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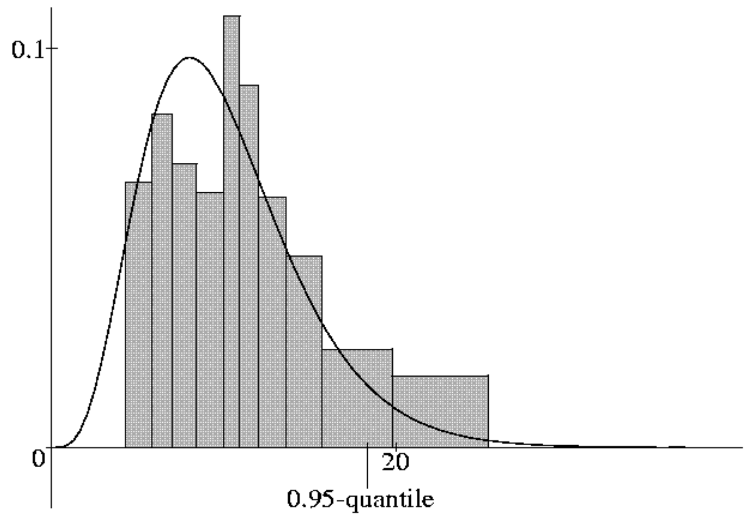

Mathematische Statistik und
Wahrscheinlichkeitstheorie

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Abstract

We consider an infinite system of hard balls in $\mathbb{R}^d$ undergoing Brownian motions and submitted to a pair potential with infinite range and quasi polynomial decay. It is modelized by an infinite-dimensional Stochastic Differential Equation with an infinite-dimensional local time term. Existence and uniqueness of a strong solution is proven for such an equation with deterministic initial condition. We also show that the set of all equilibrium measures, solution of a Detailed Balance Equation, coincides with the set of canonical Gibbs measures associated to the hard core potential.

AMS Classifications: 60H10, 60K35.
KEY-WORDS: Stochastic Differential Equation, local time, hard core potential, Gibbs measure, reversible measure.

1 Introduction

The aim of this paper is to construct and analyze an infinite system of interacting hard balls undergoing Brownian motions in $\mathbb{R}^d$ and starting from a fixed initial condition.

R. Lang ([9],[10]) constructed in a pioneer paper the reversible solution of an infinite gradient system of Brownian particles (i.e. balls with radius 0, that is reduced to points) submitted to a smooth pair interaction. It is a so-called *equilibrium dynamics* in Statistical Physics, since this process has a time-stationary distribution. J. Fritz solved some years later in [6] the non-reversible case, which occurs when the initial distribution is no more Gibbsian. For this type of systems, the main difficulty comes from a possible explosion (i.e. an infinite number of particles can enter a finite volume after a finite time).

On another side, a reversible system of infinitely many Brownian hard balls (without external potential) was studied by H. Tanemura [22]. He constructs a unique solution to an infinite-dimensional Skohorod type equation where the hard core situation – balls can not overlap – appears as a local time term in addition to the basic Brownian motion. The (reversible) initial condition is ditributed like a Gibbs measure associated to the hard core potential.

In the present paper, we deal in the dynamics $(\mathcal{E})$ with Brownian motions submitted to the sum of a hard core potential and a smooth infinite range pair potential, a model which is a mixture of both Lang's and Tanemura's models. In [5] we proved existence and uniqueness of a reversible solution of equation $(\mathcal{E})$, using at several places the time-stationarity of the solution. We propose here a new pathwise approach for the construction of a non-reversible solution of $(\mathcal{E})$: the initial condition can be any deterministic configuration in a set of allowed configurations which is clearly identified (see (18)). Furthermore, the model studied here is an important generalization of the previous works since the pair potential we consider has infinite range with a so-called quasi-polynomial decay (see condition (3)); we explain in the proof of Proposition 3.5 why this choice is almost optimal with respect to the techniques we use. The potential treated in [5] had a faster exponential decay, which is known to be much more accessible to mathematical treatment.

In Section 2 we present the infinite-dimensional equation $(\mathcal{E})$ and we state the results. We build a sequence of approximating solutions in Section 3, show their convergence and analyse the limit process, in particular its associated infinitesimal generator.

Last, we prove in Section 4 that any Gibbs measure associated with the dynamical interaction is reversible. Reciprocally, we show that any measure satisfying an equilibrium equation called *Detailed Balance Equation* is necessarily canonical Gibbs.

2 Main results

The particles we deal with in the present paper move in $\mathbb{R}^d$, for a fixed $d\geqslant 2$, endowed with the Euclidian norm denoted by $|\;|$. $B(y,\rho)$ will denote the closed ball centered in $y \in \mathbb{R}^d$ with radius $\rho \geq 0$ and more generally, for any $A \subset \mathbb{R}^d$, we define

$$B(A,\rho) = \{y \in \mathbb{R}^d \text{ such that } d(y,A)\leqslant\rho\}$$

where $d(y,A)$ denotes the Euclidian distance between y and A. The volume of a subset A in $\mathbb{R}^d$ is also denoted by $|A|$.

The modelization of point configurations may be done in two equivalent ways :
The first possibility is to represent an n-points configuration in $\mathbb{R}^d$ as a subset (with multiplicity) of cardinality n in $\mathbb{R}^d$, that is as an equivalence class on $(\mathbb{R}^d)^n$ under the action of the permutation group on $\{1,\ldots,n\}$. The second possibility is to modelize it as a point measure $\sum_{i=1}^n \delta_{\xi_i}$ on $\mathbb{R}^d$. More generally, the set of all point configurations in $\mathbb{R}^d$ will be the set $\mathcal{M}$ of all point Radon measures on $\mathbb{R}^d$:

$$\mathcal{M} = \left\{ \xi = \sum_{i\in I} \delta_{\xi_i} \text{ such that } I \subset \mathbb{N},\ \xi_i \in \mathbb{R}^d \text{ and for any } K \text{ compact in } \mathbb{R}^d,\ \xi(K) < +\infty \right\}.$$

$\mathcal{M}$ is endowed with the topology of vague convergence. By simplicity, we will identify any point measure $\xi \in \mathcal{M}$ with the subset of $\mathbb{R}^d$ $\{\xi_i, i \in I\}$ corresponding to its support and with the representants of this subset in $(\mathbb{R}^d)^I$, writing for example $\xi_\Lambda = \xi \cap \Lambda$ for the restriction of this configuration to $\Lambda \subset \mathbb{R}^d$, $\xi\eta$ for the concatenation of both configurations ξ and η. $\mathcal{M} \cap (\mathbb{R}^d)^n$ is the set of all n-point configurations.

Let us also introduce some definitions of differentiability for functions defined on the space of point configurations $\mathcal{M}$.

Definition 2.1 *A function f on $\mathcal{M}$ is* local *if there exists a compact set $K \subset \mathbb{R}^d$ such that $f(\gamma)$ only depends on $\gamma \cap K$, i.e. $\forall \gamma \in \mathcal{M} \quad f(\gamma) = f(\gamma_K)$. Such a function is called K-local.*
A local function f on $\mathcal{M}$ is called $\mathcal{C}^k$ if for any $n \in \mathbb{N}^*$ the function defined on $(\mathbb{R}^d)^n$ by $(\gamma_1, \cdots, \gamma_n) \longmapsto f(\sum_{i=1}^{n} \delta_{\gamma_i})$ is $\mathcal{C}^k$. For any $\gamma \in \mathcal{M}$, $D_x f(x\gamma)$ and $D^2_{xx} f(x\gamma)$ denote the first and second derivatives of $y \mapsto f(y\gamma)$ at $y = x$.
Our set of test functions will be $\mathcal{T} := \{$ functions $f : \mathcal{M} \to \mathbb{R},$ local and $\mathcal{C}^2\}$.

Remark that any local $\mathcal{C}^0$-function is bounded on $\mathcal{A}$ and that any local $\mathcal{C}^1$-function has a bounded derivative on $\mathcal{A}$: $\sup_{x \in \mathbb{R}^d} \sup_{\gamma \in \mathcal{A}} |D_x g(x\gamma)| < +\infty$.

We introduce some more notations.

- For $\Lambda \subset \mathbb{R}^d$, N_Λ is the counting variable on $\mathcal{M}$: $N_\Lambda(\xi) = \sharp\{i \in \mathbb{N} : \xi_i \in \Lambda\}$.
- For $\Lambda \subset \mathbb{R}^d$, $\mathcal{B}_\Lambda$ is the σ-algebra on $\mathcal{M}$ generated by the sets $\{N_A = n\}$, $n \in \mathbb{N}$, $A \subset \Lambda$, A bounded.
- π (resp. π_Λ) is the Poisson process on $\mathbb{R}^d$ (resp. on Λ) with intensity measure the Lebesgue measure dy (resp. $dy|_\Lambda$).
- For $z > 0$, π^z (resp. π^z_Λ) is the Poisson process on $\mathbb{R}^d$ (resp. on Λ) with activity z, that is with intensity measure $z\ dy$ (resp. $z\ dy|_\Lambda$).

The particles we deal with in this paper are not reduced to points but are hard balls or spheres of diameter r, for a fixed $r > 0$. Since balls can not overlap, the set of *allowed configurations* is the following subset of $\mathcal{M}$:

$$\mathcal{A} = \{\xi = \{\xi_i\}_i \in \mathcal{M} \text{ such that } \forall i \neq j\ |\xi_i - \xi_j| \geqslant r\}\,.$$

2.1 Interaction potential and (canonical) Gibbs measures

For a complete description in a general framework of the concepts introduced in this subsection, we refer the reader to [7].

We are dealing with hard balls with diameter r submitted to the action of a *pair potential*, which is a function on $\mathbb{R}^d$ of class $\mathcal{C}^2$ satisfying $\varphi(x) = \varphi(-x)$. Due to the hard core situation the values of $\varphi(x)$ may be chosen arbitrarily for $|x| < r$. In particular, one can assume without restriction that φ vanishes in a neighborhood from 0 and that $\nabla\varphi(0) = 0$. Moreover it satisfies the following assumptions (1), (2) and (3):

- **Summability of the interaction and its derivative** on $\mathcal{A}$:

$$\forall \xi \in \mathcal{A}, \sum_j |\varphi(\xi_j)| < +\infty \text{ and } \sum_j |\nabla\varphi(\xi_j)| < +\infty \tag{1}$$

- **Lipschitzianity** of $\nabla\varphi$ on finite allowed configurations :
  There exists a real number $\overline{\overline{\nabla\varphi}} > 0$ such that for each finite subset $\mathbf{J}$ of $\mathbb{N}$, each $\xi, \eta \in \mathcal{A}$ verifying $\max_{j\in\mathbf{J}} |\xi_j - \eta_j| < r/2$, one has :

$$\sum_{j\in\mathbf{J}} |\nabla\varphi(\xi_j) - \nabla\varphi(\eta_j)| \leqslant \overline{\overline{\nabla\varphi}} \max_{j\in\mathbf{J}} |\xi_j - \eta_j| \tag{2}$$

- **Quasi-polynomial decay** of the interaction : $\exists\, a, b > 0$ such that for R large enough,

$$\forall \xi \in \mathcal{A}, \sum_{\{j:|\xi_j|>R\}} |\nabla\varphi(\xi_j)| \leq g(R) := \frac{1}{R^{a(\log R)^b}} \tag{3}$$

This is obviously a stronger condition than the second summability condition in (1). The range of the pair potential φ may be finite or infinite, i.e. the support of φ may be compact or not. If the range of φ is finite, (1), (2) and (3) are trivially satisfied.

Let us also present an equivalent formulation for assumption (1). The space $\mathbb{R}^d$ can be splitten into cubes that cannot contain more than one center of hards balls with diameter r : $\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}^d} \frac{r}{\sqrt{d}}k + [0; \frac{r}{\sqrt{d}}[^d$. Let γ_k denote a point in the closed cube $\frac{r}{\sqrt{d}}k + [0; \frac{r}{\sqrt{d}}]^d$ which maximize $|\varphi|$ on this cube : $|\varphi(\gamma_k)| = \max\{|\varphi(x)|,\ x \in \frac{r}{\sqrt{d}}k + [0; \frac{r}{\sqrt{d}}]^d\}$. Clearly, for each configuration $\xi \in \mathcal{A}$, one has $\sum_j |\varphi(\xi_j)| \leqslant \sum_{k\in\mathbb{Z}^d} |\varphi(\gamma_k)|$. The configuration $\gamma = \sum_{k\in\mathbb{Z}^d} \delta_{\gamma_k}$ does not a priori belong to $\mathcal{A}$, but it is the union of at most $[\sqrt{d}+2]^d$ allowed configurations. Using a similar argument for $\nabla\varphi$, one obtains that assumption (1) is equivalent to

$$\sup_{\xi\in\mathcal{A}} \sum_j |\varphi(\xi_j)| < +\infty \quad \text{and} \quad \sup_{\xi\in\mathcal{A}} \sum_j |\nabla\varphi(\xi_j)| < +\infty$$

Moreover, using the translation invariance of the set $\mathcal{A}$, assumption (1) is equivalent to the following useful uniform summabilities :

$$\overline{\varphi} = \sup_{x\in\mathbb{R}^d} \sup_{\xi\in\mathcal{A}} \sum_j |\varphi(x - \xi_j)| < +\infty \quad \text{and} \quad \overline{\nabla\varphi} = \sup_{x\in\mathbb{R}^d} \sup_{\xi\in\mathcal{A}} \sum_j |\nabla\varphi(x - \xi_j)| < +\infty \tag{4}$$

Remark 2.2 *From a physical point of view, it is natural to assume that the pair potential $\varphi(x)$ only depends on the norm of x ($\varphi(x) = \psi(|x|)$ for some $\mathcal{C}^2$ function ψ). In this case, a sufficient condition for φ to satisfy assumptions* (1), (2) *and* (3) *is the following :*
$\exists\, a, b > 0,\ \exists\overline{\psi}$ *non-increasing function with* $\int_{\mathbb{R}^d} \overline{\psi}(|x|)\, dx < +\infty$ *such that for R large enough*

$$\forall u > R, \quad |\psi(u)| \leqslant \overline{\psi}(u), \quad |\psi'(u)| \leqslant \frac{1}{u^{a(\log u)^b}} \quad \textit{and} \quad |\psi''(u)| \leqslant \overline{\psi}(u).$$

The **energy** of a configuration $\xi \in \mathcal{M}$ submitted to the potential φ in the compact volume $\Lambda \subset \mathbb{R}^d$ with the boundary condition $\eta \in \mathcal{M}$ is given by :

$$E_\Lambda(\xi|\eta) = \begin{cases} \frac{1}{2} \sum_{\xi_i,\xi_j\in\Lambda} \varphi(\xi_i - \xi_j) + \sum_{\xi_i\in\Lambda,\eta_j\in\Lambda^c} \varphi(\xi_i - \eta_j) & \text{if } \xi_\Lambda\eta_{\Lambda^c} \in \mathcal{A} \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

(the condition $\xi_\Lambda\eta_{\Lambda^c} \in \mathcal{A}$ corresponds to configurations for which $\xi_\Lambda \in \mathcal{A}$, $\eta_{\Lambda^c} \in \mathcal{A}$ and no ball of η_{Λ^c} is overlapping a ball of ξ_Λ). The energy is well defined for $\xi_\Lambda\eta_{\Lambda^c} \in \mathcal{A}$ since the first sum contains a finite number of terms, and the second series is finite due to (4). Moreover, $e^{-E_\Lambda(\xi|\eta)}$ vanishes as soon as the configuration $\xi_\Lambda\eta_{\Lambda^c}$ is not allowed.

We now define the set $\mathcal{G}(z)$ of **Gibbs measures** on hard balls associated to the potential φ with activity parameter $z \in \mathbb{R}^+$. For each compact subset Λ of $\mathbb{R}^d$, let us define a local density function with respect to the Poisson Process π_Λ^z by :

$$f_\Lambda^z(\xi|\eta) \quad = \quad \frac{1}{Z_z^{\Lambda,\eta}} \exp(-E_\Lambda(\xi|\eta)) \tag{6}$$

where the so-called partition function $Z_z^{\Lambda,\eta}$ is the renormalizing constant :

$$Z_z^{\Lambda,\eta} = \; e^{-z|\Lambda|} \left(1 + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} \exp -E_\Lambda(y_1 \cdots y_n|\eta)\, dy_1 \cdots dy_n \right).$$

Due to the hard core, the above series is only a finite sum and $0 < Z_z^{\Lambda,\eta} < +\infty$.

Definition 2.3 *A Probability measure μ on $\mathcal{M}$ belongs to the set $\mathcal{G}(z)$ of Gibbs measures on hard balls with activity z and associated potential φ if and only if, for each compact subset $\Lambda \subset \mathbb{R}^d$,*

$$d\mu(\xi|\mathcal{B}_{\Lambda^c})(\eta) = f_\Lambda^z(\xi|\eta)\; d\pi_\Lambda^z(\xi) \qquad \text{for } \mu\text{-a.e. } \eta.$$

Remark that any Gibbs measure in $\mathcal{G}(z)$ has its support included in $\mathcal{A}$. Dobrushin proved in [1], using compactness arguments, that there exists at least one element in $\mathcal{G}(z)$ when the potential contains a hard core component. Furthermore the set $\mathcal{G}(z)$ is convex and compact.
About the cardinality of $\mathcal{G}(z)$, remarking that the sum of the hard core and the smooth potential φ is superstable and lower regular in the sense of Ruelle [15], we have :
- If z is small enough, Ruelle proved that uniqueness holds (see [14] Theorem 4.2.3). In our case, a sufficient condition would be :

$$z < z_c := \frac{\exp -(2\overline{\varphi} + 1)}{\int_{\mathbb{R}^d} |1 - \exp(-\varphi(x))| dx}.$$

- For z large enough it is conjectured (see [14] and [7]) - but still not proved - that phase transition occurs : $\sharp\mathcal{G}(z) > 1$. Moreover, it is conjectured by physicists that for z converging to infinity one can find a sequence of Gibbs measures $\mu_z \in \mathcal{G}(z)$ converging to the closest packing configurations.

See also [13] for a construction of a pure hard core Poisson Process with applications in percolation theory and [23] for the description of such a process as a Gibbs cluster process.

We now define the set $\mathcal{CG}$ of **canonical Gibbs measures** on $\mathcal{A}$ associated to the potential φ.

Definition 2.4 *A Probability measure μ on $\mathcal{A}$ belongs to the set $\mathcal{CG}$ of canonical Gibbs states on $\mathcal{A}$ for the pair potential φ if and only if, for each compact subset $\Lambda \subset \mathbb{R}^d$ and $n \in \mathbb{N}$, for μ-a.e. η,*

$$d\mu(\xi|\mathcal{B}_{\Lambda^c}, N_\Lambda)(\eta, n) = \begin{cases} \dfrac{1}{Z^{\Lambda,\eta,n}} 1\!\!1_{\{N_\Lambda(\xi)=n\}} \exp(-E_\Lambda(\xi|\eta))\; d\pi_\Lambda(\xi) & \text{if } Z^{\Lambda,\eta,n} > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where the partition function $Z^{\Lambda,\eta,n}$ for the particle number n is the finite renormalizing constant $Z^{\Lambda,\eta,n} = \frac{e^{-|\Lambda|}}{n!} \int_{\Lambda^n} \exp -E_\Lambda(y_1 \cdots y_n|\eta)\, dy_1 \cdots dy_n$.

Since the potential φ is bounded from below we deduce from (1) that the map $y \mapsto E_\Lambda(y|\eta)$ is also bounded from below on $\mathbb{R}^d$, uniformly in Λ and η. Thus Georgii's conditions (6.11) and (6.12) from [7] hold, which allows to apply Theorem 6.14 of [7] and to deduce that the set of canonical Gibbs states $\mathcal{CG}$ is obtained by mixing elements of different $\mathcal{G}(z)$, $z \in \mathbb{R}^+$: for any $\mu \in \mathcal{CG}$ there exists a probability measure θ on $\mathbb{R}^+$ such that

$$\mu = \int_{\mathbb{R}^+} \mu_z\, \theta(dz) \text{ with } \mu_z \in \mathcal{G}(z) \text{ for each } z \in \mathbb{R}^+. \tag{7}$$

2.2 The infinite-dimensional reflected stochastic equation $(\mathcal{E})$

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right continuous filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that each $\mathcal{F}_t$ contains all P-negligible sets and let $(W_i(t), t\geqslant 0)_{i\in\mathbb{N}}$ be a family of $\mathcal{F}_t$-adapted independent d-dimensional Brownian motions.

Let us denote $\mathcal{C}(\mathbb{R}^+, \mathcal{M})$ (resp. $\mathcal{C}_0(\mathbb{R}^+, \mathcal{M})$) the set of continuous $\mathcal{M}$-valued paths on $\mathbb{R}^+$ (resp. which vanish at time 0), endowed with the topology of uniform convergence on each compact time interval. $\mathcal{C}(\mathbb{R}^+, \mathcal{M})$ is the set of all possible paths, and the subset of all allowed paths is

$$\mathcal{C}(\mathbb{R}^+, \mathcal{A}) = \left\{X \in \mathcal{C}(\mathbb{R}^+, \mathcal{M}) \text{ such that } \forall t\geqslant 0 \; X(t) \in \mathcal{A}\right\}.$$

Sets $\mathcal{C}([0,T], \mathcal{M})$ and $\mathcal{C}([0,T], \mathcal{A})$ are defined similarly for any positive final time T.

Let φ be the smooth infinite range pair potential introduced in the previous subsection. We consider the following - possibly infinite - gradient system of stochastic differential equations satisfied by the Brownian balls :

$$(\mathcal{E}) \qquad \begin{cases} \text{For } i \in I \subset \mathbb{N}, t \in \mathbb{R}^+, \\ X_i(t) = X_i(0) + W_i(t) - \dfrac{1}{2}\displaystyle\sum_{j\in I}\int_0^t \nabla\varphi(X_i(s) - X_j(s))ds \\ \qquad\qquad + \displaystyle\sum_{j\in I}\int_0^t (X_i(s) - X_j(s))dL_{ij}(s) \end{cases}$$

where

- $(X_i(t), t\geqslant 0)_{i\in I} \in \mathcal{C}(\mathbb{R}^+, \mathcal{A})$, i.e. it satisfies $\forall i \neq j, \forall t\geqslant 0, \; |X_i(t) - X_j(t)| \geqslant r$;
- $(L_{ij}(t), t\geqslant 0)_{i,j\in I}$ is a family of non-decreasing $\mathbb{R}^+$-valued continuous processes satisfying :

$$L_{ij}(0) = 0, \quad L_{ij} \equiv L_{ji} \quad \text{and} \quad L_{ij}(t) = \int_0^t \mathbb{1}_{|X_i(s)-X_j(s)|=r}\, dL_{ij}(s), \quad L_{ii} \equiv 0.$$

A solution of the system $(\mathcal{E})$ with initial condition $x = (x_i)_{i\in I} \in \mathcal{A}$ is a family $(X_i^x(t), L_{ij}^x(t), t\geqslant 0, i, j \in I)$ of processes such that equation $(\mathcal{E})$ is satisfied with $X(0) = x$. The process X is infinite-dimensional as soon as x is an infinite point configuration ($\sharp I = +\infty$).

The main results of this paper are the following theorems.

Theorem 2.5 *The stochastic equation $(\mathcal{E})$ admits a solution with values in $\mathcal{A}$ for any deterministic initial configuration which belongs to the set $\underline{\mathcal{A}} = \{x \in \mathcal{A} \; : P(\Omega_x) = 1\}$, where the set Ω_x is defined in* (18)*. This solution is unique as element of $\underline{\mathcal{C}} \subset \mathcal{C}(\mathbb{R}^+, \mathcal{A})$, a subset of regular paths defined in* (40)*.*

Theorem 2.6 *Any Gibbs measure $\mu \in \mathcal{G}(z)$ with activity $z > 0$ has its support included in $\underline{\mathcal{A}}$. Furthermore, if the initial configuration of the stochastic equation $(\mathcal{E})$ is random with distribution $\mu \in \mathcal{CG}$, then this solution is time-reversible, that is its law is invariant with respect to the time reversal.*

Theorem 2.7 *Suppose that μ is a probability measure on $\mathcal{A}$ with $\mu(\underline{\mathcal{A}}) = 1$. Furthermore, suppose that for every Λ compact subset of $\mathbb{R}^d$ and μ-almost all η, $\mu(.|\mathcal{B}_{\Lambda^c})(\eta)$ is absolutely continuous with respect to π_Λ and its density $u_\Lambda(.|\eta_{\Lambda^c})$ has the following differentiability property :*

$$\begin{aligned} &\forall \xi \in \mathcal{A}_\Lambda, \text{ the map } x \mapsto u_\Lambda(x\xi|\eta_{\Lambda^c}) \text{ is } \mathcal{C}^1 \text{ on } \Lambda\smallsetminus B(\xi\eta_{\Lambda^c}, r) \text{ and its derivative} \\ &\nabla u_\Lambda(x\xi|\eta_{\Lambda^c}) \text{ verifies } \int_{\mathcal{A}}\int_{\mathcal{A}_\Lambda} \sup_{x\in\Lambda\smallsetminus B(\xi\eta_{\Lambda^c},r)} |\nabla u_\Lambda(x\xi|\eta_{\Lambda^c})| \; \pi_\Lambda(d\xi)\mu(d\eta) < +\infty \end{aligned} \tag{8}$$

If μ is an equilibrium measure for the gradient system $(\mathcal{E})$ in the sense that the Detailed Balance Equation (44) holds under μ, then μ is a canonical Gibbs measure in $\mathcal{CG}$.

3 Approximating processes and their convergence

To simplify we restrict the study of the paths on the time interval $[0,1]$. It is obvious that all the results in the sequel hold true on any time interval $[0,T]$, $T \geqslant 1$, up to a change of constants.

3.1 Construction of finite-dimensional approximations

In this whole subsection, $\ell \in \mathbb{N}^*$ is fixed. We construct the approximating process $X^{\ell,x}$ in order that it "essentially" stays in $B(0,\ell)$, the ball with radius ℓ (we use a penalization method, whose sense will be clear soon). To obtain such a behavior, we introduce in the equation $(\mathcal{E})$ an additional gradient drift $\nabla\psi^{\ell,\eta}$ which vanishes in a subset of $B(0,\ell)$ and is strongly repulsive outside of $B(0,\ell)$.

More precisely, for any allowed configuration $\eta \in \mathcal{A}$ whose support is disjoint to $B(0,\ell)$, we fix a $\mathbb{R}^+$-valued function $\psi^{\ell,\eta}$ on $\mathbb{R}^d$ which is $\mathcal{C}^2$ with bounded derivatives and vanishes on each $y \in B(0,\ell)$ such that $y\eta$ is an allowed configuration, and only on those y's (see figure 1), that is

$$\psi^{\ell,\eta}(y) = 0 \quad \Leftrightarrow \quad y \in B(0,\ell) \text{ and } y\eta \in \mathcal{A} \quad \Leftrightarrow \quad |y| \leqslant \ell \text{ and } d(y,\eta) \geqslant r.$$

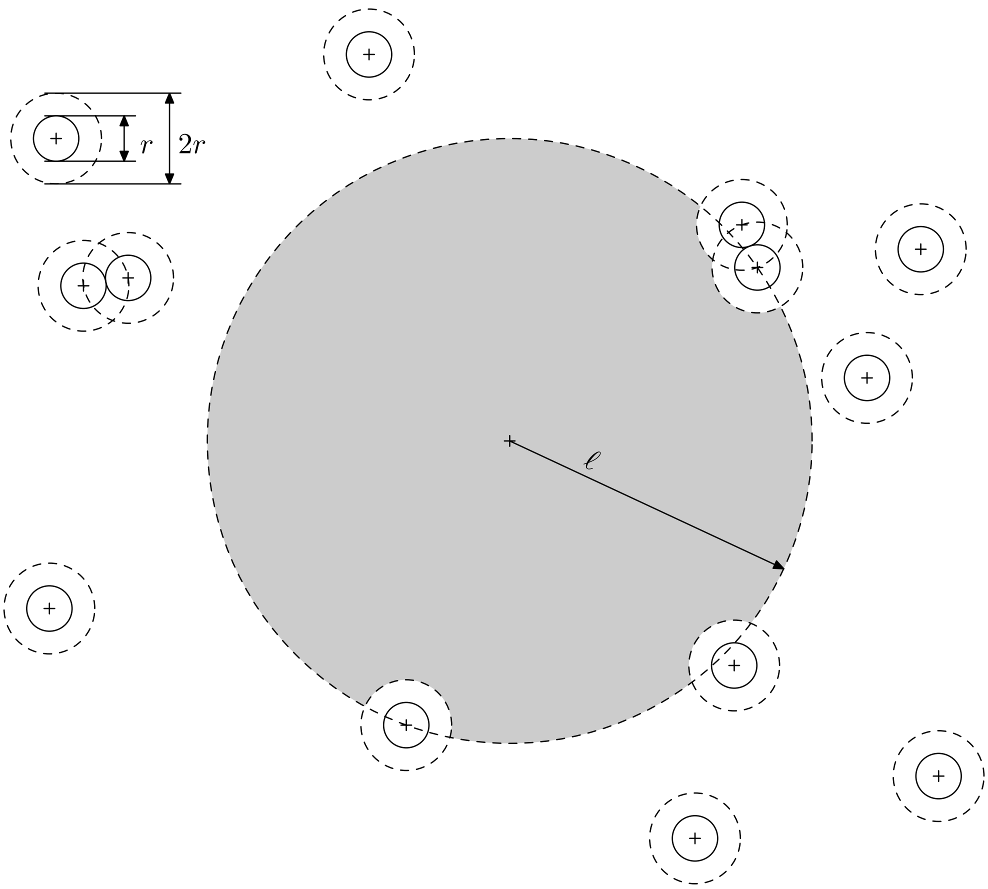

Figure 1: Particles of η are represented; the grey area is the domain where $\psi^{\ell,\eta}$ vanishes.

We extend the definition of $\psi^{\ell,\eta}$ to any configuration $\eta \in \mathcal{A}$ by taking $\psi^{\ell,\eta} = \psi^{\ell,\eta \cap B(0,\ell)^c}$. We also choose the family $(\psi^{\ell,\eta})_\ell$ such that, for every $\eta \in \mathcal{A}$,

$$\sup_{\eta \in \mathcal{A}} \sum_{\ell \in \mathbb{N}^*} \int_{\mathbb{R}^d} \mathbb{1}_{\psi^{\ell,\eta}(y)>0} \; \exp(-\psi^{\ell,\eta}(y)) \, dy \quad \leqslant \quad 1. \tag{9}$$

Such a family $(\psi^{\ell,\eta})_{\ell\in\mathbb{N}^*,\eta\in\mathcal{A}}$ exists; choose for example $\psi^{\ell,\eta}(y) = C^{\mathrm{st}}\ell^{d+1}\delta(y)$ where δ is a $\mathcal{C}^2$ function with bounded derivatives which is equivalent on $\mathbb{R}^d$ to $d(\cdot, \Lambda\setminus B(\eta_{\Lambda^c}, r))$ with $\Lambda = B(0,\ell)$ (see [20] p. 171), that is which verifies :

$$\exists c, C > 0 \text{ such that } \forall x \in \mathbb{R}^d \quad c\ d(x, \Lambda\setminus B(\eta_{\Lambda^c}, r)) \leqslant \delta(x) \leqslant C\ d(x, \Lambda\setminus B(\eta_{\Lambda^c}, r)).$$

For $\eta \in \mathcal{A}$ and $n \in \mathbb{N}^*$, let us now define the n-dimensional stochastic differential equation :

$$(\mathcal{E}_n^{\ell,\eta}) \qquad \begin{cases} \forall i \in \{1,\dots,n\}, \quad \forall t \in [0,1], \\ dX_i(t) = dW_i(t) - \dfrac{1}{2}\Bigg(\nabla\psi^{\ell,\eta}(X_i(t)) + \displaystyle\sum_{j=1,\dots,n} \nabla\varphi(X_i(t) - X_j(t)) \\ \qquad\qquad + \displaystyle\sum_{j:|\eta_j|>\ell} \nabla\varphi(X_i(t) - \eta_j)\Bigg)\, dt \\ \qquad + \displaystyle\sum_{j=1,\dots,n} (X_i(t) - X_j(t))dL_{ij}(t) \end{cases}$$

with $L_{ij} \equiv L_{ji}$ for all i and j and $L_{ij}(t) = \int_0^t \mathbb{1}_{|X_i(s)-X_j(s)|=r}\ dL_{ij}(s)$.

$(\mathcal{E}_n^{\ell,\eta})$ is a n-dimensional stochastic differential equation of Skorohod's type, reflected on the boundary of the domain

$$\mathbf{D}_n := \mathcal{A} \cap (\mathbb{R}^d)^n = \{x = (x_1,\dots,x_n) \in (\mathbb{R}^d)^n : |x_i - x_j| \geqslant r, i \neq j\}. \tag{10}$$

Its drift has a gradient form $-\frac{1}{2}\nabla\beta_n^{\ell,\eta}$ where

$$\beta_n^{\ell,\eta}(x_1,\dots,x_n) = \sum_{i=1,\dots,n}\Big(\psi^{\ell,\eta}(x_i) + \frac{1}{2}\sum_{\substack{j=1,\dots,n\\ j\neq i}} \varphi(x_i - x_j) + \sum_{j:|\eta_j|>\ell} \varphi(x_i - \eta_j)\Big). \tag{11}$$

Since the drift $-\frac{1}{2}\nabla\beta_n^{\ell,\eta}$ is bounded and Lipschitz continuous, following the results of Saisho and Tanaka (Theorem 5.1 of [17]), the equation $(\mathcal{E}_n^{\ell,\eta})$ admits a unique strong solution in the domain $\mathbf{D}_n$ for each initial n-points configuration $x \in \mathcal{A} \cap (\mathbb{R}^d)^n$. We denote this solution by $X^{\ell,\eta,n}(x,\cdot)$. For any configuration $x \in \mathcal{A}$, one can define an $\mathcal{A}$-valued finite-dimensional process with initial configuration $x \cap B(0,\ell)$ and random dynamics $(\mathcal{E}_n^{\ell,\eta})$ by

$$X^{\ell,x}(\cdot) := X^{\ell,x_{\Lambda^c},n}(x_\Lambda, \cdot) \quad \text{ with } \Lambda = B(0,\ell) \text{ and } n = \sharp(x \cap B(0,\ell)) .$$

As Kolmogorov proved in his pioneer paper [8], the solution of $(\mathcal{E}_n^{\ell,\eta})$ is reversible when one takes as initial distribution $\nu_n^{\ell,\eta}$, where $\nu_n^{\ell,\eta}$ is the finite measure defined on $(\mathbb{R}^d)^n$ by

$$d\nu_n^{\ell,\eta}(x_1,\dots,x_n) = \exp(-\beta_n^{\ell,\eta}(x_1,\dots,x_n))\ \mathbb{1}_{\mathcal{A}}(x_1,\dots,x_n)\ dx_1 \dots dx_n.$$

$Q_n^{\ell,\eta}$ denotes the time-reversible law of $X^{\ell,\eta,n}$ starting from $\nu_n^{\ell,\eta}$:

$$Q_n^{\ell,\eta} = \int P(X^{\ell,\eta,n}(x,\cdot) \in .)\ d\nu_n^{\ell,\eta}(x).$$

Remark that, like $\nu_n^{\ell,\eta}$, the finite measure $Q_n^{\ell,\eta}$ is not necessarily a Probability measure.

The Probability measure $\mu_z^{\ell,\eta}$ on $\bigcup_{n=0}^{+\infty}(\mathbb{R}^d)^n$, Poisson mixture of the $(\nu_n^{\ell,\eta})_n$, is defined by :

$$\mu_z^{\ell,\eta}(\bigcup_{n=0}^{+\infty} A_n) = \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!}\ \nu_n^{\ell,\eta}(A_n), \quad A_n \subset (\mathbb{R}^d)^n, \tag{12}$$

where $Z_z^{\ell,\eta} = e^{-z|B(0,\ell)|} \sum_{n=0}^{+\infty} \frac{z^n}{n!} \nu_n^{\ell,\eta}((\mathbb{R}^d)^n)$ (with the convention $\nu_0^{l,\eta}((\mathbb{R}^d)^0) = 1$).
Similarly, consider on the level of paths the Probability measure defined by

$$Q_z^{\ell,\eta} = \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} Q_n^{\ell,\eta}.$$

This Probability measure is time reversal invariant. Its support is included in $\mathcal{A}$, as a mixing of $\mathcal{A}$-supported measures.

3.2 Sets of irregular paths : first estimates

We first prove an estimate of the probability that a particle of a configuration following the $(\mathcal{E}_n^{\ell,\eta})$-dynamics moves with **high velocity**.
For every $\varepsilon > 0$ and $\delta \in]0,1]$, let $\tilde{\mathcal{B}}(\delta,\varepsilon)$ denote the paths for which a particle i has a δ-modulus of continuity Δ higher than ε, i.e.

$$\tilde{\mathcal{B}}(\delta,\varepsilon) = \{X \in \mathcal{C}([0,1],\mathcal{A}) : \exists i, \ \Delta(X_i,\delta) > \varepsilon\},$$

where the δ-modulus of continuity of a path w on $[0,1]$ is defined as usual by

$$\Delta(w,\delta) = \sup_{\substack{0\leqslant s,t\leqslant 1 \\ |t-s|\leqslant\delta}} |w(t) - w(s)|. \tag{13}$$

Proposition 3.1 *There exists $C_1 > 0$ (depending only on d and the interaction φ) such that the following upper bound holds : $\forall\varepsilon > 0, \forall\delta \in]0,1], \forall\ell \in \mathbb{N}^*$,*

$$\sup_{\eta\in\mathcal{A}} Q_z^{\ell,\eta}(\tilde{\mathcal{B}}(\delta,\varepsilon)) \ \leqslant \ z\, C_1 \,\frac{\ell^d}{\delta}\, \exp\left(-\frac{\varepsilon^2}{5\delta}\right).$$

Proof of Proposition 3.1

We will need the following estimate on Brownian paths, which is a consequence of Doob's inequality (The reader can find a detailed proof e.g. in the Appendix of [3]) :

Lemma 3.2 *If W is a (one-dimensional) Brownian motion on $(\Omega,\mathcal{F},P)$ then for every $\varepsilon > 0$ and every $\delta \in]0,1]$*

$$P(\Delta(W,\delta)\geqslant\varepsilon) \ \leqslant \ \frac{41}{\delta} \exp\left(-\frac{\varepsilon^2}{5\delta}\right)$$

We first compute an estimate of $Q_n^{\ell,\eta}(\tilde{\mathcal{B}}(\delta,\varepsilon))$.
Let $(X^{\ell,\eta,n}, L^{\ell,\eta,n})$ denote the unique strong solution of $(\mathcal{E}_n^{\ell,\eta})$ starting from $\nu_n^{\ell,\eta}$, and recall that the distribution $Q_n^{\ell,\eta}$ of $X^{\ell,\eta,n}$ is time reversible on $[0,1]$. By construction the processes :

$$\begin{aligned} W_i(t) &= X_i^{\ell,\eta,n}(t) - X_i^{\ell,\eta,n}(0) + \frac{1}{2}\int_0^t \nabla_i \beta_n^{\ell,\eta}(X^{\ell,\eta,n}(s))ds \\ &\quad - \int_0^t \sum_{j=1,\dots,n} (X_i^{\ell,\eta,n}(s) - X_j^{\ell,\eta,n}(s))dL_{ij}^{\ell,\eta,n}(s), \quad 1\leqslant i\leqslant n, 0\leqslant t\leqslant 1 \end{aligned}$$

and

$$\begin{aligned} \widehat{W}_i(t) &= X_i^{\ell,\eta,n}(1-t) - X_i^{\ell,\eta,n}(1) + \frac{1}{2}\int_{1-t}^1 \nabla_i \beta_n^{\ell,\eta}(X^{\ell,\eta,n}(s))ds \\ &\quad - \int_{1-t}^1 \sum_{j=1,\dots,n} (X_i^{\ell,\eta,n}(s) - X_j^{\ell,\eta,n}(s))dL_{ij}^{\ell,\eta,n}(s), \quad 1\leqslant i\leqslant n, 0\leqslant t\leqslant 1 \end{aligned}$$

are both n-dimensional Brownian motions starting from 0. Remarking that

$$\forall t\in[0,1]\quad X^{\ell,\eta,n}(t)-X^{\ell,\eta,n}(0)=\frac{1}{2}\Big(W(t)+\widehat{W}(1-t)-\widehat{W}(0)\Big)$$

and using the fact that the laws of $\widehat{W}$ and W are identical, we obtain :

$$\begin{aligned}
&Q_n^{\ell,\eta}(\tilde{\mathcal{B}}(\delta,\varepsilon))\\
&\quad=\int_{(\mathbb{R}^d)^n} P\Big(\exists i\leqslant n \text{ such that } \sup_{\substack{|t-s|\leqslant\delta\\ 0\leqslant s,t\leqslant 1}} |W_i(t)-W_i(s)+\widehat{W}_i(1-t)-\widehat{W}_i(1-s)|>2\varepsilon\Big)\,d\nu_n^{\ell,\eta}(x)\\
&\quad\leqslant P\Big(\exists i\leqslant n \text{ such that } \sup_{|t-s|\leqslant\delta}|W_i(t)-W_i(s)|>\varepsilon \text{ or } \sup_{|t-s|\leqslant\delta}|\widehat{W}_i(t)-\widehat{W}_i(s)|>\varepsilon\Big)\,\nu_n^{\ell,\eta}((\mathbb{R}^d)^n)\\
&\quad\leqslant 2P\Big(\exists i\leqslant n \text{ such that } \Delta(W_i,\delta)>\varepsilon\Big)\,\nu_n^{\ell,\eta}((\mathbb{R}^d)^n)\\
&\quad\leqslant 2\;n\;P\left(\Delta(W_1,\delta)>\varepsilon\right)\,\nu_n^{\ell,\eta}((\mathbb{R}^d)^n)
\end{aligned}$$

We know from lemma 3.2 that

$$P\left(\Delta(W_1,\delta)>\varepsilon\right)\;\leqslant\frac{41}{\delta}\;\exp\left(-\frac{\varepsilon^2}{5\delta}\right).$$

According to the definition (11) of $\beta_n^{\ell,\eta}$ and assumption (4) :

$$\begin{aligned}
\beta_n^{\ell,\eta}(x_1,\ldots,x_n) &= \psi^{\ell,\eta}(x_1)+\sum_{j=2}^{n}\varphi(x_1-x_j)+\sum_{j:|\eta_j|>\ell}\varphi(x_1-\eta_j)+\beta_{n-1}^{\ell,\eta}(x_2,\ldots,x_n)\\
&\geqslant \psi^{\ell,\eta}(x_1)-2\overline{\varphi}+\beta_{n-1}^{\ell,\eta}(x_2,\ldots,x_n)
\end{aligned}\tag{14}$$

which implies that

$$\begin{aligned}
\nu_n^{\ell,\eta}((\mathbb{R}^d)^n) &= \int_{(\mathbb{R}^d)^n}\mathbb{1}_{\mathcal{A}}(x_1,\ldots,x_n)\;e^{-\beta_n^{\ell,\eta}(x_1,\ldots,x_n)}\;dx_1\cdots dx_n\\
&\leqslant \int_{(\mathbb{R}^d)^n}\mathbb{1}_{\mathcal{A}}(x_2,\ldots,x_n)\;e^{-\beta_{n-1}^{\ell,\eta}(x_2,\ldots,x_n)}\;e^{2\overline{\varphi}}\;e^{-\psi^{\ell,\eta}(x_1)}\;dx_1\cdots dx_n\\
&\leqslant e^{2\overline{\varphi}}\;\nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})\int_{\mathbb{R}^d}e^{-\psi^{\ell,\eta}(y)}\;dy
\end{aligned}\tag{15}$$

This leads to the estimate :

$$Q_n^{\ell,\eta}(\tilde{\mathcal{B}}(\delta,\varepsilon))\quad\leqslant 2\;n\;\nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})\;\frac{41}{\delta}\;\exp\left(-\frac{\varepsilon^2}{5\delta}\right)\;e^{2\overline{\varphi}}\;\int_{\mathbb{R}^d}e^{-\psi^{\ell,\eta}}\;dy$$

After summation in n we obtain :

$$\begin{aligned}
Q_z^{\ell,\eta}(\tilde{\mathcal{B}}(\delta,\varepsilon)) &= \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}}\sum_{n=1}^{+\infty}\frac{z^n}{n!}\;Q_n^{\ell,\eta}(\tilde{\mathcal{B}}(\delta,\varepsilon))\\
&\leqslant 82\;\frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}}\;z\left(\sum_{n=1}^{+\infty}\frac{z^{n-1}}{(n-1)!}\;\nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})\right)e^{2\overline{\varphi}}\;\frac{1}{\delta}\;\exp\left(-\frac{\varepsilon^2}{5\delta}\right)\int_{\mathbb{R}^d}e^{-\psi^{\ell,\eta}}dy\\
&\leqslant 82\;z\;e^{2\overline{\varphi}}\;\frac{1}{\delta}\;\exp\left(-\frac{\varepsilon^2}{5\delta}\right)\int_{\mathbb{R}^d}e^{-\psi^{\ell,\eta}}dy\;.
\end{aligned}$$

Recalling that $\psi^{\ell,\eta}$ only vanishes into the ball $B(0,\ell)$ and using inequality (9), we get :

$$\int_{\mathbb{R}^d}e^{-\psi^{\ell,\eta}}dy=\int_{B(0,\ell)}e^{-\psi^{\ell,\eta}}dy+\int\mathbb{1}_{\psi^{\ell,\eta}>0}e^{-\psi^{\ell,\eta}}dy\leqslant\ell^d\;|B(0,1)|+1$$

which leads to the desired result with $C_1=82(|B(0,1)|+1)\;e^{2\overline{\varphi}}$. ■

In order to control the convergence of the finite-dimensional systems, we have to estimate the set of particles which are touched by a fixed particle i. If the paths have a small oscillation, this set will be finite because the particle i can not reach particles which are too far away. But we also have to avoid the bump to "propagate" along a **large chain** of neighbouring particles. We first define patterns called $(r+\varepsilon)$-chain of particles, and then prove that they are rare enough, in the sense that the probability under $\mu_z^{\ell,\eta}$ of configurations containing such a chain decreases exponentially fast as a function of the length of the chain.

Definition 3.3 *Let $x \in \mathcal{A}$ and $\varepsilon > 0$. Each subset $\{x_1, \cdots, x_n\}$ of distinct particles of x verifying*

$$|x_1 - x_2| \leqslant r+\varepsilon, \cdots, |x_{n-1} - x_n| \leqslant r+\varepsilon$$

is called an $(r+\varepsilon)$-chain of n particles of x.

For $M \in \mathbb{N}^*$ and $\varepsilon > 0$, let $Ch(M, r+\varepsilon)$ denote the set of allowed configurations containing an $(r+\varepsilon)$-chain of M particles, that is :

$$Ch(M, r+\varepsilon) = \Big\{ x \in \mathcal{A},\ \exists \{x_1, \cdots, x_M\} \subset x,\ |x_1 - x_2| \leqslant r+\varepsilon, \cdots, |x_{M-1} - x_M| \leqslant r+\varepsilon \Big\}$$

Let us now define a set of irregular paths, in which a particle belongs at some time to a large chain of interacting particles : for $m, M \in \mathbb{N}^*, \varepsilon > 0$

$$\begin{aligned} \tilde{\tilde{\mathcal{B}}}(m, M, \varepsilon) &:= \left\{ X \in \mathcal{C}([0,1], \mathcal{A}) : \begin{array}{l} \exists k \in \{0, \ldots, m-1\} \text{ such that } X(\frac{k}{m}) \text{ contains} \\ \text{one } (r+\varepsilon)\text{-chain of } M+1 \text{ particles} \end{array} \right\} \\ &= \Big\{ X \in \mathcal{C}([0,1], \mathcal{A}) :\ \exists k \in \{0, \ldots, m-1\},\ \ X(\frac{k}{m}) \in Ch(M+1, r+\varepsilon) \Big\}. \end{aligned}$$

Note that this set increases as a function of ε.
We now prove an upperbound for the $Q_z^{\ell,\eta}$-Probability of $\tilde{\tilde{\mathcal{B}}}(m, M, \varepsilon)$.

Proposition 3.4 *There exists $C_2 > 0$ (depending only on the radius of the balls, the dimension d and the interaction φ) such that, for any $m, M \in \mathbb{N}^*$ and $0 < \varepsilon < r$:*

$$\sup_{\ell \in \mathbb{N}^*} \sup_{\eta \in \mathcal{A}} Q_z^{\ell,\eta} \left(\tilde{\tilde{\mathcal{B}}}(m, M, \varepsilon) \right) \leqslant m \ (C_2 \ z \ \varepsilon)^M .$$

As corollary, for ε small enough (depending on z), the left hand side decreases exponentially fast as a function of M.

Proof of Proposition 3.4

$$Q_z^{\ell,\eta} \left(\tilde{\tilde{\mathcal{B}}}(m, M, \varepsilon) \right) = Q_z^{\ell,\eta} \left(\exists k \in \{0, \ldots, m-1\},\ X(\frac{k}{m}) \in Ch(M+1, r+\varepsilon) \right).$$

By stationarity of $Q_z^{\ell,\eta}$, this probability is smaller than

$$\leqslant \sum_{k=0}^{m-1} \mu_z^{\ell,\eta} (Ch(M+1, r+\varepsilon)) \quad = \quad m \ \mu_z^{\ell,\eta} (Ch(M+1, r+\varepsilon)) .$$

We now estimate the $\mu_z^{\ell,\eta}$-probability that such chains exist.
Each configuration in $(\mathbb{R}^d)^n \cap Ch(M+1, r+\varepsilon)$ has exactly $\frac{n!}{(n-M-1)!}$ representants in $(\mathbb{R}^d)^n$ such that $(x_{n-M}, \ldots, x_n)$ is a fixed $M+1$-uple verifying $|x_{n-M} - x_{n-M+1}| \leqslant r+\varepsilon, \cdots, |x_{n-1} - x_n| \leqslant r+\varepsilon$. In order to fix the representant of the configuration ($(x_{n-M}, \ldots, x_n) \in \mathcal{O}$), we demand that for $n - M \leqslant i \leqslant n$, x_{i+1} is defined by $|x_i - x_{i+1}| = \min\{|x_i - x_j|; i < j \leqslant n\}$. This fix the labelling of the points in the chain, except for the (negligible) set of configurations containing two points which

are exactly at the same distance of a third one. Since $\beta_n^{\ell,\eta}(x_1,\ldots,x_n)$ and $\mathbb{1}_{\mathcal{A}}(x_1,\ldots,x_n)$ do not change by permutation of the x_i's, this leads to :

$$\begin{aligned}&\nu_n^{\ell,\eta}(Ch(M+1,r+\varepsilon))\\&=\frac{n!}{(n-M-1)!}\int_{(\mathbb{R}^d)^n}\prod_{i=n-M}^{n-1}\mathbb{1}_{|x_i-x_{i+1}|\leqslant r+\varepsilon}\ \mathbb{1}_{\mathcal{O}}(x_{n-M},\ldots,x_n)\\&\qquad\qquad\mathbb{1}_{\mathcal{A}}(x_1,\ldots,x_n)\ e^{-\beta_n^{\ell,\eta}(x_1,\ldots,x_n)}dx_1\cdots dx_n\end{aligned}$$

We use inequality (14) to get :

$$\beta_n^{\ell,\eta}(x_1,\ldots,x_n)\geqslant-2\overline{\varphi}+\beta_{n-1}^{\ell,\eta}(x_1,\ldots,x_{n-1})$$

Remarking that

$$\mathbb{1}_{\mathcal{A}}(x_1,\ldots,x_n)\leqslant\mathbb{1}_{|x_n-x_{n-1}|\geqslant r}\ \mathbb{1}_{\mathcal{A}}(x_1,\ldots,x_{n-1})\ ,$$

using again inequality (14) and integrating with respect to x_n we obtain :

$$\begin{aligned}&\nu_n^{\ell,\eta}(Ch(M+1,r+\varepsilon))\\&\leqslant n\frac{(n-1)!}{(n-1-M)!}\int_{(\mathbb{R}^d)^n}\prod_{i=n-M}^{n-2}\mathbb{1}_{r\leqslant|x_i-x_{i+1}|\leqslant r+\varepsilon}\ \mathbb{1}_{\mathcal{O}}(x_{n-M},\ldots,x_{n-1})\\&\qquad\qquad\mathbb{1}_{\mathcal{A}}(x_1,\ldots,x_{n-1})\ \mathbb{1}_{r\leqslant|x_n-x_{n-1}|\leqslant r+\varepsilon}e^{2\overline{\varphi}}\ e^{-\beta_{n-1}^{\ell,\eta}(x_1,\ldots,x_{n-1})}\ dx_1\cdots dx_n\\&\leqslant n\ e^{2\overline{\varphi}}((r+\varepsilon)^d-r^d)|B(0,1)|\nu_{n-1}^{\ell,\eta}(Ch(M,r+\varepsilon))\ .\end{aligned}$$

By definition of $\mu_z^{\ell,\eta}$ (see (12)) we have

$$\begin{aligned}\mu_z^{\ell,\eta}(Ch(M+1,r+\varepsilon))&=\frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}}\sum_{n=0}^{+\infty}\frac{z^n}{n!}\ \nu_n^{\ell,\eta}(Ch(M+1,r+\varepsilon))\\&=\frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}}\sum_{n\geqslant M+1}\frac{z^n}{n!}\ \nu_n^{\ell,\eta}(Ch(M+1,r+\varepsilon))\end{aligned}\tag{16}$$

Using this, the above inequality and iterating the result on M, we obtain :

$$\begin{aligned}&\mu_z^{\ell,\eta}(Ch(M+1,r+\varepsilon))\\&\leqslant e^{2\overline{\varphi}}\ ((r+\varepsilon)^d-r^d)\ |B(0,1)|\ \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}}\ z\sum_{n\geqslant M+1}\frac{z^{n-1}}{(n-1)!}\ \nu_{n-1}^{\ell,\eta}(Ch(M,r+\varepsilon)\\&\leqslant z\ e^{2\overline{\varphi}}\ ((r+\varepsilon)^d-r^d)\ |B(0,1)|\ \mu_z^{\ell,\eta}(Ch(M,r+\varepsilon))\\&\leqslant\Big(z\ e^{2\overline{\varphi}}\ ((r+\varepsilon)^d-r^d)|B(0,1)|\Big)^M\ \mu_z^{\ell,\eta}(Ch(1,r+\varepsilon))\\&\leqslant\Big(z\ e^{2\overline{\varphi}}\ ((r+\varepsilon)^d-r^d)|B(0,1)|\Big)^M\ .\end{aligned}$$

By the binomial formula, for $\varepsilon<r$ we have :

$$(r+\varepsilon)^d\ =\ r^d+\sum_{k=1}^{d}\binom{d}{k}\ \varepsilon^k\ r^{d-k}\leqslant r^d+\varepsilon\ 2^d\ r^{d-1}$$

and then

$$\mu_z^{\ell,\eta}(Ch(M+1,r+\varepsilon))\leqslant\Big(e^{2\overline{\varphi}}\ 2^d\ r^{d-1}\ |B(0,1)|\ z\ \varepsilon\Big)^M\ .$$

The proof is completed if we take $C_2=e^{2\overline{\varphi}}\ 2^d\ r^{d-1}\ |B(0,1)|$. ■

3.3 Convergence of the approximations on the set Ω_x

Through this whole section, x denotes a fixed element of $\mathcal{A}$. The aim of this section is to prove the convergence of the sequence $(X^{\ell,x})_\ell$ to a limit process $X^{\infty,x}$.

As usual for stochastic equations, we study the dynamics only for ω's in a well choosen subset $\Omega_x \subset \Omega$. We will prove in the next section that this subset has Probability measure one.
We first choose a parameter κ such that $0 < \kappa < \frac{b}{2(b+1)}$ and define an increasing volume sequence $(\ell(m))_m$ by

$$\ell(m) := \exp(\frac{1}{6(d+1)} m^{1-2\kappa}), \quad m \in \mathbb{N}^*. \tag{17}$$

The explanations about the choice of $\ell(m)$ will be done after equation (33). From (26) we deduce the right choice of the parameter κ and from (35) we see how to choose the length M of allowed chains. From now on, we fix them. For $M > \frac{2}{\kappa}$, let us now define the set Ω_x as follows :

$$\Omega_x = \tilde{\Omega}_x \cap \tilde{\tilde{\Omega}}_x \cap \Omega_x^W$$

where

$$\begin{aligned}
\tilde{\Omega}_x &= \liminf_{m\to+\infty} \bigcap_{\ell=\ell(m)}^{\ell(m+1)} \left\{\omega \in \Omega \ : X^{\ell,x}(\omega) \notin \tilde{\mathcal{B}}(\frac{1}{m}, \frac{1}{m^\kappa})\right\} \\
\tilde{\tilde{\Omega}}_x &= \liminf_{m\to+\infty} \left\{\omega \in \Omega \ : X^{\ell(m),x}(\omega) \notin \tilde{\tilde{\mathcal{B}}}(m, M, \frac{4}{m^\kappa})\right\} \\
\Omega_x^W &= \liminf_{m\to+\infty} \bigcap_{\{i:|x_i|<\ell(m+1)\}} \left\{\omega \in \Omega \ : \forall n \geqslant m \ \Delta(W_i(\omega), \frac{1}{n}) \leqslant \frac{1}{n^\kappa}\right\}.
\end{aligned}$$

Indeed

$$\begin{aligned}
\Omega_x = \Big\{ &\omega \in \Omega \ : \exists m_0 \in \mathbb{N}, \forall m \geqslant m_0 \\
&\forall \ell \in \{\ell(m), \ldots, \ell(m+1)\} \ X^{\ell,x}(\omega,\cdot) \notin \tilde{\mathcal{B}}(\frac{1}{m}, \frac{1}{m^\kappa}) \\
&\text{and } X^{\ell(m),x}(\omega,\cdot) \notin \tilde{\tilde{\mathcal{B}}}\Big(m, M, \frac{4}{m^\kappa}\Big) \\
&\text{and } \forall i \text{ for which } |x_i| < \ell(m+1) \ \forall n \geqslant m \ \Delta(W_i(\omega,\cdot), \frac{1}{n}) \leqslant \frac{1}{n^\kappa} \Big\}
\end{aligned} \tag{18}$$

Proposition 3.5 *For every $x = (x_i, i \in I) \in \mathcal{A}$, every ω in Ω_x and every $i \in I$, the sequence of paths $(X_i^{\ell,x}(\omega,t), L_{ij}^{\ell,x}(\omega,t), j \in I, t \in [0,1])_{\ell\in\mathbb{N}^*} \in \mathcal{C}([0,1], \mathbb{R}^d \times \mathbb{R}_+^I)$ converges in the sense of the uniform convergence to a limit denoted by $(X_i^{\infty,x}(\omega,t), L_{ij}^{\infty,x}(\omega,t), j \in I, t \in [0,1])$.*

In order to prove the convergence of the sequence $(X_i^{\ell,x})_\ell$, we are looking for an upper bound for $|X_i^{\ell,x} - X_i^{\ell',x}|$ when i belongs to some subset of indices. To this aim we need general estimates to compare the solutions of two different Skorohod equations.
For $\alpha = 1, 2$, a finite set of indices $\mathbf{J}(\alpha)$, an initial condition $x^{(\alpha)} = (x_i^{(\alpha)})_{i\in\mathbb{N}}$ in $\mathcal{A}$ and a family of continuous paths $w \in \mathcal{C}_0(\mathbb{R}^+, \mathcal{M})$, let us denote by $(\xi^{(\alpha)}(t), \rho^{(\alpha)}(t))_{0\leqslant t\leqslant 1}$ the unique $\mathcal{A}$-valued process solution of the following Skorohod equation

$$\xi_i(t) = x_i^{(\alpha)} + w_i(t) + \int_0^t b_i^{(\alpha)}(\xi(s))ds + \sum_{j\in\mathbf{J}(\alpha)} \int_0^t (\xi_i(s) - \xi_j(s)) d\rho_{ij}^{(\alpha)}(s), \quad i \in \mathbf{J}(\alpha). \tag{19}$$

This process is reflected on the boundary of the domain $\mathbf{D}_{\sharp\mathbf{J}(\alpha)}$ ($\mathbf{D}_n$ was defined in (10)).

The local time processes $\rho^{(\alpha)} = (\rho_{ij}^{(\alpha)})_{i,j\in\mathbf{J}(\alpha)}$ satisfy as usually $\rho_{ij}^{(\alpha)}(0) = 0$, $\rho_{ij}^{(\alpha)} \equiv \rho_{ji}^{(\alpha)}$ and $\rho_{ij}^{(\alpha)}(t) = \int_0^t \mathbb{1}_{|\xi_i(s)-\xi_j(s)|=r} d\rho_{ij}^{(\alpha)}(s)$. The drift function on $(\mathbb{R}^d)^{\mathbf{J}(\alpha)}$ is given here by :

$$b_i^{(\alpha)}(\xi) = -\frac{1}{2} \sum_{j\in\mathbf{J}(\alpha)} \nabla\varphi(\xi_i - \xi_j) + c_i^{(\alpha)}(\xi),\ i \in \mathbf{J}(\alpha),$$

where $c^{(\alpha)} = (c_i^{(\alpha)})_{i\in\mathbf{J}(\alpha)}$ is an $(\mathbb{R}^d)^{\mathbf{J}(\alpha)}$-valued Lipschitz continuous function.

Lemma 3.6 *Assume that there exists* $M, m \in \mathbb{N}^*$, $R_0 \geqslant R' > 0$ *verifying* $mR' \leqslant R_0$ *and* $\varepsilon_0 \geqslant 0$, $\varepsilon_1 > 0$ *verifying* $\varepsilon_0 + \varepsilon_1 \leqslant \frac{r}{2M}$ *such that :*

- **(i)** $\xi^{(1)}, \xi^{(2)} \notin \tilde{\mathcal{B}}(\frac{1}{m}, \varepsilon_1)$ *and* $\xi^{(1)} \notin \tilde{\tilde{\mathcal{B}}}(m, M, 2(\varepsilon_0 + \varepsilon_1))$

  and for all indices i :
- **(ii)** $|x_i^{(1)}| < R_0$ *or* $|x_i^{(2)}| < R_0 \quad \Rightarrow \quad |x_i^{(1)} - x_i^{(2)}| \leqslant \varepsilon_0$ *and* $i \in \mathbf{J}(1) \cap \mathbf{J}(2)$
- **(iii)** *for* $\alpha = 1, 2$, $\quad |x_i^{(\alpha)}| < R_0 - R \Rightarrow \forall t \in [0, 1]\ |c_i^{(\alpha)}(\xi_i^{(\alpha)}(t))| \leqslant g(R)$, *where* $R := R' - Mr$
- **(iv)** $\forall n \geqslant m \quad \Delta(w_i, \frac{1}{n}) \leqslant \frac{1}{n^\kappa}$

Then, there exists real numbers $C_3, C_4 > 0$ *such that, for all i's for which* $|x_i^{(1)}| \leqslant R_0 - mR'$

$$\forall t \in [0, \frac{1}{m}], \qquad |\xi_i^{(1)}(t) - \xi_i^{(2)}(t)| \leqslant C_3\ \varepsilon_0 + C_4(\frac{1}{m!} + g(R)). \tag{20}$$

In the proof of Lemma 3.6, we will use the notation

$$\|w\|_t = \sup \left(\sum_{k=1}^{n} |w(t_k) - w(t_{k-1})| \ :\ 0 = t_0 < t_1 < t_2 < \cdots < t_n = t \right)$$

for the total variation of the path w on the time interval $[0, t]$.

Proof of Lemma 3.6

Due to **(i)** we note that, for any i, $\Delta(\xi_i^{(1)}, \frac{1}{m}) \leqslant \varepsilon_1$. Moreover $x^{(1)}$ does not contain any $(r + \varepsilon_2)$-chain involving more than M particles, where $\varepsilon_2 := 2(\varepsilon_0 + \varepsilon_1)$.

Remark that, for $x \in \mathcal{A}$, the relation "x_i is connected to x_j by some $\tilde{r}$-chain" (or "there is a $\tilde{r}$-chain in x containing x_i and x_j") is an equivalence relation between the particles of x.

Let i_0 be some fixed index such that $|x_{i_0}| < R_0 - R'$ and let $\mathbb{J}(i_0)$ be the set of indices defined by

$$\mathbb{J}(i_0) := \{i \in \mathbb{N} :\ x_i^{(1)} \text{ is connected to } x_{i_0}^{(1)} \text{ by some } (r + \varepsilon_2)\text{-chain in } x^{(1)}\}.$$

By construction $\mathbb{J}(i_0)$ is finite and since $\varepsilon_2 \leqslant \frac{r}{M}$ we have

$$\sharp\mathbb{J}(i_0) \leqslant (1 + \varepsilon_2/r)^d M^d \leqslant (1 + M)^d.$$

We now enlarge $\mathbb{J}(i_0)$ by introducing the index set

$$\mathbb{J}_R(i_0) := \{i \in \mathbb{N} :\ \exists j \in \mathbb{J}(i_0),\ |x_i^{(1)} - x_j^{(1)}| < R + \varepsilon_2\}.$$

It is clear that for $j \in \mathbb{J}(i_0)$, $|x_j^{(1)} - x_{i_0}^{(1)}| \leqslant (M-1)(r+\varepsilon_2)$. Then, if $i \in \mathbb{J}_R(i_0)$, using $\varepsilon_2 \leqslant \frac{r}{M}$ we have

$$|x_{i_0}^{(1)} - x_i^{(1)}| < R + \varepsilon_2 + (M-1)(r + \varepsilon_2) = R' + M\varepsilon_2 - r \leqslant R'.$$

Therefore :

$$\mathbb{J}(i_0) \subset \mathbb{J}_R(i_0) \subset \{i \in \mathbb{N} : \ |x_{i_0}^{(1)} - x_i^{(1)}| < R'\} =: \mathbf{J}_1. \tag{21}$$

By definition of $\mathbb{J}(i_0)$

$$i \in \mathbb{J}(i_0), \ j \notin \mathbb{J}(i_0) \quad \Longrightarrow \quad |x_i^{(1)} - x_j^{(1)}| > r + \varepsilon_2 \text{ and } |x_i^{(2)} - x_j^{(2)}| > r + \varepsilon_2 - 2\varepsilon_0$$

and thus since $\xi^{(2)}(\cdot) \notin \tilde{\mathcal{B}}(\frac{1}{m}, \varepsilon_1)$

$$i \in \mathbb{J}(i_0), \ j \notin \mathbb{J}(i_0) \quad \Longrightarrow \quad \forall t \in [0, \frac{1}{m}] \begin{array}{l} |\xi_i^{(1)}(t) - \xi_j^{(1)}(t)| > r + \varepsilon_2 - 2\varepsilon_1 \geqslant r \text{ and} \\ |\xi_i^{(2)}(t) - \xi_j^{(2)}(t)| > r + \varepsilon_2 - 2\varepsilon_0 - 2\varepsilon_1 = r. \end{array}$$

Then for $\alpha = 1, 2$ during the time interval $[0, \frac{1}{m}]$, a particle $\xi_i^{(\alpha)}$ with index $i \in \mathbb{J}(i_0)$ does not touch any particle with index in $\mathbb{J}(i_0)^c$. So most of the local times in the Skorohod equation disappear and, for $t \in [0, \frac{1}{m}]$ and $i \in \mathbb{J}(i_0)$,

$$\xi_i^{(\alpha)}(t) = x_i^{(\alpha)} + W_i^{(\alpha)}(t) + \sum_{j \in \mathbb{J}(i_0)} \int_0^t (\xi_i^{(\alpha)}(s) - \xi_j^{(\alpha)}(s)) d\rho_{ij}^{(\alpha)}(s),$$

where

$$W_i^{(\alpha)}(t) = w_i(t) - \frac{1}{2} \sum_{j \in \mathbf{J}(\alpha)} \int_0^t \nabla\varphi(\xi_i^{(\alpha)}(s) - \xi_j^{(\alpha)}(s)) ds + \int_0^t c_i^{(\alpha)}(\xi^{(\alpha)}(s)) ds.$$

For $x \in \mathcal{A}$ and for any $\mathbf{J} \subset \mathbb{N}$, we denote by $x_{\mathbf{J}} = (x_i, i \in \mathbf{J}) \in (\mathbb{R}^d)^{\mathbf{J}}$ the projection of x on $(\mathbb{R}^d)^{\mathbf{J}}$ and by $|x|_{\mathbf{J}} = \max_{i \in \mathbf{J}} |x_i|$ its supremum norm.

Each process $(\xi_{\mathbb{J}(i_0)}^{(\alpha)}(t))_{t \le 1/m}$ is solution of a Skorohod equation in $(\mathbb{R}^d)^{\sharp\mathbb{J}(i_0)}$. Using the bound $\sharp\mathbb{J}(i_0) \le (M+1)^d$, we embed $(\mathbb{R}^d)^{\sharp\mathbb{J}(i_0)}$ in $(\mathbb{R}^d)^{(M+1)^d}$ and consider $\xi_{\mathbb{J}(i_0)}^{(\alpha)}$ as a process with values in $(\mathbb{R}^d)^{(M+1)^d}$ (and state $\xi_i^{(\alpha)} \equiv 0$ for $i \notin \mathbb{J}(i_0)$). This process can be viewed as a solution of a Skorohod equation with values in $(\mathbb{R}^d)^N$ where $N = (M+1)^d$ particles, for $w = (W_i^{(\alpha)}, 1 \leqslant i \leqslant N)$ with $W_i^{(\alpha)} \equiv 0$ for $i \notin \mathbb{J}(i_0)$ and reflected on the boundary of $\mathbf{D}_{\sharp\mathbb{J}(i_0)} \times (\mathbb{R}^d)^{N - \sharp\mathbb{J}(i_0)}$, see Appendix. Note that

$$\forall h > 0, \quad \Delta(W_i^{(\alpha)}, h) \leqslant \Delta(w_i, h)) + \frac{\overline{\nabla\varphi}}{2} h + g(R) h \qquad \text{for indices } i \text{ for which } |x_i^{(\alpha)}| < R_0 - R;$$

thus, by **(iv)**,

$$\forall n \geqslant m \quad \Delta(W_i^{(\alpha)}, \frac{1}{n}) \leqslant \frac{1}{n^\kappa} + \frac{\overline{\nabla\varphi}}{2n} + \frac{g(R)}{n}$$

which tends uniformly in $R > 0$ and $i \in \mathbb{J}(i_0)$ to 0 when n tends to infinity.
So we can apply the result of Lemma 5.2 in the Appendix to the set of paths

$$\left\{ w \in \mathcal{C}_0([0,1], (\mathbb{R}^d)^N) : \ \text{ for } n \text{ large enough } \Delta(w, \frac{1}{n}) \leqslant \frac{1}{n^\kappa} + \frac{\overline{\nabla\varphi}}{2n} + \frac{g(R)}{n} \right\}.$$

There exists a constant C_5 (depending only on d, N and $\sharp\mathbb{J}(i_0)$) such that

$$\|(\sum_{j \in \mathbb{J}(i_0)} \int_0^t (\xi_i^{(\alpha)}(s) - \xi_j^{(\alpha)}(s)) d\rho_{ij}^{(\alpha)}(s))_{i \in \mathbb{J}(i_0)} \|_{\frac{1}{m}} \leqslant C_5.$$

From now on, $C_6, C_7, \dots$ denote non negative real numbers which depend only on d, φ and M. We now apply Lemma 5.1 of the Appendix and obtain for $t \in [0, \frac{1}{m}]$

$$\begin{aligned}|\xi_i^{(1)}(t) - \xi_i^{(2)}(t)| \quad &\leqslant |\xi_{\mathbb{J}(i_0)}^{(1)}(t) - \xi_{\mathbb{J}(i_0)}^{(2)}(t)| \\ &\leqslant \big(|x_{\mathbb{J}(i_0)}^{(1)} - x_{\mathbb{J}(i_0)}^{(2)}| + \|W_{\mathbb{J}(i_0)}^{(1)} - W_{\mathbb{J}(i_0)}^{(2)}\|_t\big) \exp\big(2C_5 C(\mathbf{D}_{\sharp\mathbb{J}(i_0)})\big) \\ &\leqslant \frac{C_6}{N}\Big(|x_{\mathbb{J}(i_0)}^{(1)} - x_{\mathbb{J}(i_0)}^{(2)}| + \|W_{\mathbb{J}(i_0)}^{(1)} - W_{\mathbb{J}(i_0)}^{(2)}\|_t\Big),\end{aligned}$$

where $C_6 := N \exp(2C_5 C(\mathbf{D}_{\sharp\mathbb{J}(i_0)}))$.

By **(i)** and **(ii)** we have $|\xi_i^{(1)}(t) - \xi_i^{(2)}(t)| \leqslant \varepsilon_0 + 2\varepsilon_1$ for all the i's such that $|x_i^{(1)} - x_{i_0}^{(1)}| \leqslant R'$. Thus using assumptions (3) and (2), we obtain

$$\begin{aligned}&\|W_{\mathbb{J}(i_0)}^{(1)} - W_{\mathbb{J}(i_0)}^{(2)}\|_t \\ &\leqslant \frac{1}{2}\sum_{i\in\mathbb{J}(i_0)}\sum_{j\in\mathbb{J}_R(i_0)}\int_0^t |\nabla\varphi(\xi_i^{(1)}(s) - \xi_j^{(1)}(s)) - \nabla\varphi(\xi_i^{(2)}(s) - \xi_j^{(2)}(s))|ds \\ &\quad + \frac{1}{2}\sum_{i\in\mathbb{J}(i_0)}\sum_{j\notin\mathbb{J}_R(i_0)}\int_0^t \big(|\nabla\varphi(\xi_i^{(1)}(s) - \xi_j^{(1)}(s))| + |\nabla\varphi(\xi_i^{(2)}(s) - \xi_j^{(2)}(s))|\big)ds \\ &\quad + \sum_{i\in\mathbb{J}(i_0)}\int_0^t \big(|c_i^{(1)}(\xi^{(1)}(s))| + |c_i^{(2)}(\xi^{(2)}(s))|\big)ds \\ &\leqslant \frac{\overline{\overline{\nabla\varphi}}}{2}\sum_{i\in\mathbb{J}(i_0)}\int_0^t \max_{j\in\mathbb{J}_R(i_0)}|\xi_i^{(1)}(s) - \xi_j^{(1)}(s) - \xi_i^{(2)}(s) + \xi_j^{(2)}(s)|ds + \ Ng(R)t + \ 2\times Ng(R)t \\ &\leqslant N\overline{\overline{\nabla\varphi}}\int_0^t |\xi^{(1)}(s) - \xi^{(2)}(s))|_{\mathbb{J}_R(i_0)}ds + \ 3Ng(R)t.\end{aligned}$$

Then we have, for $t \in [0, \frac{1}{m}]$,

$$|\xi_{i_0}^{(1)}(t) - \xi_{i_0}^{(2)}(t)| \leqslant C_6|x^{(1)} - x^{(2)}|_{\mathbb{J}(i_0)} + \overline{\overline{\nabla\varphi}}C_6\int_0^t |\xi^{(1)}(s) - \xi^{(2)}(s)|_{\mathbb{J}_R(i_0)}ds + 3C_6 g(R)t. \quad (22)$$

From (21) we see that if $|x_{i_0}^{(1)}| < R_0 - 2R'$ and $i \in \mathbb{J}_R(i_0)$, then $|x_i^{(1)}| < R_0 - R'$ and so we can apply the above computation to the i^{th} particle too : for $t \in [0, \frac{1}{m}]$,

$$\begin{aligned}&|\xi_i^{(1)}(t) - \xi_i^{(2)}(t)| \\ &\leqslant C_6|x^{(1)} - x^{(2)}|_{\mathbb{J}(i)} + \overline{\overline{\nabla\varphi}}C_6\int_0^t |\xi^{(1)}(s) - \xi^{(2)}(s)|_{\mathbb{J}_R(i)}ds + 3C_6 g(R)t.\end{aligned}$$

Since $\mathbb{J}_R(i) \subset \mathbf{J}_2 := \{j \in \mathbb{N} : |x_{i_0}^{(1)} - x_j^{(1)}| < 2R'\}$, from (21) and (22) we have for each $t \in [0, \frac{1}{m}]$:

$$\begin{aligned}|\xi_{i_0}^{(1)}(t) - \xi_{i_0}^{(2)}(t)| \quad &\leqslant \quad C_6|x^{(1)} - x^{(2)}|_{\mathbf{J}_2} \\ &\quad + \overline{\overline{\nabla\varphi}}C_6\int_0^t \Big(C_6|x^{(1)} - x^{(2)}|_{\mathbf{J}_2} + \overline{\overline{\nabla\varphi}}C_6\int_0^s |\xi^{(1)}(u) - \xi^{(2)}(u)|_{\mathbf{J}_2}du + 3C_6 g(R)s\Big)ds \\ &\quad + 3C_6 g(R)t \\ &\leqslant \quad C_6(1 + \overline{\overline{\nabla\varphi}}C_6 t)|x^{(1)} - x^{(2)}|_{\mathbf{J}_2} + \ (\overline{\overline{\nabla\varphi}}C_6)^2\int_0^t\int_0^s |\xi^{(1)}(u) - \xi^{(2)}(u)|_{\mathbf{J}_2}\, du\, ds \\ &\quad + 3C_6 g(R)(t + \overline{\overline{\nabla\varphi}}C_6\frac{t^2}{2}).\end{aligned}$$

Repeating this procedure and defining the set of indices $\mathbf{J}_k := \{j \in \mathbb{N} : |x_{i_0}^{(1)} - x_j^{(1)}| < kR'\}$, we obtain for the indices i_0 for which $|x_{i_0}^{(1)}| < R_0 - kR'$:

$$\begin{aligned}|\xi_{i_0}^{(1)}(t) - \xi_{i_0}^{(2)}(t)| \leqslant{}& C_6 \exp(\overline{\overline{\nabla\varphi}} C_6 t)|x^{(1)} - x^{(2)}|_{\mathbf{J}_k} \\ &+ (\overline{\overline{\nabla\varphi}} C_6)^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} |\xi^{(1)}(t_k) - \xi^{(2)}(t_k)|_{\mathbf{J}_k}\, dt_k \cdots dt_2\, dt_1 \\ &+ 3C_6 g(R) t \exp(\overline{\overline{\nabla\varphi}} C_6 t).\end{aligned}$$

Once more, by (i) and (ii) we have $|\xi^{(1)}(t_k) - \xi^{(2)}(t_k)|_{\mathbf{J}_k} \leqslant \varepsilon_0 + 2\varepsilon_1$. After the m^{th} iteration, we obtain for all indices i for which $|x_i^{(1)}| \leqslant R_0 - mR'$ and for all $t \in [0, \frac{1}{m}]$,

$$|\xi_i^{(1)}(t) - \xi_i^{(2)}(t)| \leqslant \varepsilon_0 \left(C_6 e^{\overline{\overline{\nabla\varphi}} C_6/m} + \frac{(\overline{\overline{\nabla\varphi}} C_6)^m}{m^m m!} \right) + 2\varepsilon_1 \frac{(\overline{\overline{\nabla\varphi}} C_6)^m}{m^m m!} + \frac{3C_6}{m} g(R) \exp(\frac{\overline{\overline{\nabla\varphi}} C_6}{m})$$

which is exactly the desired result (20) provided we define

$$C_3 := \sup_m \left(C_6 e^{\overline{\overline{\nabla\varphi}} C_6/m} + \frac{(\overline{\overline{\nabla\varphi}} C_6)^m}{m^m m!} \right) \text{ and } C_4 := \max \left(\frac{r}{M} \sup_m \frac{(\overline{\overline{\nabla\varphi}} C_6)^m}{m^m}, \sup_m \frac{3C_6}{m} \exp(\frac{\overline{\overline{\nabla\varphi}} C_6}{m}) \right).$$

■

We are now able to prove the convergence of the approximating processes $(X_i^{\ell,x}(\omega, t), L_{ij}^{\ell,x}(\omega, t), j \in I, t \in [0, 1])_{\ell \in \mathbb{N}^*}$, as stated in proposition 3.5.

Proof of Proposition 3.5

Remark first that for a finite range interaction φ, we could construct in [2] and [3] a sequence of approximations which is stationary for ℓ large enough and thus converges in the strongest way possible. This technique is no more valid for infinite range interactions φ. We will show in which follows that the sequence $(X_i^{\ell,x}(\omega))_\ell$ is a Cauchy sequence.

We fix m large enough and apply several times Lemma 3.6 to compare on the full time interval $[0, 1]$ the paths $\xi_i^{(1)} := X_i^{\ell(m),x}(\omega)$ and $\xi_i^{(2)} := X_i^{\ell,x}(\omega)$ for $\ell > \ell(m)$. All parameters will depend on m: the allowed ε_1-oscillations with $\varepsilon_1 := 1/m^\kappa$ and the domain in which the concerned particles start their motion depends on some radius $R(m)$ (increasing function of m which will be fixed later).

We also decompose the second infinite sum in the drift of $X_i^{\ell(m),x}(\omega)$ (resp. $X_i^{\ell,x}(\omega)$) in a finite sum over $\mathbf{J}(1) := \{i \in \mathbb{N} : |x_i| < \ell(m)\}$ (resp. over $\mathbf{J}(2) := \{i \in \mathbb{N} : |x_i| < \ell\}$), in such a way that the rests are given by

$$\begin{aligned} c_i^{(1)}(\gamma) &= -\frac{1}{2} \sum_{j:|x_j| \geqslant \ell(m)} \nabla\varphi(\gamma_i - x_j) - \frac{1}{2} \nabla\psi^{\ell(m),x}(\gamma_i), \\ c_i^{(2)}(\gamma) &= -\frac{1}{2} \sum_{j:|x_j| \geqslant \ell} \nabla\varphi(\gamma_i - x_j) - \frac{1}{2} \nabla\psi^{\ell,x}(\gamma_i). \end{aligned}$$

For m large enough and initial positions x_i small enough (see (23) for a precise relation), $\psi^{\ell(m),x}(X_i^{\ell(m),x}(\omega, t)) = 0$ and $\psi^{\ell,x}(X_i^{\ell,x}(\omega, t)) = 0$ at any time $t \in [0, 1]$ since $X_i^{\ell(m),x}(\omega)$ and $X_i^{\ell,x}(\omega)$ are nice paths which stay confined not too far from their initial positions. More precisely, we choose the radius $R(m)$ in such a way that

$$m^2 R'(m) + m \frac{1}{m^\kappa} + R(m) \leq \ell(m) < \ell \quad \text{ with } \quad R'(m) = R(m) + Mr. \tag{23}$$

The last inequality is in particular satisfied for $R(m) = \ell(m)/2m^2$, which is from now on fixed equal to this sequence.

Then, since $X^{\ell(m),x}(\omega) \notin \tilde{\mathcal{B}}(\frac{1}{m}, \frac{1}{m^\kappa})$ we have for indices i for which $|x_i| < m^2 R'(m)$, and for all $t \in [0,1]$ $|X_i^{\ell(m),x}(\omega,t)| \leqslant m^2 R'(m) + \frac{m}{m^\kappa}$; thus $\nabla\psi^{\ell(m),x}(X_i^{\ell(m),x}(\omega,t)) = 0$. The same result holds for $X_i^{\ell,x}$.

Thus, for each $t \in [0,1]$ and each i for which $|x_i| < m^2 R'(m)$ we have

$$|c_i^{(1)}(X_i^{\ell(m),x}(\omega,t))| \leqslant g(R(m)) \text{ and } |c_i^{(2)}(X_i^{\ell,x}(\omega,t))| \leqslant g(R(m)).$$

Note that since $\omega \in \Omega_x$, $\xi^{(1)} = X^{\ell(m),x}(\omega) \notin \tilde{\mathcal{B}}(\frac{1}{m}, \frac{1}{m^\kappa}) \cup \tilde{\tilde{\mathcal{B}}}(m, M, \frac{4}{m^\kappa})$. Also note that for $\ell(m) < \ell \leq \ell(m+1)$, $\xi^{(2)} = X^{\ell,x}(\omega) \notin \tilde{\mathcal{B}}(\frac{1}{m}, \frac{1}{m^\kappa})$ by definition of Ω_x.

For $t \in [0, \frac{1}{m}]$, we first apply Lemma 3.6 with $x^{(1)} = x^{(2)} := x$, $\varepsilon_0 = 0$, $R_0 = 2m^2 R'(m)$. Assumption **(i)** is satisfied as soon as $\varepsilon_1 = \frac{1}{m^\kappa} < \frac{r}{2M}$ which is true for m large enough. Then, for all indices i for which $|x_i| < (2m^2 - m)R'(m)$, by (20)

$$\forall t \in [0, \frac{1}{m}], \quad |X_i^{\ell(m),x}(\omega,t) - X_i^{\ell,x}(\omega,t)| \leqslant \varepsilon(m)$$

where the sequence $\varepsilon(m) := C_4 \left(\frac{1}{m!} + g(R(m))\right)$ converges to 0.

For $t \in [\frac{1}{m}, \frac{2}{m}]$, we apply Lemma 3.6 a second time with $x^{(1)} := X^{\ell(m),x}(\omega, \frac{1}{m})$, $x^{(2)} := X^{\ell,x}(\omega, \frac{1}{m})$, $\varepsilon_0 = \varepsilon(m)$ and $R_0 = (2m^2 - m)R'(m)$.

Note again that $\xi_i^{(1)} = X_i^{\ell(m),x}(\omega) \notin \tilde{\mathcal{B}}(\frac{1}{m}, \frac{1}{m^\kappa}) \cup \tilde{\tilde{\mathcal{B}}}\big(m, M, 2(\frac{1}{m^\kappa} + \varepsilon(m))\big)$ because $\varepsilon(m) \leqslant 2m^{-\kappa}$ for m large enough. We get, for all indices i for which $|x_i| < (2m^2 - 2m)R'(m)$

$$\forall t \in [\frac{1}{m}, \frac{2}{m}], \quad |X_i^{\ell(m),x}(\omega,t) - X_i^{\ell,x}(\omega,t)| \quad \leqslant \quad \varepsilon(m)(C_3 + 1).$$

Repeating m times this procedure, for the last iteration with $\varepsilon_0 = \varepsilon(m)((C_3)^{m-2} + \cdots + C_3 + 1)$ (which is bounded by $2m^{-\kappa}$ for m large enough), we have for all indices i for which $|x_i| < (2m^2 - m^2)R'(m) = m^2 R'(m)$,

$$\begin{aligned} \forall t \in [0,1], \quad |X_i^{\ell(m),x}(\omega,t) - X_i^{\ell,x}(\omega,t)| \quad &\leqslant \quad \varepsilon(m) \sum_{k=0}^{m-1} (C_3)^k \\ &\leqslant \quad C_7 \Big(\frac{1}{m!} + g(R(m))\Big)(C_3)^m. \end{aligned}$$

Hence, for any m large enough and $|x_i| < m^2 R'(m) = \ell(m)/2$ (that is for any i), the sequence $(X_i^{\ell,x}(.))_\ell$ is a Cauchy sequence for the uniform norm on $[0,1]$ as soon as the following series converges :

$$\begin{aligned} \sum_m \sup_{t \in [0,1]} |X_i^{\ell(m),x}(\omega,t) - X_i^{\ell(m+1),x}(\omega,t)| \quad &\leqslant \quad C_7 \sum_m \Big(\frac{1}{m!} + g(R(m))\Big)(C_3)^m \\ &\leqslant \quad C_7 \sum_m \frac{(C_3)^m}{m!} + C_7 \sum_m g\Big(\frac{\ell(m)}{2m^2}\Big)(C_3)^m \qquad (24) \end{aligned}$$

It suffices to control the growth of the second series in the right hand side of (24). This convergence condition determines the class of pair interaction φ we may consider between the particles. Indeed, the function $\nabla\varphi$ should decrease fast enough at infinity, in such a way that the function $g(\rho)$, introduced as an upperbound of $\rho \mapsto \sup_{\xi \in \mathcal{A}} \sum_{\{j : |\xi_j| > \rho\}} |\nabla\varphi(\xi_j)|$, satisfies

$$\sum_m (C_3)^m g\Big(\frac{1}{2m^2} \exp(\frac{1}{6(d+1)} m^{1-2\kappa})\Big) \quad < \quad +\infty. \qquad (25)$$

The quasi polynomial decay of g proposed in assumption (3) is almost optimal. For such functions g, there exists real numbers C, C' such that for large m

$$g\Big(\frac{1}{2m^2}\exp(\frac{1}{6(d+1)}m^{1-2\kappa})\Big) \leqslant C\exp(-C'm^{(1-2\kappa)(1+b)})$$

for any $b>0$ as soon as

$$0<\kappa<\frac{b}{2(1+b)}. \tag{26}$$

Furthermore, the convergence of the series (25) is very fast :

$$\exists C_8>0,\ \forall m_0\in\mathbb{N}^* \quad \sum_{m\geqslant m_0}(C_3)^m g\Big(\frac{1}{2m^2}\ell(m)\Big) \leqslant \frac{C_8}{m_0!} \tag{27}$$

Note that C_8 may be chosen such that $\sum_{m\geqslant m_0}(C_3)^m/m! \leqslant \frac{C_8}{m_0!}$ too. Thus, $(X_i^{\ell,x}(.))_\ell$ is a Cauchy sequence satisfying

$$\sum_{m\geqslant m_0}\sup_{t\in[0,1]}|X_i^{\ell(m),x}(\omega,t)-X_i^{\ell(m+1),x}(\omega,t)| \leqslant \frac{2C_8}{m_0!}.$$

It then converges uniformly in time towards a continuous path denoted by $X_i^{\infty,x}(\omega)$. Moreover, for m large enough and $\ell>\ell(m)$,

$$\sup_{t\in[0,1]}|X_i^{\ell,x}(\omega,t)-X_i^{\infty,x}(\omega,t)| < \frac{2C_8}{m!}. \tag{28}$$

We still have to prove the convergence of the local times.
By the same argument as in the proof of Lemma 3.6, there exists for any $i_0\in\mathbb{N}$, for ℓ sufficiently large and for $k=0,1,\ldots,m$ a finite subset of indices $\mathbb{J}^\ell(\frac{k}{m},i_0)$ containing i_0 such that for each $i\in\mathbb{J}^\ell(\frac{k}{m},i_0)$ and $j\notin\mathbb{J}^\ell(\frac{k}{m},i_0)$

$$|X_i^{\ell,x}(\omega,t)-X_j^{\ell,x}(\omega,t)| > r+m^{-\kappa},\ \ \forall t\in[\frac{k}{m},\frac{k+1}{m}].$$

Due to the strong convergence estimates (28), for sufficiently large ℓ, we can chose the sets $\mathbb{J}^\ell(\frac{k}{m},i_0)$ to be independent of ℓ and denote it by $\mathbb{J}(\frac{k}{m},i_0)$, $k=0,1,\ldots,m$. Then we have for $i\in\mathbb{J}(\frac{k}{m},i_0)$

$$\begin{aligned} X_i^{\ell,x}(\omega,t) &= X_i^{\ell,x}(\omega,\frac{k}{m})+W_i(\omega,t)-W_i(\omega,\frac{k}{m})-\frac{1}{2}\sum_j\int_{\frac{k}{m}}^t\nabla\varphi(X_i^{\ell,x}(\omega,s)-X_j^{\ell,x}(\omega,s))ds \\ &+\sum_{j\in\mathbb{J}(\frac{k}{m},i_0)}\int_{\frac{k}{m}}^t X_i^{\ell,x}(\omega,s)-X_j^{\ell,x}(\omega,s))dL_{ij}^{\ell,x}(\omega,s),\ \ \forall t\in[\frac{k}{m},\frac{k+1}{m}]. \end{aligned}$$

Let us denote by $\rho_i^{\ell,x}(\omega,t)$ the last term in the above decomposition of $X_i^{\ell,x}(\omega,t)$, that is its local time. From the Cauchy property of the sequence $(X_i^{\ell,x})_\ell$, we deduce that $(\rho_i^{\ell,x})_\ell$ is a Cauchy sequence too : for m large enough and $\ell(m)\leqslant\ell < \ell(m+1)$, for $i \in \mathbb{J}(\frac{k}{m}, i_0)$

$$
\begin{aligned}
|\rho_i^{\ell,x}(t) - \rho_i^{\ell+1,x}(t)| \quad &\leqslant |X_i^{\ell,x}(t) - X_i^{\ell+1,x}(t)| + |X_i^{\ell,x}(\frac{k}{m}) - X_i^{\ell+1,x}(\frac{k}{m})| \\
&+ \frac{1}{2}\sum_j \int_{\frac{k}{m}}^t |\nabla\varphi(X_i^{\ell,x}(s) - X_j^{\ell,x}(s)) - \nabla\varphi(X_i^{\ell+1,x}(s) - X_j^{\ell+1,x}(s))| ds \\
&\leqslant 4\frac{C_8}{m!} + \frac{\overline{\overline{\nabla\varphi}}}{2} \sup_{s\in[\frac{k}{m},t]} (|X_i^{\ell,x}(s) - X_i^{\ell+1,x}(s)| + \max_{j:|x_j|\leq m^2R'(m)} |X_j^{\ell,x}(s) - X_j^{\ell+1,x}(s)|) \\
&+ \frac{1}{2m} \sum_{j:|x_j|>m^2R'(m)} \sup_{s\in[\frac{k}{m},t]} (|\nabla\varphi(X_i^{\ell,x}(s) - X_j^{\ell,x}(s))| + |\nabla\varphi(X_i^{\ell+1,x}(s) - X_j^{\ell+1,x}(s))|) \\
&\leqslant \frac{C_9}{m!} + \frac{1}{m} g(R(m)) \\
&\leq \frac{C_{10}}{m!}, \qquad \forall t \in [\frac{k}{m}, \frac{k+1}{m}].
\end{aligned}
$$

Then, for indices i for which $|x_i|\leqslant m^2R'(m)$, $(\rho_i^{\ell,x}(.))_\ell$ converges uniformly in time to a process (with bounded variation) $\rho_i^{\infty,x}(.)$. Similarly we can prove that the total variation on $[0,1]$ of $\rho_i^{\ell,x}$ converges to the total variation of $\rho_i^{\infty,x}$. The proof of Proposition 3.5 is now completed. ■

3.4 The set Ω_x is of full measure

We show in this section why the set Ω_x is of full measure with respect to the Gibbs measures.

Proposition 3.7 *For any Gibbs measure $\mu \in \mathcal{G}(z), z > 0$, one has* $\int_{\mathcal{M}} P(\Omega_x)\ d\mu(x) = 1$.
As a corollary, any Gibbs measure $\mu \in \mathcal{G}(z)$ has its support included in $\underline{\mathcal{A}} := \{x \in \mathcal{A},\ P(\Omega_x) = 1\}$.

Proof of Proposition 3.7

We have to prove that $\int_{\mathcal{A}} P(\Omega \setminus \Omega_x)\ d\mu(x) = 0$

By definition of Ω_x (see (18)),

$$P(\Omega \setminus \Omega_x) \leqslant P(\Omega \setminus \tilde{\Omega}_x) + P(\Omega \setminus \tilde{\tilde{\Omega}}_x) + P(\Omega \setminus \Omega_x^W)$$

Thanks to Borel-Cantelli lemma, $\int_{\mathcal{A}} P(\Omega \setminus \Omega_x)\ d\mu(x)$ vanishes as soon as the series

$$
\begin{aligned}
\textstyle\sum_m \quad \int_{\mathcal{A}} \Big(P\Big(X^{\ell(m),x} \in \tilde{\tilde{\mathcal{B}}}\big(m, M, \frac{4}{m^\kappa})\Big) + \sum_{\ell=\ell(m)}^{\ell(m+1)} P\Big(X^{\ell,x} \in \tilde{\mathcal{B}}(\frac{1}{m}, \frac{1}{m^\kappa})\Big) \\
+ \sum_{i:|x_i|<\ell(m+1)} P\Big(\exists n \geq m : \Delta(W_i, \frac{1}{n}) > \frac{1}{n^\kappa}\Big)\Big) d\mu(x)
\end{aligned}
$$

converges. We first control the series

$$\sum_m \int_{\mathcal{A}} \sum_{i:|x_i|<\ell(m+1)} P\left(\exists n \geq m : \Delta(W_i, \frac{1}{n}) > \frac{1}{n^\kappa}\right) d\mu(x)\ .$$

This is bounded above by

$$\leqslant \quad \sum_m \frac{\ell(m+1)^d}{(r/2)^d} \sum_{n\geq m} P\left(\Delta(W_1, \frac{1}{n}) > \frac{1}{n^\kappa}\right)$$

Exchanging the sums and using the oscillation estimate for the Brownian motion (Proposition 3.2) we get

$$\begin{aligned} &\leqslant \quad 41 \sum_n \sum_{m=1}^{n} \frac{\ell(m+1)^d}{(r/2)^d} \; n \; \exp(-\frac{1}{5} n^{1-2\kappa}) \\ &\leqslant \quad \frac{41}{(r/2)^d} \sum_n n^2 \; \ell(n+1)^d \; \exp(-\frac{1}{5} n^{1-2\kappa}). \end{aligned} \tag{29}$$

Replacing the sequence $(\ell(m))_m$ by its value done in (17), and remarking that $\kappa < 1/2$, it is clear that the above series (29) converges.
We now study the convergence of

$$\sum_m \int_{\mathcal{A}} \Big(P\Big(X^{\ell(m),x} \in \tilde{\tilde{\mathcal{B}}}(m, M, \frac{4}{m^\kappa})\Big) + \sum_{\ell=\ell(m)}^{\ell(m+1)} P\Big(X^{\ell,x} \in \tilde{\mathcal{B}}(\frac{1}{m}, \frac{1}{m^\kappa})\Big)\Big) d\mu(x). \tag{30}$$

We shall show a small later (step 1) that for each $\ell \in \mathbb{N}^*$ and for $\Lambda = B(0,\ell)$, the following inequalities hold :

$$\begin{aligned} &\Big| \int_{\mathcal{A}} P(X^{\ell,x} \in \Theta) \, d\mu(x) - \int_{\mathcal{A}} Q_z^{\ell,\eta}(\Theta) \, d\mu(\eta) \Big| \\ &\qquad \leqslant \int_{\mathcal{A}} \sup_{\|f\| \leqslant 1} \Big| \int_{\mathcal{A}} f(x) \; d\mu(x|\eta_{\Lambda^c}) - \int_{\mathcal{A}} f(x) \; d\mu_z^{\ell,\eta}(x) \Big| \; d\mu(\eta) \\ &\text{and} \quad \forall \eta \in \mathcal{A} \quad \sup_{\|f\| \leqslant 1} \Big| \int_{\mathcal{A}} f(x) \; d\mu(x|\eta_{\Lambda^c}) - \int_{\mathcal{A}} f(x) \; d\mu_z^{\ell,\eta}(x) \Big| \leqslant 2 \Big(1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{\ell,\eta}} \Big) \end{aligned} \tag{31}$$

and (step 2) that

$$\forall \eta \in \mathcal{A} \quad 0 \leqslant 1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{\ell,\eta}} \leqslant z \; e^{2\overline{\varphi}} \int_{\mathbb{R}^d} \mathbb{1}_{\psi^{\ell,\eta}(y) > 0} \; \exp(-\psi^{\ell,\eta}(y)) \; dy \tag{32}$$

If inequality (32) holds, due to assumption (9) on $\psi^{\ell,\eta}$ one gets

$$\sum_{\ell=1}^{+\infty} \int_{\mathcal{A}} \Big(1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{\ell,\eta}} \Big) d\mu(\eta) < +\infty.$$

Now for each m , we use (31) twice with $\Theta = \tilde{\tilde{\mathcal{B}}}\left(m, M, \frac{4}{m^\kappa}\right)$ and with $\Theta = \tilde{\mathcal{B}}\left(\frac{1}{m}, \frac{1}{m^\kappa}\right)$.
Thanks to (31) and (32), in order to prove the convergence of the series (30), we only have to prove that

$$\sum_m \int_{\mathcal{A}} \Big(Q_z^{\ell(m),\eta}\Big(\tilde{\tilde{\mathcal{B}}}(m, M, \frac{4}{m^\kappa})\Big) + \sum_{\ell=\ell(m)}^{\ell(m+1)} Q_z^{\ell,\eta}\Big(\tilde{\mathcal{B}}(\frac{1}{m}, \frac{1}{m^\kappa})\Big)\Big) d\mu(\eta) < +\infty$$

By propositions 3.4 and 3.1, the left hand side is smaller than

$$\sum_m m \, \Big(z \; C_2 \; \frac{4}{m^\kappa} \Big)^M + z \; C_1 \; \sum_m \big(\ell(m+1) - \ell(m) + 1 \big) \; \ell(m+1)^d \; m \; \exp\big(- \frac{m^{1-2\kappa}}{5} \big).$$

It is then enough to prove that both series converge:

$$\sum_m \ell(m+1)^{d+1} \; m \; \exp\big(- \frac{m^{1-2\kappa}}{5} \big) < +\infty \tag{33}$$

$$\text{and} \sum_m \frac{1}{m^{M\kappa-1}} < +\infty. \tag{34}$$

In (24), we saw that it is convenient to choose the volume sequence $(\ell(m))_m$ as large as possible to be able to treat interactions φ with slow decay. The choice $\ell(m) := \exp(\frac{1}{6(d+1)}m^{1-2\kappa})$ is (almost) the largest possible in order to obtain (33) (and also the convergence of (29)).

On the other hand, in order that (34) holds, we get the following (sufficient) condition on the length sequence of the chains :

$$M > \frac{2}{\kappa}. \tag{35}$$

It remains to prove (31) and (32).

Step 1 : Proof of (31)

Let us fix $\ell \in \mathbb{N}^*$ and $\Lambda = B(0,\ell)$. For each event Θ on $\mathcal{C}([0,1],\mathcal{A})$, by definition of $Q_z^{\ell,\eta}$,

$$\begin{aligned} &\int_{\mathcal{A}} P(X^{\ell,x} \in \Theta)\ d\mu(x) - \int_{\mathcal{A}} Q_z^{\ell,\eta}(\Theta)\ d\mu(\eta) \\ &\leqslant \int_{\mathcal{A}}\int_{\mathcal{A}} P(X^{\ell,x\eta_{\Lambda^c}} \in \Theta)\ d\mu(x|\eta_{\Lambda^c})\ d\mu(\eta) - \int_{\mathcal{A}}\int_{\mathcal{A}} P(X^{\ell,\eta,\sharp x}(x,\cdot) \in \Theta)\ d\mu_z^{\ell,\eta}(x)\ d\mu(\eta) \end{aligned}$$

If $x\eta \in \mathcal{A}$ then $P(X^{\ell,\eta,\sharp x}(x,\cdot) \in \Theta) = P(X^{\ell,x\eta_{\Lambda^c}} \in \Theta)$ i.e. the integrated functions are equal, and since they are bounded by 1, we obtain :

$$\left| \int_{\mathcal{A}} P(X^{\ell,x} \in \Theta) d\mu(x) - \int_{\mathcal{A}} Q_z^{\ell,\eta}(\Theta) d\mu(\eta) \right| \leqslant \int_{\mathcal{A}} \sup_{\|f\|\leqslant 1} \left| \int_{\mathcal{A}} f(x)\ d\mu(x|\eta_{\Lambda^c}) - \int_{\mathcal{A}} f(x)\ d\mu_z^{\ell,\eta}(x) \right|\ d\mu(\eta)$$

Since $\mu \in \mathcal{G}(z)$, using the conditional density of μ with respect to π^z and the definition of $\mu_z^{\ell,\eta}$, one has for each $f : \mathcal{A} \to \mathbb{R}$ bounded by 1 :

$$\begin{aligned} &\left| \int_{\mathcal{A}} f(x)\ d\mu(x|\eta_{\Lambda^c}) - \int_{\mathcal{A}} f(x)\ d\mu_z^{\ell,\eta}(x) \right| \\ &= \Bigg| \frac{e^{-z|\Lambda|}}{Z_z^{\Lambda,\eta}} \Bigg(f(\eta_{\Lambda^c}) + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} f(y\eta_{\Lambda^c})\ \mathbb{1}_{\mathcal{A}}(y\eta_{\Lambda^c})\ \exp\Big(- \sum_{1\leqslant i<j\leqslant n} \varphi(y_i - y_j) - \sum_{\substack{1\leqslant i\leqslant n \\ \eta_j \in \Lambda^c}} \varphi(y_i - \eta_j)\Big) dy \Bigg) \\ &\quad - \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \Bigg(f(\eta_{\Lambda^c}) + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} f(y\eta_{\Lambda^c})\ e^{-\beta_n^{\ell,\eta}(y)}\ \mathbb{1}_{\mathcal{A}}(y)\ dy \Bigg) \Bigg| \end{aligned}$$

Note that $\beta_n^{\ell,\eta}(y) = \sum_{1\leqslant i<j\leqslant n} \varphi(y_i - y_j) + \sum_{\substack{1\leqslant i\leqslant n \\ \eta_j \in \Lambda^c}} \varphi(y_i - \eta_j)$ for the $y \in \Lambda^n$ verifying $y\eta_{\Lambda^c} \in \mathcal{A}$, because $\psi^{\ell,\eta}(y_i) = 0$ for each i in this case. Thus the above quantity is equal to

$$\begin{aligned} &\Bigg| f(\eta_{\Lambda^c}) \left(\frac{e^{-z|\Lambda|}}{Z_z^{\Lambda,\eta}} - \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \right) + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \left(\frac{e^{-z|\Lambda|}}{Z_z^{\Lambda,\eta}} - \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \right) \int_{\Lambda^n} f(y\eta_{\Lambda^c})\ \mathbb{1}_{\mathcal{A}}(y\eta_{\Lambda^c})\ e^{-\beta_n^{\ell,\eta}(y)}\ dy \\ &\quad - \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} f(y\eta_{\Lambda^c})\ e^{-\beta_n^{\ell,\eta}(y)}\ (\mathbb{1}_{\mathcal{A}}(y) - \mathbb{1}_{\mathcal{A}}(y\eta_{\Lambda^c})\ \mathbb{1}_{\Lambda^n}(y))\ dy \Bigg| \end{aligned}$$

Since $e^{-z|\Lambda|} = e^{-z|B(0,\ell)|}$, it holds

$$\begin{aligned} Z_z^{\Lambda,\eta} &= e^{-z|\Lambda|} \left(1 + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} \mathbb{1}_{\mathcal{A}}(y\eta_{\Lambda^c})\ e^{-\beta_n^{\ell,\eta}(y)}\ dy \right) \\ &\leqslant e^{-z|B(0,\ell)|} \left(1 + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} \mathbb{1}_{\mathcal{A}}(y)\ e^{-\beta_n^{\ell,\eta}(y)}\ dy \right) = Z_z^{\ell,\eta}\ . \end{aligned}$$

Since f is bounded by 1, we then obtain :

$$\left|\int_{\mathcal{A}} f(x)\ d\mu(x|\eta_{\Lambda^c}) - \int_{\mathcal{A}} f(x)\ d\mu_z^{\ell,\eta}(x)\right|$$
$$\leqslant \left|\frac{e^{-z|\Lambda|}}{Z_z^{\Lambda,\eta}} - \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}}\right| e^{z|\Lambda|} Z_z^{\Lambda,\eta} + \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \left|e^{z|B(0,\ell)|} Z_z^{\ell,\eta} - e^{z|\Lambda|} Z_z^{\Lambda,\eta}\right| = 2\left(1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{\ell,\eta}}\right)$$

and (31) is proven.

Step 2 : Proof of (32)

It is straightforward, only using the definitions of $Z_z^{\ell,\eta}$, $Z_z^{\Lambda,\eta}$ and $\psi^{\ell,\eta}$:

$$1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{\ell,\eta}} = \frac{1}{Z_z^{\ell,\eta}}(Z_z^{\ell,\eta} - Z_z^{\Lambda,\eta})$$
$$= \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} \mathbb{1}_{\mathcal{A}}(\xi_1,\ldots,\xi_n)\ e^{-\beta_n^{\ell,\eta}(\xi_1,\ldots,\xi_n)} \left(1 - \prod_{i=1}^n \mathbb{1}_{\Lambda\smallsetminus B(\eta_{\Lambda^c},r)}(\xi_i)\right) d\xi_1\cdots d\xi_n$$
$$\leqslant \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} \mathbb{1}_{\mathcal{A}}(\xi_1,\ldots,\xi_n)\ e^{-\beta_n^{\ell,\eta}(\xi_1,\ldots,\xi_n)} \left(\sum_{i=1}^n \mathbb{1}_{\psi^{\ell,\eta}(\xi_i)>0}\right) d\xi_1\cdots d\xi_n$$

and using inequality (14), this is

$$\leqslant \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!}\ n \int_{(\mathbb{R}^d)^n} \mathbb{1}_{\mathcal{A}}(\xi_1,\ldots,\xi_n)\ e^{-\psi^{\ell,\eta}(\xi_1)}\ e^{2\overline{\varphi}}\ e^{-\beta_{n-1}^{\ell,\eta}(\xi_2,\ldots,\xi_n)} \mathbb{1}_{\psi^{\ell,\eta}(\xi_1)>0}\ d\xi_1\cdots d\xi_n$$
$$\leqslant \frac{e^{-z|B(0,\ell)|}}{Z_z^{\ell,\eta}} \sum_{n=1}^{+\infty} z\ \frac{z^{n-1}}{(n-1)!}\ \nu_{n-1}^{\ell,\eta}((\mathbb{R}^d)^{n-1})\ e^{2\overline{\varphi}} \int_{\mathbb{R}^d} \mathbb{1}_{\psi^{\ell,\eta}(y)>0}\ e^{-\psi^{\ell,\eta}(y)}\ dy$$
$$\leqslant z\ e^{2\overline{\varphi}} \int_{\mathbb{R}^d} \mathbb{1}_{\psi^{\ell,\eta}(y)>0}\ e^{-\psi^{\ell,\eta}(y)}\ dy$$

■

3.5 Properties of the limit Process

Until now, we proved that the approximations converge on the set Ω_x, which is of full measure. Furthermore, for $\omega \in \Omega_x$, the convergence of the sequence $(X_i^{\ell,x}(\omega,t))_\ell$ is uniform in time. We thus derive for the limit process, denoted by $X_i^{\infty,x}$, several important properties.

Proposition 3.8 *For any $x \in \mathcal{A}$ and $\omega \in \Omega_x$ the process $X^{\infty,x}(\omega,.)$ satisfies equation $(\mathcal{E})$ with initial condition $X^{\infty,x}(\omega,0) = x$. Furthermore, let $\mathcal{T}_0 \subset \mathcal{T}$ denote the set of functions on $\mathcal{M}$ whose first derivative is orthogonal to the normal vector on the boundary of the set $\mathcal{A}$ of allowed configurations, that is :*

$$\mathcal{T}_0 = \left\{ \begin{array}{l} f : \mathcal{M} \longrightarrow \mathbb{R} \text{ local, } \mathcal{C}^2\text{-function such that} \\ \text{for each } \gamma \in \mathcal{M}, \text{ if } \gamma_i, \gamma_j \in \gamma \text{ satisfy } |\gamma_i - \gamma_j| = r \text{ then } D_{\gamma_i} f(\gamma).(\gamma_i - \gamma_j) = 0. \end{array} \right\} \quad (36)$$

For each function $f \in \mathcal{T}_0$,

$$f(X(t)) - f(X(0)) - \int_0^t \mathbf{G}f(X(s))\ ds \quad \text{is a square-integrable martingale,} \quad (37)$$

where $\mathbf{G}$, the infinitesimal generator associated to the equation $(\mathcal{E})$, is given by

$$\mathbf{G}f(\gamma) = \frac{1}{2}\sum_{\gamma_i\in\gamma} \Big(Tr(D^2_{\gamma_i\gamma_i} f(\gamma)) - D_{\gamma_i} f(\gamma). \sum_{\gamma_j\in\gamma} \nabla\varphi(\gamma_i - \gamma_j)\Big) \quad (38)$$

Remark 3.9 : Since the support of φ is not a priori compact, the function $\mathbf{G}f$ is not necessarily local even if f is local. One can also write $\mathbf{G}f(\gamma)$ as an integral under γ:

$$\mathbf{G}f(\gamma) = \frac{1}{2}\int_{\mathbb{R}^d} \big(\mathrm{Tr}(D^2_{xx}f(\gamma)) - D_x f(\gamma).(\nabla\varphi * \gamma)(x)\big)\,\gamma(dx)$$

with $(\nabla\varphi * \gamma)(x) = \int_{\mathbb{R}^d} \nabla\varphi(x-y)\ \gamma(dy)$.

Proof of Proposition 3.8

We first have to identify the process $\rho_i^{\infty,x}(.)$. For $t \in [\frac{k}{m}, \frac{k+1}{m}]$ and $j \in \mathbb{J}(\frac{k}{m}, i_0)$,

$$L_{ij}^{\ell,x}(t) = \frac{1}{r^2}\int_0^t \mathbb{1}_{|X_i^{\ell,x}(s) - X_j^{\ell,x}(s)|=r}(X_i^{\ell,x}(s) - X_j^{\ell,x}(s)).\ d\rho_i^{\ell,x}(s).$$

The uniform convergence of $(X^{\ell,x})_\ell$ and the convergence in variation of $\rho_i^{\ell,x}$ imply that the right hand side of the above equation converges to

$$L_{ij}^{\infty,x}(t) := \frac{1}{r^2}\int_0^t \mathbb{1}_{|X_i^{\infty,x}(s) - X_j^{\infty,x}(s)|=r}(X_i^{\infty,x}(s) - X_j^{\infty,x}(s)).\ d\rho_i^{\infty,x}(s).$$

We then obtain for $i \in \mathbb{J}(\frac{k}{m}, i_0)$ and $t \in [\frac{k}{m}, \frac{k+1}{m}]$,

$$\begin{aligned} X_i^{\infty,x}(\omega,t) &= X_i^{\infty,x}(\omega,\frac{k}{m}) + W_i(\omega,t) - W_i(\omega,\frac{k}{m}) - \frac{1}{2}\sum_j \int_{\frac{k}{m}}^t \nabla\varphi(X_i^{\infty,x}(\omega,s) - X_j^{\infty,x}(\omega,s))ds \\ &+ \sum_{j\in\mathbb{J}(\frac{k}{m},i_0)} \int_{\frac{k}{m}}^t (X_i^{\infty,x}(\omega,s) - X_j^{\infty,x}(\omega,s))dL_{ij}^{\infty,x}(\omega,s), \end{aligned} \tag{39}$$

which implies that $X^{\infty,x}(\cdot)$ is a solution of equation $(\mathcal{E})$.

Let us consider a fixed test function $f \in \mathcal{T}$ and the solution $X^{\infty,x}$ of equation $(\mathcal{E})$. The Itô formula holds for $X = X^{\infty,x}$ (see e.g. [12], Theorem 27.2) : for any $t > 0$,

$$\left\{ \begin{aligned} f(X(t)) = f(X(0)) &+ \sum_i \int_0^t D_{X_i(s)}f(X(s))\ dW_i(s) \\ &- \frac{1}{2}\sum_i \int_0^t D_{X_i(s)}f(X(s)).\sum_j \nabla\varphi(X_i(s) - X_j(s))ds \\ &+ \sum_i\sum_j \int_0^t D_{X_i(s)}f(X(s)).(X_i(s) - X_j(s))dL_{ij}(s) \\ &+ \frac{1}{2}\sum_i \int_0^t \mathrm{Tr}(D^2_{X_i(s)X_i(s)}f(X(s)))\ ds \end{aligned} \right.$$

If $f \in \mathcal{T}_0$, the reflection term vanishes :

$$\forall t > 0, \quad \sum_i\sum_j \int_0^t D_{X_i(s)}f(X(s)).(X_i(s) - X_j(s))dL_{ij}(s) = 0.$$

Since f is local and the first derivative of f is bounded, the quadratic variation $\sum_i \int_0^t |D_{X_i(s)}f(X(s))|^2\, ds$ of the local martingale term is bounded independently of the initial condition $X(0)$. Thus, $\sum_i \int_0^t D_{X_i(s)}f(X(s))\ dW_i(s)$ is a square-integrable martingale. ■

To complete the proof of Theorem 2.5 it remains to prove the uniqueness. To this aim we define a new set of paths $\underline{\mathcal{C}} \subset \mathcal{C}([0,1],\mathcal{A})$. It contains configurations of paths which are clearly separated on each time interval with length $1/m$ into small groups (no more than M) of paths which stay at distance ε of the others groups. This set is adapted to the uniqueness problem. For $\varepsilon > 0$ and $m, M \in \mathbb{N}$, we denote by $\mathcal{C}[\varepsilon, m, M]$ the following subset of $\mathcal{C}([0,1],\mathcal{A})$

$$\mathcal{C}[\varepsilon,m,M] \ := \ \left\{ \begin{array}{l} \xi(\cdot) = (\xi_j(\cdot))_j : \forall k = 0,1,\ldots,m-1, \exists (J_k^i)_{1\leqslant i \leqslant N} \text{ disjoint subsets of } \mathbb{N} \text{ with} \\ \{j \in \mathbb{N} : \ |\xi_j(\frac{k}{m})| < \ell(m)\} = \cup_{i=1}^N J_k^i \text{ and } \forall i,\ 1 \leqslant \sharp J_k^i \leqslant M, \text{ such that} \\ \forall i \neq i' \quad j \in J_k^i,\ j' \in J_k^{i'} \implies d(\xi_j([\frac{k}{m},\frac{k+1}{m}]),\xi_{j'}([\frac{k}{m},\frac{k+1}{m}])) > r+\varepsilon \end{array} \right\}$$

We now define the set $\underline{\mathcal{C}}$ as follows :

$$\underline{\mathcal{C}} = \limsup_m \mathcal{C}[\frac{4}{m^\kappa}, m, M], \tag{40}$$

with $\ell(m), M$, κ given by (17) and (35).

Proposition 3.10 *For any $x \in \mathcal{A}$ the process $X^{\infty,x}(.)$ is the unique one in $\underline{\mathcal{C}}$ which satisfies equation $(\mathcal{E})$ with initial condition $X^{\infty,x}(0) = x$.*

Proof Suppose that $Y(\cdot) \in \underline{\mathcal{C}}$ is also a solution of equation $(\mathcal{E})$. For each m_0 we can choose $m \geqslant m_0$ such that, on a set of full measure Ω_1, $Y(\cdot,\omega) \in \mathcal{C}[\frac{4}{m^\kappa}, m, M]$ for $\omega \in \Omega_1$.
Furthermore, for $\omega \in \Omega_x$, $X^{\infty,x}(\omega,.)$ belongs to $\liminf_m \mathcal{C}[\frac{4}{m^\kappa}, m, M]$. Then, for m_0 large enough, we can choose $m \geqslant m_0$, take $\mathbb{J}(\frac{k}{m}, i_0) \ni i_0$ for which both (39) holds for each $k = 0,1,\ldots,m$ and also

$$\begin{aligned} Y_i(t) \ &= Y_i(\frac{k}{m}) + W_i(t) - W_i(\frac{k}{m}) - \frac{1}{2}\sum_j \int_{\frac{k}{m}}^t \nabla\varphi(Y_i(s) - Y_j(s))ds \\ &+ \sum_{j\in\mathbb{J}(\frac{k}{m},i_0)} \int_{\frac{k}{m}}^t (Y_i(s) - Y_j(s))dL_{ij}^Y(s),\ i \in \mathbb{J}(\frac{k}{m}, i_0),\ t \in [\frac{k}{m}, \frac{k+1}{m}]. \end{aligned}$$

Using the same arguments which lead to (28), we obtain

$$\sup_{t\in[0,1]} |Y_i(\omega,t) - X_i^\infty(\omega,t)| < \frac{C_{11}}{m!},\ i \in \mathbb{J}(\frac{k}{m}, i_0),$$

Since we can take m as large as we want, this holds for all i and we obtain $X(t,\omega) = Y(t,\omega), t \in [0,1]$, for ω in the set of full probability $\Omega_x \cap \Omega_1$. This completes the proof of Proposition 3.10. ∎

4 Reversibility, Equilibrium equations and Canonical Gibbs measures

We first present the already known important fact that Gibbs measures are reversible and therefore, are Equilibrium measures.

4.1 Canonical Gibbs measures of $\mathcal{CG}$ are Equilibrium measures

Proposition 4.1 *The stochastic equation $(\mathcal{E})$ admits a time-reversible solution with values in $\mathcal{A}$ for any initial Gibbs distribution $\mu \in \mathcal{G}(z)$. Thus any canonical Gibbs measure $\mu \in \mathcal{CG}$ is reversible too.*

Proof of Proposition 4.1
First of all, we have to be sure that any Gibbs measure $\mu \in \mathcal{G}(z)$ has a support included in the set of admissible initial configurations $\underline{\mathcal{A}}$. This is already done in Proposition 3.7.

When the initial measure μ is Gibbsian, the solution of $(\mathcal{E})$ is approximated by reversible finite-dimensional processes solution of $(\mathcal{E}_n^{\ell,\eta})$. This implies its reversibility. More precisely, we have to prove that for any $T \in [0,1]$, for $f_1, \ldots, f_k$ bounded continuous functions on $\mathcal{M}$ with compact support and for $t_1, \ldots, t_k \in [0,T]$

$$\int\int \prod_{i=1}^{k} f_i(X^{\infty,x}(\omega,t_i))\ dP(\omega)\ d\mu(x) = \int\int \prod_{i=1}^{k} f_i(X^{\infty,x}(\omega,T-t_i))\ dP(\omega)\ d\mu(x) \tag{41}$$

But $X^{\infty,x}$ is, by construction, the weak limit of $X^{\ell,x}$. Then equality (41) holds if the following equality holds :

$$\lim_{\ell\to+\infty} \int\int \Big(\prod_{i=1}^{k} f_i(X^{\ell,x}(t_i)) - \prod_{i=1}^{k} f_i(X^{\ell,x}(T-t_i)) \Big)\ dP\ d\mu(x) = 0$$

Like in the proof of Proposition 3.7 Step 1 (see inequalities (31) and (32)), we go back to the reversible process with initial distribution $\mu_z^{\ell,\eta}$:

$$\begin{aligned} &\left| \int\int \Big(\prod_{i=1}^{k} f_i(X^{\ell,x}(t_i)) - \prod_{i=1}^{k} f_i(X^{\ell,x}(T-t_i)) \Big)\ dP\ d\mu(x) \right| \\ &\leqslant \left| \int_{\mathcal{A}}\int \Big(\prod_{i=1}^{k} f_i(X(t_i)) - \prod_{i=1}^{k} f_i(X(T-t_i)) \Big)\ dQ_z^{\ell,\eta}(X)\ d\mu(\eta) \right| \\ &\quad + 2 \prod_{i=1}^{k} \sup_{\xi\in\mathcal{A}} |f_i(\xi)| \int_{\mathcal{A}} \Big(1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{\ell,\eta}} \Big)\ d\mu(\eta) \end{aligned}$$

where $\Lambda = B(0,\ell)$. The first term of the right hand side is equal to 0 and the second term tends to zero as ℓ tends to infinity.

Any canonical Gibbs measure on $\mathcal{A}$ associated to the potential φ is also a reversible state for the process X^∞, since it is a mixture of Gibbs measures (with respect to the activity parameter z). ■

Let us now verify the foundamental symmetry property of the infinitesimal generator $\mathbf{G}$ under any measure μ which is reversible for the stochastic system $(\mathcal{E})$. We test the symmetry on $\mathcal{T}_0$, class of smooth functions for wich the Itô Formula is particularly simple.

Proposition 4.2 *Let μ be a Probability measure on $\mathcal{A}$. If the solution of the gradient-system $(\mathcal{E})$ with μ as initial distribution is time-reversible, then the infinitesimal generator $\mathbf{G}$ is symmetrical on $\mathcal{T}_0$:*

$$\forall f, g \in \mathcal{T}_0 \qquad \int_{\mathcal{M}} f\ \mathbf{G}g\ d\mu = \int_{\mathcal{M}} g\ \mathbf{G}f\ d\mu. \tag{42}$$

Proof The time-reversibility of the process X solution of $(\mathcal{E})$ implies that, for any time $t > 0$ and any $f, g \in \mathcal{T}_0$,

$$\int_{\mathcal{A}}\int \Big(g(X^{\infty,x}(0)) f(X^{\infty,x}(t)) - g(X^{\infty,x}(t)) f(X^{\infty,x}(0)) \Big)\ dP\ d\mu(x) = 0.$$

But, applying the Itô Formula and the martingale property (37) one gets

$$\begin{aligned}
&\int_{\mathcal{A}}\int \Big(g(X^{\infty,x}(0))f(X^{\infty,x}(t)) - g(X^{\infty,x}(t))f(X^{\infty,x}(0))\Big)\; dP\; d\mu(x)\\
&= \int_{\mathcal{A}}\int \Big(g(X^{\infty,x}(0))\int_0^t \mathbf{G}f(X^{\infty,x}(s))ds - f(X^{\infty,x}(0))\int_0^t \mathbf{G}g(X^{\infty,x}(s))ds\Big)\; dP\; d\mu(x)\\
&= \int_0^t\int_{\mathcal{A}}\int \big(g(X^{\infty,x}(0))\mathbf{G}f(X^{\infty,x}(s)) - f(X^{\infty,x}(0))\mathbf{G}g(X^{\infty,x}(s))\big)\; dP\; d\mu(x)ds\\
&= 0.
\end{aligned}$$

Since the paths $t \mapsto X^{\infty,x}(t)$ are continuous at time 0 and $\mathbf{G}f$ and $\mathbf{G}g$ are bounded $\mathcal{C}^0$-functions when f and g belong to $\mathcal{T}_0$, we conclude by dominated convergence

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\int_0^t\int_{\mathcal{A}}\int \big(g(X^{\infty,x}(0))\mathbf{G}f(X^{\infty,x}(s)) - f(X^{\infty,x}(0))\mathbf{G}g(X^{\infty,x}(s))\big)\; dP\; d\mu(x)ds\\
&= \int_0^t\int_{\mathcal{A}} \big(g(X^{\infty,x}(0))\mathbf{G}f(X^{\infty,x}(0)) - f(X^{\infty,x}(0))\mathbf{G}g(X^{\infty,x}(0))\big)\; dP\; d\mu(x)\\
&= \int_{\mathcal{M}} g\;\mathbf{G}f\; d\mu - \int_{\mathcal{M}} f\;\mathbf{G}g\; d\mu\\
&= 0.
\end{aligned}$$

■

By connecting the result of Proposition 4.2 to Proposition 4.1 we conclude that any canonical Gibbs measure μ is an Equilibrium measure in the sense that the infinitesimal generator $\mathbf{G}$ is symmetric under μ on the set of test functions $\mathcal{T}_0$, see (42).

In a finite-dimensional context, the Symmetry Property (42) under μ - also called Equilibrium equation - would be strong enough to characterize the reversible measures μ as Gibbs measures. Unfortunately, in our context, $\mathcal{T}_0$ is too small to generate all functions on which $\mathbf{G}$ is symmetrical. For this reason, equation (42) is a necessary condition for μ to be time-reversible, but we can not directly prove the converse statement. To overcome this difficulty, we may reason in the same way as in [4] where the case of finite range interaction was treated. We introduce below a set of localizing functions, called security functions, which control the collisions between particles in a bounded region of the space $\mathbb{R}^d$. The symmetry of $\mathbf{G}$ under μ on such functions, as stated in (44), will be a sufficient condition for μ to be reversible.
The security functions are used as "collision detectors" : they vanish for configurations containing, in a bounded region, hard balls which are too close.

Definition 4.3 *For any $R > 0$ and for $\varepsilon > 0$, we define the function S_R^ε on $\mathcal{M}$ by*

$$S_R^\varepsilon(\gamma) = \tilde{\mathbb{1}}_{]-\infty,0]}\Bigg(\sum_{\gamma_i\in\gamma}\tilde{\mathbb{1}}_{B(0,R)}(\gamma_i)\;\Big(1 - \prod_{\gamma_j\in\gamma}\tilde{\mathbb{1}}_{[2,+\infty[}(\frac{|\gamma_i-\gamma_j|^2 - r^2}{\varepsilon^2})\Big)\Bigg) \tag{43}$$

where $\tilde{\mathbb{1}}_{]-\infty,0]}$ is a $\mathcal{C}^\infty$ non-increasing function with value 1 on $]-\infty,0]$ and 0 on $[1,+\infty[$, and where $\tilde{\mathbb{1}}_{B(0,R)}$ is a $\mathcal{C}^\infty$ function from $\mathbb{R}^d$ to $[0,1]$ with value 1 on $B(0,R)$ and value 0 on the set $\mathbb{R}^d\smallsetminus B(0,R+1)$. Here $\tilde{\mathbb{1}}_{[2,+\infty[}$ denotes some fixed $\mathcal{C}^\infty$ non-decreasing fonction vanishing on $]-\infty,1]$ with value 1 on $[2,+\infty[$.

Remark that the function S_R^ε is element of $\mathcal{T}$, and is $B(0,R+1+\sqrt{r^2+2\varepsilon^2})$-local. However, S_R^ε does not belong to $\mathcal{T}_0$, since its derivative $D_{\gamma_i}S_R^\varepsilon(\gamma)$ is not orthogonal to $\gamma_i-\gamma_j$ for $\gamma_i\in B(0,R+1)\smallsetminus B(0,R)$ and $|\gamma_i-\gamma_j| = r$, as required in definition (36). The reader can find in [4] details about these functions. Their main property is the fact that they increases a.s. to 1 as ε decreases to 0.

We now can state the

Proposition 4.4 *Any canonical Gibbs measure $\mu \in \mathcal{CG}$ with support included in $\underline{\mathcal{A}}$ satisfies the following Detailed Balance Equation*

$$\forall R>0, \varepsilon>0, \forall f, g\, B(0,R)-\textit{local in } \mathcal{T}\,, \quad \int_{\mathcal{M}} f_R^\varepsilon\, \mathbf{G} g_R^\varepsilon\, d\mu = \int_{\mathcal{M}} g_R^\varepsilon\, \mathbf{G} f_R^\varepsilon\, d\mu \tag{44}$$

where $f_R^\varepsilon := f S_R^\varepsilon$ and $g_R^\varepsilon := g S_R^\varepsilon$.

Proof In Proposition 4.4 of [4], we proved a similar result for a finite-range interaction φ. The arguments can be extended to the infinite range context in a straightforward way with the difference that even if f and g are local functions, now $\mathbf{G}f, \mathbf{G}g, \mathbf{G}g_R^\varepsilon, \mathbf{G}f_R^\varepsilon$ are no more local. We refer the reader to [4] for details. ■

We will see in the next section that (44) is a sufficient condition for μ to be canonical Gibbs.

4.2 Equilibrium measures are Canonical Gibbs

The assumptions on μ in Theorem 2.7 look relatively strong, but they are physically natural in the following sense :

- As remarked by Lang ([10], Bemerkung 4) and by Georgii ([7] page 42), the finite-volume projections of any "reasonable" equilibrium measure should be absolutely continuous with respect to the (finite-volume) Poisson point process. Indeed, it seems well-known among physicists that equilibrium measures are of Gibbsian nature with respect to an unknown potential (to be identified).
- Furthermore, the existence of local conditional densities u_Λ implies that the support of the initial measure μ does not contain pathological configurations like those with collisions between two hard balls. In particular, the measure μ does not carry any closest packing configuration. Anyway, it is clear that measures carrying closest packing configurations cannot be equilibrium measures for a random dynamics containing a Brownian oscillation like equation $(\mathcal{E})$.

The arguments for the proof of Theorem 2.7 follow closely those of [4], section 5, where we proved a similar result for finite-range interactions φ. See also [7] for the case of interactions without hard-core. We only sketch the proof and refer to [4] and [7] for technical details.
Let μ be a Probability measure on $\underline{\mathcal{A}} \subset \mathcal{A}$ with smooth local conditional density $u_\Lambda(.|\eta_{\Lambda^c})$ with respect to π_Λ. One first proves that if the Detailed Balance Equation (44) is satisfied under μ, by testing it on a class of well choosen functions f and g, then the Campbell measure C_μ associated to μ satisfies a symmetry property. C_μ is defined as usual on $\mathbb{R}^d \times \mathcal{A}$, for any regular function f, by

$$\int_{\mathbb{R}^d\times\mathcal{A}} f(y,\eta) C_\mu(dy,d\eta) = \int_{\mathcal{A}} \int_{\mathbb{R}^d} f(y,\eta)\, \eta(dy)\, \mu(d\eta).$$

It satisfies for any positive measurable local function F on $\mathbb{R}^d \times \mathbb{R}^d \times \mathcal{A}$,

$$\int_{\mathbb{R}^d\times\mathcal{A}} \int_{\mathbb{R}^d} e^{-E(y'|\eta\smallsetminus y)}\, F(y,y',\eta\smallsetminus y)\, dy'\, C_\mu(dy,d\eta) = \int_{\mathbb{R}^d\times\mathcal{A}} \int_{\mathbb{R}^d} e^{-E(y'|\eta\smallsetminus y)}\, F(y',y,\eta\smallsetminus y)\, dy'\, C_\mu(dy,d\eta). \tag{45}$$

In the above formula, the function $E(y|\xi)$ denotes the one-point energy of the point $y \in \mathbb{R}^d$ with respect to the configuration ξ. Its value is equal to $\sum_{\xi_i} \varphi(y-\xi_i)$ if $y\xi \in \mathcal{A}$ and $+\infty$ otherwise.
To complete the proof of Theorem 2.7 it suffices to show that measures μ which satisfy equation (45) are elements of $\mathcal{CG}$. This was already done in Proposition 2.29 [7] for interactions without hard-core (resp. in Proposition 5.2 [4] for finite-range interactions with hard-core). These proofs can be extended to our framework without difficulty.

5 Appendix
Regularity estimates for the solution of a Skorohod equation

The existence and uniqueness of solutions of Skorohod equations were studied by many authors (Tanaka [21], Lions and Sznitman [11], Saisho [16]). The Skorohod equations we are interested in are defined on $\mathbf{D}_n = \mathcal{A} \cap (\mathbb{R}^d)^n$ with reflecting boundary $\partial\mathbf{D}_n$:

$$\partial\mathbf{D}_n = \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n : \forall i \neq i' \ |x_i - x_{i'}| \geqslant r \text{ and } \exists j, j' \ |x_j - x_{j'}| = r\}.$$

For a given $w \in \mathcal{C}_0(\mathbb{R}^+, (\mathbb{R}^d)^n)$ and $x \in \mathbf{D}_n$, the equation is :

$$\zeta_i(t) = x_i + w_i(t) + \sum_{j=1}^{n} l_{ij}(t), \quad t \geqslant 0, \quad 1 \leqslant i \leqslant n. \tag{46}$$

A solution is a pair $(\zeta, l) \in \mathcal{C}(\mathbb{R}^+, \mathbf{D}_n) \times \mathcal{C}_0(\mathbb{R}^+, (\mathbb{R}^d)^n)$ satisfying (46) and the following condition : l has bounded variation $\|l_{ij}\|_t$ on each finite time interval $[0, t]$ and satisfies

$$l_{ij}(t) = \int_0^t (\zeta_i(s) - \zeta_j(s)) d\|l_{ij}\|_s, \quad \|l_{ij}\|_t = \int_0^t \mathbb{1}_{|\zeta_i(s) - \zeta_j(s)| = r} d\|l_{ij}\|_s. \tag{47}$$

Saisho and Tanaka (see [17] theorem 4.1) proved that the domain $\mathbf{D}_n$ satisfies a uniform exterior sphere condition called Condition (A) and a uniform interior cone condition called Condition (B), and therefore Skorohod equation (46) admits a unique solution.

Moreover, Saisho (see [16] proof of proposition 4.1) proved that, under an additional Condition (D), this unique solution satisfies a Lipschitz continuity property as a function of $w(\cdot)$ and x. We proved in [5] lemma 2.2 that Condition (D) holds for any domain satisfying Conditions (A) and (B), so that the Lipschitz continuity property holds on $\mathbf{D}_n$:

Lemma 5.1 *Let $\zeta(.)$ (respectively $\zeta'(.)$) denote the unique solutions of Skorohod equation (46) for $w \in \mathcal{C}_0(\mathbb{R}^+, (\mathbb{R}^d)^n)$ and $x \in \mathbf{D}_n$ (resp. for $w' \in \mathcal{C}_0(\mathbb{R}^+, (\mathbb{R}^d)^n)$ and $x' \in \mathbf{D}_n$). Then there exists a constant $C(\mathbf{D}_n)$, depending only on the geometry of the domain $\mathbf{D}_n$, such that for each $t \geqslant 0$,*

$$|\zeta(t) - \zeta'(t)| \leqslant \Big(\|w - w'\|_t + |x - x'| \Big) \exp \Big(C(\mathbf{D}_n)(\|l\|_t + \|l'\|_t) \Big). \tag{48}$$

Remark that the constant $C(\mathbf{D}_n)$ in the above Lemma a priori depends on the number n of particles.

In [16] (Theorem 4.2) one also finds the following estimate.

Lemma 5.2 *The total variation $\|l\|_t$ of the process $l(t)$ satisfies*

$$\|l\|_t \leqslant f(\Delta(w, \delta), \sup_{s \leqslant t} |w(s)|), \qquad 0 \leqslant t \leqslant 1,$$

where the function f only depends on the geometric characteristics of the domain $\mathbf{D}_n$. Moreover, the functional $w \longrightarrow f(\Delta(w, \delta), \sup_{s \leqslant t} |w(s)|)$ is bounded on each set of paths $\mathcal{W} \subset \mathcal{C}_0(\mathbb{R}^+, (\mathbb{R}^d)^n)$ satisfying $\lim_{\delta \to 0} \sup_{w \in \mathcal{W}} \Delta(w, \delta) = 0$.

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