Institut für Mathematik

Dubrovin–rings and their connection to Hughes–free skew fields of fractions

Friedrich Jakobs

Dissertation
zur Erlangung des akademischen Grades
"doctor rerum naturalium" (Dr. rer. nat.)
in der Wissenschaftsdisziplin Algebra

eingereicht an der
Mathematisch–Naturwissenschaftlichen Fakultät
der Universität Potsdam

Betreuer: Prof. Dr. Joachim Gräter

Datum der Abgabe: 17.04.2019
Datum der Disputation: 09.09.2019

Betreuer: Prof. Dr. Joachim Gräter
Gutachter: Prof. Dr. Joachim Gräter
Prof. Dr. Nikolay I. Dubrovin
Prof. Dr. Günter Törner

Published online at the
Institutional Repository of the University of Potsdam:
https://doi.org/10.25932/publishup-43556
https://nbn-resolving.org/urn:nbn:de:kobv:517-opus4-435561

Statement of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma at any other university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying.

Potsdam, 17.04.2019
Friedrich Jakobs

Acknowledgements

My gratitude belongs to Joachim Gräter, who ignited my spark of interest for this topic and gave me the opportunity, time and help to explore this field. His influence is life changing, in more ways than only mathematical. I want to thank Stephan Naumann for being a brother in mind, a critic of ideas and an inspiration with his own special way of life. Furthermore am I thankful to Martin Weese, Winnie Krüger and Andreas Braunß, who supported me throughout the years, each of them in different ways. Finally I want to thank everyone who thinks to be deserving it.

To Bastian Carl

Contents

Introduction

One method of embedding groups into skew fields was introduced by A. I. Mal'tsev and B. H. Neumann (cf. [18, 19]). If G is an ordered group and F is a skew field, the set $F(\!(G)\!)$ of formal power series over F in G with well–ordered support forms a skew field into which the group ring $F[G]$ can be embedded. Unfortunately it is not sufficient that G is left–ordered since $F(\!(G)\!)$ is only an F–vector space in this case as there is no natural way to define a multiplication on $F(\!(G)\!)$. One way to extend the original idea onto left–ordered groups is to examine the endomorphism ring of $F(\!(G)\!)$ as explored by N. I. Dubrovin (cf. [5, 6]). It is possible to embed any crossed product ring $F[G;\eta,\sigma]$ into the endomorphism ring of $F(\!(G)\!)$ such that each non–zero element of $F[G;\eta,\sigma]$ defines an automorphism of $F(\!(G)\!)$ (cf. [5, 10]). Thus, the rational closure of $F[G;\eta,\sigma]$ in the endomorphism ring of $F(\!(G)\!)$, which we will call the Dubrovin–ring of $F[G;\eta,\sigma]$, is a potential candidate for a skew field of fractions of $F[G;\eta,\sigma]$. The methods of N. I. Dubrovin allowed to show that specific classes of groups can be embedded into a skew field. For example, N. I. Dubrovin contrived some special criteria, which are applicable on the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. These methods have also been explored by J. Gräter and R. P. Sperner (cf. [10]) as well as N.H. Halimi and T. Ito (cf. [11]).
Furthermore, it is of interest to know if skew fields of fractions are unique. For example, left and right Ore domains have unique skew fields of fractions (cf. [2]). This is not the general case as for example the free group with 2 generators can be embedded into non–isomorphic skew fields of fractions (cf. [12]). It seems likely that Ore domains are the most general case for which unique skew fields of fractions exist. One approach to gain uniqueness is to restrict the search to skew fields of fractions with additional properties. I. Hughes has defined skew fields of fractions of crossed product rings $F[G;\eta,\sigma]$ with locally indicable G which fulfill a special condition. These are called Hughes–free skew fields of fractions and I. Hughes has proven that they are unique if they exist [13, 14].
This thesis will connect the ideas of N. I. Dubrovin and I. Hughes. The first chapter contains the basic terminology and concepts used in this thesis. We present methods provided by N. I. Dubrovin such as the complexity of elements in rational closures and special properties of endomorphisms of the vector space of formal power series $F(\!(G)\!)$. To combine the ideas of N.I. Dubrovin and I. Hughes we introduce Conradian left–ordered groups of maximal rank and examine their connection to locally indicable groups. Furthermore we provide notations for crossed product rings, skew fields of fractions as well as Dubrovin–rings and prove some technical statements which are used in later parts.
The second chapter focuses on Hughes–free skew fields of fractions and their connection to Dubrovin–rings. For that purpose we introduce series representations to interpret elements of Hughes–free skew fields of fractions as skew formal Laurent series. This

allows us to prove that for Conradian left–ordered groups G of maximal rank the statement "$F[G;\eta,\sigma]$ has a Hughes–free skew field of fractions" implies "The Dubrovin ring of $F[G;\eta,\sigma]$ is a skew field". We will also prove the reverse and apply the results to give a new prove of Theorem 1 in [13]. Furthermore we will show how to extend injective ring homomorphisms of some crossed product rings onto their Hughes–free skew fields of fractions. At last we will be able to answer the open question whether Hughes–free skew fields are strongly Hughes–free (cf. [17, page 53]).

1 Basics

1.1 Rational closure and complexity

Definition 1.1.1 If R is a ring with 1, a subring S of R is called rationally closed in R, if $1 \in S$ and $s^{-1} \in S$ for every $s \in S \cap \mathrm{U}(R)$. For every subset $M \subseteq R$

$$\bigcap\{S \subseteq R \mid M \subseteq S, S \text{ is a rationally closed subring in } R\}$$

is called the rational closure of M in R.

Remark 1.1.2

1. The rational closure of M in R is the smallest rationally closed subring of R containing M.
2. If D is a skew field, each subring which is rationally closed in D is itself a skew field.

Theorem 1.1.3 ([8, Propositon 2.1]) Let Λ be an ordinal number and $\mathbb{N}(\Lambda)$ be the free abelian $\mathbb{N}_0$–monoid with basis $\{\lambda \in \mathbb{O}\mathrm{n} \mid \lambda \leq \Lambda\}$. For all $x, y \in \mathbb{N}(\Lambda)$, there are $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k \in \mathbb{O}\mathrm{n}$ and $n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N}_0$ with $\lambda_k < \cdots < \lambda_1$ and

$$\begin{aligned} x &= n_1\lambda_1 + \cdots + n_k\lambda_k, \\ y &= m_1\lambda_1 + \cdots + m_k\lambda_k. \end{aligned}$$

If there is a minimal $i \leq k$ with $n_i \neq m_i$, we define $x < y$ for $n_i < m_i$. This relation defines a well–order on $\mathbb{N}(\Lambda)$ which satisfies

$$x < y \Longrightarrow x + z < y + z.$$

for all $x, y, z \in \mathbb{N}(\Lambda)$.

Definition 1.1.4 Let R be a ring with 1, $M \subseteq R$ and D be the rational closure of M in R. We define a recursive series $(D_\alpha)_{\alpha<\gamma}$ of subsets in R with $\gamma \in \mathbb{O}\mathrm{n}$ such that the union of the series is D. We start with

$$\begin{aligned} D_0 &:= \{0\}, \\ D_1 &:= \{0, 1, -1\} \cup M \cup -M. \end{aligned}$$

If $\alpha \in \mathbb{O}\mathrm{n}$ is a limit ordinal number, we define $D_\alpha = \bigcup_{\alpha'<\alpha} D_\alpha$. Otherwise there is an $\alpha' \in \mathbb{O}\mathrm{n}$ with $\alpha = \alpha' + 1$. Here we distinguish the following cases.

Case 1: $D_{\alpha'}$ is not additively closed. Then there is a minimal $\alpha_1 + \cdots + \alpha_n \in \mathbb{N}(\alpha')$ with $D_{\alpha_1} + \cdots + D_{\alpha_n} \not\subseteq D_{\alpha'}$. We define

$$D_\alpha = D_{\alpha'} \cup (D_{\alpha_1} + \cdots + D_{\alpha_n}).$$

Case 2: $D_{\alpha'}$ is not multiplicatively closed but additively closed. Then there is a minimal $\alpha_1 + \cdots + \alpha_n \in \mathbb{N}(\alpha')$ with $D_{\alpha_1} \cdots D_{\alpha_n} \not\subseteq D_{\alpha'}$. We define

$$D_\alpha = D_{\alpha'} \cup \left(\bigcup_{\pi \in \mathrm{S}_n} D_{\alpha_{\pi(1)}} \cdots D_{\alpha_{\pi(n)}} \right).$$

Case 3: $D_{\alpha'}$ is a ring but not rationally closed in R. Then there is a minimal $\alpha_1 \leq \alpha'$ with $D_{\alpha_1}^{-1} \not\subseteq D_{\alpha'}$. We define

$$D_\alpha = D_{\alpha'} \cup D_{\alpha_1}^{-1}$$

Case 4: If $D_{\alpha'}$ is a rationally closed subring of R, we define $D_\alpha = D_{\alpha'}$.

Since this series is strictly ascending for the first three cases, there exists a minimal $\gamma \leq \operatorname{card} R$ with $D_\gamma = D_{\gamma+1}$. Therefore D_γ is the rational closure of M in R.

Definition 1.1.5 ([8, Definitions 2.2, 2.3]) With the notation like in Definition 1.1.4 we define $\operatorname{cp}(a) := \min\{\alpha < \gamma \mid a \in D_\alpha\}$ as the complexity of $a \in D$. Furthermore we define

$$a \trianglelefteq b \Longleftrightarrow \operatorname{cp}(a) \leq \operatorname{cp}(b),$$
$$a \triangleleft b \Longleftrightarrow \operatorname{cp}(a) < \operatorname{cp}(b)$$

for all $a, b \in D$.

Remark 1.1.6 It is important to note that the complexity depends on M and not purely the rational closure of M. If M and M' have the same rational closure they may define different complexities.

Definition 1.1.7 ([8, page 38]) If $a \in D$ is not a sum of elements with lesser complexity, we call a (additively) indecomposable. Otherwise it is called additively decomposable and there are $a_1, \ldots, a_n \in D$ with $a = a_1 + \cdots + a_n$ and $cp(a_1) + \cdots + \operatorname{cp}(a_n)$ minimal in $\mathbb{N}(\gamma)$. This representation as a sum is called a complete additive decomposition of a. If $a \in D$ is additively indecomposable, we call a a complete additive decomposition of a itself.

Remark 1.1.8 ([8, Proposition 3.1]) If $a \in D$, $a \neq 0$ is additively indecomposable, $\{b \in D \mid b \triangleleft a\}$ is an abelian group with respect to $+$.

Theorem 1.1.9 ([8, Theorem 3.6]) If $a \in D$ is additively decomposable and $a_1+\cdots+a_n$ is a complete additive decomposition of a as well as

$$x = a_1 + \cdots + a_j,$$
$$y = a_{j+1} + \cdots + a_n$$

for some $j \in \{1, \ldots, n-1\}$, the following statements hold true.

i) The sums $a_1 + \cdots + a_j$ and $a_{j+1} + \cdots + a_n$ are complete additive decompositions of x and y respectively.

ii) For $x', y' \in D$ with $x' \trianglelefteq x$ and $y' \trianglelefteq y$ we have $x' + y' \trianglelefteq a$. If additionally $x' \triangleleft x$ or $y' \triangleleft y$ holds, then $x' + y' \triangleleft a$.

Remark 1.1.10

1. Each additively decomposable element in D is a sum of two elements with lesser complexity.
2. The above theorem can be generalized for any finite sum.

Definition 1.1.11 Let $a \in D$ be additively indecomposable. If $a \in D$ is not a product of elements with lesser complexity, we call a an atom. Otherwise it is called multiplicatively decomposable and there are $a_1, \ldots, a_n \in D$ with $a = a_1 \cdots a_n$ and $cp(a_1) + \cdots + \mathrm{cp}(a_n)$ minimal in $\mathbb{N}(\gamma)$. This representation as a product is called a complete multiplicative decomposition of a. If $a \in D$ is an atom, we call a a complete multiplicative decomposition of a itself and for $a \notin D_1$ we call a a proper atom.

Theorem 1.1.12 ([8, Proposition 4.1]) If $a \in D$ is a proper atom, it is a unit in D and $a^{-1} \triangleleft a$. Furthermore $\{b \in D \mid b \triangleleft a\}$ is a subring in D and for each unit $b \in D$ the following holds:

$$b^{-1} \trianglelefteq a^{-1} \Longrightarrow b \trianglelefteq a,$$
$$b^{-1} \triangleleft a^{-1} \Longrightarrow b \triangleleft a.$$

Theorem 1.1.13 ([8, Theorem 4.6]) If $a \in D$ is multiplicatively decomposable and $a_1 \cdots a_n$ is a complete multiplicative decomposition of a as well as

$$x = a_1 \cdots a_j,$$
$$y = a_{j+1} \cdots a_n$$

for some $j \in \{1, \ldots, n-1\}$, the following statements hold true.

i) The products $a_1 \cdots a_j$ and $a_{j+1} \cdots a_n$ are complete multiplicative decompositions of x and y respectively.

ii) For $x', y' \in D$ with $x' \trianglelefteq x$ and $y' \trianglelefteq y$ we have $x'y' \trianglelefteq a$. If additionally $x' \triangleleft x$ or $y' \triangleleft y$, then $x'y' \triangleleft a$.

Remark 1.1.14

1. Each multiplicatively decomposable element in D is a product of two elements with lesser complexity.
2. The above theorem can be generalized for any finite products.

Theorem 1.1.15 ([8, Proposition 4.8, Theorem 4.9]) Let M as in Definition 1.1.4 be a subgroup of the group of units in R. Then the following statements hold.

i) If $a \in D$ and $g \in M \cup -M$, then $\operatorname{cp}(ag) = \operatorname{cp}(ga) = \operatorname{cp}(a)$.

ii) If $a \in D \setminus \{0\}$ is additively indecomposable, then a is a unit in D. If additionally $\operatorname{cp}(a) > 1$, then $a^{-1} \lhd a$ and a^{-1} is additively decomposable.

Proposition 1.1.16 Let S, R_1, R_2 be rings with 1 and $\varphi : R_1 \longrightarrow R_2$ as well as $\iota_i : S \longrightarrow R_i$ be injective ring homomorphisms with $\varphi(1) = 1$ and $\iota_i(1) = 1$ for $i \in \{1, 2\}$ such that

$$\begin{tikzcd} S \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & R_1 \arrow[d, "\varphi"] \\ & R_2 \end{tikzcd}$$

is a commutative diagram. If R_1 is the rational closure of $\iota_1(S)$ in R_1, then φ is uniquely determined by the commutative diagram.

Proof. Let $\varphi' : R_1 \longrightarrow R_2$ be an injective ring homomorphism such that

$$\begin{tikzcd} S \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & R_1 \arrow[d, "\varphi'"] \\ & R_2 \end{tikzcd}$$

is a commutative diagram. We will show $\varphi(r) = \varphi'(r)$ for all $r \in R_1$ by induction on $\operatorname{cp}(r)$. The induction basis is $r = \iota_1(s)$ for some $s \in S$. Since the above diagrams are commutative we conclude

$$\varphi(r) = \varphi(\iota_1(s)) = \iota_2(s) = \varphi'(\iota_1(s)) = \varphi'(r).$$

If r is additively decomposable there are $r_1, r_2 \in R_1$ with $r = r_1 + r_2$ and $r_1, r_2 \lhd r$ as seen in Remark 1.1.10. Thus,

$$\varphi(r) = \varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2) \overset{\text{IH}}{=} \varphi'(r_1) + \varphi'(r_2) = \varphi'(r_1 + r_2) = \varphi'(r).$$

If r is multiplicatively decomposable there are $r_1, r_2 \in R_1$ with $r = r_1 r_2$ and $r_1, r_2 \lhd r$ as seen in Remark 1.1.14. Thus,

$$\varphi(r) = \varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2) \overset{\text{IH}}{=} \varphi'(r_1)\varphi'(r_2) = \varphi'(r_1 r_2) = \varphi'(r).$$

If r is a proper atom, then $r^{-1} \lhd r$ as seen in Theorem 1.1.12. Thus,

$$\varphi(r) = \varphi(r^{-1})^{-1} \stackrel{\text{IH}}{=} \varphi'(r^{-1})^{-1} = \varphi'(r).$$

$\square$

Proposition 1.1.17 Let S, R be rings with 1, D be a skew field and $\varphi : D \longrightarrow R$, $\iota_1 : S \longrightarrow D$ as well as $\iota_2 : S \longrightarrow R$ be injective ring homomorphisms with $\varphi(1) = 1$ and $\iota_i(1) = 1$ for $i \in \{1, 2\}$ such that

$$\begin{tikzcd} S \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & D \arrow[d, "\varphi"] \\ & R \end{tikzcd}$$

is a commutative diagram. If D is the rational closure of $\iota_1(S)$ in D, then $\varphi(D)$ is the rational closure of $\iota_2(S)$ in R.

Proof. Since D is a skew field and φ is injective, $\varphi(D)$ is also a skew field and as such rationally closed in R. Furthermore, $\iota_1(S) \subseteq D$ implies $\iota_2(S) = \varphi(\iota_1(S)) \subseteq \varphi(D)$. Therefore, if R' is the rational closure of $\iota_2(S)$ in R, then $R' \subseteq \varphi(D)$. Let D' be the inverse image of R' under φ. If $d \in D'$ is a unit in D, then $\varphi(d) \in R'$ is a unit in R which implies $\varphi(d^{-1}) = \varphi(d)^{-1} \in R'$ and thus $d^{-1} \in D'$. Therefore, D' is rationally closed in D. As $\varphi(\iota_1(S)) = \iota_2(S) \subseteq R'$ implies $\iota_1(S) \subseteq D'$ and D is the rational closure of $\iota_1(S)$ in D, we conclude $D \subseteq D'$. Therefore $\varphi(D) \subseteq \varphi(D') = R'$. $\square$

1.2 Conradian left–ordered groups and locally indicable groups

Definition 1.2.1 If G is a group and $<$ is a total order on G, then $<$ is called left–order of G and G is called left–ordered with respect to $<$ if

$$a < b \Longrightarrow ca < cb$$

for all $a, b, c \in G$. Analogously one defines right–orders. If $<$ is a left–order and a right–order at the same time it is called an order of G and G is called ordered group with respect to $<$.

Remark 1.2.2

1. If G is a left–ordered group and nothing more is said, we will use $<$ as the symbol for the corresponding left–order. Even if there are multiple left–ordered groups we will use the same symbol if there is no danger of confusion.
2. If G is abelian and $<$ is a left–order of G, then $<$ is an order of G.

3. If G is a left–ordered group it is torsion–free since $e < g$ implies $e < g < g^2 < \cdots < g^n$ and therefore $e \neq g^n$ for all $n \in \mathbb{N}$.

Definition 1.2.3 If G is a group then a subset $P \subseteq G$ is called a positive cone of G, if the following properties are fulfilled:

i) $P \cdot P \subseteq P$,

ii) $P \cap P^{-1} = \emptyset$,

iii) $G = P \cup P^{-1} \cup \{e\}$.

Theorem 1.2.4 (cf. [4, page 267][16, section 1.5.]) If G is a left–ordered group then $P_< := \{a \in G \mid e < a\}$ is a positive cone of G such that

$$a < b \Longleftrightarrow a^{-1}b \in P_<$$

for all $a, b \in G$.

Theorem 1.2.5 (cf. [4, page 267][16, section 1.5.]) If G is a group and P is a positive cone of G then

$$a < b :\Longleftrightarrow a^{-1}b \in P$$

for all $a, b \in G$ defines a left–order of G such that $P = P_<$.

Remark 1.2.6

1. As seen above each left–order admits a corresponding positive cone and vice versa. Therefore we will use both terms interchangeable.
2. If G is a right–ordered group, then $P_< := \{a \in G \mid e < a\}$ is a positive cone such that $a < b$ is equivalent to $ba^{-1} \in P_<$ for all $a, b \in G$. Conversely, each positive cone P of a group G defines a right–order $<$ on G such that $P = P_<$. Thus, each left–order of a group has a corresponding right–order and vice versa. This way one can translate statements about left–orders and right–orders of groups into each other. This comes in handy as we will mainly use left–orders even though most of the literature is about right–orders.

Definition 1.2.7 If G is a left–ordered group and $C \subseteq G$ is a subset (subgroup) of G then C is called convex subset (subgroup) of G, if

$$a < b < c \Longrightarrow b \in C$$

for all $b \in G$ and $a, c \in C$.

Remark 1.2.8 For a subgroup C of the left–ordered group G it is sufficient to prove that $e < a < b$ implies $a \in C$ for all $a \in G$ and $b \in C$ to prove that C is a convex subgroup of G.

Theorem 1.2.9 ([16, Theorem 2.1.1]) If G is a left–ordered group the following statements hold:

i) The set of all convex subgroups is totally ordered with respect to $\subset$.

ii) The intersection as well as the union of any nonempty family of convex subgroups is a convex subgroup.

Definition 1.2.10 If G is a left–ordered group and $a \in G$, $a \neq e$ we define

$$\begin{aligned} C_a^- &:= \bigcup \{C \subseteq G \mid C \text{ is a convex subgroup of } G \text{ with } a \notin C\}, \\ C_a^+ &:= \bigcap \{C \subseteq G \mid C \text{ is a convex subgroup of } G \text{ with } a \in C\}, \\ C_a &:= C_a^- \langle a \rangle. \end{aligned}$$

Remark 1.2.11

1. We define $C_e^- := C_e^+ := C_e := \{e\}$.
2. Theorem 1.2.9 shows that C_a^- and C_a^+ are convex subgroups of G. Furthermore C_a^+ (C_a^-) is the smallest (biggest) convex subgroup of G (not) containing a (for $a \neq e$).

Definition 1.2.12 Let G and H be left–ordered groups. A group homomorphism $\varphi : G \longrightarrow H$ is called order–preserving if

$$\varphi(a) < \varphi(b) \Longrightarrow a < b$$

for all $a, b \in G$.

Remark 1.2.13

1. The above property is equivalent to $a \leq b \Longrightarrow \varphi(a) \leq \varphi(b)$ for all $a, b \in G$.
2. The kernel $\ker \varphi$ is convex in G.
3. If G and H are left–ordered groups with corresponding positive cones P_G and P_H then a group homomorphism $\varphi : G \longrightarrow H$ is order–preserving if and only if $\varphi(a) \in P_H$ implies $a \in P_G$ for all $a \in G$.

Theorem 1.2.14 ([4, 3.5 and Remark before 3.6]) Let G be a left–ordered group and C be a convex normal subgroup of G. Then G/C is a left–ordered group with respect to the positive cone

$$P := \{gC \in G/C \mid g > e \text{ and } g \notin C\}$$

and the canonical homomorphism $\varphi : G \longrightarrow G/C$ is order–preserving.

Remark 1.2.15

1. The above order of G/C is called canonical left–order of G/C.
2. If C is a convex normal subgroup of the left–ordered group G, then there is an order–preserving correspondence between the convex subgroups of G/C and the convex subgroups of G containing C. [4, after 3.6]

Theorem 1.2.16 (cf. [4, Lemma 4.1]) Let G be a left–ordered group. The following properties are equivalent.

i) For all $a, b \in G$ with $e < a, b$ there exists an $n \in \mathbb{N}$ with $ab < (ba)^n$.

ii) For all $a, b \in G$ with $e < a < b$ there exists an $n \in \mathbb{N}$ with $b < a^{-1}b^n a$.

iii) For all $a, b \in G$ with $e < a, b$ there exists an $n \in \mathbb{N}$ with $a < ba^n$.

Definition 1.2.17 A left–order $<$ is called Conradian left–order if it has one of the properties in Theorem 1.2.16 and a left–ordered group is called Conradian left–ordered group if its left–order is a Conradian left–order.

Remark 1.2.18 Each ordered group is also Conradian left–ordered.

Definition 1.2.19 A left–ordered group G is called Archimedean left–ordered group if for all $a, b \in G$ with $e < b$ there exists an $n \in \mathbb{N}$ with $a < b^n$.

Theorem 1.2.20 (cf. [4, 3.8][16, Theorem 2.2.1]) If G is a left–ordered group, then G is Archimedean left–ordered if and only if there exists an order–preserving isomorphism of G onto a subgroup of the additive group $\mathbb{R}$.

Remark 1.2.21 Since $\mathbb{R}$ is commutative, each Archimedean left–order is also an order. Therefore we will also use the term Archimedean order.

Theorem 1.2.22 ([4, 4.1]) Let G be a left–ordered group with respect to $<$. The following statements are equivalent.

i) The left–order $<$ is a Conradian left–order.

ii) For every $a \in G$ the subgroup C_a^- is a normal subgroup of C_a^+ and C_a^+/C_a^- is an Archimedean ordered group with respect to the canonical order.

Corollary 1.2.23 If G is a Conradian left–ordered group and $a \in G$ then C_a is a subgroup of C_a^+ and C_a^- is a normal subgroup of C_a.

Definition 1.2.24 Let G be a group and Λ be a set of subgroups of G. We call Λ a subnormal system if the following holds:

i) $\{e\}, G \in \Lambda$.

ii) Λ is totally ordered with respect to $\subset$.

iii) If $\Lambda' \subseteq \Lambda$, $\Lambda' \neq \emptyset$, then $\bigcap \Lambda', \bigcup \Lambda' \in \Lambda$.

iv) If $\Delta, \Delta' \in \Lambda$ such that Δ is the direct successor of Δ' in Λ, then Δ' is a normal subgroup of Δ. We call Δ/Δ' factor of Λ.

Remark 1.2.25

1. If Λ is a subnormal system and $a \in G$, $a \neq e$, we define $\Delta_a^- := \bigcup\{\Delta \in \Lambda \mid a \notin \Delta\}$ and $\Delta_a^+ := \bigcap\{\Delta \in \Lambda \mid a \in \Delta\}$. As one can easily see, Δ_a^+ is the direct successor of Δ_a^- in Λ. Furthermore Δ_a^+ (Δ_a^-) is the smallest (biggest) element of Λ (not) containing a.
2. Theorem 1.2.22 shows that a left–ordered group G is Conradian left–ordered if and only if the set of all convex subgroups of G is a subnormal system such that the canonical left–orders of its factors are Archimedean orders.

Lemma 1.2.26 If G is a group and Λ is a subnormal system in G such that each factor of Λ admits a Conradian left–order, there is a Conradian left–order on G so that the canonical homomorphisms of the factors of Λ are order preserving. Especially each element of Λ is a convex subgroup of G.

Proof. We define $P := \{a \in G \setminus \{e\} \mid a\Delta_a^- > \Delta_a^-\}$. If $a, b \in P$ one can assume that $b \in \Delta_a^+$. We examine the following cases

Case 1: If $b \in \Delta_a^-$ then $ab \in \Delta_a^+\Delta_a^- \subseteq \Delta_a^+$ and $ab\Delta_a^- = a\Delta_a^- > \Delta_a^-$ which implies $ab \notin \Delta_a^-$. Hence $\Delta_{ab}^- = \Delta_a^-$ and therefore $ab \in P$.

Case 2: If $b \notin \Delta_a^-$ then $\Delta_b^- = \Delta_a^-$ and $\Delta_b^+ = \Delta_a^+$. Since Δ_a^+/Δ_a^- is left–ordered we conclude $ab\Delta_a^- > \Delta_a^-$ which also implies $ab \notin \Delta_a^-$. Since $ab \in \Delta_a^+\Delta_b^+ = \Delta_a^+$ we have $\Delta_{ab}^- = \Delta_a^-$ and therefore $ab \in P$.

These cases prove $P \cdot P \subseteq P$. If $a \in P$ then $a\Delta_a^- > \Delta_a^-$. This implies $a^{-1}\Delta_a^- < \Delta_a^-$ and since $\Delta_a^- = \Delta_{a^{-1}}^-$ we conclude $a \notin P$. Thus $P \cap P^{-1} = \emptyset$. If $a \in G$ with $a \neq e$, then $a\Delta_a^- > \Delta_a^-$ or $a^{-1}\Delta_a^- > \Delta_a^-$. Since $\Delta_a^- = \Delta_{a^{-1}}^-$ this proves $G = P \cup P^{-1} \cup \{e\}$. Hence P is a positive cone and defines a left–order $<$ of G.
Let Δ/Δ' be a factor of Λ and $\varphi : \Delta \longrightarrow \Delta/\Delta'$ be the corresponding canonical homomorphism. If $a \in \Delta$ with $\varphi(a) > e$, then $a\Delta_a^- > \Delta_a^-$ and therefore $a > e$. Thus φ is order–preserving according to Remark 1.2.13. Furthermore this shows that Δ' is convex in G as it is the kernel of φ.
If $\Delta \in \Lambda$ with $\Delta = G$ it is obviously convex. Otherwise Δ is convex since

$$\Delta = \bigcap_{a \in G\setminus\Delta} \Delta_a^-,$$

where each Δ_a^- is convex as seen above and the nonempty intersection of convex subgroups is itself convex according to Theorem 1.2.9.
For $a, b \in G$ with $e < a, b$ we can assume that $b \in \Delta_a^+$. Since $a > e$ we have $a\Delta_a^- > \Delta_a^-$. Let $\varphi : \Delta_a^+ \longrightarrow \Delta_a^+/\Delta_a^-$ be the canonical homomorphism. We examine the following cases.

Case 1: If $b \in \Delta_a^-$ then $ab\Delta_a^- = ba\Delta_a^- = a\Delta_a^- < a^2\Delta_a^- = (ab)^2\Delta_a^- = (ba)^2\Delta_a^-$. Thus $ab < (ba)^2$ and $ba < (ab)^2$ since φ is order–preserving.

Case 2: If $b \notin \Delta_a^-$ then $\Delta_a^- = \Delta_b^-$ and therefore $b\Delta_a^- > \Delta_a^-$ as $b > e$. Since Δ_a^+/Δ_a^- is a Conradian left–ordered group there exists an $n \in \mathbb{N}$ with $ab\Delta_a^- = a\Delta_a^- b\Delta_a^- < (b\Delta_a^- a\Delta_a^-)^n = (ba)^n\Delta_a^-$. Thus $ab < (ba)^n$ since φ is order–preserving.

This proves that $<$ is a Conradian left–order. □

Theorem 1.2.27 If G is a group and Λ is a subnormal system in G such that each factor of Λ is abelian and torsion–free, there exists a Conradian left–order of G with the following properties:

i) Each element of Λ is a convex subgroup of G.

ii) For each $a \in G$ the finitely generated subgroups of C_a^+/C_a^- are cyclic.

Proof. Because of Lemma 1.2.26 it is sufficient to prove this statement for torsion–free abelian groups, whereas the second property is obtained by considering the following diagram and Remark 1.2.15.

$$C_a^+/C_a^- \twoheadleftarrow C_a^+/\Delta_a^- \subseteq \Delta_a^+/\Delta_a^-$$

Let H be an additively written abelian torsion–free group. Then $\sim$ defined by

$$(a, m) \sim (b, n) :\Longleftrightarrow na = mb$$

for all $a, b \in H$ and $m, n \in \mathbb{N}$ is an equivalence relation on $H \times \mathbb{N}$. If one defines $\frac{a}{n} := \{(b, m) \in H \times \mathbb{N} \mid (a, n) \sim (b, m)\}$ and $H' := \{\frac{a}{n} \mid a \in H, n \in \mathbb{N}\}$ then H' equipped with the operation

$$\frac{a}{n} + \frac{b}{m} := \frac{ma + nb}{nm}$$

for all $\frac{a}{n}, \frac{b}{m} \in H'$ is an abelian torsion–free group such that $H \longrightarrow H', h \longmapsto \frac{h}{1}$ is an injective group homomorphism. Furthermore H' is divisible and can therefore be viewed as a $\mathbb{Q}$–vector space. It has a $\mathbb{Q}$ basis B which we assume to be well–ordered. Let $B = \{v_\alpha \mid \alpha < \gamma\}$ for an ordinal number $\gamma \in \mathbb{O}\mathrm{n}$. For each $\beta \leq \gamma$ we define H_β as the subspace of H' with basis $\{v_\alpha \mid \alpha < \beta\}$. Thus $\Lambda := \{H_\beta \mid \beta \leq \gamma\}$ is a subnormal system in H' with factors which are isomorphic to $\mathbb{Q}$. Lemma 1.2.26 implies that there exists a Conradian left–order on H' such that the elements of Λ are convex subgroups of H'. This induces a Conradian left–order on H such that the factors of the subnormal system of its convex subgroups are isomorphic to subgroups of $\mathbb{Q}$. Thus each finitely generated subgroup of such a factor is cyclic. □

Definition 1.2.28 A Conradian left–ordered group G has maximal rank if for each $a \in G$ the finitely generated subgroups of C_a^+/C_a^- are cyclic.

Lemma 1.2.29 If G is a Conradian left–ordered group with respect to $<$, there exists a Conradian left–order $<'$ on G such that G with $<'$ has maximal rank and each convex subgroup of G with respect to $<$ is also a convex subgroup with respect to the $<'$.

Proof. Let Λ be the set of all convex subgroups of G with respect to $<$. According to Remark 1.2.25 Λ is a subnormal system with Archimedian ordered factors. Since Archimedian ordered groups are abelian and torsion–free we can apply Theorem 1.2.27 which proves the claim. $\square$

Definition 1.2.30 ([16, page 50]) A group G is called locally indicable if for every finitely generated nontrivial subgroup U of G there is a nontrivial homomorphism from U onto $\mathbb{Z}$.

Theorem 1.2.31 ([21, Theorem 4.1.][15]) A group is locally indicable if and only if there exists a Conradian left–order of the group.

1.3 Group extensions and crossed product rings

Remark 1.3.1 Details about crossed product groups (group extensions) can be found in [22] and [1, Chapter 4.1].

Definition 1.3.2 Let H and N be groups and $\sigma : H \longrightarrow \mathrm{Aut}N, a \longmapsto \sigma_a$ as well as $\eta : H^2 \longrightarrow N$ be functions. We call (N, H, η, σ) a factor system if the following is true for all $a, b, c \in H$ and $u \in N$:

i) $\eta(a, e) = \eta(e, a) = e$,

ii) $\sigma_a\sigma_b(u) = \eta(a, b)\sigma_{ab}(u)\eta(a, b)^{-1}$,

iii) $\sigma_a (\eta(b, c)) \eta(a, bc) = \eta(a, b)\eta(ab, c)$.

Remark 1.3.3 If $u \in N$ then $\sigma_e\sigma_e(u) = \eta(e, e)\sigma_{e^2}(u)\eta(e, e)^{-1} = e \cdot \sigma_e(u) \cdot e = \sigma_e(u)$ and since σ_e is an automorphism we conclude $\sigma_e = \mathrm{id}_N$.

Definition 1.3.4 Let (N, H, η, σ) be a factor system. Then $N \rtimes_{\eta,\sigma} H$ is defined as the set $N \times H$ equipped with the operation

$$(u, a)(v, b) := (u\sigma_a(v)\eta(a, b), ab).$$

This set is called crossed product group of N and H with respect to the factor system (N, H, η, σ).

Theorem 1.3.5 If (N, H, η, σ) is a factor system, $N \rtimes_{\eta,\sigma} H$, the crossed product group of N and H with respect to (N, H, η, σ), is a group,

$$\iota : N \longrightarrow N \rtimes_{\eta,\sigma} H \ , \ u \longmapsto (u, e)$$

is an injective homomorphism and

$$\pi : N \rtimes_{\eta,\sigma} H \longrightarrow H \ , \ (u, a) \longmapsto a$$

is a surjective homomorphism with kernel $\ker \pi = \iota(N)$.

Theorem 1.3.6 Let G, H be groups, N be a normal subgroup of G and $G/N \cong H$. Then there exists a factor system (N, H, η, σ) such that $G \cong N \rtimes_{\eta,\sigma} H$.

Remark 1.3.7 Details about crossed product rings can be found in [20, Chapter 1].

Theorem 1.3.8 Let F be a skew field, G be a group and $(F^\times, G, \eta, \sigma)$ be a factor system. For any fixed set X and bijective map $x : G \longrightarrow X \ , \ g \longmapsto x_g$, define $F[G; \eta, \sigma]$ as the left F–vector space with basis X. Each element of $F[G; \eta, \sigma]$ has a unique representation in the form

$$\sum_{g \in G} a_g x_g$$

with $a_g \neq 0$ for only finitely many $g \in G$ and

$$\left(\sum_{g \in G} a_g x_g \right) \left(\sum_{h \in G} b_h x_h \right) := \sum_{g \in G} \sum_{h \in G} a_g x_g \cdot b_h x_h$$

with

$$a_g x_g \cdot b_h x_h := a_g \sigma_g(b_h) \eta(g, h) x_{gh}$$

defines a multiplication on $F[G; \eta, \sigma]$ such that $F[G; \eta, \sigma]$ is a ring with 1. We call $F[G; \eta, \sigma]$ a crossed product ring.

Remark 1.3.9 Let $F[G; \eta, \sigma]$ be a crossed product ring.

1. There are canonical embeddings of F and $F^\times \rtimes_{\eta,\sigma} G$ into $F[G; \eta, \sigma]$ and the group of units of $F[G; \eta, \sigma]$ respectively. These are

$$\begin{aligned} \pi_1 :& F \longrightarrow F[G; \eta, \sigma] \ , \ a \longmapsto a x_e, \\ \pi_2 :& F^\times \rtimes_{\eta,\sigma} G \longrightarrow \mathrm{U}(F[G; \eta, \sigma]) \ , \ (a, g) \longmapsto a x_g. \end{aligned}$$

   We will view F and $F^\times \rtimes_{\eta,\sigma} G$ as subsets of $F[G; \eta, \sigma]$.

2. For $T \subseteq F$ and $U \subseteq G$ we will write $TU := \{a x_g \mid a \in T, g \in U\}$. Thus, $F^\times G = F^\times \rtimes_{\eta,\sigma} G$.

Theorem 1.3.10 If G is a left–ordered group and $F[G;\eta,\sigma]$ is a crossed product ring, then the group of units in $F[G;\eta,\sigma]$ is equal to $F^\times G$.

Proof. As noted in Remark 1.3.9 each $ax_g \in F^\times G$ is a unit in $F[G;\eta,\sigma]$. Now let $b_1x_{h_1} + \cdots + b_mx_{h_m}$ be a unit in $F[G;\eta,\alpha]$ with $m \in \mathbb{N}$, $b_j \in F^\times$ and $h_j \in G$ for $j \leq m$. Without loss of generality one can assume that $h_1 < \cdots < h_m$. If $a_1x_{g_1} + \cdots + a_nx_{g_n}$ is the inverse of $b_1x_{h_1} + \cdots + b_mx_{h_m}$ with $n \in \mathbb{N}$, $a_i \in F^\times$ and $g_i \in G$ for $i \leq n$, as well as pairwise different g_i, we have

$$x_e = (a_1x_{g_1} + \cdots + a_nx_{g_n})\,(b_1x_{h_1} + \cdots + b_mx_{h_m})$$

and therefore

$$0 = \sum_{\substack{i\leq n, j\leq m\\ g_ih_j=g}} a_ix_{g_i}b_jx_{h_j} = \sum_{\substack{i\leq n, j\leq m\\ g_ih_j=g}} \underbrace{(a_i\sigma_{g_i}(b_j)\eta(g_i,h_j))}_{\neq 0}x_g$$

for all $g \in G$ with $g \neq e$. We choose $i', i'' \leq n$ with $g_{i'}h_1$ minimal and $g_{i''}h_m$ maximal in G. If $j \leq m$ and $i \leq n$, we observe

$$\left.\begin{array}{ll} g_{i'}h_1 \leq g_ih_1 < g_ih_j, & \text{for } j \neq 1\\ g_{i'}h_1 < g_ih_1 \leq g_ih_j, & \text{for } i \neq i' \end{array}\right\} \Longrightarrow g_{i'}h_1 < g_ih_j, \text{ for } (i,j) \neq (i',1).$$

Thus there is only one pair (i,j) with $g_ih_j = g_{i'}h_1$. For this we conclude $e = g_ih_j = g_{i'}h_1$. Similarly $g_{i''}h_m = e$ which implies $m = 1$. Hence $b_1x_{h_1} + \cdots + b_mx_{h_m} = b_1x_{h_1} \in F^\times G$. □

Theorem 1.3.11 Let G_1, G_2 be left–ordered groups and $F_1[G_1;\eta_1,\sigma_1], F_2[G_2;\eta_2,\sigma_2]$ be crossed product rings as well as

$$\varphi : F_1[G_1;\eta_1,\sigma_1] \longrightarrow F_2[G_2;\eta_2,\sigma_2]$$

be a ring homomorphism with $\varphi(1) = 1$. Then $\varphi(F_1) \subseteq F_2$ and there exists a unique group homomorphism $\psi : G_1 \longrightarrow G_2$, such that for every $g \in G_1$ there is an $a \in F_2^\times$ with $\varphi(x_g) = ax_{\psi(g)}$.

Proof. If $a \in F_1$ with $a = 0$ or $a = 1$, then $\varphi(a) = 0 \in F_2$ or $\varphi(a) = 1 \in F_2$. Now let us choose an $a \in F_1$ with $a \neq 0, 1$. Then a as well as $a - 1$ are units in $F_1[G_1,\eta_1,\alpha_1]$. Therefore $\varphi(a)$ and $\varphi(a-1)$ are units in $F_2[G_2,\eta_2,\alpha_2]$. Theorem 1.3.10 implies that there are $b \in F_2$, $g \in G_2$ with $\varphi(a) = bx_g$. Thus $bx_g - x_e = \varphi(a) - \varphi(1) = \varphi(a-1) \in F_2^\times G_2$. Theorem 1.3.10 now implies $g = e$ and therefore $\varphi(a) = bx_e \in F_2$.
For each $g \in G_1$ there are $a \in F_2^\times$ and $g' \in G_2$ with $\varphi(x_g) = ax_{g'}$. Let $\psi : G_1 \longrightarrow G_2$ be defined by $\psi(g) := g'$. If $g, h \in G_1$, there are $a, b \in F_2^\times$ and $g', h' \in G_2$ with $\varphi(x_g) = ax_{g'}$ and $\varphi(x_h) = bx_{h'}$. Since

$$\begin{aligned}\varphi(x_{gh}) &= \underbrace{\varphi(\eta_2(g,h))^{-1}}_{=:c\in F_2^\times}\varphi(\eta_2(g,h))\varphi(x_{gh}) = c\varphi(\eta_2(g,h)x_{gh}) = c\varphi(x_gx_h) = c\varphi(x_g)\varphi(x_h)\\ &= cax_{g'}bx_{h'} = \underbrace{ca\sigma_{2g'}(b)\eta_2(g',h')}_{\neq 0}x_{g'h'}\end{aligned}$$

we conclude $\psi(gh) = g'h' = \psi(g)\psi(h)$. Thus ψ is a group homomorphism. At last, ψ is uniquely defined by φ since for each $g \in G_1$ there is an $a \in F_2^\times$ with $\varphi(x_g) = ax_{\psi(g)}$. □

Remark 1.3.12 If $F[G;\eta,\sigma]$ is a crossed product ring and H is a subgroup of G, then $F[H;\eta|_{H\times H},\sigma|_H]$ is a crossed product ring and

$$\iota : F[H;\eta|_{H\times H},\sigma|_H] \longrightarrow F[G;\eta,\sigma]\,,\ a_1x_{h_1} + \cdots + a_nx_{h_n} \longmapsto a_1x_{h_1} + \cdots + a_nx_{h_n}$$

is an injective ring homomorphism with $\iota(1) = 1$. We will write $F[H;\eta,\sigma]$ instead of $F[H;\eta|_{H\times H},\sigma|_H]$ and can interpret $F[H;\eta,\sigma]$ as a subring of $F[G;\eta,\sigma]$.

Proposition 1.3.13 Let $F[G;\eta,\sigma]$ be a crossed product ring. Then the following statements hold.

i) If $a_1,\ldots,a_n \in F$ and $g_1,\ldots,g_n \in G$ for some $n \in \mathbb{N}$, then there is an $a \in F$ with

$$\prod_{i=1}^{n} a_ix_{g_i} = ax_{g_1\cdots g_n}.$$

Furthermore $a = 0$ implies $a_i = 0$ for some $i \le n$.

ii) If $g \in G$ then there is an $a \in F^\times$ with $x_g^{-1} = ax_{g^{-1}}$.

Proof.

i) We will prove this by induction on n. For $n = 1$ there is nothing to show. If $n > 1$ we have

$$\prod_{i=1}^{n} a_ix_{g_i} = a_1x_{g_1}\prod_{i=2}^{n} a_ix_{g_i} \overset{\text{IH}}{=} a_1x_{g_1}a'x_{g_2\cdots g_n} = \underbrace{a_1\sigma_{g_1}(a')\eta(g_1, g_2\cdots g_n)}_{=:a\in F}x_{g_1\cdots g_n}.$$

If $a = 0$ then $a_1 = 0$ or $\sigma_{g_1}(a') = 0$, where the latter implies $a' = 0$ and therefore $a_i = 0$ for some $i \le n$ with $i \ne 0$ by induction hypothesis.

ii) Since

$$x_{g^{-1}}x_g = \eta(g^{-1}, g)x_{g^{-1}g} = \eta(g^{-1}, g)x_e,$$

we have $x_g^{-1} = \eta(g^{-1}, g)^{-1}x_{g^{-1}}$. □

1.4 Formal power series

Remark 1.4.1 In the following section F will be a skew field and Γ a totally ordered nonempty set without maximal or minimal elements. Furthermore we define $\hat{\Gamma} := \Gamma \cup \{\infty\}$ together with $\gamma < \infty$ for all $\gamma \in \Gamma$.

Definition 1.4.2 Let $m : \Gamma \longrightarrow F$ be a function. Then $\operatorname{supp} m := \{\gamma \in \Gamma \mid m(\gamma) \neq 0\}$ is called the support of m. If $\operatorname{supp} m$ is a well–ordered subset of Γ we call m a formal power series (over Γ with coefficients in F). Furthermore $F((\Gamma))$ denotes the set of all formal power series over Γ with coefficients in F.

Remark 1.4.3

1. $F((\Gamma))$ is a right F–vector space with respect to the operations

$$m + m' : \Gamma \longrightarrow F\,,\ \gamma \longmapsto m(\gamma) + m'(\gamma),$$
$$ma : \Gamma \longrightarrow F\,,\ \gamma \longmapsto m(\gamma)a$$

   for $m, m' \in F((G))$ and $a \in F$.

2. For $m \in F((\Gamma))$ we define $m_\gamma := m(\gamma)$ for all $\gamma \in \Gamma$ and write m as the formal sum $\sum_{\gamma\in\Gamma} \gamma m_\gamma$ or just $\sum \gamma m_\gamma$ if there is no ambiguity. Then we have

$$\sum \gamma m_\gamma + \sum \gamma m'_\gamma = \sum \gamma \left(m_\gamma + m'_\gamma\right),$$
$$\left(\sum \gamma m_\gamma\right) \cdot a = \sum \gamma (m_\gamma a).$$

3. If $\operatorname{supp} m = \{\gamma_1, \ldots, \gamma_n\}$ for some $n \in \mathbb{N}_0$ we also write $m = \gamma_1 m_{\gamma_1} + \cdots + \gamma_n m_{\gamma_n}$.
4. We write γ instead of $\gamma 1$. Thus, we can treat Γ as a subset of $F((\Gamma))$.

Definition 1.4.4 For $\{m_i \mid i \in I\} \subseteq F((\Gamma))$ the formal sum $\sum_{i\in I} m_i$ is called convergent if for every $\gamma \in \Gamma$ there are only finitely many $i \in I$ with $\gamma \in \operatorname{supp} m_i$ and $\bigcup_{i\in I} \operatorname{supp} m_i$ is well–ordered.

Remark 1.4.5

1. If $\sum_{i\in I} m_i$ is convergent then

$$m : \Gamma \longrightarrow F\,,\ \gamma \longmapsto \sum_{i\in I} m_i(\gamma)$$

   is a well–defined function and $\operatorname{supp} m \subseteq \bigcup_{i\in I} \operatorname{supp} m_i$ is well–ordered. Thus, $m \in F((\Gamma))$. We write $\sum_{i\in I} m_i := m$.

2. If $\sum_{i\in I} m_i$ is convergent then $\sum_{i\in I'} m_i$ is convergent for each $I' \subseteq I$.
3. We will write $\sum m_i$ instead of $\sum_{i\in I} m_i$ if there is no ambiguity.

4. For $m \in F((\Gamma))$ we can interpret γm_γ as an element of $F((\Gamma))$. In this case the sum $\sum \gamma m_\gamma$ is convergent and therefore the formal sum coincides with the convergent sum.

5. If $\sum m_i$ is convergent then $\{m_i \mid i \in I\}$ is called summable in [5, §4 Definition 2].

Lemma 1.4.6 Let J, I be index sets $m_j \in F((\Gamma))$ for all $j \in J$ such that $\sum_{j\in J} m_j$ converges and $J_i \subseteq J$ for all $i \in I$ such that $J = \bigcup_{i\in I} J_i$ is a disjoint union. Then $\sum_{j\in J_i} m_j$ converges for all $i \in I$. If $m_i' := \sum_{j\in J_i} m_j$ then $\sum_{i\in I} m_i'$ converges and $\sum_{j\in J} m_j = \sum_{i\in I} m_i'$.

Proof. Since $J_i \subseteq J$ for each $i \in I$ there is nothing to show for the first convergence. Because of $\operatorname{supp} m_i' \subseteq \bigcup_{j\in J_i} \operatorname{supp} m_j$ we have

$$\bigcup_{i\in I} \operatorname{supp} m_i' \subseteq \bigcup_{i\in I} \bigcup_{j\in J_i} \operatorname{supp} m_j \subseteq \bigcup_{j\in J} \operatorname{supp} m_j$$

which implies that $\bigcup_{i\in I} \operatorname{supp} m_i'$ is well–ordered. If $\gamma \in \Gamma$ with $\gamma \in \operatorname{supp} m_i'$ for some $i \in I$ then there exists a $j \in J_i$ with $\gamma \in \operatorname{supp} m_j$. Since $J = \bigcup_{i\in I} J_i$ is a disjoint union and there are only finitely many $j \in J$ with $\gamma \in \operatorname{supp} m_j$ there are only finitely many $i \in I$ with $\gamma \in \operatorname{supp} m_i'$. Thus $\sum_{i\in I} m_i'$ converges.
For a fixed $\gamma \in \Gamma$ the set $J' := \{j \in J \mid \gamma \in \operatorname{supp} m_j\}$ is finite. If $J_i' := J' \cap J_i$ for all $i \in I$ then $J' = \bigcup_{i\in I} J_i'$ is a disjoint union. Hence

$$\sum_{j\in J} m_j(\gamma) = \sum_{j\in J'} m_j(\gamma) = \sum_{i\in I} \sum_{j\in J_i'} m_j(\gamma) = \sum_{i\in I} m_i'(\gamma)$$

and $\sum_{j\in J} m_j = \sum_{i\in I} m_i'$ is proven. □

Remark 1.4.7 ([9]) Let I, J be index sets and $m_{ij} \in F((\Gamma))$ for all $i \in I$ and $j \in J$. If $\sum_{(i,j)\in I\times J} m_{ij}$ converges then $\sum_{j\in J} m_{ij}$ converges for all $i \in I$, $\sum_{i\in I} m_i$ converges for $m_i = \sum_{j\in J} m_{ij}$ and $\sum_{(i,j)\in I\times J} m_{ij} = \sum_{i\in I} m_i$. This is proven by Lemma 1.4.6 if one chooses $J' := I \times J$ and $J_i' := \{i\} \times J$ for all $i \in I$.

Theorem 1.4.8 (cf. [7, 9]) For $\Delta \subseteq \Gamma$ anti–well–ordered we define

$$U_\Delta := \{m \in F((\Gamma)) \mid \forall \gamma \in \Delta : m(\gamma) = 0\}$$

and

$$\mathfrak{U} := \{U_\Delta \mid \Delta \subseteq \Gamma \text{ is anti–well–ordered}\}.$$

There is a topology on $F((\Gamma))$ such that $F((\Gamma))$ is a topological F–vector space and $\mathfrak{U}$ forms a basis for the neighborhood filter of $\mathcal{O}$. Hereby we will consider F as topological field with respect to the discrete topology.

Remark 1.4.9

1. We will always consider $F((\Gamma))$ as a topological F–vector space with respect to this topology.
2. An endomorphism f of $F((\Gamma))$ is continuous if and only if for every $U_\Delta \in \mathfrak{U}$ there is a $U_{\Delta'} \in \mathfrak{U}$ with $f(U_{\Delta'}) \subseteq U_\Delta$.
3. The set of all continuous endomorphisms of $F((\Gamma))$ forms a subalgebra with 1 of the endomorphism algebra of $F((\Gamma))$.
4. For details about topological vector spaces see [24].

Theorem 1.4.10 For $m \in F((\Gamma))$ and $m_i \in F((\Gamma))$ with $i \in I$ the following statements are equivalent.

i) The sum $\sum_{i\in I} m_i$ is convergent with $\sum_{i\in I} m_i = m$.

ii) For every anti–well–ordered $\Delta \subseteq \Gamma$ there is a finite $I' \subseteq I$ such that

$$\sum_{i\in I'} m_i \in m + U_\Delta$$

and $m_i \in U_\Delta$ for each $i \in I \setminus I'$.

Proof. "i)$\Longrightarrow$ii)": Let $\Delta \subseteq \Gamma$ be anti–well–ordered. Since $\sum_{i\in I} m_i$ is convergent, $\bigcup_{i\in I} \operatorname{supp} m_i$ is well–ordered and thus $\Delta' := \Delta \cap \bigcup_{i\in I} \operatorname{supp} m_i$ is finite. For each $\gamma \in \Gamma$ we define $I_\gamma = \{i \in I \mid \gamma \in \operatorname{supp} m_i\}$. Since $\sum_{i\in I} m_i$ is convergent, each I_γ is finite. Furthermore $I_\gamma = \emptyset$ for $\gamma \in \Delta \setminus \Delta'$. Now we define

$$I' := \{i \in I \mid m_i \notin U_\Delta\} = \bigcup_{\gamma\in\Delta} I_\gamma = \bigcup_{\gamma\in\Delta'} I_\gamma.$$

Since Δ' and each I_γ are finite, I' is finite. Furthermore $m_i \in U_\Delta$ for each $i \in I \setminus I'$ by the definition of I'. At last, if $\gamma \in \Delta$, then

$$m(\gamma) = \sum_{i\in I_\gamma} m_i(\gamma) = \sum_{i\in I'} m_i(\gamma)$$

and thus $\sum_{i\in I'} m_i \in m + U_\Delta$.
"ii)$\Longrightarrow$i)": Let $\Delta \subseteq \bigcup_{i\in I} \operatorname{supp} m_i$ be a strictly decreasing sequence. As such it is anti–well–ordered and therefore there is a finite $I' \subseteq I$ like in the premise. For $i \in I \setminus I'$ we have $m_i \in U_\Delta$ and thus $\operatorname{supp} m_i \cap \Delta = \emptyset$. Therefore, we conclude that $\Delta \subseteq \bigcup_{i\in I'} \operatorname{supp} m_i$ and since $\bigcup_{i\in I'} \operatorname{supp} m_i$ is a finite union of well–ordered subsets of Γ it is itself well–ordered. As Δ is well–ordered and anti–well–ordered it has to be finite which shows that $\bigcup_{i\in I} \operatorname{supp} m_i$ is well–ordered.
For any $\gamma \in \Gamma$ we define $\Delta := \{\gamma\}$. Thus, there is a finite $I' \subseteq I$ like in the premise. For each $i \in I$, if $\gamma \in \operatorname{supp} m_i$, then $m_i \notin U_\Delta$ and therefore $i \in I'$. As I' is finite there are only finitely many $i \in I$ with $\gamma \in \operatorname{supp} m_i$. Hence $\sum_{i\in I} m_i$ is convergent. Furthermore,

we have

$$\left(\sum_{i\in I} m_i\right)(\gamma) = \sum_{i\in I'} m_i(\gamma) = m(\gamma),$$

since $\sum_{i\in I'} m_i \in m + U_\Delta$. This shows $\sum_{i\in I} m_i = m$. □

Theorem 1.4.11 (cf. [7, 9]) Let f be an endomorphism of $F((\Gamma))$. The following statements are equivalent:

i) The endomorphism f is continuous.

ii) If $\sum m_i$ is convergent, then $\sum f(m_i)$ is convergent and $f\left(\sum m_i\right) = \sum f(m_i)$.

Proof. "i)$\Longrightarrow$ii)": Let $\sum_{i\in I} m_i$ be convergent with $m = \sum_{i\in I} m_i$ and $\Delta \subseteq \Gamma$ be an anti–well–ordered set. Since f is continuous there is an anti–well–ordered set $\Delta' \subseteq \Gamma$ with $f(U_{\Delta'}) \subseteq U_\Delta$. According to Theorem 1.4.10, since $\sum_{i\in I} m_i$ is convergent there is a finite set $I' \subseteq I$ such that

$$\sum_{i\in I'} m_i \in m + U_{\Delta'}$$

and $m_i \in U_{\Delta'}$ for each $i \in I \setminus I'$. Thus, we have

$$\sum_{i\in I'} f(m_i) = f\left(\sum_{i\in I'} m_i\right) \in f(m + U_{\Delta'}) = f(m) + f(U_{\Delta'}) \subseteq f(m) + U_\Delta,$$

and $f(m_i) \in U_\Delta$ for each $i \in I \setminus I'$. This shows that $\sum_{i\in I} f(m_i)$ is convergent with $\sum_{i\in I} f(m_i) = f(m) = f\left(\sum_{i\in I} m_i\right)$.
"ii)$\Longrightarrow$i)": For any anti–well–ordered $\Delta \subseteq \Gamma$ we define

$$\Delta' := \{\gamma \in \Gamma \mid \operatorname{supp} f(\gamma) \cap \Delta \neq \emptyset\}.$$

Let $\{\gamma_i \mid i \in I\} \subseteq \Delta'$ be a strictly increasing sequence. Then $\sum \gamma_i$ converges in $F((\Gamma))$. According to the premise this implies that $\sum f(\gamma_i)$ converges with $\sum f(\gamma_i) = f\left(\sum \gamma_i\right)$. Therefore, there is a finite $I' \subseteq I$, such that

$$\sum_{i\in I'} f(\gamma_i) \in f\left(\sum_{i\in I} \gamma_i\right) + U_\Delta$$

and $f(\gamma_i) \in U_\Delta$ for each $i \in I \setminus I'$. Thus $\operatorname{supp} f(\gamma_i) \cap \Delta = \emptyset$ for each $i \in I \setminus I'$. Since $\operatorname{supp} f(\gamma_i) \cap \Delta \neq \emptyset$ for each $i \in I$ this implies $I = I'$. Hence I is finite. As each strictly increasing sequence in Δ' is finite, Δ' is anti–well–ordered.
For $m \in U_{\Delta'}$ we can write $m = \sum_{\gamma\in\Gamma} \gamma m_\gamma$. According to the premise we know that $\sum_{\gamma\in\Gamma} f(\gamma m_\gamma)$ is convergent with $\sum_{\gamma\in\Gamma} f(\gamma m_\gamma) = f(m)$. Thus

$$f(m) = \sum_{\gamma\in\Gamma} \underbrace{f(\gamma m_\gamma)}_{\in U_\Delta} \in U_\Delta$$

and therefore $f(U_{\Delta'}) \subseteq U_\Delta$. □

Remark 1.4.12 In [5, §5 Definition 1] the notion of σ–linearity is introduced. Theorem 1.4.11 shows that σ–linear endomorphisms and continuous endomorphisms are exactly the same.

Definition 1.4.13 A map $f : F(\!(\Gamma)\!) \longrightarrow F(\!(\Gamma)\!)$ is called v–compatible if

$$v(m) < v(m') \Longleftrightarrow v(f(m)) < v(f(m'))$$

holds for all $m, m' \in F(\!(\Gamma)\!)$, where

$$v : F(\!(\Gamma)\!) \longrightarrow \hat{\Gamma}\,,\; m \longmapsto \begin{cases} \min \operatorname{supp} m & \text{for } m \neq \mathcal{O} \\ \infty & \text{else.} \end{cases}$$

Remark 1.4.14

1. If f is a v–compatible endomorphism of $F(\!(\Gamma)\!)$ then f is injective.
2. The v–compatible automorphisms of $F(\!(\Gamma)\!)$ form a group with respect to the composition.
3. In [5, §5 Definition 2] v–compatible endomorphisms are called monotone.

Theorem 1.4.15 ([6, Lemma 3][9, 23]) If f is a v–compatible continuous automorphism of $F(\!(\Gamma)\!)$ then f^{-1} is also a v–compatible continuous automorphism.

Definition 1.4.16 A map $f : F(\!(\Gamma)\!) \longrightarrow F(\!(\Gamma)\!)$ is called v–compatible on Γ if

$$\gamma < \gamma' \Longleftrightarrow v(f(\gamma)) < v(f(\gamma'))$$

for all $\gamma, \gamma' \in \Gamma$. Furthermore f is called surjective on Γ if for every $\gamma \in \Gamma$ there exists a $\gamma' \in \Gamma$ such that $\gamma = v(f(\gamma'))$.

Remark 1.4.17 A mapping $f : F(\!(\Gamma)\!) \longrightarrow F(\!(\Gamma)\!)$ which is v–compatible on Γ (surjective on Γ) is called locally monotone (locally surjective) in [5, §5 Definition 2, §5 Definition 4].

Theorem 1.4.18 (cf. [5, §5 Theorem 1][9]) If f is a continuous endomorphism of $F(\!(\Gamma)\!)$ then the following statements are equivalent:

i) f is a v–compatible automorphism,

ii) f is v–compatible on Γ and surjective on Γ.

Definition 1.4.19 An endomorphism f of $F(\!(\Gamma)\!)$ is called monomial if for every $\gamma \in \Gamma$ there exist $\gamma' \in \Gamma$ and $m_{\gamma'} \in F$ such that $f(\gamma) = \gamma' m_{\gamma'}$.

Lemma 1.4.20 (cf. [5, §7 Proposition 1][9]) Let $\mathcal{G}$ be a group of monomial continuous v–compatible automorphisms of $F((\Gamma))$. If D is the rational closure of $\mathcal{G}$ in the endomorphism ring of $F((\Gamma))$ then

$$\operatorname{supp} f(m) \subseteq \{v(g(\gamma)) \mid g \in \mathcal{G}, \gamma \in \operatorname{supp} m\}$$

for all $f \in D$ and $m \in F((\Gamma))$.

1.5 Skew fields of fractions of crossed product rings

Remark 1.5.1 For the remainder of this chapter F will be a skew field, G a Conradian left–ordered group and $F[G;\eta,\sigma]$ a crossed product ring.

Definition 1.5.2 If R is a ring and D a skew field then D is called skew field of fractions of R if there is an injective homomorphism $\varphi : R \longrightarrow D$ with $\varphi(1) = 1$ such that D is the rational closure of $\varphi(R)$ in D. The map φ is called the associated embedding of R into D.

Remark 1.5.3 Since φ is injective one can interpret R as a subring of D.

Definition 1.5.4 Let R be a ring with 1 containing $F[G;\eta,\sigma]$ as a subring such that both have the same 1. If U is a subgroup of G and $g \in G$, then R_U is the rational closure of $F^\times U$ (or $F[U;\eta,\sigma]$) in R and $R_g^+ := R_{C_g^+}$, $R_g^- := R_{C_g^-}$ as well as $R_g := R_{C_g}$.

Remark 1.5.5 If we are using complexity as in Definition 1.1.5 for R_U or its derived rings as in Definition 1.5.4, we will consider $F^\times U$ as the starting set M if nothing else is specified.

Proposition 1.5.6 If D is a skew field of fractions of $F[G;\eta,\sigma]$, then $x_g^{-k} D_g^- x_g^k = D_g^-$ for all $g \in G$ and $k \in \mathbb{Z}$.

Proof. At first we will show $x_g^{-k} r x_g^k \subseteq D_g^-$ for all $r \in D_g^-$, using induction on the complexity of $r \in D_g^-$. For $r \in FC_g^-$ there are $a \in F$ and $h \in C_g^-$ with $r = ax_h$. Applying Proposition 1.3.13 we have $x_g^{-k} a x_h x_g^k = a' x_{g^{-k}hg^k}$ for some $a' \in F$. Since C_g^- is a normal subgroup of C_g^+ we conclude $g^{-k}hg^k \in C_g^-$ and therefore $x_g^{-k} a x_h x_g^k = a' x_{g^{-k}hg^k} \in FC_g^- \subseteq D_g^-$.
If r is additively decomposable there are $r_1, \ldots, r_n \in D_g^-$ with $r = r_1 + \cdots + r_n$ and $r_1, \ldots, r_n \lhd r$. Applying the induction hypothesis we have

$$x_g^{-k} r x_g^k = x_g^{-k}(r_1 + \cdots + r_n) x_g^k = \underbrace{x_g^{-k} r_1 x_g^k}_{\in D_g^-} + \cdots + \underbrace{x_g^{-k} r_n x_g^k}_{\in D_g^-} \in D_g^-.$$

If r is multiplicatively decomposable there are $r_1, \ldots, r_n \in D_g^-$ with $r = r_1 \cdots r_n$ and

$r_1, \dots, r_n \lhd r$. Applying the induction hypothesis we have

$$x_g^{-k} r x_g^k = x_g^{-k}(r_1 \cdots r_n) x_g^k = \underbrace{(x_g^{-k} r_1 x_g^k)}_{\in D_g^-} \cdots \underbrace{(x_g^{-k} r_n x_g^k)}_{\in D_g^-} \in D_g^-.$$

If r is a proper atom, there is a $r_1 \in D_g^-$ with $r_1 \lhd r$ and $r = r_1^{-1}$. Applying the induction hypothesis we have

$$x_g^{-k} r x_g^k = x_g^{-k} r_1^{-1} x_g^k = \underbrace{(x_g^{-k} r_1 x_g^k)}_{\in D_g^-}{}^{-1} \in D_g^-.$$

This shows $x_g^{-k} D_g^- x_g^k \subseteq D_g^-$ for all $k \in \mathbb{Z}$, and therefore also $x_g^k D_g^- x_g^{-k} \subseteq D_g^-$ for all $k \in \mathbb{Z}$, which completes the proof. $\square$

Corollary 1.5.7 If D is a skew field of fractions of $F[G; \eta, \sigma]$, $g \in G$, $n_1, \dots, n_k \in \mathbb{Z}$ and $a_1, \dots, a_k \in D_g^-$ for some $k \in \mathbb{N}$, then there is an $a \in D_g^-$ with

$$\prod_{i=1}^{k} a_i x_g^{n_i} = a x_g^{n_1 + \cdots + n_k}.$$

Proof. We use induction on k and apply Proposition 1.5.6. If $k = 1$ there is nothing to show. For $k > 1$, by induction hypothesis, there is some $a' \in D_g^-$ with

$$\prod_{i=2}^{k} a_i x_g^{n_i} = a' x_g^{n_2 + \cdots + n_k}.$$

Therefore

$$\prod_{i=1}^{k} a_i x_g^{n_i} = a_1 x_g^{n_1} a' x_g^{n_2 + \cdots + n_k} = a_1 \underbrace{x_g^{n_1} a' x_g^{-n_1}}_{\in D_g^-} x_g^{n_1 + n_2 + \cdots + n_k} = a x_g^{n_1 + \cdots + n_k}$$

with $a = a_1(x_g^{n_1} a' x_g^{-n_1}) \in D_g^-$ by Proposition 1.5.6. $\square$

Definition 1.5.8 Let D be a skew field and x an indeterminate over D. If $a_n \in D$ for all $n \in \mathbb{Z}$ and $a_n = 0$ for all $n < N$ and some $N \in \mathbb{Z}$ then the formal sum

$$\sum_{n \in \mathbb{Z}} a_n x^n$$

is called (skew) formal Laurent series over D in x.

Remark 1.5.9 If $f = \sum_{n \in \mathbb{Z}} a_n x^n$ is a (skew) formal Laurent series over D in x as in Definition 1.5.8 we will also write $f = \sum_{n \geq N} a_n x^n$ or $f = \sum a_n x^n$.

Theorem 1.5.10 (cf. [3, Chapter 1.5]) Let D be a skew field, σ an automorphism of D, x an indeterminate over D and $D[[x;\sigma]]$ the set of all skew formal Laurent series over D in x. Then $D[[x;\sigma]]$ is a skew field with respect to the canonical addition and multiplication defined by

$$\left(\sum a_n x^n\right) \cdot \left(\sum b_n x^n\right) := \sum c_n x^n$$

whereas

$$c_n := \sum_{k+l=n} a_k \sigma^k(b_l).$$

We call $D[[x;\sigma]]$ the ring of skew formal Laurent series over D in x.

Remark 1.5.11 For $f \in D[[x;\sigma]]$ with

$$f = \sum a_n x^n$$

and $a_n = 0$ for all $n \in \mathbb{Z}$, $n \leq 0$ the inverse of $1 - f$ in $D[[x;\sigma]]$ can be calculated by applying the geometric series. The idea is to use $(1-f)^{-1} = 1 + f + f^2 + \dots$, although it is not formally defined. If

$$(1-f)^{-1} = \sum b_n x^n,$$

then $b_n = a_{0,n} + \dots + a_{n,n}$ whereas $a_{k,n}$ is defined by

$$f^k = \sum_{n \in \mathbb{Z}} a_{k,n} x^n$$

for each $k \in \mathbb{N}_0$. Thus,

$$a_{k,n} x^n = \sum_{n_1 + \dots + n_k = n} \prod_{i=1}^{k} a_{n_i} x^{n_i}$$

or

$$a_{k,n} = \left(\sum_{n_1 + \dots + n_k = n} \prod_{i=1}^{k} a_{n_i} x^{n_i} \right) x^{-n}$$

for all $n, k \in \mathbb{Z}$, $k \geq 0$. If $k \in \mathbb{N}_0$ and $n_1, \dots, n_k \in \mathbb{Z}$, then $\prod_{i=1}^{k} a_{n_i} x^{n_i} \neq 0$ implies $a_{n_i} \neq 0$ and therefore $n_i \geq 1$ for all $i \leq k$. Hence $n_1 + \dots + n_k \geq k$ if $\prod_{i=1}^{k} a_{n_i} x^{n_i} \neq 0$. By contraposition $a_{k,n} = 0$ for all $k, n \in \mathbb{Z}$ with $k > n$. We can write

$$b_n = \sum_{k \geq 0} \left(\sum_{n_1 + \dots + n_k = n} \prod_{i=1}^{k} a_{n_i} x^{n_i} \right) x^{-n},$$

since the occurring sums have only finitely many non–zero summands. For $\hat{f} \in D[[x;\sigma]]$, $\hat{f} \neq 0$ with

$$\hat{f} = \sum \hat{a}_n x^n$$

such that $N \in \mathbb{Z}$ is minimal with $\hat{a}_N \neq 0$ we write

$$\hat{f} = (\hat{a}_N x^N) \sum (\hat{a}_N x^N)^{-1} \hat{a}_n x^n = (\hat{a}_N x^N) \sum c_n x^n$$

with $c_n := (\hat{a}_N x^N)^{-1} \hat{a}_{n+N} x^N$. Then $c_n = 0$ for all $n \in \mathbb{Z}$, $n < 0$ and $c_0 = 1$. Applying the above results leads to

$$\left(\sum c_n x^n\right)^{-1} = \sum \hat{c}_n x^n$$

with $\hat{c}_n \in D$ defined by

$$\hat{c}_n = \sum_{k \geq 0} \left(\sum_{\substack{n_1,\dots,n_k \in \mathbb{N} \\ n_1+\dots+n_k=n}} \prod_{i=1}^{k} (-c_{n_i}) x^{n_i} \right) x^{-n}$$

for all $n \in \mathbb{Z}$. Now

$$\hat{f}^{-1} = \left(\sum c_n x^n\right)^{-1} (\hat{a}_N x^N)^{-1} = \sum \hat{c}_n x^n (\hat{a}_N x^N)^{-1} = \sum \hat{b}_n x^n$$

with $\hat{b}_n := \hat{c}_{n+N} x^{n+N} (\hat{a}_N x^N)^{-1} x^{-n} \in D$. Thus,

$$\begin{aligned} \hat{b}_n &= \sum_{k \geq 0} \left(\sum_{\substack{n_1,\dots,n_k \in \mathbb{N} \\ n_1+\dots+n_k=n+N}} \prod_{i=1}^{k} (-c_{n_i}) x^{n_i} \right) x^{-(n+N)} x^{n+N} (\hat{a}_N x^N)^{-1} x^{-n} \\ &= \sum_{k \geq 0} \left(\sum_{\substack{n_1,\dots,n_k \in \mathbb{N} \\ n_1+\dots+n_k=n+N}} \prod_{i=1}^{k} -(\hat{a}_N x^N)^{-1} \hat{a}_{n_i+N} x^{n_i+N} \right) (\hat{a}_N x^N)^{-1} x^{-n}. \end{aligned}$$

Proposition 1.5.12 (cf. [3, page 88]) Let D be a skew field, σ an automorphism of D and x an indeterminate over D. For the skew Laurent polynomial ring $D[x,x^{-1};\sigma]$ there is a unique injective ring homomorphism $\iota : D[x,x^{-1};\sigma] \longrightarrow D[[x;\sigma]]$ such that $\iota(a_k x^k + \dots + a_l x^l) = \sum a_n x^n$ for all $k,l \in \mathbb{Z}$, $k \leq l$ and $a_n \in D$ for $n \in \mathbb{Z}$ with $a_n = 0$ for $n \notin \{k,\dots,l\}$. We call ι the canonical embedding of $D[x,x^{-1};\sigma]$ into $D[[x;\sigma]]$ and view $D[x,x^{-1};\sigma]$ as a subring of $D[[x;\sigma]]$.

Remark 1.5.13 Let D_i be a skew field, σ_i an automorphism of D_i, x_i an indeterminate over D_i for $i \in \{1,2\}$ and $\varphi : D_1[x_1, x_1^{-1}; \sigma_1] \longrightarrow D_2[x_2, x_2^{-1}; \sigma_2]$ an injective ring homomorphism such that $\varphi(D_1) \subseteq D_2$ and $\varphi(x_1) = dx_2^l$ for some $d \in D_2$, $d \neq 0$ and $l \in \mathbb{N}$. Then

$$\psi : D_1[[x_1; \sigma_1]] \longrightarrow D_2[[x_2; \sigma_2]]\,,\ \sum a_n x_1^n \longmapsto \sum \hat{a}_n x_2^n$$

with $\hat{a}_n := a_m (dx_2^l)^m x_2^{-n}$ if there is some $m \in \mathbb{Z}$ with $n = lm$ and $\hat{a}_n = 0$ else, is a well–defined injective ring homomorphism such that

$$\begin{tikzcd}
D_1[x_1, x_1^{-1}; \sigma_1] \arrow[r, "\iota_1"] \arrow[d, "\varphi"'] & D_1[[x_1; \sigma_1]] \arrow[d, "\psi"] \\
D_2[x_2, x_2^{-1}; \sigma_2] \arrow[r, "\iota_2"] & D_2[[x_2; \sigma_2]]
\end{tikzcd}$$

is a commutative diagram, whereas ι_1, ι_2 are the canonical embeddings. Furthermore, ψ is uniquely defined by φ. This allows us to view $D_1[[x_1; \sigma_1]]$ as a subring of $D_2[[x_2; \sigma_2]]$.

1.6 Dubrovin–rings

Theorem 1.6.1 (cf. [10]) For $g \in G$ and $a \in F$, $a \neq 0$

$$f_{ax_g} : F(\!(G)\!) \longrightarrow F(\!(G)\!)\,,\ \sum_{\gamma \in G} \gamma m_\gamma \longmapsto \sum_{\gamma \in G} g\gamma \underbrace{\left(\sigma_{g\gamma}(a\eta(g,\gamma))m_\gamma\right)}_{\in F}$$

is a monomial, continuous, and v–compatible automorphism of $F(\!(G)\!)$. The map

$$f : F[G; \eta, \sigma] \longrightarrow \mathrm{End}(F(\!(G)\!))\,,\ a_1 x_{g_1} + \cdots + a_n x_{g_n} \longmapsto f_{a_1 x_{g_1}} + \cdots + f_{a_n x_{g_n}}$$

is a well–defined injective ring homomorphism.

Proof. Let $g \in G$ and $a \in F$, $a \neq 0$ be fixed. For $m \in F(\!(G)\!)$ and $\gamma \in G$ we define

$$\hat{m}_\gamma := g\gamma \left(\sigma_{g\gamma}(a\eta(g,\gamma))m_\gamma\right) \in F(\!(G)\!).$$

If $\gamma \in G$, then $\operatorname{supp} \hat{m}_\gamma = \{g\gamma\}$ for $\gamma \in \operatorname{supp} m$ and $\operatorname{supp} \hat{m}_\gamma = \emptyset$ else, since $\hat{m}_\gamma = 0$ is equivalent to $m_\gamma = 0$. If $\gamma' \in G$, then $\gamma' \in \operatorname{supp} \hat{m}_\gamma$ implies $\gamma' = g\gamma$ and therefore $\gamma = g^{-1}\gamma'$. Thus, there is only one $\gamma \in G$ with $\gamma' \in \operatorname{supp} \hat{m}_\gamma$. Since $\operatorname{supp} m$ is well–ordered and G is a left–ordered group, $g \operatorname{supp} m$ is well–ordered. Hence

$$\bigcup_{\gamma \in G} \operatorname{supp} \hat{m}_\gamma = \bigcup_{\gamma \in \operatorname{supp} m} g\gamma = g \operatorname{supp} m$$

implies that $\bigcup_{\gamma \in G} \operatorname{supp} \hat{m}_\gamma$ is well–ordered. Thus $\sum_{\gamma \in G} \hat{m}_\gamma$ converges and f_{ax_g} is well–defined. Furthermore, we have shown that $\operatorname{supp} f_{ax_g}(m) = g \operatorname{supp} m$. If $\gamma \in G$, then

$\operatorname{supp} f_{ax_g}(\gamma) = \{g\gamma\}$, hence f_{ax_g} is monomial.
Let $\Delta \subseteq G$ be anti–well–ordered. Then $g^{-1}\Delta \subseteq G$ is anti–well–ordered. If $m \in U_{g^{-1}\Delta}$, then $\gamma \notin \operatorname{supp} m$ for all $\gamma \in g^{-1}\Delta$. Thus, $g\gamma \notin g \operatorname{supp} m = \operatorname{supp} f_{ax_g}(m)$ for all $\gamma \in g^{-1}\Delta$ and therefore, $\gamma' \notin \operatorname{supp} f_{ax_g}(m)$ for all $\gamma' \in \Delta$. Hence $f_{ax_g}(U_{g^{-1}\Delta}) \subseteq U_\Delta$ and f_{ax_g} is continuous.
If $\gamma, \gamma' \in G$ with $\gamma < \gamma'$, then $v(f_{ax_g}(\gamma)) = g\gamma < g\gamma' = v(f_{ax_g}(\gamma'))$, since G is a left–ordered group. Hence f_{ax_g} is v–compatible on G. Furthermore, if $\gamma \in G$, then $v(f_{ax_g}(g^{-1}\gamma)) = gg^{-1}\gamma = \gamma$ with $g^{-1}\gamma \in G$. Hence f_{ax_g} is surjective on G. Since f_{ax_g} is continuous, v–compatible on G and surjective on G we can apply Theorem 1.4.18 which proves that f_{ax_g} is a v–compatible automorphism.
For $g \in G$ we define $f'_g : F \longrightarrow \operatorname{End}(F(\!(G)\!))\,,\ a \longmapsto f_{ax_g}$ whereas $f_0 := 0$. If $a_1, a_2 \in F$ and $m \in F(\!(G)\!)$, then

$$\begin{aligned} f'_g(a_1 + a_2)(m) &= \sum_{\gamma \in G} g\gamma\left(\sigma_{g\gamma}((a_1+a_2)\eta(g,\gamma))m_\gamma\right) \\ &= \sum_{\gamma \in G} g\gamma\left(\sigma_{g\gamma}(a_1\eta(g,\gamma))m_\gamma\right) + \sum_{\gamma \in G} g\gamma\left(\sigma_{g\gamma}(a_2\eta(g,\gamma))m_\gamma\right) \\ &= f'_g(a_1)(m) + f'_g(a_2)(m). \end{aligned}$$

Hence f'_g is a group homomorphism. Since $F[G;\eta,\sigma]$ is a left vector space with basis $\{x_g \mid g \in G\}$ it is also a direct sum of copies of the additive group F. We define the group homomorphisms $\iota_g : F \longrightarrow F[G;\eta,\sigma]\,,\ a \longmapsto ax_g$ for all $g \in G$. According to the universal property of direct sums there exists a unique group homomorphism $f : F[G;\eta,\sigma] \longrightarrow \operatorname{End}(F(\!(G)\!))$ such that

$$\begin{tikzcd} F \arrow[r, "\iota_g"] \arrow[dr, "f'_g"'] & F[G;\eta,\sigma] \arrow[d, "f"] \\ & \operatorname{End}(F(\!(G)\!)) \end{tikzcd}$$

is a commutative diagram for each $g \in G$. If $g, \gamma \in G$ and $a, m_\gamma \in F$, then

$$x_{g\gamma}^{-1} a x_g x_\gamma = x_{g\gamma}^{-1} a\eta(g,\gamma) x_{g\gamma} = \sigma_{g\gamma}(a\eta(g,\gamma))$$

and thus

$$f(ax_g)(\gamma m_\gamma) = f_{ax_g}(\gamma m_\gamma) = g\gamma\left(\sigma_{g\gamma}(a\eta(g,\gamma))m_\gamma\right) = g\gamma\left(x_{g\gamma}^{-1} a x_g x_\gamma m_\gamma\right).$$

Therefore,

$$\begin{aligned} f((a_1x_{g_1})(a_2x_{g_2}))(\gamma m_\gamma) &= f\big((\underbrace{a_1x_{g_1}a_2x_{g_2}x_{g_1g_2}^{-1}}_{\in F})x_{g_1g_2}\big)(\gamma m_\gamma) \\ &= (g_1g_2)\gamma\big(x_{(g_1g_2)\gamma}^{-1} a_1x_{g_1}a_2x_{g_2}x_{g_1g_2}^{-1}x_{g_1g_2}x_\gamma m_\gamma\big) \\ &= g_1(g_2\gamma)\big(x_{g_1(g_2\gamma)}^{-1} a_1x_{g_1}x_{g_2\gamma}x_{g_2\gamma}^{-1}a_2x_{g_2}x_\gamma m_\gamma\big) \\ &= f(a_1x_{g_1})\left(g_2\gamma\left(x_{g_2\gamma}^{-1}a_2x_{g_2}x_\gamma m_\gamma\right)\right) \\ &= f(a_1x_{g_1})f(a_2x_{g_2})(\gamma m_\gamma). \end{aligned}$$

for all $a_1, a_2 \in F^\times$ and $g_1, g_2 \in G$. Since all occurring endomorphisms are continuous, this proves $f((a_1x_{g_1})(a_2x_{g_2})) = f(a_1x_{g_1})f(a_2x_{g_2})$. This is sufficient to prove that f is a ring homomorphism. If $g_1, \ldots, g_n \in G$ are pairwise different and $a_1, \ldots, a_n \in F$ for some $n \in \mathbb{N}_0$, then $f(a_1x_{g_1} + \cdots + a_nx_{g_n}) = 0$ implies

$$g_1\sigma_{g_1}(a_1) + \cdots + g_n\sigma_{g_n}(a_n) = f_{a_1x_{g_1}}(e) + \cdots + f_{a_nx_{g_n}}(e) = 0.$$

Hence $\sigma_{g_1}(a_1) = \cdots = \sigma_{g_n}(a_n) = 0$, since $g_1, \ldots, g_n$ are linearly independent in $F(\!(G)\!)$. Therefore $a_1 = \cdots = a_n = 0$ which means that f is injective. $\square$

Definition 1.6.2 The rational closure of $f(F^\times G)$ in $\operatorname{End}(F(\!(G)\!))$ with f as in Theorem 1.6.1 is called the Dubrovin–ring of $F[G; \eta, \sigma]$.

Remark 1.6.3 The Dubrovin–ring R of $F[G; \eta, \sigma]$ is the rational closure of $f(F[G; \eta, \sigma])$ in $\operatorname{End}(F(\!(G)\!))$. Since f is injective, we can interpret $F[G; \eta, \sigma]$ as a subring of R.

Lemma 1.6.4 Let R be the Dubrovin–ring of $F[G; \eta, \sigma]$ and I a set. If $g \in G$ with $g > e$ and $a_i \in R_g^-$, $n_i \in \mathbb{Z}$ for all $i \in I$ such that for each $n \in \mathbb{Z}$ there are only finitely many $i \in I$ with $n_i \leq n$ and $a_i \neq 0$, then

$$\sum_{i \in I} a_i x_g^{n_i} m$$

converges for all $m \in F(\!(C_g^+)\!)$ such that $\sum_{i \in I} a_i x_g^{n_i} m \in F(\!(C_g^+)\!)$.

Proof. Since G is a Conradian left–ordered group, the factor group C_g^+/C_g^- is Archimedian ordered. Because of Lemma 1.4.20 we have

$$\begin{aligned}
\operatorname{supp} a_i x_g^{n_i} m &\subseteq \{v(ax_h\gamma) \mid ax_h \in F^\times C_g^-, \gamma \in \operatorname{supp} x_g^{n_i} m\} \\
&\subseteq \{h\gamma \mid h \in C_g^-, \gamma \in g^{n_i} \operatorname{supp} m\} \\
&= C_g^- g^{n_i} \operatorname{supp} m \subseteq C_g^- C_g^+ C_g^+ \subseteq C_g^+
\end{aligned}$$

for each $i \in I$. Let $\gamma \in G$ be fixed. If $i \in I$ with $\gamma \in \operatorname{supp} a_i x_g^{n_i} m$ then $a_i \neq 0$ and there are $\gamma' \in \operatorname{supp} m \subseteq C_g^+$ and $c \in C_g^-$ with $\gamma = cg^{n_i}\gamma'$. Since $v(m) \leq \gamma'$, we have $v(m)C_g^- \leq \gamma' C_g^-$. Hence

$$\gamma C_g^- = cg^{n_i}\gamma' C_g^- = g^{n_i}\gamma' C_g^- \geq g^{n_i} v(m) C_g^-$$

which implies $\gamma v(m)^{-1} C_g^- \geq g^{n_i} C_g^-$. Since C_g^+/C_g^- is Archimedian ordered there is an $n \in \mathbb{N}$ with

$$g^{n_i} C_g^- \leq \gamma v(m)^{-1} C_g^- < (gC_g^-)^n = g^n C_g^-$$

and thus $n_i < n$. Hence there are only finitely many $i \in I$ with $\gamma \in \operatorname{supp} a_i x_g^{n_i} m$.
Let M be a nonempty subset of $\bigcup_{i \in I} \operatorname{supp} a_i x_g^{n_i} m$ and $\gamma \in M$. As seen above there is

an $n \in \mathbb{N}$ with $\gamma v(m)^{-1}C_g^- < g^n C_g^-$. For all $i \in I$ and $\gamma' \in G$ with $n_i > n$ and $\gamma' \leq \gamma$ we have

$$\gamma' v(m)^{-1}C_g^- \leq \gamma v(m)^{-1}C_g^- < g^n C_g^- < g^{n_i} C_g^-$$

and therefore $\gamma' \notin \operatorname{supp} a_i x_g^{n_i} m$. We define $I' := \{i \in I \mid n_i \leq n, a_i \neq 0\}$, which is a finite set by assumption. Since $\bigcup_{i\in I'} \operatorname{supp} a_i x_g^{n_i} m$ is a finite union of well–ordered sets it is well–ordered itself. Because of $\gamma \in M \cap \bigcup_{i\in I'} \operatorname{supp} a_i x_g^{n_i} m$, there is a smallest element γ_0 in $M \cap \bigcup_{i\in I'} \operatorname{supp} a_i x_g^{n_i} m$. If $\gamma' \in M$ with $\gamma' \leq \gamma_0$, then $\gamma' \leq \gamma_0 \leq \gamma$ and $\gamma' \in \operatorname{supp} a_i x_g^{n_i} m$ for some $i \in I$. The argumentation above shows $n_i \leq n$ and thus $i \in I'$ by definition of I', which means $\gamma' \in M \cap \bigcup_{i\in I'} \operatorname{supp} a_i x_g^{n_i} m$ and therefore $\gamma_0 \leq \gamma'$. Hence γ_0 is the smallest element of M and $\bigcup_{i\in I} \operatorname{supp} a_i x_g^{n_i} m$ is well–ordered.
Since $\operatorname{supp} a_i x_g^{n_i} m \subseteq C_g^+$ for all $i \in I$ we conclude $\operatorname{supp} \sum a_i x_g^{n_i} m \subseteq C_g^+$. Hence $\sum a_i x_g^{n_i} m \in F(\!(C_g^+)\!)$. □

Corollary 1.6.5 Let R be the Dubrovin–ring of $F[G;\eta,\sigma]$, $g \in G$ with $g > e$ and $a_n \in R_g^-$ for all $n \in \mathbb{Z}$. If there exists an $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n < N$, then

$$\sum_{n\in\mathbb{Z}} a_n x_g^n m$$

converges for all $m \in F(\!(C_g^+)\!)$ and $\sum_{n\in\mathbb{Z}} a_n x_g^n m \in F(\!(C_g^+)\!)$.

Lemma 1.6.6 Let R be the Dubrovin–ring of $F[G;\eta,\sigma]$, $g \in G$ with $g > e$ and $r_1, \ldots, r_k \in R$, $N_1, \ldots, N_k \in \mathbb{Z}$ for some $k \in \mathbb{N}$. If $a_{i,n} \in R_g^-$ are continuous for all $n \in \mathbb{Z}$, $i \in \{1, \ldots, k\}$ and $a_{i,n} = 0$ for all $n < N_i$, $i \in \{1, \ldots, k\}$ such that

$$r_i m = \sum_{n\in\mathbb{Z}} a_{i,n} x_g^n m$$

for all $m \in F(\!(C_g^+)\!)$, then

$$\sum_{n\in\mathbb{Z}} \left(\sum_{\substack{n_1,\ldots,n_k\in\mathbb{Z} \\ n_1+\cdots+n_k=n}} \left(\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i} \right) m \right)$$

converges for all $m \in F(\!(C_g^+)\!)$ and

$$r_1 \cdots r_k m = \sum_{n\in\mathbb{Z}} \left(\sum_{\substack{n_1,\ldots,n_k\in\mathbb{Z} \\ n_1+\cdots+n_k=n}} \left(\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i} \right) m \right). \tag{1.1}$$

Proof. As seen in Corollary 1.5.7, if $n_1, \ldots, n_k \in \mathbb{Z}$, there is an $a_{n_1,\ldots,n_k} \in R_g^-$ with $\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i} = a_{n_1,\ldots,n_k} x_g^{n_1+\cdots+n_k}$ which means that

$$\left(\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i} \right) x_g^{-(n_1+\cdots+n_k)} = a_{n_1,\ldots,n_k} \in R_g^-.$$

Let $n \in \mathbb{Z}$ be fixed for now. If $n_1, \dots, n_k \in \mathbb{Z}$ with $a_{n_1,\dots,n_k} \neq 0$ and $n_1 + \dots + n_k \leq n$ then $a_{i,n_i} \neq 0$ for all $i \in \{1, \dots, k\}$, which implies $n_i \geq N_i$ for all $i \in \{1, \dots, k\}$. Therefore,

$$N_i \leq n_i \leq n - \sum_{j \neq i} n_j \leq n - \sum_{j \neq i} N_j$$

for all $i \in \{1, \dots, k\}$. Thus, there are only finitely many $(n_1, \dots, n_k) \in \mathbb{Z}^k$ with $a_{n_1,\dots,n_k} \neq 0$ and $n_1 + \dots + n_k \leq n$. According to Lemma 1.6.4 the sum

$$\sum_{(n_1,\dots,n_k) \in \mathbb{Z}^k} \underbrace{\left(\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i} \right) m}_{= a_{n_1,\dots,n_k} x_g^{n_1 + \dots + n_k} m} \tag{1.2}$$

converges for all $m \in F(\!(C_g^+)\!)$. Because of Lemma 1.4.6 and

$$\mathbb{Z}^k = \bigcup_{n \in \mathbb{Z}} \{(n_1, \dots, n_k) \in \mathbb{Z}^k \mid n_1 + \dots + n_k = n\}$$

being a disjoint union,

$$\sum_{n \in \mathbb{Z}} \left(\sum_{n_1 + \dots + n_k = n} \left(\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i} \right) m \right)$$

converges for all $m \in F(\!(C_g^+)\!)$ and is equal to (1.2).
To prove (1.1), we will use induction on k. For $k = 1$ prerequisite and claim are identical and there is nothing to show. If $k > 1$ we can apply the induction hypothesis on $r_1 \cdots r_{k-1}$. Thus

$$\begin{aligned}
r_1 \cdots r_{k-1} (r_k m) &\overset{\text{IH}}{=} \sum_{n \in \mathbb{Z}} \left(\sum_{n_1 + \dots + n_{k-1} = n} \left(\prod_{i=1}^{k-1} a_{i,n_i} x_g^{n_i} \right) r_k m \right) \\
&= \sum_{n \in \mathbb{Z}} \left(\sum_{n_1 + \dots + n_{k-1} = n} \left(\prod_{i=1}^{k-1} a_{i,n_i} x_g^{n_i} \right) \left(\sum_{n_k \in \mathbb{Z}} a_{k,n_k} x_g^{n_k} m \right) \right) \\
&= \sum_{n \in \mathbb{Z}} \left(\sum_{n_k \in \mathbb{Z}} \left(\sum_{n_1 + \dots + n_{k-1} = n} \left(\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i} \right) m \right) \right).
\end{aligned}$$

Hereby we use, that the $a_{i,n_i} x_g^{n_i}$ are continuous. Using the convergence in (1.2) as well as applying Lemma 1.4.6 and the facts that

$$\mathbb{Z}^k = \bigcup_{n \in \mathbb{Z}} \underbrace{\{(n_1, \dots, n_k) \in \mathbb{Z}^k \mid n_1 + \dots + n_{k-1} = n\}}_{=: M_n}$$

is a disjoint union and

$$M_n = \{(n_1, \dots, n_{k-1}) \in \mathbb{Z}^{k-1} \mid n_1 + \cdots + n_{k-1} = n\} \times \mathbb{Z}$$

for all $n \in \mathbb{Z}$ we observe that

$$\begin{aligned}
\sum_{(n_1,\dots,n_k)\in\mathbb{Z}^k} \left(\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i}\right) m &= \sum_{n\in\mathbb{Z}} \left(\sum_{\substack{(n_1,\dots,n_k)\in\mathbb{Z}^k \\ n_1+\cdots+n_{k-1}=n}} \left(\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i}\right) m \right) \\
&= \sum_{n\in\mathbb{Z}} \left(\sum_{n_k\in\mathbb{Z}} \left(\sum_{n_1+\cdots+n_{k-1}=n} \left(\prod_{i=1}^{k} a_{i,n_i} x_g^{n_i}\right) m \right)\right).
\end{aligned}$$

Thus we have proven (1.1). $\square$

2 Hughes–free embeddings

2.1 Hughes–free skew fields of fractions

Definition 2.1.1 (cf. [13]) Let $F[G;\eta,\sigma]$ be a crossed product ring and G a locally indicable group. A skew field D is called a Hughes–free skew field of fractions of $F[G;\eta,\sigma]$ if D is a skew field of fractions of $F[G;\eta,\sigma]$ and the following holds. For each finitely generated subgroup H of G and each normal subgroup N of H such that H/N is an infinite cyclic group with hN as a generating element of H/N every $t \in F^{\times}x_h$ is left transcendental over the rational closure D_N of $F[N;\eta,\sigma]$ in D, that is, $a_nt^n + \cdots + a_1t + a_0 = 0$ implies $a_n = \cdots = a_0 = 0$ for all $a_0, \ldots, a_n \in D_N$ and all $n \in \mathbb{N}_0$. The associated embedding is called Hughes–free embedding.

Definition 2.1.2 Let $F[G;\eta,\sigma]$ be a crossed product ring and G a Conradian left–ordered group with maximal rank. A skew field D is called a free skew field of fractions of $F[G;\eta,\sigma]$ if D is a skew field of fractions of $F[G;\eta,\sigma]$ and any $t \in F^{\times}x_g$ is left transcendental over the rational closure D_g^- of $F[C_g^-;\eta,\sigma]$ in D for each $g \in G \setminus \{e\}$. The associated embedding is called free embedding.

Proposition 2.1.3 Let $F[G;\eta,\sigma]$ be a crossed product ring and G a Conradian left–ordered group with maximal rank. If D is a Hughes–free skew field of fractions of $F[G;\eta,\sigma]$ then it is also a free skew field of fraction of $F[G;\eta,\sigma]$ with respect to the same embedding.

Proof. According to Theorem 1.2.31, G is locally indicable. For any $g \in G$, $g \neq e$ let $a_0, \ldots, a_n \in D_g^-$ and $t \in F^{\times}x_g$ be such that

$$a_nt^n + \cdots + a_1t + a_0 = 0.$$

Since $a_0, \ldots, a_n \in D_g^-$ there are $g_1, \ldots, g_k \in C_g^-$ for some $k \in \mathbb{N}$ such that $a_0, \ldots, a_n \in D_{\langle g_1,\ldots,g_n\rangle}$. If we define $U := \langle g, g_1, \ldots, g_k\rangle \subseteq C_g^+$ and $N := U \cap C_g^-$ then N is a normal subgroup of U such that U/N is infinite cyclic and gN is a generating element of U/N. Because of $g_1, \ldots, g_k \in U \cap C_g^- = N$ we know that $a_0, \ldots, a_n \in D_N$ and since D is a Hughes–free skew field of fractions of $F[G;\eta,\sigma]$ this implies $a_0 = \cdots = a_n = 0$. Thus we have proven that D is a free skew field of fractions of $F[G;\eta,\sigma]$. $\square$

2.2 Series Representations

Remark 2.2.1 For the remainder of this chapter we will assume that $F[G;\eta,\sigma]$ is a crossed product ring whereas G is a Conradian left–ordered group with maximal rank with respect to $<$ and D is a free skew field of fractions of $F[G;\eta,\sigma]$ with $\iota : F[G;\eta,\sigma] \longrightarrow D$ as the associated embedding. Furthermore, in most cases we will consider $F[G;\eta,\sigma]$ to be a subring of D.
If $g \in G$ then x_g induces an automorphism on $F[G;\eta,\sigma]$ by conjugation. It is an extension of σ_g and can also be extended onto D_g^- as seen in Proposition 1.5.6. We will denote all these automorphisms by σ_g. If $g \neq e$ then x_g is left transcendental over D_g^- since D is a free skew field of fractions of $F[G;\eta,\sigma]$ and therefore an indeterminate over D_g^-. The skew Laurent polynomial ring $D_g^-[x_g,x_g^{-1};\sigma_g]$ is an Ore–domain (cf. [1, Chapter 1.1 especially Proposition 1.1.4.]) with the unique skew field of fractions $D_g^-(x_g;\sigma_g)$ which is isomorphic to D_g. Since $D_g^-[[x_g;\sigma_g]]$ contains a skew field of fractions of $D_g^-[x_g,x_g^{-1};\sigma_g]$ there is a unique embedding of D_g into $D_g^-[[x_g;\sigma_g]]$ such that

$$\begin{tikzcd} D_g^-(x_g;\sigma_g) \arrow[r] \arrow[dr] & D_g \arrow[d] \\ D_g^-[x_g,x_g^{-1};\sigma_g] \arrow[u] \arrow[r] & D_g^-[[x_g;\sigma_g]] \end{tikzcd}$$

is a commutative diagram [1, page 88]. We will consider D_g to be a subring of $D_g^-[[x_g;\sigma_g]]$.

Remark 2.2.2 We formally define $D_e^-[[x_e;\sigma_e]] := F$ and write each $d \in D_e^-[[x_e;\sigma_e]]$ in the form

$$d = \sum_{n\in\mathbb{Z}} a_n x_e^n$$

with $a_0 = d$ and $a_n = 0$ for $n \in \mathbb{Z}$, $n \neq 0$.

Definition 2.2.3 If $d \in D$ and $g \in G$, $g \geq e$ such that $d \in D_g \subseteq D_g^-[[x_g;\sigma_g]]$, we call

$$d = \sum_{n\in\mathbb{Z}} a_n x_g^n \in D_g^-[[x_g;\sigma_g]]$$

series representation of d. If additionally $a_n \lhd d$ holds for all $n \in \mathbb{Z}$ or $d \in F^\times$ and $g = e$, we call the series representation proper. If there is an $h \in G$ such that $x_h^{-1}d$ or dx_h^{-1} has a (proper) series representation, it is called a (proper) left or right series representation respectively.

Theorem 2.2.4 If $d \in D$ and $h \in G$ such that

$$x_h^{-1} d = \sum_{n \in \mathbb{Z}} a_n x_g^n \in D_g^-[[x_g; \sigma_g]]$$

is a left series representation and $\hat{g}, \hat{h} \in G$ with $g, \hat{h}^{-1} h \in C_{\hat{g}}$ as well as $\hat{g} \geq e$, then there exist $\hat{a}_n \in D_{\hat{g}}^-$ for all $n \in \mathbb{Z}$ such that

$$x_{\hat{h}}^{-1} d = \sum_{n \in \mathbb{Z}} \hat{a}_n x_{\hat{g}}^n \in D_{\hat{g}}^-[[x_{\hat{g}}; \sigma_{\hat{g}}]],$$

is a left series representation such that one of the following alternatives holds.

i) If $C_g^+ = C_{\hat{g}}^+$, then for each $\hat{n} \in \mathbb{Z}$ there exist $n \in \mathbb{Z}$ and $b, c \in FG$ such that $\hat{a}_{\hat{n}} = b a_n c \in D_{\hat{g}}^-$ or $\hat{a}_{\hat{n}} = 0$.

ii) If $C_g^+ \subset C_{\hat{g}}^+$, then there are $b, c \in FG$ and $\hat{N} \in \mathbb{Z}$ with $\hat{a}_n = bdc \in D_{\hat{g}}^-$ for $n = \hat{N}$ and $\hat{a}_n = 0$ for $n \neq \hat{N}$.

Proof. Since $g, \hat{h}^{-1} h \in C_{\hat{g}}$ there are $l, m \in \mathbb{Z}$ and $g', h' \in C_{\hat{g}}^-$ with $g = g' \hat{g}^l$ as well as $\hat{h}^{-1} h = h' \hat{g}^m$. Hence there exist $b', c' \in FC_{\hat{g}}^-$ with $x_g = c' x_{\hat{g}}^l$ and $x_{\hat{h}}^{-1} x_h = b' x_{\hat{g}}^m$. As seen in Remark 1.5.13 this allows us to view $D_g^-[[x_g; \sigma_g]]$ as a subring of $D_{\hat{g}}^-[[x_{\hat{g}}; \sigma_{\hat{g}}]]$. We examine the following cases.

Case 1: $C_g^+ = C_{\hat{g}}^+$. For $g = \hat{g} = e$ there is nothing to show. If $g, \hat{g} > e$ then $l > 0$. For each $n \in \mathbb{Z}$ there exists a $c_n' \in FC_{\hat{g}}^-$ with $x_g^n = \left(c' x_{\hat{g}}^l\right)^n = c_n' x_{\hat{g}}^{ln}$. Therefore we have

$$\begin{aligned} x_{\hat{h}}^{-1} d = x_{\hat{h}}^{-1} x_h x_h^{-1} d = x_{\hat{h}}^{-1} x_h \sum_{n \in \mathbb{Z}} a_n x_g^n &= b' x_{\hat{g}}^m \sum_{n \in \mathbb{Z}} a_n \left(c_n' x_{\hat{g}}^{ln}\right) \\ &= \sum_{n \in \mathbb{Z}} b' x_{\hat{g}}^m a_n c_n' x_{\hat{g}}^{-m} x_{\hat{g}}^{ln+m}. \end{aligned}$$

We define $\hat{a}_{\hat{n}} := b' x_{\hat{g}}^m a_n c_n' x_{\hat{g}}^{-m} \in D_{\hat{g}}^-$ for $\hat{n}, n \in \mathbb{Z}$ with $\hat{n} = ln + m$ and $\hat{a}_{\hat{n}} = 0$ else, to get

$$x_{\hat{h}}^{-1} d = \sum_{n \in \mathbb{Z}} \hat{a}_n x_{\hat{g}}^n.$$

Because of $b', a_n, c_n' \in D_{\hat{g}}^-$ we have $\hat{a}_{ln+m} = b' x_{\hat{g}}^m a_n c_n' x_{\hat{g}}^{-m} \in D_{\hat{g}}^-$ as well as $b' x_{\hat{g}}^m, c_n' x_{\hat{g}}^{-m} \in FG$.

Case 2: $C_g^+ \neq C_{\hat{g}}^+$. Then $C_g^+ \subseteq C_{\hat{g}}^-$, which implies $x_h^{-1} d \in D_g \subseteq D_{\hat{g}}^-$ as well as

$$x_{\hat{h}}^{-1} d = x_{\hat{h}}^{-1} x_h x_h^{-1} d = b' x_{\hat{g}}^m x_h^{-1} d = \big(\underbrace{b' x_{\hat{g}}^m x_h^{-1}}_{\in FG} d \underbrace{x_{\hat{g}}^{-m}}_{\in FG}\big) x_{\hat{g}}^m.$$

Since $x_h^{-1} d, b' \in D_{\hat{g}}^-$, we have $b' x_{\hat{g}}^m x_h^{-1} d x_{\hat{g}}^{-m} \in D_{\hat{g}}^-$. We define $\hat{N} := m$. $\square$

Remark 2.2.5 The left series representation of $x_{\hat{h}}^{-1}d$ as given in Theorem 2.2.4 is proper if and only if the left series representation of $x_h^{-1}d$ is proper and $C_g^+ = C_{\hat{g}}^+$.

Theorem 2.2.6 Let

$$x_{h_i}^{-1}d_i = \sum_{n\in\mathbb{Z}} a_{i,n}x_{g_i}^n \in D_{g_i}^-[[x_{g_i};\sigma_{g_i}]]$$

be left series representations of $d_1,\dots,d_k \in D$. There exist $g,h \in G$ with $g \geq e$ and

$$C_g^+ = C_{g_1}^+ \cup \dots \cup C_{g_k}^+ \cup C_{h^{-1}h_1}^+ \cup \dots \cup C_{h^{-1}h_k}^+$$

as well as

$$g_1,\dots,g_k,h^{-1}h_1,\dots,h^{-1}h_k \in C_g.$$

For $d := d_1 + \dots + d_k$ there exists a left series representation

$$x_h^{-1}d = \sum_{n\in\mathbb{Z}} a_n x_g^n \in D_g^-[[x_g;\sigma_g]]$$

with $a_n = \hat{a}_{1,n} + \dots + \hat{a}_{k,n}$. Hereby for each $n \in \mathbb{Z}$ and $i \in \{1,\dots,k\}$ there are $n' \in \mathbb{Z}$ and $b,c \in FG$ with $\hat{a}_{i,n} = ba_{i,n'}c \in D_g^-$, $\hat{a}_{i,n} = bd_ic \in D_g^-$ or $\hat{a}_{i,n} = 0$. Furthermore, for each $n \in \mathbb{Z}$ there exist $i \in \{1,\dots,k\}$, $n' \in \mathbb{Z}$ and $b,c \in FG$ with $\hat{a}_{i,n} = ba_{i,n'}c \in D_g^-$ or $\hat{a}_{i,n} = 0$.

Proof. One can choose any $h \in h_1C_{g_1}^+ \cup \dots \cup h_kC_{g_k}^+$. Since the convex subgroups of G are ordered with respect to $\subseteq$ one of the convex subgroups $C_{g_1}^+,\dots,C_{g_k}^+,C_{h^{-1}h_1}^+,\dots,C_{h^{-1}h_k}^+$ is maximal, which we will denote by C^+. As G has maximal rank there is a $g \in C^+$, $g \geq e$ with $C_g^+ = C^+$ and $g_1,\dots,g_k,h^{-1}h_1,\dots,h^{-1}h_k \in C_g$. According to Theorem 2.2.4 there exists a left series representation

$$x_h^{-1}d_i = \sum_{n\in\mathbb{Z}} \hat{a}_{i,n}x_g^n \in D_g^-[[x_g;\sigma_g]],$$

for each $i \in \{1,\dots,k\}$, such that for every $n \in \mathbb{Z}$ there are $n_i \in \mathbb{Z}$ and $b,c \in FG$ with $\hat{a}_{i,n} = ba_{i,n_i}c \in D_g^-$, $\hat{a}_{i,n} = bd_ic \in D_g^-$ or $\hat{a}_{i,n} = 0$.
We examine the following cases.

Case 1: There is an $i \in \{1,\dots,k\}$ with $C_g^+ = C_{g_i}^+$. Then for each $n \in \mathbb{Z}$ there are $n_i \in \mathbb{Z}$ and $b,c \in FG$ with $\hat{a}_{i,n} = ba_{i,n_i}c \in D_g^-$ or $\hat{a}_{i,n} = 0$.

Case 2: $C_g^+ \neq C_{g_i}^+$ for all $i \in \{1,\dots,k\}$. Then there is a $j \in \{1,\dots,k\}$ with $C_g^+ = C_{h^{-1}h_j}^+$. Furthermore, for each $i \in \{1,\dots,k\}$ there are $N_i \in \mathbb{Z}$ as well as $b,c \in FG$ with $\hat{a}_{i,n} = bd_ic \in D_g^-$ for $n = N_i$ and $\hat{a}_{i,n} = 0$ else. To show that for each $n \in \mathbb{Z}$ there exists an $i \in \{1,\dots,k\}$ with $\hat{a}_{i,n} = 0$, it is sufficient to prove, that $N_1, \ldots, N_k$ are not all the same. According to Theorem 2.2.4 we have $h^{-1}h_i \in g^{N_i}C_g^-$ for all $i \in I$. Since $C_g^+ = C_{h^{-1}h_j}^+$, we know that $h^{-1}h_j \notin C_g^-$ and therefore $N_j \neq 0$. By choice of h there is a $j' \in \{1, \ldots, k\}$ with $h \in h_{j'}C_{g_{j'}}^+$. Hence $h^{-1}h_{j'} \in C_{g_{j'}}^+ \subseteq C_g^-$, which implies $N_{j'} = 0$. $\square$

Theorem 2.2.7 Let $d_1, \ldots, d_k \in D$ and $h_1, \ldots, h_{k+1} \in G$, such that each $d_i x_{h_{i+1}}$ with $i \in \{1, \ldots, k\}$ has a left series representation

$$x_{h_i}^{-1}(d_i x_{h_{i+1}}) = \sum_{n \in \mathbb{Z}} a_{i,n} x_{g_i}^n \in D_{g_i}^-[[x_{g_i}; \sigma_{g_i}]].$$

Then there are $g, h \in G$ with $g \geq e$ and

$$C_g^+ = C_{g_1}^+ \cup \cdots \cup C_{g_k}^+$$

as well as

$$g_1, \ldots, g_k, h^{-1}h_1 \in C_g,$$

such that $d := d_1 \cdots d_k x_{h_{k+1}}$ has a left series representation

$$x_h^{-1} d = \sum_{n \in \mathbb{Z}} a_n x_g^n \in D_g^-[[x_g; \sigma_g]]$$

with

$$a_n = \sum_{n_1 + \cdots + n_k = n} \left(\prod_{i=1}^{k} \hat{a}_{i,n_i} x_g^{n_i} \right) x_g^{-n}.$$

Hereby, for each $i \in \{1, \ldots, k\}$ and $n_i \in \mathbb{Z}$ there exist $n' \in \mathbb{Z}$ and $b, c \in FG$ such that $\hat{a}_{i,n_i} = b a_{i,n'} c \in D_g^-$, $\hat{a}_{i,n_i} = b d_i c \in D_g^-$ or $\hat{a}_{i,n_i} = 0$. Furthermore, there is an $i \in \{1, \ldots, k\}$ such that for each $n_i \in \mathbb{Z}$ there exist $n' \in \mathbb{Z}$ as well as $b, c \in FG$ with $\hat{a}_{i,n_i} = b a_{i,n'} c \in D_g^-$ or $\hat{a}_{i,n_i} = 0$.

Proof. Since the convex subgroups of G are totally ordered with respect to $\subseteq$, one of the convex subgroups $C_{g_1}^+, \ldots, C_{g_k}^+$, is maximal, which will be denoted by C^+. We may choose any $h \in h_1 C^+$ and since G has maximal rank there is a $g \in C^+$, $g \geq e$ with $C_g^+ = C^+$ and $g_1, \ldots, g_k, h^{-1}h_1 \in C_g$. According to Theorem 2.2.4 and since $h_2^{-1}h_2, \ldots, h_k^{-1}h_k \in C_g$ there exist left series representations

$$x_h^{-1}(d_1 x_{h_2}) = \sum_{n_1 \in \mathbb{Z}} \hat{a}_{1,n_1} x_g^{n_1} \in D_g^-[[x_g; \sigma_g]]$$

for $d_1 x_{h_2}$ and

$$x_{h_i}^{-1}(d_i x_{h_{i+1}}) = \sum_{n_i \in \mathbb{Z}} \hat{a}_{i,n_i} x_g^{n_i} \in D_g^-[[x_g; \sigma_g]]$$

for $d_i x_{h_{i+1}}$ with $i \in \{2, \ldots, k\}$, such that for every $n_i \in \mathbb{Z}$ there are $n' \in \mathbb{Z}$ and $b, c \in FG$ with $\hat{a}_{i,n_i} = b a_{i,n'} c \in D_g^-$, $\hat{a}_{i,n_i} = b d_i x_{h_{i+1}} c \in D_g^-$ or $\hat{a}_{i,n} = 0$.
Now we have

$$x_h^{-1} d = x_h^{-1} d_1 \cdots d_k x_{h_{k+1}} = x_h^{-1} d_1 x_{h_2} \prod_{i=2}^{k} x_{h_i}^{-1} d_i x_{h_{i+1}} = \prod_{i=1}^{k} \sum_{n_i \in \mathbb{Z}} \hat{a}_{i,n_i} x_g^{n_i} = \sum_{n \in \mathbb{Z}} a_n x_g^n$$

and

$$a_n = \sum_{n_1 + \cdots + n_k = n} \left(\prod_{i=1}^{k} \hat{a}_{i,n_i} x_g^{n_i} \right) x_g^{-n}.$$

Since $C_g^+ = C_{g_i}^+$ holds for at least one $i \in \{1, \ldots, k\}$ there exist $n' \in \mathbb{Z}$ and $b, c \in FG$ for each $n_i \in \mathbb{Z}$, such that $\hat{a}_{i,n_i} = b a_{i,n'} c \in D_g^-$ or $\hat{a}_{i,n_i} = 0$. □

Remark 2.2.8 The analogous statements about right series representations for Theorems 2.2.4, 2.2.6, 2.2.7 are also true.

Theorem 2.2.9 Each element $d \in D$ with $\operatorname{cp}(d) \geq 1$ has a proper left series representation. If

$$x_h^{-1} d = \sum_{n \in \mathbb{Z}} a_n x_g^n \in D_g \subseteq D_g^-[[x_g; \sigma_g]]$$

is a proper left series representation then C_g^+ and the left coset hC_g^+ are uniquely determined by d. The analogous statement for proper right series representations and right cosets also holds.

Proof. We will only prove the statements about left series representations as the respective statements about right series representations can be proven similarly. We will use induction on the complexity of d. The induction basis is $\operatorname{cp}(d) = 1$. Then $d \in FG$ and there are $g \in G$, $b \in F$ with $d = b x_g = x_{g^{-1}}^{-1}(x_{g^{-1}} b x_g)$. By choosing $h = g^{-1}$ and $a = x_{g^{-1}} b x_g \in F$ we are done. For the induction step we can assume $d \in D$ with $d \notin FG$.
If d is additively decomposable let $d = d_1 + \cdots + d_k$ with $d_1, \ldots, d_k \lhd d$ be a complete additive decomposition of d. Applying the induction hypothesis there are proper left series representations

$$x_{h_i}^{-1} d_i = \sum_{n \in \mathbb{Z}} a_{i,n} x_{g_i}^n \in D_{g_i}^-[[x_{g_i}; \sigma_{g_i}]]$$

for $d_1, \ldots, d_k$. Since $d \notin FG$ we know that $h_1 = \cdots = h_k$ and $g_1 = \cdots = g_k = e$ are not both true. Because of Theorem 2.2.6 there are $g, h \in G$ with $g > e$

$$C_g^+ = C_{g_1}^+ \cup \cdots \cup C_{g_k}^+ \cup C_{h^{-1}h_1}^+ \cup \cdots \cup C_{h^{-1}h_k}^+$$

as well as

$$g_1, \dots, g_k, h^{-1}h_1, \dots, h^{-1}h_k \in C_g,$$

such that for $d := d_1 + \cdots + d_k$ there exists a left series representation

$$x_h^{-1}d = \sum_{n\in\mathbb{Z}} a_n x_g^n \in D_g^-[[x_g;\sigma_g]]$$

with $a_n = \hat{a}_{1,n} + \cdots + \hat{a}_{k,n}$. Hereby, for each $n \in \mathbb{Z}$ and $i \in \{1, \dots, k\}$ there are $n' \in \mathbb{Z}$ and $b, c \in FG$ with $\hat{a}_{i,n} = ba_{i,n'}c$, $\hat{a}_{i,n} = bd_ic$ or $\hat{a}_{i,n} = 0$. Applying Theorem 1.1.15 leads to

$$\begin{aligned}\hat{a}_{i,n} &= ba_{i,n'}c \trianglelefteq a_{i,n'} \trianglelefteq d_i,\\ \hat{a}_{i,n} &= bd_ic \trianglelefteq d_i\end{aligned}$$

or $\hat{a}_{i,n} = 0 \trianglelefteq d_i$. Furthermore for each $n \in \mathbb{Z}$ there exist $i \in \{1, \dots, k\}$, $n' \in \mathbb{Z}$ and $b, c \in FG$ with $\hat{a}_{i,n} = 0 \triangleleft d_i$ or $\hat{a}_{i,n} = ba_{i,n'}c \triangleleft d_i$ if $d_i \notin F^\times G$.
Let us assume that there is an $n \in \mathbb{Z}$ such that no $i \in \{1, \dots, k\}$ satisfies $\hat{a}_{i,n} \triangleleft d_i$. Then $d_i \in F^\times G$ for all $i \in \{1, \dots, k\}$. Thus, there are $N_i \in \mathbb{Z}$ and $b, c \in FG$ for each $i \in \{1, \dots, k\}$ with $\hat{a}_{i,n} = bd_ic$ if $n = N_i$ and $\hat{a}_{i,n} = 0$ else. If $N_1 = \cdots = N_k$ then

$$d = x_h \left(\sum_{i=1}^{k} \hat{a}_{i,n}\right) x_g^n \in FG$$

for an $n \in \mathbb{Z}$, which contradicts $\mathrm{cp}(d) > 1$. Therefore, for each $n \in \mathbb{Z}$ there is an $i \in \{1, \dots, k\}$ with $\hat{a}_{i,n} = 0 \triangleleft d_i$, a contradiction to the above assumption.
Thus, we have proven that $\hat{a}_{i,n} \trianglelefteq d_i$ for each $i \in \{1, \dots, k\}$ and $\hat{a}_{i,n} \triangleleft d_i$ for some $i \in \{1, \dots, k\}$ if $n \in \mathbb{Z}$. Therefore, $a_n = \hat{a}_{1,n} + \cdots + \hat{a}_{k,n} \triangleleft d_1 + \cdots + d_k = d$ by Theorem 1.1.9. This implies that the left series representation

$$x_h^{-1}d = \sum_{n\in\mathbb{Z}} a_n x_g^n \in D_g^-[[x_g;\sigma_g]]$$

is proper.
If d is additively indecomposable and multiplicatively decomposable then d admits a complete multiplicative decomposition $d = d_1 \cdots d_k$ with $d_1, \dots, d_k \triangleleft d$. We will define $h_{k+1}, \dots, h_1$ in such a way, that $d_i x_{h_{i+1}}$ has a proper left decomposition

$$x_{h_i}^{-1}(d_i x_{h_{i+1}}) = \sum_{n\in\mathbb{Z}} a_{i,n} x_{g_i}^n \in D_{g_i}^-[[x_{g_i};\sigma_{g_i}]]$$

for $i \in \{1, \dots, k\}$ and we choose $h_{k+1} := e$. If $h_{k+1}, \dots, h_{i+1}$ are chosen, we can apply the induction hypothesis because of $\mathrm{cp}(d_i x_{h_{i+1}}) = \mathrm{cp}(d_i)$. Hence $d_i x_{h_{i+1}}$ has a proper left series representation of the required kind. As the d_i are proper atoms, we know that $g_1, \dots, g_k > e$ for all $i \in \{1, \dots, k\}$. According to Theorem 2.2.7 there are $g, h \in G$ with $g > e$ and

$$C_g^+ = C_{g_1}^+ \cup \cdots \cup C_{g_k}^+$$

as well as

$$g_1,\dots,g_k,h^{-1}h_1 \in C_g$$

such that $d := d_1\cdots d_k x_{h_{k+1}}$ has a left series representation

$$x_h^{-1}d = \sum_{n\in\mathbb{Z}} a_n x_g^n \in D_g^-[[x_g;\sigma_g]]$$

with

$$a_n = \sum_{n_1+\cdots+n_k=n}\left(\prod_{i=1}^k \hat{a}_{i,n_i}x_g^{n_i}\right)x_g^{-n}.$$

Hereby, for each $i\in\{1,\dots,k\}$ and $n_i\in\mathbb{Z}$ there are $n'\in\mathbb{Z}$ and $b,c\in FG$ with $\hat{a}_{i,n_i} = ba_{i,n'}c \trianglelefteq a_{i,n'} \trianglelefteq d_i$, $\hat{a}_{i,n_i} = bd_ic \trianglelefteq d_i$ or $\hat{a}_{i,n_i} = 0 \trianglelefteq d_i$. Furthermore, there is an $i\in\{1,\dots,k\}$ such that for each $n_i\in\mathbb{Z}$ there exist $n'\in\mathbb{Z}$ as well as $b,c\in FG$ with $\hat{a}_{i,n_i} = ba_{i,n'}c \trianglelefteq a_{i,n'} \lhd d_i$ or $\hat{a}_{i,n_i} = 0 \lhd d_i$.
If $n_i\in\mathbb{Z}$ is fixed for each $i\in\{1,\dots,k\}$ we know that $\hat{a}_{i,n_i}x_g^{n_i} \trianglelefteq \hat{a}_{i,n_i} \trianglelefteq d_i$ holds for all $i\in\{1,\dots,k\}$ and $\hat{a}_{i,n_i}x_g^{n_i} \trianglelefteq \hat{a}_{i,n} \lhd d_i$ is true for some $i\in\{1,\dots,k\}$. Therefore, we can apply Theorems 1.1.15 and 1.1.13 to show

$$\left(\prod_{i=1}^k \hat{a}_{i,n_i}x_g^{n_i}\right)x_g^{-n} \trianglelefteq \prod_{i=1}^k \underbrace{\hat{a}_{i,n_i}x_g^{n_i}}_{\trianglelefteq d_i} \lhd d.$$

Since d is additively indecomposable we can furthermore apply Remark 1.1.8 and conclude

$$a_n = \underbrace{\sum_{n_1+\cdots+n_k=n}\left(\prod_{i=1}^k \hat{a}_{i,n_i}x_g^{n_i}\right)x_g^{-n}}_{\lhd d} \lhd d,$$

which shows that the associated left series representation is proper.
If d is a proper atom then $d^{-1}\lhd d$ according to Theorem 1.1.12, which implies that d^{-1} has a proper right series representation

$$d^{-1}x_h^{-1} = \sum_{n\in\mathbb{Z}} \hat{a}_n x_g^n \in D_g^-[[x_g;\sigma_g]].$$

Then $\hat{a}_n \lhd d^{-1} \lhd d$ and therefore $\hat{a}_n x_g^n \lhd d^{-1}\lhd d$ for each $n\in\mathbb{Z}$. Let $N\in\mathbb{Z}$ be minimal with $\hat{a}_N\neq 0$. Since d is a proper atom $((\hat{a}_N x_g^N)^{-1})^{-1}\lhd d^{-1}$ implies $\left(\hat{a}_N x_g^N\right)^{-1}\lhd d$ according to Theorem 1.1.12. Now we have $x_{h^{-1}}^{-1}d = x_{h^{-1}}^{-1}(d^{-1}x_h^{-1}x_h)^{-1} = (x_{h^{-1}}^{-1}x_h^{-1})(d^{-1}x_h^{-1})^{-1}$. Hence d has a left series representation

$$x_{h^{-1}}^{-1}d = \sum_{n\in\mathbb{Z}} a_n x_g^n \in D_g^-[[x_g;\sigma_g]],$$

with

$$a_n = \sum_{k\geq 0} \sum_{\substack{n_1,\dots,n_k\in\mathbb{N}\\ n_1+\dots+n_k=n+N}} \left(x_{h^{-1}}^{-1}x_h^{-1}\right) \left[\prod_{i=1}^{k} - \underbrace{\left(\hat{a}_N x_g^N\right)^{-1}}_{\lhd d} \underbrace{\hat{a}_{n_i+N} x_g^{n_i+N}}_{\lhd d}\right] \underbrace{\left(\hat{a}_N x_g^N\right)^{-1}}_{\lhd d} x_g^{-n} \lhd d$$

according to Theorem 1.1.12 for the complexity and Remark 1.5.11 for calculating the inverse of $d^{-1}x_h^{-1}$.
To prove the uniqueness we take a $d \in D$ with $\operatorname{cp}(d) > 1$ as well as $h_1, h_2, g_1, g_2 \in G$, $g_1, g_2 > e$ such that d has proper left series representations

$$x_{h_i}^{-1} d = \sum_{n\in\mathbb{Z}} a_{i,n} x_{g_i}^n$$

for $i \in \{1,2\}$. As the series representations are proper, there are at least two $n \in \mathbb{Z}$ for each i with $a_{i,n} \neq 0$.
We first show $h_1 C_{g_1}^+ \cap h_2 C_{g_2}^+ \neq \emptyset$ and assume equality. Without loss of generality let $g := h_1^{-1}h_2 > e$. Then $h_1^{-1}h_2 \notin C_{g_1}^+, C_{g_2}^+$ and therefore $C_{g_1}^+, C_{g_2}^+ \subseteq C_g^-$. Hence

$$x_{h_1}^{-1}d, x_{h_2}^{-1}d \in D_g^-.$$

There is a $c \in F^\times$ with $x_{h_1}^{-1}x_{h_2} = c x_{h_1^{-1}h_2} = c x_g$. Therefore we know that

$$\begin{aligned} 0 = x_{h_1}^{-1}(d-d) &= x_{h_1}^{-1}d - \left(x_{h_1}^{-1}x_{h_2}\right) x_{h_2}^{-1}d \\ &= x_{h_1}^{-1}d - c x_g \left(x_{h_2}^{-1}d\right) x_g^{-1} x_g = a_0 + a_1 x_g \end{aligned}$$

with $a_0 = x_{h_1}^{-1}d \in D_g^-$ and $a_1 = -c x_g \left(x_{h_2}^{-1}d\right) x_g^{-1} \in D_g^-$. Since D is free, we conclude that $0 = a_0 = x_{h_1}^{-1}d$ and especially $d = 0$, which contradicts $\operatorname{cp}(d) > 1$. It remains to show that $C_{g_1}^+ = C_{g_2}^+$ since then $h_1 C_{g_1}^+ \cap h_2 C_{g_2}^+ \neq \emptyset$ would imply $h_1 C_{g_1}^+ = h_2 C_{g_2}^+$. We assume $C_{g_1}^+ \neq C_{g_2}^+$. Without loss of generality let $C_{g_2}^+ \subseteq C_{g_1}^-$. As G has maximal rank, there is a $g \in G$, $g > e$ with $C_g^+ = C_{g_1}^+$ and $g_1, h_1^{-1}h_2 \in C_g$. Without loss of generality we can assume that $g_1 = g$. Furthermore, we define $h := h_1$. Since $x_{h_2}^{-1}d \in D_{g_2} \subseteq D_g^-$ and $x_{h_1}^{-1}x_{h_2} \in FC_g$ there is an $N \in \mathbb{Z}$ and a $b_N \in D_g^-$ with $b_N x_g^N = x_{h_1}^{-1}x_{h_2}x_{h_2}^{-1}d$. Hence

$$0 = x_h^{-1}(d-d) = x_{h_1}^{-1}d - x_{h_1}^{-1}x_{h_2}x_{h_2}^{-1}d = \left(\sum_{n\in\mathbb{Z}} a_{1,n}x_g^n\right) - b_N x_g^N = \sum_{n\in\mathbb{Z}} a_n x_g^n$$

with $a_n = a_{1,n} \in D_g^-$ for all $n \in \mathbb{Z}$ with $n \neq N$ as well as $a_N = a_{1,N} - b_N \in D_g^-$. Since at least two of the $a_{1,n}$ are not zero some of the a_n are not zero. This is a contradiction, since x_g is an indeterminate over D_g^-. $\square$

2.3 Embedding Hughes–free skew fields of fractions into Dubrovin–rings

Theorem 2.3.1 If $d \in D$, $d \neq 0$,

$$x_h^{-1} d = \sum_{n \in \mathbb{Z}} a_n x_g^n$$

is a proper left series representation and $g' \in G$ then $d \in D_{g'}^+$ if and only if $g, h \in C_{g'}^+$.

Proof. "$\Longleftarrow$": For $x_h^{-1} d \in D_g$ we have $d \in x_h D_g \subseteq D_h^+ D_g^+ \subseteq D_{g'}^+$ because of $g, h \in C_{g'}^+$.
"$\Longrightarrow$": We will prove this statement by contradiction. Without loss of generality we can assume $h = e$ if $h \in C_g^+$.

Case 1: $h \notin C_{g'}^+$. Then $g' \in C_h^-$ and therefore $d \in D_{g'}^+ \subseteq D_h^-$. Furthermore we have $D_g^+ \subseteq D_h^-$, since $h \neq e$ and hence $g \in C_h^-$. Thus

$$0 = \underbrace{x_h^{-1} d}_{\in x_h^{-1} D_h^-} - \underbrace{x_h^{-1} d}_{\in D_g^+ \subseteq D_h^-} .$$

Since D is a free skew field of fractions we conclude $x_h^{-1} d = 0$ and $d = 0$, which contradicts $d \neq 0$.

Case 2: $h \in C_{g'}^+$ and $g \notin C_{g'}^+$. Then $C_h^+ \subseteq C_{g'}^+ \subseteq C_g^-$ and hence $d \in D_{g'}^+ \subseteq D_g^-$ as well as $h = e$. As such we get

$$\sum_{n \in \mathbb{Z}} a_n x_g^n = x_h^{-1} d = d \in D_g^-$$

and hence the contradiction $d = a_0 x_g^0 = a_0 \lhd d$. □

Corollary 2.3.2 For $g' \in G$ and $d \in D_{g'}^+$ ($d \in D_{g'}^-$) the following statements hold.

i) If d is additively decomposable in D there are $d_1, \ldots, d_k \in D_{g'}^+$ ($d_1, \ldots, d_k \in D_{g'}^-$) such that $d = d_1 + \cdots + d_k$ is a complete additive decomposition of d in D.

ii) If d is additively indecomposable and multiplicatively decomposable there are $d_1, \ldots, d_k \in D_{g'}^+$ ($d_1, \ldots, d_k \in D_{g'}^-$) such that $d = d_1 \cdots d_k$ is a complete multiplicative decomposition of d in D.

Proof. Since $D_{g'}^- = \bigcup \{D_{g''}^+ | g'' \in C_{g'}^-\}$ it is sufficient to prove the statements for $d \in D_{g'}^+$.

i) Because d is additively decomposable, it has a complete additive decomposition $d = d_1 + \cdots + d_k$ in D. Similar to the proof of Theorem 2.2.6 one gets a proper left series representation of d from proper series representations of $d_1, \ldots, d_k$ with

$$C_g^+ = C_{g_1}^+ \cup \cdots \cup C_{g_k}^+ \cup C_{h^{-1}h_1}^+ \cup \cdots \cup C_{h^{-1}h_k}^+ .$$

According to Theorem 2.3.1 we know that $g, h \in C_{g'}^+$ and as such

$$g_1, \ldots, g_k, h^{-1}h_1, \ldots, h^{-1}h_k \in C_g^+ \subseteq C_{g'}^+,$$

which also leads to $h_i \in hC_{g'}^+ \subseteq C_h^+ C_{g'}^+ \subseteq C_{g'}^+$ for each $i \in \{1, \ldots, k\}$. Hence $d_1, \ldots, d_k \in D_{g'}^+$.

ii) Since d is additively indecomposable and multiplicatively decomposable, it has a complete multiplicative decomposition $d = d_1' \cdots d_k'$ in D. Similar to the proof of Theorem 2.2.7 there are $h_1, \ldots, h_{k+1}$ and one gets a proper left series representation of d from proper series representations of $d_1' x_{h_2}, \ldots, d_k' x_{h_{k+1}}$ with

$$C_g^+ = C_{g_1}^+ \cup \cdots \cup C_{g_k}^+.$$

According to Theorem 2.3.1 we know that $g, h \in C_{g'}^+$ and as such

$$g_1, \ldots, g_k \in C_g^+ \subseteq C_{g'}^+.$$

Hence

$$d = \underbrace{\left(x_h x_{h_1}^{-1} d_1' x_{h_2}\right)}_{\in x_h D_{g_1}^+ \subseteq D_{g'}^+} \underbrace{\left(x_{h_2}^{-1} d_2' x_{h_3}\right)}_{\in D_{g_2}^+ \subseteq D_{g'}^+} \cdots \underbrace{\left(x_{h_k}^{-1} d_k' x_{h_{k+1}}\right)}_{\in D_{g_k}^+ \subseteq D_{g'}^+}.$$

We define $d_1 = x_h x_{h_1}^{-1} d_1' x_{h_2}$ and $d_i = x_{h_i}^{-1} d_i' x_{h_{i+1}}$ for $i \in \{2, \ldots, k\}$. By Theorem 1.1.15 this implies $\operatorname{cp}(d_i) = \operatorname{cp}(d_i')$ for all $i \in \{1, \ldots, k\}$. Thus, $d = d_1 \cdots d_k$ is a complete multiplicative decomposition of d in D with $d_1, \ldots, d_k \in D_{g'}^+$, since $d = d_1' \cdots d_k'$ is a complete multiplicative decomposition of d in D. □

Theorem 2.3.3 Let R be the Dubrovin-ring of $F[G; \eta, \sigma]$. There exists a unique ring isomorphism $\varphi : D \longrightarrow R$ such that

$$\begin{tikzcd} F[G;\eta,\sigma] \arrow[r, "\iota"] \arrow[dr, "f"'] & D \arrow[d, "\varphi"] \\ & R \end{tikzcd}$$

is a commutative diagram. Furthermore, every non–zero element of R is a v–compatible continuous automorphism of $F((G))$.

Proof. Let κ be the supremum of the complexities of the elements in D. We will construct a series of maps $\varphi_\alpha : D_\alpha \longrightarrow R$ for $1 \leq \alpha \leq \kappa$ with the following properties.

i) If $\beta < \alpha$, then $\varphi_\alpha|_{D_\beta} = \varphi_\beta$.

ii) If $d \in D_\alpha$ and $b, c \in FG$, then $\varphi_\alpha(bdc) = b\varphi_\alpha(d)c$.

iii) If $g \in G$, then $\varphi_\alpha(D_g^- \cap D_\alpha) \subseteq R_g^-$ and $\varphi_\alpha(D_g^+ \cap D_\alpha) \subseteq R_g^+$.

iv) Each non–zero element of $\varphi_\alpha(D_\alpha)$ is a continuous, v–compatible automorphism.

v) If $d \in D_\alpha$ with $\mathrm{cp}(d) > 1$ and

$$d = \sum_{n\in\mathbb{Z}} a'_n x_{g'}^n$$

is a proper series representation as well as $m \in F((C_{g'}^+))$, then

$$\varphi_\alpha(d)m = \sum_{n\in\mathbb{Z}} \varphi_\alpha(a'_n) x_{g'}^n m,$$

whereas the right sum is convergent.

vi) If $d_{ij} \in D_\alpha$ for all $i \leq k$ and $j \leq l_i$ then

$$\sum_{i=1}^{k}\prod_{j=1}^{l_i} d_{ij} = 0 \Longrightarrow \sum_{i=1}^{k}\prod_{j=1}^{l_i} \varphi_\alpha(d_{ij}) = 0.$$

We define $\varphi_1 : D_1 \longrightarrow R, ax_g \longmapsto ax_g$ and as such the above properties are fulfilled, whereas iv) was shown in Theorem 1.6.1. Assume that $\alpha > 1$ is fixed and the maps φ_β are defined for all $\beta < \alpha$. If α is a limit ordinal number we will define $\varphi_\alpha := \bigcup_{\beta<\alpha} \varphi_\alpha$ and the properties are obviously fulfilled. For the remainder we assume that α is a successor ordinal number. If $d \in D$ with $\mathrm{cp}(d) < \alpha$ we define $\varphi_\alpha(d) := \varphi_{\alpha-1}(d)$, which implies i) trivially. Thus, let $d \in D$ be with $\mathrm{cp}(d) = \alpha$. For $b, c \in FG$ with $b = 0$ or $c = 0$ and independent of the definition of $\varphi_\alpha(d)$, we have

$$\varphi_\alpha(bdc) = \varphi_\alpha(0) = \varphi_{\alpha-1}(0) = 0 = b\varphi_\alpha(d)c$$

and thus ii) holds for $b = 0$ or $c = 0$. Because of vi), if $a_1, \dots, a_k, b_1, \dots, b_l \in D_{\alpha-1}$ then

$$\begin{aligned}
&a_1 + \cdots + a_k = b_1 + \cdots + b_l \\
&\quad\Longrightarrow a_1 + \cdots + a_k + (-b_1) + \cdots + (-b_l) = 0 \\
&\quad\overset{vi)}{\Longrightarrow} \varphi_{\alpha-1}(a_1) + \cdots + \varphi_{\alpha-1}(a_k) + \varphi_{\alpha-1}(-b_1) + \cdots + \varphi_{\alpha-1}(-b_l) = 0 \\
&\quad\overset{ii)}{\Longrightarrow} \varphi_{\alpha-1}(a_1) + \cdots + \varphi_{\alpha-1}(a_k) - \varphi_{\alpha-1}(b_1) - \cdots - \varphi_{\alpha-1}(b_l) = 0 \\
&\quad\Longrightarrow \varphi_{\alpha-1}(a_1) + \cdots + \varphi_{\alpha-1}(a_k) = \varphi_{\alpha-1}(b_1) + \cdots + \varphi_{\alpha-1}(b_l)
\end{aligned}$$

and

$$\begin{aligned}
&a_1 \cdots a_k = b_1 \cdots b_l \\
&\quad\Longrightarrow a_1 \cdots a_k + (-1) \cdot b_1 \cdots b_l = 0 \\
&\quad\overset{vi)}{\Longrightarrow} \varphi_{\alpha-1}(a_1) \cdots \varphi_{\alpha-1}(a_k) + \varphi_{\alpha-1}(-1)\varphi_{\alpha-1}(b_1) \cdots \varphi_{\alpha-1}(b_l) = 0 \\
&\quad\overset{ii)}{\Longrightarrow} \varphi_{\alpha-1}(a_1) \cdots \varphi_{\alpha-1}(a_k) = \varphi_{\alpha-1}(b_1) \cdots \varphi_{\alpha-1}(b_l).
\end{aligned}$$

To define $\varphi_\alpha(d)$ we will examine the following cases.

Case 1: Let d be additively decomposable. Then d has a complete additive decomposition $d = d_1 + \cdots + d_k$ and we define

$$\varphi_\alpha(d) = \varphi_{\alpha-1}(d_1) + \cdots + \varphi_{\alpha-1}(d_k).$$

To prove that $\varphi_\alpha(d)$ is well–defined we have to show that it is independent of the complete additive decomposition. Let $d = d'_1 + \cdots + d'_{k'}$ be another complete additive decomposition of d. Then

$$d_1 + \cdots + d_k = d'_1 + \cdots + d'_{k'}$$

and therefore

$$\varphi_{\alpha-1}(d_1) + \cdots + \varphi_{\alpha-1}(d_k) = \varphi_{\alpha-1}(d'_1) + \cdots + \varphi_{\alpha-1}(d'_{k'})$$

according to vi). As seen in Corollary 2.3.2 we can assume that $d_1, \ldots, d_k \in D_g^-$ if $d \in D_g^-$ and therefore

$$\varphi_\alpha(d) = \underbrace{\varphi_{\alpha-1}(d_1)}_{\in R_g^-} + \cdots + \underbrace{\varphi_{\alpha-1}(d_k)}_{\in R_g^-} \in R_g^-,$$

which proves the first part of iii) for this case. The second part can be proven similarly. Furthermore, $\varphi_\alpha(d)$ is continuous, since it is a sum of continuous endomorphisms. If $b, c \in F^\times G$, then $\mathrm{cp}(bd_ic) = \mathrm{cp}(d_i)$ for all $i \in \{1, \ldots, k\}$ and $\mathrm{cp}(bdc) = \mathrm{cp}(d)$ by Theorem 1.1.15. Hence $bdc = bd_1c + \cdots + bd_kc$ is a complete additive decomposition of bdc in D. Therefore

$$\begin{aligned}\varphi_\alpha(bdc) &= \varphi_{\alpha-1}(bd_1c) + \cdots + \varphi_{\alpha-1}(bd_kc) \overset{ii)}{=} b\varphi_{\alpha-1}(d_1)\,c + \cdots + b\varphi_{\alpha-1}(d_k)\,c \\ &= b\left(\varphi_{\alpha-1}(d_1) + \cdots + \varphi_{\alpha-1}(d_k)\right)c = b\varphi_\alpha(d)c,\end{aligned}$$

which proves ii) for this case.
First we will prove v) only for one proper series representation and treat the general later on. As in the proof of Theorem 2.2.9 we see that $d_1, \ldots, d_k$ have left series representations

$$x_h^{-1}d_i = \sum_{n\in\mathbb{Z}} \hat{a}_{i,n}x_g^n.$$

Each of them is either proper or has only one non–zero summand. Thus, we can either apply v) or use the fact that the sum is finite to obtain

$$\varphi_{\alpha-1}(x_h^{-1}d_i) = \sum_{n\in\mathbb{Z}} \varphi_{\alpha-1}(\hat{a}_{i,n})x_g^n m$$

for each $m \in F(\!(C_g^+)\!)$ and $i \in \{1, \ldots, k\}$. Furthermore, d has a a proper left series representation

$$x_h^{-1}d = \sum_{n\in\mathbb{Z}} a_n x_g^n,$$

with $a_n = \hat{a}_{1,n} + \cdots + \hat{a}_{k,n}$ for each $n \in \mathbb{Z}$. By assumption, d has a proper series representation, which is a proper left series representation. Hence $h \in C_g^+$ by Theorem 2.2.9. Therefore, without loss of generality we can assume that $h = e$. For $m \in F((C_g^+))$ we conclude

$$\begin{aligned}
\varphi_\alpha(d)m &= (\varphi_{\alpha-1}(d_1) + \cdots + \varphi_{\alpha-1}(d_k))\, m = \varphi_{\alpha-1}(d_1)m + \cdots + \varphi_{\alpha-1}(d_k)m \\
&= \sum_{n\in\mathbb{Z}} \varphi_{\alpha-1}(\hat{a}_{1,n})x_g^n m + \cdots + \sum_{n\in\mathbb{Z}} \varphi_{\alpha-1}(\hat{a}_{k,n})x_g^n m \\
&= \sum_{n\in\mathbb{Z}} \big(\varphi_{\alpha-1}(\hat{a}_{1,n})x_g^n m + \cdots + \varphi_{\alpha-1}(\hat{a}_{k,n})x_g^n m\big) \\
&= \sum_{n\in\mathbb{Z}} \left(\varphi_{\alpha-1}(\hat{a}_{1,n}) + \cdots + \varphi_{\alpha-1}(\hat{a}_{k,n})\right) x_g^n m \\
&\overset{vi)}{=} \sum_{n\in\mathbb{Z}} \big(\varphi_{\alpha-1}(\hat{a}_{1,n} + \cdots + \hat{a}_{k,n})x_g^n m\big) \\
&= \sum_{n\in\mathbb{Z}} \varphi_{\alpha-1}(a_n)x_g^n m = \sum_{n\in\mathbb{Z}} \varphi_\alpha(a_n)x_g^n m.
\end{aligned}$$

The convergence of the sums is obtained by Lemmata 1.6.4 and 1.4.6.

Case 2: Let d be additively indecomposable and multiplicatively decomposable. Then d has a complete multiplicative decomposition $d = d_1 \cdots d_k$ and we define

$$\varphi_\alpha(d) = \varphi_{\alpha-1}(d_1) \cdots \varphi_{\alpha-1}(d_k).$$

To prove that $\varphi_\alpha(d)$ is well–defined we have to show, that it is independent of the complete multiplicative decomposition. Let $d = d'_1 \cdots d'_{k'}$ be another complete multiplicative decomposition of d. Then

$$d_1 \cdots d_k = d'_1 \cdots d'_{k'}$$

and therefore

$$\varphi_{\alpha-1}(d_1) \cdots \varphi_{\alpha-1}(d_k) = \varphi_{\alpha-1}(d'_1) \cdots \varphi_{\alpha-1}(d'_{k'})$$

according to vi). Because of iv) we know that $\varphi_{\alpha-1}(d_1), \ldots, \varphi_{\alpha-1}(d_k)$ are continuous, v–compatible automorphisms. Since $\varphi_{\alpha-1}(d)$ is a product of these automorphisms it is a continuous (Remark 1.4.9), v–compatible (Remark 1.4.14) automorphism as well, which proves iv) for this case. As seen in Corollary 2.3.2 we can assume $d_1, \ldots, d_k \in D_g^-$ if $d \in D_g^-$. Therefore

$$\varphi_\alpha(d) = \underbrace{\varphi_{\alpha-1}(d_1)}_{\in R_g^-} \cdots \underbrace{\varphi_{\alpha-1}(d_k)}_{\in R_g^-} \in R_g^-,$$

which proves the first part of iii) for this case. The second part can be proven similarly.

If $b, c \in FG$ with $b, c \neq 0$ then $\mathrm{cp}(bd_1) = \mathrm{cp}(d_1)$, $\mathrm{cp}(d_k c) = \mathrm{cp}(d_k)$ and $\mathrm{cp}(bdc) = \mathrm{cp}(d)$. Hence $bdc = (bd_1)d_2 \cdots d_{k-1}(d_k c)$ is a complete multiplicative decomposition of bdc in D. Therefore,

$$\begin{aligned}\varphi_\alpha(bdc) &= \varphi_{\alpha-1}(bd_1)\,\varphi_{\alpha-1}(d_2)\cdots\varphi_{\alpha-1}(d_{k-1})\varphi_{\alpha-1}(d_k c)\\ &\overset{ii)}{=} b\varphi_{\alpha-1}(d_1)\cdots\varphi_{\alpha-1}(d_k)\,c\\ &= b\varphi_\alpha(d)c,\end{aligned}$$

which proves ii) for this case.
We will prove v) only for one proper series representation and treat the general later on. As in the proof of Theorem 2.2.9 we define $h_{k+1}, \ldots, h_1 \in G$ with $h_{k+1} = e$ such that each $x_{h_i}^{-1} d_i x_{h_{i+1}}$ has a series representation

$$x_{h_i}^{-1} d_i x_{h_{i+1}} = \sum_{n\in\mathbb{Z}} \hat{a}_{i,n} x_g^n$$

for $i \in \{1, \ldots, k\}$. Each of them is either proper or has only one non–zero summand. Furthermore, d has a proper left series representation

$$x_h^{-1} d = \sum_{n\in\mathbb{Z}} a_n x_g^n,$$

with

$$a_n = \sum_{n_1+\cdots+n_k=n} \left(\prod_{i=1}^{k} \hat{a}_{i,n_i} x_g^{n_i}\right) x_g^{-n}$$

for each $n \in \mathbb{Z}$ and $h = h_1$. By assumption, d has a proper series representation, which is a proper left series representation. Hence $h \in C_g^+$ by Theorem 2.2.9. Therefore, without loss of generality we can assume that $h = e$. This means $h_1 \in C_g^+$ since $h^{-1}h_1 \in C_g \subseteq C_g^+$ as seen in the proof of Theorem 2.2.9. By the same argument as above we can therefore assume that $h_1 = e$. Applying ii), vi) and Lemma 1.6.6 (∗) we get

$$\begin{aligned}\varphi_\alpha(d)m &= \varphi_{\alpha-1}(d_1)\cdots\varphi_{\alpha-1}(d_k)m\\ &= (x_{h_1}^{-1}\varphi_{\alpha-1}(d_1)x_{h_2})\cdots(x_{h_k}^{-1}\varphi_{\alpha-1}(d_k)x_{h_{k+1}})m\\ &\overset{ii)}{=} \varphi_{\alpha-1}(x_{h_1}^{-1}d_1x_{h_2})\cdots\varphi_{\alpha-1}(x_{h_k}^{-1}d_kx_{h_{k+1}})m\\ &\overset{*}{=} \sum_{n\in\mathbb{Z}}\left(\sum_{n_1+\cdots+n_k=n}\left(\prod_{i=1}^{k}\varphi_{\alpha-1}(\hat{a}_{i,n_i})x_g^{n_i}\right)m\right)\\ &\overset{vi)}{=} \sum_{n\in\mathbb{Z}}\varphi_{\alpha-1}\left(\sum_{n_1+\cdots+n_k=n}\left(\prod_{i=1}^{k}\hat{a}_{i,n_i}x_g^{n_i}\right)\right)m\\ &= \sum_{n\in\mathbb{Z}}\varphi_{\alpha-1}(a_n)x_g^n m = \sum_{n\in\mathbb{Z}}\varphi_\alpha(a_n)x_g^n m.\end{aligned}$$

Case 3: Let d be a proper atom. Then d has an inverse $d^{-1} \in D$. Since d^{-1} has a proper left series representation according to Theorem 2.2.9 we can apply ii) and v) and conclude that $\varphi_{\alpha-1}(d^{-1}) \neq 0$. As such $\varphi_{\alpha-1}(d^{-1})$ is an automorphism according to iv) and we can define

$$\varphi_\alpha(d) := \left(\varphi_{\alpha-1}(d^{-1})\right)^{-1}.$$

According to Theorem 1.4.15, since $\varphi_{\alpha-1}(d^{-1})$ is a continuous, v–compatible automorphism, $\varphi_\alpha(d)$ is also a continuous, v–compatible automorphism, which proves iv). Furthermore, if $d \in D_g^-$, then $d^{-1} \in D_g^-$ and thus $\varphi_{\alpha-1}(d^{-1}) \in R_g^-$, which implies $\varphi_\alpha(d) := \left(\varphi_{\alpha-1}(d^{-1})\right)^{-1} \in R_g^-$. This proves the first part of iii) for this case. The second part can be proven similarly.

If $b, c \in FG$ with $b, c \neq 0$ then $\mathrm{cp}(bdc) = \mathrm{cp}(d)$, which implies that bdc is a proper atom. Therefore,

$$\begin{aligned}\varphi_\alpha(bdc) &= \varphi_{\alpha-1}((bdc)^{-1})^{-1} = \varphi_{\alpha-1}(c^{-1}d^{-1}b^{-1})^{-1} \\ &\overset{ii)}{=} \left(c^{-1}\varphi_{\alpha-1}(d^{-1})b^{-1}\right)^{-1} = b\varphi_{\alpha-1}(d^{-1})^{-1}c = b\varphi_\alpha(d)c,\end{aligned}$$

which proves ii) for this case.

We will prove v) only for one proper series representation and treat the general later on. As in the proof of Theorem 2.2.9 we construct a proper left series representation

$$x_h^{-1}d = \sum_{n\in\mathbb{Z}} a_n x_g^n$$

of d by using a proper right series representation

$$d^{-1}x_{h^{-1}}^{-1} = \sum_{n\in\mathbb{Z}} \hat{a}_n x_g^n$$

of d^{-1}. By assumption, d has a proper series representation, which is a proper left series representation. Hence $h \in C_g^+$ by Theorem 2.2.9. Therefore, without loss of generality we can assume that $h = e$. For these series representations we obtain

$$\begin{aligned}1 = dd^{-1} &= \sum_{n\in\mathbb{Z}} a_n x_g^n \sum_{n\in\mathbb{Z}} \hat{a}_n x_g^n \\ &= \sum_{n\in\mathbb{Z}} \underbrace{\left(\sum_{\substack{n_1,n_2\in\mathbb{Z}\\ n_1+n_2=n}} \left(a_{n_1}x_g^{n_1}\right)\left(\hat{a}_{n_2}x_g^{n_2}\right)x_g^{-n}\right)}_{\in D_g^-} x_g^n,\end{aligned}$$

which implies

$$\sum_{n_1+n_2=0} \left(a_{n_1}x_g^{n_1}\right)\left(\hat{a}_{n_2}x_g^{n_2}\right) = 1$$

and

$$\sum_{n_1+n_2=n} \left(a_{n_1} x_g^{n_1}\right) \left(\hat{a}_{n_2} x_g^{n_2}\right) x_g^{-n} = 0$$

for $n \neq 0$ whereas the left sums are all finite. Applying vi) and ii) we get

$$\sum_{n_1+n_2=0} \varphi_{\alpha-1}(a_{n_1}) x_g^{n_1} \varphi_{\alpha-1}(\hat{a}_{n_2}) x_g^{n_2} = 1$$

and

$$\sum_{n_1+n_2=n} \varphi_{\alpha-1}(a_{n_1}) x_g^{n_1} \varphi_{\alpha-1}(\hat{a}_{n_2}) x_g^{n_2} x_g^{-n} = 0$$

for $n \neq 0$.
For $m \in F((C_g^+))$ we define $m' := \varphi_\alpha(d)m$. Since $d \in D_g^+$, Lemma 1.4.20 and iii) imply

$$\begin{aligned} \operatorname{supp} m' = \operatorname{supp} \underbrace{\varphi_\alpha(d)}_{\in R_g^+} m &\subseteq \{v(ax_h(\gamma)) \mid ax_h \in F^\times C_g^+, \gamma \in \operatorname{supp} m\} \\ &= \{h\gamma \mid h \in C_g^+, \gamma \in \operatorname{supp} m \subseteq C_g^+\} \subseteq C_g^+ \end{aligned}$$

and therefore $m' \in F(\!(C_g^+)\!)$. Thus

$$m = \varphi_\alpha(d)^{-1} m' = \varphi_{\alpha-1}(d^{-1}) m' = \sum_{n \in \mathbb{Z}} \varphi_{\alpha-1}(\hat{a}_n) x_g^n m'.$$

Hence

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \varphi_\alpha(a_n) x_g^n m &= \sum_{n_1 \in \mathbb{Z}} \varphi_{\alpha-1}(a_{n_1}) x_g^{n_1} \left(\sum_{n_2 \in \mathbb{Z}} \varphi_{\alpha-1}(\hat{a}_{n_2}) x_g^{n_2} m' \right) \\ &= \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \varphi_{\alpha-1}(a_{n_1}) x_g^{n_1} \varphi_{\alpha-1}(\hat{a}_{n_2}) x_g^{n_2} m' \\ &= \sum_{n \in \mathbb{Z}} \left(\sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1+n_2=n}} \varphi_{\alpha-1}(a_{n_1}) x_g^{n_1} \varphi_{\alpha-1}(\hat{a}_{n_2}) x_g^{n_2} x_g^{-n} \right) x_g^n m' \\ &\overset{vi)}{=} m' = \varphi_\alpha(d) m. \end{aligned}$$

The convergence of the sums is obtained by Lemma 1.6.4.

Next we will prove that v) is independent of the choice of the proper series representation of d. Therefore, we take two proper series representations

$$d = \sum_{n\in\mathbb{Z}} a_n x_g^n,$$
$$d = \sum_{n\in\mathbb{Z}} \hat{a}_n x_{\hat{g}}^n$$

and show

$$\varphi_\alpha(d)m = \sum_{n\in\mathbb{Z}} \varphi_\alpha(a_n)x_g^n m = \sum_{n\in\mathbb{Z}} \varphi_\alpha(\hat{a}_n)x_{\hat{g}}^n m$$

for all $m \in F(\!(C_g^+)\!)$. Without loss of generality we can assume that $\hat{g}C_g^- = g^k C_g^-$ for some $k \in \mathbb{N}$. Remark 1.5.13 shows that we can view $D_{\hat{g}}^-[[x_{\hat{g}};\sigma_{\hat{g}}]]$ as a subring of $D_g^-[[x_g;\sigma_g]]$. Thus,

$$\sum_{n\in\mathbb{Z}} a_n x_g^n = d = \sum_{\hat{n}\in\mathbb{Z}} \hat{a}_{\hat{n}} x_{\hat{g}}^{\hat{n}} = \sum_{\hat{n}\in\mathbb{Z}} \underbrace{\left(\hat{a}_{\hat{n}} x_{\hat{g}}^{\hat{n}} x_g^{-k\hat{n}}\right)}_{\in D_g^-} x_g^{k\hat{n}}$$

implies $a_n = \hat{a}_{\hat{n}} x_{\hat{g}}^{\hat{n}} x_g^{-k\hat{n}}$ for all $n, \hat{n} \in \mathbb{Z}$ with $n = k\hat{n}$ and $a_n = 0$ else. Applying i), ii) and vi) we get $\varphi_\alpha(a_n) = \varphi_{\alpha-1}(a_n) = \varphi_{\alpha-1}(\hat{a}_{\hat{n}} x_{\hat{g}}^{\hat{n}} x_g^{-k\hat{n}}) = \varphi_\alpha(\hat{a}_{\hat{n}}) x_{\hat{g}}^{\hat{n}} x_g^{-k\hat{n}}$. Therefore

$$\sum_{n\in\mathbb{Z}} \varphi_\alpha(a_n)x_g^n m = \sum_{\hat{n}\in\mathbb{Z}} \left(\varphi_\alpha(\hat{a}_{\hat{n}}) x_{\hat{g}}^{\hat{n}} x_g^{-k\hat{n}}\right) x_g^{k\hat{n}} m = \sum_{\hat{n}\in\mathbb{Z}} \varphi_\alpha(\hat{a}_{\hat{n}}) x_{\hat{g}}^{\hat{n}} m$$

which proves that v) is independent of the choice of the proper series representation of d.

Now we will show statement iv) for all $d \in D_\alpha$ with $\mathrm{cp}(d) = \alpha$ which are additively decomposable. According to Theorem 1.4.18 since $\varphi_\alpha(d)$ is continuous it is sufficient to show that it is v–compatible on G and surjective on G to prove that it is a continuous, v–compatible automorphism.

If $\gamma, \gamma' \in G$ with $\gamma < \gamma'$ then $e < \gamma^{-1}\gamma'$. Since $\mathrm{cp}(dx_\gamma) = \mathrm{cp}(d) > 1$ there exists a proper left series representation

$$x_h^{-1}(dx_\gamma) = \sum_{n\in\mathbb{Z}} a_n x_g^n \in D_g$$

of dx_γ. Let $N \in \mathbb{Z}$ be minimal with $a_N \neq 0$. Since $x_h^{-1}(dx_\gamma) \in D_g \subseteq D_g^+$ we know that $x_h^{-1}\varphi_\alpha(dx_\gamma) \overset{ii)}{=} \varphi_\alpha(x_h^{-1}(dx_\gamma)) \overset{iii)}{\in} R_g^+$. This implies

$$\begin{aligned}
\varphi_\alpha(d)\gamma &= \varphi_\alpha(dx_\gamma x_\gamma^{-1})\gamma = \varphi_\alpha(dx_\gamma)x_\gamma^{-1}\gamma = \varphi_\alpha(dx_\gamma)e,\\
\varphi_\alpha(d)\gamma' &= \varphi_\alpha(dx_\gamma x_\gamma^{-1})\gamma' = \varphi_\alpha(dx_\gamma)x_\gamma^{-1}\gamma' = \varphi_\alpha(dx_\gamma)k\gamma^{-1}\gamma'
\end{aligned}$$

for some $k \in F^\times$. We examine the following two cases.

Case 1: $\gamma^{-1}\gamma' \notin C_g^+$. Applying Lemma 1.4.20 shows

$$\begin{aligned}
\operatorname{supp}\varphi_\alpha(d)\gamma &= \operatorname{supp} x_h\varphi_\alpha(x_h^{-1}dx_\gamma)e = h\operatorname{supp}\varphi_\alpha(x_h^{-1}dx_\gamma)e \subseteq hC_g^+e,\\
\operatorname{supp}\varphi_\alpha(d)\gamma' &= \operatorname{supp} x_h\varphi_\alpha(x_h^{-1}dx_\gamma)k\gamma^{-1}\gamma'\\
&= h\operatorname{supp}\varphi_\alpha(x_h^{-1}dx_\gamma)k\gamma^{-1}\gamma' \subseteq hC_g^+\gamma^{-1}\gamma'.
\end{aligned}$$

This implies $v(\varphi_\alpha(d)\gamma) < v(\varphi_\alpha(d)\gamma')$ as otherwise there would be $s, s' \in C_g^+$ with $hs\gamma^{-1}\gamma' \leq hs'$ and therefore $e < \gamma^{-1}\gamma' < s^{-1}s' \in C_g^+$ which would lead to $\gamma^{-1}\gamma' \in C_g^+$ since C_g^+ is convex.

Case 2: $\gamma^{-1}\gamma' \in C_g^+$. Then

$$\varphi_\alpha(d)\gamma = x_h \varphi_\alpha(x_h^{-1} d x_\gamma) e = x_h \sum_{n\in\mathbb{Z}} \varphi_\alpha(a_n) x_g^n e,$$
$$\varphi_\alpha(d)\gamma' = x_h \varphi_\alpha(x_h^{-1} d x_\gamma) k\gamma^{-1}\gamma' = x_h \sum_{n\in\mathbb{Z}} \varphi_\alpha(a_n) x_g^n (k\gamma^{-1}\gamma')$$

and hence

$$v(\varphi_\alpha(d)\gamma) = hv([\varphi_\alpha(a_N)x_g^N]e),$$
$$v(\varphi_\alpha(d)\gamma') = hv([\varphi_\alpha(a_N)x_g^N]k\gamma^{-1}\gamma').$$

Since $a_N \lhd d$ we can apply i) and iv) which shows that $\varphi_\alpha(a_N) = \varphi_{\alpha-1}(a_N)$ is v–compatible. Thus,

$$v([\varphi_\alpha(a_N)x_g^N]e) < v([\varphi_\alpha(a_N)x_g^N]k\gamma^{-1}\gamma')$$

and therefore

$$v(\varphi_\alpha(d)\gamma) = hv([\varphi_\alpha(a_N)x_g^N]e) < hv([\varphi_\alpha(a_N)x_g^N]k\gamma^{-1}\gamma') = v(\varphi_\alpha(d)\gamma').$$

The case study shows that $\varphi_\alpha(d)$ is v–compatible on G.
To prove the surjectivity on G we take a $\gamma \in G$ and have to show that there exists a $\gamma' \in G$ with $v(\varphi_\alpha(d)\gamma') = \gamma$. This is equivalent to $v(x_\gamma^{-1}\varphi_\alpha(d)\gamma') = e$. According to Theorem 2.2.9 there is an $h \in G$ such that $x_\gamma^{-1} d x_h^{-1}$ has a proper series representation

$$x_\gamma^{-1} d x_h^{-1} = \sum_{n\in\mathbb{Z}} a_n x_g^n.$$

Let $N \in \mathbb{Z}$ be minimal with $a_N \neq 0$. By i) we know that $\varphi_\alpha(a_N) = \varphi_{\alpha-1}(a_N)$. Hence $\varphi_\alpha(a_N)x_g^N$ is surjective on G and according to iv) there is a $\gamma' \in G$ with $v(\varphi_{\alpha-1}(a_N)x_g^N\gamma') = e$. Since

$$e \in \operatorname{supp} \varphi_\alpha(a_N)x_g^N\gamma' \subseteq C_g^- g^N \gamma' \subseteq C_g^+\gamma',$$

we conclude, that $\gamma' \in C_g^+$. Now we can apply v) which leads to

$$\varphi_\alpha(x_\gamma^{-1} d x_h^{-1})\gamma' = \sum_{n\in\mathbb{Z}} \varphi_\alpha(a_n) x_g^n \gamma'$$

and therefore $v(\varphi_\alpha(x_\gamma^{-1} d x_h^{-1})\gamma') = v(\varphi_\alpha(a_N)x_g^N\gamma') = e$. Thus, $\varphi_\alpha(d)$ is surjective on G.
All together we have shown that $\varphi_\alpha(d)$ is a continuous, v–compatible automorphism.
At last we have to prove statement vi). Therefore we will use the well–ordered set $\mathbb{N}(\mathbb{N}(\kappa))$ for a transfinite induction. Elements in $\mathbb{N}(\mathbb{N}(\kappa))$ will be written as

$$m_1\mu_1 \oplus \cdots \oplus m_n\mu_n = \bigoplus_{i=1}^{n} m_i\mu_i$$

with $\mu_1, \ldots, \mu_n \in \mathbb{N}(\kappa)$ and $m_1, \ldots, m_n \in \mathbb{N}_0$. The induction will run on

$$\bigoplus_{i=1}^{k} \sum_{j=1}^{l_i} \mathrm{cp}(d_{ij}) \in \mathbb{N}(\mathbb{N}(\kappa)).$$

As induction base we choose sums of the form $\bigoplus_{i=1}^{k} \sum_{j=1}^{l_i} \mathrm{cp}(d_{ij})$ with $\mathrm{cp}(d_{ij}) < \alpha$ for $i \leq k \in \mathbb{N}$ and $j \leq l_i$. Here we can use i) and the fact that $vi)$ holds for $\varphi_{\alpha-1}$.
In our induction step we will assume $d_{ij} \neq 0$ for all $i \leq k$ and $j \leq l_i$ without loss of generality as the corresponding products could be disregarded. Since $\varphi_\alpha(d_{ij})$ is continuous for all $i \leq k$ and $j \leq l_i$ we know that

$$\sum_{i=1}^{k} \prod_{j=1}^{l_i} \varphi_\alpha(d_{ij})$$

is also continuous and it is sufficient to show that

$$\sum_{i=1}^{k} \left(\prod_{j=1}^{l_i} \varphi_\alpha(d_{ij}) \right) x_\gamma e = \left(\sum_{i=1}^{k} \prod_{j=1}^{l_i} \varphi_\alpha(d_{ij}) \right) \gamma = \mathcal{O}$$

holds for all $\gamma \in G$.
Let $i \leq k$ be arbitrarily fixed for now. We can choose $h_{i,1}, \ldots, h_{i,l_i+1} \in G$ recursively such that each $d_{ij} x_{h_{i,j+1}}$ has a proper left series representation

$$x_{h_{i,j}}^{-1}(d_{ij} x_{h_{i,j+1}}) = \sum_{n \in \mathbb{Z}} a_{ij,n} x_{g_{ij}}^n.$$

Hereby we define $h_{i,l_i+1} := \gamma$ and generate all $h_{i,j}$ in ascending order of j. We define $h_i := h_{i,1}$ and apply Theorem 2.2.7 such that that

$$\left(\prod_{j=1}^{l_i} d_{ij} \right) x_\gamma = x_{h_{i,1}} \prod_{j=1}^{l_i} x_{h_{i,j}}^{-1} d_{ij} x_{h_{i,j+1}}$$

has a left series representation

$$x_{h_i}^{-1} \left(\prod_{j=1}^{l_i} d_{ij} \right) x_\gamma = \sum_{n \in \mathbb{Z}} a_{i,n} x_{g_i}^n$$

with

$$a_{i,n} = \sum_{n_1 + \cdots + n_{l_i} = n} \left(\prod_{j=1}^{l_i} \hat{a}_{ij,n_j} x_{g_i}^{n_j} \right) x_{g_i}^{-n}$$

for suitable $\hat{a}_{ij,n_j} \in D_{g_i}^-$ with $\hat{a}_{ij,n_j} \trianglelefteq d_{ij}$ for all $j \leq l_i$ and $\hat{a}_{ij,n_j} \triangleleft d_{ij}$ for at least one $j \leq l_i$.
We define $C := C_{g_1}^+ \cup \cdots \cup C_{g_k}^+$ and consider 2 cases.

Case 1: If $\bigcap_{i=1}^k h_iC = \emptyset$, we set $h := h_1$ and choose a $g \in G$ with

$$C^+_{h_1^{-1}h_2} \cup \cdots \cup C^+_{h_1^{-1}h_k} = C^+_g,$$
$$h_1^{-1}h_2, \ldots, h_1^{-1}h_k \in C_g.$$

Since $\bigcap_{i=1}^k h_iC = \emptyset$, there is an $i' \leq k$ with $h_1C \neq h_{i'}C$. Hence $h_1^{-1}h_{i'} \notin C$ and therefore $C^+_{g_i} \subseteq C \subseteq C^-_{h_1^{-1}h_{i'}} \subseteq C^-_g$. Furthermore there is an $n_i \in \mathbb{Z}$ for each $i \leq k$ with $h_1^{-1}h_iC^-_g = g^{n_i}C^-_g$. Therefore

$$x_h^{-1}\left(\prod_{j=1}^{l_i} d_{ij}\right) x_\gamma x_g^{-n_i} = \underbrace{(x_{h_1}^{-1}x_{h_i})}_{\in F^\times x_g^{n_i}} \underbrace{x_{h_i}^{-1}\left(\prod_{j=1}^{l_i} d_{ij}\right) x_\gamma\, x_g^{-n_i}}_{\in D_{g_i} \subseteq D_g^-} \in D_g^-$$

according to Proposition 1.5.6. $C^-_g = h_1^{-1}h_1C^-_g = g^{n_1}C^-_g$ implies $n_1 = 0$ and $n_i \neq 0$ for all $i \leq k$ with $C^+_{h_1^{-1}h_i} = C^+_g \neq C^-_g$. Therefore $n_1, \ldots, n_k$ are not all identical. For $n \in \mathbb{Z}$ we define $I_n := \{i \in \{1, \ldots, k\} \mid n_i = n\}$. We examine the series representations

$$x_h^{-1}\left(\sum_{i=1}^k \prod_{j=1}^{l_i} d_{ij}\right) x_\gamma = \sum_{i=1}^k x_h^{-1}\left(\prod_{j=1}^{l_i} d_{ij}\right) x_\gamma x_g^{-n_i} x_g^{n_i} = \sum_{n\in\mathbb{Z}} a_n x_g^n,$$

with

$$a_n = \sum_{i\in I_n} x_h^{-1}\left(\prod_{j=1}^{l_i} d_{ij}\right) x_\gamma x_g^{-n_i} = \sum_{i\in I_n}(x_h^{-1}d_{i1})\left(\prod_{j=2}^{l_i-1} d_{ij}\right)(d_{il_1}x_\gamma x_g^{-n_i}).$$

Since $n_1, \ldots, n_k$ are not all identical, we get

$$\begin{aligned}\bigoplus_{i\in I_n}\left(\mathrm{cp}(x_h^{-1}d_{i1}) + \sum_{j=2}^{l_i-1}\mathrm{cp}(d_{ij}) + \mathrm{cp}(d_{il_1}x_\gamma x_g^{-n_i})\right) &= \bigoplus_{i\in I_n}\sum_{j=1}^{l_i}\mathrm{cp}(d_{ij})\\ &< \bigoplus_{i=1}^{k}\sum_{j=1}^{l_i}\mathrm{cp}(d_{ij}).\end{aligned}$$

Because of $x_h^{-1}\left(\sum_{i=1}^k \prod_{j=1}^{l_i} d_{ij}\right) x_\gamma = 0$, we have $a_n = 0$ for all $n \in \mathbb{Z}$. Hence we can apply the induction hypothesis on

$$\sum_{i\in I_n}(x_h^{-1}d_{i1})\left(\prod_{j=2}^{l_i-1} d_{ij}\right)(d_{il_1}x_\gamma x_g^{-n_i}) = 0$$

and get

$$\begin{aligned}\sum_{i\in I_n} x_h^{-1}\left(\prod_{j=1}^{l_i}\varphi_\alpha(d_{ij})\right) x_\gamma x_g^{-n_i} &= \sum_{i\in I_n}\varphi_\alpha(x_h^{-1}d_{i1})\left(\prod_{j=2}^{l_i-1}\varphi_\alpha(d_{ij})\right)\varphi_\alpha(d_{il_1}x_\gamma x_g^{-n_i})\\ &= 0.\end{aligned}$$

By defining $M := \{n_1, \dots, n_k\}$ we conclude

$$\begin{aligned} x_h^{-1}\left(\sum_{i=1}^{k}\prod_{j=1}^{l_i}\varphi_\alpha(d_{ij})\right)x_\gamma &= \sum_{i=1}^{k} x_h^{-1}\left(\prod_{j=1}^{l_i}\varphi_\alpha(d_{ij})\right)x_\gamma x_g^{-n_i}x_g^{n_i} \\ &= \sum_{n\in M}\underbrace{\left(\sum_{i\in I_n} x_h^{-1}\left(\prod_{j=1}^{l_i}\varphi_\alpha(d_{ij})\right)x_\gamma x_g^{-n_i}\right)}_{=0} x_g^n \\ &= 0. \end{aligned}$$

Case 2: If $\bigcap_{i=1}^{k} h_i C \neq \emptyset$ we choose a $g \in G$ with

$$\begin{aligned} C_{g_1}^{+} \cup \dots \cup C_{g_k}^{+} &= C_g^{+}, \\ g_1, \dots, g_k &\in C_g, \\ h_1^{-1}h_2, \dots, h_1^{-1}h_k &\in C_g. \end{aligned}$$

Without loss of generality we can assume $g_i = g$ for all $i \leq k$ with $C_{g_i}^{+} = C_g^{+}$. If $i \leq k$ with $C_{g_i}^{+} \subseteq C_g^{-}$, then

$$x_{h_{i,j}}^{-1}(d_{ij}x_{h_{i,j+1}}) \in D_{g_{ij}} \subseteq D_{g_i} \subseteq D_g^{-}.$$

Therefore, we obtain left series representations for $\left(\prod_{j=1}^{l_i} d_{ij}\right) x_\gamma$ in the form of

$$x_{h_i}^{-1}\left(\prod_{j=1}^{l_i} d_{ij}\right)x_\gamma = \sum_{n\in\mathbb{Z}} a_{i,n}x_g^n$$

with

$$a_{i,n} = \sum_{n_1+\dots+n_{l_i}=n}\left(\prod_{j=1}^{l_i}\hat{a}_{ij,n_j}x_g^{n_j}\right)x_g^{-n},$$

where $\hat{a}_{ij,N_{ij}} = (x_{h_{i,j}}^{-1}d_{ij}x_{h_{i,j+1}})x_g^{-N_{ij}}$ and $\hat{a}_{ij,n_j} = 0$ holds for all $j \leq l_i$ with $n_j \neq N_{ij}$ for some $N_{ij} \in \mathbb{Z}$.
We define $h := h_1$. Without loss of generality one can assume $h = h_1 = \dots = h_k$ by simply adjusting the series representations since $h_1^{-1}h_i \in C_g$ for all $i \leq k$. Thus, we obtain the following series representation for 0

$$\begin{aligned} 0 = x_h^{-1}0x_\gamma = x_h^{-1}\left(\sum_{i=1}^{k}\prod_{j=1}^{l_i} d_{ij}\right)x_\gamma &= \sum_{i=1}^{k} x_h^{-1}\left(\prod_{j=1}^{l_i} d_{ij}\right)x_\gamma \\ = \sum_{i=1}^{k}\sum_{n\in\mathbb{Z}} a_{i,n}x_g^n = \sum_{n\in\mathbb{Z}} a_n x_g^n \end{aligned}$$

with $a_n = a_{1,n} + \cdots + a_{k,n}$ for all $n \in \mathbb{Z}$. Since x_g is an indeterminate over D_g^- we obtain

$$0 = a_n = \sum_{i=1}^{k} \sum_{n_{i1}+\cdots+n_{il_i}=n} \left(\prod_{j=1}^{l_i} \hat{a}_{ij,n_{ij}} x_g^{n_{ij}} \right) x_g^{-n}$$

and may apply the induction hypothesis. Hereby we will only consider the non–zero summands which means we will only use the $i \leq k$ and $n_{i1}, \ldots, n_{il_i} \in \mathbb{Z}$ with $n_{i1} + \cdots + n_{il_i} = n$ and $\hat{a}_{ij,n_{ij}} \neq 0$ for all $j \leq l_i$. Therefore, we define

$$T_n^i := \{(n_{i1}, \ldots, n_{il_i}) \in \mathbb{Z}^{l_i} \mid n_{i1} + \cdots + n_{il_i} = n \text{ and } \hat{a}_{ij,n_{ij}} \neq 0 \text{ for all } j \leq l_i\}.$$

If $i \leq k$ with $g_i \in C_g^-$, we have

$$\bigoplus_{(n_{i1},\ldots,n_{il_i}) \in T_n^i} \left(\left(\sum_{j=1}^{l_i-1} \mathrm{cp}(\hat{a}_{ij,n_{ij}} x_g^{n_{ij}}) \right) + \mathrm{cp}(\hat{a}_{ij,n_{il_i}} x_g^{n_{il_i}} x_g^{-n}) \right) = \sum_{j=1}^{l_i} \mathrm{cp}(d_{ij}),$$

since $\hat{a}_{ij,N_{ij}} = (x_{h_{i,j}}^{-1} d_{ij} x_{h_{i,j+1}}) x_g^{-N_{ij}}$ and $\hat{a}_{ij,n_{ij}} = 0$ for all $j \leq l_i$ and $n_{ij} \neq N_{ij}$. Thus, there is at most one tupel in T_n^i.
If $i \leq k$ with $g \in C_{g_i}$, then

$$\left(\sum_{j=1}^{l_i-1} \mathrm{cp}(\hat{a}_{ij,n_{ij}} x_g^{n_{ij}}) \right) + \mathrm{cp}(\hat{a}_{ij,n_{il_i}} x_g^{n_{il_i}} x_g^{-n}) < \sum_{j=1}^{l_i} \mathrm{cp}(d_{ij})$$

for all $n_{i1}, \ldots, n_{il_i} \in \mathbb{Z}$, since $\hat{a}_{ij,n_{ij}} \trianglelefteq d_{ij}$ for all $j \leq l_i$ and $\hat{a}_{ij,n_{ij}} \triangleleft d_{ij}$ for at least one $j \leq l_i$. Therefore, we obtain

$$\bigoplus_{(n_{i1},\ldots,n_{il_i}) \in T_n^i} \left(\left(\sum_{j=1}^{l_i-1} \mathrm{cp}(\hat{a}_{ij,n_{ij}} x_g^{n_{ij}}) \right) + \mathrm{cp}(\hat{a}_{ij,n_{il_i}} x_g^{n_{il_i}} x_g^{-n}) \right) < \sum_{j=1}^{l_i} \mathrm{cp}(d_{ij}).$$

This means

$$\bigoplus_{i=1}^{k} \bigoplus_{(n_{i1},\ldots,n_{il_i}) \in T_n^i} \left(\left(\sum_{j=1}^{l_i-1} \mathrm{cp}(\hat{a}_{ij,n_{ij}} x_g^{n_{ij}}) \right) + \mathrm{cp}(\hat{a}_{ij,n_{il_i}} x_g^{n_{il_i}} x_g^{-n}) \right) < \bigoplus_{i=1}^{k} \sum_{j=1}^{l_i} \mathrm{cp}(d_{ij}).$$

Applying the induction hypothesis we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{k} \sum_{(n_{i1},\ldots,n_{il_i}) \in T_n^i} \left(\prod_{j=1}^{l_i-1} \varphi_\alpha(\hat{a}_{ij,n_{ij}} x_g^{n_{ij}}) \right) \varphi_\alpha(\hat{a}_{il_i,n_{il_i}} x_g^{n_{il_i}} x_g^{-n}) \\ &= \sum_{i=1}^{k} \sum_{n_{i1}+\cdots+n_{il_i}=n} \left(\prod_{j=1}^{l_i} \varphi_\alpha(\hat{a}_{ij,n_{ij}}) x_g^{n_{ij}} \right) x_g^{-n}. \end{aligned}$$

Furthermore,

$$\begin{aligned}\left(\sum_{i=1}^{k}\left(\prod_{j=1}^{l_i}\varphi_\alpha(d_{ij})\right)x_\gamma\right)e=\mathcal{O} &\iff x_h^{-1}\left(\sum_{i=1}^{k}\left(\prod_{j=1}^{l_i}\varphi_\alpha(d_{ij})\right)x_\gamma\right)e=\mathcal{O}\\ &\iff \sum_{i=1}^{k}x_h^{-1}\left(\prod_{j=1}^{l_i}\varphi_\alpha(d_{ij})\right)x_\gamma e=\mathcal{O}\\ &\iff \sum_{i=1}^{k}\left(\prod_{j=1}^{l_i}\varphi_\alpha(x_{h_{i,j}}^{-1}d_{ij}x_{h_{i,j+1}})\right)e=\mathcal{O}.\end{aligned}$$

We will examine the summands separately and fix $i \leq k$. From the above considerations we gain a series representation of $x_{h_{i,j}}^{-1}d_{ij}x_{h_{i,j+1}}$ for each $j \leq l_i$

$$x_{h_{i,j}}^{-1}d_{ij}x_{h_{i,j+1}}=\sum_{n\in\mathbb{Z}}\hat{a}_{ij,n}x_g^n$$

and these are proper or satisfy $\hat{a}_{ij,N_{ij}}=x_{h_{i,j}}^{-1}d_{ij}x_{h_{i,j+1}}x_g^{-N_{ij}}$ and $\hat{a}_{ij,n}=0$ for all $n\in\mathbb{Z}$, $n\neq N_{ij}$ and some $N_{ij}\in\mathbb{Z}$. Furthermore, $x_{h_{i,j}}^{-1}d_{ij}x_{h_{i,j+1}}\in D_g$. Applying Theorem 1.6.6

$$\begin{aligned}\left(\prod_{j=1}^{l_i}\varphi_\alpha(x_{h_{i,j}}^{-1}d_{ij}x_{h_{i,j+1}})\right)e&=\sum_{n\in\mathbb{Z}}\left(\sum_{n_{i1}+\dots+n_{il_i}=n_i}\left(\prod_{j=1}^{l_i}\varphi_\alpha(\hat{a}_{ij,n_{ij}})x_g^{n_{ij}}\right)e\right)\\ &=\sum_{n\in\mathbb{Z}}\left(\sum_{n_{i1}+\dots+n_{il_i}=n}\left(\prod_{j=1}^{l_i}\varphi_\alpha(\hat{a}_{ij,n_{ij}})x_g^{n_{ij}}\right)x_g^{-n}\right)x_g^ne.\end{aligned}$$

All together we have shown that

$$\begin{aligned}&\sum_{i=1}^{k}\left(\prod_{j=1}^{l_i}\varphi_\alpha(x_{h_{i,j}}^{-1}d_{ij}x_{h_{i,j+1}})\right)e\\ =&\sum_{i=1}^{k}\sum_{n\in\mathbb{Z}}\left(\sum_{n_{i1}+\dots+n_{il_i}=n}\left(\prod_{j=1}^{l_i}\varphi_\alpha(\hat{a}_{ij,n_{ij}})x_g^{n_{ij}}\right)x_g^{-n}\right)x_g^ne\\ =&\sum_{n\in\mathbb{Z}}\underbrace{\sum_{i=1}^{k}\left(\sum_{n_{i1}+\dots+n_{il_i}=n}\left(\prod_{j=1}^{l_i}\varphi_\alpha(\hat{a}_{ij,n_{ij}})x_g^{n_{ij}}\right)x_g^{-n}\right)}_{=0}x_g^ne\\ =&\mathcal{O}.\end{aligned}$$

The convergence of the sums is secured by Lemma 1.6.4. We conclude

$$\left(\sum_{i=1}^{k}\left(\prod_{j=1}^{l_i}\varphi_\alpha(d_{ij})\right)x_\gamma\right)e=0.$$

Let $\varphi : D \longrightarrow R$ be defined by $\varphi(d) = \varphi_\alpha(d)$ for $d \in D$ with $\alpha = \mathrm{cp}(d)$. Because of i) we get $\varphi(d) = \varphi_\alpha(d)$ for all $\alpha < \kappa$ with $\alpha \geq \mathrm{cp}(d)$. Therefore, we can apply properties ii), iii), iv), v) and vi) on φ by replacing φ_α with φ. Since

$$d_1 + d_2 = (d_1 + d_2) \overset{vi)}{\Longrightarrow} \varphi(d_1) + \varphi(d_2) = \varphi(d_1 + d_2),$$
$$d_1 d_2 = (d_1 d_2) \overset{vi)}{\Longrightarrow} \varphi(d_1)\varphi(d_2) = \varphi(d_1 d_2)$$

for all $d_1, d_2 \in D$, we have shown that φ is a homomorphism. By definition φ_1 is not trivial and since D is a skew field φ has to be injective. Because R is the rational closure of $f(F[G;\eta,\sigma])$ in R and D is a skew field of fractions of $F[G;\eta,\sigma]$, we conclude $\varphi(D) = R$ according to Proposition 1.1.17. Therefore, φ is surjective. This together with iv) also shows that each nonzero element of R is a continuous, v–compatible automorphism. At last, the uniqueness of φ is obtained by Proposition 1.1.16. □

Corollary 2.3.4 ([13, page 182]) Let G be locally indicable and $F[G;\eta,\sigma]$ be a crossed product ring with Hughes–free skew fields of fractions D_1 and D_2 and accompanying injective ring homomorphisms ι_1 and ι_2. Then there is a unique ring isomorphism $\varphi : D_1 \longrightarrow D_2$ such that

$$\begin{tikzcd} F[G;\eta,\sigma] \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & D_1 \arrow[d, "\varphi"] \\ & D_2 \end{tikzcd}$$

is a commutative diagram.

Proof. According to Theorem 1.2.31 and Lemma 1.2.29 since G is locally indicable it admits a Conradian left–order $\leq$ of maximal rank. Let R be the Dubrovin–ring of $F[G;\eta,\sigma]$ with respect to $\leq$ and $f : F[G;\eta,\sigma] \longrightarrow R$ the associated embedding. Since D_i is a Hughes–free skew field of fractions for $i \in \{1,2\}$ it is, according to Proposition 2.1.3, also a free skew field of fractions and according to Theorem 2.3.3 there is a uniquely determined ring isomorphism $\varphi_i : D_i \longrightarrow R$ such that

$$\begin{tikzcd} F[G;\eta,\sigma] \arrow[r, "\iota_i"] \arrow[dr, "f"'] & D_i \arrow[d, "\varphi_i"] \\ & R \end{tikzcd}$$

is a commutative diagram for $i \in \{1,2\}$. We define $\varphi := \varphi_2^{-1}\varphi_1$. Since $\varphi_1\iota_1 = f = \varphi_2\iota_2$ we have $\varphi\iota_1 = \varphi_2^{-1}\varphi_1\iota_1 = \iota_2$. The uniqueness of φ is obtained by Proposition 1.1.16. □

Theorem 2.3.5 Let G_1, G_2 be locally indicable groups, F_1, F_2 skew fields and let $F_1[G_1;\eta_1,\sigma_1]$, $F_2[G_2;\eta_2,\sigma_2]$ be crossed product rings as well as

$$\varphi : F_1[G_1;\eta_1,\sigma_1] \longrightarrow F_2[G_2;\eta_2,\sigma_2]$$

be an injective ring homomorphism such that the accompanying group homomorphism $\psi : G_1 \longrightarrow G_2$ (see Theorem 1.3.11) is injective. If $F_1[G_1;\eta_1,\sigma_1]$ and $F_2[G_2;\eta_2,\sigma_2]$ have Hughes–free skew fields of fractions D_1 and D_2 there exists a unique injective ring homomorphism $\varphi' : D_1 \longrightarrow D_2$ such that

$$\begin{tikzcd} F_1[G_1;\eta_1,\sigma_1] \arrow[r, "\iota_1"] \arrow[d, "\varphi"'] & D_1 \arrow[d, "\varphi'"] \\ F_2[G_2;\eta_2,\sigma_2] \arrow[r, "\iota_2"] & D_2 \end{tikzcd}$$

is a commutative diagram.

Proof. Let D' be the rational closure of $\iota_2\varphi\left(F_1[G_1;\eta_1,\sigma_1]\right)$ in D_2. We want to show, that D' is a Hughes–free skew field of fractions of $F_1[G_1;\eta_1,\sigma_1]$. If U is a finitely generated subgroup of G_1 and N a normal subgroup of U such that U/N is infinitely cyclic, there is a $g \in G_1$ with $U/N = \langle gN \rangle$. Since ψ is injective, $\psi(N)$ is a normal subgroup of $\psi(U)$ and $\psi(U)/\psi(N) = \langle \psi(g)\psi(N) \rangle$ is infinitely cyclic. If $D_{2\psi(N)}$ is the rational closure of $\iota_2\left(F_2^\times \psi(N)\right)$ in D_2, then $\iota_2(ax_{\psi(g)})$ is transcendental over $D_{2\psi(N)}$ for each $a \in F_2^\times$, since D_2 is a Hughes–free skew field of fractions of $F_2[G_2;\eta_2,\sigma_2]$. We want to show, that $\iota_2\varphi(x_g)$ is transcendental over D'_N, where D'_N is the rational closure of $\iota_2\varphi(F_1^\times N)$ in D'. There is an $a \in F_2^\times$ with $\varphi(x_g) = ax_{\psi(g)}$ and therefore $\iota_2\varphi(x_g) = \iota_2(ax_{\psi(g)})$. For $b \in F_1^\times$ and $h \in N$ there is a $b' \in F_2^\times$ with

$$\varphi(bx_h) = \varphi(b)\varphi(x_h) = \underbrace{\varphi(b)b'}_{\in F_2^\times} x_{\psi(h)} \in F_2^\times \psi(N).$$

Thus $\iota_2\varphi(F_1^\times N) \subseteq \iota_2\left(F_2^\times\psi(N)\right)$ and therefore $D'_N \subseteq D_{2\psi(N)}$. Since $\iota_2\varphi(x_g)$ is transcendental over $D_{2\psi(N)}$, it is also transcendental over the subskew field D'_N. Thus D' is a Hughes–free skew field of fractions of $F_1[G_1;\eta_1,\sigma_1]$.
According to Corollary 2.3.4 there is a unique ring isomorphism $\varphi'' : D_1 \longrightarrow D'$ such that

$$\begin{tikzcd} F_1[G_1;\eta_1,\sigma_1] \arrow[r, "\iota_1"] \arrow[dr, "\iota_2\varphi"] & D_1 \arrow[d, "\varphi''"] \\ & D' \end{tikzcd}$$

is a commutative diagram. We define $\varphi' : D_1 \longrightarrow D_2\,,\ d \longmapsto \varphi''(d)$ and

$$\begin{tikzcd} F_1[G_1;\eta_1,\sigma_1] \arrow[r, "\iota_1"] \arrow[d, "\varphi"'] & D_1 \arrow[d, "\varphi'"] \\ F_2[G_2;\eta_2,\sigma_2] \arrow[r, "\iota_2"] & D_2 \end{tikzcd}$$

is a commutative diagram. Furthermore, φ' is injective, since φ'' is an isomorphism. The uniqueness of φ' is obtained by applying Proposition 1.1.16. $\square$

Corollary 2.3.6 Let G be locally indicable and let $F[G;\eta,\sigma]$ be a crossed product ring with a Hughes–free skew field of fractions D and accompanying injective ring homomorphism ι. Each automorphism φ of $F[G;\eta,\sigma]$ can be uniquely extended to an automorphism φ' of D, such that

$$\begin{tikzcd} F[G;\eta,\sigma] \arrow[r, "\iota"] \arrow[d, "\varphi"'] & D \arrow[d, "\varphi'"] \\ F[G;\eta,\sigma] \arrow[r, "\iota"] & D \end{tikzcd}$$

is a commutative diagram.

Proof. Let ψ and ψ' be the associated group homomorphisms for φ and φ^{-1} respectively according to Theorem 1.3.11. Then $\psi'\psi$ is the associated unique group homomorphism for $\varphi^{-1}\varphi = \mathrm{id}$ and has to be the identity. This implies that ψ is injective and the injectivity of ψ' follows similarly. Hence, we can apply Theorem 2.3.5 twice and there exist unique injective ring homomorphisms $\varphi' : D \longrightarrow D$ and $\varphi'' : D \longrightarrow D$ such that

$$\begin{tikzcd} F[G;\eta,\sigma] \arrow[r, "\iota"] \arrow[d, bend right, "\varphi"'] & D \arrow[d, bend right, "\varphi'"'] \\ F[G;\eta,\sigma] \arrow[r, "\iota"] \arrow[u, bend right, "\varphi^{-1}"'] & D \arrow[u, bend right, "\varphi''"'] \end{tikzcd}$$

is a commutative diagram. Thus, we get the following commutative diagram

$$\begin{tikzcd} F[G;\eta,\sigma] \arrow[r, "\iota"] \arrow[dr, "\iota"'] & D \arrow[d, "\varphi''\varphi'"] \\ & D \end{tikzcd}$$

and applying the uniqueness in Corollary 2.3.4 we observe $\varphi''\varphi' = \mathrm{id}_D$. Analogously we can show $\varphi'\varphi'' = \mathrm{id}_D$, which proves, that φ' is an automorphism. □

2.4 Strongly Hughes-free skew fields of fractions

Definition 2.4.1 Let $F[G;\eta,\sigma]$ be a crossed product ring and G a locally indicable group. A skew field D is called strongly Hughes–free skew field of fractions of $F[G;\eta,\sigma]$ if D is a skew field of fractions of $F[G;\eta,\sigma]$ and the following holds. For each subgroup H of G and each normal subgroup N of H

$$a_1 x_{h_1} + \cdots + a_n x_{h_n} = 0 \Longrightarrow a_1 = \cdots = a_n = 0$$

holds for all $h_1, \ldots, h_n \in H$ generating pairwise different N–cosets and $a_1 \ldots, a_n \in D_N$ whereas D_N is the rational closure of $F[N;\eta,\sigma]$ in D. The associated embedding is called strongly Hughes–free embedding.

Remark 2.4.2 As is easily seen, each strongly Hughes–free skew field of fractions is also a Hughes–free skew field of fractions.

Theorem 2.4.3 If the Dubrovin–ring R of $F[G;\eta,\sigma]$ is a skew field then R is a strongly Hughes–free skew field of fractions of $F[G;\eta,\sigma]$, whereas the canonical embedding f is also the associated strongly Hughes–free embedding.

Proof. Let H be a subgroup of G and N a normal subgroup of H. If $h_1,\ldots,h_n \in H$ are generating pairwise different N–cosets and $a_1,\ldots,a_n \in R_N$ then

$$\operatorname{supp} a_i(h_i) \subseteq Nh_i$$

for each $i \leq n$ by Lemma 1.4.20. Hence $Nh_1,\ldots,Nh_n$ being pairwise disjoint implies that $\operatorname{supp} a_1(h_1),\ldots,\operatorname{supp} a_n(h_n)$ are pairwise disjoint. Now if $a_1x_{h_1}+\cdots+a_nx_{h_n}=0$ then

$$\begin{aligned}\emptyset &= \operatorname{supp}\left(\left(a_1x_{h_1}+\cdots+a_nx_{h_n}\right)(e)\right) = \operatorname{supp}\left(a_1(h_1)+\cdots+a_n(h_n)\right)\\ &= \bigcup_{i=1}^{n} \operatorname{supp} a_i(h_i)\end{aligned}$$

and thus $\operatorname{supp} a_i(h_i)=\emptyset$ for each $i \leq n$. Since R is a skew field, each element of R is either an automorphism or 0, which implies $a_i = 0$ for each $i \leq n$. Hence R is a strongly Hughes–free skew field of fractions of $F[G;\eta,\sigma]$. □

Theorem 2.4.4 Let G be locally indicable and $F[G;\eta,\sigma]$ be a crossed product ring with a Hughes–free skew field of fractions D and accompanying embedding ι. Then D is a strongly Hughes–free skew field of fractions of $F[G;\eta,\sigma]$ with respect to the embedding ι.

Proof. Since D is Hughes–free skew field of fractions of $F[G;\eta,\sigma]$, the Dubrovin–ring R of $F[G;\eta,\sigma]$ is a skew field according to Theorem 2.3.3. By Theorem 2.4.3 this implies that R is a strongly Hughes–free skew field of fractions of $F[G;\eta,\sigma]$ with respect to the canonical embedding. This transfers to D by applying Theorem 2.3.3. □

Remark 2.4.5 Theorem 2.4.4 answers Problem 4.8. in [17, page 53].

Bibliography

[1] P. M. Cohn. *Algebra. Volume 3.* Chichester etc.: John Wiley & Sons, 2nd edition, 1991.

[2] P. M. Cohn. *Skew fields. Theory of general division rings.* Cambridge: Cambridge Univ. Press, 1995.

[3] P. M. Cohn. *Free Ideal Rings and Localization in General Rings.* New Mathematical Monographs. Cambridge University Press, 2006.

[4] P. Conrad. Right-ordered groups. *Mich. Math. J.*, 6:267–275, 1959.

[5] N. I. Dubrovin. Rational closures of group rings of left-ordered groups. *Russian Academy of Sciences. Sbornik Mathematics*, 79(2):231–263, feb 1994.

[6] N. I. Dubrovin. Formal sums and power series over a group. *Sbornik: Mathematics*, 191(7):955–971, aug 2000.

[7] N. I. Dubrovin. Rational operators of the space of formal series. *Journal of Mathematical Sciences*, 149:1191–1223, 03 2008.

[8] N. I. Dubrovin, J. Gräter, and T. Hanke. Complexity of elements in rings. *Algebr. Represent. Theory*, 6(1):33–45, 2003.

[9] J. Gräter. Left–orderings, power series, valuations. *Potsdam, unpublished*, 2015.

[10] J. Gräter and R. P. Sperner. On embedding left-ordered groups into division rings. *Forum Math.*, 27(1):485–518, 2015.

[11] N. H. Halimi and T. Ito. Cones of certain isolated left orderings and chain domains. *Forum Math.*, 27(5):3027–3051, 2015.

[12] D. Herbera and J. Sánchez. Computing the inversion height of some embeddings of the free algebra and the free group algebra. *J. Algebra*, 310(1):108–131, 2007.

[13] I. Hughes. Division rings of fractions for group rings. *Commun. Pure Appl. Math.*, 23:181–188, 1970.

[14] I. Hughes. Division rings of fractions for group rings. II. *Commun. Pure Appl. Math.*, 25:127–131, 1972.

[15] F. Jakobs. Linksgeordnete Gruppen von Conrad–Typ. Diplomarbeit, Universität Potsdam, 2011.

[16] V. M. Kopytov and N. Ya. Medvedev. *Right-ordered groups. Transl. from the Russian by Scientific Book.* New York, NY: Consultants Bureau, 1996.

[17] P. A. Linnell. Noncommutative localization in group rings. In A. Ranicki, edi-

tor, *Noncommutative Localization in Algebra and Topology*, number 330 in London Mathematical Society Lecture Note Series, pages 40–59. Cambridge University Press, 2006.

[18] A. I. Mal'tsev. On the embedding of group algebras in division algebras. *Dokl. Akad. Nauk SSSR*, 60:1499–1501, 01 1948.

[19] B. H. Neumann. On ordered division rings. *Trans. Am. Math. Soc.*, 66:202–252, 1949.

[20] D. S. Passman. *Infinite crossed products.* Boston, MA etc.: Academic Press, Inc., 1989.

[21] A. Rhemtulla and D. Rolfsen. Local indicability in ordered groups: braids and elementary amenable groups. *Proc. Am. Math. Soc.*, 130(9):2569–2577, 2002.

[22] O. Schreier. Über die Erweiterung von Gruppen I, II. *Monatsh. Math. Phys.*, 34:165–180, 1926.

[23] M. Simon. Stetige und v–verträgliche Automorphismen. Diplomarbeit, Universität Potsdam, 2009.

[24] S. Warner. *Topological fields.*, volume 157. New York etc.: North-Holland, 1989.