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A Lefschetz Fixed Point Formula for Elliptic Quasicomplexes

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Abstract. In a recent paper, the Lefschetz number for endomorphisms (modulo trace class operators) of sequences of trace class curvature was introduced. We show that this is a well defined, canonical extension of the classical Lefschetz number and establish the homotopy invariance of this number. Moreover, we apply the results to show that the Lefschetz fixed point formula holds for geometric quasiendomorphisms of elliptic quasicomplexes.

Keywords. Elliptic complexes, Fredholm complexes, Lefschetz number.

1. Introduction

The concept of quasicomplexes goes back at least as far as the early 1980s, when those objects were introduced as essential complexes in [7]. These are sequences of bounded linear operators on Banach spaces whose curvature is compact. So they generalise the concept of complexes which have vanishing curvature. A main result was the fact that Fredholm quasicomplexes of Hilbert spaces are compact perturbations of Fredholm complexes. This was the base for the definition of Euler characteristic of Fredholm quasicomplexes in [9]. In the sequel these ideas were used to introduce elliptic quasicomplexes on compact manifolds with boundary [5] and compact closed manifolds [11]. As described in [11], it was an open question how the concept of Lefschetz number can be extended to quasicomplexes. The main idea how this problem can be solved is given in [10].

In the present paper we show that the main result of [10] can be obtained with a weaker definition of Fredholm property along more classical lines. In the third section we show that the definition of Lefschetz number is correct and prove some properties of this number. Note that these results were proved independently in [2] in the more general context of Banach spaces. Our method based on Hilbert space techniques has the advantage of providing

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explicit formulas. Finally, we prove that the Lefschetz fixed point formula is still valid for geometric quasiendomorphisms of elliptic quasicomplexes. This generalises the classical result of [1] for geometric endomorphisms of elliptic complexes.

2. $\mathcal{I}$-Quasicomplexes

In this paper we consider sequences of the form

$$(V^{\cdot}, A):\ 0 \to V^0 \overset{A^0}{\to} V^1 \overset{A^1}{\to} \cdots \overset{A^{N-1}}{\to} V^N \to 0$$

where V^i are Hilbert spaces and A^i are linear bounded operators. Such a sequence is called complex if its curvature $A^{i+1}A^i$ vanishes and it is called quasicomplex if its curvature is compact.

As is shown in [10], it also makes sense to consider sequences whose curvatures belong to some operator ideal $\mathcal{I} \subset \mathcal{K}$, where $\mathcal{K}$ is the class of all compact operators. We will write $\mathcal{F}$ for the class of operators of finite rank and $\mathfrak{S}_p$, with $p \geq 1$, for the Schatten classes. Note that $\mathcal{F} \subset \mathcal{I}$ holds, if $\mathcal{I} \neq 0$ (cf. [6]).

Definition 2.1. A sequence $(V^{\cdot}, A)$ of operators $A^i \in \mathcal{L}(V^i, V^{i+1})$ is called $\mathcal{I}$-quasicomplex if $A^{i+1}A^i \in \mathcal{I}(V^i, V^{i+2})$ holds for all $i = 0, 1, \ldots, N-2$.

A 0-quasicomplex is obviously a complex and a $\mathcal{K}$-quasicomplex is just called quasicomplex.

Definition 2.2. Let $(V^{\cdot}, A)$ be a quasicomplex. By an $\mathcal{I}$-parametrix of this quasicomplex is meant any sequence of operators $P^i \in \mathcal{L}(V^i, V^{i-1})$ which satisfy

$$P^{i+1}A^i + A^{i-1}P^i = Id_{V^i} - R^i$$

for all $i = 0, 1, \ldots, N$, with $R^i \in \mathcal{I}(V^i)$.

With this definition, a $\mathcal{K}$-parametix is a parametrix in the classical sense, i.e. there are operators $K^i \in \mathcal{K}(V^i)$, such that $P^{i+1}A^i + A^{i-1}P^i = Id_{V^i} - K^i$ for all $i = 0, 1, \ldots, N$.

It is well known that a complex $(V^{\cdot}, D)$ of Hilbert spaces is Fredholm (i.e. the cohomology $H^i(V^{\cdot}, D) := \ker D^i / \operatorname{im} D^{i-1}$ is finite dimensional at each step $i = 0, 1, \ldots, N$) if and only if it has a parametrix. For this reason a quasicomplex is said to be Fredholm if it possesses a parametrix. An equivalent definition of the Fredholm property can be given using the notion of Calkin algebra.

Let $(V^{\cdot}, A)$ be an $\mathcal{I}$-quasicomplex. The so-called adjoint quasicomplex is given by

$$(V^{\cdot}, A^*):\ 0 \leftarrow V^0 \overset{A^{0*}}{\leftarrow} V^1 \overset{A^{1*}}{\leftarrow} \cdots \overset{A^{N-1*}}{\leftarrow} V^N \leftarrow 0,$$

where $A^{i*} \in \mathcal{L}(V^{i+1}, V^i)$ stands for the adjoint of A^i in the sense of Hilbert spaces. Obviously the operators $A^{i*}A^{i+1*}$ are compact again. The operators

$$\Delta^i = A^{i-1}A^{i-1*} + A^{i*}A^i$$

are called the Laplacians of the quasicomplex. As mentioned in [11], $(V^\cdot, A)$ is Fredholm if and only if all Laplacians Δ^i of $(V^\cdot, A)$ are Fredholm. In this case, we denote by $H^i \in \mathcal{F}(V^i)$ the orthogonal projection of V^i onto the null-space of Δ^i and introduce the Green operator

$$G^i := (\Delta^i \restriction_{(\ker \Delta^i)^\perp})^{-1}(Id_{V^i} - H^i).$$

Then $Id_{V^i} = H^i + \Delta^i G^i$ holds. It is easy to see that the Laplacians fulfill

$$A^i \Delta^i - \Delta^{i+1} A^i \in \mathcal{I}(V^i, V^{i+1}).$$

Multiplying this operator by G^{i+1} from the left and by G^i from the right we obtain

$$A^i G^i - G^{i+1} A^i \in \mathcal{I}(V^i, V^{i+1}),$$

since $H^{i+1}A^i = A^i H^i = 0$, if $\mathcal{I} = 0$, and $H^i \in \mathcal{I}(V^i)$, if $\mathcal{I} \neq 0$. Hence it follows that the operators

$$P^i := A^{i-1*} G^i$$

yield a parametrix P for $(V^\cdot, A)$.

If $\mathcal{I} = 0$, this is a special $\mathcal{F}$-parametrix of the (quasi-)complex. If $\mathcal{I} \neq 0$, then $\mathcal{F} \subset \mathcal{I}$ implies that P is a special $\mathcal{I}$-parametrix in this case.

Obviously, when perturbing the operators of a Fredholm complex by operators of $\mathcal{I}$, we obtain a Fredholm $\mathcal{I}$-quasicomplex. It turns out that the inverse theorem is also true. This follows from the main theorem in [10] and the fact that each Fredholm $\mathcal{I}$-quasicomplex possesses an $\mathcal{I}$-parametrix, provided that $\mathcal{I} \neq 0$.

Theorem 2.3. *Let $(V^\cdot, A)$ be a Fredholm $\mathcal{I}$-quasicomplex. Then there exist operators $D^i \in \mathcal{L}(V^i, V^{i+1})$, such that $D^i - A^i \in \mathcal{I}(V^i, V^{i+1})$ and $D^{i+1}D^i = 0$.*

An $\mathcal{I}$-quasiendomorphism of a quasicomplex $(V^\cdot, A)$ is a sequence of linear maps $E^i \in \mathcal{L}(V^i)$ which makes the diagram

$$\begin{array}{ccccccccc} 0 \to & V^0 & \overset{A^0}{\to} & V^1 & \overset{A^1}{\to} \cdots & \overset{A^{N-1}}{\to} & V^N & \to 0 \\ & \downarrow E^0 & & \downarrow E^1 & & & \downarrow E^N & \\ 0 \to & V^0 & \overset{A^0}{\to} & V^1 & \overset{A^1}{\to} \cdots & \overset{A^{N-1}}{\to} & V^N & \to 0 \end{array}$$

commutative modulo operators of $\mathcal{I}$, i.e. $E^{i+1}A^i - A^i E^i \in \mathcal{I}(V^i, V^{i+1})$ for all $i = 0, 1, \ldots, N-1$.

As above, $\mathcal{K}$-quasiendomorphisms are called quasiendomorphisms and 0-quasiendomorphisms are endomorphisms.

Theorem 2.4. *Let $(V^\cdot, A)$ be a Fredholm $\mathcal{I}$-quasicomplex, E an $\mathcal{I}$-quasiendomo rphism of this quasicomplex and $(V^\cdot, D)$ any complex with the property that $D^i - A^i \in \mathcal{I}(V^i, V^{i+1})$. Then, there is an endomorphism $\tilde{E}$ of $(V^\cdot, D)$ satisfying $\tilde{E}^i - E^i \in \mathcal{I}(V^i)$.*

Proof. The case $\mathcal{I} = 0$ is trivial. For $\mathcal{I} \neq 0$, let P be an $\mathcal{I}$-parametrix of $(V^\cdot, A)$, i.e. $P^{i+1}A^i + A^{i-1}P^i = Id_{V^i} - R^i$ with $R^i \in \mathcal{I}(V^i)$. Now it is easy to see that

$$\tilde{E}^i := D^{i-1}E^{i-1}P^i + E^i P^{i+1} D^i$$

is an endomorphism of $(V^\cdot, D)$.

Setting

$$\begin{aligned} T^i &:= E^{i+1}D^i - D^i E^i \\ &= E^{i+1}A^i - A^i E^i + E^{i+1}(D^i - A^i) - (D^i - A^i)E^i \\ &\in \mathcal{I}(V^i, V^{i+1}) \end{aligned}$$

we obtain

$$\begin{aligned} E^i - \tilde{E}^i &= E^i - (D^{i-1}E^{i-1}P^i + E^i P^{i+1} D^i) \\ &= E^i - E^i(D^{i-1}P^i + P^{i+1}D^i) + T^{i-1}P^i \\ &= E^i R^i + T^{i-1}P^i \\ &\in \mathcal{I}(V^i), \end{aligned}$$

as desired. □

Two quasiendomorphisms E and F of $(V^\cdot, A)$ are said to be homotopic if there exists a sequence of bounded linear operators $h^i : V^i \to V^{i-1}$ with the property that

$$E^i - F^i = A^{i-1}h^i + h^{i+1}A^i$$

for all $i = 0, 1, \ldots, N$.

3. Lefschetz Number

Suppose $E = \{E^i\}$ is an endomorphism of a Fredholm complex $(V^\cdot, D)$. Then the mapping

$$HE^i : H^i(V^\cdot, D) \to H^i(V^\cdot, D),$$

given by $[v] \mapsto [E^i v]$, is an endomorphism of the finite-dimensional space $H^i(V^\cdot, D)$, and so the trace $\operatorname{tr} HE^i$ is well defined, for each i. The alternating sum

$$L(E, D) := \sum_i (-1)^i \operatorname{tr} HE^i$$

is called the Lefschetz number of the endomorphism.

If $E^i = Id_{V^i}$ are the identity maps, then the trace $\operatorname{tr} HE^i$ just amounts to the dimension of $H^i(V^\cdot, D)$ whence $L(Id_{V^\cdot}, D) = \chi(V^\cdot, D)$, where $\chi(V^\cdot, D)$ is the Euler characteristic of the complex.

If E and F are homotopic endomorphisms of a Fredholm complex $(V^\cdot, D)$ then $L(E, D) = L(F, D)$ holds, as is easy to check.

The elements of the Schatten class $\mathfrak{S}_1$ are called trace class operators. Such operators possess a trace, if they are selfmappings. This trace has the following important property which is a consequence of a well-known theorem of V. B. Lidskii.

Theorem 3.1. *Let V, W be Hilbert spaces and $A \in \mathcal{L}(V, W)$, $B \in \mathcal{L}(W, V)$ be such that $BA \in \mathfrak{S}_1(V)$ and $AB \in \mathfrak{S}_1(W)$. Then* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ *holds.*

It turns out that the Lefschetz number can be extended to $\mathfrak{S}_1$-quasiendomorphisms of $\mathfrak{S}_1$-quasicomplexes. To show this we need an auxiliary result which is usually referred to as Euler's identity, see [1] or Theorem 19.1.15 in [4].

Lemma 3.2. *Let E be an endomorphism of a Fredholm complex $(V^\cdot, D)$, such that $E^i \in \mathfrak{S}_1(V^i)$ for all $i = 0, 1, \ldots, N$. Then*

$$L(E, D) = \sum_{i=0}^{N} (-1)^i \operatorname{tr} E^i.$$

Note that Lemma 3.2 is valid not only for trace class operators E^i but also for all operators E^i, for which the wave front calculus allows one to define the trace by restricting the Schwartz kernel to the diagonal, see Theorem 19.4.1 of [4].

The following definition is of crucial importance in this paper. As mentioned, it stems from [10] by direct calculation.

Definition 3.3. Let $(V^\cdot, A)$ be a Fredholm $\mathfrak{S}_1$-quasicomplex and E an $\mathfrak{S}_1$-quasiendomorphisms of this quasicomplex. Then the Lefschetz number is defined as

$$L(E, A) = L(\tilde{E}, D) + \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - \tilde{E}^i),$$

where $(V^\cdot, D)$ is a complex, such that $D^i - A^i \in \mathfrak{S}_1(V^i, V^{i+1})$, and $\tilde{E}$ is an endomorphism of $(V^\cdot, D)$, such that $\tilde{E}^i - E^i \in \mathfrak{S}_1(V^i)$.

Obviously, $L(E, A)$ coincides with the classical Lefschetz number, if $(V^\cdot, A)$ is a Fredholm complex and E is an endomorphism of $(V^\cdot, A)$.

We have to show that the definition is independent of the particular choice of D and $\tilde{E}$. For this purpose we choose an arbitrary $\mathfrak{S}_1$-parametrix P. Then $\tilde{E}$ and

$$\tilde{E}^i - D^{i-1} \tilde{E}^{i-1} P^i - \tilde{E}^i P^{i+1} D^i \in \mathfrak{S}_1(V^i)$$

are homotopic endomorphisms of $(V^\cdot, D)$. By Lemma 3.2,

$$L(\tilde{E}, D) = \sum_{i=0}^{N} (-1)^i \operatorname{tr}(\tilde{E}^i - D^{i-1} \tilde{E}^{i-1} P^i - \tilde{E}^i P^{i+1} D^i)$$

and therefore

$$\begin{aligned} L(E,A) &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - D^{i-1}\tilde{E}^{i-1}P^i - \tilde{E}^i P^{i+1} D^i) \\ &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - D^{i-1}E^{i-1}P^i - E^i P^{i+1} D^i) \\ &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - A^{i-1}E^{i-1}P^i - E^i P^{i+1} A^i), \end{aligned}$$

the second and third equalities being due to Theorem 3.1. Indeed, the differences of the right-hand sides and the left-hand sides of these equalities just amount to

$$\sum_{i=0}^{N-1} (-1)^i \operatorname{tr}((E^i - \tilde{E}^i)P^{i+1}D^i - D^i(E^i - \tilde{E}^i)P^{i+1}),$$
$$\sum_{i=0}^{N-1} (-1)^i \operatorname{tr}(E^i P^{i+1}(A^i - D^i) - (A^i - D^i)E^i P^{i+1}),$$

respectively, where each summand vanishes by Theorem 3.1. This shows the independence of $\tilde{E}$ and D.

Definition 3.3 implies in particular that $L(E,A) = L(E,D)$, and so we obtain immediately

$$L(Id_{V^\cdot}, A) = L(Id_{V^\cdot}, D) = \chi(V^\cdot, D) =: \chi(V^\cdot, A),$$

cf. [9].

Corollary 3.4. *Let $(V^\cdot, A)$ be a Fredholm $\mathfrak{S}_1$-quasicomplex and E an $\mathfrak{S}_1$-quasiendomorphism of this quasicomplex. Then*

$$L(E,A) = \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - A^{i-1}E^{i-1}P^i - E^i P^{i+1} A^i)$$

for each $\mathfrak{S}_1$-parametrix P of $(V^\cdot, A)$.

The corollary above can also be used as a definition of Lefschetz number. This was precisely our approach in [10].

Choosing $\tilde{E}^i := D^{i-1}E^{i-1}P^i + E^i P^{i+1} D^i$ as in the proof of Theorem 2.4, we get $L(\tilde{E}, D) = 0$, for $\tilde{E}$ and 0 are homotopic endomorphisms of $(V^\cdot, D)$. Hence it follows that

$$L(E,A) = \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - \tilde{E}^i)$$

in this special case.

Theorem 3.5. *Let $(V^\cdot, A)$ be a Fredholm $\mathfrak{S}_1$-quasicomplex and E, F homotopic $\mathfrak{S}_1$-quasiendomorphisms of this quasicomplex. Then $L(E,A) = L(F,A)$ holds.*

Proof. Choose a complex $(V^\cdot, D)$, such that $T^i := A^i - D^i \in \mathfrak{S}_1(V^i, V^{i+1})$. Set

$$\begin{aligned} G^i &:= E^i - F^i - T^{i-1}h^i - h^{i+1}T^i \\ &= D^{i-1}h^i + h^{i+1}D^i. \end{aligned}$$

Then G is an endomorphism of the complex $(V^\cdot, D)$ homotopic to 0, and we find

$$\begin{aligned} L(E,A) - L(F,A) &= L(E,D) - L(F,D) \\ &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - F^i - D^{i-1}(E^{i-1} - F^{i-1})P^i - (E^i - F^i)P^{i+1}D^i) \\ &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(G^i - D^{i-1}G^{i-1}P^i - G^iP^{i+1}D^i) \\ &= L(G,D) \\ &= 0, \end{aligned}$$

the third equation being a consequence of Theorem 3.1. □

Remark 3.6. The equivalence of Definition 3.3 and Corollary 3.4 was shown recently by Eschmeier in [2]. Moreover, he proved Theorem 2.4 in the case of $\mathfrak{S}_p$-quasicomplexes in Banach spaces.

4. Fixed Point Formula

Let X be a C^∞ compact closed manifold of dimension n and F^i smooth vector bundles over X.

By a quasicomplex of pseudodifferential operators on X is meant any sequence of the form

$$(C^\infty(X,F^\cdot),A):\ 0 \to C^\infty(X,F^0) \stackrel{A^0}{\to} C^\infty(X,F^1) \stackrel{A^1}{\to} \cdots \stackrel{A^{N-1}}{\to} C^\infty(X,F^N) \to 0$$

with $A^i \in \Psi_{\mathrm{cl}}^{m_i}(X;F^i,F^{i+1})$ satisfying $A^{i+1}A^i \in \Psi^{-\infty}(X;F^i,F^{i+2})$. In other words, the curvature of $C^\infty(X,F^\cdot)$ is a smoothing operator in the operator algebra under study. The quasicomplex is called elliptic if the complex of principal symbols

$$\pi^*F^\cdot:\ 0 \to \pi^*F^0 \stackrel{\sigma^{m_0}(A^0)}{\to} \pi^*F^1 \stackrel{\sigma^{m_1}(A^1)}{\to} \cdots \stackrel{\sigma^{m_{N-1}}(A^{N-1})}{\to} \pi^*F^N \to 0$$

is exact away from the zero section of T^*X.

By a parametrix of a quasicomplex $C^\infty(X,F^\cdot)$ is meant any sequence of pseudodifferential operators $P^i \in \Psi_{\mathrm{cl}}^{-m_{i-1}}(X;F^i,F^{i-1})$ satisfying the homotopy equations

$$A^{i-1}P^i + P^{i+1}A^i = Id_{F^i} - S^i$$

with smoothing operators $S^i \in \Psi^{-\infty}(X;F^i)$ for all $i = 0,1,\ldots,N$.

Theorem 4.1. *For a quasicomplex $(C^\infty(X,F^\cdot),A)$ to possess a parametrix it is necessary and sufficient that it is elliptic.*

Proof. See [11]. □

Let $s \in \mathbb{R}$. We may extend the quasicomplex $C^\infty(X, F^\cdot)$ to a quasicomplex of Sobolev spaces, i.e.

$$(H^{s_\cdot}(X, F^\cdot), A) : \; 0 \to H^{s_0}(X, F^0) \xrightarrow{A^0} H^{s_1}(X, F^1) \xrightarrow{A^1} \cdots \xrightarrow{A^{N-1}} H^{s_N}(X, F^N) \to 0$$

where s_i are given by $s_0 := s$ and $s_{i+1} := s_i - m_i$. This is a quasicomplex in the context of Hilbert spaces. More precisely, it is an $\mathfrak{S}_p$-quasicomplex for all $p \geq 1$.

Theorem 4.2. *Assume that $(C^\infty(X, F^\cdot), A)$ is an elliptic quasicomplex. Then the extended quasicomplex $(H^{s_\cdot}(X, F^\cdot), A)$ is Fredholm.*

Proof. See [11]. □

A quasiendomorphism of $(C^\infty(X, F^\cdot), A)$ is a family $E = \{E^i\}$ of bounded linear selfmaps E^i of $C^\infty(X, F^i)$, such that $E^{i+1}A^i = A^i E^i$ modulo smoothing operators $\Psi^{-\infty}(X; F^i, F^{i+1})$ for all $i = 0, 1, \ldots, N-1$. By Theorem 7.6 of [11] there is a perturbation D of the differential A by smoothing operators, such that $(C^\infty(X, F^\cdot), D)$ is a complex. Moreover, a slight change in the proof of Lemma 2.4 shows that there is an endomorphism $\tilde{E} = \{\tilde{E}^i\}$ of $(C^\infty(X, F^\cdot), D)$, such that $\tilde{E}^i - E^i \in \Psi^{-\infty}(X; F^i)$ for all $i = 0, 1, \ldots, N$. Note that each smoothing operator $S \in \Psi^{-\infty}(X; F^i)$ belongs to the ideal $\mathfrak{S}_1(H^s(X, F))$ for each $s \in \mathbb{R}$ and its trace in the sense of Sobolev spaces just amounts to the trace obtained by restricting the (smooth) Schwartz kernel of S to the diagonal of $X \times X$, evaluating the matrix trace of the restriction and integrating it over the diagonal. For a quasiendomorphism E of an elliptic quasicomplex $(C^\infty(X, F^\cdot), A)$ we introduce the Lefschetz number $L(E, A)$ by Definition 3.3, i.e.

$$L(E, A) = L(\tilde{E}, D) + \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - \tilde{E}^i),$$

where the traces are evaluated in Sobolev spaces. Clearly, this definition is independent of the particular choice of s. The cohomology of the Fredholm complex $(H^{s_\cdot}(X, F^\cdot), D)$ does not depend on the particular choice of s, too, and just amounts to that of $(C^\infty(X, F^\cdot), D)$. Hence, if every map $E^i \in \mathcal{L}(C^\infty(X, F^i))$ extends to a bounded linear selfmap of $H^{s_i}(X, F^i)$ for s large enough, then the same is true for $\tilde{E}^i$ and so the Lefschetz number $L(\tilde{E}, D)$ can be also evaluated for the complex $(H^{s_\cdot}(X, F^\cdot), D)$ of Hilbert spaces. However, the geometric quasiendomorphisms E to be considered fail to be of trace class, hence the Euler identity of Lemma 3.2 no longer applies. Even so, using the facts that $\tilde{E}^i$ and

$$\tilde{E}^i - D^{i-1}\tilde{E}^{i-1}P^i - \tilde{E}^i P^{i+1} D^i \in \mathcal{L}(C^\infty(X, F^i))$$

are homotopic and $Id - D^{i-1}P^i - P^{i+1}D^i \in \Psi^{-\infty}(X; F^i)$ holds, we can exploit Theorem 19.4.1 of [4] and obtain

$$\begin{aligned} L(\tilde{E}, D) &= L(\tilde{E} - D\tilde{E}P - \tilde{E}PD, D) \\ &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(\tilde{E}^i - D^{i-1}\tilde{E}^{i-1}P^i - \tilde{E}^i P^{i+1} D^i) \\ &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(\tilde{E}^i - A^{i-1}E^{i-1}P^i - E^i P^{i+1} A^i), \end{aligned}$$

where the traces are evaluated by restricting the (not necessarily smooth) kernels. Hence, we can compute the Lefschetz number by the explicit formula of Corollary 3.4

$$L(E, A) := \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - A^{i-1}E^{i-1}P^i - E^i P^{i+1} A^i)$$

where P is a parametrix of the quasicomplex $(C^\infty(X, F^\cdot), A)$, cf. Lemma 7.2 of [8].

Let f be a smooth selfmap of the manifold X and f^*F^i the induced bundles. The maps $f^* : C^\infty(X, F^i) \to C^\infty(X, f^*F^i)$ given by $(f^*u)(x) = u(f(x))$ are linear. Moreover, we consider smooth bundle homomorphisms $h^i : f^*F^i \to F^i$. For the induced maps $h^i : C^\infty(X, f^*F^i) \to C^\infty(X, F^i)$ we also write h^i. Then the compositions $E^i := h^i \circ f^*$ are obviously selfmaps of $C^\infty(X, F^i)$. More precisely, we define

$$E^i u(x) = h^i(x) u(f(x))$$

for $u \in C^\infty(X, F^i)$.

Definition 4.3. The family $E = \{h^i \circ f^*\}$ is called geometric quasiendomorphism of $(C^\infty(X, F^\cdot), A)$ if $A^i E^i = E^{i+1} A^i$ holds modulo smoothing operators for all $i = 0, 1, \ldots, N-1$.

The following theorem presents a natural generalisation of the Lefschetz fixed point formula for elliptic complexes on a compact closed manifold due to [1].

Theorem 4.4. *Assume $E = \{h^i \circ f^*\}_{i=0,1,\ldots}$ is a geometric quasiendomorphism of an elliptic quasicomplex $(C^\infty(X, F^\cdot), A)$ and f has only simple fixed points. Then*

$$L(E, A) = \sum_{p \in \operatorname{Fix}(f)} \nu(p)$$

with

$$\nu(p) = \frac{\sum (-1)^i \operatorname{tr} h^i(p)}{|\det(1 - df(p))|}.$$

Proof. The proof follows the scheme suggested by Fedosov in [3]. We pick a partition of unity (ϕ_ν) on X with the property that each ϕ_ν either vanishes or is equal to 1 in a neighbourhood of any fixed point of f. Let further ψ_0 be a function of compact support on T^*X such that $\psi_0(\xi) \equiv 1$ near $\xi = 0$, and let $\psi_\infty = 1 - \psi_0$. In local coordinates on X, we introduce operators $\Psi_{0,\nu}$ and $\Psi_{\infty,\nu}$ by

$$\begin{aligned}\Psi_{0,\nu} u &= F^{-1}_{\xi\mapsto x}\psi_0(h\xi)F_{x\mapsto\xi}(\phi_\nu u),\\ \Psi_{\infty,\nu} u &= F^{-1}_{\xi\mapsto x}\psi_\infty(h\xi)F_{x\mapsto\xi}(\phi_\nu u),\end{aligned}$$

F being the Fourier transform and h a positive number. These operators decompose the identity operator; moreover, the operators $\Psi_{0,\nu}$ are smoothing and hence of trace class on each Sobolev space. We can assert, by the Lidskii theorem, that

$$\operatorname{tr} A^i E^i P^{i+1}\,\Psi_{0,\nu} = \operatorname{tr} E^i P^{i+1}\,\Psi_{0,\nu} A^i$$

whence

$$\begin{aligned}&\sum_{i=0}^{N}(-1)^i \operatorname{tr}\left(E^i - A^{i-1}E^{i-1}P^i - E^iP^{i+1}A^i\right)\\ &= \sum_\nu \sum_{i=0}^{N}(-1)^i \operatorname{tr} E^i\,\Psi_{0,\nu}\\ &+ \sum_\nu \sum_{i=0}^{N}(-1)^i \operatorname{tr}\left(E^i - A^{i-1}E^{i-1}P^i - E^iP^{i+1}A^i\right)\Psi_{\infty,\nu}\\ &- \sum_\nu \sum_{i=0}^{N-1}(-1)^i \operatorname{tr} E^iP^{i+1}[A^i,\,\Psi_{0,\nu}],\end{aligned} \tag{1}$$

$[A^i,\,\Psi_{0,\nu}]$ being the commutator of A^i and $\Psi_{0,\nu}$.

In a local chart close to a fixed point of f, the operator $E^i\,\Psi_{0,\nu}$ is given by the iterated integral

$$E^i\,\Psi_{0,\nu}u\,(x) = \frac{1}{(2\pi h)^n}\int\!\!\int e^{\imath(\xi/h)(f_M(x)-y)}\,h^i(x)\psi_0(\xi)\,\phi_\nu(y)u(y)\,dyd\xi,$$

and consequently

$$\operatorname{tr}E^i\,\Psi_{0,\nu} = \frac{1}{(2\pi h)^n}\int\!\!\int e^{\imath(\xi/h)(f_M(x)-x)}\operatorname{tr} h^i(x)\,\psi_0(\xi)\phi_\nu(x)\,d\xi dx.$$

For $h\to 0$, the limit of the integral on the right-hand side of this equality can be evaluated by the method of stationary phase. Moreover, the stationary points are just the points where $\xi = 0$ and $f(x) - x = 0$. In the principal part independent of h the contribution of a fixed point p is equal to

$$\frac{\operatorname{tr} h^i(p)}{|\det(Id - df(p))|}.$$

On the other hand, the remaining terms on the right side of (1) are oscillatory integrals whose exponent has no critical points. Indeed,

$$\begin{aligned}[A^i,\,\Psi_{0,\nu}] &= [A^i,\,\Psi_{0,\nu} - Id]\\ &= -[A^i,\,\Psi_{\infty,\nu}]\end{aligned}$$

close to each fixed point and the function ψ_∞ vanishes in a neighbourhood of $\xi = 0$. Hence it follows that the remaining summands in (1) are rapidly decreasing as $h \to 0$. Since the left-hand side of (1) is actually independent of h, we arrive at the desired formula. □

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