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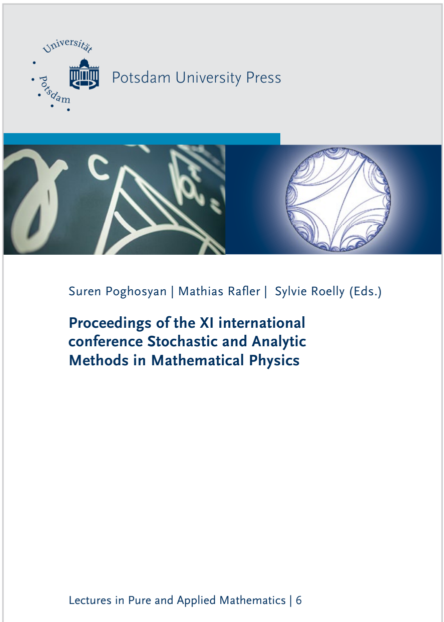

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Semi-recursive algorithm of piecewise linear approximation of two-dimensional function by the method of worst segment dividing

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1 Introduction

In the numerical solution of two-dimensional non-linear boundary value problems of mathematical physics, the finite element method is often used. This method assumes that the domain of the boundary problem is divided into small sub-domains (elements) within which the desired function is assumed to be linear. Thus, the desired function is approximated by a piecewise linear function. Its graph consists of triangles, the projections of which on the OXY plane form a triangular mesh.

In recent years, meshes with variable number of nodes are often used, i.e. the process of successive approximations extends not only to the approximated function, but also to the corresponding grid. At the same time, additional nodes are sequentially added in the worst (in the sense of approximation error) sub-domains. Thus, the mesh is successively improved and the approximation error is minimised.

In [6], an algorithm for an automatic construction of piecewise linear approximations of one-dimensional continuous functions was proposed. The algorithm minimised the approximation error for a given number of lattice points and was based on the principle

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of the worst segment dividing. In [2], the one-dimensional algorithm proposed in [6] was generalised for the two-dimensional case.

For practical reasons, recursive algorithms of mesh generation are preferable. In applications they are very convenient because they are easily programmed using cycle operators. However, the algorithm proposed in [2] was not recursive.

Recursive algorithms. An algorithm is called *recursive*, if each subsequent step does not lead to changes in the parameters obtained in the past. In the case of the construction of meshes with variable number of nodes, the algorithm will be recursive if the addition of each new node leaves the old nodes and old edges (the connections between nodes) in place.

Any one-dimensional uniform lattice cannot be constructed using a recursive algorithm, since when you add a new node, all old lattice nodes are shifted. The algorithm for automatically constructing a one-dimensional piecewise linear approximation with a non-uniform lattice, proposed in [6], is recursive.

In the present paper, a recursive algorithm is proposed for the automated construction of piecewise linear approximations of a two-dimensional continuous function by dividing the worst segment. An improved (but not recursive) algorithm for the automated construction of a two-dimensional piecewise linear approximation by dividing the worst segment was also studied. The improved algorithm can be called semi-recursive, since the addition of each new node leaves in place all the old nodes and almost all edges.

We have constructed a semi-recursive algorithm for constructing a piecewise linear approximation of a two-dimensional function by dividing the worst segment. When adding a new vertex, all previous vertices and almost all edges remain in their places. The edge may change if the “flip” operation is applicable to it: replacing a longer diagonal with a shorter one in a tetragon.

2 Delaunay triangulation

Let $M_n = \{P_i\}_{i=1}^n$ be a finite set of points in the plane. The interior of a domain D we denote by $\operatorname{int} D$. A set $\{D_j\}_{j=1}^m$ of triangles is called *triangle mesh* or *triangle tessellation* with knots M_n, if the following conditions are fulfilled:

1. The interiors of triangles are pairwise disjoint:

$$\operatorname{int} D_j \cap \operatorname{int} D_k = \emptyset, \quad j \neq k.$$

2. The set of all vertices of triangles is the set $\{P_i\}_{i=1}^n$.

3. The union of triangles fills the whole of convex hull of the knots:

$$\bigcup_{j=1}^{m} D_j = \operatorname{conv}\{P_i\}_{i=1}^n.$$

A triangle mesh $\{D_j\}_{j=1}^m$ is called *Delaunay triangulation* with knots $M_n = \{P_i\}_{i=1}^n$, if the following condition is fulfilled (see [1]):

4. For any triangle D_j

$$\operatorname{int} K(D_j) \cap \{P_i\}_{i=1}^n = \emptyset, \quad j = 1, \ldots, m,$$

where $K(D)$ is the circumscribing circle of triangle D.

The Delaunay triangulation with the system of knots M_n we denote by $\mathscr{D}(M_n)$. In [1] it was proved that for any finite set of points M_n there exists a Delaunay triangulation $\mathscr{D}(M_n)$ (not necessarily unique).

3 The method of dividing the worst segment

Let $F(x,y)$ be a two-dimensional continuous function on the plane, the domain of definition of which is the rectangle $[a,b] \times [c,d]$. Our goal is to construct a recursive algorithm for the automated construction of a piecewise linear approximation to the function $F(x,y)$.

First, consider the following recursive algorithm for constructing a sequence of knots $M_n = \{P_1, P_2, \ldots, P_n\}$ and the corresponding mesh S_n. On the rectangle $[a,b] \times [c,d]$, consider the primary set of knots M_4, consisting of 4 vertices: $P_1 = (a,c)$, $P_2 = (b,c)$, $P_3 = (b,d)$, $P_4 = (a,d)$ and the primary mesh S_4 consisting of two triangles $\Delta P_1 P_2 P_3$ and $\Delta P_1 P_3 P_4$ (see Figure 3.1).

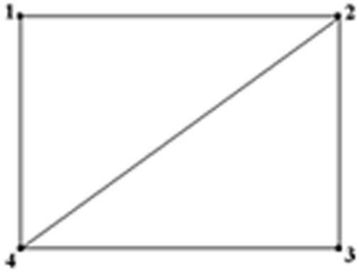

Figure 3.1: The primary mesh S_4.

Note that the mesh S_4 is a Delaunay triangulation with four knots $P_1,\dots,P_4$, i.e. $S_4 = \mathscr{D}(M_4)$. Suppose that the set of nodes $M_{n-1} = \{P_1,P_2,\dots,P_{n-1}\}$ and the corresponding triangle mesh S_{n-1} have already been built. Construct the next point P_n as follows.

Two knots P_i and P_j are called neighbouring and denoted by $P_i \sim P_j$, if they are the endpoints of a side of some triangle of the mesh. Note that in the primary mesh S_4, all pairs of nodes are neighbouring, except for the pair P_2 and P_4.

The pairs of neighbouring knots we call *the edges* of the mesh. Denote by $R(S_{n-1})$ the set of edges of S_{n-1}, i.e.

$$R(S_{n-1}) = \{(P_i,P_j):\ P_i \sim P_j\}.$$

For any edge $e = (P_i,P_j)$ from $R(S_{n-1})$ with vertices P_i and P_j we calculate the difference $|z_i - z_j|$, where $z_i = F(x_i,y_i)$ is the value of our function F at the point P_i, and (x_i,y_i) are Cartesian coordinates of vertex P_i.

The edge $e = (P_i,P_j) \in R(S_{n-1})$ we call *the worst edge*, if it gives the maximum value of differences $|z_i - z_j|$. In the middle of this edge we add a new node, this is the desired node P_n.

We determine what new edges will appear at the mesh S_n when adding a new vertex P_n. Denote by $\mathrm{conv}(M_n)$ the convex hull of the points $M_n = \{P_1,P_2,\dots,P_n\}$. Two cases are possible:

1) the new vertex P_n belongs to the interior of the hull $\mathrm{conv}(M_n)$. In this case we call P_n an internal knot, and the corresponding worst edge is called diagonal. In this case, when adding a new vertex, two new edges appear (see Figure 3.2);

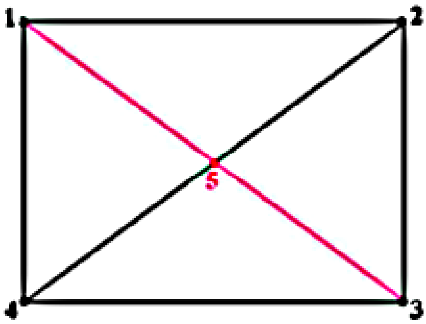

Figure 3.2: The case of an internal knot, the worst edge is P_2P_4.

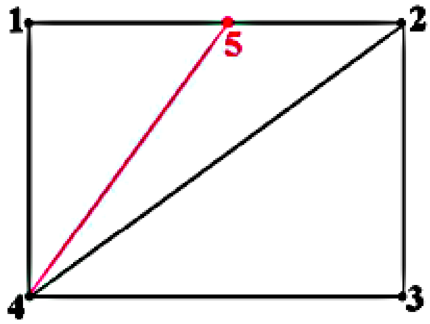

Figure 3.3: The case of a boundary knot, the worst edge is P_1P_2.

2) the new vertex P_n lies on the boundary of the hull $\text{conv}(M_n)$. In this case we call P_n a boundary knot, and the corresponding worst segment is called a boundary edge. In this case, when adding a new vertex, one new edge appears (see Figure 3.3).

We will continue to add knots by dividing in half the worst edge P_i, P_j corresponding to the largest of the values of the differences $|z_i - z_j|$ until the number of knots reaches a given value n. Note that the resulting mesh from the application of the proposed recursive algorithm will not necessarily be a Delaunay triangulation, it depends on the form of the approximate function $F(x,y)$. So the mesh in Figure 3.2 is a Delaunay triangulation, while the mesh in Figure 3.3 is not a Delaunay triangulation.

Having a triangular mesh S_n with a system of knots $M_n = \{P_1, P_2, \ldots, P_n\}$, we construct the approximation $F_n(x,y)$ to the function $F(x,y)$ as follows. The approximation $F_n(x,y)$ is a piecewise linear function; its graph consists of flat triangles whose projections onto the coordinate plane OXY form the mesh S_n. At the points $P_1, P_2, \ldots, P_n$ the values of the functions $F(x,y)$ and $F_n(x,y)$ coincide. By E_n we denote the approximation error:

$$E_n = \max_{a \le x \le b,\, c \le y \le d} |F(x,y) - F_n(x,y)|.$$

The error E_n can be estimated using the variation V_n, i.e. the largest difference of the values of the function $F(x,y)$ at the knots of the mesh S_n:

$$V_n = \max_{P_i \sim P_j} |z_i - z_j|,$$

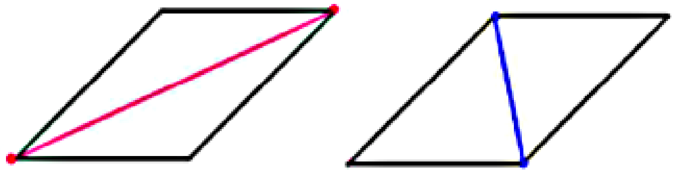

Figure 3.4: Flip operation: long diagonal is replaced by shorter.

where $z_i = F(x_i, y_i)$ and the maximum is taken over all pairs of neighbouring knots of the mesh S_n. It is easy to see that for the proposed recursive algorithm, the sequence V_n monotonically decreases with increasing number of knots n.

Observe that if the sequence V_n tends to zero, then the error E_n may not tend to zero. To ensure that the error tends to zero, we must additionally require the condition of monotonicity.

Proposition 3.1 Let $F(x,y)$ be a smooth function monotonic in both variables. Then for sufficiently large n we have $E_n \approx V_n$, therefore the recursive algorithm leads to vanishing approximation error E_n.

4 Semi-recursive algorithm

Now we modify the proposed recursive algorithm for automated construction of a piecewise-linear approximation of a two-dimensional continuous function by dividing the worst segment by allowing the flip operation, if applicable.

The operation of replacing a longer diagonal in a tetragon with a shorter one is called a "flip" (Figure 3.4). In the mesh in Figure 3.2 there are not four vertices for which the flip operation is applicable, while in Figure 3.3 there is one such set for four knots – these are the vertices P_2, P_3, P_4, P_5. Applying the flip operation to this set of four knots, we obtain the triangulation shown in Figure 3.5. Here the old edge P_2P_4 is replaced by the shorter P_3P_5.

The modified algorithm can be called semi-recursive, since adding each new vertex leaves all old knots and almost all old edges in place, except for one. It is easy to see that the largest edge of the mesh obtained as a result of the semi-recursive algorithm with

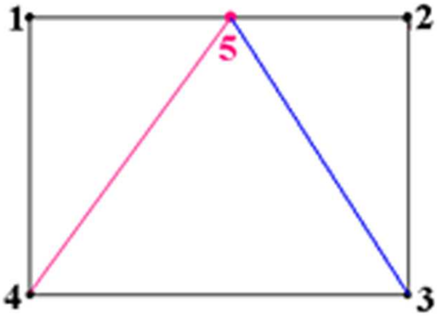

Figure 3.5: The triangulation from Figure 3.3 after the flip operation.

the addition of the flip operation is less than that of the same mesh obtained as a result of the recursive algorithm without the flip operation. Since for smooth monotonic functions $F(x,y)$, the approximation error E_n decreases with decreasing length of the largest edge, we can make the following conjecture.

Conjecture 3.2 As increasing number of knots, the approximation error, resulting from the work of a semi-recursive algorithm for automated construction of a piecewise-linear approximation of a two-dimensional monotonic function by dividing the worst segment with the addition of the flip operation, tends to zero faster than the error obtained as a result of the recursive algorithm without flip operation.

Conjecture 3.2 intuitively seems true and successfully passed the test with numerous practical examples, however the rigorous proof remains open.

It was shown in [3, 4] that the flip operation reduces the sum of the cotangents of the inner angles of a triangular mesh. It is also proved that Delaunay triangulation minimises the sum of the cotangents of the inner angles of the triangulation. From this we obtain the following statement.

Theorem 3.3 For any approximated function and any number of vertices, the mesh resulting from the operation of the semi-recursive algorithm is a Delaunay triangulation.

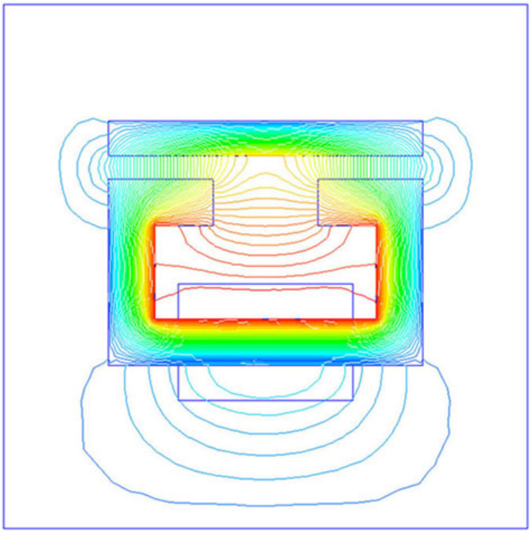

Figure 3.6: The solution of a magnetic field problem.

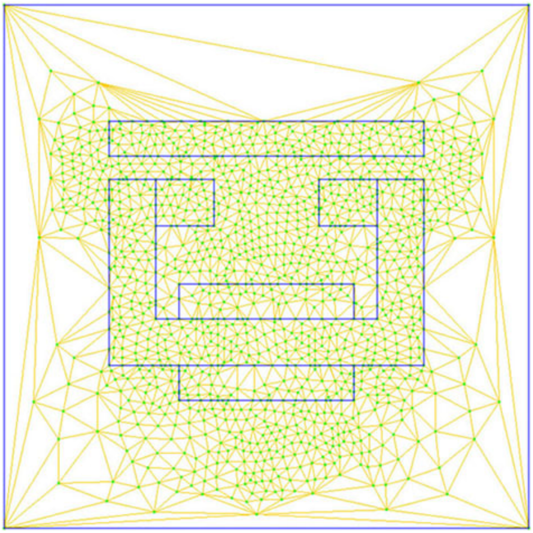

Figure 3.7: The corresponding mesh is Delaunay triangulation.

5 Applications to mathematical physics

Let $F(x,y)$ be an unknown solution of a boundary value problem

$$\begin{cases} L\big(F(x,y)\big) = 0, & (x,y) \in D, \\ F(x,y) = f, & (x,y) \in \partial D, \end{cases}$$

where L is a differential operator acting on rectangle D with boundary ∂D.

Assume that we can construct the piecewise linear approximation of F for given mesh S_n, and we can determine the worst segment of the mesh S_n. Then using the semi-recursive algorithm we obtain a new method of solution of boundary value problem using the meshes with variable number of vertices.

Theorem 3.4 ([5]) For given points, the best mesh for the finite elements approximation for the Maxwell equation of the magnetic field is a Delaunay triangulation.

By Theorems 3.3 and 3.4 we obtain

Theorem 3.5 For the construction of a mesh for the numerical approximation of the Maxwell equation of the magnetic field by the finite elements method, the semi-recursive algorithm is better than algorithms that do not lead to Delaunay triangulation.

Example 3.6 We solve the Maxwell equation for magnetic field using the semi-recursive algorithm. Figure 3.6 shows the solution, while Figure 3.7 shows the corresponding mesh.

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