eta, X^m), Z^{(0)}(\eta) = 1,$$

be the corresponding total measures. We assume now that π *is integrable* in the sense that

$$0 < \Xi(\eta) = \sum_{m\geq 0} \frac{Z^{(m)}(\eta)}{m!} < +\infty. \tag{4.2}$$

In this situation we can construct a finite point process $\mathcal{P}_\pi^\eta$ in X by means of a given kernel π and a given *environment* η in two steps:

(1) Realize a natural number $m \in \mathbb{N}_0$ according to the law

$$\frac{1}{\Xi(\eta)} \cdot \frac{Z^{(m)}(\eta)}{m!};$$

(2) realize then a configuration $\delta_{a_1} + \cdots + \delta_{a_m}$ of m *particles* or *quanta* according to the image of the law

$$\frac{1}{Z^{(m)}(\eta)} \cdot \pi^{(m)}(\eta, .)$$

under the transformation $M_m : (a_1, \ldots, a_m) \longrightarrow \delta_{a_1} + \cdots + \delta_{a_m}$.

Formally this finite point process is given by

(4.3)

$$\mathcal{P}_\pi^\eta(\varphi) = \frac{1}{\Xi(\eta)} \cdot \sum_{m=0}^{+\infty} \frac{1}{m!} \sum_{(a_1,\ldots,a_m)} \varphi(\delta_{a_1} + \cdots + \delta_{a_m})\pi^{(m)}(\eta; a_1, \ldots, a_m), \varphi \in F_+.$$

(Here we write $\pi^{(m)}(\eta; a_1, \ldots, a_m)$ instead $\pi^{(m)}(\eta, (a_1, \ldots, a_m))$ for simplicity.)

We call this process $\mathcal{P}_\pi^\eta$ the (finite) *Papangelou process with environment η and kernel π.*

We then make another assumption which will be of fundamental importance in the sequel. We assume that π satisfies the socalled *cocycle condition*, i.e.:

(**cc**) the kernel π is symmetric in the sense that
$(a, b) \longmapsto \pi(\eta, a)\pi(\eta + \delta_a, b), \eta \in \mathcal{M}^{\cdot\cdot}(X)$,
define symmetric measures on X^2.

Under these conditions we call the point process $\mathcal{P}_\pi^\eta$ the finite *Papangelou process with environment η and kernel π.*

Exercise 18 * *Consider the above examples (1) and (2) for a given ϱ.*

$\natural$ *Show first that for both examples the above two conditions are fullfilled.*

ad (1) *Show that in the first example one obtains the Poisson process with intensity measure ϱ. Moreover, ζ_X has a Poisson distribution for the parameter $\varrho(X)$.*

ad (2) *Show that in the second case*

(a) $Z^{(m)}(\eta) = z^m \cdot |\varrho+\eta|^{[m]}$. *Here* $a^{[m]} = a(a+1)\dots(a+m-1)$ *and* $|\varrho| = \varrho(X)$.

(b) $\Xi(\eta) = \exp(|\varrho+\eta|\kappa(z))$, *where* $\kappa(z) = \sum_{j\geq 1}\frac{z^j}{j} = -\ln(1-z)$.

Thus the total particle number ζ_X is distributed according to

$$\mathcal{P}_\pi^\eta\{\zeta_X = m\} = (1-z)^{|\varrho+\eta|}\cdot\frac{z^m\cdot|\varrho+\eta|^{[m]}}{m!}, m\geq 0.$$

In case that the environment is void, i.e. $\eta = 0$, we call $\mathcal{P}_\pi^0$ the *Polya sum process for the parameters* (z,ϱ) and write $\mathcal{P}_{z,\varrho}$. Note, if for this process ϱ is a probability, then

$$\mathcal{P}_\pi^0\{\zeta_X = m\} = (1-z)\cdot z^m, m\geq 0.$$

Thus the counting variable ζ_X has a geometric distribution with parameter $1-z$.

We now show that the Campbell measure of the Papangelou process for the parameters (η,π) solves an integration by parts formula.

Theorem 5 *Under the conditions* (4.2) *and* (**cc**) $\mathcal{P}_\pi^\eta, \eta\in\mathcal{M}_f^{\cdot\cdot}(X)$ *is a solution* $P^\eta\in\mathcal{PM}^{\cdot\cdot}(X)$ *of the equation*

$$\mathcal{C}_{P^\eta}(h) = \sum_\mu\sum_a h(a,\mu+\delta_a)\pi(\mu+\eta,a)P^\eta(\mu), h\in F_+. \tag{4.4}$$

Proof. Given $h\in F_+$ the symmetrie of π

$$\begin{aligned}
\mathcal{C}_{P_\pi^\eta}(h) = &\quad \frac{1}{\Xi(\eta)}\cdot\sum_{m\geq 0}\frac{1}{m!}\sum_{j=1}^m\sum_{(a_1,\dots,a_m)} h(a_j,\delta_{a_1}+\dots+\delta_{a_m})\pi^{(m)}(\eta;a_1,\dots,a_m)\\
= &\quad \frac{1}{\Xi(\eta)}\cdot\sum_{m\geq 1}\frac{1}{(m-1)!}\sum_{(a_1,\dots,a_{m-1})}\sum_a h(a,\delta_{a_1}+\dots+\delta_{a_{m-1}}+\delta_a)\cdot\\
&\quad \cdot\pi(\eta+\delta_{a_1}+\dots+\delta_{a_{m-1}},a)\pi^{(m-1)}(\eta;a_1,\dots,a_{m-1})\\
= &\quad \sum_\mu\sum_a h(a,\mu+\delta_a)\pi(\eta+\mu,a)P_\pi^\eta(\mu).
\end{aligned}$$

qed

Equation (4.4) can equivalently stated as

$$\mathcal{C}_{P^\eta_\pi}((a,\mu)) = \pi(\mu+\eta-\delta_a,a)\cdot P^\eta_\pi(\mu-\delta_a), \mu(a)\geq 1.$$

Example 12 *In case of example* (1) *we obtain here Mecke's equation. In case of example* (2) *the equation is given by*

$$\mathcal{C}_{\mathcal{P}_{z,\varrho}}(h) = \sum_\mu\sum_a h(a,\mu+\delta_a)z(\varrho+\mu)(a)\mathcal{P}_{z,\varrho}(\mu), h\in F_+. \tag{4.5}$$

We now deduce by means of this integration by parts formula the main properties of the Polya sum process.

Lemma 3 *For any $B\subset X$ and any event $\mathcal{N}$ which happens outside B, i.e. $\mathcal{N}$ depends only on ζ_{B^c},*

$$\mathcal{P}_{z,\varrho}(\{\zeta_B=k\}\cap\mathcal{N}) = \frac{z^k}{k!}\cdot\varrho(B)^{[k]}\cdot\mathcal{P}_{z,\lambda}(\{\zeta_B=0\}\cap\mathcal{N}). \tag{4.6}$$

Proof. Let $k\geq 1$. Given B and $\mathcal{N}$ we obtain by means of (3.31)

$$\begin{aligned}
\mathcal{P}_{z,\varrho}(\{\zeta_B=k\}\cap\mathcal{N}) \quad &= \frac{1}{k}\cdot\sum_\mu\sum_{a\in B} 1_{\{\zeta_B=k\}}(\mu)1_{\mathcal{N}}(\mu)\mu(a)\mathcal{P}_{z,\varrho}(\mu)\\
&= \frac{1}{k}\cdot\sum_\mu\sum_{a\in B} 1_{\{\zeta_B=k-1\}}(\mu)1_{\mathcal{N}}(\mu)z(\varrho+\mu)(a)\mathcal{P}_{z,\varrho}(\mu)\\
&= \frac{z}{k}\cdot[\varrho(B)+(k-1)]\mathcal{P}_{z,\varrho}(\{\zeta_B=k-1\}\cap\mathcal{N}).
\end{aligned}$$

Iterating this step shows the lemma. **qed**

Exercise 19 *

(1) Consider $k_1,\ldots,k_n\geq 1$ and pairwise disjoint subsets $B_1,\ldots,B_n$ of X. Show by means of the lemma that

$$\mathcal{P}_{z,\varrho}\{\zeta_{B_1}=k_1,\ldots,\zeta_{B_n}=k_n\} = \mathcal{P}_{z,\varrho}\{\zeta_{B_1}=0,\ldots,\zeta_{B_n}=0\}\cdot\prod_{j=1}^n\frac{z^{k_j}}{k_j!}\cdot\varrho(B_j)^{[k_j]}.$$

(2) Calculate then $\mathcal{P}_{z,\varrho}\{\zeta_{B_1}=0,\ldots,\zeta_{B_n}=0\}$. (This is given by $\exp(-\varrho(B_1\cup\ldots B_n)\cdot\kappa(z))$.)

(3) Deduce that the distribution of the counting variables $\zeta_B, B \subset X$, is negative binomial:

$$\mathcal{P}_{z,\varrho}\{\zeta_B = k\} = \exp(-\varrho(B)\kappa(z)) z^k \frac{\varrho(B)^{[k]}}{k!}, k \geq 0. \tag{4.7}$$

In particular, if ϱ is the counting measure λ then any ζ_x has a geometric distribution with parameter $1 - z$.

(4) Deduce that the field variables $\zeta_x, x \in X$, of the Polya sum process are independent.

(5) Show that the intensity measure of the Polya sum process is given by

$$\nu^1_{\mathcal{P}_{z,\varrho}} = \frac{z}{1-z}\varrho. \tag{4.8}$$

Remark 1 *The above considerations used only that $\mathcal{P}_{z,\varrho}$ is a solution of the integration by parts formula. And we saw that this equation determines completely the distribition of the whole process. Thus, looking at the equation, we see that the Polya sum process is completely determined by the socalled Papangelou kernel $\pi(\mu, \cdot) = z \cdot (\varrho + \mu)(\cdot)$.*

We next show that the Polya sum process for the parameters (z, cd) is symmetric in the sense of the last chapter. Here cd denotes the counting measure on X. (Recall that $d = cd(X) = |X|$.)

Theorem 6 *$\mathcal{P} = \mathcal{P}_{z,cd}$ is invariant under the action of any permutation $\sigma \in \mathcal{S}(X)$.*

Proof. We use the method of the Campbell measure. By means of equation (3.32) we have

$$\begin{aligned} \mathcal{C}_{\sigma\mathcal{P}}(h) &= \sum_\mu \sum_a h(\sigma a, \sigma\mu)\mu(a)\mathcal{P}(\mu) \\ &= \sum_\mu \sum_a h(\sigma a, \sigma\mu + \delta_{\sigma a}) z(cd + \mu)(a)\mathcal{P}(\mu) \\ &= \sum_\mu \sum_a h(a, \mu + \delta_a) z(cd + \mu)(a)\sigma\mathcal{P}(\mu). \end{aligned}$$

Thus we see that $\sigma\mathcal{P}$ solves the equation (3.32). Since $\mathcal{P}$ is the unique solution we obtain the result. **qed**

Combining this theorem with Krickeberg's disintegration we obtain the corresponding point process in $\mathbb{N}_0$ of d particles.

Theorem 7

$$R_{z,cd}(\gamma) = (1-z)^d \cdot \binom{d}{\gamma} \cdot z^{N(\gamma)}, \gamma \in \mathcal{M}_{\ddot{d}}(\mathbb{N}_0). \tag{4.9}$$

Here $N(\gamma) = \sum_{j\geq 1} j \cdot \gamma(j)$.

Proof. By formula (3.4) one has

$$R_{z,cd}(\gamma) = \binom{d}{\gamma} \cdot \mathcal{P}_{z,cd}(\mu(\gamma)), \tag{4.10}$$

where $\mu(\gamma)$ is any element chosen in Y_γ. To calculate $\mathcal{P}_{z,cd}(\mu(\gamma))$, recall that $\mu = \mu(\gamma)$ is an element of Y_γ iff $cd\{\mu(.) = j\} = \gamma(j), j \geq 0$. By construction one has

$$\mathcal{P}_{z,cd}(\mu) = \exp(-d \cdot \kappa(z)) \cdot \prod_{a \in X} \frac{z^{\mu(a)}}{\mu(a)!} \cdot 1^{[\mu(a)]}.$$

But $1^{[\mu(a)]} = \mu(a)!$ by definition and $\prod_{a\in X} z^{\mu(a)} = z^{N(\gamma)}$. **qed**

Exercise 20 *The aim is to calculate the distribution of the counting variable* N.

(1) *Show that*

$$R_{z,cd}(N = n) = (1-z)^d \cdot z^n \cdot \tilde{Q}_n(d), n \geq 0, \tag{4.11}$$

where $\tilde{Q}_n(d) = \sum_{\gamma : N(\gamma)=n} \binom{d}{\gamma}$.

(2) *Calculate the terms* $\tilde{Q}_n(d)$. *Show that* $\tilde{Q}_0(d) = 1$

$$\tilde{Q}_n(d) = \frac{1}{n!} \cdot (d+n-1), n \geq 1, \tag{4.12}$$

Exercise 21 *Denoting by* M *the identity, deduce that* $R_{z,cd}\{M = (.)|N = 0\} = \Delta_{\gamma_0}$ *with* $\gamma_0 = d \cdot \delta_0$*; and if* $n \geq 1$

(4.13)

$$R_{z,cd}\{M = \gamma | N = n\} = \frac{n!}{d+n-1} \cdot \binom{d}{\gamma} \cdot 1_{\{N=n\}}(\gamma), \gamma \in \mathcal{M}_{\ddot{d}}(\mathbb{N}_0) \setminus \{\gamma_0\}.$$

Comment 7 *The Polya sum process for the parameters* (z, cd) *is also a model for the light emission as it has been discussed in the early days of quantum mechanics.(See [4])* ζ_x *counts the quanta in cell* x*. This random variable has a geometric distribution for the parameter* $1 - z$*. As a consequence, its expectation is* $\frac{z}{1-z}$ *and its variance is given by Einstein's fluctuation formula in exercise 17 as*

$$V_{\mathcal{P}_{z,cd}}(\zeta_x) = (\frac{z}{1-z})^2 + \frac{z}{1-z} = E_{\mathcal{P}_{z,cd}}(\zeta_x)^2 + E_{\mathcal{P}_{z,cd}}(\zeta_x). \tag{4.14}$$

This does not depend on x*. Thus for any* $B \subset X$ *we obtain more generally by using Bienayme's formula the fluctuation formula*

$$V_{\mathcal{P}_{z,cd}}(\zeta_B) = [E_{\mathcal{P}_{z,cd}}(\zeta_{a_0})^2 + E_{\mathcal{P}_{z,cd}}(\zeta_{a_0})] \cdot |B|. \tag{4.15}$$

Here a_0 *is any chosen element of* X*.*

Comment 8 *We remain in the context of the last comment and consider the statistical aspects. Assume that* z *is an unknown parameter. Equation* (4.8) *shows that each* $\hat{r} = \frac{\zeta_B}{|B|}, B \subset X,$ *is an unbiased estimator for the parameter* $r = \frac{z}{1-z}$*. We thus consider for the statistical scheme* $(\mathcal{P}_{z,cd})_{z \in]0,1[}$ *the problem of estimating the unknown parameter* r*. Equation* (4.10) *then shows that this estimator* $\hat{r}$ *is consistent in the sense that its variance*

$$V_{\mathcal{P}_{z,cd}}(\frac{\zeta_B}{|B|}) = \frac{r^2 + r}{|B|} \tag{4.16}$$

is small if $|B|$ *is large, in particular if* X *is large. This then implies that the following weak law of large numbers holds true: For any* z *and* $\varepsilon > 0$

$$\lim_{d \to +\infty} \mathcal{P}_{z,cd}\{|\frac{\zeta_X}{d} - \frac{z}{1-z}| > \varepsilon\} = 0. \tag{4.17}$$

To summarize, we see that in the symmetric Polya sum process one can estimate the unknown parameter r *and thereby* z *in a reasonable way if the random field is large enough.*

Scholion: Two representations of the Poya sum process

We first identify the Polya sum process as a discrete Poisson-Gamma process and then as a Cox process with directing measure given by a continuous

Poisson-Gamma process. Both representations are due to Mathias Rafler [20, 21].

Let X be finite, non-empty and ϱ a non-trivial, finite measure on it. Consider also a locally finite measure τ on $]0,+\infty[$. Call L the image of the product measure $\gamma = \varrho \otimes \tau$ under the transformation

$$\chi : X \times]0,+\infty[\longrightarrow \mathcal{M}_1(X), (x,t) \longmapsto t \cdot \delta_x. \tag{4.18}$$

Here $\mathcal{M}_1(X)$ is the range of χ.

We assume that L is a *locally finite* measure.

Example 13 *(1) Here τ is given on $\mathbb{N}$ by $\tau(j) = \frac{z^j}{j}, 0 < z < 1$. Note that in this case L is finite: $|L| = |\varrho| \cdot \kappa(z)$.*

(2) Now τ is given on $]0,+\infty[$ by $\tau(dt) = \frac{1}{t}\exp(-\alpha \cdot t)dt$. This measure is locally finite in the following sense: It is finite on all intervals $[a,b[, 0 < a < b \leq +\infty$, but infinite on their complements. The corresponding measure L is called Kingman's measure here. ([13])

Thus we can consider in both cases the Poisson process P_L. (We remark that P_L is a point process in a space which is not finite as above but a continuum. Thus we need the general theory of Poisson processes in this scholion.) It realizes in the first case configurations of finitely many Dirac measures of the form $t\delta_x$, whereas in the second case countably infinitely many of them.

In a next step we transform P_L by means of the transformation

$$\psi : \mu \longmapsto \nu_\mu = \sum_{y \in supp\,\mu} \mu(y) \cdot y. \tag{4.19}$$

Here $supp\,\mu$ denotes the support of the measure μ. In the first case this image process $Q_L = \psi P_L$ is called here *the discrete Poisson-Gamma process* for the parameters (z, ϱ) whereas in the second we speak of the *continuous Poisson-Gamma process* for (z, ϱ). (The terminology differs in the literature. In [1] the continuous Poisson-Gamma process is called a negative binomial process whereas its directing measure is called a Gamma process.) The latter process had been constructed by Kingman [13]. He can show that Q_L realizes locally finite measures μ for which $\zeta_B(\mu)$ has a $\Gamma-$distribution.

Exercise 22 *Give an interpretation of $\nu_\mu(B)$ resp. $\nu_\mu(f)$.*

Theorem 8 (Rafler [20]) *Let L be the discrete measure of the first example. Then the discrete Poisson-Gamma process for (z, ϱ) is the Polya sum process $\mathcal{P}_{z,\varrho}$.*

Proof. We use the method of the Campbell measure. On one hand one has

$$\begin{aligned}
\mathcal{C}_{Q_L}(h) &= \sum_{\mu} \sum_{(x,j)} j \cdot h(x, \nu_\mu) \mu(j\delta_x) P_L(\mu) \\
&= \sum_{\mu} \sum_{(x,j)} j \cdot h(x, \nu_\mu + j\delta_x) \frac{z^j}{j} \varrho(x) P_L(\mu) \\
&= \sum_{j} z^j \sum_{\mu} \sum_{x} h(x, \mu + j\delta_x) \varrho(x) Q_L(\mu).
\end{aligned}$$

On the other hand the Campbell measure of the Polya process coincides with this expression which follows from iteration of (3.32): For any $N \in \mathbb{N}$ we get

$$\begin{aligned}
\mathcal{C}_{\mathcal{P}_{z,\varrho}} &= \sum_{j=1}^{N} z^j \sum_{\mu} \sum_{x} h(x, \mu + j\delta_x) \varrho(x) \mathcal{P}_{z,cd}(\mu) + \\
&+ z^N \sum_{\mu} \sum_{x} h(x, \mu + N\delta_x) \mu(x) \mathcal{P}_{z,cd}(\mu) \\
&\longrightarrow_{N \to +\infty} \sum_{j=1}^{\infty} z^j \sum_{\mu} \sum_{x} h(x, \mu + j\delta_x) \varrho(x) \mathcal{P}_{z,cd}(\mu).
\end{aligned}$$

qed

The following result is very surprising in the context above. We do not give a proof here because it is beyond the elementary framework of this lecture.

Theorem 9 (Rafler [21]) *Let L' denote Kingman's measure for the following choice of the parameter α: $\alpha = \frac{1-z}{z}$. Then the socalled Cox process directed by the continuous Poisson-Gamma process $Q_{L'}$, defined by*

$$P_{Q_{L'}} = \int P_\kappa Q_{L'}(d\kappa), \tag{4.20}$$

coincides with the Poya sum process $\mathcal{P}_{z,\varrho}$.

Comment 9 *The Cox process $P_{Q_{L'}}$ represents the following law: First an intensity measure κ is realized according to the law $Q_{L'}$ of a continuous Poisson-Gamma process. This random measure fluctuates locally according to some $\Gamma-$distribution. And then, in a second step, a configuration of quanta or particles is realized in the phase space X according to the Poisson process P_κ with the random intensity measure κ. (For an interpretation in quantum optics and other fields of application of Cox processes see the following scholion.)*

Scholion: Quantum Optics and other fields of application

Here we collect some quotations from seminal papers of Odile Macchi [17] and of Klaus Krickeberg [15] to document the importance of the model of a Cox process for quantum optics or other fields of application as well as the statistical problems involved. They are marked by means of the sign ð. We mention in passing that a third important field where Cox processes are used is the theory of statistical communication. (See Middleton [18].)

We begin to quote from [17].

ð This paper is concerned with problems from the statistical theory of estimation occurring in the optical communication field (...). The objective is to measure the light intensity of a weak flux, detected by a photomultiplier. The characteristic context is as follows: necessarily the information used for estimation is enclosed in a point process, that of the instants when the detector delivers a photoelectron.

ð Dans tous les cas que nous envisageons, il s'agit d'extraire l'information contenue dans ce flux lumineux, en mesurant son intensite $I(t)$.

Dans cette etude nous n'avons considere que le cas des flux faibles, ou la nature corpusculaire du rayonnement electromagnetique (...) joue un role determinant. [Un flux lumineux faible] est constitue par une suite d'impulsions tres breves, de formes pratiquement identique correspondant a l'absorption d'un photon et a l'emission d'un photoelectron par le detecteur. Les epoches t_i de ces impulsions sont aleatoires. La seule information accessible sur le flux lumineux est donc constituee par le processus ponctuel $\mathcal{P}$ forme par les t_i.

ð La nature du processus ponctuel $\mathcal{P}$ est maintenant bien connue (...): il s'agit d'un processus de Poisson compose, c'est a dire d'un processus pontuelle qui, conditionellement a une realisation d'une fonction aleatoire positive $\lambda(t)$, appelee densite, est poissonnien de densite $\lambda(t)$. Avec cette definition, l'etude des grandeurs statistiques du processus de Poisson compose se fait tres simplement a partir de l'etude similaire du processus de Poisson pur. Cependant, la grande difficulte de la theorie des lois de probabilite conditionelles nous conduit a donner une autre definition du processus de Poisson compose par sa loi temporelle.

ð Le caractere aleatoire de la densite $\lambda(t)$ du processus ponctuel $\mathcal{P}$ est du au caractere aleatoire de la densite $\rho(t)$ des photoelectrons. En effet $\rho(t)$ est donnee par la relation

$$\rho(t) = s \cdot I(t), \tag{4.21}$$

en fonction de l'intensite lumineuse $I(t)$, qui est une fonction aleatoire de par nature meme du champ electromagnetique.

Le coefficient s est une constante positive, caracteristique de l'efficacite du detecteur. En plus des photoelectrons, le detecteur delivre des thermoelectrons, correspondant a son bruit interne, et indiscernables des premier electrons. On peut admettre que ceux-ci forment un processus de Poisson pur stationaire de densite b (certaine), et independent du processus des photoelectrons. On voit alors aisement que le processus ponctuel $\mathcal{P}$ resultant de la superposition des deux, est encore un processus de Poisson compose de densite

$$\lambda(t) = \rho(t) + b = s \cdot I(t) + b. \tag{4.22}$$

(...) nous traitons d'une impulsion lumineuse de forme connue $I_0(t)$. On a alors

$$I(T) = a \cdot I_0(t), \tag{4.23}$$

ou a est un parametre inconnu qui traduit l'energie du champ lumineux et qu'il s'agit d'estimer a partir de l'observation $\{t_n\}$. (...) dans l'etude des radars optiques, se pose ce genre de probleme.

Tenant compte de (3.43), (3.44) devient alors

$$\lambda(t) = a \cdot \rho_0(t) + b. \tag{4.24}$$

La fonction $\rho_0(t)$ est connue, et nous supposons que a est une variable aleatoire , possedant une densite de probabilite $q(a)$, continument derivable. Dans le cas ou a n'est pas une variable aleatoire, mais une valeur

> certaine inconnue, le processus ponctuel de detection est une processus de Poisson pur (...). Nous adoptons successivement les critere (Q) de l'erreur quadratique moyenne et (V) du maximum de vraisemblance a posteriori, qui sont les deux critere statistiques les plus employes.

We terminate with quotations from [15].

ð Apart from the [problem] that gave rise to the concept of a Cox process $P_W(d\mu) = \int P_\varrho(d\mu)W(d\varrho)$, where ϱ represents the strength of a thread running through a loom and μ the sequence of instances where this thread breaks, let us only mention the following one. A substance is injected into the veins of a guinea-pig. Its concentration in the animal's blood and in the course of time is subject to chance, hence it may be regarded as a realization ϱ of a random measure W. Thus ϱ is the object of our interest but it cannot be observed. We can, however, "mark" the original substance with a radioactive substance, and the coordinates in space and time of the emission represent the points of the corresponding realization μ of the Cox process P_W. Quite often μ is observable, hence we have the problem of finding ϱ from μ.

Chapter 5

Symmetric random permutations

In this final more abstract lecture we present a famous symmetric point process which has been found independently in theoretical biology by Warren Ewens [11] and in quantum statistical mechanics by Andras Sütö [22]. It is a random permutation which is invariant under socalled conjugation.

We consider the group $\mathcal{G}$ all permutations of a finite set E. Thus $\mathcal{G} = \mathcal{S}(E)$. Set $n = |E|$. (Take for example $E = [n]$. Don't mix E with a state space.)

The group $\mathcal{G}$ operates on itself by *conjugation*, i.e. by means of the action

$$\phi : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}, (g, \sigma) \longmapsto g\sigma g^{-1}. \tag{5.1}$$

The corresponding equivalence classes are the *conjugacy classes*.

Exercise 23 * *One can show that the associated equivalence relation can be represented by the following pair* (Γ, r)*:*

$$\Gamma = \mathcal{M}^{\cdot\cdot}_{(n)}(\mathbb{N}) = \{\gamma \in \mathcal{M}^{\cdot\cdot}_{f}(\mathbb{N}) | N(\gamma) = n\};$$
$$r : \mathcal{G} \longrightarrow \Gamma, \sigma \longmapsto \gamma.$$

Here $\gamma(j) = r(\sigma)(j)$ *is the number of cycles of length* j *in the cycle decomposition of* σ*.*

We give some indications of a proof. For the details we refer to [5]. Every permutation σ *is a product of cycles. (Recall exercise 6.) Let* $\gamma(j)$ *be the number of cycles in* σ *with length* j*. Then* $\sum_j j \cdot \gamma(j) = n$*. Now a conjugate class consists of those permutations having the same cycle numbers* $\gamma(j), j \in \mathbb{N}$*. There is therefore a one-to-one correspondence between the conjugacy classes and the counting measures* γ *having the property* $\sum_j j \cdot \gamma(j) = n$*.*

Exercise 24 * *Show that*

$$|Y_\gamma| = n! \cdot \prod_{j\geq 1} \frac{1}{\gamma(j)! \cdot j^{\gamma(j)}}. \tag{5.2}$$

[Hint: The proof is difficult. Consult the books on group theory, for instance [5].]

Thus again the cardinality of the equivalence classes Y_γ is finite, and we have as a unique symmetric law on them the uniform distribution. (Symmetry now means invariance under the *inner automorphisms* $\chi_g : \sigma \longmapsto g.\sigma = g\sigma g^{-1}, g \in \mathcal{G}$.) Denoting

$$q(\gamma) = \prod_{j\geq 1} \frac{1}{\gamma(j)! \cdot j^{\gamma(j)}}, \tag{5.3}$$

we see that the uniform distribution on Y_γ is defined by

$$\Lambda_\gamma(\sigma) = \frac{1}{n! \cdot q(r(\sigma))}, \sigma \in Y_\gamma. \tag{5.4}$$

To summarize, by Krickeberg's theorem the formula

$$P(\sigma) = \Lambda_{r(\sigma)} \cdot R(r(\sigma)), \sigma \in \mathcal{S}(X). \tag{5.5}$$

yields a one to one correspondence between symmetric random permutations $P \in \mathcal{P}(\mathcal{G})$ and point processes $R \in \mathcal{PM}^{\cdot\cdot}_{(n)}(\mathbb{N})$.

We obtain a simple first *example*, if we choose for R the Dirac measure Δ_γ for some $\gamma \in \Gamma$. The corresponding symmetric random permutation is Λ_γ. The first main example is

5.1 The Ewens-Sütö process

Here we construct a point process R which in connection with formula (5.5) yields the Ewens-Sütö process. We consider the following finite Poisson process P_ρ on the set of natural numbers $\mathbb{N}$. Its intensity measure ρ is defined by

$$\rho(j) = d(j) \cdot \frac{z^j}{j}, j \epsilon \mathbb{N}, \tag{5.6}$$

where $0 < z < 1$ and $d(.) > 0$. We assume that ρ is a *finite* measure on $\mathbb{N}$, i.e. $\rho(\mathbb{N}) = \sum_{j=1}^{\infty} \rho(j) < \infty$. This is a condition on the function d. P_ρ is a law on the collection $\mathcal{M}^{\cdot\cdot}_f(\mathbb{N})$ of all finite point measures μ on $\mathbb{N}$.

Example 14 *Examples for ρ are:*

(1) d is a constant d given by some natural number. In this case $\rho(\mathbb{N}) = -d \cdot ln(1-z)$.

(2) $d(j) = C \cdot j^{-\frac{\nu}{2}}$ where $\nu \in \mathbb{N}$ and C is a positive constant.

We denote as above by $(\zeta_j)_{j\in\mathbb{N}}$ the field variables $\zeta_j(\mu) = \mu(j)$. We know from exercise 14 that these variables are independent if P_ρ is the underlying law; moreover ζ_j has a Poisson distribution with parameter $\rho(j)$. This implies immediately that

$$P_\rho(\mu) = \exp(-\rho(\mathbb{N})) \cdot z^{N(\mu)} \cdot d(\mu) \cdot q(\mu),$$

where

$$q(\mu) = \prod_{j\geq 1} \frac{1}{\mu(j)! \cdot j^{\mu(j)}},$$
$$d(\mu) = \prod_{j\geq 1} d(j)^{\mu(j)} \ \ and$$
$$N(\mu) = \sum_{j\geq 1} j \cdot \mu(j).$$

We remark that here the products resp. the sum terminate after finitely many steps because μ is finite. The range of N therefore is $\mathbb{N}_0$, the collection of natural numbers augmented by 0. Denoting by M the identity on $\mathcal{M}_{\dot{f}}(\mathbb{N})$, we have for any $\mu \in \mathcal{M}_{\dot{f}}(\mathbb{N}), n \geq 0$

$$(5.7) \qquad P_\rho\{M = \mu, N = n\} = \exp(-\rho(\mathbb{N})) \cdot z^n \cdot d(\mu) \cdot q(\mu) \cdot 1_{\{N=n\}}(\mu).$$

Summing over all μ and n we obtain

$$(5.8) \qquad exp(\rho(\mathbb{N})) = \sum_{n\geq 0} Q_n(d) \cdot z^n,$$

where

$$(5.9) \qquad Q_n(d) = \sum_{\mu:N(\mu)=n} d(\mu) \cdot q(\mu)$$

denotes the socalled *canonical partition function of the ideal Bose gas* in quantum statistical mechanics, where it is the starting point of all investigations.

Example 15 *Show in case d is a constant natural number that $Q_n(d) = \binom{d+n-1}{n}$ which contains Cauchy's formula for $d = 1$:*

$$\sum_{\mu : N(\mu)=n} q(\mu) = 1. \tag{5.10}$$

Exercise 25 * *Show that formula (5.7) implies that the random variable N is distributed according to the following form of the negative binomial distribution:*

$$P_\rho\{N = n\} = \exp(-\rho(\mathbb{N})) \cdot z^n \cdot Q_n(d), n \geq 0. \tag{5.11}$$

Exercise 26 * *Deduce from the above considerations that*

$$P_\rho\{M = \mu | N = n\} = \frac{1}{Q_n(d)} \cdot d(\mu) \cdot q(\mu) \cdot 1_{\{N=n\}}(\mu), n \geq 0, \mu \in \mathcal{M}_{\ddot{f}}(\mathbb{N})$$

We are now in the position to choose a point process R in $\mathbb{N}$ which in connection with formula (5.5) will give us the random permutation we are looking for. Let

$$\mathbb{S}_d^{(n)} = P_\rho(.|N = n), n \geq 1; \mathbb{S}_\rho^{(0)} = \delta_0;$$
$$\mathcal{M}_{\ddot{(n)}}(\mathbb{N}) = \{N = n\}.$$

Observe that $\mathbb{S}_\rho^{(n)}$ no longer depends on z.

Consider now the point process $\mathbb{S}_d^{(n)}$ in $\mathbb{N}$. The associated symmetric random permutation is given by

$$\mathcal{S}_d^{(n)}(\sigma) = \frac{1}{n! Q_n(d)} \cdot d(r(\sigma)) \cdot 1_{\mathcal{M}_{\ddot{(n)}}}(r(\sigma)), \sigma \in \mathcal{G}. \tag{5.12}$$

If we now mix the family $(\mathcal{S}_d^{(n)})_{n \geq 0}$ with respect to the law determined by equation (5.12) we obtain as another symmetric process the celebrated *Ewens-Sütö process for the parameters (d, z):*

$$\mathcal{S}_{d,z} = \exp(-\sum_j d(j) \cdot \frac{z^j}{j}) \cdot \sum_{n \geq 0} z^n \cdot Q_n(d) \cdot \mathcal{S}_d^{(n)}. \tag{5.13}$$

Thus

$$\mathcal{S}_{d,z}(\sigma) = \exp(-\sum_j d(j) \cdot \frac{z^j}{j}) \cdot \frac{z^{N(r(\sigma))}}{N(r(\sigma))!}, \sigma \in \mathcal{G}.$$

Exercise 27 *Show that*

(1) *the image of* $\mathcal{S}_{d,z}$ *under* r *is the Poisson process* P_ρ.

(2) *Using the notation* $N(\sigma) = N(r(\sigma))$, *one has*

$$\mathcal{S}_{d,z}\{N = n\} = \exp(-\rho(\mathbb{N})) \cdot z^n \cdot Q_n(d), n \geq 0. \tag{5.14}$$

This means that N*, the length of a random permutation, is distributed according to the negative binomial distribution for the Ewens-Sütö cycle process* $\mathcal{S}_{d,z}$.

5.2 Symmetric random Polya permutations

We here present another symmetric random permutation which is induced via Krickeberg's theorem by the Polya sum process and which we call the symmetric Polya random permutation.

We use for its construction the process $R_d^{(n)} = R_{z,\lambda}\{.|N = n\}$. This is a point process in $\mathbb{N}_0$. To be more precise, $R_d^{(n)}$ is a probability on the following configuration space:

$$\mathcal{M}^{\cdot\cdot}_{(n,d)}(\mathbb{N}_0) = \{N = n\} \cup \mathcal{M}^{\cdot\cdot}_d(\mathbb{N}_0).$$

So it cannot be used directly because we have to start with a process in $\mathbb{N}$. But it is evident that there is a 1-1-correspondence between this space and the configuration space

$$\mathcal{M}^{\cdot\cdot}_{(n,d)}(\mathbb{N}) = \{\gamma \in \mathcal{M}^{\cdot\cdot}_{(n)}(\mathbb{N}) : |\gamma| \leq d\}.$$

Therefore $R_d^{(n)}$ can be represented as follows as a point process in $\mathbb{N}$:

$$S_d^{(n)}(\gamma) = \frac{n!}{d+n-1} \cdot \left(\begin{array}{c} d \\ d - |\gamma|, \gamma \end{array} \right) \cdot 1_{M^{\cdot\cdot}_{(n,d)}(\mathbb{N})}(\gamma), \gamma \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}).$$

By means of (5.5) we then obtain the following symmetric random permutation which we call *symmetric random Polya permutation* for the parameters (n, d).

$$\pi_d^{(n)}(\sigma) = \frac{1}{q(r(\sigma))} \cdot \frac{1}{d+n-1} \cdot \left(\begin{array}{c} d \\ d - |r(\sigma)|, r(\sigma) \end{array} \right) \cdot 1_{M^{\cdot\cdot}_{(n,d)}(\mathbb{N})}(r(\sigma)), \sigma \in \mathcal{S}(E).$$

A simple computation then shows that this random permutation is given by
(5.15)
$$\pi_d^{(n)}(\sigma) = \prod_{j\geq 1} j^{r(\sigma)(j)} \cdot \frac{1}{d+n-1} \cdot \frac{d!}{(d-|r(\sigma)|)!} 1_{\mathcal{M}_{\ddot{(n,d)}}(\mathbb{N})}(r(\sigma)), \sigma \in \mathcal{S}(X).$$

Exercise 28 *Deduce the following combinatorial identity:*

$$\sum_{k=1}^{d} \sum_{\sigma:N(r(\sigma))=n,|r(\sigma)|=k} (d-k+1)\cdots d \cdot \prod_{j\geq 1} j^{r(\sigma)(j)} \cdot = d+n-1. \tag{5.16}$$

We finally mix the family $(\pi_d^{(n)})_{n\geq 0}$ with respect to the distribution (4.11) of N and obtain the following symmetric random permutation, which we call also *symmetric random Polya permutation for* (d,z):

$$\pi_{d,z} = (1-z)^d \cdot \sum_{n\geq 0} z^n \cdot \tilde{Q}_n(d) \cdot \pi_d^{(n)}. \tag{5.17}$$

Final comments

We finish with some remarks. We used above the following properties of $\mathcal{G} = \mathcal{S}_n$: Every permutation $\sigma \in \mathcal{G}$ can be decomposed in a unique way into disjoint cycles. Denoting by $r_j(\sigma)$ the number of cycles of length j one has $\sum_j j \cdot r_j(\sigma) = n$, a conjugacy class consists of those permutations σ having the same numbers $(r_j(\sigma))_j$. For this reason the elements of $\mathcal{M}_{\ddot{(n)}}(\mathbb{N})$ can be considered as these classes.

Furthermore, it is very useful to consider σ as a simple point measure of disjoint cycles, including the trivial ones. Symbolically one could write

$$\sigma = \sum_{x\in\sigma} \delta_x, \tag{5.18}$$

where the sum is taken over all cycles x of the cycle decomposition of σ. This suggests to consider a random permutation as a point process of disjoint cycles. The Ewens-Sütö process is an example. It can be interpreted as follows: First realize a natural number n according to the negative binomial distribution (4.10); then realize according to the law $\mathcal{S}_d^{(n)}$ a permutation σ decomposed into its cycles x. A detailed analysis of $\mathcal{S}_{d,z}$ can be found in [19].

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