Sebastian Reich

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On a Geometrical Interpretation of Differential-Algebraic Equations*

Sebastian Reich[1]

Abstract. The subject of this paper is the relation of differential-algebraic equations (DAEs) to vector fields on manifolds. For that reason, we introduce the notion of a regular DAE as a DAE to which a vector field uniquely corresponds. Furthermore, a technique is described which yields a family of manifolds for a given DAE. This so-called family of constraint manifolds allows in turn the formulation of sufficient conditions for the regularity of a DAE, and the definition of the index of a regular DAE. We also state a method for the reduction of higher-index DAEs to lower-index ones that can be solved without introducing additional constants of integration. Finally, the notion of realizability of a given vector field by a regular DAE is introduced, and it is shown that any vector field can be realized by a regular DAE. Throughout this paper the problem of path-tracing is discussed as an illustration of the mathematical phenomena.

1. Introduction

Two important mathematical concepts are currently applied to modeling of lumped physical systems such as electronic circuits and mechanical systems. These concepts are, on the one hand, differential-algebraic equations (DAEs) and, on the other hand, vector fields on manifolds. While DAEs have been used for a long time, the vector-field approach to modeling is still rather young. For example, the vector-field approach to modeling of electronic circuits was described for the first time in a paper by Smale [1] in 1972. Vector fields have become an essential tool whenever a global study of lumped physical systems is undertaken (see, for instance, [2] and [3]). In line with this, it is hardly surprising that in recent years interest has increased in the development of numerical methods for the computational analysis of

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[1] Technische Universität Dresden, Sektion Informationstechnik, Bereich Bauelemente und Systeme, Mommsenstrasse 13, Dresden, DDR-8027.

vector fields. On the other hand, with the improvement of numerical integration methods for DAEs during the seventies, DAE formulation has been used nearly exclusively for the computational analysis of lumped physical systems. For a long time, DAEs were simply regarded as implicitly written ordinary differential equations. Only in about 1981, after the failure of numerical integration methods applied to certain DAEs had been repeatedly reported, did Gear and Petzold [4], Newcomb [5], Campbell [6], Rheinboldt [7], and Griepentrog and März [8] inspire discussion about the theoretical foundations of DAEs and the numerical treatment of DAEs.

In this context, it is rather surprising that until now, with the exception of Rheinboldt [7], little research has been done on the investigation of the relation of DAEs to vector fields. It seems that further progress in global, as well as in computational, analysis of lumped physical systems can be achieved by a close interaction of both these mathematical concepts. We want to establish some foundations for this here. Our aim is to formulate conditions under which a DAE is equivalent to a vector field and vice versa. This provides us, in turn, with new insights into the properties of DAEs and vector fields. In this paper special emphasis is put on the higher-index DAEs [4] (algebraic incomplete in [7], nontransferable in [8] DAEs, respectively).

More specifically, after a summary of necessary definitions, we introduce, in Section 3, the notion of a regular DAE as a DAE to which a vector field uniquely corresponds. In Section 4 we describe a technique which yields, for a given DAE, a family of manifolds. With this so-called family of constraint manifolds, we hope to provide a better geometrical insight into the algebraic incomplete DAEs. Furthermore, the family of constraint manifolds allows the formulation of sufficient conditions for the regularity of a DAE and the definition of the index of a regular DAE. In Section 5 we state a method for the reduction of higher-index DAEs to lower-index ones that can be solved although not introducing any constant of integration. Moreover, by means of this reduction method, we can show that our definition of the index of a DAE is equivalent in essence to the definition of the global index by Gear, Lötstedt, and Petzold [4], [9], [10]. In Section 6 we show that a regular DAE corresponds to any vector field such that the solutions of the DAE and the vector field are in one-to-one correspondence. This DAE is then called the realization of the given vector field. As a running example, the problem of path-tracing is used throughout this paper. Path-tracing is needed in various fields of applications, for example in the computation of characteristics of resistive circuits [11] and elsewhere [7]. Based on ideas of Haase [11] and Gear [12], we propose a DAE for the computation of a given path and investigate its properties.

2. Notation

We restrict our study to quasi-linear, time-invariant DAEs of the form

$$A(z)z' = f(z)$$

This restriction simplifies the results significantly and is motivated by the fact that analysis of many lumped physical systems leads to DAEs of this type. Furthermore, we can easily generalize the results obtained in this paper to the time-varying case. This can be done by the well-known techniques for reducing a time-varying DAE into a time-invariant one [7].

Definition 1. By a *differential-algebraic equation* (abbreviated DAE), we mean a triple (E, A, f) where E is a finite-dimensional real Banach space and $A: E \to L(E, E)$, $f: E \to E$ are arbitrary mappings.

A *solution* of the DAE (E, A, f) is a differentiable mapping $c: I \to E$ (where I denotes an interval in the real line) such that, for all $t \in I$,

$$A(c(t)) \frac{dc}{dt}(t) = f(c(t)).$$

Throughout this paper we use the following notations. If M is a differentiable manifold, then TM denotes the tangent bundle of M and T_xM is the tangent space of M at $x \in M$. A vector field on a differentiable manifold M is a mapping $v: M \to TM$ such that $v(x) \in T_xM$ for all $x \in M$. By a solution of a vector field $v: M \to TM$, we mean a differentiable mapping $w: I \to M$ (where I denotes an interval in the real line) such that, for all $t \in I$,

$$\frac{dw}{dt}(t) = v(w(t)).$$

Let $f: M \to E$ be a differentiable mapping from a manifold M into a real Banach space E. We denote the derivative of the mapping f by $Df: TM \to E$ and the derivative at $x \in M$ by $Df(x) \in L(T_xM, E)$. By definition, a differentiable and bijective mapping $f: E \to E$ is a diffeomorphism if the inverse mapping f^{-1} is differentiable too. Additionally, if M is a manifold with $M \subseteq E$, then let $j: M \to E$ be the natural injection. (For an introduction in the theory of manifolds and vector fields we refer to [13].)

As we will see, many properties of DAEs discussed in this paper are invariant with respect to a coordinate transformation $u: E \to E$ and a scaling by a mapping $S: E \to L(E, E)$. Consequently, we arrive at the following definition.

Definition 2. Let (E, A, f) and $(E, A^{\#}, f^{\#})$ be DAEs over the same space E. We say these DAEs are *equivalent* if there exists a diffeomorphism $u: E \to E$ and a mapping $S: E \to L(E, E)$ such that

(i) $S(z) \in L(E, E)$ is a diffeomorphism for all $z \in E$ and
(ii) $f(z) = S(z) f^{\#}(u(z))$, $\quad A(z) = S(z) A^{\#}(u(z)) Du(z) \quad$ for all $z \in E$.

Let (E, A, f) and $(E, A^{\#}, f^{\#})$ be equivalent DAEs. Then it is readily seen from Definition 2 that the mapping $c: I \to E$ is a solution of the DAE

(E, A, f) if and only if the composed mapping $c^{\#} := u \circ c$ is a solution of the DAE $(E, A^{\#}, f^{\#})$.

Because we like to give a geometrical interpretation of DAEs, the set defined in the next definition turns out to be of special importance.

Definition 3. Let (E, A, f) be a DAE. Then we call the set

$$N = \{(z, z') \in E \times E: A(z)z' = f(z)\}$$

the *corresponding set* of the DAE (E, A, f).

Note, in Definition 3, the DAE

$$A(z)z' = f(z)$$

is considered as a nonlinear system of equations in the variables z and z'. For that reason, the corresponding set N of a DAE (E, A, f) is a subset of $E \times E$.

Clearly, a differentiable mapping $c: I \to E$ is a solution of this DAE if and only if $(c(t), (dc/dt)(t)) \in N$ for all $t \in I$. Therefore, with respect to the solutions of a DAE, we can instead consider the corresponding set. This fact is explored further in Section 4.

3. Regular DAEs

In [7] Rheinboldt developed the idea of considering DAEs as vector fields on manifolds. Inspired by this, we propose the following definition of a regular DAE.

Definition 4. Let (E, A, f) be a DAE. Then this DAE is called a *regular* DAE if there is a differentiable manifold $M \subseteq E$ and a vector field $v: M \to TM$ such that a differentiable mapping $w: I \to M$ is a solution of the vector field if and only if the mapping $c: = j \circ w: I \to E$ is a solution of the DAE (E, A, f).

The manifold M is then called the *configuration space* and the vector field v the *corresponding vector field* of the DAE (E, A, f).

In the following we focus our attention on the investigation of regular DAEs. But it should also be mentioned that nonregular DAEs provide an interesting tool for the study of lumped physical systems with jump behavior [14]. Evidently, the next proposition may be of interest in practical applications and in the development for a theory of normal forms for regular DAEs.

Proposition 1. *Let (E, A, f) and $(E, A^{\#}, f^{\#})$ be two equivalent DAEs. Then the DAE (E, A, f) is regular if and only if the DAE $(E, A^{\#}, f^{\#})$ is a regular DAE too.*

Proof. Let $u: E \to E$ be a coordinate transformation according to Definition 2. As already mentioned, a differentiable mapping $c: I \to E$ is a solution of the DAE (E, A, f) if and only if the mapping $c^{\#} := u \circ c$ is a solution of the DAE $(E, A^{\#}, f^{\#})$. Therefore, if one of the two DAEs is regular, the other DAE is regular as well. □

We would like to illustrate the concept of a regular DAE by the problem of path-tracing. In the following example we suggest a vector field and a DAE for the computation of a given path S.

Example 1. Let us assume that the given path S is a one-dimensional differentiable manifold. Consequently, we can propose the following vector field $v: S \to TS$ for path-tracing this path S.

The vector field v on M is defined by the correspondence $x \mapsto 1$ where 1 denotes the unity element of T_xS according to a given orientation on S. Let $c: I \to S$ be any solution of this vector field v. Then the image of the interval I by the mapping c is a segment of S.

Clearly, this approach is not convenient for numerical computations. Accordingly, we are led to consider a DAE for the path-tracing. Therefore, let us assume that the path S can be defined by a smooth mapping $g: \mathbb{R}^2 \to \mathbb{R}$ by

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \colon g(x_1, x_2) = 0\}.$$

In the following we assume that $Dg(x_1, x_2)$ has full rank for all $(x_1, x_2) \in \mathbb{R}^2$, and that the orientation on S was chosen such that the unity element $1 \in T_xS$ satisfies

$$-D_2 g(x_1, x_2)x_1' + D_1 g(x_1, x_2)x_2' > 0$$

for all $x \in S$ where $(x_1, x_2) = j(x)$, $(x_1', x_2') = Dj(x)1$, and $j: S \to \mathbb{R}^2$ denotes the natural injection. Furthermore, we introduce the two abbreviations

$$a = D_1 g(x_1, x_2) \quad \text{and} \quad b = D_2 g(x_1, x_2).$$

Using ideas of Haase [11] and Gear [12], we now propose the DAE (E, A, f) with

(i) $E := \mathbb{R}^3$, $z =: (x_1, x_2, y) \in E$,

(ii) $A(z) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

(iii) $f(z) := \begin{bmatrix} -b/[a^2 + b^2]^{1/2} + ay \\ a/[a^2 + b^2]^{1/2} + by \\ g(x_1, x_2) \end{bmatrix}$.

We show that this DAE is a regular DAE. Let $j: S \to E$ be the natural injection and let 1 be the unity element of $T_x S$. Clearly, because

$$f(j(x)) = Dj(x)1$$

for any $x \in S$, we have, for all $x \in S$,

$$A(j(x))Dj(x)v(x) = f(j(x)).$$

Consequently, for any solution w of the vector field v, the mapping $c: = j \circ w$ is a solution of the DAE (E, A, f). Furthermore, using results of [12], we can show that, for any solution c of the DAE (E, A, f), $c(t) \in S$ for all $t \in I$. From this, we conclude that w is a solution of the vector field v if and only if the mapping $c: = j \circ w$ is a solution of the DAE (E, A, f). Therefore, the DAE (E, A, f) is a regular DAE, the vector field v is the corresponding vector field, and S is the configuration space of the DAE.

4. Constraint manifolds and regular DAEs

In this section we derive sufficient conditions for the regularity of a DAE. Furthermore, we state a technique by means of which we can obtain the configuration space M and the corresponding vector field v for a given regular DAE. To start we make the following observation.

Observation 1. Let N be the corresponding set of a given DAE (E, A, f). We now consider the projection of this set onto the first component, that is the set

$$M_1 := p_1(N) \subseteq E.$$

Let us assume that M_1 is a differentiable manifold. Clearly, under this assumption, a differentiable mapping $c: I \to E$ is a solution of the DAE (E, A, f) if and only if

$$\left(c(t), \frac{dc}{dt}(t) \right) \in (TM_1 \cap N)$$

for all $t \in I$. In general, we will have $N \supset (TM_1 \cap N)$. Therefore, we consider the set

$$M_2 = p_1(N \cap TM_1) \subseteq M_1.$$

If, now, the set M_2 is a differentiable manifold as well, we again conclude that a differentiable mapping $c: I \to E$ is a solution of the DAE (E, A, f) if and only if

$$\left(c(t), \frac{dc}{dt}(t) \right) \in (TM_2 \cap N)$$

for all $t \in I$. This process can be continued as long as the set

$$M_i = p_1(N \cap TM_{i-1})$$

is a differentiable manifold and may be stopped if $M_{i+1} = M_i$.

This observation leads us in a natural way to the following definition.

Definition 5. Let N be the corresponding set of a given DAE (E, A, f). Then we define a family $(M_i)_{i=0,\dots,s}$ of manifolds M_i by the following recursion:

(i) $M_0 = E$,
(ii) $M_{i+1} = p_1(N \cap TM_i)$ $(i = 0, \dots, s-1)$,

where s is the largest integer such that the sets M_i are differentiable manifolds and $M_{s-1} \neq M_s$. In case $M_1 = E$, we define $s = 0$.

We call the family (M_i) the *family of constraint manifolds* and the integer s the *degree* of the DAE.

Remark. From (ii) in Definition 5, we conclude $\dim(M_{i+1}) \leq \dim(M_i)$ for all $i = 0, \dots, s-1$. If $\dim(M_{i+1}) = \dim(M_i) > 0$, we have

$$TM_i = TM_{i+1} \cup \left(\bigcup_{x \in M_i/M_{i+1}} T_x M_i \right)$$

and consequently

$$N \cap TM_i = N \cap TM_{i+1}.$$

This allows the important conclusion that the degree s of an arbitrary DAE (E, A, f) satisfies $s \leq \dim(E)$.

Remark. In [7] Rheinboldt describes a procedure for obtaining a family of manifolds as the solutions of an overdetermined system of equations. We can show that this family is identical with our family of constraint manifolds. Therefore, our definition can be considered as a geometrical interpretation of the procedure given in [7].

By means of the family of constraint manifolds, we can state sufficient conditions for the regularity of a DAE.

Theorem 1. *Let N be the corresponding set, let (M_i) be the family of constraint manifolds, and let s be the degree of a DAE (E, A, f). Then this DAE is regular if the condition*

C1: *$N \cap T_x M_s$ contains exactly one element for each $x \in M_s$*

is satisfied.

Under this condition, the configuration space of the DAE (E, A, f) is given by $M = M_s$, *and the corresponding vector field* $v: M \to TM$ *is, for all* $x \in M$, *defined by*

$$\{v(x)\} = N \cap T_x M_s.$$

Proof. Condition C1 ensures that the mapping $v: M_s \to TM_s$ defined by

$$\{v(x)\} = N \cap T_x M_s.$$

for all $x \in M_s$ is a vector field on M_s. Furthermore, from Observation 1, it is easily derived that a differentiable mapping $w: I \to M_s$ is a solution of the vector field v if and only if the mapping $c: = j \circ w: I \to E$ is a solution of the DAE (E, A, f). □

Clearly, Theorem 1 allows the development of a solution theory for regular DAEs by "translating" related results of vector fields to regular DAEs. To do this is beyond the scope of this paper, and we refer instead to [15].

In several papers [4], [9], [10] the properties of DAEs are characterized by an integer called the index. We want to state here a definition of an index of a DAE as well and show in Section 5 that our definition is in essence equivalent to the one given in [4], [9], and [10].

Definition 6. Let N be the corresponding set, let (M_i) be the family of constraint manifolds, and let s be the degree of a DAE (E, A, f). Then this DAE is called a DAE of *index s* if it satisfies Condition C1 of Theorem 1.

Remark. We like to emphasize that any DAE, for which an index can be defined, is a regular DAE according to Theorem 1.

It should be mentioned that the index of a DAE is invariant under coordinate transformations and scalings of the DAE. Once again, this may be of interest in practical applications and in the development of a theory for normal forms.

Proposition 2. *Let* (E, A, f) *and* $(E, A^{\#}, f^{\#})$ *be two equivalent DAEs. Then* (E, A, f) *is a DAE of index i if and only if* $(E, A^{\#}, f^{\#})$ *is a DAE of index i too.*

Proof. Let $u: E \to E$ be a coordinate transformation according to Definition 2. Then, for the corresponding sets of both DAEs, we have

$$N = (u, Du)(N^{\#})$$

and, for the constraint manifolds,

$$M_i = u(M_i^{\#}) \qquad (i = 0, \ldots, s).$$

Because, u is a diffeomorphism, it follows that the DAE (E, A, f) is a DAE of index i if and only if the DAE $(E, A^{\#}, f^{\#})$ has index i too. □

At the end of this section we want to illustrate the results obtained so far by the problem of path-tracing.

Example 2. Let us consider the DAE (E, A, f) of Example 1. Clearly, according to Definition 5 the set M_1 is given by

$$\begin{aligned} M_1 &= \{(x_1, x_2, y) \in \mathbb{R}^3: g(x_1, x_2) = 0\} \\ &= S \times \mathbb{R}. \end{aligned}$$

Clearly, M_1 is a differentiable manifold and the tangent bundle TM_1 can be identified with the solutions of the following nonlinear system of equations:

$$\begin{aligned} g(x_1, x_2) &= 0, \\ Dg(x_1, x_2)(x_1', x_2')^T &= 0. \end{aligned}$$

In turn, this yields that $N \cap TM_1$ is given by the solutions of the nonlinear system of equations:

$$\begin{aligned} g(x_1, x_2) &= 0, \\ Dg(x_1, x_2)(x_1', x_2')^T &= 0, \\ x_1' &= -b/[a^2 + b^2]^{1/2} + ay, \\ x_2' &= a/[a^2 + b^2]^{1/2} + by, \\ 0 &= g(x_1, x_2), \end{aligned}$$

where the last three equations are the DAE of Example 1 with the abbreviations used there. Elimination of x_1' and x_2' in the above nonlinear system of equations yields

$$\begin{aligned} M_2 &= \{(x_1, x_2, y) \in \mathbb{R}^3: g(x_1, x_2) = 0 \text{ and } y = 0\} \\ &= S. \end{aligned}$$

Clearly, the set $N \cap T_z M_2$ contains exactly one element for each $z \in M_2$. This element $z' = (x_1', x_2', y')$ is given by

$$\begin{aligned} x_1' &= -b/[a^2 + b^2]^{1/2}, \\ x_2' &= a/[a^2 + b^2]^{1/2}, \\ y' &= 0. \end{aligned}$$

Consequently, the DAE satisfies Condition C1 of Theorem 1 with $s = 2$, and, therefore, the DAE (E, A, f) is a DAE of index two. The configuration space is given by $M = M_2 = S$, and the corresponding vector field is equivalent to the vector field v of Example 1.

5. Reduction of higher-index DAEs

Within the framework of regular DAEs, the DAEs of index one are especially well understood. For such DAEs, the configuration space is identical with the constraint manifold M_1 and can therefore be easily determined. Moreover, the properties and the methods for the numerical treatment of DAEs of index one are well investigated [8].

But while, in the past, DAEs of index one were in the centre of attention, in the future, regular DAEs, which are not of index one, may take the focus of interest. For these DAEs, the configuration space is identical with some constraint manifold M_i with $i > 1$. Therefore they are called algebraic incomplete in [7].

Our approach here is to characterize algebraic incomplete DAEs geometrically by a reduction proposed in [16] for the first time. The idea behind this reduction method is to pass over from a given DAE of index i ($i > 1$) to a new one of index $i - 1$ such that the solutions of both DAEs are identical. Consequently, during the process of reduction, no additional constant of integration is introduced. Clearly, by means of this reduction method, any higher-index DAE can be reduced to a DAE of index one. This enables us to investigate a much more simple DAE of index one instead of a higher-index DAE.

Remark. Another reduction method is stated in [10] for the reduction of higher-index problems to lower ones that can be solved while not introducing any additional constant of integration. But in general, by this method, we can reduce a given higher-index DAE only to a DAE of index two.

Before we state the reduction method, we make the following observation.

Observation 2. Let (E, A, f) be a DAE, let (M_i) be the family of constraint manifolds, let N be the corresponding set, and let s be the degree of this DAE. By definition, we have

$$M_1 = \{z \in E\colon \text{there is a } z' \in E \text{ such that } A(z)z' = f(z)\}$$

and

$$M_2 = \{z \in E\colon \text{there is a } z' \in T_zM_1 \text{ such that } A(z)z' = f(z)\}.$$

This is equivalent to

$$M_1 = \{z \in E\colon f(z) \in \mathrm{im}(A(z))\}$$

and

$$M_2 = \{z \in E\colon f(z) \in A(z)T_zM_1\},$$

respectively. Now let $Q\colon E \to L(E, E)$ be an arbitrary mapping such that, for all $z \in M_1$, $Q(z)$ is a projection onto T_zM_1. Then this yields

$$M_2 = \{z \in E\colon f(z) \in \operatorname{im}(A(z)Q(z))\}.$$

This leads us to the investigation of a new DAE $(E^{\#}, A^{\#}, f^{\#})$ with

(i) $E^{\#} = E$,
(ii) $A^{\#} = A \circ Q$,
(iii) $f^{\#} = f$,

where the composition is to be interpreted like $A \circ Q(z) = A(z)Q(z)$ for all $z \in E$.

Let $(M_i^{\#})$ be the family of constraint manifolds, let $N^{\#}$ be the corresponding set, and let $s^{\#}$ be the degree of this new DAE. Clearly, we have

$$M_1^{\#} = M_2.$$

From this, we conclude

$$N^{\#} \cap TM_1^{\#} = N \cap TM_2$$

and therefore

$$s^{\#} = s - 1,$$
$$M_i^{\#} = M_{i+1} \qquad (i = 1, \ldots, s-1).$$

This observation leads us in a natural way to the definition of the reduction of a DAE of index i $(i > 1)$.

Definition 7. Let (E, A, f) be a DAE of index i with $i > 1$ and let (M_i) be the family of constraint manifolds of this DAE. Furthermore, let $Q\colon E \to L(E, E)$ be an arbitrary mapping such that $Q(z)$ is a projection onto T_zM_1 for all $z \in M_1$. Then we call the DAE $(E, A^{\#}, f)$ with $A^{\#} = A \circ Q$ the *reduced DAE* of the DAE (E, A, f). By a *reduction*, we mean the transition from the DAE (E, A, f) to the DAE $(E, A^{\#}, f)$.

We now formulate the main result of this section.

Theorem 2. *Let (E, A, f) be a DAE of index i with $i > 1$ and let $(E, A^{\#}, f)$ be a reduced DAE of this DAE. Then the reduced DAE is a DAE of index $i - 1$ and the corresponding vector fields of both DAEs are identical.*

Proof. This follows immediately from Observation 2, the definition of an index, and the definition of the corresponding vector field of a DAE. □

By using results of [17], the proposed reduction method also allows comparison of the definition of an index in the sense of Gear and Petzold [4] with our definition.

Proposition 3. *Let (E, A, f) be a DAE of index i in the sense of Gear and Petzold* [4]. *Then this DAE is a DAE of index i in the sense of Definition* 6.

Proof. In [17] it is shown that any DAE of index i $(i > 1)$ in the sense of Gear and Petzold can be reduced $(i - 1)$ times by the reduction given in Definition 7. Furthermore, it follows from Definitions 6 and 7 that a DAE (E, A, f) has the index i in our sense if and only if $(i - 1)$ times reductions of the DAE (E, A, f) yield a DAE of index one. □

Finally, we illustrate our reduction method with the DAE of Example 1.

Example 3. The DAE (E, A, f) of Example 1 has the following constraint manifold M_1:

$$M_1 = \{(x_1, x_2, y) \in \mathbb{R}^3 \colon g(x_1, x_2) = 0\}.$$

Consequently, the mapping $Q\colon E \to L(E, E)$ in Definition 7 can be chosen such that

$$Q(x_1, x_2, y) = [a^2 + b^2]^{-1} \begin{bmatrix} b^2 & -ab & 0 \\ -ab & a^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for all $(x_1, x_2, y) \in \mathbb{R}^3$, where we use the abbreviations given in Example 1.

Thus we get the reduced DAE $(E, A^{\#}, f)$ with

$$A^{\#}(z) = A(z)Q(z) = [a^2 + b^2]^{-1} \begin{bmatrix} b^2 & -ab & 0 \\ -ab & a^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for all $(x_1, x_2, y) \in \mathbb{R}^3$. The constraint manifold $M_1^{\#}$ of this DAE is given by

$$M_1^{\#} = \{(x_1, x_2, y) \in \mathbb{R}^3 \colon g(x_1, x_2) = 0 \text{ and } y = 0\}.$$

Because the DAE (E, A, f) is a DAE of index two, the DAE $(E, A^{\#}, f)$ is a DAE of index one, and the configuration space M of both DAEs is identical with the manifold $M_1^{\#} = S$.

6. Realization of vector fields by regular DAEs

In the previous sections we investigated conditions under which a given DAE is a regular one. In this section we now address the question under what conditions a regular DAE corresponds to a given vector field. Therefore, we introduce the notion of realizability of a vector field.

Definition 8. A regular DAE (E, A, f) is called a *realization* of a given vector field $v{:}M \to TM$ if the vector field is the corresponding vector field to the DAE (E, A, f).

We make the following observation.

Observation 3. Let $v{:}\ M \to TM$ be an arbitrary vector field on a differentiable manifold M. In the following we construct a DAE of index two, which realizes this given vector field v.

By a theorem of Whitney (see, for example, [13]), any k-dimensional manifold can be embedded in an n-dimensional Euclidean space $\mathbb{R}^n$ with $n > 2k$. Therefore, let us assume that the manifold M has been embedded in an $\mathbb{R}^n$. Then there exists a mapping $g{:}\ \mathbb{R}^n \to \mathbb{R}^{n-k}$ such that

$$M = \{x \in \mathbb{R}^n{:}\ g(x) = 0\}$$

and a mapping $h{:}\ \mathbb{R}^n \to \mathbb{R}^n$ such that, for all $x \in M$,

$$h(x) = v(x).$$

Using an idea of Gear [12], we now consider the DAE

$$\begin{aligned} x' &= h(x) + R(x)y, \\ 0 &= g(x), \end{aligned}$$

where $R{:}\ \mathbb{R}^n \to L(\mathbb{R}^{n-k}, \mathbb{R}^n)$ is a mapping with (i) $\operatorname{rank}(R(x)) = n - k$ and (ii) $\operatorname{im}(R(x)) \oplus T_xM = \mathbb{R}^n$ for all $x \in M$. This DAE is a DAE of index two because

$$M_1 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{n-k}{:}\ g(x) = 0\},$$

$$M_2 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{n-k}{:}\ g(x) = 0 \text{ and } y = 0\},$$

and

$$N \cap T_zM_2 = \{(x', y') \in \mathbb{R}^n \times \mathbb{R}^{n-k}{:}\ x' = h(x) \text{ and } y' = 0\}$$

for all $z = (x, y) \in M_2$. In addition, the corresponding vector field of this DAE is identical with the vector field v. Thus, this DAE is a realization of the vector field v.

Remark. The mapping g in the above observation can, for example, be defined by

$$g(x) = \begin{cases} 0 & \text{for } \ x \in M, \\ 1 & \text{for } \ x \notin M. \end{cases}$$

But, in general, g can be chosen to be differentiable. Furthermore, if $Dg(x)$ has full rank for all $x \in M$, then the mapping R can be given by

$$R(x) = Dg(x)^T$$

for all $x \in \mathbb{R}^n$.

With Observation 3 in mind, we are able to state the main result of this section.

Theorem 3. *Any vector field can be realized by a DAE of index i for any $i > 0$.*

Proof. The DAE of index two in Observation 3 can be constructed for an arbitrary vector field v, and can be reduced to a DAE of index one according to the results of Section 5. At the same time, we can add to this DAE of index one a higher-index DAE, which posses only the trival solution $c(t) = 0$. Consequently, we have shown that any vector field can be realized by a DAE of index i for any $i > 0$. The analysis of many practical problems, like the analysis of electronic circuits and mechanical systems, can lead to semi-explicit DAEs [5], [8]. Consequently, here we investigate this class of DAEs in more detail. □

Definition 9. A DAE (E, A, f) is called a *semiexplicit* DAE if $A(z) = A_0$ with $A_0 \in L(E, E)$ for all $z \in E$.

Proposition 4. *Let M be a differentiable manifold. Then an arbitrary vector field $v\colon M \to TM$ on M can be realized by a semiexplicit DAE of index one if and only if there exists a coordinate splitting $E\colon = E_1 \times E_2$, with the projection $p_1\colon E \to E_1$, such that the restriction of p_1 to M is a diffeomorphism.*

Proof. We first show the conditions of Proposition 4 to be sufficient. Clearly, if the conditions of Proposition 4 are satisfied, we have a mapping $d := p_2 \circ j \circ (p_1|M)^{-1}$ where $j\colon M \to E$ is the natural injection and $p_2\colon E \to E_2$ is a projection. Therefore, the manifold M can be described by

$$M = \{(x, y) \in E_1 \times E_2\colon y = d(x)\}.$$

Furthermore, let $h\colon E \to E_1$ be an arbitrary mapping such that

$$h(z) = p_1 \circ v(z)$$

for all $z \in M$. We now consider the DAE

$$\begin{aligned} x' &= h(x, y), \\ 0 &= y - d(x). \end{aligned}$$

Clearly, this DAE is a semiexplicit DAE of index one, and the corresponding vector field is identical with the vector field v.

It remains to show that the conditions of Proposition 4 are necessary too. According to Definition 6, for any semiexplicit DAE of index one

$$\{v(z)\} = \{z' \in \mathbb{R}^n\colon z' \in T_zM \text{ and } A_0 z' = f(z)\}$$

must hold for all $z \in M$. Thus, a necessary condition for us to be able to find a semiexplicit DAE of index one, which satisfies the above equality for an arbitrary vector field v, is the existence of a mapping $A_0 \in L(E, E)$ such that

$$z' \in T_z M,$$
$$z' \in \ker(A_0)$$

has a unique solution for each $z \in M$. Now, such a mapping A_0 can be found for a given manifold M if and only if the conditions of Proposition 4 are satisfied. □

Remark. A necessary condition, for the existence of a coordinate splitting with the properties given in Proposition 4, is the existence of an atlas for the manifold M, which consists of only one chart (see [13]). Some manifolds which do not satisfy this property are the torus, the circle, and the n-dimensional sphere.

Remark. Clearly, any vector field v can be realized by a semiexplicit DAE of index two. This becomes obvious from Observation 3.

The surprising fact about semiexplicit DAEs is that not all vector fields can be realized by a semiexplicit DAE of index one. We illustrate this again by the example of path-tracing.

Example 4. Let us assume that the path S of Example 1 can be described by

$$M = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 = d(x_1)\}.$$

Clearly, there exists a semiexplicit DAE of index one for the computation of this path. Consider, for example, the semiexplicit DAE

$$x_1' = 1,$$
$$0 = x_2 - d(x_1).$$

In general, the path S cannot be described by an explicit function, but only by an implicit one [11]. Then, according to the last proposition, there is, in general, no semiexplicit DAE of index one for the computation of the path M and we have to use other DAEs like, for example, the DAE of Example 1. This DAE is a realization of the vector field v given in Example 1 by the method described in Observation 3. The mapping h of Observation 3 is in this case given by

$$h(z) = \begin{bmatrix} -b/[a^2 + b^2]^{1/2} \\ a/[a^2 + b^2]^{1/2} \end{bmatrix}$$

for all $z \in E$.

7. Concluding remarks

This paper has described a number of theoretical results concerning the relation of DAEs to vector fields. We have found that both regular DAEs and vector fields can be used for modeling lumped physical systems. While the vector-field approach allows global analysis, regular DAEs are needed for computational analysis. Accordingly, it seems that both mathematical concepts are very important for a complete understanding of the behavior of lumped physical systems. Further studies about the relation of DAEs and vector fields on special fields, such as bifurcation and stability theory, are still needed. We hope that the twofold way of modeling lumped physical systems turns out to be useful.

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