Sebastian Reich

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Differential-Algebraic Equations and Applications in Circuit Theory

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Differential-Algebraic Equations and Applications in Circuit Theory
Technical and physical systems, especially electronic circuits, are frequently modeled as a system of differential and nonlinear implicit equations. In the literature such systems of equations are called differential-algebraic equations (DAEs). It turns out that the numerical and analytical properties of a DAE depend on an integer called the index of the problem. For example, the well-known BDF method of Gear can be applied, in general, to a DAE only if the index does not exceed one. In this paper we give a geometric interpretation of higher-index DAEs and indicate problems arising in connection with such DAEs by means of several examples.

Algebrodifferentialgleichungen in der Netzwerktheorie
Die mathematische Modellierung technisch physikalischer Systeme wie elektrische Netzwerke, führt häufig auf ein System von Differentialgleichungen und nichtlinearen impliziten Gleichungen sogenannten Algebrodifferentialgleichungen (ADGL). Es zeigt sich, daß die numerischen und analytischen Eigenschaften von ADGL durch den Index des Problems charakterisiert werden können. Insbesondere können die bekannten Integrationsformeln von Gear im allgemeinen nur auf ADGL mit dem Index eins angewendet werden. In diesem Beitrag wird eine geometrische Interpretation von ADGL mit einem höheren Index gegeben sowie auf Probleme im Zusammenhang mit derartigen ADGL an Hand verschiedener Beispiele hingewiesen.

1. Introduction

Many physical and technical systems are described mathematically by *differential-algebraic equations* (DAEs) of the type

$$x' = f(x,y), \quad 0 = h(x,y) \qquad (1)$$

where $f : X \times Y \to \mathbf{R}^q$, $h : X \times Y \to \mathbf{R}^p$ are sufficiently smooth mappings, and $X \subseteq \mathbf{R}^q$, $Y \subseteq \mathbf{R}^p$ are open sets in $\mathbf{R}^q$, $\mathbf{R}^p$ respectively. The analysis of electronic circuits by means of the most familiar and easily programmed (*modified*) *nodal analysis* [1] leads to a system of equations of type (1).

Frequently (see, e.g. [2]) a DAE (1) is considered as the limiting system of a *stiff* (or *singularly perturbed*) *ordinary differential equation* (ODE) of type

$$x' = f(x,y), \quad \epsilon\, y' = h(x,y)$$

where $\epsilon \in \mathbf{R}$ is a small parameter. This approach has been proved to be extremely useful whenever the partial derivative of the mapping h with respect to the second variable is nonsingular; i.e. $\det[\mathrm{D}_2 h(x,y)] \neq 0$, for all $(x,y) \in X \times Y$. Especially, as shown by Gear in [3], numerical methods for stiff ODEs can be applied to DAEs under this assumption.

This result marked a break-through for the numerical transient analysis of large-scale electronic circuits. While beforehand complicated algorithms [1] had to be used to formulate the circuit equations in *state space form* [1], the circuit equations obtained by means of nodal analysis could now be integrated directly [4]. Based on this approach, codes like SPICE and NAP2 were developed for the transient analysis of large-scale integrated circuits.

However, as reported recently in several publications (see, e.g., [5] and [6]), this approach to the numerical and analytical analysis of DAEs is not suitable in general. Instead it could be shown by means of differential-geometric techniques [7], [8], that rather general DAEs induce ODEs *on* suitable *submanifolds* of $X \times Y$. Indeed, already in [9] Desoer and Wu could show that the circuit equations of a nonlinear RLC circuit can be considered, under certain conditions, as an implicit description of an ODE on a submanifold of the space of branch voltages and branch currents. Thus, following these results, we have introduced in [8] the notion of a *regular* DAE as a DAE to which uniquely corresponds an ODE on a manifold. This ODE is then called the *corresponding* ODE of the problem. Furthermore, it could be shown (see, e.g., [5] and [6]) that the numerical properties of a regular DAE depend on an integer called the index of the DAE. For example, a regular DAE (1) is of index one if $\mathrm{D}_2 h(x,y)$ remains nonsingular for all $(x,y) \in X \times Y$ and is of higher index otherwise. Problems in circuit theory, control theory, and mechanical engineering, which give rise to higher index formulations, are discussed in Section 3. Now, as shown, e.g., in [5] and [6], DAEs can be solved by stiff ODE methods in general only if the index does not exceed one. To make higher index problems numerically tractable, recent investigations have focused on numerical methods for ODEs on manifolds (see, e.g., [10], [11], and [6]).

In this paper we summarize the differential-geometric approach to higher index DAEs as developed in [8] and [12]. We state necessary conditions for the regularity of a DAE which allow also for computation of the corresponding ODE of a regular DAE. Throughout this paper we illustrate our approach by

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Dr.-Ing. S. Reich, Karl-Weierstrass-Institute of Mathematics, Mohrenstrasse 39, O-1086, Germany.

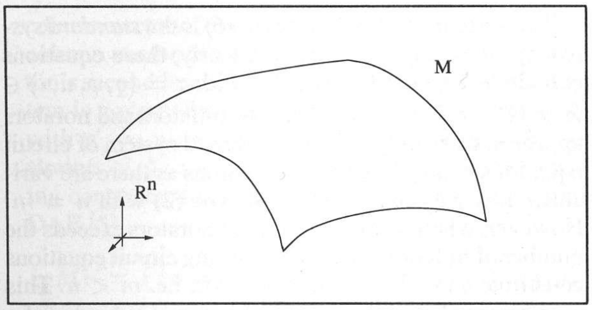

Fig. 1. Illustration of a differentiable submanifold of $\mathbf{R}^n$.

means of several examples. Finally we suggest a generalization of DAEs of type (1) with $X = \mathbf{R}^q$ and $Y = \mathbf{R}^p$ to DAEs where X and Y are arbitrary Banach spaces [13]. It is shown that Maxwell's field equations and the semiconductor device equations can be considered as special cases of such generalized DAEs.

2. Mathematical Background

In this paper we restrict our study to quasilinear and time-invariant DAEs of type

$$A(z)z' = g(z) \tag{2}$$

where $g : Z \to \mathbf{R}^m, A : Z \to L(\mathbf{R}^n, \mathbf{R}^m)$ are sufficiently smooth mappings and $Z \subseteq \mathbf{R}^n$ is an open set in $\mathbf{R}^n$. If $m > n$ ($m < n$), then the DAE (2) is called an *overdetermined* (*underdetermined*) DAE. The restriction to time-invariant DAEs simplifies the results significantly and is motivated by the fact that one can easily generalize the results obtained in this paper to time-varying DAEs [12].

By a *solution* of a DAE (2) we mean a differentiable mapping $z : I \to Z$ (where I denotes an open interval in the real line with $0 \in I$) such that, for all $t \in I$, $A(z(t))z'(t) = g(z(t))$.

We summarize now a few concepts from differential geometry. For more detailed results we refer the reader to standard texts like [13] and [14]. Let Z be an open set in $\mathbf{R}^n$ and let $f : Z \subseteq \mathbf{R}^n \to \mathbf{R}^m, n > m$, be a differentiable mapping with $\operatorname{rank}[\mathrm{D}f(z)] = m$ for all $z \in Z$. Then the set M, defined by

$$M := \{z \in Z : f(x) = 0\},$$

has the structure of a differentiable *submanifold* of Z (Fig. 1). Let $\bar{M}$ denote the closure of the set M in $\mathbf{R}^n$. Then the differentiable submanifold M, as defined above, satisfies $\bar{M} \cap Z = M$. For each $z \in M$, let $T_z M$ denote the set of all tangent vectors of M at z, called the *tangent space* at z (Fig 2). Now the tangent space $T_z M$ can be associated in a natural way (Fig. 2) with a linear subspace of $\mathbf{R}^n$ by means of an embedding $j_z : T_z M \to \mathbf{R}^n$; i.e.

$$j_z(T_z M) = \{\hat{z} \in \mathbf{R}^n : \mathrm{D}f(z)\hat{z} = 0\}.$$

Let TM denote the disjoint union of all $T_z M$ with $z \in M$, called the *tangent bundle* of M. Again, the tangent bundle TM can be associated in a natural way with a submanifold of $Z \times \mathbf{R}^n$ by means of an embedding $j : TM \to Z \times \mathbf{R}^n$; i.e.

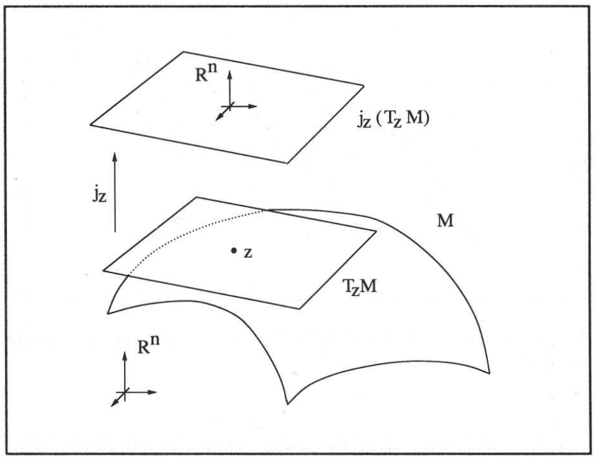

Fig. 2. For each point z on a differentiable submanifold M of $\mathbf{R}^n$, the plane tangent to M at z is denoted by $T_z M$. By means of the embedding j_z, the tangent plane $T_z M$ can be associated with a linear subspace of $\mathbf{R}^n$.

$$j(TM) = \{(z, \hat{z}) \in Z \times \mathbf{R}^n : \\ 0 = f(z)\,,\, 0 = \mathrm{D}f(z)\hat{z}\}.$$

In the sequel we will not distinguish between the sets $T_z M$ and $j_z(T_z M)$ (TM and $j(TM)$, respectively) and denote both sets simply by $T_z M$ (TM, respectively).

By a *vector field* on a differentiable submanifold M of Z we mean a mapping $v : M \to \mathbf{R}^n$ such that $v(z) \in T_z M$ for all $z \in M$ (Fig. 3). With any vector field v on M we associate an ODE on the *manifold* M by means of

$$z' = v(z), \qquad z \in M. \tag{3}$$

By a *solution* of an ODE (3) on M we mean a differentiable mapping $z : I \to Z$ such that $z(t) \in M$ and $z'(t) = v(z(t))$ for all $t \in I$. Now let $\Omega \subseteq \mathbf{R} \times M$ be an open set in $\mathbf{R} \times M$, then a differentiable mapping $g : \Omega \to Z$ is called a *flow* on M if the mapping g satisfies $g(0, x) = x$, $x \in M$, and $g(t + s, x) = g(t, g(s, x))$ for all $(s, x) \in \Omega$ and all $t \in \mathbf{R}$ with $(t, g(s, x)) \in \Omega$. Flows are associated to ODEs on manifolds in the following way [14]: If $v : M \to Z$ defines an ODE on M and if the mapping v is differentiable, then there is a unique open set $\Omega \subseteq \mathbf{R} \times M$ and a unique flow $g : \Omega \to Z$ such that a differentiable mapping $z : I \to Z$ is a solution of the ODE if and only if $I \times \{z(0)\} \subseteq \Omega$ and $z(t) = g(t, z(0))$ for all $t \in I$.

Finally, we restate the notion of regularity for a DAE (2) as introduced in [8] and [12]. A DAE (2) is called a *regular* DAE if there is a unique differentiable submanifold M of Z and a unique ODE (3) on M such that a differentiable mapping $z : I \to Z$ is a solution of the

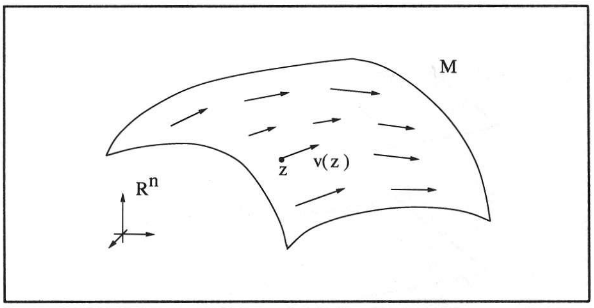

Fig. 3. Illustration of a vector field v on a differentiable submanifold M.

DAE if and only if z is a solution of the ODE (3). The manifold M is then called the *configuration space* and the ODE (3) the *corresponding* ODE of the DAE. Let $\bar{M}$ denote the closure of the configuration space M in $\mathbf{R}^n$. Then we call a regular DAE *strongly regular* if its configuration space satisfies $\bar{M} \cap Z = M$. In Section 4 we will formulate sufficient conditions for the regularity of a DAE (2). In this section an example of a DAE, which is regular but is not strongly regular, will be discussed as well.

3. Examples

In this section we display some of the variety of DAEs encountered in applications, and hopefully justify the study of general DAEs of type (2).

Example 1. We consider a nonlinear autonomous RLC circuit $\mathcal{N}$ which, for simplicity, is assumed to have a connected graph with b branches. Electrical coupling among branches of the same kind is allowed; thus controlled sources are viewed as coupled resistors. Let $v \in \mathbf{R}^b$ denote the column vector of branch voltages and let $i \in \mathbf{R}^b$ denote the column vector of branch currents, partitioned as (v_c, v_r, v_l) and (i_c, i_r, i_l) respectively, where subscript c (r, l, respectively) denotes the variable corresponding to capacitors (resistors, inductors, respectively). Let us assume that the circuit $\mathcal{N}$ consists of n_r resistors, n_c capacitors, and n_l inductors and that nullators and norators are considered, for notational simplicity, as resistors.

Given any tree of $\mathcal{N}$, KVL is expressed by

$$0 = Bv, \qquad 0 = Qi \tag{4}$$

where B and Q are the corresponding fundamental loop and cutset matrices, respectively. Let us assume that the constitute relations of the resistors, capacitors, and inductors are given by the following system of equations

$$0 = f_r(u_r, i_r), \; 0 = f_c(u_c, q), \; 0 = f_l(i_l, \phi) \tag{5}$$

where the charge q, the current i_c, the flux ϕ, and the voltage u_l are related by

$$q' = i_c, \qquad \phi' = u_l. \tag{6}$$

The system of equations (4) to (6) is the *standard system of circuit equations* [15]. Clearly, these equations constitute a nonlinear DAE (2) with $z = (q, \phi, u, i) \in Z = \mathbf{R}^{n_c} \times \mathbf{R}^{n_l} \times \mathbf{R}^b \times \mathbf{R}^b$. If nullators and norators appear in pairs only, then the standard system of circuit equations contains as many equations as there are variables; i.e., we obtain a DAE of type (2) with $n = m$. However, whenever the number of norators exceeds the number of nullators, then the resulting circuit equations constitute an underdetermined DAE; i.e. $m < n$. This case occurs, for example, if the terminal behavior of a N-pole is computed by connecting norators to the N terminals of the N-pole in an appropriated way [16].

We call an RLC circuit $\mathcal{N}$ *well-posed* if the standard system of circuit equations constitutes a stronly regular DAE. The consequences of this definition will be discussed in Section 4 by means of several examples.

In several papers (see, e.g., [9], [17], and [15]) conditions have been stated so that the solution set of the nonlinear equations (4) and (5), denoted by M_1, has the structure of a differentiable submanifold of Z and that the differential equations (6) define a unique ODE on this submanifold M_1 (see, e.g., [9], [17], and [15]). Thus, under these conditions, the standard system of circuit equations (4) to (6) is a stronly regular DAE of index one and therefore the circuit $\mathcal{N}$ is well-posed. However, there are various examples of circuits (see, e.g., [15]) which are well-posed but for which the standard system of circuit equations is not a DAE of index one. For example, circuits containing capacitor loops and/or inductor cutsets yield index two formulations (compare Example 6 in Section 4). However, it is even possible to obtain examples of well-posed circuits containing operational amplifiers for which the standard system of circuit equations is of arbitrary high index (see, e.g., [5]). □

Example 2. In a control problem we usually have a differential equation of the form $x' = f(x, u)$, where $u \in \mathbf{R}$ represents a control, together with an output equation $y = h(x)$, where $y \in \mathbf{R}$ denotes the output. Now the problem of output tracking [5]; i.e., problems where a control u must be applied such that the output y satisfies some constraint $0 = g(y)$, is naturally cast as a DAE. Let us demonstrate this for the problem of zeroing the output; i.e. $y = 0$. This problem leads to the DAE formulation

$$x' = f(x, u), \quad 0 = h(x).$$

Now, if $\text{rank}[\text{D}h(x)\text{D}_2 f(x, u)] = 1$ for all $(x, y) \in \mathbf{R}^q \times \mathbf{R}$, then this DAE is strongly regular and of index two. Furthermore, the corresponding ODE is defined on the manifold M_2 which is given by

$$M_2 = \{(x, u) \in \mathbf{R}^k \times \mathbf{R}^l : \\ 0 = h(x), \, 0 = \text{D}h(x) f(x, u)\}.$$ □

Example 3. An interesting class of DAEs appears in modelling of constrained mechanical systems. Let us first consider a system of N free particles with mass matrix M on which a force F acts. Let $p \in \mathbf{R}^{3N}$ be the column vector of position coordinates and $v \in \mathbf{R}^{3N}$ the column vector of velocities. Then, according to Newton's second law, the motion of this unconstrained system is governed by the ODE $Mv' = F(v,p)$ together with $p' = v$. Suppose now that we have some constraints $g(p) = 0$ on our movement. Then the resulting constrained mechanical system is governed by the DAE [5]

$$Mv' = F(p,v) - \mathrm{D}g(p)\lambda, \quad p' = v, \quad 0 = g(p)$$

where λ denotes the Lagrange multipliers corresponding to the constraints $g(p) = 0$. It can be shown (see, e.g., [5]) that this DAE is a strongly regular DAE of index three. □

4. Regular DAEs

In this section we discuss DAEs of type (2) by means of a differential-geometric approach. Let us introduce the sets $N \in Z \times \mathbf{R}^n$ and $N_z \in \mathbf{R}^n$, $z \in Z$, by

$$N := \{(z,\hat{z}) \in Z \times \mathbf{R}^n : A(z)\hat{z} = g(z)\},$$
$$N_z := \{\hat{z} \in Z : A(z)\hat{z} = g(z)\}.$$

We call the set N the *corresponding set* of the DAE. Consider now the projection of this set onto the first component; i.e. the set

$$M_1 := \mathrm{pr}_1(N).$$

Obviously we have

$$N = \bigcup_{z \in M_1} \{z\} \times N_z.$$

The set M_1 is a subset of Z and reflects algebraic constraints on the solutions of (2); i.e., any solution $z : I \to Z$ of the DAE has to satisfy $z(t) \in M_1$. Let us assume that M_1 is a differentiable submanifold of Z. Clearly, under this assumption, a differentiable mapping $z : I \to Z$ is a solution of the given DAE if and only if, for all $t \in I$, $z(t) \in M_1$ and $z'(t) \in (N_z \cap T_z M_1)$. This is equivalent to $(z(t), z'(t)) \in (N \cap TM_1)$. In general, $(N \cap TM_1)$ will be a subset of N. Therefore, we consider the set

$$M_2 := \mathrm{pr}_1(N \cap TM_1).$$

This implies however that a point $z \in M_1$ is an element of M_2 if and only if the point z satisfies $(N_z \cap T_z M_1) \neq \emptyset$ (Fig. 4). If now the set M_2 is a differentiable submanifold of Z as well, we conclude again that a differentiable mapping $z : I \to \mathbf{R}^n$ is a solution of the DAE if and only if $(z(t), z'(t)) \in (N \cap TM_2)$ for all $t \in I$. This process can be continued as long as the set

$$M_i := \mathrm{pr}_1(N \cap TM_{i-1})$$

is a differentiable submanifold of Z and may be stopped whenever $M_i = M_{i-1}$. In accordance with

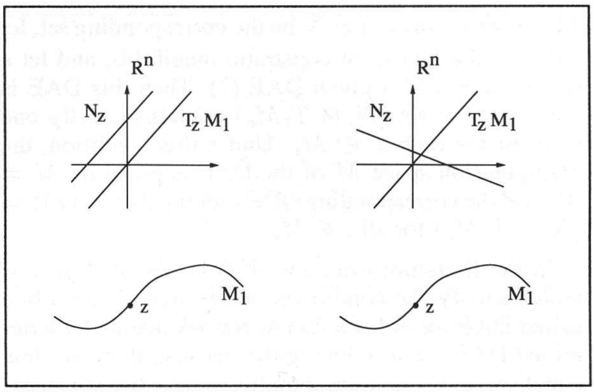

Fig. 4. Schematic representation of the set $N_z \cap T_z M_1$, $z \in M_1$. The picture on the left illustrates the case $(N_z \cap T_z M_1) = \emptyset$ and thus $z \notin M_2$. The picture on the right illustrates the opposite case.

this, we call the family $(M_i)_{i=0,\ldots,s}$ the *family of constraint manifolds* and the integer s the *degree of the DAE* where s is the largest integer such that the sets M_i are differentiable submanifolds of Z and $M_{s-1} \neq M_s$. In case $M_1 = \mathbf{R}^n$, we define $s = 0$.

The family (M_i) of constraint manifolds can be obtained directly from a DAE (2) of degree s in the following way: Let $R(z)$ be a projection along $\mathrm{im}[A(z)]$ for all $z \in Z$. Then the constraint manifold M_1 is given by the solutions $z \in Z$ of

$$Rg(z) := R(z)g(z) = 0.$$

Thus the set $N \cap TM_1$ is given by the solutions $(z,\hat{z}) \in Z \times \mathbf{R}^n$ of the following system of equations

$$A(z)\,\hat{z} = g(z), \quad \mathrm{D}(Rg)(z)\,\hat{z} = 0.$$

By substituting the variable $\hat{z}$ through z', this system of equations can be rewritten as an overdetermined DAE of type (2). This new DAE is of degree $\tilde{s} = s - 1$ and has the corresponding set $\tilde{N} = (N \cap TM_1)$. Thus the constraint manifold M_2 of the DAE (2) is now given by $M_2 = \tilde{M}_1$ where the constraint manifold $\tilde{M}_1$ is obtained from the overdetermined DAE, by means of an appropriated chosen projector $\tilde{R}(z)$, in the same way as the constraint manifold M_1 from the DAE (2). This process can be continued and one obtains successively the constraint manifolds M_i for $i = 3, \ldots, s$. In the literature (see, e.g., [5]) such recursive definitions of DAEs of lower degree are called *index transformations*.

In [8] we have derived another technique for computation of the constraint manifolds (M_i) by means of an index transformation which, in contrast to the above technique, does not increase the number of equations. For linear DAEs of type (2) this technique can be identified with the well-known algorithm of Dervisoglu and Desoer [18] which can be carried out by means of standard linear algebra (Gauss algorithm).

By means of the family of constraint manifolds, we can state sufficient conditions for the regularity of a DAE.

Theorem 1 ([8]). Let N be the corresponding set, let (M_i) be the family of constraint manifolds, and let s be the degree of a given DAE (2). Then this DAE is regular if the set $(N \cap T_z M_s)$ contains exactly one element for each $z \in M_s$. Under this condition, the configuration space M of the DAE is given by $M = M_s$ and the corresponding ODE is defined by $\{v(x)\} = (N \cap T_z M_s)$ for all $z \in M_s$.

In the literature (see, e.g., [5]) DAEs of degree s, which satisfy the conditions of Theorem 1, are often called DAEs of index s. Let us remark that underdetermined DAEs cannot be regular because there are less equations then variables. Finally, we like to state an existence and uniqueness result for regular DAEs.

Proposition 1 ([12]). Let as assume that a DAE (2) of degree s satisfies the conditions of Theorem 1 and that the set $(N \cap TM_s)$ is a differentiable submanifold of $Z \times \mathbf{R}^n$. Then there is a unique open set $\Omega \subseteq \mathbf{R} \times M_s$ and a unique flow $g : \Omega \to Z$ on M_s such that a differentiable mapping $z : I \to Z$ is a solution of the DAE if and only if $I \times \{z(0)\} \subseteq \Omega$ and $z(t) = g(t, z(t))$ for all $t \in I$.

Example 4. Let us rewrite the standard system of circuit equations (4) to (6) in the form

$$q' = i_c, \quad \phi' = u_l, \quad 0 = F(q, \phi, u, i) \qquad (7)$$

with $F : Z = \mathbf{R}^{n_c} \times \mathbf{R}^{n_l} \times \mathbf{R}^b \times \mathbf{R}^b \to \mathbf{R}^{2b}$. We assume that the mapping F is sufficiently smooth and that $\operatorname{rank}[\mathrm{D}F(q, \phi, u, i)] = 2b$ for all $(q, \phi, u, i) \in Z$. Then the constraint manifold M_1 is given by the solutions $(q, \phi, u, i) \in Z$ of

$$0 = F(q, \phi, u, i),$$

and the set $(N_z \cap T_z M_1)$ is characterized for all $z = (q, \phi, u, i) \in M_1$ by the solutions $\hat{z} = (\hat{q}, \hat{\phi}, \hat{u}, \hat{i}) \in Z$ of

$$\mathrm{D}F(q, \phi, u, i) \begin{pmatrix} \hat{q} \\ \hat{\phi} \\ \hat{u} \\ \hat{i} \end{pmatrix} = 0, \qquad \begin{matrix} \hat{q} = i_c \\ \hat{\phi} = u_l. \end{matrix} \qquad (8)$$

Now, if

$$\operatorname{rank}[\mathrm{D}_3 F(q, \phi, u, i) \mid \mathrm{D}_4 F(q, \phi, u, i)] = 2b \quad (9)$$

for all $(q, \phi, u, i) \in M_1$, then (8) has a unique solution and thus the set $(N_z \cap T_z M_1)$ contains exactly one element for all $z = (q, \phi, u, i) \in M_1$. Therefore, if (9) holds, then (7) is a strongly regular DAE of index one. □

Example 5. In this example we discuss circuits for which (9) fails for some but not all $(q, \phi, u, i) \in M_1$. Let us introduce the set

$$S := \{(q, \phi, u, i) \in M_1 : \\ \operatorname{rank}[\mathrm{D}_3 F(q, \phi, u, i) \mid \mathrm{D}_4 F(q, \phi, u, i)] < 2b\}.$$

We assume that the set S is a differentiable submanifold of M_1 with $\dim[S] < \dim[M_1]$ and that, for all $z = (q, \phi, u, i) \in S$, (8) has no solution; i.e. $(N_z \cap T_z M_1) = \emptyset$. These assumptions imply that the constraint manifold M_2 is given by $M_2 = M_1 \setminus S$ and that (7) is regular on M_2.

Now the constraint manifold M_2 satisfies $\dim[M_2] = \dim[M_1]$ and $\bar{M}_2 = M_1$ where $\bar{M}_2$ denotes the closure of M_2 in M_1. Thus the DAE (7) is regular but not strongly regular. Furthermore, there are solutions $z : (0, T) \to Z$, $T < \infty$, of (7) with $\lim_{t \to T-0} = z_T \in S$; i.e., the solution $z : (0, T) \to Z$ cannot be extended beyond the point $z_T \in S$. Therefore points $z_T \in S$ are called impasse points [15], [1]. Impasse points give rise to the so called jump behavior of circuits [1]. A simple example of this phenomenon is given by the van der Pol oscillator

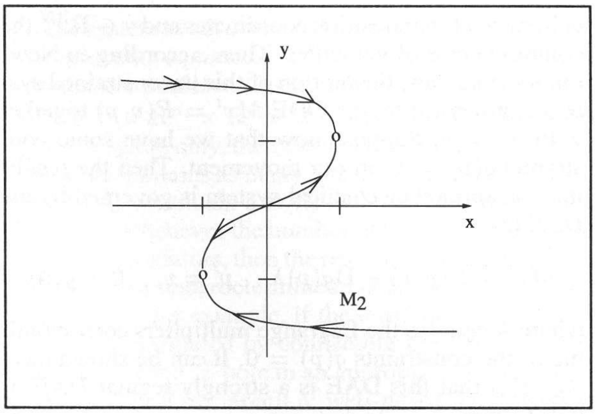

Fig. 5. Illustration of the corresponding vector field of the degenerate van der Pol oscillator.

$$x' = y, \quad \epsilon y' = 3y - y^3 - 2x$$

for $\epsilon = 0$ [1], [12]. For $\epsilon = 0$, the constraint manifold M_1 is given by the solutions $(x, y) \in \mathbf{R}^2$ of

$$0 = 3y - y^3 - 2x$$

and the constraint manifold M_2 is given by $M_2 = M_1 \setminus \{(1, 1), (-1, -1)\}$. Thus the two points $(1, 1)$ and $(-1, -1)$ are impasse points. The vector field $v : M_2 \to \mathbf{R}^2$ is depicted schematically in Fig. 5.

Let us finally remark that impasse points cannot occure for strongly regular DAEs. This fact also motivates our notion of a well-posed circuit. □

Example 6. Let us consider now the 3-pole of Fig. 6 which is given by a capacitor only loop consisting of the three capacitors C_1, C_2, and C_3. For analysing the terminal behavior of this 3-pole, we transform the 3-pole into the circuit of Fig. 6 by connecting two norators N_1 and N_2 to the terminals of the 3-pole. The standard system of circuit equations of the resulting circuit is given by (after eliminating the charges q_1, q_2, and q_3)

$$0 = u_{c_1} - u_{c_2} + u_{c_3}, \; 0 = i_{n_1} - i_{c_3} + i_{c_1}, \\ 0 = u_{c_1} - u_{n_1}, \; 0 = i_{n_2} + i_{c_2} + i_{c_3}, \\ 0 = u_{c_2} - u_{n_2}, \qquad (10)$$

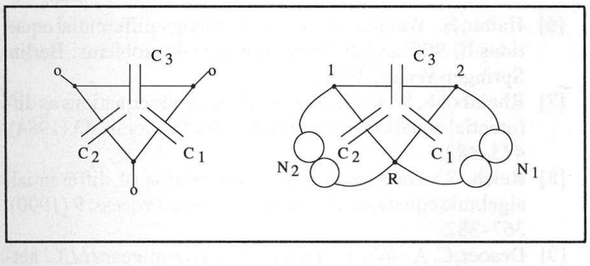

Fig. 6. 3-pole consisting of a capacitor loop and the resulting circuit for computation of its terminal behavior.

$$C_1 u'_{c_1} = i_{c_1},\ C_2 u'_{c_2} = i_{c_2},\ C_3 u'_{c_3} = i_{c_3}. \quad (11)$$

These eight equations constitute an underdetermined DAE for the ten unknown branch voltages and currents $(u, i) \in \mathbf{R}^{10}$. The constraint manifold M_1 of this DAE is given by the solutions (u, i) of the linear equations (10). Now differentiating the constraints (10) with respect to time yields, among others, the equation

$$0 = u'_{c_1} - u'_{c_2} + u'_{c_3}$$

which implies together with the differential equations (11) that the constraint manifold M_2 is given by

$$M_2 = \{(u, i) \in M_1 : i_{c_1}/C_1 - i_{c_2}/C_2 + i_{c_3}/C_3 = 0\}$$

Straightforward calculation yields now that the equations (10) to (11) constitute an underdetermined DAE of degree two (which, of course, is not regular). Furthermore, any well-posed network containing the 3-pole of Fig. 6 gives a standard system of circuit equations with index $s \geq 2$.

Let us now show that different techniques for analysing circuits can lead to DAEs of different degree. For that reason let us analyse the circuit of Fig. 6 by means of the *nodal analysis* [1]. If we chose the node R as the reference node, then we obtain the following underdetermined DAE:

$$\begin{pmatrix} C_1 + C_3 & -C_3 \\ -C_3 & C_2 + C_3 \end{pmatrix} \begin{pmatrix} u'_{c_1} \\ u'_{c_2} \end{pmatrix} + \begin{pmatrix} i_{n_1} \\ i_{n_2} \end{pmatrix} = 0.$$

This system of equations does not contain algebraic equations and thus the DAE is of degree zero with the constraint manifold $M_0 = \mathbf{R}^4$.

The fact that different techniques for analysis of circuits leads to DAE formulations of different degree (index) seems to be important for the numerical transient analysis. This is due to the fact that, in general, only DAEs of index one can be solved numerically by standard integration codes for stiff ODEs [5]. This example seems to imply that the *(modified) nodal analysis* [1] is more suited for the numerical transient analysis of circuits containing capacitor loops than the *sparse tableau approach* [1] which leads to the standard system of circuit equations. This corresponds to the fact that the modified nodal analysis is used in almost all current network analysis programs like SPICE and NAP2.

In several papers (see, e.g., [15]) a circuit transformation technique to eliminate capacitor loops has been described. For the 3-pole of Fig. 6 this results in the new 3-pole of Fig. 7. If the terminal behavior of this 3-pole is analysed in the same way as for the 3-pole of Fig. 6, then it is easy to show that the resulting standard system of circuit equations is now a DAE of degree one. Thus, this circuit transformation technique can be considered as an index transformation for the standard circuit eqs. (10), (11) in the sense of Section 3. □

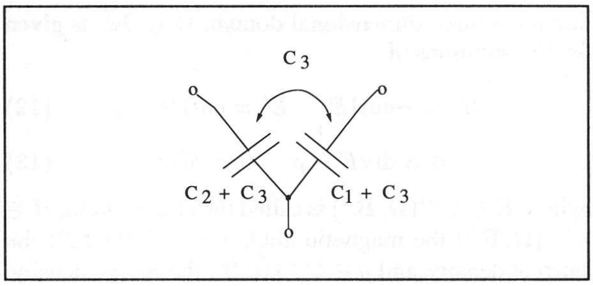

Fig. 7. Circuit transformation for the removal of capacitor loops.

5. Infinite Dimensional DAEs

In several books (see, e.g., [19]) the notion of a ODE on a submanifold of $\mathbf{R}^n$ has been generalized to ODEs on submanifolds of arbitrary finite or infinite dimensional Banach spaces [13]. This generalization has been proved to be very useful in analysis of parabolic partial differential equations (see, e.g., [19]). Here we like to generalize the concept of a DAE (2), as introduced in Section 2, to the case that the space Z is now an arbitrary finite or infinite dimensional Banach space. It can be shown that the results of Section 4 remain valid without further modifications. However it must be taken into consideration that the constraint manifolds M_i are no longer finite dimensional submanifolds of $Z = \mathbf{R}^n$, but submanifolds of (in general infinite dimensional) Banach spaces [13]. Furthermore, the corresponding ODE is now, in general, an infinite dimensional ODE on a infinite dimensional manifold. Finally we like to remark that Proposition 1 is very useful in any study of DAEs over finite dimensional Banach spaces but has only very rare application for DAEs over infinite dimensional Banach spaces. This is due to the fact that for those DAEs the set $(N \cap TM_s)$ is, in general, not a differentiable submanifold of $Z \times Z$.

We like to discuss our concept of a DAE over an infinite dimensional Banach space by means of the following example.

Example 7. Classical electromagnetism is governed by Maxwell's field equation. The form of these equations depends on the physical units chosen, and changing these units introduce factors like 4π, c (speed of light), ϵ_0 (dielectric constant), and μ_0 (magnetic permeability). This discussion assumes that ϵ_0, μ_0 are constants; the choice of units is such that the equations take the simplest form; thus $c = \epsilon_0 = \mu_0 = 1$ and factors 4π disappear. Therefore we can identify E with D, and B with H and the electromagnetic field in a smoothly bounded three dimensional domain $\Omega \subseteq \mathbf{R}^3$ is given by the solutions of

$$H' = -\mathrm{curl} E, \quad E' = \mathrm{curl} H - j, \qquad (12)$$

$$0 = \mathrm{div} E - \rho, \quad 0 = \mathrm{div} H \qquad (13)$$

where $E \in C^\infty(\bar{\Omega}, \mathbf{R}^3)$ is called the electric field, $H \in C^\infty(\bar{\Omega}, \mathbf{R}^3)$ the magnetic field, $j \in C^\infty(\bar{\Omega}, \mathbf{R}^3)$ the current density, and $\rho \in C^\infty(\bar{\Omega}, \mathbf{R})$ the charge density. The current density and the charge density satisfy the continuity equation

$$\rho' = -\mathrm{div}\, j. \qquad (14)$$

For simplicity, let us assume that we consider initial boundary value problems

$$\begin{array}{ll} E(0) = E_0, E_{|\partial\Omega}(t) = 0, & \\ H(0) = H_0, H_{|\partial\Omega}(t) = 0, & (t > 0) \\ \rho(0) = \rho_0, \rho_{|\partial\Omega}(t) = 0, & \end{array}$$

where $\bar{\Omega}$ denotes the closure of Ω in $\mathbf{R}^3$, $\partial\Omega$ the boundary of $\bar{\Omega}$, and subscript $|\partial\Omega$ the restriction of a function on $\bar{\Omega}$ to the boundary $\partial\Omega$. The differential equations (12) and (14) together with the constraints (13) constitute an overdetermined infinite dimensional DAE in the four unknowns E, H, j, and ρ. The space Z is given by the Banach space of smooth functions on $\bar{\Omega}$ which are identical zero on the boundary $\partial\Omega$ of $\bar{\Omega}$; i.e., $(E, H, j, \rho) \in Z = V \times V \times V \times W$ where we used the abbreviations $V := C_0^\infty(\bar{\Omega}, \mathbf{R}^3)$ and $W := C_0^\infty(\bar{\Omega}, \mathbf{R})$. A detailed discussion of this infinite dimensional DAE can be found in [20]. □

A similar example of this type are the systems of partial differential equations describing the flow of electrons and holes in a semiconductor [21].

Let us finally remark that, whenever an infinite dimensional DAEs is discretized with respect to the spatial variable by means of a finite difference method or by a finite element method as in the *method of lines* [22], then the resulting system of equation is a finite dimensional DAE.

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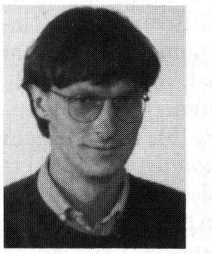

Sebastian Reich studied at the Technical University Dresden from 1982 until 1986, where he received his master degrees in electrical engineering and mathematics. He earned his Ph.D. in 1990. His research interests include system theory, nonlinear dynamical systems, and signal processing. He has held research positions at the Technical Universities of Prague and Budapest, and at the Karl-Weierstrass-Institute of Mathematics Berlin. Currently he is with the Simon Fraser University Burnaby where he is doing research in the field of numerical analysis.