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Traces of pseudo-differential operators on compact and Hausdorff groups

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Abstract We give a characterization of and a trace formula for trace class pseudo-differential operators on compact Hausdorff groups.

Keywords Pseudo-differential operators · Compact groups

Mathematics Subject Classification (2000) Primary 22A10 · 43A77 · 47G30; Secondary 22C05

1 Introduction

Pseudo-differential operators, first developed by Kohn and Nirenberg [8] in 1965 and then is used by Hörmander [7] and others for problems in partial differential equations. Weyl transforms which are a class of pseudo-differential operators have applications in Quantization due to Hermann Weyl [18]. In [21] Weyl transforms on compact Lie groups are introduced and the heat kernels of the Laplacian on the compact Lie group is obtained. In this paper, we look at pseudo-differential operators on compact and Hausdorff groups with L^2 symbols. The L^p, $1 \leq p \leq \infty$ conditions on the symbols allowing singularities are ideal for a broad spectrum of disciplines ranging from functional analysis to operator algebras to quantization. An analogue of the results in this paper is studied for the compact Lie group $\mathbb{S}^{n-1}$, i.e., the unit sphere

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with center at the origin in [5]. Pseudo-differential operators on the unit sphere are studied extensively in [1–11,13,14,16,17,22].

The aim of this paper is to give a characterization of trace class pseudo-differential operators on compact and Hausdorff groups. We give a formula for the trace of pseudo-differential operators in the trace class. The main technique is to obtain a formula for the symbol of the product of two pseudo-differential operators on a compact and Hausdorff group.

In Sect. 2, we give a brief recall of Hilbert-Schmidt and trace class operators. We define pseudo-differential operators on compact and Hausdorff groups by using the space of all unitary and irreducible representations in Sect. 3. A product formula for pseudo-differential operators is given. Then we give a characterization of trace class pseudo-differential operators.

2 Hilbert-Schmidt and trace class operators

Let $\mathcal{H}$ be a complex and separable Hilbert space in which the inner product and norm are denoted by $(,)_{\mathcal{H}}$ and $\| \|_{\mathcal{H}}$. Let $A : \mathcal{H} \to \mathcal{H}$ be a compact operator. Then the absolute value of A denoted by $|A|$ is defined by

$$|A| = (A^* A)^{1/2}.$$

The operator $|A|$ is compact and positive. So, by the spectral theorem, there exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of eigenvectors of $|A|$ with the corresponding eigenvalues $\{s_j\}_{j=1}^{\infty}$ of real numbers. The operator A is said to be in the Hilbert-Schmidt class S_2 if

$$\sum_{j=1}^{\infty} s_j^2 < \infty,$$

and we define its Hilbert-Schmidt norm by

$$\|A\|_{HS} = \left(\sum_{j=1}^{\infty} s_j^2 \right)^{1/2}.$$

The operator A is said to be in the trace class S_1 if

$$\sum_{j=1}^{\infty} s_j < \infty,$$

and we define its trace by

$$tr(A) = \sum_{j=1}^{\infty} s_j.$$

We have the following theorem which shows that the Hilbert-Schmidt norm and trace is independent of the choice of orthonormal basis for $\mathcal{H}$, for details see [12] by Reed and Simon.

Theorem 2.1 *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then*

- *$A \in S_2$ if and only if there exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for $\mathcal{H}$ such that*

$$\|A\|_{HS} = \left(\sum_{j=1}^{\infty} \|A\varphi_j\|_{\mathcal{H}}^2 \right)^{1/2} < \infty.$$

- *$A \in S_1$ if and only if there exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for $\mathcal{H}$ such that*

$$tr(A) = \sum_{j=1}^{\infty} (A\varphi_j, \varphi_j)_{\mathcal{H}} < \infty.$$

The following theorem will be useful in the next section.

Theorem 2.2 *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then $A \in S_1$ if and only if $A = BC$, where B and C are in S_2.*

3 Pseudo-differential operators on compact and Hausdorff groups

Let G be a locally compact and Hausdorff group. Let $\mathcal{H}$ be a separable and complex Hilbert space. We denote the group of all unitary operators by $U(\mathcal{H})$. A group homomorphism $\pi : G \to U(\mathcal{H})$ is said to be a unitary representation of G on $\mathcal{H}$ if it is strongly continuous, i.e., the mapping

$$g \mapsto \pi(g)h$$

is a continuous mapping for all $h \in \mathcal{H}$. The dimension of $\mathcal{H}$ is knows as the degree or the dimension of the representation π.

A closed subspace M of $\mathcal{H}$ is said to be invariant with respect to the unitary representation $\pi : G \to U(\mathcal{H})$ if

$$\pi(g)M \subseteq M, \quad \forall g \in G.$$

A unitary representation $\pi : G \to U(\mathcal{H})$ is called irreducible if it has only the trivial invariant subspaces, i.e., $\{0\}$ and $\mathcal{H}$. The following theorem is crucial in defining pseudo-differential operators on compact Hausdorff groups. For its proof see [6,20].

Theorem 3.1 *Let G be a compact and Hausdorff group. Then any irreducible and unitary representation of G on a complex and separable Hilbert space is finite dimensional.*

Let $\mathbb{G}$ be a compact and Hausdorff group with the left (and right) Haar measure is denoted by μ. Let $\hat{\mathbb{G}}$ be the set of all irreducible and unitary representations of $\mathbb{G}$.

Let $\pi \in \hat{\mathbb{G}}$. Then π is finite dimensional. Let a_π be the degree of π and let $\mathcal{H}_\pi$ be its representation space. We denote the inner product and the norm in $\mathcal{H}_\pi$ by $(\ ,\ )_{\mathcal{H}_\pi}$ and $\|\ \|_{\mathcal{H}_\pi}$ respectively. Let $\{\psi_1, \psi_2, \ldots, \psi_{a_\pi}\}$ be an orthonormal basis for $\mathcal{H}_\pi$. Then for all $j, k = 1, 2, \ldots,$ we define π_{jk} by

$$\pi_{jk}(g) = \sqrt{a_\pi}\left(\pi(g)\psi_k, \psi_j\right)_{\mathcal{H}_\pi}$$

Theorem 3.2 (Peter-Weyl Theorem) $\{\pi_{jk} : \pi \in \hat{\mathbb{G}},\ j, k = 1, 2, \ldots, a_\pi\}$ *forms an orthonormal basis for* $L^2(\mathbb{G})$.

By Peter-Weyl theorem, every $f \in L^2(\mathbb{G})$ can be expressed as

$$f = \sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \left(f, \pi_{jk}\right)_{L^2(\mathbb{G})} \pi_{jk}.$$

have the following Plancheral theorem.

Theorem 3.3 *For all* $f \in L^2(\mathbb{G})$,

$$\|f\|_{L^2(\mathbb{G})} = \left\{\sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} |\left(f, \pi_{jk}\right)_{L^2(\mathbb{G})}|^2\right\}^{1/2}.$$

Let σ be a measurable function on $\mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N}$. For all measurable functions f on $\mathbb{G}$, we define $T_\sigma f$ on $\mathbb{G}$ formally by

$$(T_\sigma f)(g) = \sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \sigma(g, \pi, j, k)\left(f, \pi_{jk}\right)_{L^2(\mathbb{G})} \pi_{jk}(g), \quad g \in \mathbb{G}.$$

T_σ is called the pseudo-differential operator on $\mathbb{G}$ corresponding to the symbol σ. For pseudo-differential operators on Euclidean spaces see for example [15,19].

Let $L^2(\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N})$ be the space all measurable functions σ on $\mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N}$ for which

$$\|\sigma\|_{L^2(\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N})} = \left\{\sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \int_{\mathbb{G}} \left|\sigma(g, \pi, j, k)\pi_{jk}(g)\right|^2 d\mu(g)\right\}^{1/2} < \infty.$$

We have the following result on the L^2-boundedness of pseudo-differential operators on $\mathbb{G}$.

Theorem 3.4 *Let* $\sigma \in L^2(\mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N})$. *Then* $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ *is a bounded operator and*

$$\|T_\sigma\|_* \leq \|\sigma\|_{L^2(\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N})},$$

where $\|\ \|_*$ *is the norm in the* C^**-algebra of all bounded linear operators from* $L^2(\mathbb{G})$ *into* $L^2(\mathbb{G})$.

Proof Let $f \in L^2(\mathbb{G})$. By Minkowski's inequality and the Schwarz inequality, we have

$$
\begin{aligned}
&\|T_\sigma f\|_{L^2(\mathbb{G})} \\
&= \left(\int_{\mathbb{G}} |(T_\sigma f)(g)|^2 \, d\mu(g) \right)^{1/2} \\
&= \left(\int_{\mathbb{G}} \left| \sum_{\pi \in \hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \sigma(g, \pi, j, k) \left(f, \pi_{jk}\right)_{L^2(\mathbb{G})} \pi_{jk}(g) \right|^2 \, d\mu(g) \right)^{1/2} \\
&\le \sum_{\pi \in \hat{\mathbb{G}}} \left(\int_{\mathbb{G}} \left| \sum_{j,k=1}^{a_\pi} \sigma(g, \pi, j, k) \left(f, \pi_{jk}\right)_{L^2(\mathbb{G})} \pi_{jk}(g) \right|^2 \, d\mu(g) \right)^{1/2} \\
&\le \sum_{\pi \in \hat{\mathbb{G}}} \left(\int_{\mathbb{G}} \left(\sum_{j,k=1}^{a_\pi} \left| \left(f, \pi_{jk}\right)_{L^2(\mathbb{G})} \right|^2 \right) \left(\sum_{j,k=1}^{a_\pi} \left| \sigma(g, \pi, j, k) \pi_{jk}(g) \right|^2 \right) d\mu(g) \right)^{1/2} \\
&= \sum_{\pi \in \hat{\mathbb{G}}} \left(\sum_{j,k=1}^{a_\pi} \left| \left(f, \pi_{jk}\right)_{L^2(\mathbb{G})} \right|^2 \right)^{1/2} \left(\int_{\mathbb{G}} \sum_{j,k=1}^{a_\pi} \left| \sigma(g, \pi, j, k) \pi_{jk}(g) \right|^2 d\mu(g) \right)^{1/2} \\
&\le \left(\sum_{\pi \in \hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \left| \left(f, \pi_{jk}\right)_{L^2(\mathbb{G})} \right|^2 \right)^{1/2} \left(\sum_{\pi \in \hat{\mathbb{G}}} \int_{\mathbb{G}} \sum_{j,k=1}^{a_\pi} \left| \sigma(g, \pi, j, k) \pi_{jk}(g) \right|^2 d\mu(g) \right)^{1/2} \\
&= \|f\|_{L^2(\mathbb{G})} \|\sigma\|_{L^2(\mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N})}.
\end{aligned}
$$

□

Now we are ready to give a necessary and sufficient condition on σ to guarantee Hilbert-Schmidt properties of T_σ.

Theorem 3.5 *Let σ be a measurable function on $\mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N}$. Then T_σ is a Hilbert-Schmidt operator if and only if $\sigma \in L^2(\mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N})$ and*

$$\|T_\sigma\|_{HS} = \|\sigma\|_{L^2(\mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N})}.$$

Proof For $\tilde{\pi} \in \hat{\mathbb{G}}$ and $j_0, k_0 = 1, 2, \ldots, a_{\tilde{\pi}}$, we have

$$
\begin{aligned}
\left(T_\sigma \tilde{\pi}_{j_0 k_0}\right)(g) &= \sum_{\pi \in \hat{\mathbb{G}}} \sum_{j,k=0}^{a_\pi} \sigma(g, \pi, j, k) \left(\tilde{\pi}_{j_0 k_0}, \pi_{jk}\right)_{L^2(\mathbb{G})} \pi_{jk}(g) \\
&= \sum_{\pi \in \hat{\mathbb{G}}} \sum_{j,k=0}^{a_\pi} \sigma(g, \pi, j, k) \left(\tilde{\pi}_{j_0 k_0}, \pi_{jk}\right)_{L^2(\mathbb{G})} \pi_{jk}(g) \\
&= \sigma(g, \tilde{\pi}, k_0, j_0) \tilde{\pi}_{j_0 k_0}(g), \quad g \in \mathbb{G}.
\end{aligned}
$$

Hence

$$\begin{aligned}
&\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\|T_\sigma\pi_{jk}\|^2_{L^2(\mathbb{G})}\\
&=\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\left(\int_{\mathbb{G}}|T_\sigma\pi_{jk}(g)|^2\,d\mu(g)\right)\\
&=\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\int_{\mathbb{G}}|\sigma(g,\pi,j,k)\pi_{jk}(g)|^2\,d\mu(g)\\
&=\|\sigma\|^2_{L^2(\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N})}.
\end{aligned}$$

and the proof is complete. □

Let σ and τ measurable functions on $\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N}$. Then we define $\sigma\circledast\tau$ by

$$\begin{aligned}
&\sigma\circledast\tau(g,\xi,l,m)\\
&=\left(\int_{\mathbb{G}}\tau(w,\xi,l,m)\xi_{lm}(w)\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\sigma(g,\pi,j,k)\overline{\pi_{jk}(w)}\pi_{jk}(g)\,d\mu(w)\right)(\xi_{lm}(g))^{-1}
\end{aligned}\tag{3.1}$$

for all $g\in\mathbb{G}$, $\xi\in\hat{\mathbb{G}}$ and $l,m=1,\ldots,a_\xi$. The following theorem shows that that the composition of two pseudo-differential operators on $\mathbb{G}$ is again a pseudo-differential operator.

Theorem 3.6 *Let σ and τ be measurable functions on $\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N}$. Then*

$$T_\sigma T_\tau = T_\lambda$$

where $\lambda=\sigma\circledast\tau$.

Proof Let f be in $L^2(\mathbb{G})$. Then for all $g\in\mathbb{G}$

$$\begin{aligned}
&(T_\sigma T_\tau f)(g)\\
&=\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\sigma(g,\pi,j,k)\left(T_\tau f,\pi_{jk}\right)_{L^2(\mathbb{G})}\pi_{jk}(g)\\
&=\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\sigma(g,\pi,j,k)\left(\int_{\mathbb{G}}(T_\tau f)(w)\overline{\pi_{jk}(w)}\,d\mu(w)\right)\pi_{jk}(g)\\
&=\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\sigma(g,\pi,j,k)\left(\int_{\mathbb{G}}\sum_{\xi\in\hat{\mathbb{G}}}\sum_{l,m=1}^{a_\xi}\tau(w,\xi,l,m)\,(f,\xi_{lm})_{L^2(\mathbb{G})}\,\xi_{lm}(w)\overline{\pi_{jk}(w)}\,d\mu(w)\right)\pi_{jk}(g)
\end{aligned}$$

$$= \sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \sigma(g,\pi,j,k) \left(\sum_{\xi\in\hat{\mathbb{G}}} \sum_{l,m=1}^{a_\xi} (f,\xi_{lm})_{L^2(\mathbb{G})} \int_{\mathbb{G}} \tau(w,\xi,l,m)\xi_{lm}(w)\overline{\pi_{jk}(w)}\, d\mu(w) \right) \pi_{jk}(g)$$

$$= \sum_{\xi\in\hat{\mathbb{G}}} \sum_{l,m=1}^{a_\xi} \left(\int_{\mathbb{G}} \tau(w,\xi,l,m)\xi_{lm}(w) \sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \sigma(g,\pi,j,k)\overline{\pi_{jk}(w)}\pi_{jk}(g)\, d\mu(w) \right)$$

$$\times (\xi_{lm}(g))^{-1} (f,\xi_{lm})_{L^2(\mathbb{G})}\, \xi_{lm}(g)$$

$$= \sum_{\xi\in\hat{\mathbb{G}}} \sum_{l,m=1}^{a_\xi} \lambda(g,\xi,l,m)\, (f,\xi_{lm})_{L^2(\mathbb{G})}\, \xi_{lm}(g), \tag{3.2}$$

where $\lambda = \sigma \circledast \tau$. Thus,

$$T_\lambda = T_\sigma T_\tau$$

□

Now we are ready to give a characterization of trace class pseudo-differential operators on $\mathbb{G}$.

Theorem 3.7 *Let λ be a measurable function in $\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N}$. Then $T_\lambda \in S_1$ if and only if there exist symbols σ and τ in $L^2(\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N})$ such that*

$$\lambda = \sigma \circledast \tau,$$

and

$$tr(T_\lambda) = \sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \int_{\mathbb{G}} \lambda(g,\pi,j,k)|\pi_{jk}(g)|^2\, d\mu(g)$$

$$= \sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \sum_{\xi\in\hat{\mathbb{G}}} \sum_{l,m=1}^{a_\xi} \big(\tau(\cdot,\xi,l,m)\xi_{lm},\pi_{jk}\big)_{L^2(\mathbb{G})} \big(\sigma(\cdot,\pi,j,k)\pi_{jk},\xi_{lm}\big)_{L^2(\mathbb{G})}.$$

Proof The first part of the theorem follows from Theorem 2.2, Theorem 3.5 and Theorem 3.6.Now we check the absolute convergence of the series

$$\sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \sum_{\xi\in\hat{\mathbb{G}}} \sum_{l,m=1}^{a_\xi} \big(\tau(\cdot,\xi,l,m)\xi_{lm},\pi_{jk}\big)_{L^2(\mathbb{G})} \big(\sigma(\cdot,\pi,j,k)\pi_{jk},\xi_{lm}\big)_{L^2(\mathbb{G})}$$

By the Schwarz inequality and Plancheral's theorem,

$$\sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \sum_{\xi\in\hat{\mathbb{G}}} \sum_{l,m=1}^{a_\xi} \left|\big(\tau(\cdot,\xi,l,m)\xi_{lm},\pi_{jk}\big)_{L^2(\mathbb{G})}\right| \left|\big(\sigma(\cdot,\pi,j,k)\pi_{jk},\xi_{lm}\big)_{L^2(\mathbb{G})}\right|$$

$$\leq \left\{ \sum_{\pi\in\hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \sum_{\xi\in\hat{\mathbb{G}}} \sum_{l,m=1}^{a_\xi} \left|\big(\tau(\cdot,\xi,l,m)\xi_{lm},\pi_{jk}\big)_{L^2(\mathbb{G})}\right|^2 \right\}^{1/2}$$

$$\left\{\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\sum_{\xi\in\hat{\mathbb{G}}}\sum_{l,m=1}^{a_\xi}\left|\left(\sigma(\cdot,\pi,j,k)\pi_{jk},\xi_{lm}\right)_{L^2(\mathbb{G})}\right|^2\right\}^{1/2}$$

$$=\left\{\sum_{\xi\in\hat{\mathbb{G}}}\sum_{l,m=1}^{a_\xi}\int_{\mathbb{G}}|\tau(g,\xi,l,m)\xi_{lm}(g)|^2\,d\mu(g)\right\}^{1/2}$$

$$\left\{\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\xi}\int_{\mathbb{G}}\left|\sigma(g,\pi,j,k)\pi_{jk}(g)\right|^2\,d\mu(g)\right\}^{1/2}$$

$$=\|\sigma\|_{L^2(\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N})}\|\tau\|_{L^2(\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{N}\times\mathbb{N})}<\infty.$$

Since $\bigcup_{\pi\in\hat{\mathbb{G}}}\{\pi_{jk}:\ j,k=1,2,\ldots,a_\pi\}$ forms an orthonormal basis for $L^2(\mathbb{G})$, it follows that

$$\begin{aligned}
tr(T_\lambda)&=\sum_{\xi\in\hat{\mathbb{G}}}\sum_{l,m=1}^{a_\xi}(T_\lambda\xi_{lm},\xi_{lm})_{L^2(\mathbb{G})}\\
&=\sum_{\xi\in\hat{\mathbb{G}}}\sum_{l,m=1}^{a_\xi}\int_{\mathbb{G}}\lambda(g,\xi,l,m)\xi_{lm}(g)\overline{\xi_{lm}(g)}\,d\mu(g)\\
&=\sum_{\xi\in\hat{\mathbb{G}}}\sum_{l,m=1}^{a_\xi}\int_{\mathbb{G}}\left(\int_{\mathbb{G}}\tau(w,\xi,l,m)\xi_{lm}(w)\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\sigma(g,\pi,j,k)\overline{\pi_{jk}(w)}\pi_{jk}(g)\,d\mu(w)\right)\\
&\quad\times\overline{\xi_{lm}(g)}\,d\mu(g)\\
&=\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\sum_{\xi\in\hat{\mathbb{G}}}\sum_{l,m=1}^{a_\xi}\int_{\mathbb{G}}\tau(w,\xi,l,m)\xi_{lm}(w)\overline{\pi_{jk}(w)}\,d\mu(w)\int_{\mathbb{G}}\sigma(g,\pi,j,k)\pi_{jk}(g)\\
&\quad\times\overline{\xi_{lm}(g)}\,d\mu(g)\\
&=\sum_{\pi\in\hat{\mathbb{G}}}\sum_{j,k=1}^{a_\pi}\sum_{\xi\in\hat{\mathbb{G}}}\sum_{l,m=1}^{a_\xi}\left(\tau(\cdot,\xi,l,m)\xi_{lm},\pi_{jk}\right)_{L^2(\mathbb{G})}\left(\sigma(\cdot,\pi,j,k)\pi_{jk},\xi_{lm}\right)_{L^2(\mathbb{G})}.
\end{aligned}$$

□

We can now look at the special case when $\mathbb{G}=\mathbb{S}^1$.

Example 3.8 Let $\mathbb{S}^1$ be the unit circle centered at the origin. For all $n\in\mathbb{Z}$, we define $\pi_n:\mathbb{S}^1\to U(1)$ by

$$\pi_n(\theta)=(2\pi)^{-1/2}e^{in\theta},\quad \theta\in[-\pi,\pi].$$

Then

$$\widehat{\mathbb{S}^1}=\{\pi_n:\ n\in\mathbb{Z}\},$$

and $\widehat{\mathbb{S}} \simeq \mathbb{Z}$. Hence for any symbol σ on $\mathbb{S}^1 \times \mathbb{Z}$ we can define pseudo-differential operator T_σ on $L^2(\mathbb{S}^1)$ by

$$(T_\sigma f)(\theta) = \sum_{n\in\mathbb{Z}} \sigma(\theta, n)(f, \pi_n)_{L^2(\mathbb{S}^1)}\pi_n(\theta), \quad f \in L^2(\mathbb{S}^1),\ \theta \in [-\pi, \pi].$$

Hence,

$$(T_\sigma f)(\theta) = (2\pi)^{-1} \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{in(\theta-\varphi)}\sigma(n, \theta) f(\varphi)\, d\varphi.$$

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