4

Pinned Gibbs processes

Mathias Rafler[*]

Abstract. *Finite Gibbs processes are conditioned on the barycentre of the point configurations being at a certain location. An integration-by-parts formula is derived from a classical one for such a pinned Gibbs process along with a characterisation. This entails a stochastic domination result for the total number of points as well as a simulation scheme for conditioned point processes.*

1 Pinning Gibbs processes

Classical integration-by-parts formulas for point processes give a dynamic view on point processes identifying certain ones as reversible laws of spatial birth-and-death processes with given birth and death rates. Usually existing points are chosen to disappear independently of each other at rate 1, see e. g. [2] for further possible choices. If, in addition, new points appear independent of the current point configuration, this characterises a Poisson process with intensity given by the birth rate. This property is summarised in Mecke's formula. If this intensity is modified with a term depending on the configuration of points, this yields more general Gibbs processes.

Conforti et al. studied (finite) Poisson processes subject to a pinning of the first moment of the point configurations, i. e. the Poisson process conditioned on an event of probability zero [1]. Clearly, such a conditioned Poisson process cannot satisfy Mecke's

[*]Justus-Liebig-Universität Gießen, Mathematisches Institut, Arndtstr. 2, 35392 Gießen, Germany; *mathias.rafler @math.uni-giessen.de*

formula, since typically removing or adding a point changes the first moment of a point configuration. However, they introduced a new kind of dynamic keeping the given condition invariant: They merge three steps of a birth-and-death process, which, suitably chosen, conserve the given condition. Either one point is removed in favour of two new ones, or two points are removed in favour of one new point. Hence, the total number of points increases or decreases by one only.

The idea presented in [1] may be generalised in several ways. Firstly, points may be allowed to interact with each other, i. e. the finite Poisson process is replaced by a finite Gibbs process. Secondly, point configurations are conditioned on a fixed barycentre. Such a condition fails to be linear and transformations need to chosen more carefully.

2 Transformations and invariance

For some $d \in \mathbb{N}$, denote by $\mathscr{M}^{\cdot\cdot}$ the set of locally finite point measures on $\mathbb{R}^d$ as well as $\mathscr{M}_\mathrm{f}^{\cdot\cdot}$ its subset of finite point measures, both are Polish spaces. For any atom $x \in \mathbb{R}^d$ of a point measure $\mu \in \mathscr{M}_\mathrm{f}^{\cdot\cdot}$, i. e. $\mu(\{x\}) > 0$, write $x \in \mu$.

Denote by $\mathfrak{b} : \mathscr{M}_\mathrm{f}^{\cdot\cdot} \to \mathbb{R}^d$ the functional assigning the barycentre to a finite point configuration, i. e.

$$\mathfrak{b}\mu := \frac{1}{\mu(X)} \int x\,\mu(\mathrm{d}x).$$

The barycentre of the empty configuration will be understood as some special element o not contained in $\mathbb{R}^d$.

Of particular interest are transformations replacing a single point by two new ones while leaving the barycentre invariant. A short computation shows that for $z \in \mu$,

$$\mathfrak{b}(\mu + \delta_x + \delta_y - \delta_z) = \mathfrak{b}\mu \qquad \Longleftrightarrow \qquad z = x + y - \mathfrak{b}\mu. \tag{4.1}$$

Whenever a functional is invariant under such a transformation, this pair of functional and transformation is called compatible.

3 Integration by parts

3.1 Unpinned Gibbs processes

Let $\phi_1 : \mathbb{R}^d \to (-\infty, \infty]$ and $\phi_2 : \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, \infty]$ be potentials of single points and pairs of points, both being bounded from below. Then the energy U of a finite point configuration μ is given by

$$U(\mu) = \int \phi_1(x)\mu(\mathrm{d}x) + \frac{1}{2}\int \phi_2(x,y)\mu^{(2)}(\mathrm{d}x, \mathrm{d}y),$$

where $\mu^{(2)}(\mathrm{d}x, \mathrm{d}y) := \big(\mu - \delta_x\big)(\mathrm{d}y)\mu(\mathrm{d}x)$ is the second factorial measure of μ. Integration with respect to $\mu^{(2)}$ means to sum with respect to all pairs of distinct points respecting possible multiplicities. Observe that if $\mu(\mathbb{R}^d) < 2$, then $\mu^{(2)} = 0$, and there is no contribution of a pair interaction. The energy of a point $x \in \mathbb{R}^d$ given a point configuration $\mu \in \mathscr{M}_{\mathrm{f}}^{\cdot\cdot}$ is

$$U(x\,|\,\mu) = \phi_1(x) + \int \phi_2(x,y)\mu(\mathrm{d}y).$$

A finite point process N is a random element in $\mathscr{M}_{\mathrm{f}}^{\cdot\cdot}$, and for a measurable set $B \subseteq \mathbb{R}^d$, N_B is the number of points of N in B. Here both, N and its law $\mathbb{P}$, are called point process. A finite point process N with law $\mathbb{P}$ is called Gibbs process with potentials ϕ_1 and ϕ_2, if its Campbell measure $C_{\mathbb{P}}$ satisfies the (classical) integration-by-parts formula

$$\int h\,\mathrm{d}C_{\mathbb{P}} := \iint h(x,\mu)\mu(\mathrm{d}x)\mathbb{P}(\mathrm{d}\mu) = \iint h(x,\mu+\delta_x)\exp\big(-U(x\,|\,\mu)\big)\,\mathrm{d}x\,\mathbb{P}(\mathrm{d}\mu) \quad (4.2)$$

for all non-negative, measurable functions h. Equivalently, Equation (4.2) may be written as

$$\mathbf{E}\left[\int h(x,N)N(\mathrm{d}x)\right] = \mathbf{E}\left[\int h(x,N+\delta_x)\exp\big(-U(x\,|\,N)\big)\,\mathrm{d}x\right].$$

Equation (4.2) has a solution if e. g. $\exp(-\phi_1)$ is integrable and ϕ_2 is stable [3]. Subsequently, the potentials shall be chosen such that a finite point process exists as a solution. Note that there is a natural choice of a spatial birth-and-death process on $\mathbb{R}^d$ such that $\mathbb{P}$ is a reversible distribution: Points die independently of each other at rate one, while new points appear at a rate given by the exponential, see [2] for a discussion. An explicit representation of $\mathbb{P}$ in terms of the kernel is available and given in [4].

Of particular interest is the second order reduced Campbell measure given by

$$\int h \, \mathrm{d}C^{(2)}_{\mathbb{P}} := \mathbf{E}\left[\int h(x,y,N-\delta_x-\delta_y) N^{(2)}(\mathrm{d}x,\mathrm{d}y)\right].$$

Note that a finite point process $\mathbb{P}$ can be reconstructed from $C^{(2)}_{\mathbb{P}}$ only on $\{N_{\mathbb{R}^d} \geq 2\}$.

Subsequently, simplify notation and denote by $\mathfrak{b}$ the barycentre of the point configuration μ. Assume that

$$U(z-y+\mathfrak{b}\,|\,\mu-\delta_z+\delta_y)+U(y\,|\,\mu-\delta_z)<\infty \quad\Longrightarrow\quad U(z\,|\,\mu-\delta_z)<\infty, \qquad z\in\mu,$$

which means that allowing to add points at y and $z-y+\mathfrak{b}$ with a finite energy needs to allow adding a point at z with a finite energy as well.

Proposition 4.1 Let N be a finite Gibbs process with potentials ϕ_1 and ϕ_2. Then

$$\mathbf{E}\left[\int F(x,y,N)N^{(2)}(\mathrm{d}x,\mathrm{d}y)\right] = \int \mathbf{E}\left[\int F\big(z-y+\mathfrak{b},y,N+\delta_{z-y+\mathfrak{b}}+\delta_y-\delta_z\big)\sigma(N,y,z)N(\mathrm{d}z)\right]\mathrm{d}y \quad (4.3)$$

for all non-negative, measurable functions F, where

$$\sigma(\mu,y,z) := \exp\big(-U(z-y+\mathfrak{b}\,|\,\mu-\delta_z+\delta_y)-U(y\,|\,\mu-\delta_z)+U(z\,|\,\mu-\delta_z)\big), \qquad z\in\mu.$$

Proof. The statement is proven straight forward by an application of integration by parts followed by the substitution $x=z-y+\mathfrak{b}$ and an integration by parts backwards. □

3.2 Pinned Gibbs processes

For a finite point process N with law $\mathbb{P}$ denote by $\tau := \mathbb{P}\circ\mathfrak{b}^{-1}$ the distribution of $\mathfrak{b}$ under $\mathbb{P}$. τ is a distribution on $\mathbb{R}^d\cup\{o\}$ and concentrated on $\mathbb{R}^d$ if and only if N does not charge the empty configuration. Let $\mathbb{P}^a := \mathbb{P}(\,\cdot\,|\,\mathfrak{b}=a)$ be law of the pinned point process

N^a. Note that in any case, if $a \in \mathbb{R}^d$, then necessarily $N^a \neq 0$ almost surely. By the Equivalence (4.1), x, y and z are chosen such that the mapping

$$\mathscr{M}_{\mathrm{f}}^{\cdot\cdot} \to \mathscr{M}_{\mathrm{f}}^{\cdot\cdot}, \qquad \mu \mapsto \mu + \delta_y + \delta_{z-y+\mathfrak{b}\mu} - \delta_z, \qquad z \in \mu,$$

keeps the barycentre invariant.

Proposition 4.2 Let $\mathbb{P}$ be a finite Gibbs process with potentials ϕ_1 and ϕ_2. Then for τ-a.e. $a \in \mathbb{R}^d$, $\mathbb{P}^a$ solves (4.3).

Proof of Proposition 4.2. Let $f : \mathbb{R}^d \cup \{o\} \to \mathbb{R}$ be τ-integrable. Then the statement follows from

$$\int f(a) \int h(x,y,\mu) C^{(2)}_{\mathbb{P}^a}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}\mu)\tau(\mathrm{d}a) = \int f(\mathfrak{b}\mu) h(x,y,\mu) C^{(2)}_{\mathbb{P}}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}\mu)$$

by applying Equation (4.3) respecting the invariance of $\mathfrak{b}$, and the disintegration with respect to τ. □

The pinned point processes inherit the property of solving the second order integration-by-parts formula (4.3). Hence, Equation (4.3) has besides the unpinned Gibbs process at least the parametric family of pinned Gibbs processes as solutions. However, the main statement is that Equation (4.3) together with this pinning characterises the pinned Gibbs processes.

Theorem 4.3 Let Q be a finite point point process such that

1) Q solves (4.3),

2) $Q(\mathfrak{b} = a) = 1$ for some $a \in \mathbb{R}^d$.

Then Q is a pinned Gibbs process subject to the pinning $\mathfrak{b} = a$.

The main steps to prove Theorem 4.3 may be of interest and shall be given and commented on here without too many details.

For a finite point process N, let N^- be the diminished point process, that is N with a uniformly chosen point removed. Since the empty configuration cannot be reduced, it is mapped to a tomb Δ as an extra state added to $\mathscr{M}_{\mathrm{f}}^{\cdot\cdot}$. Note that since $N^a \neq 0$ a.s, $(N^a)^- \neq \Delta$ a.s. and its distribution $(\mathbb{P}^a)^-$ is a probability measure concentrated on $\mathscr{M}_{\mathrm{f}}^{\cdot\cdot}$.

The surprising result is that the law of the diminished pinned Gibbs process has a density with respect to the law of the unpinned Gibbs process.

Proposition 4.4 Let N be a finite Gibbs process with distribution $\mathbb{P}$. Then for τ-a.e. $a \in \mathbb{R}^d$, $(\mathbb{P}^a)^- \ll \mathbb{P}$, and the density is given by

$$\frac{\exp\Big[-U\Big(\big(\mu(\mathbb{R}^d)+1\big)\cdot a - \mu(\mathbb{R}^d)\cdot \mathfrak{b}(\mu)\,\Big|\,\mu\Big)\Big]}{\tau(a)}, \tag{4.4}$$

where $0 \cdot \mathfrak{b}(0) = 0$.

Observe that the density is essentially an energy, and the argument in the potential shall be interpreted as follows: μ is a diminished point configuration, hence $\big(\mu(\mathbb{R}^d)+1\big)\cdot a$ is the first moment of a point configuration of the Gibbs process pinned at a. Since $\mu(\mathbb{R}^d)\cdot \mathfrak{b}(\mu)$ is the first moment of the diminished point configuration, their difference is the location of the removed point. Consequently, each point configuration is weighted with a factor containing the energy to add the point to get the correct barycentre.

In [1], an approximation argument is used to derive this statement for a certain class of Poisson processes with an absolutely continuous intensity measure. The following sketch of proof uses conditional expectations allowing to replace the absolute continuity with respect to the Lebesgue measure with some reference measure.

Sketch of proof. Let $f : \mathbb{R}^d \cup \{o\} \to \mathbb{R}$ be τ-integrable with $f(o) = 0$ and F be $\mathbb{P}$-integrable, then mixing with respect to τ, integration by parts followed by a disintegration yields

$$\begin{aligned}\int f(a)\mathbf{E}\big[F\big((N^a)^-\big)\big]\,\tau(\mathrm{d}a) = &\int f\big(\mathfrak{b}(\mu+\delta_x)\big)\frac{F(\mu)}{\mu(X)+1}\exp\big(-U(x\,|\,\mu)\big)\mathbb{P}^a(\mathrm{d}\mu)\tau(\mathrm{d}a)\,\mathrm{d}x\\ &+\int f\big(\mathfrak{b}(\mu+\delta_x)\big)\frac{F(\mu)}{\mu(X)+1}\exp\big(-U(x\,|\,\mu)\big)\mathbb{P}^o(\mathrm{d}\mu)\tau(o)\,\mathrm{d}x.\end{aligned}$$

Since $\mathbb{P}^o$ charges the empty configuration only, $f\big(\mathfrak{b}(\mu+\delta_x)\big) = f(x)$ $\mathbb{P}^o$-a.s, and replacing x by b yields that the second integral may be turned into

$$\iint f(b)F(\mu)\frac{\exp\big(-U(b\,|\,\mu)\big)}{\tau(b)}\mathbb{P}^o(\mathrm{d}\mu)\tau(o)\tau(b)\,\mathrm{d}b.$$

The first integral is evaluated with the aid of two transformations, where the first one pins $\mathfrak{b}(\mu+\delta_x)$ at some $b \in \mathbb{R}^d$ and thus expresses a in terms of b and x, and the second one shifts this expression to a new variable y replacing x. □

An essential observation is that the density (4.4) is positive under the diminished law.

Lemma 4.5 Under the reduced pinned law $(\mathbb{P}^a)^-$,

$$U\Big(\big(\mu(\mathbb{R}^d)+1\big)\cdot a - \mu(\mathbb{R}^d)\cdot \mathfrak{b}(\mu)\,\Big|\,\mu\Big) < \infty$$

for $(\mathbb{P}^a)^- \otimes \tau(\mathrm{d}a)$-a.e. (μ, a), hence the energy is positive almost surely.

Proof. This is shown by proving that the event

$$A := \big\{(y,\mu) \in \mathbb{R}^d \times \ddot{\mathscr{M}_{\mathrm{f}}} \mid U(y|\mu) = +\infty\big\}$$

has measure 0. Since N^a is pinned at a, $N^a_X \cdot a - (N^a_X - 1)\cdot \mathfrak{b}(N^a - \delta_y) = y$. Mixed with respect to τ

$$\begin{aligned}\int (\mathbb{P}^a)^-(A)\tau(\mathrm{d}a) &= \mathbf{E}\left[\frac{1}{N_X}\int 1_A\Big(y, N-\delta_y\Big)N(\mathrm{d}y)\cdot 1_{N>0}\right]\\ &= \mathbf{E}\left[\frac{1}{N_X+1}\int 1_{\pi(N,y)=0}\pi(N,y)\,\mathrm{d}y\right].\end{aligned}$$

Since the inner integral vanishes, the claim follows. □

Let Q be a solution of Equation (4.3) such that $Q(\mathfrak{b} = a) = 1$ for some $a \in \mathbb{R}^d$. Then one shows that $Q^- \ll \mathbb{P}$ with the density given in (4.4).

Proposition 4.6 Let Q be a finite point process which solves (4.3) and satisfies $Q(\mathfrak{b} = a) = 1$ for some $a \in \mathbb{R}^d$. Then $\tilde{Q}$ given by

$$\tilde{Q}(\mathrm{d}\mu) := \frac{\tau(a)}{\exp\Big\{\Big(\big(\mu(\mathbb{R}^d)+1\big)\cdot a - \mu(\mathbb{R}^d)\cdot\mathfrak{b}(\mu)\Big|\mu\Big)\Big\}}Q^-(\mathrm{d}\mu)$$

is a Gibbs process with potentials ϕ_1 and ϕ_2.

The proof is straightforward by showing that the reduced Campbell measure of $\tilde{Q}$ satisfies a classical integration-by-parts formula (4.2). Note that the Campbell measure of $\tilde{Q}$ turns into a second order Campbell measure for Q allowing the application of (4.3).

What remains is to reconstruct the finite point process Q, or equivalently its reduced Campbell measure from Q^-. Its proof is already given in [1].

Proposition 4.7 Let Q be a finite point process such that $Q(\mathfrak{b} = a) = 1$ for some $\in \mathbb{R}^d$. If $Q^- = \mathbb{P}^-$, where $\mathbb{P}$ is a Gibbs process with potentials ϕ_1 and ϕ_2 conditioned on $\{\mathfrak{b} = a\}$, then $Q = \mathbb{P}$.

4 Applications

4.1 Stochastic domination

A distribution p dominates a distribution q, if the tails of p are heavier than those of q. As shown in [1], a sufficient condition for positive law p on $\mathbb{N}$ dominating q is

$$q(k+1)p(k) \leq q(k)p(k+1) \qquad \text{for all } k \geq 1.$$

Such an inequality can be shown for the law of the total number of points of a pinned Gibbs process N^a and a Poisson distribution conditioned to be positive once

$$\mathbf{E}\big[F(N^a_{\mathbb{R}^d}) \cdot N^a_{\mathbb{R}^d}\big] \leq K \cdot \mathbf{E}\big[F(N^a_{\mathbb{R}^d} + 1)\big] \tag{4.5}$$

is shown for some $K > 0$. The domination then follows in choosing the indicators $F(n) := 1_{\{n+1\}}$ for any $n \in \mathbb{N}$ and completing the conditioned Poisson weights with parameter K.

Equation (4.5) follows from the integration-by-parts formula by choosing functions depending on its parameters via $N^a_{\mathbb{R}^d}$ only, i. e.

$$\mathbf{E}\big[g(N^a_{\mathbb{R}^d})(N^a_{\mathbb{R}^d} - 1)N^a_{\mathbb{R}^d}\big] = \mathbf{E}\left[g(N^a_{\mathbb{R}^d} + 1) \iint \sigma(N^a, y, z)\, \mathrm{d}y N^a(\mathrm{d}z)\right],$$

and the innermost integral can shown to be bounded from above uniformly by K. For a hard-core interaction of particles inside some bounded box, this is satisfied automatically since

$$U(z - y + \mathfrak{b} \,|\, \mu - \delta_z + \delta_y) + U(y \,|\, \mu - \delta_z) - U(z \,|\, \mu - \delta_z) \geq 0.$$

A domination result with the roles exchanged can be shown as soon as the innermost integral is bounded from below by a positive constant.

4.2 Monte Carlo subject to conditions

The integration-by-parts formula allows a dynamic approach to sample approximately from the law of the conditioned point process N^a by running a continuous time Markov chain starting from a single point at $a \in \mathbb{R}^d$. If μ is the current point configuration, then the following jumps occur at the following rates:

1) With rate one each, a pair of distinct points $x, y \in \mu$ is chosen and removed in favour of a single point at $x + y - a$.

2) A rejection method is applied to remove one point in favour of two new points, so assume that $\sigma(\mu, y, z) \leq K$ for some constant K. At rate $K \cdot \mu(\mathbb{R}^d)$, choose a point y uniformly. Choose $z \in \mu$ uniformly, and toss a coin with success probability

$$\frac{\sigma(\mu, y, z)}{K}.$$

In case of success, remove z from μ and add points at y and $z - y + a$. Otherwise, reject any jump.

The barycentre at a is a conserved quantity and does not need to be computed at each step.

Bibliography

[1] Conforti, G., Kosenkova, T., Rœlly, S.: *Conditioned point processes with application to Lévy bridges*. J. Theoret. Probab. **32**(4), 2111–2134 (2019).

[2] Garcia, N., Kurtz, T.: *Spatial birth and death processes as solutions of stochastic equations*. Alea **1**, 281–303 (2006).

[3] Ruelle, D.: *Superstable interactions in classical statistical mechanics*. Commun. Math. Phys. **18**, 127–159 (1970).

[4] Zessin, H.: *Der Papangelou-Prozess*. J. Contemp. Math. Anal. **44**(1), 36–44 (2009).