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first published in:
Circuits, Systems, and Signal Processing 14 (1995), 4, p. 427-443

Postprint published at the Institutional Repository of the Potsdam University:
In: Postprints der Universität Potsdam
Mathematisch-Naturwissenschaftliche Reihe ; 159
http://opus.kobv.de/ubp/volltexte/2010/4673/
http://nbn-resolving.de/urn:nbn:de:kobv:517-opus-46739

Postprints der Universität Potsdam
Mathematisch-Naturwissenschaftliche Reihe ; 159

On the Local Qualitative Behavior of Differential-Algebraic Equations*

Sebastian Reich[1]

Abstract. A theoretical framework for the investigation of the qualitative behavior of differential-algebraic equations (DAEs) near an equilibrium point is established. The key notion of our approach is the notion of regularity. A DAE is called regular locally around an equilibrium point if there is a unique vector field such that the solutions of the DAE and the vector field are in one-to-one correspondence in a neighborhood of this equilibrium point. Sufficient conditions for the regularity of an equilibrium point are stated. This in turn allows us to translate several local results, as formulated for vector fields, to DAEs that are regular locally around a given equilibrium point (e.g. Local Stable and Unstable Manifold Theorem, Hopf theorem). It is important that these theorems are stated in terms of the given problem and not in terms of the corresponding vector field.

1. Introduction

Differential-algebraic equations (DAEs) are frequently identified as implicit equations

$$F(x, x') = 0 \tag{1}$$

for which x' cannot be expressed explicitly as a function of x. Such equations arise in many areas of science and engineering. In particular, constrained mechanical systems and electronic circuits may be modelled using equations of type (1) [2]. But the investigation of control systems described by ordinary differential equations with certain outputs required to be identically zero can be done in terms of DAEs as well [14]. In recent years the literature on existence and uniqueness results has been growing rapidly (see, e.g., [2], [16], [18], [17], [3]). However, up to now, results on the qualitative behavior are available only for a few selected classes of DAEs that are characterized by restrictive conditions on the form of the mapping F in (1) (see, e.g., [6], [13], [14], [21]).

* Received October 25, 1992; revised July 22, 1993.

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In this paper we investigate the qualitative behavior of a DAE (1) near an equilibrium point x_0; i.e., a point satisfying $F(x_0, 0) = 0$. The key notion of our approach is regularity. We call a DAE regular locally around an equilibrium point if there is a unique vector field such that the solutions of the vector field and the DAE are in one-to-one correspondence in a neighborhood of the equilibrium point under consideration. By associating to the DAE (1) a family of overdetermined DAEs, we derive in Section 3 sufficient conditions for the regularity of the DAE near an equilibrium point. Note that our approach is close to an approach taken, e.g., in [3] where the equation (1) is differentiated several times with respect to time to obtain the so-called derivative array equations. However, in contrast to the approach taken in [3], we eliminate the second order derivative x'' after each differentiation and thus avoid having higher order derivatives of x enter into the equations. We show that this elimination step can be done by simple manipulations on matrix valued functions. In Section 4 we discuss the linearization of a nonlinear DAE in terms of the linearized DAE and the corresponding vector field. We also show in this section how the spectrum of the linearized vector field can be obtained in a direct way from the linearized DAE. These results are used in Section 5 to "translate" the Local Stable and Unstable Manifold Theorem and the Hopf theorem [10], as formulated for vector fields, to DAEs that are regular locally around the equilibrium point under consideration. It is important that these theorems are formulated in terms of the given DAE rather than in terms of the corresponding vector fields.

2. Mathematical background

In this section we collect some basic material needed throughout the remainder of this presentation. We begin with some standard terminology (see, e.g., [1] for details).

Throughout this presentation we consider only submanifolds of $\mathbb{R}^n$, $n > 0$, which, for simplicity of exposition, are assumed to be of class C^∞.

If $M \subseteq \mathbb{R}^n$ is a *submanifold* of $\mathbb{R}^n$, then TM denotes the *tangent bundle* of M and T_xM denotes the *tangent space* of M at $x \in M$. Now the tangent space T_xM and the tangent bundle TM can be associated canonically with a linear subspace of $\mathbb{R}^n$ and a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$, respectively, by means of canonical embeddings $j_x : T_xM \to \mathbb{R}^n$, and $j : TM \to \mathbb{R}^n \times \mathbb{R}^n$. In the sequel we will not distinguish between T_xM and $j_x(T_xM)$ (TM and $j(TM)$) and denote both spaces by T_xM (TM). Let $\mathrm{pr}_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denote the projection of $\mathbb{R}^n \times \mathbb{R}^n$ onto the first component and let $TM \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be the tangent bundle of a submanifold $M \subseteq \mathbb{R}^n$. Then $M = \mathrm{pr}_1(TM)$.

As in the case of manifolds we assume for simplicity that all the mappings under consideration will be of class C^∞ although, once again, this condition can be reduced easily. By a *vector field* on a submanifold $M \subseteq \mathbb{R}^n$ we mean a mapping $v : M \to \mathbb{R}^n$ such that $v(x) \in T_xM$ for all $x \in M$. A mapping $x : I \to \mathbb{R}^n$ on the open interval $I \subseteq \mathbb{R}$ is called a *solution* of a given vector field $v : M \to \mathbb{R}^n$

if, for all $t \in I$, $x(t) \in M$ and $x'(t) = v(x(t))$. Given a point $(x_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^n$, $U(x_0, p_0) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ will always denote a sufficiently small neighborhood of (x_0, p_0) in $\mathbb{R}^n \times \mathbb{R}^n$. We will often write U instead of $U(x_0, p_0)$ to further simplify notations.

In Section 3 we will focus on DAEs of the type

$$G(x, x') = 0$$

where $G : U \to \mathbb{R}^m$ with $U = U(x_0, p_0) \subseteq \mathbb{R}^n \times \mathbb{R}^n$, $n \leq m \leq 2n$, and $(x_0, p_0) \in G^{-1}(0)$ in some fixed points in $\mathbb{R}^n \times \mathbb{R}^n$. For the analysis of such DAEs we will need the following results.

We call G *admissible* (locally around (x_0, p_0)) if rank $DG(x_0, p_0) = m$ and rank $D_pG(x, p) = r \leq m$ is constant for all $(x, p) \in G^{-1}(0)$. The following result for admissible mappings G was proven in [17]: If $G : U \to \mathbb{R}^m$ is admissible, then the sets $N = G^{-1}(0)$ and $M = \mathrm{pr}_1(N)$ are submanifolds of $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{R}^n$ respectively. Furthermore, it was shown in [16] that for admissible mappings G the tangent space T_xM of M at $x \in T_xM$ can be characterized by

$$T_xM = \left\{\bar{x} \in \mathbb{R}^n : D_xG(x, p)\bar{x} \in \mathrm{rge}\, D_pG(x, p)\right\}$$

where $(x, p) \in N$. However, to obtain a characterization of T_xM in terms of an implicit system of equations we need the following.

Proposition 1. *Let* $G : U \to \mathbb{R}^m$ *be an admissible mapping with* rank $D_pG(x, p) = r$ *for all* $(x, p) \in G^{-1}(0)$. *Then there exists a smooth mapping* $S : U \to L(\mathbb{R}^m, \mathbb{R}^{m-r})$ *such that* $S(x, p)$ *has full rank for all* $(x, p) \in U$ *and* $S(x, p)D_pG(x, p) = 0$ *for all* $(x, p) \in N$. *We call the mapping* S *a mapping associated to* G.

Proof. We give a constructive proof in three steps:

Step 1. There are permutations $P_1, P_2 \in L(\mathbb{R}^m)$ such that

$$P_1 D_pG(x, p)P_2 = V(x, p) = \begin{pmatrix} V_{11}(x, p) & V_{12}(x, p) \\ V_{21}(x, p) & V_{22}(x, p) \end{pmatrix}$$

with $V_{11}(x_0, p_0)$ nonsingular and $r \times r$.

Step 2. Define

$$L = \begin{pmatrix} I & 0 \\ -V_{21}V_{11}^{-1} & I \end{pmatrix}$$

where L is defined for all (x, p) in a sufficiently small neighborhood $\tilde{U}$ of (x_0, p_0), $\tilde{U} \subseteq U$. Let us assume, for simplicity, that U was chosen such that $\tilde{U} = U$. Then we obtain on U

$$LV = \begin{pmatrix} V_{11} & V_{12} \\ 0 & V_{22} - V_{21}V_{11}^{-1}V_{12} \end{pmatrix}.$$

Now, because rank $V_{11}(x, p) =$ rank $D_pG(x, p)$ for all $(x, p) \in G^{-1}(0)$, $V_{22} - V_{21}V_{11}^{-1}V_{12} = 0$ on $G^{-1}(0)$.

Step 3. Now the mapping S can be defined by

$$S = \begin{pmatrix} -V_{21}V_{11}^{-1} & I \end{pmatrix} P_1$$

and satisfies $S(x, p)D_pG(x, p) = S(x, p)D_pG(x, p)P_2 = 0$ for all $(x, p) \in G^{-1}(0)$ as required. $\square$

As a consequence of Proposition 1 we obtain that T_xM is given by

$$T_xM = \ker[S(x, p)D_xG(x, p)]$$

where $(x, p) \in N$. Note that in contrast to results in [16], the mapping S in Proposition 1 is of class C^∞ although the rank of $D_pG(x, p)$ is not constant on the whole set U but only on $N = G^{-1}(0)$.

Remark. The construction described in the proof of Proposition 1 might not be numerically robust unless the rank of the derivatives involved is constant in a whole neighborhood of (x_0, p_0). Hence our formulation in terms of constant rank conditions on submanifolds can be exploited mainly in an analytical or symbolic analysis.

Example 1. Consider the mapping $G : U \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$G(x, p) = \begin{pmatrix} p_3 + p_2 - x_1 \\ (x_1 + x_2)p_2 + x_3 \\ x_1 + x_2 \end{pmatrix}.$$

Because rank $DG(x, p) = 3$ and rank $D_pG(x, p) = 1$ for $(x, p) \in G^{-1}(0)$, G is admissible. Now, as

$$D_pG(x, p) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & x_1 + x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we put $P_1 = I$ and

$$P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus

$$V = \begin{pmatrix} 1 & 1 & 0 \\ x_1 + x_2 & 0 & \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$S = \begin{pmatrix} -(x_1 + x_2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $G : U(x_0, p_0) \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ be an admissible mapping. Then we call the DAE

$$G(x, x') = 0 \tag{2}$$

overdetermined if $m > n$. By a *solution* of a DAE (2) we mean a mapping $x : I \to \mathbb{R}^n$ on the open interval $I \subseteq \mathbb{R}$ such that, for all $t \in I$, $(x(t), x'(t)) \in U$ and $G(x(t), x'(t)) = 0$. Furthermore, we call the DAE *regular* (locally around (x_0, p_0)) if there exists a unique submanifold W of $\mathbb{R}^n$ and a unique vector field $v : W \to \mathbb{R}^n$ such that a mapping $x : I \to \mathbb{R}^n$ is a solution of the DAE if and only if x is a solution of the vector field v. The vector field v is then called the *corresponding vector field* and the manifold W the *configuration space*.

Example 2. A pendulum with mass m attached at the end of a rigid massless wire with length l attached at the origin in the (q_1, q_2) plane satisfies the DAE

$$\begin{aligned} q_1' &= v_1 \\ q_2' &= v_2 \\ mv_1' &= -\lambda q_1 \\ mv_2' &= -\lambda q_2 - mg \\ 0 &= q_1^2 + q_2^2 - \ell^2 \end{aligned}$$

where λ is the (unknown) tension of the wire and g is the gravity constant. If we introduce the new variables $x = (q_1, q_2, v_1, v_2, \lambda) \in \mathbb{R}^5$, $p = (q_1', q_2', v_1', v_2', \lambda') \in \mathbb{R}^5$, then the problem assumes the form (2) with

$$G(x, p) = \begin{pmatrix} p_1 - x_3 \\ p_2 - x_4 \\ mp_3 + x_5 x_1 \\ mp_4 + x_5 x_2 + mg \\ x_1^2 + x_2^2 - \ell^2 \end{pmatrix}.$$

On the other hand it is well known that in polar coordinates (r, ϕ) the motion of the pendulum is described by the second order differential equation

$$\phi'' + \frac{g}{\ell} \sin(\phi) = 0. \tag{3}$$

We will see in Section 3 that (3) is the corresponding vector field of the above DAE written as a second order differential equation.

Let us finally introduce the following notations. Let $v^l : M \to \mathbb{R}^n$ be a linear vector field on a linear subspace M of $\mathbb{R}^n$; i.e., $v^l \in L(M, \mathbb{R}^n)$. Then the *spectrum* of v^l, denoted by $\rho(v^l)$, is the set of all $\mu \in \mathbb{C}$ such that there exists a $y \in M$, $y \neq 0$, with

$$(\mu j - v^l)y = 0$$

where $j : M \to \mathbb{R}^n$ is the canonical embedding of M into $\mathbb{R}^n$. Consider now the *matrix pencil* (A, B) [8] with $A, B \in L(\mathbb{R}^n)$. Then the set of all $\mu \in \mathbb{C}$ satisfying $\det(\mu A + B) = 0$, denoted by $\rho(A, B)$, is called the *spectrum of the matrix pencil.*

3. Existence and uniqueness of solutions near an equilibrium point

Throughout this paper we consider DAEs

$$F(x, x') = 0, \tag{4}$$

$F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, with an *equilibrium point* $x_0 \in \mathbb{R}^n$; i.e.,$F(x_0, 0) = 0$. We assume that

$$\text{A1 : rank } D_x F(x_0, 0) = n,$$

which implies that x_0 is an isolated equilibrium point of (4), and that

$$\text{A2 : rank } D_p F(x, p) = \text{const}$$

in a neighborhood of $(x_0, 0)$ in $F^{-1}(0)$. Both assumptions together imply that the mapping F is admissible locally around $(x_0, 0)$.

In this section we derive results on the existence and uniqueness of solutions near the equilibrium point x_0, although most of the results of this section can be easily generalized to arbitrary points $(x, p) \in F^{-1}(0)$. For that reason we consider (4) only on a sufficiently small neighborhood $U = U(x_{,0}, 0)$ of the equilibrium point $(x_0, 0)$. In contrast to other approaches on the existence and uniqueness of solutions (see, e.g., [3], [17]) our results are based on a family of overdetermined DAEs which we define as follows:

Definition 1. Let a DAE (4) with an equilibrium point x_0 be given. Assume that assumptions A1 and A2 hold. Let U be a sufficiently small neighborhood of $(x_0, 0)$. Then we define the family (G_i) of mappings $G_i : U \to \mathbb{R}^{m_i}$ by:

$$\begin{aligned} G_0(x, p) &:= F(x, p) \\ G_{i+1}(x, p) &:= \begin{pmatrix} G_0(x, p) \\ S_i(x, p) D_x G_i(x, p) p \end{pmatrix} \quad (i = 1, \ldots, s-1) \end{aligned}$$

where S_i is a mapping associated to G_i (see Proposition 1) and s is the largest integer such that the G_i's are admissible and $G_s^{-1}(0) \neq G_{s-1}^{-1}(0)$. We say that the DAE (4) is of *degree s locally around* the equilibrium point x_0.

Example 3. The DAE associated to the mapping G, as considered in Example 1, is of degree one. The mapping G_1 is given by

$$G_1(x, p) = \begin{pmatrix} G(x, p) \\ (x_1 + x_2) p_1 + p_2(p_1 + p_2) + p_3 \\ p_1 + p_2 \end{pmatrix}$$

and $G_2 = G_1$ because rank $D_p G_1(x, p) = 3$.

Example 4. Let us consider quasilinear DAEs

$$F(x, x') := A(x)x' + f(x) = 0$$

i.e., DAEs for which x' enters linearly into the equations. Let us assume that rank $A(x) = k$ for all $x \in V$, where V is a sufficiently small neighborhood of x_0 in $\mathbb{R}^n$, $f(x_0) = 0$, and rank $Df(x_0) = n$. In this case F is admissible and we simply have $D_p F = A$. If S is a mapping associated to F; i.e., $S(x)A(x) = 0$, then we obtain

$$M = \{x \in \mathbb{R}^n : g(x) := S(x)f(x) = 0\}$$

and

$$T_x M = \ker Dg(x).$$

Thus the mapping G_1 in Definition 1 can be modified to

$$G_1(x, p) = \begin{pmatrix} A(x)p + f(x) \\ Dg(x)p \end{pmatrix}$$

which is again linear in the variable p and, consequently, similar modifications can be made for all mappings G_i.

In the following proposition we use the special structure of the DAE (4) near x_0 to state sufficient conditions for the admissibility of the mappings G_i.

Proposition 2. *Assume that G_{i-1} is admissible. Then G_i is admissible if* rank $D_p G_i(x, p) =$ const. *for all* $(x, p) \in G_i^{-1}(0)$.

Proof. We have to show that the admissibility of G_{i-1} implies that $DG_i(x_0, 0)$ has full rank. By definition (and using the fact that $p_0 = 0$)

$$DG_i(x_0, 0) = \begin{pmatrix} D_x G_0(x_0, 0) & D_p G_0(x_0, 0) \\ 0 & S_{i-1}(x_0, 0) D_x G_{i-1}(x_0, 0) \end{pmatrix}.$$

Now rank $D_x G_0(x_0, 0) = n$ by assumption A1 and $S_{i-1}(x_0, 0) D_x G_{i-1}(x_0, 0)$ has full rank by the admissibility of G_{i-1}. □

We associate now with each of the mappings G_i a DAE of type (2). Comparing their solution behavior we obtain

Proposition 3. *Let* (4) *be a DAE of degree s locally around x_0, then the DAEs*

$$G_i(x, x') = 0 \quad (i = 0, \ldots, s)$$

have identical solution behavior.

Proof. Let us introduce the sets $N_i = G_i^{-1}(0)$, $M_i = \mathrm{pr}_1(N_i)$. By the admissibility of the mappings G_i, the sets N_i, M_i are manifolds and satisfy $N_{i+1} \subseteq N_i$, $M_{i+1} \subseteq M_i$. Thus, by the definition of the mappings G_i:

$$N_{i+1} = \bigcup_{x \in M_i} [N_0 \cap \{x \times T_x M_i\}]$$

and thus $N_{i+1} = N_0 \cap TM_i$. Now because of $N_0 \cap TM_i = N_0 \cap TM_{i-1} \cap TM_i = N_i \cap TM_i$, we obtain $N_{i+1} = N_i \cap TM_i$. As already shown in [18], a mapping $x : I \to \mathbb{R}^n$ satisfies $(x(t), x'(t)) \in N_i$ if and only if $(x(t), x'(t)) \in N_i \cap TM_i$. □

Remark. Our approach is close to an approach taken, e.g., in [3] where the equation (4) is differentiated several times with respect to t to obtain the so-called derivative array equations. However, in contrast to the approach taken in [3], we eliminate the second order derivative x'' after each differentiation and thus avoid having higher order derivatives of x enter into the equations. The advantage of the approach taken in [3] is that it does not involve constant rank conditions in the intermediate steps.

We are now ready to state sufficient conditions for the regularity of a DAE (4) locally around an equilibrium point x_0.

Theorem 1. *Let* (4) *be a DAE of degree s locally around* $(x_0, 0)$. *If*

$$\mathrm{rank}\, D_p G_s(x_0, 0) = n$$

then (4) *is regular locally around* $(x_0, 0)$. *The corresponding vector field* $v : W \to \mathbb{R}^n$ *is characterized by*

$$\mathrm{graph}\, v = G_s^{-1}(0).$$

Proof. Rank $D_p G_s(x_0, 0) = n$ implies that $N_s = G_s^{-1}(0)$ is the graph of a function $v : M_s \to \mathbb{R}^n$. Because $N_s \cap TM_s = N_s$, $v(x) \in T_x M_s$ for all $x \in M_s$. Thus Proposition 3 implies the regularity of (4) locally around $(x_0, 0)$. □

Definition 2. In correspondence with the definitions given, e.g., in [2], [17], we call a DAE (4) of *index s locally around* $(x_0, 0)$ if the conditions of Theorem 1 are satisfied.

Example 5. The DAE considered in Example 2 is of index three. The DAE corresponding to the mapping G_3 can be rewritten as

$$
\begin{aligned}
q_1' &= v_1 \\
q_2' &= v_2 \\
mv_1' &= -\lambda q_1 \\
mv_2' &= -\lambda q_2 - mg \\
\lambda' &= \frac{1}{l^2}(3mv_2 g + 2v_1 q_1 \lambda + 2v_2 q_2 \lambda) \\
0 &= q_1^2 + q_2^2 - l^2 \\
0 &= q_1 v_1 + q_2 v_2 \\
0 &= \frac{l^2}{m}\lambda + q_2 g - (v_1^2 + v_2^2).
\end{aligned}
$$

Introducing polar coordinates (r, ϕ) in the (q_1, q_2)-plane and denoting r' and ϕ' by ρ, θ, respectively, this DAE reduces to (after elimination of the variable λ)

$$
\begin{aligned}
\phi' &= \theta \\
r' &= \rho \\
l\theta' &= -g\ \sin(\phi) \\
\rho' &= 0 \\
0 &= r^2 - l^2 \\
0 &= r\rho
\end{aligned}
$$

and the resulting ODE

$$
\begin{aligned}
\phi' &= \theta \\
\theta' &= -\frac{g}{l}\ \sin(\phi)
\end{aligned}
$$

represents the corresponding vector field of the DAE on the manifold $M = \{(q, v) : q_1^2 + q_2^2 = r^2 = l^2, q_1 p_1 + q_2 p_2 = r\rho = 0\}$.

4. Linearization of a DAE in an equilibrium point

Many local qualitative results for vector fields are based on the linearization of a vector field $v : M \to \mathbb{R}^n$ along a given solution $x : I \to \mathbb{R}^n$. In the sequel we focus on problems where this solution is an *equilibrium point* $x_0 \in M$; i.e., $v(x_0) = 0$. The investigation of the resulting *linear vector field* $v^l \in L(T_{x_0}M, \mathbb{R}^n)$, given by

$$v^l := Dv(x_0),$$

allows us in many cases to determine the qualitative solution properties of the vector field v in a sufficiently small neighborhood of the equilibrium point [10]. Of course one would like to apply this strategy to regular DAEs as well. For that reason let $x_0 \in \mathbb{R}^n$ be an equilibrium point of the DAE (4); i.e., assume that

$$0 = F(x_0, 0).$$

Then the linearization of the DAE (4) along the solution $x(t) = x_0$ results in the linear DAE

$$A\bar{x}' + B\bar{x} = 0 \tag{5}$$

where $A = D_pF(x_0, 0)$ and $B = D_xF(x_0, 0)$.

It seems reasonable to conjecture that, whenever the DAE (4) is regular locally around an equilibrium point x_0, the linearized DAE (5) is regular too and that the corresponding vector field of (5) is given by the linearization of the corresponding vector field of (4) at x_0. However, as shown by Example 6, this conjecture does not always hold.

Example 6. Consider the DAE

$$xx' = -x^2, \tag{6}$$

$x \in \mathbb{R}$. Note that this DAE does not satisfy assumptions A1 and A2 of Section 3 at $(0, 0)$ and that therefore no degree can be defined for the DAE locally around $(0, 0)$. However, the DAE is regular locally around $(0, 0)$ and the corresponding vector field is given by $x' = -x$. Now, this is already a linear differential equation and we expect linearization of (6) at $(0, 0)$ to yield a regular DAE with $x' = -x$ as the corresponding vector field. However, what we obtain really is the "DAE" $0 = 0$, which is obviously nonregular. Note that this situation can never occur for a DAE (4) that satisfies assumption A1.

Assumption A1 implies that the linearized DAE (5) is regular whether or not the given DAE (4) is regular locally around the equilibrium. But even if the DAE (4) is regular this does not imply that the corresponding vector field of the linear DAE (5) is equivalent to the linearization of the corresponding vector field of the nonlinear problem at x_0.

Example 7. Consider the DAE

$$x' = -y \tag{7}$$

$$0 = y^3 - x \tag{8}$$

with $(x, y) \in \mathbb{R} \times \mathbb{R}$. This DAE satisfies assumptions A1 and A2 but does not satisfy the conditions of Theorem 1. However, because $(x(t), y(t)) = (0, 0)$ is the only solution with $(x(0), y(0)) = (0, 0)$, the corresponding vector field of (7) is given by

$$(x', y') = \begin{cases} (-y, -1/3y) & \text{for } (x, y) \in M_1 \setminus \{(0, 0)\} \\ (0, 0) & \text{for } (x, y) = (0, 0). \end{cases}$$

Because this vector field has a discontinuity at the equilibrium point $(0, 0)$ no linearization of the vector field at the equilibrium point exists. However, linearization of the DAE (7) at $(0, 0)$ yields the linear DAE

$$\begin{aligned} \bar{x}' &= \bar{y} \\ 0 &= \bar{x} \end{aligned} \tag{9}$$

which is of index two and possesses only the trivial solution $(\bar{x}(t), \bar{y}(t)) = (0, 0)$.

In the following theorem we state sufficient conditions under which the linearized DAE (5) properly reflects the linearized solution behavior of the given nonlinear problem (4).

Theorem 2. *Let* (4) *be a DAE of index s locally around an equilibrium point x_0. Assume that the DAE satisfies assumption A1. Denote the corresponding vector field of* (4) *by $v : M \to \mathbb{R}^n$ and the corresponding vector field of the linearized DAE* (5) *by $v^l : M^l \to \mathbb{R}^n$. Then we have*

$$Dv(x_0) = v^l$$

where $Dv(x_0) \in L(T_{x_0}M, \mathbb{R}^n)$ is the linearization of v at x_0 and $M^l = T_{x_0}M$.

Proof. As in the proof of Proposition 3 we introduce the sets $N_i = G_i^{-1}(0)$, $M_i = \mathrm{pr}_1(N_i)$, $i = 0, \ldots, s$. We denote the corresponding sets of the linearized DAE by N_i^l, M_i^l, respectively. We prove the theorem by induction. By definition we have $N_0^l = T_{(x_0,0)}N_0$. Let us assume that $N_i^l = T_{(x_0,0)}N_i$ holds for some i, $s > i \geq 0$. Then $M_i^l = T_{x_0}M_i$ and $TM_i^l = T_{(x_0,0)}TM_i$ as well. Now, as shown in the proof of Proposition 3, we have

$$N_{i+1} = N_i \cap TM_i$$

which implies that

$$T_{(x_0,0)}N_{i+1} = T_{(x_0,0)}(N_i \cap TM_i).$$

Now the admissibility of the mapping G_i gives

$$\begin{aligned} T_{(x_0,0)}N_{i+1} &= T_{(x_0,0)}N_i \cap T_{(x_0,0)}TM_i \\ &= N_i^l \cap TM_i^l \\ &= N_{i+1}^l. \end{aligned}$$

Finally we observe that graph $v = N_s$ and graph $v^l = N_s^l$. □

In the following proposition we show that the regularity of the linearized DAE simplifies the conditions for the regularity of the nonlinear problem as stated in Theorem 1.

Proposition 4. *Let x_0 be an equilibrium point of a DAE* (4) *and let* (5) *be the linearization of* (4) *in x_0. Assume that* (5) *is of index s and that the DAE* (4) *is at least of degree $s - 1$ locally around x_0. Then* (4) *is of index s locally around the equilibrium point x_0.*

Proof. We use the same notation as in the proof of Theorem 2. Because (4) is at least of degree $s - 1$, G_{s-1} is defined and admissible. Thus G_s is defined too and, as shown in the proof of Proposition 2, $DG_s(x_0, 0)$ has full rank. Therefore $N_s = N_{s-1} \cap TM_{s-1}$ is a manifold and $T_{(x_0,0)}N_s = N_s^l$. This implies rank $D_pG_s(x_0, 0) = n$. Thus G_s is admissible and satisfies the conditions of Theorem 1. □

Example 8. Let x_0 be an equilibrium point of a quasilinear DAE

$$F(x, x') = A(x)x' + f(x) = 0.$$

Assume that rank $A(x) = k$ for all $x \in V$, where V is a sufficiently small neighborhood of x_0 in $\mathbb{R}^n$, and that rank $Df(x_0) = n$. This implies that F is admissible. Furthermore, $F_p = A$. Let $S(x)$ be the mapping associated to F and define the mapping $g : V \to \mathbb{R}^{n-k}$ by $g(x) = S(x)f(x)$. Then N_0 and M_0 are given by

$$N_0 = \{(x, p) \in V \times \mathbb{R}^n : A(x)p = f(x)\}$$
$$M_0 = \{x \in V : g(x) = 0\}.$$

If now the linearized DAE is of index one, then Proposition 4 implies that the DAE is of index one locally around the equilibrium point x_0. More generally, if

$$\operatorname{rank}\begin{pmatrix} A(x) \\ Dg(x) \end{pmatrix} = \text{const.}$$

for all $x \in V$, then the mapping G_1 given by

$$G_1(x, p) = \begin{pmatrix} A(x)p + f(x) \\ Dg(x)p \end{pmatrix}$$

is admissible too. Thus, if the linearized DAE is of index two, then Proposition 4 yields in this case that the DAE is of index two locally around the equilibrium point x_0.

It is well known that the spectrum $\rho(v^l)$ of a linear vector field v^l plays a crucial role in computing the solution behavior. If v^l is the corresponding vector field of a linear DAE (5), then, as shown in the next proposition, the spectrum $\rho(v^l)$ can be obtained directly from the matrix pencil (A, B) of the linear DAE.

Proposition 5. *Let* (5) *be a DAE of index s and let $v^l : M^l \to \mathbb{R}^n$ be the corresponding vector field of this DAE. Then the spectrum of v^l and the spectrum of (A, B) satisfy*

$$\rho(v^l) = \rho(A, B).$$

Proof. If $\mu \in \rho(v^l)$ then there is a $y \in M^l$, $y \neq 0$, such that

$$(\mu j - v^l)y = 0.$$

Furthermore, by definition of the corresponding vector field we have

$$(Av^l + B)y = 0$$

for all $y \in M^l$. Thus, if $\mu \in \rho(v^l)$ then $\mu \in \rho(A, B)$ because of

$$(\mu A + B)y = 0. \tag{10}$$

On the other hand, if $\mu \in \rho(A, B)$, then the corresponding $y \in \mathbb{R}^n$ in (10) has to satisfy $y \in M^l$. This implies $\rho(v^l) = \rho(A, B)$. □

5. Applications

The preceding results, especially Theorem 1, Theorem 2, and Proposition 5, can be used to discuss the solution behavior of DAEs near an equilibrium point $x_0 \in \mathbb{R}^n$. According to the results of Section 4 one can hope that the linearized DAE plays a crucial role in determining the qualitative behavior of (4) near an equilibrium point x_0. For that reason let (5) be the linearization of (4) at x_0 and assume that (5) is regular. Then, according to Proposition 5 the solution properties of the linear DAE (5) can be obtained from the spectrum $\rho(A, B)$ with

$$\begin{aligned}
\rho^s &:= \{\mu \in \rho(A, B) : \text{Re}\ \mu < 0\} \\
\rho^c &:= \{\mu \in \rho(A, B) : \text{Re}\ \mu = 0\} \\
\rho^u &:= \{\mu \in \rho(A, B) : \text{Re}\ \mu > 0\}.
\end{aligned}$$

Especially, then are *invariant subspaces* of the DAE (5) associated to ρ^*, $\star = s, c, u$, call these X^*, and $M^l = X^s \oplus X^c \oplus X^u$ where M^l is the configuration space of the DAE (5). The association is that X^* is spanned by the eigenvectors $v_i \in \mathbb{R}^n$ corresponding to the eigenvalues $\mu_i \in \rho^*$, i.e.,

$$(\mu_i A + B)v_i = 0.$$

Any solution $\bar{x}$ of (5) with $\bar{x}(0) \in X^s$ tends exponentially to 0 as $t \to \infty$ and any solution with $\bar{x}(0) \in X^u$ to 0 as $t \to -\infty$. As long as the eigenvalues of ρ^c are simple, the solutions with $\bar{x}(0) \in X^c$ are periodic.

We turn now to the nonlinear problem (4) and assume that the conditions of Theorem 1 are satisfied. Then, in accordance with corresponding definitions for vector fields, we call x_0 a *hyperbolic equilibrium point* if the linearized DAE satisfies $\rho^c = \emptyset$. Furthermore, let $x(\cdot; x_a) : I \to \mathbb{R}^n$ denote solutions of (4) in a sufficiently small neighborhood U of $(x_0, 0)$ with $x(0, x_a) = x_a$, then we define [10]

$$W^s := \{x_a \in \mathbb{R}^n : \exists x(\cdot; x_a) : [0, \infty) \to \mathbb{R}^n \\ \text{suchthat}\ x(t; x_a) \to x_0 \text{ for } t \to \infty\}$$

$$W^u := \{x_a \in \mathbb{R}^n : \exists x(\cdot; x_a) : [0, -\infty) \to \mathbb{R}^n \\ \text{suchthat}\ x(t; x_a) \to x_0 \text{ for } t \to -\infty\}.$$

For hyperbolic equilibrium points the solutions of (5) resemble the solution behavior of (4) in a sufficiently small neighborhood of x_0. Specifically:

Theorem 3 (Local Stable and Unstable Manifold). *Let x_0 be an equilibrium point of the DAE (4). Assume that (4) is of index s, $s \geq 1$, locally around x_0 and that x_0 is a hyperbolic equilibrium point. Consider the DAE (4) on a sufficiently small neighborhood U of $(x_0, 0)$. Then the sets W^s and W^u are submanifolds of the configuration space W that are tangent to X^s, X^u, respectively, at x_0; i.e., $T_{x_0} W^s = X^s$ and $T_{x_0} W^u = X^u$.*

Proof. By assumption the DAE (4) is regular on U. Furthermore, by Theorem 2 and Proposition 5, the point x_0 is a hyperbolic equilibrium point of the corresponding vector field. Thus one can apply the stable and unstable manifold theorem as formulated for vector fields in [10] to the DAE (4). □

Remarks. The set W^s (W^u) is called the *local stable (unstable) manifold.* These manifolds can be used to give a coordinate system on the configuration space in a neighborhood of x_0. Furthermore, if $\rho^u = \emptyset$, then the equilibrium point x_0 is *asymptotically stable* in the sense of Lyapunov [10].

Example 9. A Lagrangian formulation of the equations describing a constrained multibody system may be written as

$$\begin{aligned} q' &= v \\ M(q)v' &= f(q, v) - G^T(q)\lambda \\ 0 &= g(q) \end{aligned}$$

where $M(q)$ is a regular (symmetric, positive definite) mass matrix, f is a vector of applied forces, and λ represents the Lagrange multipliers or constraint forces coupled to the system by the constraint matrix $G(q) := Dg(q)$, which is assumed to have full rank. This implies that the above DAE is of index three. Let us assume that $f(q_0, 0) = 0$, then the point $(q_0, 0)$ is an equilibrium solution of the corresponding mechanical systems. To determine the stability properties of $(q_0, 0)$ we can make use of Theorem 3. Linearization yields the following linear DAE:

$$\begin{pmatrix} I & 0 & 0 \\ 0 & M(q_0) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{q}' \\ \bar{v}' \\ \bar{\lambda}' \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ D_p f(q_0, 0) & D_v f(q_0, 0) & -G^T(q_0) \\ G(q_0) & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{q} \\ \bar{v} \\ \bar{\lambda} \end{pmatrix}.$$

From this one gets the matrices A and B and finally one has to compute the generalized eigenvalues of the matrix pencil (A, B) [7].

The study of fixed points becomes more delicate when they are not hyperbolic. Nonhyperbolic fixed points often arise in bifurcation problems. Here we restrict ourselves to the case in which ρ^c contains two simple eigenvalues $\pm j\omega$ on the imaginary axis. For vector fields the resulting bifurcation is called the *Hopf bifurcation* [10], [11]. A first result on Hopf bifurcations for DAEs of index one can be found in [21].

Let

$$F(x, x'; \lambda) = 0 \quad (\lambda \in \mathbb{R}) \tag{11}$$

be a one-parametric family of DAEs with an equilibrium point x_0 for $\lambda = 0$. Assume that assumption A1 holds for $\lambda = 0$. Then the implicit function theorem implies that there are neighborhoods $\Lambda \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}^n$ of 0 and x_0 such that the system of equations

$$F(x, 0; \lambda) = 0$$

has exactly one solution $x \in V$ for all $\lambda \in \Lambda$; i.e., there is a one-parametric family $x_0(\lambda)$ of equilibrium points with $x_0(0) = x_0$.

Assume that (11) is of index s locally around $x_0(\lambda)$ for all $\lambda \in \Lambda$ and that the mapping $G_s(.,.;\lambda)$ depends smoothly on λ. Then the corresponding vector field $v(.;\lambda)$ also depend smoothly on λ. Furthermore, the linearized DAEs

$$A(\lambda)\bar{x}' + B(\lambda)\bar{x} = 0 \quad (\lambda \in \Lambda) \tag{12}$$

with $A(\lambda) := D_p F(x_0(\lambda), 0; \lambda)$ and $B(\lambda) := D_x F(x_0(\lambda), 0; \lambda)$ are regular. Assume now that the spectrum of the matrix pencil $(A(\lambda), B(\lambda))$ contains exactly one pair of complex eigenvalues $\mu_{1,2} = \rho(\lambda) \pm j\omega(\lambda)$ such that

$$\text{A3}: \qquad \begin{aligned} \rho(0) = 0, \ \omega(0) \neq 0 \\ D\rho(0) \neq 0 \end{aligned}$$

and that (12) has no other eigenvalues on the imaginary axis for $\lambda = 0$. This implies that, for $\lambda = 0$, the linearized DAE satisfies $\rho^c = \{\pm\omega(0)\}$ and that the equilibrium points $x_0(\lambda)$ are hyperbolic for $\lambda \neq 0$. Using results in [10], a Hopf theorem for DAEs of the above type follows easily. Specifically:

Theorem 4 (Hopf). *Let us consider the one-parametric family of DAEs* (11) *where* x_0 *is an equilibrium point of* (11) *for* $\lambda = 0$ *satisfying assumption A1. Assume that* (11) *is of index s locally around* $x_0(\lambda)$ *for all* $\lambda \in \Lambda$ *and that the mapping* $G_s(.,.;\lambda)$ *depends smoothly on* λ. *Let us assume furthermore that assumption A3 holds. Then, in any neighborhood U of* x_0 *in* $\mathbb{R}^n$ *and any given* λ_0 *there is a* $\bar{\lambda}$ *with* $|\bar{\lambda}| < \lambda_0$ *such that* (11) *with* $\lambda = \bar{\lambda}$ *has a nontrivial periodic solution in U.*

Remark. The theorem in its above formulation does not allow us to draw any conclusion on whether the periodic solution is locally unique or not. To obtain such results we would need a theory of normal forms for DAEs. First results in that direction can be found in [5].

As an application of Theorem 4, let us consider the numerical computation of Hopf points for DAEs in terms of an augmented system of nonlinear equations. This so-called direct approach has been very successfully applied in the ODE context [20]. Here we give a brief outline on how to generalize these results to DAEs.

Let us assume that all the conditions of Theorem 4 are satisfied. Then the Hopf point is characterized (locally) by the unique solution of the following augmented system of nonlinear equations:

$$\begin{aligned} 0 &= F(x, 0; \lambda) \\ 0 &= \alpha A(x; \lambda)r + B(x; \lambda)s \\ 0 &= \alpha A(x; \lambda)s - B(x; \lambda)r \\ 0 &= N(r, s) \end{aligned} \tag{13}$$

where $N : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^2$ is a normalization function and $A(x;\lambda) = F_p(x, 0; \lambda)$, $B(x; \lambda) = F_x(x, 0; \lambda)$. Following the same arguments as given in [12] and [20] for the ODE case, one can show that (13) has (locally) an isolated solution provided N is chosen properly. For example, one could chose the following normalization function N:

$$0 = y^t r$$
$$0 = r^t r - 1$$

with $y \in \mathbb{R}^n$ suitably selected [20].

Example 10. A nonlinear autonomous RLC circuit [4] which, for simplicity, is assumed to have a connected graph with b branches can be described by the DAE

$$0 = Bv \tag{14}$$
$$0 = Qi \tag{15}$$
$$q' = i_c \tag{16}$$
$$0 = f_c(u_c, q) \tag{17}$$
$$\phi' = u_l \tag{18}$$
$$0 = f_l(i_l, \phi) \tag{19}$$
$$0 = f_r(u_r, i_r). \tag{20}$$

Here $v \in \mathbb{R}^b$ denotes the column vector of branch voltages and $i \in \mathbb{R}^b$ denotes the column vector of branch currents, partitioned, respectively, as (v_c, v_l, v_r) and (i_c, i_l, i_r) where subscript c (l, r, respectively) denotes the variable corresponding to capacitors (inductors, resistors, respectively). Equations (16) through (20) represent the constitute relations of the capacitors, inductors, and resistors. Equations (14) and (15) represent the interconnection of all circuit elements. The system of equations (14) through (20) is the standard system of circuit equations [4]; modifications of this system, such as the nodal analysis, are widely used in circuit analysis packages [4].

In [15], conditions for the existence of oscillations in oscillator circuits [4], such as the Wien bridge oscillator, were derived via a state-space analysis and the standard Hopf theorem for ODEs. The results in this paper show that the same results can be obtained by a direct analysis of the corresponding standard system of circuit equations, and a corresponding Hopf point can be computed by using (13). Such an approach is preferable especially for large circuits where the state-space form is difficult to obtain.

Acknowledgments

We would like to thank the anonymous referee for pointing out a simplification in the proof of Proposition 1.

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