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Algebraic foundation of Group Field Theory

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Contents

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Potsdam, 07.05.2018 (Alexander Kegeles)

Acknowledgments

The year I started my PhD was coined by two significant changes in my life: the start of the PhD, and a proposal to a beautiful woman. Now, five years later, my life is on the brink of changes once again as I finally finish the PhD and finally get married to — the same woman — Dorina. It didn't take her five years to say "yes" but her understanding, care and helpful support allowed me to focus on my PhD and postpone the wedding preparations every year by a year. For the last 15 years and especially over the time period of my PhD she always stood by my side and was responsible for most of the beautiful things that happened to me. Thank you for this!

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I want especially thank Christian for his corrections he provided for the earlier version of this manuscript. Even though his area of research can be considered very different from the one presented in this thesis, he careful and constructive criticism definitely improved the presentation of the content.

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Summary

Most quantum field theories (QFT) that are defined on space-time can be formulated in two conceptually different ways: the covariant — in terms of (Euclidean) functional integrals — and the canonical — in terms of operators on Hilbert spaces. Hereby, covariant formulation is conceptually easier and often better suited for (numerical) calculations, but it often lacks an intuitive physical interpretation. The canonical formulation, on the other hand, provides a better physical intuition of particles, conservation laws and observables but is sometimes less convenient for explicit calculations. However, both formalism are related and we can go from one to the other by Wick rotation and a subsequent time slicing. Evidently, for this relation the time variable is fundamentally important. But can we relate these two formulation if the time variable does not exist? This question appears in group field theory.

Group field theory (GFT) is one candidate theory for quantum gravity, formulated as quantum field theory. The theory is, however, not defined on space-time; but instead it is constructed to produce space-time as one of its outcomes. Nevertheless, just as the QFT's on space-time, GFT exists in two different formulations — the functional and the operator one. The functional formalism is the statistical formulation in terms of a functional integral and a generating partition function. It is used for renormalization analysis and relates to spin foam models and tensor models. Its ingredients are, however, not easily interpreted in physical terms, which permits an intuitive guidance for theory building. The operator formulation, on the other hand, is given in terms of operators on Hilbert spaces. It provides a physically intuitive definition of GFT-particles and describes the whole theory in the language of many body quantum physics. Its relation to the functional approach and to other related formulations of quantization of gravity, however, is not clear. An explicit relation between the two formalisms would allow us to combine the best of each.

Unfortunately, the relation via Wick rotation and time-slicing can not be applied in the GFT case. This is because GFT does not have any notion of time, since it suppose to describe the emergence of space-time only as one of the outputs of the formalism but does not presume time

as a fundamental ingredient of the theory.

In this thesis we provide a construction of the operator framework starting from the functional formulation of GFT. We define operator algebras on Hilbert spaces whose expectation values in specific states provide correlation functions of the functional formulation. Our construction allows us to give a direct relation between the ingredients of the functional GFT and its operator formulation in a perturbative regime. Using this construction we provide an example of GFT states that can not be formulated as states in a Fock space and lead to mathematically inequivalent representations of the operator algebra. We show that such inequivalent representations can be grouped together by their symmetry properties and sometimes break the left translation symmetry of the GFT action. We interpret these groups of inequivalent representations as phases of GFT, similar to the classification of phases that we use in QFT's on space-time.

For our construction we need to find minima of the GFT action on the space of tempered distributions. To simplify this task we perform the symmetry analysis of several models in GFT and fully classify their point-symmetry groups in a structured way.

The structure of the thesis is as follows: in chapter 1 we introduce a general problem of quantum gravity and present some direct complications that appear when we try to combine the principles of general relativity and quantum field theory; in chapter 2, we motivate and discuss the framework of group field theory in its functional and operator formulation; in chapter 3 we derive an algebraic formulation of group field theory and construct operator algebras starting from the functional formulation; in chapter 4 we develop a local symmetry analysis for multi-local actions of GFT and apply it to simplicial and geometric GFT models; in chapter 5 we summarize and combine the results of previous chapters, arguing that a classification of phases in GFT can be given in terms of symmetry breaking of left translation on the base manifold.

1 Introduction to Quantum Gravity

Our contemporary understanding of physics is governed by two theories: the classical theory of gravitation — *general relativity* (GR) — and the quantum theory of strong, weak and electromagnetic interactions — the *standard model* (SM). Together they cover all known fundamental forces and give extremely precise predictions for all present-day experiments. And yet, as we will see below, these two formalisms seem to be incompatible and the fundamental principles of one appear to contradict those of the other.

The reason why we do not experience any such contradictions in experiments lies in the fact that gravitational and quantum mechanical interactions operate at different scales. Gravitation is by far the weakest of the four fundamental forces. The gravitational fine structure constant[1] α_g is about 33 orders of magnitude smaller than that of the strong interaction, 31 orders of magnitude smaller than that of electro-magnetic interactions and 27 orders of magnitude smaller than that of the weak interaction.

[1] *The fine structure constant of gravity α_g is given by [1]*

$$\alpha_g = \frac{G m_{pr}^2}{\hbar c},$$

where m_{pr} denotes the proton mass.

Interactions	Coupling constants	
Strong	α_s	≈ 1
Electromagnetic	α	$\approx 10^{-2}$
Weak	α_w	$\approx 10^{-6}$
Gravitational	α_g	$\approx 10^{-45}$

Table 1.1: Estimate of the fine structure constants for the four fundamental interactions of nature.

For that reason we need substantial gravitational sources to get into regimes in which quantum field theory (QFT) and general relativity become equally important. At low energy scales gravitation can be generated either by the gravitational mass or by its energy density. However:

Massive objects are composed of a large number of particles and for that reason have very short decoherence times [2]. Even if we were able to create a very massive quantum object, its quantum nature would disappear within a fraction of a second and we would be left with classical systems subjected only to laws of general relativity, one example of which is classical cosmology.

Highly energetic objects do not need to have large gravitational masses (for example collision experiments at the Large Hadron Collider), and therefor may be more relevant for quantum and gravity experiments. However, the amount of energy needed to overcome that immense gap between the strength of coupling constants exceeds all our current resources by many orders of magnitude.

In other words: our current detectors are not sensible enough to capture the effects of quantum mechanics and general relativity in the same measurement. This is the reason why we do not encounter any experimental violations of seemingly contradictory theories.

A contemporary technical inability of having sensible detectors, however, does not provide a satisfactory resolution to our incomplete understanding of nature. To complete our understanding we need to construct a theory that reconciles the geometrical description of general relativity with fundamental principles of quantum mechanics — the theory of *quantum gravity* (QG).

The aim of quantum gravity is to provide a quantum mechanical description for a theory of gravity coupled to matter. A successful theory would be an extension of the standard model by an additional force, but will not yet unify all four fundamental forces to a single one and hence should not be confused with the so called theory of everything[2]. The scale at which this theory becomes dominant is called *the scale of quantum gravity* [3].

[2] *The difference between the theory of everything and quantum gravity is similar to that between standard model and grand unified theories.*

1.1 The scale of quantum gravity

Retrospectively we attribute the first encounter with quantum gravity to Max Planck [4], who realized that one can use the speed of light c, the Planks constant $\hbar$ and the gravitational constant G to define universal unites of length, time and mass — *the Planck unites* ,

$$l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}\,\mathrm{m}, \tag{1.1}$$

$$t_p = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-44}\,\mathrm{s}, \tag{1.2}$$

$$m_p = \sqrt{\frac{\hbar c}{G}} \approx 10^{32}\,\mathrm{eV} \approx 10^{-5}\,\mathrm{g}. \tag{1.3}$$

But it was Bronstein [5, 6] [3] who first observed, that the Planck unites provide a scale at which quantum mechanical and general relativistic principles mingle and impose fundamental bounds on the notion of distance. Roughly his argument goes as follows (from [7]):

[3] *His discussion dates back to 1936.*

Let us say we want to localize a particle within a region L. By the Heisenberg uncertainty principle the variances of particles position and momentum are related by

$$\Delta x > \frac{\hbar}{\Delta p}. \tag{1.4}$$

In order to resolve its position with accuracy, $L > \Delta x$, we need to satisfy $\Delta p > \hbar L^{-1}$ and hence “inject” energy in the particle,

$$p^2 > \Delta p^2 > \left(\frac{\hbar}{L}\right)^2.$$

The higher the precision of our measurement, the more energetic the particle gets. In GR, however, energy provides a gravitational source, $E = mc^2$, and therefore distorts space-time. In conclusion a higher space resolution implies a larger space-time distortion. This process has a natural bound when space-time curves strongly enough to produces a black hole. This happens when the Schwarzschild radius (up to the factor of 2)

$$R \propto \frac{Gm}{c^2}, \tag{1.5}$$

becomes larger than the region that we resolve with our measurement. In this case L becomes hidden beyond the black hole event horizon and any further specification of particles position loses its meaning. Using the uncertainty principle for particle's position and momentum we can relate the mass and the precision of the measurement by

$$m = \frac{p}{c} > \frac{\hbar}{Lc}, \tag{1.6}$$

And the critical resolution length L_C becomes the Planck length,

$$L_C = l_p. \tag{1.7}$$

Bronstein realized that beyond this scale the notion of length loses its meaning and the quantum mechanical uncertainty of space-time becomes dominant.

At this scale we expect new physics to take place, due to the incompatible nature of two theories. Currently, however, we do not have a complete theory of quantum gravity and hence we do not know what is the physics at the Planck scale. The problem of not having predictions for experimental outcomes of extremely energetic gravitational systems is sometimes referred to as the *problem of quantum gravity* [8].

1.2 *Is quantum gravity observable?*

The energy scale of quantum gravity seems to be beyond any reach (fig. 1.1), and we need to ask ourselves the question how to test a possible candidate for a theory of quantum gravity.

To answer this question in a structured way we need to distinguish between different limits of quantum gravity, the *classical*, the *semi-classical*, *the weak field* and *the strong field limits*:

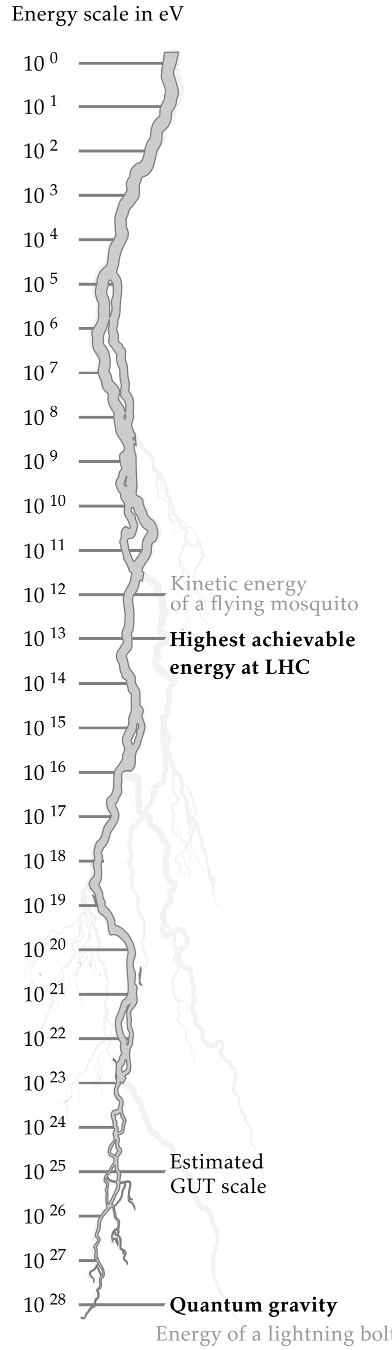


Figure 1.1: Typical energies related to the scale of quantum gravity. The Planck energy is of the same order of magnitude as the energy of a lightning bolt concentrated at a volume of an elementary particle. The relation between the Planck energy and the currently highest achievable energy at LHC is roughly the same as that of a lightning bolt compared to the kinetic energy of a flying mosquito.

THE CLASSICAL LIMIT where the gravitational field is described by GR or Newtonian gravity:

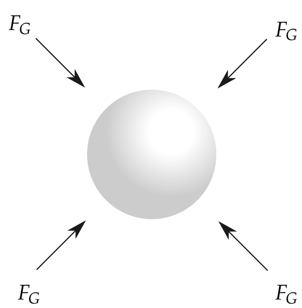
Every theory of quantum gravity should reduce to the classical theory of gravitation in an appropriate limit. We call this limit the *classical* *limit of quantum gravity*. Perturbative approaches to QG [9–11] naturally satisfy this criteria. However, in non-perturbative approaches, for example those that we will present in the next chapter, a clear derivation of the correct classical limit is still unknown[4]. For that reason this theoretical test of quantum gravity provides a very non-trivial consistency check for any non-perturbative theory of quantum gravity.

[4] *See [12, 13] for recent progress.*

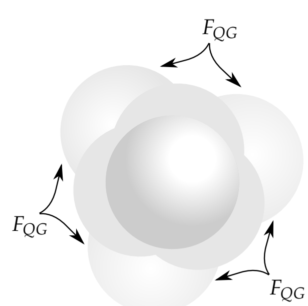
The semi-classical limit in which gravity is introduced as an external classical field in a quantum system:

For example we can consider a neutron beam send through an interferometer, whose arms are rotated about 90 degrees such that each of them experiences a slightly different gravitational force. The outcome of such an experiment would give first insights into a regime in which quantum and gravitational interactions interfere.

This experiment was realized by Colella, Overhauser and Werner (COW-experiment [14]) and the results turned out to be in full agreement with quantum mechanical predictions. According to calculations, the phase shift of the neutron beam is proportional to the experimental result. The proportionality constant is the ratio between the gravitational and inertial mass of the neutron [15], which has to be equal to 1 in order to agree with experiments. Hence, this result is understood as a verification of the weak equivalence principle in the quantum regime — equivalence between the gravitational and inertial masses.



(a) Classical gravitation source.



(b) Quantum gravitational source in a superposition state.

Figure 1.2: Classical vs. quantum gravitational source.

The weak field limit (WFL) where gravity is quantized but the gravitational field is considered small:

In this regime, the gravitational field is weak and can be reliably treated with perturbation theory. The perturbations around a background metric are called *gravitons*.

In principle this regime can be tested by various experiments and recently many table top experiments have been suggested for a potential verification of that limit in the near future:

Gravitational quantum source: The simplest way to access this regime in a table top experiment is to measure the gravitational field of a massive quantum object that is in a superposition between two different position states (fig. 1.2). Practically, however, this approach is quite challenging since one needs to create very massive quantum systems with very long decoherence times. The requirement on the mass is dictated by the sensibility of the current detectors that measure the gravitational pull. And at the present moment the gap between the size of a quantum mechanical system and the sensibility of the detector is about 12 orders of magnitude [16]. However, it has been shown, that one can use modern nano-technologies [16] and improve the isolation methods for quantum systems [17, 18] to bring the gap down to 5 orders of magnitude within the next few years. With further advances in technology the gap is expected to close within the next few decades.

Gravity as a source of coherence: A very recent experiment suggests to probe the quantum nature of gravity by a coherence experiment. The idea hereby is to look at the entanglement between two, initially uncorrelated, massive spin particles after they have been subjected to each others gravitational field [19]. The amount of resulting correlations between the spins will be due to the quantum nature of gravitation which can be measured by interference. The necessary amount of isolation and the size of required quantum systems is claimed to be achievable in separate experiments even today. For that reason it is estimated that such an experiment can be carried out in the near future.

STRONG FIELD LIMIT, (SFL) where the gravitational field plays a dominant role:

This is the regime of non-perturbative quantum gravity, in which the space-time geometry has to be fully quantized. The strong quantum fluctuations of the gravitational field prohibit a reliable use of perturbation techniques. It is this regime that is described by the Planck scale, and whose direct observations are very difficult to achieve. Here the hope of having a reliable laboratory experiment within our life time rapidly decreases. Nevertheless, indirect tests of this regime might exist even today. The most natural realm of such experiments is cosmology, and the laboratory is the Universe. This makes the problem of reproducible experiments a bit difficult:

Cosmological varification of SFL: The fact that we can not produce experiments for testing the SFL does not imply that processes on that scale never happen in nature. More specifically, we know that this regime was dominant during the Big Bang. Our goal is to understand the implications of that regime on our todays observations of the sky — the so-called *imprints of quantum gravity*. There are a number of proposals for the indirect tests of QG, reaching from observation of gamma ray bursts [20], to the search of Lorentz violations in the spectrum of the cosmic microwave background [21].

For example, some theories of quantum gravity predict a breakdown of Lorentz symmetry at the Planck scale [22]. This violation, if present, could lead to observable effects even in the low energy regime [23]. These theories, predict a variation of the speed of light, c, depending on the photon energy according to the modification [22],

$$c = 1 \pm a \frac{E}{E_p}, \tag{1.8}$$

where E is the energy of the photon, E_p is a Planck energy and a is a constant real number. This modified dispersion relation can be tested for high-energy-photons traveled over very long distances. Gamma ray bursts are the most suitable physical phenomenon to check for this modification and put experimental bounds on the parameter a [20].

However, to this day no conclusive evidence of truly quantum gravitational effects has been found. This may change soon due to the modern developments in the gravitational wave and 21 cm astronomy[5].

[5] *The Hawking radiation of black holes is believed to be a good candidate for the direct observations of SFL. The Hawking radiation of contemporary black holes is orders of magnitude smaller than the cosmic microwave background and therefore is currently not detectable. However, the radiation of primordial black holes of about* 10^{18} *g should currently result in observable x-ray bursts. Some of the candidate theories for QG predict a discrete spectrum of such x-ray bursts. An observation of the cosmic x-ray spectrum would be a direct evidence for the QG regime. These observations may become accessible due to 21 cm astronomy [24].*

The problem of reproducible experiments in cosmology complicates the detection of such imprints and sometimes we need to be lucky enough to look at the right spot of the sky at the right time. Recently, however, also table top experiments have been suggested that, if not testing SFL directly, can at least provide new bounds for any future theory of quantum gravity:

Table top SFL gravity: Many theories of quantum gravity suggest a fundamental discreteness of space-time at the Planck scale, which leads to modified commutation relation [25–29]. In [30] the authors suggest to use coherent states of light to check for the violation of such commutation relations between conjugate operators at the Planck scale. They show that a number of photons in a light beam can be used to amplify the tiny corrections of the commutator, resulting in a measurable effect. Such measurements could potentially put a bound on the violation of the commutation relations at the Planck scale.

Theoretical improvement: Contemporary fundamental physics is based on the assumption of separation of scales. This assumption says that processes at largely separated energy scales do not affect each other and therefore can be treated independently. However, a phenomenological discussion of gravitational theories suggests that this principle may not work for the gravitational interaction and physics at the Planck scale could affect to macroscopic scales. If this is true, quantum gravitational effects may be observable in every day life without us realizing it. An example for such effects in cosmology could be the phenomenon of dark matter [31], the small value of the cosmological constant [32] or the need for inflation in the description of our universe [33]. A theoretical development of the framework may, for that reason, eventually lead to an experimental verification of the theory.

As we can see a measurement of a quantum gravitational effect is difficult but may not be as hopeless as it seems at the first sight. After all, the history of physics is full of experiments that have been long considered impossible; the latest being the direct detection of gravitational waves.

1.3 Conceptual problems of quantum gravity

General relativity and quantum field theory rely on very different mathematical structures and physical assumptions. For that reason, a combination of their principles in one framework leads to a number of very challenging conceptual and technical issues. We will discuss some explicit issues of GR quantization in the next chapter, but here we want to emphasize the major differences between quantum field theory and general relativity that make the development of QG so complicated.

General relativity is a covariant theory, which complicates its quantization. Before we discuss the explicit problems of quantization, we want to clarify what we mean by the covariance of GR.

1.3.1 Diffeomorphism invariance of GR

Very often covariance of GR is described as invariance under local chart transformations. Even though this statement is not wrong, it is a little confusing, since the principle of invariance under chart transformations is a general feature of differential geometry. In fact chart independence is in the very definition of the concept of manifolds, and therefore does not sound as a particularly strong physical requirement. Any physical theory that can be formulated on a manifold is invariant under chart transformations[6]. For that reason this can not be the distinctive feature of gravitation.

[6] *For example electro-dynamics can be formulated in terms of forms on Minkowski space-time and hence independent of charts.*

Charts and their transformations simplify our comprehension of curved manifolds but they themselves do not have a physical meaning. Diffeomorphisms of space-time, on the other hand, (sometimes called active diffeomorphisms) do have a physical meaning, at least in cases when the space-time manifold is considered a physical object. For example, chart transformations in electro-dynamics leave the theory invariant, whereas local diffeomorphisms of space-time do not.

In general relativity, covariance implies that local diffeomorphisms of space-time are indistinguishable from chart transformations and can be used interchangeably. This statement is not about invariance of the theory under unphysical chart changes, it is about the fact that diffeomorphisms are just as much unphysical [34]. In other words it says that the space-time manifold does not have a physical meaning.

This problem was recognized by Einstein during his construction of the theory. Apparently, even though he had the formulation of GR already by 1912, he discarded it due to the above realization [35]. It took

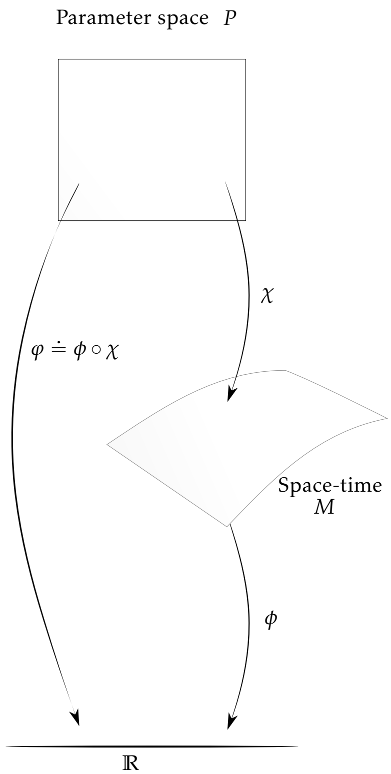


Figure 1.3: Field of the non-covariant theory ϕ can be reparametrized by artificial coordinates χ. The resulting theory for the fields $\varphi = \phi \circ \chi$ is then covariant under diffeomorphisms on the parameter space.

him three more years to understand that this is not a bug of his formulation but rather a feature of gravitation. His argument is nowadays known as *the hole argument* [35, 36] .

The above discussion, however, also does not fully capture the full meaning of diffeomorphism invariance. Indeed any (non-covariant) theory can be formulated in a covariant way when we parametrize space-time with some other four dimensional parameter-manifold (fig 1.3). Such parametrization may or may not simplify the formulation of the theory but it does not contain physical information and because of this the physics is invariant under reparametrizations.

Conversely, if we started with the parametrized theory we would observe its symmetry under reparametrizations and from this conclude that the parameter manifold is unphysical. However, if the theory in question is just a parametrized, initially non-covariant, theory it can be reduced back to a non-covariant formulation.

In GR the space-time itself plays a role of a parameter space and its truly distinctive feature is that it can not be formulated in a non-covariant way [34, 37]. This meaning of diffeomorphism invariance is what distinguishes GR from an artificially parametrized theory.

In summary: diffeomorphism invariance of general relativity says, that the space-time manifold does not have a physical interpretation — it is a parameter space that parametrizes a theory of geometry.

1.3.2 Gravity vs. quantum mechanics

Diffeomorphism invariance is a very difficult technical complication that obscures the procedure of quantization of gravity. The resulting technical problems accompany fundamental issues that come up when we start thinking about the quantum nature of space-time. For example:

Geometry instead of metric field: Due to the diffeomorphism invariance, the metric field does not have a physical meaning. What is physical instead, is the equivalence class of metric fields, where any two of them are called equivalent if they can be transformed into each other by a diffeomorphism. This equivalence class is what we call *geometry*. This implies that the actual physical degrees of freedom of quantum gravity should be the geometry and not the metric field. However, we are not used to a purely geometrical formulation of physical theories. Moreover, the diffeomorphism group of four dimensional manifolds is very large and even a complete characterization of the above equivalence classes is practically unknown.

QFT on many curved space-times at once: In a rigorous formulation, quantum field theory is formulated in terms of five axioms, called *the Wightman axioms* [38]. Despite the fact that it is very difficult to find a non-trivial theory that satisfies all five of them, Wightman's formulation is generally accepted as a definition of quantum field theory on Minkowski space-time. Unfortunately, there is no such agreement for quantum field theories on curved space-times[7]. It turns out that quantum mechanical structures, such as the definition of fields, particles or Hilbert spaces are extremely sensible even to slightest changes in the geometry of space-time. A single quantum theory that describes interactions of matter on any geometry seems to be incompatible with our understanding of QFT's.

[7] *Recent developments in algebraic quantum field theory provide a rigorous mathematical description of quantum field theory on curved space-times. This formalism, however, has only recently been applied to the case of quantum gravity [10, 39].*

THE PROBLEM OF TIME: Another closely related problem is the so-called *problem of time* [40, 41]. Due to diffeomorphism invariance, general relativity does not posses a conventional time evolution. Even though the Einstein field equations can be recast in the Hamiltonian form, this Hamiltonian takes the form of a pure constraint, meaning that it generates the orbits of the gauge symmetry and hence implements diffeomorphism invariance. In the quantum formulation such constraint operators have to be implemented in the Dirac way such that the physical states satisfy[8]

[8] *Here and throughout the thesis we use $|\cdot\rangle$ for the usual Dirac-Bra-Ket notation.*

$$H|\psi\rangle = 0, \tag{1.9}$$

and the relevant quantum operators fulfill

$$[\phi, H] = 0. \tag{1.10}$$

From the point of view of quantum field theory this formulation is completely static, since all Heisenberg equations of motion vanish. In a covariant theory this is to be expected, since there is no distinct time direction, with respect to which we can define a dynamical evolution. The resulting Hilbert space does not describe states at particular time but rather states at all times. The problem of time is the fact that we do not know how to interpret operators in/or extract physical information from such a quantum mechanical theory.

CAUSALITY: The so-called *locality* or *micro causality* principle is at the core of any quantum field theory [38, 39, 42]. In its non-rigorous formulation, it reads as follows: let x and y be two space-like separated points of space-time, than all commutators between observables at these points have to vanish, i.e.

$$[g(x), g(y)] = [\pi(x), \pi(y)] = [g(x), \pi(y)] = 0, \tag{1.11}$$

where $g(x)$ is the (would be) operator for the gravitational field and $\pi(x)$ is its canonical conjugate. Yet, without a fixed causal structure we can not define space-like separation and the above commutation relations can not be formulated.

CAUSALITY 2.0: Another fundamental issue is the understanding of the causal structure in cases when the gravitational field obtains quantum fluctuations. In this case it can happen, that two points are time-like and also space-like separated, due to the fluctuations of the causal cone of the metric field. A classical outcome of such theory could randomly assign time- or space-like properties to curves and hence mess up causality[9].

[9] *See [43] for recent results.*

The above listed problems form only a very small portion of difficulties one encounters on the way to quantum gravity, and many more mathematical issues and physical complications arise when we begin to think about the gravitational field in the quantum way. Some of the above mentioned problems are answered in the context of background independent theories that we are going to introduce in the next section and some of them still remain unsolved. But the "take away message" is this: in order to provide a consistent and sensible theory of quantum gravity we need to develop radically new ideas. This is what makes the quest of searching for this theory so exciting — we can be pretty sure of the fact, that any theory that will resolve all of the above problems will be nothing like what we already know, and will drastically change our understanding of nature once again.

In this thesis we are focusing on the development of a specific approach coming from the direction of the so-called background independent quantization of the gravitational field. But to provide a small impression on the diversity of models we conclude this chapter with a very nice flow chart, kindly provided us by Lisa Glaser.

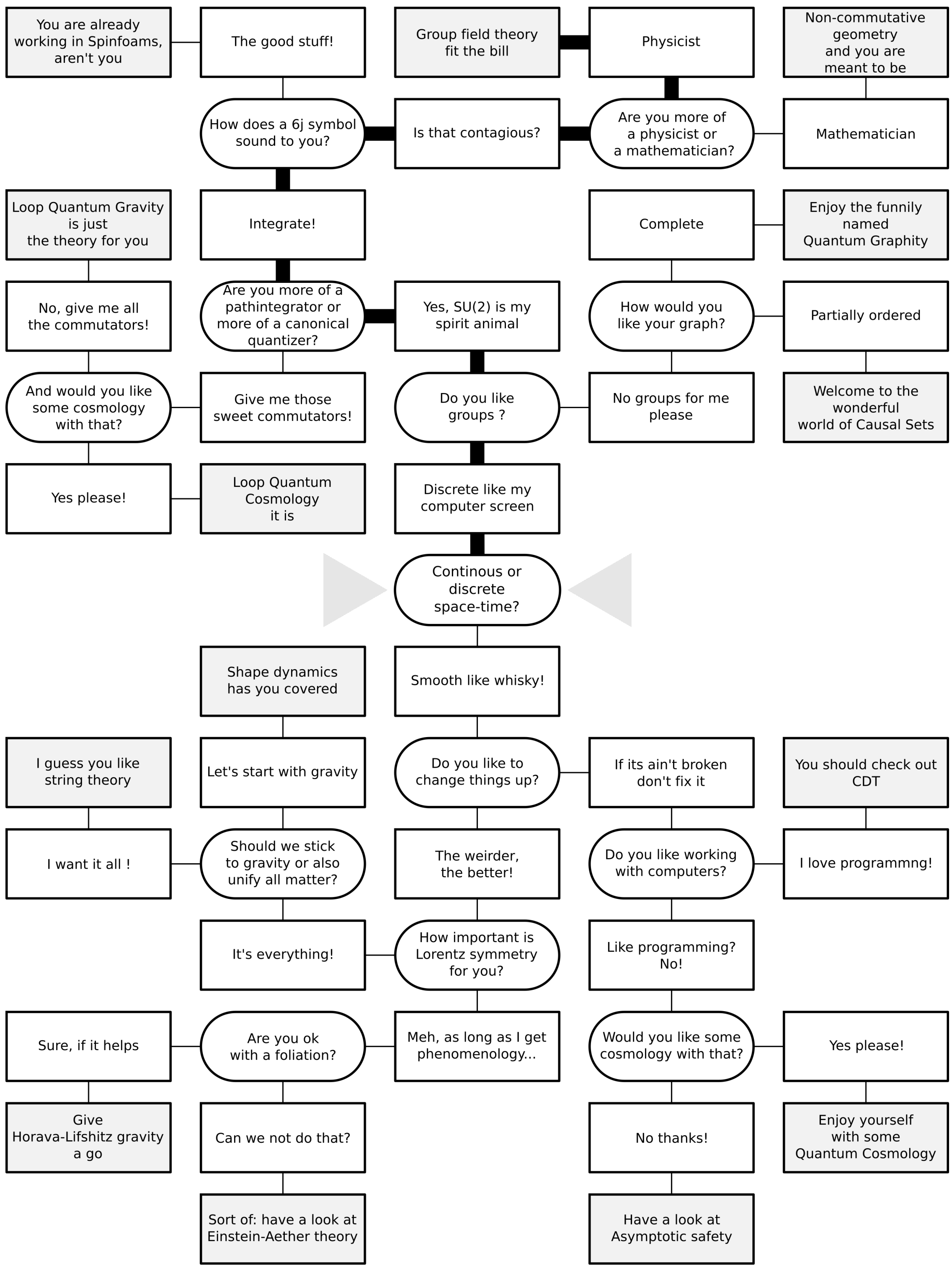


Figure 1.4:
A mind map of quantum gravity (cc Lisa Glaser). Start in the middle of the chart and find your favorite theory of quantum gravity. The road we prefer and the one the reader should be interested in order to find the following work appealing is outlined with the bold path that amounts in group field theory.

2 Group field theory

In this chapter we are going to discuss *group field theory* (GFT), its conceptual ideas and some of its technical details. It is a challenging task to describe a very technical and ill-understood subject without being drowned in mathematical nomenclature. To prevent this from happening we will leave out most of the mathematical details that are not needed for the understanding of the following work. Nevertheless, even in this version it is inevitable that some of the following discussion still remains a little technical.

Apart from *matrix*- and *tensor-models* [44–46], group field theory [47–49] is historically rooted in *loop quantum gravity* (LQG) [50–53] and *spin foam models* [54, 55] — sometimes called covariant LQG. Even though the contemporary development of group field theory suggests to treat it as an independent, self-contained quantum field theory [56–58], we feel that its ideas, motivations, and intuition are still best presented in the context of covariant LQG and for that matter, we choose this path for our introduction of the subject. Nevertheless, since our focus is on group field theory, covariant loop quantum gravity is introduced only for the sake of better conceptual understanding and for that reason is not discussed in details. For further information on LQG we refer to [3, 50, 59, 60] for the canonical approach and to [7, 54, 61, 62] for the covariant formulation.

In this chapter we will introduce and motivate the concepts of GFT starting with the basic idea of a path integral for quantum gravity. The chapter is structured in the following way[1]:

We will begin with the conceptually simple but mathematically challenging idea of Misner that suggests to define quantum gravity via the

[1] *In QFT's on space-time we distinguish between the covariant and canonical formulation of QFT. The first is written in terms of functional integrals, whereas the second is given in terms of operators on Hilbert spaces. In GFT we call these formulations the* functional GFT *and the* operator GFT, *respectively — to prevent any confusion with the underling symmetry-association of the name giving.*

usual path integral [63],

$$Z = \int \mathcal{D}g \, e^{\frac{i}{\hbar} S_{\mathrm{HE}}[g]},$$

with the Hilbert-Einstein action S_{HE}, eq. (2.1), and the formal integral measure, $\mathcal{D}g$, over the space of all geometries. Quantum gravity can than be summarized as an attempt to provide meaning to this, rather formal, expression. This turns out to be a very daring endeavor.

From the point of view of mathematics, the difficulties to define the above expression are severe and require a development of new computational tools; from the point of view of physics, the problems of understanding the fundamental concepts of the resulting theory are also very challenging and urge the need for new ideas. Some of those ideas and mathematical techniques we will present in this chapter.

2.1 *Misner quantization of GR*

Misner's idea for quantization of gravity suggests to use covariant formulation of quantum field theories for GR [63]. In this formulation the starting point is the classical theory given by the Hilbert-Einstein action S_{HE} in four dimensions,

$$S_{\mathrm{HE}}\left[g\right] = -\frac{c^4}{16\pi G} \int_M \mathrm{vol}_M \, R\left(g\right), \tag{2.1}$$

where g is the metric field, $R\left(g\right)$ is the scalar curvature, and vol_M denotes the volume density on the Lorentzian manifold given in chart coordinates by

$$\mathrm{vol}_M = \mathrm{d}^4 x \sqrt{-\det\left(g_{\mu\nu}\right)}. \tag{2.2}$$

The corresponding quantum theory is then defined by the formal integral

$$\int_q \mathcal{D}g \, e^{\frac{i}{\hbar} S_{\mathrm{HE}}[g]}, \tag{2.3}$$

where the integration is performed over all geometries of a four dimensional (4D), differentiable manifold M. The boundary value $q = g\left(\partial\Omega\right)$ specifies the metric, or rather, the geometry[2] on the boundary of Ω — a three dimensional (3D) sub-manifold of M (fig. 2.1). The value of the integral (2.3) gives the probability amplitude for the transition between the fixed boundary geometries of M. Heuristically, in the classical limit, $\hbar \to 0$, one finds by the argument of steepest descent that the integral is dominated by the classical extrema of the Hilbert-Einstein action, leading to field configurations that solve the Einstein field equations with fixed boundary conditions.

[2] *Reminder*

We refer to the space of geometries as the space of equivalence classes of metric fields $[g]$, where two metric fields are considered equivalent if they differ by a diffeomorphism,

$$g \sim \tilde{g} \quad \Leftrightarrow \quad \tilde{g} = \varphi^\star g, \tag{2.4}$$

where φ is a local diffeomorphism on M and $\varphi^\star$ denotes the pull back.

Overall, this prescription is intuitive and provides a straightforward adaptation of the covariant quantization methods of ordinary field theories. And just as in these more familiar cases, it is purely formal. The typical problem of this formulation is the lack of any measure theoretical notion of the above integral. And in the above case this notion is even more complicated due to the diffeomorphism invariance of GR.

The Hilbert-Einstein action is invariant under local diffeomorphisms and their field equations are covariant. As we already mentioned in the previous chapter, this implies that the true degree of freedom of GR is the geometry and not the metric.

Since physical quantities are invariant under the group of diffeomorphisms, gravity can be seen as a peculiar type of a gauge theory. In contrast to usual gauge theories, however, the gauge group of gravity is not finite dimensional and is poorly characterized. Because of this we can not specify the equivalence classes of the metric field. Practically, this means that we can not decide (except for some special cases) if any two solutions of the Einstein field equations describe the same geometry or not.

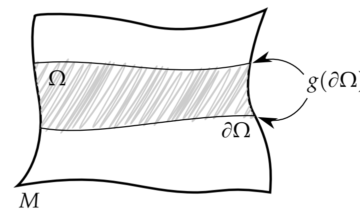


Figure 2.1: Functional integral of gravity would provide a probability for the space-time geometry in Ω with boundary geometries given by $g\,(\partial\Omega)$.

Unfortunately, geometry without any reference to coordinate system may be unintuitive and the concept of the space of geometries is even less clear. This complicates our understanding of the already ill defined measure $\mathcal{D}\,[g]$ (the measure over the space of metric classes) even more. Moreover, one could assume, that the integration in (2.3) should include not only geometries $[g]$ but also topologies of a four dimensional manifold [64], which can not be done even in principle [65].

The definition of the measure is a serious issue and it is only the tip of the problems that we begin to encounter when we embark on the quantization of the gravitational field. For example, in ordinary quantum field theories on Minkowksi space-time the above functional integral is not well defined either and is used only for a structured derivation of perturbation theory. Many physical phenomena, however, are not captured within the perturbative approach and require a non-perturbative definition of the theory. To manage these problems in particle physics we typically perform *the Wick rotation*, by going from the real to imaginary time. Due to this rotation, the action S transforms to the so called *Euclidean action* $\imath S_E$ and the exponent in the functional integral obtains some nice convergent properties

$$\int \mathcal{D}\phi\, e^{\frac{\imath}{\hbar}S[\phi]} \quad \xrightarrow{\text{Wick}} \quad \int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S_E[\phi]}. \tag{2.5}$$

The resulting integral can sometimes be given a rigorous meaning [66–69] or at least can be numerically approximated [70] and provides access to non-perturbative effects. Wick-rotating the final results back to the Lorentzian time captures non-perturbative effects of the integral (2.3) [70, 71].

In gravity, Wick rotation is complicated. Contrary to the ordinary case, a generic Lorentzian metric g with the signature $(-,+,+,+)$ can not be rotated to the Riemannian one without obtaining complex values in some of its components [72]. Hence, under such rotations the Hilbert-Einstein action will become complex-valued and the problem of the oscillating factor in the exponent can not be resolved [73, 74].

In order to avoid this problem we can try to define our theory of gravity directly in the Euclidean language, with the hope that the resulting equations can be eventually Wick rotated. However, even in this case we encounter problems; the Euclidean Hilbert-Einstein action S_{EHE}[3] is not bounded from below [74, 75], sometimes called *bottomless*. This means that we can find field configurations that make the exponent of the integrand in equation (2.5) arbitrary large. Clearly, these configurations dominate the integral — as a result it can not be given any rigorous meaning again. If the action S_{EHE} had a local minimum we could define a perturbative theory, however, the Euclidean Hilbert-Einstein action does not even have those. The only extreme points of S_{EHE} are saddle points, due to the presence of the conformal mode in the metric. This is known as the *conformal-factor problem* [75].

3 The Euclidean Hilbert-Einstein action has the same form as the original Hilbert-Einstein action,

$$S_{EHE}\left[g\right] = -\frac{c^4}{16\pi G}\int_M vol_M\, R\left(g\right),$$

but defined on the space of Riemannian metric fields g.

This fact was realized by Hawking and Gibbons in the late 70s [75, 76] when they suggested a regularization scheme to bound the Hilbert-Einstein action, but up to now there is no common agreement on this procedure. In [77] the authors claim that the conformal divergence can be canceled by contributions in the definition of the functional measure, but also this still needs to be clarified more carefully.

We see that techniques from ordinary QFT seem to be inapplicable for gravity and we need to develop new mathematical tools in order to get the quantization of GR under control. Some of the new tools were introduced in the framework of loop quantum gravity in its canonical and covariant formulation. But before we move on to this, we summarize the variations of the Misner integral that we already encountered and that will reappear in what follows.

LORENTZIAN QUANTUM GRAVITY: Quantization of the Hilbert-Einstein action with a Lorentzian metric and an oscillating integrand,

$$\int \mathcal{D}g\, e^{\imath S_{\mathrm{HE}}[g]}.$$

Typically, this is just a formal expression that can not be used for rigorous non-perturbative calculations.

RIEMANNIAN QUANTUM GRAVITY: A version of quantum gravity, in which the Lorentzian metric is replaced by the Riemannian one,

$$\int \mathcal{D}g\, e^{\imath S_{\mathrm{HE}}[g]}.$$

This provides a technical simplification of the above case where the non-compact Lorentz symmetry group of the Lorentz metric is replaced by the compact rotational group of the Riemannian one. It is not clear how the results from Riemannian quantum gravity can be used for the Lorentzian case, but we are still interested in this formulation because we hope to gain useful intuition for quantization of covariant systems even in this case.

EUCLIDEAN QUANTUM GRAVITY: A "would be" Wick rotated version of Lorentzian quantum gravity with better convergence properties

$$\int \mathcal{D}g\, e^{-S_{\mathrm{EHE}}[g]}.$$

The action S_{EHE} has the same form as the usual Hilbert-Einstein action but is defined on the space of Riemannian metric fields. This integral should be much better under control, but it diverges due to the conformal mode of the metric field. Even if the conformal-factor problem can be overcome, it is still not fully understood how much this theory will relate to the Lorentzian quantum gravity since the Wick rotation back to the Lorentzian signature in general does not exist.

2.2 *Covariant loop quantum gravity*

Spin foam models — sometimes called covariant loop quantum gravity — regularize the Lorentzian quantum gravity path integral using a different set of variables other than the metric field. The idea is to discretize the Misner integral in a gauge invariant way and then subsequently remove the regulator. The spirit of the construction can be captured in the following equation[4]

$$\int_{q_1}^{q_2} \mathcal{D}g\, e^{\imath S_{\mathrm{HE}}[g]} = \sum_{\mathcal{F}} A\left[\mathcal{F}_{q_1,q_2}\right], \tag{2.6}$$

where the left hand side is the transition amplitude between two 3D geometries specified by q_1 and q_2 and the right hand side provides a corresponding definition of a spin foam model that we will explain below. But first we spend a few words on the classical reformulation of GR.

[4] *From now on we will assume units for with $\hbar = 1$. If powers of $\hbar$ will be needed for calculations we will explicitly state this in the text.*

2.2.1 *Classical reformulation of GR — the Plebanski action*

It is well known that the choice of the right classical degrees of freedom can significantly simplify quantization. In the Hilbert-Einstein formulation of GR the degree of freedom is the metric field. But it is sometimes not the most convenient one for canonical quantization. It turns out that for covariant quantization there is a more suitable set of variables — the *frame field* and the *connection* [78].

Mathematically, the frame field is a local trivialization[5] of the tan-

[5] *In general the local trivialization is not global. But, here for the purpose of better readability, we restrict ourselves to the simple case in which the tangent bundle is diffeomorphisc to the trivial bundle.*

gent bundle, or alternatively a vector valued one form. That is,

$$e : TM \to M \times \mathbb{R}^4.$$

Its physical interpretation is that of a reference frame of a free falling observer at each point of space-time. In this frame the observer does not feel any gravitational force and to him the metric at the given point looks flat. This can be expressed by the following relation,

$$g = e^{\star}\eta, \tag{2.7}$$

where η is the Minkowski metric, e is the frame field, g is a solution of the Einstein field equations and $\star$ denotes the pull back. We can imagine e as a local "flattening" of space-time; to keep track of geometry on the resulting flat space we then need to introduce an additional variable — the connection field A.

The connection field $A : TM \to \mathfrak{so}(3,1)$ is a differential one-form on M with values in the Lie algebra of the symmetry group $SO(3,1)$. To vividly understand its role take two points[6] on $M \times \mathbb{R}^4$, say (x_1, v_1) and (x_2, v_2), and compare the vectors v_1 and v_2 (fig. 2.2). Since we can only relate vectors at the same base point we need to transport v_1 from the base point x_1 to the base point x_2. For this we use the connection A and a curve γ on the manifold along which we transport the vector. Since A is valued in the Lie algebra $\mathfrak{so}(3,1)$ its transport along γ is an element of the Lie group $SO(3,1)$, given by $g_\gamma = \mathcal{P}\left(e^{\int_\gamma A}\right)$, where $\mathcal{P}$ is a path ordering[7]. Hence, we transport the vector v_1 along γ by rotating it with the group element g_γ in a suitable representation of the Lie group. The vector v_1 at the base point x_2 takes the form

$$\bar{v}_1 = \pi\left(g_\gamma\right) \cdot v_1, \tag{2.8}$$

where $\pi\left(g_\gamma\right)$ is the four dimensional representation of the Lie group $SO(3,1)$ and $\cdot$ denotes the matrix multiplication. Such rotational matrices $\pi\left(g_\gamma\right)$ for (all) curves on the space-time M encode the geometry. A trivial connection $A(x) = 0$, for example, leads to a trivial rotation $g = \mathbb{1}$ along any curve and therefore corresponds to a flat geometry. The idea of the generalized connection A is hence very similar to that of the Levi-Civita connection on TM. However, for a generic manifold, the Levi-Civita connection is not Lie algebra-valued and in general coordinate dependent.

The role of the generalized connection field A is to keep track of the curvature of space-time by *imitating* the Levi-Civita connection on the trivial bundle $M \times \mathbb{R}^4$ [40]. For that reason A is sometimes called the *imitation connection*. In summary we can say, that: the field e flattens the space-time, whereas A imitates the Levi-Civita connection.

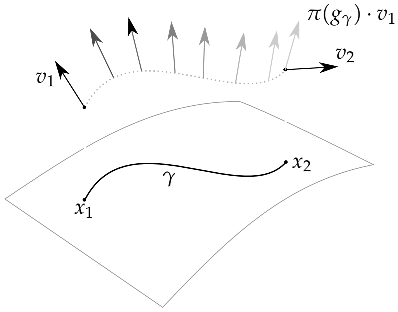


Figure 2.2: Transport of the vector v_1 along a curve γ using connection A. The idea of that transport is the same as that of the usual parallel transport of the Levi-Civita connection, however, in this setting there is no natural structure to define the "parallel" part.

[6] *Products of the form $M \times V$, with M a differential manifold and V a vector space are called* trivial vector bundles, *see chapter 4 for a brief introduction to vector bundles*

[7] *Path ordering is a way of ordering the exponential integrals such that*

$$\mathcal{P}\left(e^{\int_\gamma A,}\right) = \sum_{n=0}^{\infty} \int_{1 \geqslant t \geqslant t_1 \geqslant \cdots \geqslant t_n \geqslant 0} A(\dot{\gamma})\, dt_n \cdots dt_1,$$

where $A(\dot{\gamma}) \doteq \langle A(\gamma(t)), \dot{\gamma}(t)\rangle$ and γ is a curve parametrized by $t \in (0,1)$.

Using e and A as an independent pair of variables the Hilbert-Einstein action can be rewritten in the form of a gauge theory, called the *Plebanski action* [79],

$$S\left[e,A,B\right] = \frac{1}{8\pi G}\int_M \left(\sum_{ijkl}\sum_{\alpha,\beta,\gamma,\delta} \epsilon_{ijkl}\epsilon_{\alpha\beta\gamma\delta}B_{ij}^{\alpha\beta}F_{kl}^{\gamma\delta}\left(A\right) + \lambda C\left(e,B\right)\right)\mathrm{d}x. \tag{2.9}$$

where, $i,j,k,l \in \{1,2,3,4\}$ are the indexes of the $\mathbb{R}^4$ fiber, $\alpha,\beta,\gamma,\delta \in \{1,2,3,4\}$ are space-time indices, $F\left(A\right)$ is the curvature two form much like the electro-magnetic field tensor in Maxwell's and Yang-Mills theories[8], A is the imitating connection, and B is an auxiliary field that is related to the frame field e by the constraint $C\left(e,B\right)$ with a Lagrange multiplier λ. The factor ϵ_{ijkl} is the Levi-Civita symbol in four dimensions.

[8] *In a suitable chart x^μ the curvature of the connection A is given by*

$$F_{\mu\nu}^{ij} = \partial_\mu A_\nu^{ij} - \partial_\nu A_\mu^{ij} + \sum_k A_\mu^{ik}A_\nu^{kj} - A_\nu^{ik}A_\mu^{kj},$$

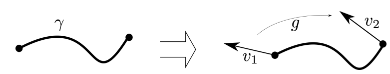
where A_μ denote the generators of the four dimensional representation of the Lie algebra.

Solutions of equations of motion of the Plebanski action impose the dynamical relations of GR on the variables e and A and become equivalent to solutions of the Einstein field equations [80].

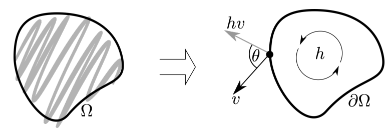
We already mentioned that the variables e and A are differential forms on M. This fact becomes especially useful in the covariant formalism. This is because differential forms can be naturally integrated. Locally, around any point of space-time and in some suitable chart x^μ, the variables e and A can be written as

$$e\left(x\right) = e_\mu^I\left(x\right)\,\mathrm{d}x^\mu \qquad A\left(x\right) = A_\mu^{IJ}\left(x\right)\,\mathrm{d}x^\mu, \tag{2.10}$$

where $e_\mu^I\left(x\right)$ and $A_\mu^{IJ}\left(x\right)$ are their components in the chart, and $\mathrm{d}x^\mu$ denotes a chart-induced one-form on the tangent space of space-time. In this sense they bring their own notion of the integral measure, $\mathrm{d}x^\mu$, as opposed to functions that require the volume form, $\mathrm{vol} = \mathrm{d}^4x\,\sqrt{-\det\left(g\right)}$.



(a) Smearing of the connection along a path γ leads to a parallel transport $g \in G$. Where G is the symmetry group of the metric field.



(b) Smearing of the curvature around an area Ω gives a holonomy $h \in G$ around the loop $\partial\Omega$.

Figure 2.3: Smearing of the connection-one-form A and the curvature-two-form $F\left(A\right)$.

In general we say, one-forms can be integrated (or *smeared*) over 1D lines, two-forms can be integrated (smeared) over 2D surfaces and so on. Such smearing does not require the notion of a metric and hence we call them *natural*. In particular, in the case of the connection form A, we can define its integral along curves, which leads to the definition of the "parallel" transport (fig. 2.2 and fig. 2.3a). We can do the same with the curvature $F\left(A\right)$ if we smear it over surfaces. The result of this integral is a "parallel" transport along the boundary of the integral surface, which is called *holonomy* (fig. 2.3b). In this sense the "smearing" of forms on extended objects can be understood as a discretization procedure.

Hence, by this "natural" smearing, the BF part of the action[9] (2.9) can be regularized, once we fix the line elements (called edges) and surfaces (called faces) used for the smearing of A and F, respectively. This combination of edges and faces in a mathematical jargon is called a *two-complex*. It turns out that also the B field can be discretized in the

[9] *The part of the action that is proportional to the B and F fields. This is known in the literature as the BF-action. We adopt this notation for rest of the thesis.*

same way and corresponds to a choice of representation for each link of the two complex [62]. In order to implement the regularization of the B field we need also to label the faces of this two-complex by representations of the symmetry group of the metric field. For further reasons, that we do not want to discuss here, the edges of the two-complex have to be labeled by intertwiners between the representations of adjacent faces; the resulting labeled two-complex is called a *spin foam* [54]. Hence if we fix a spin foam over the manifold M, we can give a discrete meaning to almost all ingredients of the Plebanski action — all except for $C(e,B)$. The constraint $C(e,B)$ is more complicated to deal with but in this text we will not need its explicit implementation.

2.2.2 Covariant quantization

We come back to the discussion of the quantization and write the covariant integral for gravity in the new variables e and A using the Plebanski action, eq. (2.11),

$$\int \mathcal{D}B\,\mathcal{D}e\,\mathcal{D}A\; e^{\imath S[e,A,B]}. \tag{2.11}$$

At this point we will deal with the Riemannian gravity in order to avoid additional technical complications. That is, we assume the metric to be Riemannian with a compact symmetry Lie group, $SO(4)$, which implies that the connection field A is valued in the Lie algebra $\mathfrak{so}(4)$ instead of $\mathfrak{so}(3,1)$. The above integral is the Riemannian version of Misner's definition, formulated in terms of the connection, the frame and the auxiliary fields. Using the discretization we described above we can give it a rigorous meaning. Fixing a spin foam $\mathcal{F}$, whose faces are labeled with representations of $SO(4)$ and whose edges are labeled with intertwiners, the functional integral reduces to products of Haar measure integrals and for the BF part of the action, can be calculated in the closed form[10] [54]. This regularization is very similar to the discretization used in lattice gauge theories, where the role of a lattice is replaced by a labeled two-complex, $\mathcal{F}$.

The result of the Haar measure integrals is denoted

$$\mathscr{A}[\mathcal{F}], \tag{2.12}$$

and called the *spin foam amplitude*.

In order to remove the notion of the two-complex from the final result, that is to remove the discretization, we need to sum over all possible two-complexes that discretize space-time. This defines the spin foam model[11]

$$\sum_{\mathcal{F}} \mathscr{A}[\mathcal{F}]. \tag{2.13}$$

[10] *The constraint $C(e,B)$, however, complicates the expressions such that the discretized version of (2.9) can not be easily integrated. For that reason we first formulate the spin foam amplitudes for the BF part of the action and subsequently modify them according to our physical interpretation of the constraint $C(e,B)$.*

[11] *Alternatively, a spin foam model can be defined by some suitable refinement limit procedure [81].*

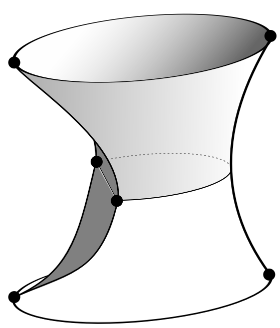


Figure 2.4: Spin foam with faces shaded in gray, edges shown with dark lines, and vertices shown with dark dots.

If we specify the boundary geometry q_1 and q_2 on some boundary of a space-time region Ω, as we did above, we would have to sum only over those two-complexes that fix the discretized boundary geometry (fig. 2.4). In this case we write

$$\sum_{\mathcal{F}_{q_1,q_2}} \mathscr{A}\left[\mathcal{F}_{q_1,q_2}\right] \tag{2.14}$$

So far goes the theoretical construction of a spin foam model. In practice, the situation is not as simple as we presented above. The problem is the constraint function $C(e,B)$, that typically prevents the reduction of eq. (2.11) to closed expression of Haar measure integrals. For that reason we first perform the regularization steps for the BF-action, obtain the amplitudes $\mathscr{A}[\mathcal{F}]$, and then modify the amplitudes according to our understanding of the constraint $C(e,B)$. At this point there are different suggestions for the modification of the amplitudes that lead to different spin foam models [82–84]. These different models suppose to capture different features of gravity and the decision for one model over the other has to be done based on the results. Especially, the classical continuum limit of the theory has to reduce to classical GR. Unfortunately, we lack the necessary technical control to check explicitly for this condition in any of the realistic spin foam models[12].

[12] *Instead of performing the classical limit of the continuous theory, we can take the classical limit of its discrete version, that is without performing the sum over complexes. In this case it was shown that some models [85] relate to a discrete version of the Hilbert-Einstein action evaluated at the solutions of the discrete Einstein field equations.*

In order to check for the continuum limit we need to remove the regulator, which implies that we need to calculate the sum in equation (2.13). More specifically, for a meaningful definition of a theory this sum should converge. This, however, is not the case in most of the existing models, and the question arises if it is possible to renormalize the model in a similar spirit as it happens in lattice gauge theory. This is a current, active field of research [86–88].

A problem of the spin foam quantization is the unordered structure of the sum (2.13). It is difficult to define the domain of summation [55]. Even if we restrict ourselves only to a specific type of spin two-complexes, namely to those that appear as dual complexes of triangulations of space-time, we are still missing a practical prescription to perform the sum. Usually, we would try to use perturbation theory and truncate the sum with some regulator, however, in the general prescription (2.13) it is not clear which configurations contribute most to the sum. Especially, since we are missing a structured way to sum over the two-complexes, we cannot order the amplitudes by their importance, and thus we cannot trivially truncate the theory.

At this point the formalism of group field theory enters the game and provides a prescription to sort the amplitudes in a particularly nice way such that we can deal with the sum (2.13) using the methods of quantum field theory [47, 89, 90].

REMARK In three dimensions, the Plebanski action takes the form

$$S[B,A] = \int_M \sum_{i,j,k} \sum_{\alpha\beta,\gamma} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} B_i^\alpha F_{\beta\gamma}^{jk}(A)\, \mathrm{d}x, \tag{2.15}$$

without additional constraints [54]. In this case the above procedure can be carried out without much ambiguities and we obtain a spin foam model of 3D gravity [54].

2.3 *Group field theory — functional formulation*

Group field theory [47, 48, 91, 92] provides a structured way to sum over two-complexes such that the resulting theory can be formulated in the language of quantum field theory. For the *BF* action in 3D it has been realized that one obtains a sum over 2 complexes with the same spin foam amplitudes, starting from a scalar field theory defined on 3 copies of the group $SU(2)$, that is with the fields $\phi : SU(2)^{\times 3} \to \mathbb{R}$. The action for this field theory is known as the *Boulatov* action [93] defined as

$$\begin{aligned} S[\phi] = &\int_{SU(2)^{\times 3}} \mathrm{d}g\, \phi(g_1,g_2,g_3)\, \phi(g_3,g_2,g_1) \\ &+ \lambda \int_{SU(2)^{\times 6}} \mathrm{d}g\, \phi(g_1,g_2,g_3)\, \phi(g_1,g_4,g_5)\, \phi(g_6,g_2,g_5)\, \phi(g_6,g_4,g_3), \end{aligned} \tag{2.16}$$

where $\mathrm{d}g$ denotes the Haar measure on all the group elements involved in the integrand. The spin foam model for the 3D Plebanski action (2.15) is then defined in terms of the perturbative expansion of the partition function

$$Z = \int \mathcal{D}\phi\, e^{-S[\phi]}. \tag{2.17}$$

We will describe the detailed relation below, but to see the formal connection we recall that Feynman diagrams are graphs made of lines (propagators), vertices (interaction vertices) and faces (loops of the diagram) and hence can be seen as two-complexes. If, in addition, the fields ϕ satisfy a symmetry such that for any $h \in SU(2)$ the fields are invariant under the diagonal action of h, that is,

$$\phi(g_1,g_2,g_3) = \phi(g_1h, g_2h, g_3h) \tag{2.18}$$

at any point $(g_1,g_2,g_3) \in SU(2)^{\times 3}$, these two-complexes obtain a required labeling by intertwiners and hence define spin foams. For reasons that we will mention below (see margin note on page 38) this symmetry of the fields is called the *closure constraint*.

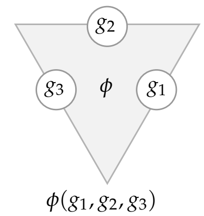
The Feynman amplitudes — evaluation of the Feynman diagrams — define the spin foam amplitudes of the 3D Plebanski action discretized over the two complex defined by the Feynman diagrams. Moreover, the sum over Feynman diagrams provides a sum over two complexes, such one formally expects an expression like

$$\sum_{\mathcal{F}} A\left[\mathcal{F}\right] = \int \mathcal{D}\phi\, e^{-S[\phi]} = \sum_{n} \sum_{Diag(n)} \mathrm{Sym}_{Diag(n)}\, F\left(Diag\left(n\right)\right). \tag{2.19}$$

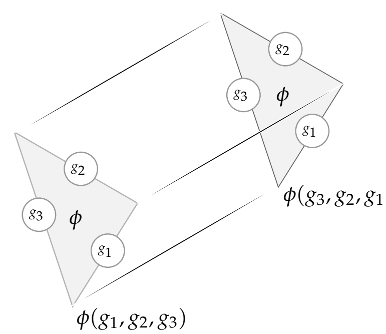
The left hand side is formulated in terms of spin foam amplitudes, whereas the right hand side is a perturbative, diagrammatic expansion of a GFT where the second sum ranges over all inequivalent Feynman diagrams with n vertices, $\mathrm{Sym}_{Diag(n)}$ refers to the symmetry factor of that diagram and $F\left(Diag\left(n\right)\right)$ denotes its Feynman amplitude.

The action in eq.(2.16) has a very specific structure that is quite different from the usual cases in QFT. The peculiarity is twofold:

1. the quadratic part of the action does not involve any differential operator, we say *it has a trivial propagator*, and
2. the interaction part is a non-local combination of fields with mixed variables. We call this type of interactions *combinatorially multi-local*.



(a) Representation of the field as a triangle with labeled edges.



(b) Multiplication of two fields. The order of variables need to be reversed in order to maintained the same orientation.

Figure 2.5: Interpretation of the field as a triangle.

2.3.1 *Geometric interpretation of the Boulatov interaction*

In order to understand the interaction of the Boulatov model, it is enlightening to introduce a pictorial association to fields.

Let us picture the field ϕ by a triangle, whose edges are labeled with the variables of the field as in figure 2.5a. If the variables of the two fields coincide, the triangles touch each other edge to edge. If all of the variables coincide, all three edges of the triangles overlap, we say: *the triangles are glued together*.

With this association the quadratic part of the action represents two triangles glued together face to face as shown in figure (2.5b).

The interaction

$$\phi\left(g_1, g_2, g_3\right)\phi\left(g_1, g_4, g_5\right)\phi\left(g_6, g_2, g_5\right)\phi\left(g_6, g_4, g_3\right) \tag{2.20}$$

then corresponds to four triangles glued along their edges to form a tetrahedron shown in figure 2.6.

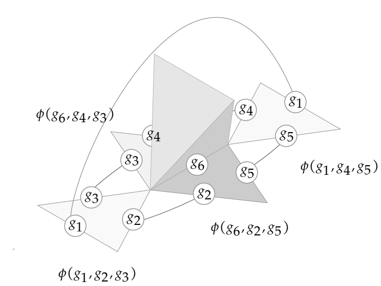


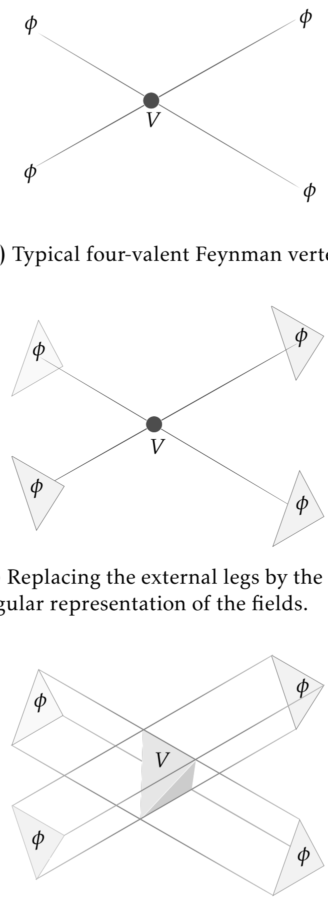
Figure 2.6: Fields represented as triangles close together to form a tetrahedron.

Let us look at the quantum case and perturbatively expand the four point function $\left\langle \phi\left(g^1\right)\phi\left(g^2\right)\phi\left(g^3\right)\phi\left(g^4\right)\right\rangle$, with $g^i \in SU\left(2\right)^{\times 3}$ for $i \in \{1, 2, 3, 4\}$. The series that we get from this expansion can be formally written as

$$\langle\phi\phi\phi\phi\rangle = \langle\phi\phi\phi\phi\rangle_0 + \langle V\phi\phi\phi\phi\rangle_0 + \langle VV\phi\phi\phi\phi\rangle_0 + \cdots, \tag{2.21}$$

where $\langle\cdot\rangle_0 = \int \mathcal{D}\phi \cdot e^{-\int\phi\phi}$ denotes the expectation value with respect to the quadratic part of the action and that can be calculated using the usual Wick contraction. The Feynman diagram for the $\langle V\phi\phi\phi\phi\rangle_F$ term can now be understood as follows:

1. begin with the usual Feynman diagram (figure 2.7a),
2. replace the fields by triangles (figure (2.7b)),
3. replace the vertex by the tetrahedron (figure (2.7c)).



(a) Typical four-valent Feynman vertex.

(b) Replacing the external legs by the triangular representation of the fields.

(c) Replacing the vertex according to the combinatorial pattern of a tetrahedron triangles.

Figure 2.7: Beginning with the usual Feynman diagram and subsequently replacing their ingredients according to the above prescription we obtain a Feynman diagram of GFT.

The resulting diagram is a combination of triangles and tetrahedra. Diagrams with nth power in the interaction, correspond to n tetrahedra glued together face to face. Such constructions are called *simplicial complexes*, and we can interpret them as discrete pictures of space-time. Sometimes it can happen, that tetrahedra are glued to themselves leading to some non-geometrical structures. To avoid this, we can color the faces of the tetrahedron in four different colors and require that only two triangles with the same color can be glued together [44, 94]. Such models are called *colored models* and are much better behaved in the perturbative expansion. But for the further conceptual understanding these details are not important. The above pictorial identification shows, that we can view Feynman diagrams of GFT as graphs, that are dual to simplicial complexes. Such graphs define two-complexes, and due to the closure constraint of the fields they become labeled in the correct way to produce spin foams.

All in all we can understand the relation between spin foam models and group field theory as follows: spin foams rely on the two-complexes that can be understood as discretization of space-time, whereas group field theory provides a construction of the same two complexes from more fundamental degrees of freedom described by the fields (triangles).

So far goes the discussion of the 3D Plebanski gravity. In the 4D case the BF part of Plebanski action can be formulated in terms of scalar fields in the similar way, with minor modifications: the fields become functions on four copies of $Spin\,(4)$ and can be pictured as tetrahedra (instead of triangles). The interaction part is then a combination of 5 tetrahedra to a pentatope — a four dimensional tetrahedron. But unlike in the 3D case, the 4D Plebanski action has an additional constraint term, that needs to be implemented in the model on the level of the action $S\,[\phi]$. Just as in the spin foam formulation, it is an open field of research, to understand the modifications that need to be made. This freedom inspires us to develop different models, some of which are more physically motivated than others [93, 95, 96]. Such variety of models allows us to experiment with the theoretical structure of combinatorially non-local theories and to develop appropriate mathematical techniques. The variations in the models consider different choices of the Lie group, different types of the quadratic and the interaction terms in the action or the real of complex type of fields. For example for the 4D case we often consider the Lie group to be $SL\,(2,\mathbb{C})$ [97], $Spin\,(4)$

[98] or $SO(4) \times SU(2)$ [99, 100]. We will introduce some 4D models in chapter 4, when we discuss their symmetries.

In summary: we can reformulate a spin foam model of Plebanski action in terms of a perturbative expansion of a scalar field theory that has the following features:

1. The action of the theory is combinatorially multi-local.
2. Feynman diagrams are associated with two complexes and Feynman amplitudes define spin foam amplitudes.
3. In models whose diagrams should have the right labeling by intertwiners, the scalar fields need to be symmetric under the diagonal action of the group G — in short satisfy the closure constraint (2.18).

The relation between spin foam models and group field theory happens therefore on the perturbative level. More precisely, it is the perturbative expansion around the trivial expectation values of the field, $\langle \phi \rangle = 0$, that relates to spin foams. Being a statistical field theory, GFT, however, may have a non-perturbative definition as well. A non-perturbative definition could be understood as a sum over all two-complexes and hence resolve the regularization dependence problem of spin foams.

We want to remark here, that this step presents a non-trivial shift in the interpretation of group field theory. The original idea of GFT was to repackage spin foam amplitudes in a structured way. Giving a meaning to a non-perturbative definition of GFT, however, puts it on the level of an independent, self contained quantum (statistical) field theory, that relates to spin foam models only in the perturbative regime.

In order to have at least some hope for a non-perturbative definition of the theory, the partition function (2.17) has to be renormalizable. In fact this is one of the first criteria for any fundamental quantum field theory — even the perturbative one. The problem of perturbative as well as non-perturbative renormalization of GFT is one of the most active research areas in the community [56, 101–110].

We will not touch on this part of research, but want at least mention that a requirement of renormalizability can imply some non-trivial changes to GFT models. For example in the Boulatov action the requirement of perturbative renormalizability generates a non-trivial dynamical operator, adding a Laplace term to the quadratic one in the action [56]. For that reason the action gets modified and becomes

$$\begin{aligned} S[\phi] = &\int_{G^{\times 3}} \mathrm{d}g\, \phi(g_1, g_2, g_3) \left(-\Delta + m^2\right) \phi(g_3, g_2, g_1) \\ &+ \lambda \int_{G^{\times 6}} \mathrm{d}g\, \phi(g_1, g_2, g_3)\, \phi(g_1, g_4, g_5)\, \phi(g_6, g_2, g_5)\, \phi(g_6, g_4, g_3), \end{aligned} \tag{2.22}$$

where $G = SU(2)$ and Δ denotes a Laplace-Beltrami operator on $SU(2)^{\times 3}$. This new quadratic term provides a non-trivial kinetic part to the action that modifies the propagator in the Feynman diagrams. This affects the mathematics of the model but it does not change much the pictorial interpretation we presented above.

2.3.2 *Complications of the formalism*

Perturbative GFT provides a structured way to sum over two-complexes and to evaluate the amplitudes of spin foam models. But as quantum field theory with a possible non-perturbative meaning, the original models can get modified due to QFT principles, for example due to renormalizability requirements. As a result the Feynman amplitudes of renormalized models may differ from the original spin foam amplitudes, and may not relate to Plebanski action. Instead, the theory may relate to some quantum corrected gravity action and needs to be interpreted in a quantum mechanical way — in terms of expectation values of observables.

Apart from the technical problems of renormalizability GFT encounters some conceptual issues that need to be understood for a meaningful model building process. Two of these issues we mention below.

Theory space of GFT

In ordinary field theories the theory space is the space of possible interaction terms that are compatible with the fundamental symmetry principles. In non-perturbative renormalization this space defines the space of possible coupling constants in which the renormalization group flow takes place. In ordinary field theories the theory space is the space of all interaction terms that respect the Lorentz symmetry and do not violate causality. In algebraic renormalization, this principle is then modified by model dependent symmetries and is used to prove renormalizability of models without explicit calculations of divergent counter terms [111].

In group field theory the theory space is not known. One approach to restrict the theory space is first to understand the symmetries of unrenormalized GFT models and subsequently try to generalize them to some fundamental principles. But GFT models are combinatorially multi-local and a symmetry analysis for such actions is not fully understood[13]. In chapter 4 we will develop a symmetry analysis for multi-local actions and apply it to GFT models.

[13] *See [112, 113] for recent results.*

Bottomless action

Another issue is the bottomless action. Even in 3D models, where the interaction part of S is quartic in the fields, it can be shown that the action is unbounded from below [114]. That means that for any real number C there are field configurations ϕ^+ and ϕ^- such that

$$S\left[\phi^+\right] > C \qquad\qquad S\left[\phi^-\right] < -C \tag{2.23}$$

This makes the expressions in the functional integral ill defined because such ϕ^- configurations dominate the integral,

$$\int \mathcal{D}\phi \, e^{-S[\phi]} \approx \lim_{\phi \to \phi^-} e^{-S[\phi]} \to \infty. \tag{2.24}$$

Perturbativelly, around local minima of the action S, the above expression still makes sense in the form of a formal power series. But non-perturbativelly it may appear that the quantum effective action does not have a global minimum. The problem of Euclidean field theory with an unbounded action is known for a long time even in ordinary field theories and especially in the case of gravity [75]. In the next chapter we will discuss the implications of this problem on the operator formulation of GFT, but the main message is that bottomless action presents a serious obstacle in the formulation of non-perturbative quantum field theories.

2.4 *Group field theory — operator formulation*

The operator formulation of GFT is a quite recent development that suggests to formulate a theory based on Hilbert spaces and operators to reproduce the expectation values of the functional formulation of GFT. Structurally, we would write,

$$\int \mathcal{D}\phi \, q\left(\phi\right) q'\left(\phi\right) e^{-S[\phi]} = (\Omega|\hat{P}_{q,q'}|\Omega). \tag{2.25}$$

The left hand side defines an expectation value of qq' in the functional formulation and the right hand side suppose to reproduce the same expectation values in the operator approach. Hereby the ingredients on the right hand side can be summarized as follows:

The Hilbert space H_{ph} with a *cyclic*[14] *state* $|\Omega) \in H_{ph}$. This Hilbert space is the space of states that satisfy the dynamical laws for the GFT degrees of freedom. Because these degrees of freedom correspond to "atoms" of space-time, the dynamical equations are not those of GR. For the same reason states of this Hilbert space do not necessarily correspond to geometrical states (smooth, continuous or discrete), just as not every state of a collection of atoms corresponds to a fluid[15].

[14] *A state is called cyclic with respect to some operator algebra, if every other state can be reached by an application of a suitable algebra element on it. We will discuss the precise definition of cyclic states in the next chapter.*

[15] *The Hilbert space H_{ph} is closely related to the physical Hilbert space of LQG, however, it is structured in a different way and the relation between these two theories is not fully understood [92, 115].*

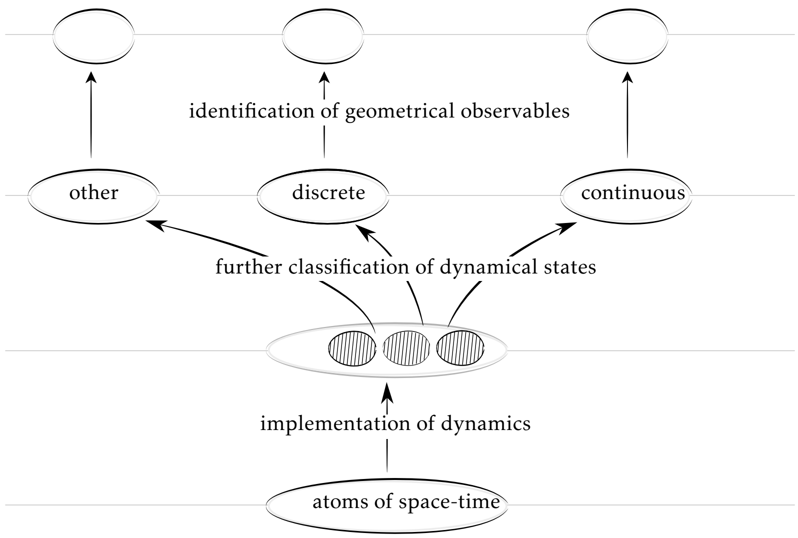


Figure 2.8: Conceptual construction of operator group field theory.

The algebra of self-adjoint operators $\left\{P_{q,q'}\right\}$ on the Hilbert space H_{ph}. These operators suppose to encode the information about the insertions in the functional integral in the sense that

$$P_q|\Omega) \tag{2.26}$$

creates a state $|q)$ such that the expectation value of the insertion q can be understood as a transition amplitude $(\Omega|q)$. Ideally, a subclass of these operators would correspond to geometrical measurements and create geometrical states once applied on $|\Omega)$.

As we will discuss in the following, the explicit construction of the Hilbert space and the related algebra closely follows the spirit of second quantization of non-relativistic many body quantum physics and statistical field theories. The particles of GFT are considered as building blocks of space-time that arrange themselves in different phases. Some of these phases would then contain states that relate to continuous geometry, some of them may correspond to discrete geometry and some of them may correspond to no geometry at all. But in any case the space-time becomes an effective quantity that emerges only in some of the phases of GFT.

The conceptual idea of the construction is summarized in figure 2.8, and can be understood in four steps:

step i: kinematical hilbert space

The space of GFT degrees of freedom without dynamical information. It is a Fock space, whose one particle Hilbert space is that of a quantized polyhedron[16]. The creation and annihilation operators $\phi^\dagger$ and ϕ describe creation of polyhedra over the Fock vacuum — the

[16] *Quantization of a triangle*

(see for example [82, 116]) To quantize a triangle we consider the normal vectors to each of its edges, which are three, 3D vectors. These vectors we associate with elements of the Lie algebra of the rotational symmetry group of the triangle, $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$. *Then we can associate a phase space of a triangle to the dual of the Lie algebra,* $\mathfrak{su}^\star(2)$. *The Lie bracket on* $\mathfrak{su}(2)$ *gives a natural way to define a Poisson bracket on the space of smooth functions on the phase space. By deformation quantization this Poisson algebra gets deformed to a non-commutative algebra of observables on the Hilbert space* $L^2\left(SU(2)^{\times 3}/SU(2)\right)$ *of square integrable functions on three copies of* $SU(2)$ *modulo the diagonal action of* $SU(2)$. *This diagonal symmetry is the closure constraint and it appears from the fact that the faces of a triangle have to close in order to form a geometrical figure.*

From this single particle Hilbert space H, *we then construct the Fock space in the usual way as a collection of symmetric n-fold tensor products of the single particle Hilbert space,*

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{S}\left(H^{\otimes n}\right).$$

state of no atoms of space-time — and satisfy canonical commutation relations,

$$\left[\phi\left(g\right),\phi^{\dagger}\left(h\right)\right]=\delta\left(g,h\right), \tag{2.27}$$

with $g, h \in G$, for a suitable Lie group G. This space sets the stage for the further construction of the physical Hilbert space.

Step II: Physical Hilbert space

We identify a set of constraint operators $C_1, \cdots, C_n$ that should implement the dynamical relations of the GFT degrees of freedom and reduce the kinematical Hilbert space down to the physical Hilbert space $\mathcal{H}_{ph}$ by

$$\mathcal{H}_{ph}=\{|\psi\rangle \mid C_i|\psi\rangle=0\}. \tag{2.28}$$

The zero eigenvalue of C_i's may not belong to the discrete part of their spectrum. In this case the Hilbert space $\mathcal{H}_{ph}$ will not be a subspace of $\mathcal{H}_{kin}$ but should be constructed in a more complicated way[17]. However, for the understanding of the conceptual idea this technical details do not matter.

[17] For example following the Gelfand procedure [117].

The algebra of operators has to be filtered by the constraint operators as well, such that the relevant algebra is given by

$$\mathcal{A}=\left\{P\left(\phi,\phi^{\dagger}\right) \mid \left[C_i,P\left(\phi,\phi^{\dagger}\right)\right]=0\right\}, \tag{2.29}$$

where P stands for polynomials of creation and annihilation operators[18]. The resulting physical Hilbert space is not yet a space of states that encode information about geometries, but rather a space of dynamical space-time atoms. However, because the degrees of freedom in GFT are assumed to be more fundamental than the space-time itself, the dynamics is not that of GR and there might be states in $\mathcal{H}_{ph}$ that do not correspond to any geometrical space-times.

[18] This procedure can be heuristically understood in ordinary quantum mechanics if we choose the constraint operator to be

$$C=\imath\partial_t-H. \tag{2.30}$$

The space of physical states is then the space of states that satisfy the relation

$$C|\psi(t)\rangle=\imath\partial_t|\psi(t)\rangle-H|\psi(t)\rangle=0, \tag{2.31}$$

which is the usual Schroedinger equation. The dynamical equation for the operators is given by

$$[C,\mathcal{O}]=\imath\partial_t\mathcal{O}-[H,\mathcal{O}]=0, \tag{2.32}$$

and is the usual Heisenberg equation of motion for the operators.

In principle this step can include some idealization procedures such as a requirement of infinite number of particles or removal of any possible cut offs on the constraint operators (if they were needed for a rigorous definition). Due to this, the resulting physical Hilbert space is expected to split in a direct sum of Hilbert spaces each of which corresponds to a different phase of GFT. Without such idealization the states may not have a clear-cut distinction between phases and the question of phases becomes mathematically difficult to handle, just as it is the case in many body quantum mechanics [118].

Step III: Phases of GFT

We expect that the space $\mathcal{H}_{ph}$ will split in sub-sectors. We call these sub-sectors phases of GFT. These phases correspond to inequivalent representations of the algebra $\mathcal{A}$ from equation (2.29). However, within this algebra, some operators may be more relevant for physical observations than others, and the actual relevant measurements could be a very complicated composition of observables of the fundamental GFT degrees of freedom. This is similar to situations in many body theories in which effective observables of a composed system lead to more (or even the only) relevant observations. For example in a gas phase of a many particle system a one-particle velocity may not be relevant whereas the temperature is. We call the effective, relevant observables *geometric*.

STEP IV: QUANTUM CORRECTED GR

The last step is an extraction of physical information from expectation values of geometrical observables. Hereby, it can happen, that some observables will directly correspond to geometrical quantities, for example curvature, volume or area of the state, but it is also expected that some geometrical quantities should be extracted from relations between expectation values in the same spirit as emergent quantities appear in macroscopic physics (for example viscosity, susceptibility etc.). For those states that contain information about smooth geometries, resulting expectation values should correspond to degrees of freedom of GR with additional quantum corrections. Such states may belong to different phases, and hence the set of geometric observables may differ.

At this stage we need to point out, that this construction is quite heuristic. Partially, it follows the construction of condensed matter physics and partially that of loop quantum gravity. However, there does not exist any model of GFT that utilizes all the above steps.

The major difficulty is to define the operators $C_1, \cdots, C_n$. Here is where the relation to the functional formulation should enter. The operators $C_1, \cdots, C_n$ have to be defined such that each correlation function of the functional formulation can be uniquely identified with an operator $P \in \mathcal{A}$ and a state $|\Omega)$ that produces the same expectation values by the relation (2.25). However, so far there is no formulation of operator GFT that successfully satisfies this relation.

Without any correspondence between the functional and operator formulation these two formalisms really describe two different theories. And wheras the functional approach is still motivated by spin foam models the operator formulation is independent of it. For that reason it is very important to provide a relation between the functional and operator formulation of GFT.

In ordinary quantum field theories, the relation between the operator and statistical formulation are given by the Osterwalder-Schrader axioms [119] that, however, rely on the Minkowski space-time structure and the Lorentz invariance. Already for quantum field theories on curved space-times this relation was not known for a long time [120]. In GFT the abstraction level is even higher, as the base manifold does not represent space-time and therefore does not assume any Lorentz symmetry.

An explicit relation between the functional and operator formalism of GFT is desirable, since the dynamics of GFT is only understood in the functional formulation, whereas the particle interpretation is more intuitive from the physical point of view. Hence, a relation may boost our intuition of GFT models as well as its technical and conceptual development.

In the next chapter we will provide a construction of the physical Hilbert space directly from the dynamical definition of the functional theory circumventing an explicit construction of the constraint operators. Our construction will not resolve the problem of finding the right dynamics, but it will provide a direct relation between the functional and the operator formulation of GFT on the perturbative level.

3 Algebraic formulation of group field theory

In the previous chapter we introduced the GFT framework in its functional and operator formulation. However, as we also discussed, there is yet no explicit relation between these formulations and with this regard they should be considered as two different theories. A suggestion of a possible relation between the formalisms was put forward in [92] but an explicit relation, as well as a rigorous proof of its existence was not shown.

A connection between the operator and functional formulation should be given by a dictionary; a dictionary that relates the dynamical evolution of the functional formulation — given by the action S in the functional integral — and states in the operator formulation that provide the same correlation functions in terms of expectation values of field operators.

In this section we will suggest such a relation, construct an operator theory that matches the perturbative expansion of the partition function and explicitly provide the desired dictionary. At the end of this chapter we will argue that this construction can be used as a definition of an effective operator formalism of GFT. And in chapter 5 we will discuss a possible characterization of GFT phases in terms of symmetry breaking.

We will begin our discussion with a brief reminder of field theories on space-time and their corresponding relation between the canonical and covariant formulation. Then (on page 46) we introduce the concept of *algebraic quantum field theory* (AQFT) and apply this formalism to GFT (on page 54). At the end of this chapter we provide an explicit example for our construction based on a simplified model of the Boulatov action.

3.1 Covariant and canonical formulation of QFT

Formally, covariant QFT is defined by the generating functional

$$Z[J] = \int \mathcal{D}\phi \, e^{\imath S[\phi] + \int J\phi}. \tag{3.1}$$

The n-particle correlators can be derived from this expression by functional derivatives with respect to J — the *external source field*. For example the Feynman propagator $G(x,y)$ is given by

$$G(x,y) = \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} Z[J] \,|_{J=0.} \tag{3.2}$$

In the canonical formulation of QFT — a theory of field operators on Hilbert spaces with existing notion of time — the n-particle correlators are given by time-ordered expectation values of n-field operators in the vacuum, that is we can write (see for example [121])

$$\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} Z[J] \,|_{J=0.} = G(x,y) = (0|T[\phi(x)\phi(y)]|0). \tag{3.3}$$

This formal equality provides the connection between the covariant and canonical formulation of QFT on Minkowski space-time.

As we already discussed above, the formulation of QFT in terms of path integrals is not rigorous, since we do not have a good notion of the integral measure for the space of fields. For that reason we usually use the Wick rotated version of the integral, in which the generating functional becomes a statistical partition function

$$Z[J] = \int \mathcal{D}\phi \, e^{-S_E[\phi] + \int J\phi}. \tag{3.4}$$

Derivatives of that partition function with respect to J define the *n-point correlation functions*, and after Wick rotation, back to Lorentzian time, they define time-ordered expectation values of field operators. This gives a relation between the (more or less) rigorously defined covariant and canonical theory.

This relation, however, relies on the time-notion of space-time; we use it in the definition of the time-ordered products in the canonical theory and also in the definition of the Wick rotations in the covariant integral.

In GFT — a theory without the notion of time nor space — the formulation of time-ordered products and Wick rotations is not clear, and hence the usual relation that we described above, does not hold. Moreover, due to the bottomless nature of the GFT action (at least for most common models), the statistical functional formulation may be defined only perturbatively. For that reason we need more care in formulating the operator GFT and requiring a match between the formulations.

To avoid problems stemming from the bottomless nature of the action, we will assume a perturbative definition of $Z[J]$ and use the background field method (see for example [121, 122]) to expand the theory around different (if they exist) local minima of the action S. Even though this is a common technique in quantum and statistical field theory, we briefly discuss it here for self consistency.

If φ is a local minimum of the action S the first derivative of the action at φ — that we denote S'_φ — vanishes, $S'_\varphi = 0$. Hence, the Taylor series of S around φ reads

$$S[\varphi] + \frac{1}{2} S''_\varphi (\phi, \phi) + \mathcal{O}\left(\phi^3\right), \tag{3.5}$$

where ϕ is a (small) fluctuation around the *background field* φ, and $S''_\varphi(\phi,\phi) \doteq \int \mathrm{d}x\,\mathrm{d}y\, \phi(x) \left[\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} S(\varphi)\right] \phi(y)$. Shifting the integral variable in eq. (3.4) as $\phi \mapsto \varphi + \sqrt{\hbar}\phi$, and using the Taylor expansion for S we obtain

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S[\varphi] - \frac{1}{2}S''_\varphi(\phi,\phi) - \mathcal{O}\left(\hbar^{\frac{3}{2}}\right) + \sqrt{\hbar}\int J\phi + \int J\varphi} \\ &\approx e^{-\frac{1}{\hbar}S[\varphi] + \int J\varphi} \sum_n^\infty \frac{(-1)^n}{n!} \int \mathcal{D}\phi\, e^{-\frac{1}{2}S''_\varphi(\phi,\phi) + \int J\phi}\, \mathcal{O}\left(\hbar^{\frac{3}{2}n}\right), \end{aligned} \tag{3.6}$$

where we explicitly wrote out all $\hbar$ appearances in the exponent. We neglect for now the perturbation terms in $\hbar$ — since we can deal with them as insertions — and calculate the integral

$$Z_\varphi[J] = e^{-\frac{1}{\hbar}S[\varphi] + \int J\varphi} \int \mathcal{D}\phi\, e^{-\frac{1}{2}\int \phi\left(S''_\varphi\right)\phi + \sqrt{\hbar}\int J\phi}. \tag{3.7}$$

This generating functional defines a free theory with modified dynamics given by S''_φ (see figure (3.1)). If φ is a local minimum of the action S, then $S''_\varphi > 0$, and the integral can be performed with standard Gaussian integral techniques leading to the closed expression for the partition function,

$$Z_\varphi[J] := e^{\int J\varphi}\, e^{\frac{1}{2}\int JCJ}, \tag{3.8}$$

where C is the Green's function for the operator S''_φ. If the action S has more than one minimum labeled φ_i we will have different theories given by partition functions $Z_{\varphi_i}[J]$. This implies that we have to expect different operator theories each of which relating to the covariant formulation with the partition function $Z_{\varphi_i}[J]$: some of these operator theories will differ only by the choice of the vacuum state but as we will show below, generally, not even this is true.

In the following we will take the expression (3.8) as a definition of the statistical expectation values and construct an operator theory that reproduces the same n-point correlation functions[1]. We will see that if the action S has more than one local minimum the relation between the

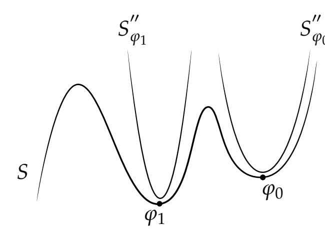


Figure 3.1: Approximation of the classical action by quadratic term of the Taylor expansion. The actual shape of the quadratic expansion is different for different minima φ, which leads to different quantum fluctuations described by $Z_\varphi[J]$.

[1] *of course at this level the action S has to be the renormalized one. However, throughout this work we do not discuss the issue of renormalizability and whenever necessary, we assume that the action can be perturbatively renormalized to finite order in perturbation theory.*

functional and operator approach will be of the type:

$$\left.\begin{array}{c} Z_{\varphi_1}[J] \\ Z_{\varphi_2}[J] \\ \vdots \end{array}\right\} \leftrightarrow \begin{cases} (\mathcal{A}^1, |\varphi_1), \mathcal{H}_1) & \text{for the local minimum at } \varphi_1 \\ (\mathcal{A}^2, |\varphi_2), \mathcal{H}_2) & \text{for the local minimum at } \varphi_2 \; . \\ \vdots & \vdots \end{cases} \tag{3.9}$$

where we denote the collection of observables by $\mathcal{A}^i$, the vacuum state by $|\varphi_i)$ and the corresponding Hilbert spaces by $\mathcal{H}_i$. In this formulation we can interpret the right hand side as different phases of GFT.

The states $|\varphi_i)$ may belong to different Hilbert spaces and we need a formalism, that captures all those Hilbert spaces at once. Ever more, there is no reason to believe that the local minima of S will be square integrable functions and therefore the scalar product of $\mathcal{H}_i$ may be non-trivial.

We will use the framework of non-relativistic algebraic quantum field theory [118, 123, 124] to construct the operator theory of GFT with the above properties. We begin with the brief introduction of the algebraic framework.

3.2 Non-relativistic algebraic quantum field theory

Algebraic quantum field theory [42, 123–125] is based on two major ingredients: the *set of observables* and *the set of algebraic states*.

The observables represent measurements that we can perform on a system and, mathematically, are described as elements of a $C^\star$-algebra[2] (see for example [126]). Unlike the canonical quantum field theory, where observables are given by linear operators on some Hilbert space, the algebraic formulation does not require a Hilbert space and uses only the algebraic relations between observables. However, the Hilbert space and the usual formulation in terms of field operators can be derived from the abstract algebraic relations.

The states — called *algebraic states* — are continuous positive and normalized functionals (see the margin note on page 50) from the $C^\star$-algebra to complex numbers (see for example [126]). We give an explicit definition of algebraic states below, when we discuss coherent states of GFT but it is important to remark already here that algebraic states play two roles in algebraic field theory:

First, an algebraic state fixes all expectation values of all observables.

Being a functional on the algebra of observables, an algebraic state assigns a number to each physical measurement. This number is understood as the expectation value of that measurement. Since an algebraic state is supported on the whole algebra it fully specifies a physical state, by fixing all polynomials of each observable [126].

[2] *$C^\star$-algebra*

A $C^\star$-algebra is a Banach star algebra such that, for any $A \in \mathfrak{A}$, the norm $\|\cdot\|$ satisfies,

$$\|A^\star A\| = \|A\|\|A^\star\|.$$

$\star$-algebra

A $\star$-algebra is an algebra with involution, that is a map $\star : \mathfrak{A} \ni A \to A^\star \in \mathfrak{A}$, with the following properties:

1. *for all $A \in \mathfrak{A}$: $A^{\star\star} = A$,*
2. *for all $A, B \in \mathfrak{A}$: $(A+B)^\star = A^\star + B^\star$ and $(A \cdot B)^\star = B^\star \cdot A^\star$,*
3. *for every complex number $\lambda \in \mathbb{C}$ and every $A \in \mathfrak{A}$: $(\lambda A)^\star = \bar{\lambda} A^\star$.*

Banach algebra

A Banach algebra is a Banach space (complex of real) and at the same time an algebra such that the multiplication is continuous. That is for any $A, B \in \mathfrak{A}$, the norm $\|\cdot\|$ satisfies,

$$\|A \cdot B\| \leqslant \|A\| \cdot \|B\|.$$

Second, an algebraic state defines a representation for the $C^\star$-algebra.

Due to the theorem by Gelfand, Naimark and Segal [127, 128] (GNS theorem see margin note on page 50) any algebraic state provides a notion of a Hilbert space on which the algebra of observables acts as an algebra of linear bounded operators. This theorem connects the abstract algebraic framework with the canonical description of quantum field theory — the theory of operators on Hilbert spaces.

Due to the GNS theorem, the Hilbert space becomes a derived concept of a more fundamental underling algebraic structure. Moreover, different algebraic states can correspond to different, mathematically inequivalent, representations of the observable algebra and provide a consistent treatment of several representations within one framework; this becomes unavoidable when dealing with systems with infinitely many degrees of freedom [42]. Inequivalent representations of the observable algebra often correspond to different phases of a model and provide a mathematically rigorous construction for the theory of phase transitions [118, 129, 130] and symmetry breaking.

It is also important to remark that the set of algebraic states does not form a Hilbert space — it does not have a natural notion of a scalar product. This gives a more general formulation of states, especially in field theories in which the volume of the base manifold is infinite [131].

Motivated by statistical field theory we choose the Weyl algebra as the algebra of observables in GFT. In the following we will discuss a detailed construction of that Weyl algebra and algebraic states in GFT.

3.2.1 *Weyl algebra of GFT*

We begin with the definition of the Weyl algebra in GFT. The procedure is quite common in the algebraic approach, but we recall it here to make the construction more accessible for the unfamiliar reader.

The Weyl construction is a map from the phase space, or more generally symplectic space of the classical theory, to a $C^\star$-algebra. We, therefore, need to start with the formulation of a suitable symplectic space in our case.

Symplectic space of GFT

The construction of the Weyl algebra begins with the definition of the symplectic space. Even though this space can be understood as the phase space of the theory, it is only its closure that has an interpretation of the one particle Hilbert space. For that reason we have a certain freedom in its choice.

In our case we choose that space to be $\mathcal{S}_\infty$ — the space of smooth, complex valued functions on the base manifold $M = SU(2)^{\times d}$ with the topology induced by the family of semi-norms[3]

[3] *Pull-backs and Lie derivatives*

Let L_h and R_h denote the left and right multiplication on $SU(2)$ by $h \in SU(2)$, i.e. for any $g \in SU(2)$

$$L_h(g) = hg \qquad R_h(g) = gh, \qquad (3.10)$$

The pull back of a smooth function f by the left/right multiplication is given by

$$(L_h^\star f)(g) = f(hg) \qquad (R_h^\star f)(g) = f(gh).$$

Let $X \in \mathfrak{su}(2)$ and $t \in I \subset \mathbb{R}$ a real parameter then we define the action of the Lie algebra on the space of functions as

$$(Xf) := \partial_t R^\star_{e^{tX}} f\,|_{t=0}.$$

$$\left\{\|f\|_{k,\infty} = \|X_1 \cdots X_k f\left(g\right)\|_{\infty} \ : \ k \in \mathbb{N}; \ X_1, \cdots, X_k \in \mathfrak{su}\left(2\right)\right\},$$

where X_i's denote the Lie algebra elements and act on f as Lie derivatives. With this topology $\mathcal{S}_\infty$ is a complete, topological, locally convex, vector space [132]. When the topology will be not important in our discussion we will denote the space of smooth functions on M simply $\mathcal{S}$. Further we choose a symplectic form[4] $\mathfrak{s} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$. As we will see in the following the symplectic form will be defined by the GFT action, but in general it will always be of the form

$$\mathfrak{s}\left(f, g\right) = \Im\left(f, \mathcal{O}g\right), \tag{3.12}$$

where $f, g \in \mathcal{S}$ are integrable functions and $\mathcal{O}$ is a continuous operator on $\mathcal{S}$, $\Im$ refers to the imaginary part of the expression and $(\cdot, \cdot)$ is the L^2-scalar product with respect to the Haar measure, such that for any $f, g \in \mathcal{S}$,

$$\left(f, g\right)_{L^2} = \int_M \mathrm{d}x\, \bar{f}\left(x\right) g\left(x\right). \tag{3.13}$$

Here and in the following we will denote the points of M by x and y, however, keeping in mind that they belong to a non-commutative Lie group. Also $\mathrm{d}x$ will refer to the Haar measure integral on M whereas $\mathrm{d}h$ will refer to the Haar measure on the single copy of $SU\left(2\right)$.

Since M is compact, every smooth function is integrable and the scalar product induces the norm

$$\|f\|_{L^2} = \int_M \mathrm{d}x\, \bar{f}\left(x\right) f\left(x\right). \tag{3.14}$$

We denote $\mathcal{S}$ with the topology induced by this norm as $\mathcal{S}_{L^2}$. This space is not complete and its completion is the space of square integrable functions $L^2\left(M, \mathrm{d}x\right)$ [132]. The space $\mathcal{S}$ splits in a direct sum as $\mathcal{S} = \mathcal{S}_\mathbb{R} + \imath\mathcal{S}_\mathbb{R}$, where $\mathcal{S}_\mathbb{R}$ is the restriction of $\mathcal{S}$ to real valued functions.

The space $\mathcal{S}$ has two important features. It is:

Closed under translations: Let $L_y : M \to M$ denote the left multiplication on M by some $y \in M$, and let $L_y^\star f \doteq f \circ L_y$ denote the pull back of f, then $L_y f \in \mathcal{S}_\infty$ [132].

Direct sum: Let $h \in SU\left(2\right)$ and $D : SU\left(2\right) \to SU\left(2\right)^{\times n}$ be a diagonal map such that $D\left(h\right) = \left(h, \cdots, h\right)$. Then $f \in \mathcal{S}$ satisfies the closure constraint (see chapter 2 eq. (2.18)) if

$$R^\star_{D(h)} f = f \qquad\qquad \forall h \in SU\left(2\right). \tag{3.15}$$

We denote the space of functions that satisfy the closure constraint by $\mathcal{S}_G$. Then the space $\mathcal{S}_\infty$ splits in an (internal) direct sum as $\mathcal{S}_\infty = \mathcal{S}_G \oplus \mathcal{S}_{NG}$ where $\mathcal{S}_{NG}$ is the complement of $\mathcal{S}_G$ in $\mathcal{S}_\infty$. See appendix A.4 for the proof.

The space $\mathcal{S}_G$ is, however, not closed under right multiplications. That is in general for $f \in \mathcal{S}_G$ and $y \in M$, $R_y^\star f \notin \mathcal{S}_G$.

[4] *Symplectic form*

A symplectic form $\mathfrak{s}$ on a vector space V is a two-form which is:

1. *real bi-linear: for all $v, w, z \in V$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ we have*
$$\mathfrak{s}\left(\lambda_1 v + z, \lambda_2 w\right) = \lambda_1\lambda_2 \mathfrak{s}\left(v, w\right) + \lambda_2 \mathfrak{s}\left(z, w\right),$$
2. *skew-symmetric: for all $v, w \in V$*
$$\mathfrak{s}\left(v, w\right) = -\mathfrak{s}\left(w, v\right),$$
3. *non-degenerate: if for any $v \in V$*
$$\mathfrak{s}\left(v, w\right) = 0, \tag{3.11}$$
then $w = 0$.

We will call the space $\mathcal{S}_\infty$ the space of *smearing* or *test functions*.

With this definition of the symplectic space we can now construct the Weyl algebra of GFT.

Weyl algebra of GFT

The construction of the Weyl algebra from the symplectic space is a standard procedure presented for example in [133, 134]. However, to make our discussion easy readable, we provide the construction in the margin note[5] and summarize the most important properties of the Weyl algebra in plain text below.

In the following we will denote the Weyl algebra as $\mathfrak{A}(\mathcal{S})$ or simply $\mathfrak{A}$, when no confusion is possible. It is a $C^\star$-algebra generated by the Weyl elements $W_{(f)}$ for $f \in \mathcal{S}$. The product is defined by

$$W_{(f)} W_{(g)} = e^{-\frac{i}{2}\mathfrak{s}(f,g)} W_{(f+g)} \quad , \tag{3.17}$$

and the involution $\star : \mathfrak{A} \to \mathfrak{A}$ is given by

$$W^\star_{(f)} = W_{(-f)}. \tag{3.18}$$

Since $\mathcal{S}$ is closed under left and right translations the maps $\alpha_y : \mathfrak{A} \to \mathfrak{A}$ and $\beta_y : \mathfrak{A} \to \mathfrak{A}$

$$\alpha_y \left(W_{(f)} \right) \doteq W_{\left(L^\star_y f\right)}, \qquad \beta_y \left(W_{(f)} \right) \doteq W_{\left(R^\star_y f\right)} \tag{3.19}$$

define algebra automorphisms of $\mathfrak{A}$ (see appendix A.5 for the proof).

We can represent $\mathfrak{A}$ in terms of bounded linear operators on some Hilbert space, and denote the corresponding operators $W^\pi_{(f)}$. Due to the Hilbert space structure we get access to the operator norm[6] on the Hilbert space, which is weaker than the $C^\star$-norm [123], and hence makes a larger number of sequences convergent. Because of this the Weyl algebra $\mathfrak{A}$ is not closed in the operator norm. Its closure is called the *von Neumann algebra* that contains operators $W^\pi_{(f)}$ for f in the closure of $\mathcal{S}_{L^2}$.

Due to the direct sum decomposition of $\mathcal{S}_\infty$ we can define a gauge invariant Weyl algebra $\mathfrak{A}_G = \mathfrak{A}(\mathcal{S}_G)$, by restricting the space of test functions to $\mathcal{S}_G$. $\mathfrak{A}_G$ is the maximal $C^\star$-sub-algebra of $\mathfrak{A}$ that satisfies the closure constraint (for the precise statement and the proofs see appendix A.5). In the following, we will not distinguish the gauge invariant and the gauge variant algebra and simply write $\mathfrak{A}$. Whenever an explicit distinction will be in order we will denote this in the text.

3.2.2 Algebraic states

An algebraic state is a linear, positive, normalized functional on the Weyl algebra $\mathfrak{A}$,

$$\omega : \mathfrak{A} \to \mathbb{C}.$$

[5] *Weyl algebra*

First we define the space $\mathcal{A}(\mathcal{S})$ such that:

1. *The elements of $\mathcal{A}(\mathcal{S})$ are complex valued functions on $\mathcal{S}$ with support consisting of a finite subset of $\mathcal{S}$. Obviously, $\mathcal{A}(\mathcal{S})$ is a vector space.*
2. *Define a ℓ^1 norm on $\mathcal{A}(\mathcal{S})$ by*
$$\|A\|_1 = \sum_{f \in \mathcal{S}} |A(f)|.$$
3. *Functionals of the form $W_{(f)}$ such that*
$$W_f(g) = \begin{cases} 1 & \text{if } f = g \text{ pointwise} \\ 0 & \text{otherwise} \end{cases},$$
form a dense linear basis for $\mathcal{A}(\mathcal{S})$.
4. *Define the multiplication law on the basis of $\mathcal{A}(\mathcal{S})$ as*
$$W_{(f)} \cdot W_{(g)} = e^{-\frac{i}{2}\mathfrak{s}(f,g)} W_{(f+g)}.$$
and extend it to the full $\mathcal{A}(\mathcal{S})$ by linearity.
5. *Define the involution $W^\star(f) = W(-f)$. With this, $\mathcal{A}(\mathcal{S})$ becomes a $\star$-algebra.*

Closing $\mathcal{A}(\mathcal{S})$ in the ℓ^1 norm provides a Banach*-algebra that we denote $\mathbb{A}(\mathcal{S})$. This algebra can be represented by bounded linear operators on some Hilbert space. Denoting the space of all non degenerate representation by Rep, we define the Weyl algebra.

Definition 1. The Weyl $C^\star$-algebra over $\mathcal{S}$ is the completion of $\mathbb{A}(\mathcal{S})$ in the norm

$$\|W\| = \sup_{\pi \in Rep} \|\pi(W)\|. \tag{3.16}$$

We denote it by $\mathfrak{A}(\mathcal{S})$ and call it the Weyl algebra.

[6] *Operator norm*

Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$. An operator norm of A is given by

$$\|A\| = \sup_{x \in \mathcal{H}} \frac{\|Ax\|}{\|x\|}.$$

The set of algebraic states — that we denote by $\mathfrak{S}$ — is a convex subset of the Banach space of continuous, positive, linear functionals on $\mathfrak{A}$[7] [135].

By the GNS construction [8], every algebraic state provides:

a Hilbert space $\mathcal{H}_\omega$ on which the algebra elements act as bounded linear operators,

a unique (up to isomorphism) *representation* $\pi_\omega : \mathfrak{A} \to \mathcal{L}(\mathcal{H}_\omega)$ of the Weyl algebra on $\mathcal{H}_\omega$,

a state vector $|\Omega) \in \mathcal{H}_\omega$ that is cyclic. That means for any state $|\psi) \in \mathcal{H}_\omega$ and any $\epsilon > 0$ we can find a finite number of test functions $\{f_n\}$ and complex coefficients $\{a_n\}$ such that the polynomial

$$Pol_\psi(W^\pi) := \sum_{n=1}^{N} a_n W^\pi_{(f_n)}$$

satisfies

$$\||\psi) - Pol_\psi(W^\pi)|\Omega)\| < \epsilon.$$

The triple, consisting of the Hilbert space, representation and a cyclic state vector is sometimes called *the GNS triple* and is denoted

$$(\mathcal{H}_\omega, \pi_\omega, |\Omega)) . \tag{3.20}$$

Informally, we can say that the GNS construction implies that physical observations uniquely determine the "Bra-Ket" notation by relation,

$$\omega(A) = (\Omega|\pi(A)|\Omega) \ . \tag{3.21}$$

The concept of the Hilbert space becomes derived from expectation values of measurements.

A state ω is called regular if for some parameter $t \in I \subset \mathbb{R}$, with I an interval containing zero, and any fixed $f \in \mathcal{S}$ the function $\Omega(t) := \omega\left(W_{(tf)}\right) : I \to \mathbb{C}$ is smooth. For a regular ω the generators of $W_{(tf)}$ exist in the corresponding GNS representation and can be defined by [136]

$$(\Omega|\Phi(f)|\Omega) := (-\imath\partial_t)\,\omega\left(W_{(tf)}\right)|_{t=0}. \tag{3.22}$$

The generator $\Phi(f)$ is an unbounded operator defined on the dense domain $D(\Phi(f)) \subset \mathcal{H}_\omega$. From this definition we can obtain properties of generators in the strong operator topology. That is for any $|\psi) \in D(\Phi(f))$,

$$\begin{aligned}\|\Phi(f)|\psi)\|^2 &:= \left(\Omega|Pol_\psi(W^\star)\,\Phi(f)\,\Phi(f)\,Pol_\psi(W)|\Omega\right) \\ &= \left(-\partial_t^2\right)\omega\left(Pol_\psi(W^\star)\,W_{(tf)}Pol_\psi(W)\right).\end{aligned}$$

[7] *Positive functional*

A positive functional ω on a $C^\star$-algebra $\mathfrak{A}$ is a functional such that for all $A \in \mathfrak{A}$

$$\omega(A^\star A) \geqslant 0.$$

It is normalized if

$$\sup_{A\in\mathfrak{A}} |\omega(A)| = 1.$$

If the algebra is unital, that means it contains the identity element with respect to the product, then the normalization condition reads $\omega(\mathbb{1}) = 1$.

The Weyl algebra is unital with the identity element $W_{(0)}$.

[8] *GNS Theorem*

Given an unital $C^\star$-algebra $\mathfrak{A}$ and an algebraic state ω, there is a Hilbert space $\mathcal{H}_\omega$ and a representation $\pi_\omega : \mathfrak{A} \to \mathcal{L}(\mathcal{H}_\omega)$, such that

1. *$\mathcal{H}_\omega$ contains a cyclic vector $|\Omega)$,*
2. *$\omega(A) = (\Omega|\pi(A)|\Omega)$ for any $A \in \mathfrak{A}$,*
3. *every other representation π in a Hilbert space $\mathcal{H}_\pi$ with a cyclic vector $|o)$ such that for any $A \in \mathfrak{A}$*

$$\omega(A) = (o|\pi(A)|o),$$

is unitarily equivalent to π_ω, i.e. there exists an isometry $U : \mathcal{H}_\pi \to \mathcal{H}_\omega$ such that

$$U\pi(A)U^{-1} = \pi_\omega(A), \qquad U|o) = |\Omega).$$

Fock state and the Fock representation

To get acquainted with the formalism and to show the relation between the algebraic verses operator formulation of GFT we discuss the Fock representation of the Weyl algebra.

The *Fock algebraic state* that, by GNS theorem, leads to the Fock representation is given by [130],

$$\omega_F\left(W_{(f)}\right) = e^{-\frac{\|f\|^2}{4}}\,. \tag{3.23}$$

By continuity and linearity of ω_F and the product of the Weyl algebra (eq. (3.17)) this equation defines the acton of ω_F on the whole algebra $\mathfrak{A}$. For example, the action of ω_F on $W_{(f)}W_{(g)}$ is

$$\omega_F\left(W_{(f)}W_{(g)}\right) = \omega_F\left(W_{(f+g)}\right)e^{-\frac{\imath}{2}\mathfrak{s}(f,g)} = e^{-\frac{\|f+g\|^2}{4}}e^{-\frac{\imath}{2}\mathfrak{s}(f,g)}. \tag{3.24}$$

We denote the GNS triple for the Fock representation by $(\mathcal{H}_F, \pi_F, |0\rangle)$. Clearly the algebraic Fock state is regular since the function

$$\Omega\left(t\right) = e^{-t^2\frac{\|f\|^2}{4}}, \tag{3.25}$$

is smooth in t. Hence, we can define generators of Weyl operators by differentiation, as described above. We call these generators $\Phi_F\left(f\right)$ and write the represented Weyl element as

$$W^F_{(f)} \doteq \pi_F\left(W_{(f)}\right) = e^{\imath\Phi_F(f)}. \tag{3.26}$$

By construction $\Phi_F\left(f\right)$ is a self-adjoint, unbounded operator, defined on a dense domain $D\left(\Phi_F\left(f\right)\right) \subset \mathcal{H}_F$ [136]. It is *real linear*; that is for $\lambda \in \mathbb{R}$ and $f, g \in \mathcal{S}$,

$$\Phi_F\left(\lambda f + g\right) = \lambda\Phi_F\left(f\right) + \Phi_F\left(g\right). \tag{3.27}$$

The action of $\Phi_F\left(f\right)$ on $D\left(\Phi_F\left(f\right)\right)$ can be determined by derivation as we discussed above

$$\langle 0|\Phi_F\left(f\right)|\psi\rangle := \left(-\imath\partial_t\right)|_{t=0}\,\omega\left(W^F_{(tf)}\,Pol_\psi\left(W^F\right)\right). \tag{3.28}$$

In the similar fashion we can derive the commutators between $\Phi_F\left(f\right)$ and $\Phi_F\left(g\right)$ for $f, g \in \mathcal{S}$,

$$\left[\Phi_F\left(f\right), \Phi_F\left(g\right)\right] = \imath\Im\left(f, g\right). \tag{3.29}$$

We can also define creation and annihilation operators by

$$\psi_F\left(f\right) := \frac{1}{\sqrt{2}}\left[\Phi_F\left(f\right) + \imath\Phi_F\left(\imath f\right)\right] \tag{3.30a}$$

$$\psi_F^\dagger\left(f\right) := \frac{1}{\sqrt{2}}\left[\Phi_F\left(f\right) - \imath\Phi_F\left(\imath f\right)\right], \tag{3.30b}$$

with $\psi_F(f)^\dagger = \psi_F^\dagger(f)$. It follows that $\psi(f)$ is anti-linear[9] in f, $\psi^\dagger(f)$ is linear in f, both are closed on the domain $D(\phi)$ of ϕ and fulfill the canonical commutation relations [137],

$$[\psi_F(f), \psi_F(g)] = \left[\psi_F^\dagger(f), \psi_F^\dagger(g)\right] = 0 \tag{3.31}$$

and

$$\left[\psi_F(f), \psi_F^\dagger(g)\right] = (f,g)\,\mathbb{1}\,, \tag{3.32}$$

where $\mathbb{1}$ is the identity on the Fock space. It follows from the definition of the state (3.23) that[10]

$$\psi_F(f)\,|0) = 0, \tag{3.33}$$

for any $f \in \mathcal{S}$. Hence, $|0)$ corresponds to the Fock vacuum on which ψ_F and $\psi_F^\dagger$ act as creation and annihilation operators and we obtain the GFT Fock space from the algebraic Fock state ω_F.

Coherent states and non-Fock representations

As we have seen above, the algebraic Fock state corresponds to the Fock vacuum state $|0)$. In this section we want to construct algebraic states ω_φ that correspond to coherent states $|\varphi)$. Using the algebraic construction we will be able to construct coherent states, whose "order parameter" is a tempered distribution, and hence not necessarily normalizable in L^2. Such states will lead to Fock inequivalent representations.

Usually coherent states are characterized by the condition to be eigenstates of the annihilation operator in the Fock representation [138]. That is for $x \in M$,

$$\psi_F(x)\,|\varphi) = \varphi(x)\,|\varphi). \tag{3.34}$$

This definition does, however, require the Fock space and therefore needs to be modified for a representation independent, algebraic formulation.

In the algebraic approach coherent states can be introduced in a representation independent way. This has been done for example in [137, 139], where the authors provide a classification of algebraic coherent states in Fock and non-Fock coherent states — those that lead to the Fock representation and those that do not. The definition goes as follows.

Definition 2. Let $\varphi : \mathcal{S}_\infty \to \mathbb{C}$ be a continuous linear functional on the space of test functions $\mathcal{S}_\infty$. A state ω of the form

$$\omega_\varphi\left(W_{(f)}\right) = \omega_F\left(W_{(f)}\right)\, e^{\imath\sqrt{2}\Re[\varphi(f)]}, \tag{3.35}$$

where $\Re$ denotes the real part of the expression, is called a coherent state[11]. It is pure and regular [137].

[9] *For $\lambda \in \mathbb{R}$ we have*

$$\begin{aligned}\psi(\imath\lambda f) &= \frac{1}{\sqrt{2}}\left[\Phi(\imath\lambda f) + \imath\Phi(-\lambda f)\right]\\ &= \frac{\lambda}{\sqrt{2}}\left[\Phi(\imath f) - \imath\Phi(f)\right]\\ &= \frac{-\imath\lambda}{\sqrt{2}}\left[+\imath\Phi(\imath f) + \Phi(f)\right]\\ &= -\imath\lambda\,\psi(f).\end{aligned}$$

[10] *Explicitly, we get*

$$\|\psi_F(f)\,|0)\|^2 = (0|\psi_F^\dagger(f)\,\psi_F(f)\,|0).$$

Rewriting this expectation value in terms of the field $\Phi(f)$ using the definition (3.30) the above expression is up to the global factor of $\frac{1}{2}$

$$\begin{aligned}&(0|\Phi(f)\,\Phi(f)\,|0)\\ &+(0|\Phi(\imath f)\,\Phi(\imath f)\,|0)\\ &-\imath(0|\left[\Phi(f), \Phi(\imath f)\right]|0).\end{aligned}$$

Using the commutation relations for

$$[\Phi(f), \Phi(\imath f)] = -\|f\|^2,$$

the above expectation values become

$$\begin{aligned}&-\partial_t^2\,\omega\left(W_{(tf)}\right)|_{t=0}\\ &-\partial_t^2\,\omega\left(W_{(t\imath f)}\right)|_{t=0}\\ &-\|f\|^2.\end{aligned}$$

And by the definition of the state we obtain

$$\|\psi_F(f)\,|0)\|^2 = \frac{1}{2}\left(\|f\|^2 - \|f\|^2\right) = 0.$$

[11] *There exist an even more general definitions of a coherent state provided by the same authors in [139], but the definition we use here is the one that most closely reflect the condition of being eigenfunction to the annihilation operator, and will fit our needs for the reconstruction of the n-point correlation functions.*

We see that the Fock state is the special case of the above family of coherent states for $\varphi = 0$.

Any continuous linear functional φ on $\mathcal{S}_\infty$ corresponds to a well defined coherent state [137].

Proposition 3 ([139, Proposition 2.5]). *The state ω of the above form is equivalent to the Fock representation, iff*[12] *φ is continuous on $\mathcal{S}_{L^2}$.*

The non-Fock coherent states are hence classified by functionals φ which are continuous on $\mathcal{S}_\infty$ — given by the space of (tempered) distributions — but discontinuous (or unbounded) on $\mathcal{S}_{L^2}$. This implies that non-Fock coherent states are characterized by the quotient space $\mathcal{S}'\backslash L^2(M, \mathrm{d}x)$, sometimes called the space of *tempered micro-functions*. Here and in the following S' denotes the topological dual of $\mathcal{S}_\infty$.

By Riesz-Markow theorem [136] every functional φ on $\mathcal{S}_\infty$ is of the form

$$\varphi(f) = \int_M f\,\mathrm{d}\nu, \tag{3.36}$$

for some Baire measure ν. And we get the following corollary.

Corollary 4. *If φ is invariant under left multiplication i.e. $\varphi(L_x^\star f) = \varphi(f)$ for any $f \in \mathcal{S}$ and $x \in SU(2)^{\times d}$, then the coherent state ω_φ is Fock*[13].

By this corollary, translation invariant coherent states are always Fock. In order to have a rich phase structure in GFT we would like to have non-Fock representations as well, which — if coherent — can not be translation invariant.

Physical remark

For φ to be a tempered micro-function the integral measure in eq. (3.36) has to be singular with respect to the Haar measure on M. On a compact manifold this can happen only due to local behavior of the measure, for example, when ν develops pure points. From the physical point of view, such point singularities of the measure correspond to states in which an infinite number of particles is concentrated in a local region of M. For field theories on space(-time), this situation is clearly not desirable since an infinite number of particles in a finite region corresponds to infinite energy density. Accordingly, in QFT's on compact space(-time) we require a finite particle number. This requirement is usually captured in the statement that no phase transition can occur in field theories on finite volume [118, 121, 140].

In GFT, on the other hand, there is no reason to prohibit states with divergent particle density. This is because the base manifold of GFT is not space-time and should be more generally understood as a set of particle labels (in the sense, that $\varphi^\dagger(x)$ creates a particle with label x). From this perspective an infinite number of particles with the same

[12] *An intuition of proposition 3*

If the representation is Fock the particle number expectation value is given by the L^2-norm of the order parameter

$$(\Omega|N|\Omega) = \|\varphi\|_{L^2}^2 .$$

If φ is, however, not continuous on $\mathcal{S}_{L^2}$ its formal L^2-norm is unbounded and, heuristically, we can interpret it as an indication for the divergent particle number.

[13] *Proof*

Let φ be invariant under left translations. Then for any $f \in \mathcal{S}$ we have

$$\varphi\left(L_g^\star f\right) = \int_M L_g^\star f\, d\mu = \varphi(f) = \int_M f\, d\mu,$$

hence the measure μ is a left invariant measure on $SU(2)$, which is identical with the Haar measure up to rescaling,

$$\mu = c \cdot \mu_H, \tag{3.37}$$

for some $c \in \mathbb{R}$. But then we have

$$|\varphi(f)| \leqslant c\|f\|,$$

and φ is continuous on $L^2(M, dx)$.

label is not problematic if x does not have a meaning of a space-time point. The micro-local behavior of φ could even be desirable, or at least reasonable, from the point of view of the interpretation of GFT particles as "building blocks of space-time and geometry". This is because, intuitively, we need an infinite number of discrete building blocks to define smooth objects.

3.3 *Algebraic group field theory*

We turn to the construction of an operator theory from functional GFT. Our treatment is based on known and well developed concepts of Euclidean field theory (for example [66, 124]) but to our knowledge they have not yet been applied to GFT.

Our procedure will be to use algebraic coherent states that correspond to the local minima of the action S. The concurrent GNS representation of the Weyl algebra will lead to field operators, whose expectation values provide the n-point correlation functions of the functional approach.

Before we start with the details of the construction we summarize the idea of the following procedure:

The correlation function of the functional formulation of GFT include the dynamics of the model. This is because they are defined by the functional integral whose weights are given by the action S. Hence, an operator formulation that produces the same correlation functions needs to follow the same dynamical relations. Such dynamical relations should be encoded in the algebra of observables, and hence, we need to modify the Weyl algebra of GFT. We do this by putting the linearized dynamics from the action S in the symplectic structure on the space of smearing function $\mathcal{S}$. After this construction the algebra will capture the dynamics of the functional formulation of GFT, which is one necessary step in order to reproduce the correct correlation function.

The other necessary step is the definition of the state. Our construction is perturbative around a classical field configuration and hence, coherent states seem to be appropriate, since they can be seen as the "most classical" states of a quantum theory. Indeed, as we will show, coherent states provide exactly the right structure in order to obtain the correct correlation functions.

We begin our construction with some realizations about the minimal field configurations of the action that will be needed to implement the linearized dynamical relations in the algebra.

Let φ be a real valued distribution on $\mathcal{S}_{\mathbb{R}}$, which is an extremum of S on $\mathcal{S}_{\mathbb{R}}$, meaning that $\varphi \in \mathcal{S}_{\mathbb{R}}'$ satisfies

$$S_{\varphi}'(f) \doteq \partial_{\epsilon} S[\varphi + \epsilon f]|_{\epsilon=0} = 0 \qquad \forall f \in \mathcal{S}_{\mathbb{R}}. \tag{3.38}$$

For a local action this condition would not make sense if φ is a delta distribution. However, in our case we assume that this expression is well defined for suitable $\varphi \in \mathcal{S}'_{\mathbb{R}}$ due to the multi-local structure of S. The second derivative of the action is a distribution on $\mathcal{S}_{\mathbb{R}} \times \mathcal{S}_{\mathbb{R}}$, such that for $f, g \in \mathcal{S}_{\mathbb{R}}$

$$S''_{\varphi}(f,g) = \int \mathrm{d}x\mathrm{d}y\, f(x) \left[\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)} S[\phi]\right]_{\phi=\varphi} g(y). \tag{3.39}$$

We can extend this distribution to the whole $\mathcal{S}$ by defining

$$S''_{\varphi}(f,g) = \int \mathrm{d}x\mathrm{d}y\, \overline{f(x)} \left[\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)} S[\phi]\right] g(y). \tag{3.40}$$

Since functional derivatives commute, S''_{φ} is symmetric on $\mathcal{S}_{\mathbb{R}}$, i.e.

$$S''_{\varphi}(f,g) = S''_{\varphi}(g,f), \tag{3.41}$$

and on $\mathcal{S}$ it satisfies

$$S''_{\varphi}(f,g) = \overline{S''_{\varphi}(g,f)}. \tag{3.42}$$

If φ is a local minimum it is positive on $\mathcal{S}_{\mathbb{R}}$, such that for any $f \in \mathcal{S}_{\mathbb{R}}$,

$$S''_{\varphi}(f,f) > 0, \tag{3.43}$$

and it follows that S''_{φ} is positive on the whole $\mathcal{S}$[14]. By Schwartz-Kernel theorem [38], S''_{φ} defines a continuous linear map $\mathcal{K}_{\varphi} : \mathcal{S} \to \mathcal{S}'$ such that

$$S''_{\varphi}(f,g) = \mathcal{K}_{\varphi}(g)[f] = \int \bar{f}(x) \left[K_{\varphi} g\right](x). \tag{3.44}$$

We will assume that K_{φ} is an elliptic differential operator on $\mathcal{S}$. It follows that K_{φ} is self-adjoint on $\mathcal{S}_{L^2}$[15], moreover, K_{φ} is positive and non-degenerate due to eq. (3.43) and therefore can be inverted on $\mathcal{S}$. The Green's function of K_{φ} denoted by $C_{\varphi} : \mathcal{S} \to \mathcal{S}$ satisfies

$$K_{\varphi} \circ C_{\varphi}|_{\mathcal{S}} = \mathbb{1}_{\mathcal{S}} \qquad C_{\varphi} \circ K_{\varphi}|_{\mathcal{S}} = \mathbb{1}_{\mathcal{S}}, \tag{3.45}$$

where $\mathbb{1}_{\mathcal{S}}$ is the identity operator on $\mathcal{S}$. We call C_{φ} the *propagator*, it follows from the properties of K_{φ}, that C_{φ} is positive and self-adjoint on $\mathcal{S}$. By functional calculus we can define the square roots, $K_{\varphi}^{\frac{1}{2}}$ and $C_{\varphi}^{\frac{1}{2}}$, which are both self-adjoint operators on $\mathcal{S}$. In the following we will drop the subscript φ for clearer notation but we shell keep in mind that K as well as the propagator C depend on the minimum φ.

Since C is positive and non-degenerate on $\mathcal{S}$ it defines an inner product[16]

$$(f,g)_C := (f, Cg)_{L^2}. \tag{3.47}$$

We denote the corresponding norm $\|.\|_C$, and define the resulting symplectic form on $\mathcal{S}_{\infty}$ by

$$\mathfrak{s}(f,g) = \Im(f,g)_C. \tag{3.48}$$

[14] *This is because for any $f \in \mathcal{S}$ we can find an $h, g \in \mathcal{S}_{\mathbb{R}}$ such that $f = h + \imath g$. Then*

$$S''_{\varphi}(f,f) = S''_{\varphi}(h,h) + S''_{\varphi}(g,g) \geqslant 0.$$

[15] *This is because for any $f, g \in \mathcal{S}$*

$$S''_{\varphi}(f,g) = (f, K_{\varphi} g)_{L^2} = \overline{S''_{\varphi}(g,f)} = \overline{(g, K_{\varphi} f)_{L^2}} = (K_{\varphi} f, g)_{L^2}.$$

[16] *The two-form $(\circ, C\circ)$ is linear in the second and anti-linearity in the first component by definition. Since C is invertible on $\mathcal{S}$ it does not have non-trivial zero modes and*

$$(f, Cg) = 0 \quad \forall f \in \mathcal{S}, \tag{3.46}$$

implies $g = 0$. Since C is positive the two-form is positive, $(f, Cf) \geqslant 0$.

With this definition the multiplication in the Weyl algebra $\mathfrak{A}(\mathcal{S})$ becomes

$$W_{(f)} \cdot W_{(g)} = e^{-\frac{\imath}{2}\Im((f,g)_C)} W(f+g). \qquad (3.49)$$

On this algebra we choose a coherent state[17]

$$\omega_\varphi\left(W_{(f)}\right) = e^{-\frac{\|f\|_C^2}{4}} e^{\imath\Re(\varphi(f))}. \qquad (3.50)$$

The one-particle Hilbert space in the corresponding GNS representation is then given by the completion of $\mathcal{S}$ in the norm[18] $\|\cdot\|_C$.

The generators of the Weyl algebra are defined in the same way as we did in the Fock representation; we denote them $\Phi_\varphi(f)$. It follows that they satisfy the following commutation relations

$$\left[\Phi_\varphi(f), \Phi_\varphi(g)\right] = \imath\mathfrak{s}(f,g). \qquad (3.51)$$

As above, we can also define the creation and annihilation operators by

$$\Phi_\varphi(f) = \frac{1}{\sqrt{2}}\left(\psi_\varphi(f) + \psi_\varphi^\dagger(f)\right). \qquad (3.52)$$

However, the commutation relations for $\psi_\varphi(f)$ and $\psi_\varphi^\dagger(f)$ now read

$$\left[\psi_\varphi(f), \psi_\varphi^\dagger(g)\right] = (f, Cg), \qquad (3.53)$$

and therefore ψ_φ and $\psi_\varphi^\dagger$ can not be interpreted as creation and annihilation operators on a Fock space.

To see the particle content we introduce the operators

$$A\left(\check{f}\right) = \sum_{J,\alpha,\beta} A_{J,\alpha,\beta}\, d_J\, \bar{\check{f}}_{J,\alpha,\beta} = \psi_\varphi\left(K^{\frac{1}{2}} f\right) \qquad (3.54a)$$

$$A^\dagger\left(\check{f}\right) = \sum_{J,\alpha,\beta} A^\dagger_{J,\alpha,\beta}\, d_J\, \check{f}_{J,\alpha,\beta} = \psi_\varphi^\dagger\left(K^{\frac{1}{2}} f\right) \qquad (3.54b)$$

where $\check{f}$ denotes the Fourier transform of f (for details and properties of the Fourier transform on $SU(2)$ see appendix A.2). It follows from (3.53) that A and $A^\dagger$ satisfy the usual commutation relations, since for all $f, g \in \mathcal{S}$,

$$\left[A\left(\check{f}\right), A^\dagger\left(\check{g}\right)\right] = \left[\psi\left(K^{\frac{1}{2}} f\right), \psi^\dagger\left(K^{\frac{1}{2}} g\right)\right] = (f,g).$$

Denoting with $\check{C}$ the Peter-Weyl transform of C, the field operator can be written in terms of A and $A^\dagger$ as[19]

$$\Phi(x) = \sum_{J,\alpha,\beta}\sum_{K,\gamma,\delta} \check{C}^{\frac{1}{2}}_{J,\alpha,\beta;K,\gamma,\delta}\, d_J\, d_K \left(A^\dagger_{J,\alpha,\beta}\, \check{f}_{K,\gamma,\delta} + A_{K,\gamma,\delta} \bar{\check{f}}_{J,\alpha,\beta}\right), \qquad (3.55)$$

[17] This state differs from those defined above by the factor of $\sqrt{2}$ in the phase factor. This is because the function in the phase of the coherent state has to be $\frac{1}{\sqrt{2}}\varphi$ if the minimum of S is given by φ, if we want to match the expectation values in ω_φ with the n-point correlation functions of $Z_\varphi[J]$.

[18] The closure of $\mathcal{S}$ in the correlator norm can be larger then L^2 and consists of distributions.

[19] We define the Peter-Weyl transform of C as the kernel of the two from $(\cdot, C\cdot)$ in the Peter-Weyl representation. Dropping the magnetic indices in the notation we obtain

$$\int dxdy\, \bar{f}(x)\, C(x,y)\, g(y) = \sum_{J,K} f_K\, d_K\, \check{C}_{k;J}\, d_J\, g_J.$$

Inserting the magnetic indices $\check{C}_{K,J}$ is given by

$$\check{C}_{K,\gamma,\delta;J,\alpha,\beta} = \int dxdy\, \bar{D}^K_{\gamma,\delta}(x)\, C(x,y)\, D^J_{\alpha,\beta}(y).$$

For the product of the field operators at the same point, $\Phi(f)^n$, we use the Wick product, which is given (neglecting magnetic indices) by the normal ordered product

$$
\begin{aligned}
:\Phi(f)^n: = &\sum_{J,K} d_{K_1} d_{J_1} \cdots d_{K_n} d_{J_n} \; \breve{C}^{\frac{1}{2}}_{J_1,K_1} \cdots \breve{C}^{\frac{1}{2}}_{J_n,K_n} \\
&\times \sum_{s=1}^{n} \begin{pmatrix} n \\ s \end{pmatrix} A^{\dagger}_{J_1} \cdots A^{\dagger}_{J_s} A_{K_{s+1}} \cdots A_{K_n} \; \breve{f}_{K_1} \cdots \breve{f}_{K_s} \breve{\breve{f}}_{J_{s+1}} \cdots \breve{\breve{f}}_{J_n},
\end{aligned}
$$

We want to remark here, that due to the multi-local structure of the interactions, it is not obvious that we will need this product for the calculation of perturbative corrections. This is because in the combinatorially multi-local interaction all fields are evaluated at different points.

Restricting $\mathcal{S}$ to the subspace of real valued functions, $\mathcal{S}_{\mathbb{R}}$, gives a (maximal) abelian[20] $C^{\star}$-sub-algebra $\mathfrak{A}(\mathcal{S}_{\mathbb{R}})$ and the algebraic state ω_{φ} corresponds to a probability measure on $\mathfrak{A}(\mathcal{S}_{\mathbb{R}})$ [135]. This probability measure is equivalent to the Gaussian measure of the functional integral around the field configuration φ in the sense that it provides the same correlation functions. To see this we calculate the expectation value of the generator $\Phi_{\varphi}(f)$,

$$(\varphi|\Phi_{\varphi}(f)|\varphi) = (-\imath\partial_t)\,\omega_{\varphi}\left(W_{(tf)}\right)|_{t=0} = \varphi(f)\,, \tag{3.56}$$

for all $f \in \mathcal{S}_{\mathbb{R}}$. Using eq. (3.8) and formally evaluating the fields Φ at single points $x \in M$ we get

$$(\varphi|\Phi_{\varphi}(x)|\varphi) = \varphi(x) = \delta_{J(x)} Z[J]\,|_{J=0}. \tag{3.57}$$

In the same way we obtain for the two point function

$$(\varphi|\Phi(x)\,\Phi(y)|\varphi) = \frac{1}{2} C(x,y) + \varphi(x)\,\varphi(y) = \delta_{J(x)}\delta_{J(y)} Z[J]\,|_0. \tag{3.58}$$

It is straightforward to convince oneself that the following equality holds (see Appendix A.6)

$$
\begin{aligned}
&\partial_{t_1} \cdots \partial_{t_n} \omega_{\varphi}\left(W_{(t_1 f_1)} \cdots W_{(t_n f_n)}\right)|_{t=0} \\
&= \int \mathrm{d}x^1\, f_1\left(x^1\right) \delta_{J(x^1)} \cdots \int \mathrm{d}x^n\, f_n\left(x^n\right) \delta_{J(x^n)} Z[J]\,|_{J=0}.
\end{aligned}
$$

This equality holds true if products of fields at the same point do not appear. If they do, we have to use Wick products, that define the renormalized expectation values of the partition function. This concludes our construction of the operator GFT.

Summary:

Choosing the coherent state ω_{φ} and the Weyl algebra $\mathfrak{A}(\mathcal{S})$ with the symplectic form given by the second variation of the GFT action we

[20] *If $f, g \in \mathcal{S}_{\mathbb{R}}$ the imaginary part of the scalar product (f, Cg) is zero and we obtain an abelian $C^{\star}$-algebra $\mathfrak{A}(\mathcal{S}_{\mathbb{R}})$*

$$W_{(f)} W_{(g)} = W_{(f+g)} = W_{(g)} W_{(f)}.$$

That this algebra is maximal follows from the fact that the space $\mathcal{S}$ can be decomposed as

$$\mathcal{S} = \mathcal{S}_{\mathbb{R}} + \imath \mathcal{S}_{\mathbb{R}}.$$

obtain an operator description of GFT, with operators ψ_φ and $\psi_\varphi^\dagger$ that satisfy the commutation relations

$$\left[\psi_\varphi(x), \psi_\varphi^\dagger(y)\right] = C(x,y). \tag{3.59}$$

The expectation values of polynomials of field operators

$$\Phi_\varphi(x) = \frac{1}{\sqrt{2}}\left(\psi_\varphi(x) + \psi_\varphi^\dagger(y)\right),$$

are equal to the correlation functions from the perturbative canonical approach around the minimum φ. Hereby the minimum φ does not need to be a smooth function but can be extended to a distribution on $\mathcal{S}$.

3.4 *Application of the algebraic formulation*

A rigorous analysis of local minima of combinatorially multi-local actions such as those that we presented in chapter 2 is yet to be performed. The problem is that the variational equations for such functionals reduce to non-linear integro-differential equations, that are very difficult to solve in general. In order to avoid this problem here, we use a simplified action on $M = SU(2) \times SU(2)$,

$$\begin{aligned} S[\phi] = &- m\int \mathrm{d}x\,\phi(x)\,\phi(x) \\ &+ \lambda\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}v\,\phi(x)\,\phi(y)\,\phi(z)\,\phi(v) \\ &\times \delta\left(x_1 y_1^{-1}\right)\delta\left(x_2 v_2^{-1}\right)\delta\left(y_2 z_2^{-1}\right)\delta\left(z_1 v_1^{-1}\right), \end{aligned} \tag{3.60}$$

where $\delta\left(x_1 \cdot x_2^{-1}\right)$ denotes the Dirac-Delta distribution on $SU(2)$ that satisfies $\int \mathrm{d}x\, f(x)\,\delta\left(x_1 \cdot x_2^{-1}\right) = \int \mathrm{d}x_1\, f(x_1, x_1)$. This action arises from the Boulatov equation that we introduced earlier in eq. (2.16) if we assume that the fields are constant in the middle variable,

$$\phi(x_1, x_2, x_3) = \phi(x_1, x_3). \tag{3.61}$$

Even in this simplified case that represents a matrix model the discussion of saddle points is not trivial and our arguments are rather heuristic. Nevertheless, we do not go into details of the variational problem, since this is not the point of discussion here. Instead we will show, that some distributional configurations can be seen as local minima. And use those for our explicit construction of an operator theory.

The variation of the action with respect to the field ϕ leads to the following extremal condition

$$S'_\phi(x,y) = -2m\phi(x) + 4\lambda\int \mathrm{d}y\,\phi(x_1,y_1)\,\phi(y_1,y_2)\,\phi(y_2,x_2) = 0.$$

And the following field configuration can be seen as a solution of that integral equation

$$\varphi(x) = \sqrt{\frac{m}{2\lambda}}\,\delta\left(x_1 \cdot x_2^{-1}\right). \tag{3.62}$$

The second variation of S at φ reads

$$S''|_\varphi = 4m\,\delta\left(x_1 y_1^{-1}\right)\delta\left(x_2 y_2^{-1}\right), \tag{3.63}$$

and K becomes, $K = 4m\,\mathbb{1}_{\mathcal{S}}$, where $\mathbb{1}_{\mathcal{S}}$ is the identity operator on $\mathcal{S}$ and $C = \frac{1}{4m}\mathbb{1}$.

This operator is diagonal, positive and invertible and hence we can use our construction.

3.4.1 *Coherent δ states*

The field configuration that we take as a local minimum of S is given in equation (3.62). An algebraic state that corresponds to φ is defined by equation (3.35) with $\varphi(f) = \frac{1}{\sqrt{2}}\int \mathrm{d}x\,\varphi(x)\, f(x)$,

$$\omega_\varphi\left(W_{(f)}\right) = e^{-\frac{\|f\|_C^2}{4}}\, e^{-\imath\sqrt{\frac{m}{2\lambda}}\Re\left[\int \mathrm{d}h\,\delta\left(x_1\cdot x_2^{-1}\right) f(x)\right]}. \tag{3.64}$$

Where the new scalar product is simple the rescaled L^2 product given by

$$(f,g)_C = \frac{1}{4m}\int \mathrm{d}x\,\bar{f}(x)\, g(x). \tag{3.65}$$

In order to construct an explicit representation that corresponds to this algebraic state we use the construction by Araki and Woods [131] that we applied to GFT in [141]. We take a one dimensional space of real numbers $\mathbb{R}$ with multiplication as a scalar product and introduce commutative operators as

$$P(f)\,r = F_D\, r \qquad\qquad Q(f)\,r = \overline{F}_D\, r, \tag{3.66}$$

where $F_D = \int \mathrm{d}x_1\, f(x_1, x_1)$ denotes the diagonal integral of f and $r \in \mathbb{R}$.

Let $\psi_F(f)$ and $\psi_F^\dagger(f)$ be the Fock creation and annihilation operators from Eq. (3.30) and let $|0)$ denote their Fock vacuum. We define unitary operators on the Hilbert space, $\mathcal{H}_F \otimes \mathbb{R}$ by

$$W^\delta_{(f)} = e^{\frac{\imath}{\sqrt{8m}}\left[\psi_F(f)+\psi_F^\dagger(f)\right]} \otimes e^{\frac{\imath}{2}\sqrt{\frac{m}{2\lambda}}[P(f)+Q(f)]}, \tag{3.67}$$

and a state

$$|\delta) = |0) \otimes 1 \in \mathcal{H}_F \otimes \mathbb{R}. \tag{3.68}$$

where $1 \in \mathbb{R}$ is a normalized state on $\mathbb{R}$. We can readily verify that the expectation values of $W_{(f)}$ on $|\delta)$ are given for any $f \in \mathcal{S}$ by,

$$\begin{aligned}(\delta|W^\delta_{(f)}|\delta) &= e^{-\frac{\|f\|_C^2}{4}}\, e^{\imath\sqrt{\frac{m}{2\lambda}}\Re[F_D]} \\ &= \omega_\varphi\left(W_{(f)}\right).\end{aligned} \tag{3.69}$$

By linearity this equality extends to the whole algebra, and hence, it provides a representation of the Weyl algebra that is equivalent to the GNS representation given by ω_φ. Irreducibility and cyclicity of this representation are inherited from the Fock representation since $\mathbb{R}$ is one-dimensional.

The generators of Weyl operators are isomorphic to

$$\Phi_\delta(f) = \left(\frac{1}{\sqrt{4m}}\psi_F(f) + \sqrt{\frac{m}{4\lambda}}Q(f)\right) + \left(\frac{1}{\sqrt{4m}}\psi_F^\dagger(f) + \sqrt{\frac{m}{4\lambda}}P(f)\right),$$

that act on the Fock space. And the creation and annihilation operators can be read of as

$$\psi_\delta(f) = \frac{1}{\sqrt{4m}}\psi_F(f) + \sqrt{\frac{m}{4\lambda}}Q(f) \quad \psi_\delta^\dagger(f) = \frac{1}{\sqrt{4m}}\psi_F^\dagger(f) + \sqrt{\frac{m}{4\lambda}}P(f).$$

From eq. (3.54) the Euclidean creation and annihilation operators follow,

$$A^\dagger_{J,\alpha,\beta} = \psi_\delta^\dagger\left(\sqrt{4m}\mathfrak{D}^J_{\alpha,\beta}\right) = \psi_F^\dagger\left(\mathfrak{D}^J_{\alpha,\beta}\right) + \frac{m\,d_J}{\sqrt{\lambda}}\delta_{\alpha,\beta} \tag{3.70}$$

$$A_{J,\alpha,\beta} = \psi_\delta\left(\sqrt{4m}\mathfrak{D}^J_{\alpha,\beta}\right) = \psi_F\left(\mathfrak{D}^J_{\alpha,\beta}\right) + \frac{m\,d_J}{\sqrt{\lambda}}\delta_{\alpha,\beta}. \tag{3.71}$$

Where $\mathfrak{D}^J_{\alpha,\beta}(g)$ are the product Wigner-Matrix representations (see appendix A.1 and A.3) used to smear the Fock creation and annihilation operators. The operators A and $A^\dagger$ satisfy the canonical commutation relations, and hence can be used as a definition of a particle.

3.5 Conclusion

As we pointed out at the end of the last chapter, the general idea of the operator GFT formalism is to construct a theory of operators and Hilbert spaces that provide a reformulation of functional GFT. However, despite the use of the operator framework in numerous applications [57, 58, 97, 142–148], an explicit relation between functional and operator GFT remains an open issue. The problem of this relation stems from the fact that group field theory does not have a concept of time and for that reason can not rely on the usual relation between covariant and canonical formulation of QFT. In our work we suggested a solution to this problem at the perturbative level and explicitly provided a relation between the two theories.

We can try to apply our construction to ordinary field theories on space-time, however, there we face the following problems:

The action S of QFT on space-time is typically required to be bounded from below to avoid vacuum instability problems. In this case the global minimum of the classical theory exists, and we can perform our construction around this minimum. The quantum effective action — denoted Γ — will in this case be a convex function [118, 125] and for that reason will have a unique global minimum — the true vacuum of the quantum theory. If this minimum coincides with that of the classical action we again can use our construction for perturbative formulation of the operator theory. However, if this minimum is different, the second derivative of the action, S'', evaluated at the true vacuum of the quantum theory will not be a positive operator, and our construction will fail. In this case we should expand our theory around the minimum of Γ, which in principle is possible but in practice requires the knowledge of the whole quantum effective action. Moreover, in this case there is no reason to assume that the vacuum state of the theory will be adequately described by a coherent state which would lead to additional complications in our construction. For that reason, our construction works best, when we define a perturbative theory around the classical action S. The bottomless structure of the GFT action, however, does not allow us to do anything else even in the functional formulation. For that reason our construction seems adequate[21].

[21] *In AQFT the same construction suggests a rigorous reformulation of the Lorentzian path integral. However, the algebra of observables in AQFT is much more complicated to construct and is missing a clear notion of a norm. The resulting formulation leads to a rigorous, perturbative definition of QFT on Lorentizan space-time. Our suggested formulation is strongly motivated by this treatment.*

of course a regularization of the bottomless action S needs to be investigated in future works and non-perturbative definition of the theory should be provided.

Our explicit construction provides a definition of an operator theory directly from functional GFT. Tempered micro-functions that are the minima of the action S correspond to Fock-inequivalent operator theories. These theories could be understood as effective theories in different phases of GFT, which we mentioned at the end of the previous chapter. The operator algebras define suitable operators in the corresponding phase, and the coherent states play the role of the vacuum. To see how our construction fits the conceptual idea of the operator GFT, which we presented in the previous chapter, we summarize the steps according to figure 3.2:

STEP I: KINEMATICAL HILBERT SPACE

Instead of the Hilbert space, we deal with the more general space of algebraic states. In our construction, the kinematical degrees of freedom are captured by the choice of the Weyl algebra, that is motivated by the quantization of a simplex and that relates to the GFT Fock space by the choice of an appropriate state.

STEP II: PHYSICAL HILBERT SPACE

A restriction of the kinematical space by dynamical relations is not explicitly present in our formulation. Instead, the dynamics is directly encoded in the algebraic product and in the choice of the coherent state

$$\omega_{\varphi}\left(W_{(f)}\right) = e^{-\frac{\|f\|^2}{4}} e^{\imath \Re[\int \varphi f]}, \tag{3.72}$$

The product of the algebra results from the choice of symplectic structure on $\mathcal{S}$, which (in suitable representations) leads to modified commutation relations between creation and annihilation operators

$$\left[\psi\left(f\right), \psi^{\dagger}\left(g\right)\right] = \left(f, Cg\right). \tag{3.73}$$

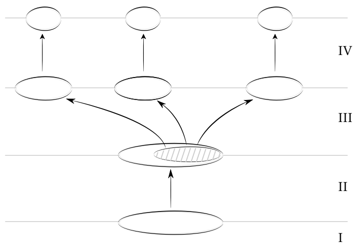


Figure 3.2: Reminder of the conceptual idea of operator GFT.

An explicit equality between n-point correlation functions then directly shows that our theory knows the dynamical relations of the functional formulation. But it remains to be understood if and how this construction can be recast in the language of constraints as introduced in the previous chapter.

In our case the idealization step is the infinite number of particles in the state ω_{φ}. If the particle number were finite all coherent states would be equivalent to Fock and the distinction between phases would be not clear. This does not imply that the phases with different physical properties would not be present in the model, but their distinction would be much more complicated. On the other hand the assumption of infinitely many particles could be a real physical requirement that is needed in order to describe states that correspond to smooth geometries using discrete particles.

step iii: phases of gft

A notion of different phases is very clearly presented by the choice of the local minimum of S. Moreover, if that minima are given by tempered micro-functions, the phases will not be Fock. On the other hand, algebraic coherent states with finite particle number will always lead to the Fock phase. Hence, in order to have other phases, the particle number has to be infinite, which is in direct relation to the usual description of phase transitions in statistical field theory.

step iv: quantum corrected gt

The final step of effective definition of approximate gravitational equations is absent at the current stage and has to be done in future works. A possible test would be to follow the cosmological calculations in GFT [148]. However, whereas the existing calculations approximate the dynamics of GFT by neglecting the interaction part, our approach will lead to calculations that include the full linearized dynamics and whose corrections can be included in the perturbative series.

Most of our steps were based on known and developed methods in the context of algebraic field theory [39, 123, 124], but needed some modifications due to the conceptual and technical differences between group field theory and ordinary QFT on space-time.

Appendix

A

A.1 Group theory of $SU(2)$ in a nutshell

In this chapter, we used some basic features of group theory on $SU(2)$. Here, we provide the properties of the $SU(2)$ group and its irreducible representations that we used throughout the chapter. For more details we refer to the literature [132, 149].

The group $SU(2)$

1. The group $SU(2)$ is a compact, connected, simply connected three dimensional Lie group. The Haar measure μ_H on $SU(2)$ is a unique normalized, left and right translation invariant Borel measure. That is, for any integrable function $f : SU(2) \to \mathbb{R}$ and for any $R, L \in SU(2)$ we have

$$\int_{SU(2)} \mathrm{d}\mu_H \, f(LgR) = \int_{SU(2)} \mathrm{d}\mu_H(g) \; f(g), \tag{A.1}$$

and

$$\int_{SU(2)} \mathrm{d}\mu_H = 1. \tag{A.2}$$

We denote this measure $\mathrm{d}h$.

2. The unitary, irreducible representations of $SU(2)$ are labeled by their dimension that we denote $d_j = 2j + 1$ for $j \in \frac{\mathbb{N}}{2}$. Any two unitary irreducible representations with the same dimension are equivalent. Therefore it is enough to provide one irreducible representation for each dimension.

Wigner matrices

1. The Wigner matrix representation of $SU(2)$ is an unitary irreducible representation of $SU(2)$ in terms of matrices of rank d_j. We write

$$D^j_{\alpha,\beta}(g) \in Mat\left(d_j \times d_j, \mathbb{C}\right), \tag{A.3}$$

where j is called the spin, and $\alpha, \beta \in \{-j, \cdots, j\}$ are the spin-z components of the representation. Usually α and β are referred to as *magnetic indices.*

2. Wigner matrix coefficients are smooth, complex valued functions on $SU(2)$.

3. The complex conjugation of the Wigner matrices satisfy

$$\overline{D}^{j}_{\alpha,\beta}(g) = (-1)^{\alpha-\beta} D^{j}_{-\alpha,-\beta}(g), \tag{A.4}$$

   and their adjoint is

$$\left(D^{\dagger}\right)^{j}_{\alpha,\beta}(g) = D^{j}_{\alpha,\beta}\left(g^{-1}\right) = \overline{D}^{j}_{\beta,\alpha}(g). \tag{A.5}$$

4. By Peter-Weyl theorem the Wigner matrix coefficients are dense in L^2. They form an orthogonal system in L^2 normalized to the inverse of dimension of the representation (Schur lemma)

$$\int \mathrm{d}g\, \overline{D}^{j}_{\alpha,\beta}(g)\, D^{\tilde{j}}_{\tilde{\alpha},\tilde{\beta}}(g) = \frac{1}{d_j}\delta_{j,\tilde{j}}\delta_{\alpha,\tilde{\alpha}}\delta_{\beta,\tilde{\beta}}, \tag{A.6}$$

   moreover, this system is complete

$$\sum_{j,\alpha,\beta} d_j D^{j}_{\alpha,\beta}(g)\, \overline{D}^{j}_{\alpha,\beta}(h) = \sum_{J,\alpha} d_j D^{j}_{\alpha,\alpha}\left(gh^{-1}\right) = \delta(g,h), \tag{A.7}$$

   where $\delta(g,h)$ denotes a Dirac-Delta distribution such that for any smooth function f we have

$$\int \mathrm{d}g\, f(g)\,\delta(g,h) = f(h). \tag{A.8}$$

   Hence, the set $\left\{\sqrt{d_j} D^{j}_{\alpha,\beta}(g)\right\}_{j,\alpha,\beta}$ is an orthonormal basis in $L^2(SU(2))$ and it follows

$$\lim_{N\to\infty} \| f - \sum_{J}^{N} d_J \sum_{\alpha,\beta=-j}^{j} \left(D^{j}_{\alpha,\beta}, f\right)_{L^2} D^{j}_{\alpha,\beta} \| = 0. \tag{A.9}$$

5. By the Peter-Weyl theorem the set of Wigner matrix coefficients is also dense in $\mathcal{S}_\infty$ (see for example [149]), i.e.

$$\lim_{N\to\infty} \| f - \sum_{j}^{N} d_j \sum_{\alpha,\beta=-j}^{j} f_{j,\alpha,\beta}\, \overline{D}^{j}_{\alpha,\beta}(g) \|_{k,\infty} = 0. \tag{A.10}$$

A.2 Fourier transform on $SU(2)$

1. There exists a notion of Fourier transform $\mathcal{F}$ on $\mathcal{S}$ — that we call the *Peter-Weyl transform* — defined by

$$\mathcal{F}(f)(j,\alpha,\beta) = \left(D^{j}_{\alpha,\beta} f\right)_{L^2} \doteq \check{f}_{j,\alpha,\beta}. \tag{A.11}$$

Meaning a point wise equality we define the function $\check{f}$ as

$$\check{f} \doteq \mathcal{F}(f), \tag{A.12}$$

and write

$$\check{f}(j, \alpha, \beta) = \check{f}_{j,\alpha,\beta}. \tag{A.13}$$

2. The inverse of $\mathcal{F}$ is given by (A.9)

$$\mathcal{F}^{-1}\left(\check{f}\right)(g) = \sum_{j \in \frac{\mathbb{N}}{2}} d_j \sum_{\alpha,\beta=-j}^{j} \check{f}_{j,\alpha,\beta} D^j_{\alpha,\beta}(g) \overset{\text{(A.9)}}{=} f(g). \tag{A.14}$$

3. $\mathcal{F}$ is a topological isomorphism between $\mathcal{S}$ and the space of rapidly decreasing sequences [132], that we denote $\mathcal{S}(\mathbb{N})$. A sequence $\{f_{j,\alpha,\beta}\}$ with $j \in \frac{\mathbb{N}}{2}$ and $\alpha, \beta \in \{-j, \cdots, j\}$ is called rapidly decreasing if for any $n \in \mathbb{N}$

$$\lim_{j \to \infty} |j^n| \, \|f_j\| < \infty, \tag{A.15}$$

with $\|f_j\|^2 = \sum_{\alpha,\beta=-j}^{j} \bar{f}_{j,\alpha,\beta} f_{j,\alpha,\beta}$.

4. The Peter-Weyl transform can be extended to the space of square integrable functions on which it defines an isomorphisms between L^2 and the space of square summable sequences ℓ^2. Moreover, it can be extended to the dual space of $\mathcal{S}_\infty$ which is the space of distributions $\mathcal{S}'_\infty$

A.3 Notation suitable for GFT

Since the domain of fields in GFT is usually an n-fold product of Lie groups, we will use product representations with the following properties, to simplify the notation:

1. The group $SU(2)^{\times n}$ is a connected (in the product topology), simply connected, compact Lie group, with the Haar measure defined as

$$\mu_H\left(SU(2)^{\times n}\right) = \mu_H(SU(2)) \times \mu_H(SU(2)) \times \cdots \times \mu_H(SU(2)).$$

We call the domain of fields the *base manifold of GFT*, and denote it with $M \doteq SU(2)^{\times n}$. To make our notation as close as possible to ordinary cases we will denote the points of M by x and y. Nevertheless, we should always keep in mind that x and y are elements of a non-commutative Lie group. When we need to refer to single components of x or y we use the subscript notation, such that $x_j \in SU(2)$ and

$$x = (x_1, \cdots, x_n) \in M = SU(2)^{\times n}. \tag{A.16}$$

With this notation we denote the Haar measure on M by $\mathrm{d}x$. When we need to refer to n points on M we use the superscript notation such that $x^j \in M$.

2. The product Wigner matrix representations of $SU(2)^{\times n}$ are defined as products such that

$$\mathfrak{D}^{(j_1,\cdots,j_n)}_{(\alpha_1,\cdots,\alpha_n),(\beta_1,\cdots,\beta_n)}(x) = D^{j_1}_{\alpha_1,\beta_1}(x_1)\cdots D^{j_n}_{\alpha_n,\beta_n}(x_n). \tag{A.17}$$

To shorten the notation we use the multi index notation, such that $J = (j_1, j_2, \cdots, j_n)$ with each $j_i \in \frac{\mathbb{N}}{2}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$, where $\alpha_i, \beta_i \in \{-j_i, \cdots, j_i\}$. Hence for any $x \in M$ we write

$$\mathfrak{D}^{J}_{\alpha,\beta}(x) = D^{j_1}_{\alpha_1,\beta_1}(x_1)\cdots D^{j_n}_{\alpha_n,\beta_n}(x_n). \tag{A.18}$$

The dimension of the product representation $\mathfrak{D}^J$ is

$$d_J = (2j_1+1)\cdots(2j_n+1). \tag{A.19}$$

3. It is straightforward to verify that the above properties of Wigner coefficients and Peter-Weyl transform translate one-to-one to the product representation $\mathfrak{D}$. In particular, the Peter-Weyl coefficients for a field φ can be written as

$$\check{\varphi}(J,\alpha,\beta) = \int \mathrm{d}x\, \overline{\mathfrak{D}}^{J}_{\alpha,\beta}(x)\ \varphi(x) \tag{A.20}$$

and the inverse is given by

$$\varphi(x) = \sum_{J,\alpha,\beta} d_J\, \check{\varphi}^{J}_{\alpha,\beta}\, \mathfrak{D}^{J}_{\alpha,\beta}(x). \tag{A.21}$$

A.4 Proof of lemmas for $\mathcal{S}$

Lemma 5. *$\mathcal{S}$ is closed under translations; that is for any $y \in M$ and $f \in \mathcal{S}$ the functions $L^\star_y f$ and $R^\star_y f$ are again in $\mathcal{S}$. Moreover, $L^\star_y$ and $R^\star_y$ leave the L^2-bracket, $(\cdot,\cdot)_{L^2}$, invariant.*

Proof. The first statement follows from smoothness of the maps L_x and R_x. The second statement is a direct consequence of the left (respectively right) invariance of the Haar measure $\mathrm{d}x$. That is for $f, g \in \mathcal{S}$ and $y \in M$,

$$\begin{aligned}\left(L^\star_y f, L^\star_y g\right)_{L^2} &= \int_M \overline{f}(yx)\, g(yx)\ \mathrm{d}x \\ &= \int_M \overline{f}(x)\, g(x)\ \mathrm{d}x \\ &= (f,g)_{L^2}.\end{aligned}$$

And similar for $R^\star_y f$. □

Let $X_i \in \mathfrak{m}$ be a Lie algebra element of M, then X_i acts as a derivation on smooth functions such that for $f \in \mathcal{S}$, $I \subset \mathbb{R}$ an interval containing zero and $t \in I$,

$$X_i f(x) \doteq \partial_t f\left(e^{tX_i}\, x\right)|_{t=0}, \tag{A.22}$$

where e^{tX_i} denotes the exponential map on M [132].

Lemma 6. *$\mathcal{S}$ equipped with topology induced by the family of semi-norms*

$$\left\{\|f\|_{k,\infty} = \|X_1 \cdots X_k f\left(g\right)\|_\infty : X_1, \cdots, X_k \in \mathfrak{m}; \forall k \in \mathbb{N}\right\},$$

is a complete, topological, locally convex, vector space.

Proof. See reference[1] [132]. □

When the topology of $\mathcal{S}$ will be important in our discussion we will denote this topological space by $\mathcal{S}_\infty$.

Since M is compact, every smooth function on it is finite integrable and we can equip $\mathcal{S}$ with the norm-topology induced by the norm,

$$\|f\|_{L^2}^2 = \int_M \bar{f}\left(x\right) f\left(x\right)\, \mathrm{d}x. \tag{A.23}$$

Lemma 7. *$\mathcal{S}$ equipped with the norm topology is not complete and its completion is the space of square integrable functions on M.*

Proof. See reference [132]. □

Let $h \in SU\left(2\right)$ and $D : SU\left(2\right) \to M$ be a diagonal map such that $D_h \equiv D\left(h\right) = \left(h, \cdots, h\right)$. We say f satisfies the closure constraint (or f is gauge invariant) if

$$R^\star_{D_h} f = f \qquad\qquad \forall h \in G. \tag{A.24}$$

We denote the space of functions that satisfy the closure constraint by $\mathcal{S}_G$.

Proposition 8. *$\mathcal{S}$ can be decomposed in complementary subspaces $\mathcal{S}_G$ and $\mathcal{S}_{NG}$ such that*

$$\mathcal{S}_\infty = \mathcal{S}_G + \mathcal{S}_{NG}, \tag{A.25}$$

and $\mathcal{S}_G \cap \mathcal{S}_{NG} = \{0\}$. Where $\mathcal{S}_G$ is a space of gauge invariant functions and $\mathcal{S}_{NG}$ is a space of functions that do not satisfy the closure constraint.

Proof. Let P define an operator on $\mathcal{S}$ point wise by

$$\left(Pf\right)\left(x\right) = \int_G \left(R^\star_{D_h} f\right)\left(x\right)\, \mathrm{d}h.$$

P is linear since it is a combination of linear operators, $R^\star_{D_h}$ and $\int_G \left(\cdot\right)\, \mathrm{d}h$. We show that the image of P is in $\mathcal{S}_\infty$. By [132, lemma 2.1] it is enough to show that $\|Pf\|_{k,\infty} < \infty$ for any $k \in \mathbb{N}$. For an arbitrary fixed k we get

$$\begin{aligned}\|Pf\|_{k,\infty} &= \sup_{x\in M} \left|X_1 \cdots X_k \left(Pf\right)\left(x\right)\right| \\ &= \sup_{x\in M} \left|X_1 \cdots X_k \int_G \left(R^\star_{D_h} f\right)\left(x\right)\, \mathrm{d}h\right|.\end{aligned}$$

[1] *In this reference the authors define the Lie algebra by left invariant vector fields as opposed to our definition as right invariant vector fields. For that reason in the original paper eq.(A.22) is defined by right multiplication with the exponential map. This small change, however, does not change the results of the paper.*

By lemma (5) the integrand is a smooth function and can be upper bounded by $\sup_{x\in M}\left|\left(R^{\star}_{D_h}f\right)(x)\right|$. Hence, by dominant convergence theorem

$$\|Pf\|_{k,\infty} \leqslant \int_G \sup_{x\in M}\left|X_1\cdots X_k\,\left(R^{\star}_{D_h}f\right)(x)\right|\,\mathrm{d}h$$

For any fixed $h\in G$ we have

$$X_1\cdots X_k\,\left(R^{\star}_{D_h}f\right)(x) = \partial_{t_1}\cdots\partial_{t_k} f\left(e^{t_1X_1}\cdots e^{t_kX_x}\,x\,D_h\right),$$

where all derivatives are taken at zero. Since $x\,D_h\in M$ it follows that

$$\sup_{x\in M}\left|X_1\cdots X_k\,\left(R^{\star}_{D_h}f\right)(x)\right| = \sup_{x\in M}\left|X_1\cdots X_k\,f(x)\right|.$$

and we obtain

$$\|Pf\|_{k,\infty} \leqslant \|f\|_{k,\infty}.$$

Therefore, $P:\mathcal{S}_\infty\to\mathcal{S}_\infty$, is a continuous linear operator on $\mathcal{S}$.

Further, by right invariance of the Haar measure it follows that $P^2f = Pf$. By [135, theorem 1.1.8] it follows that $\mathcal{S}_\infty$ can be decomposed as

$$\mathcal{S}_\infty = \mathcal{S}_G + \mathcal{S}_{NG},$$

where $\mathcal{S}_G = P\mathcal{S}_\infty$ and $\mathcal{S}_{NG} = (1-P)\,\mathcal{S}_\infty$ and $\mathcal{S}_G\cap\mathcal{S}_{NG} = \{0\}$. □

Lemma 9. *P is an orthogonal projector on* $L^2(M,dx)$.

Proof. P is bounded on $\mathcal{S}_{L^2}$ since for any $f\in\mathcal{S}$ we have by right invariance of the Haar measure

$$\|Pf\|_{L^2} = \int_M\int_G f(x\,D_h)\,\mathrm{d}h\,\mathrm{d}x = \int_M f(x)\,\mathrm{d}x = \|f\|_{L^2}.$$

Let $f,g\in\mathcal{S}$. Then by Fubini and the invariance of the Haar measure under right multiplication and inversion, we have

$$\begin{aligned}(f,Pg)_{L^2} &= \int_M \overline{f}(x)\left(\int_G\left(R^{\star}_{D_h}g\right)(x)\,\mathrm{d}h\right)\mathrm{d}x\\ &= \int_M\left(\int_G\overline{\left(R^{\star}_{D_h}f\right)}(x)\,\mathrm{d}h\right)g(x)\,\mathrm{d}x\\ &= (Pf,g)_{L^2}.\end{aligned}$$

And for $h_1,h_2\in G$ we have

$$\begin{aligned}(PPf)(x) &= \int_G\int_G\left(R^{\star}_{D(h_1)}R^{\star}_{D(h_2)}f\right)(x)\,\mathrm{d}h_1\,\mathrm{d}h_2\\ &= \int_G\int_G\left(R^{\star}_{D(h_1)}R^{\star}_{D(h_2)}f\right)(x)\,\mathrm{d}h_1\,\mathrm{d}h_2\\ &= \int_G\int_G\left(\left(R_{D(h_1h_2)}\right)^{\star}f\right)(x)\,\mathrm{d}h_1\,\mathrm{d}h_2\\ &= \int_G\left(\left(R_{D_h}\right)^{\star}f\right)(x)\,\mathrm{d}h = (Pf)(x)\ .\end{aligned}$$

Therefore, P is an orthogonal projection on the dense domain of $L^2(M,\mathrm{d}x)$ and extends uniquely to the whole $L^2(M,\mathrm{d}x)$ by continuity. □

Theorem 10. *The space $\mathcal{S}_G = P\mathcal{S}$ is dense in $PL^2(M, dx)$ — the image of the orthogonal projection P on $L^2(M, dx)$.*

Proof. Since $PL^2(M, \mathrm{d}x)$ is given by the projection P, it is a closed subspace of $L^2(M, \mathrm{d}x)$. By lemma (7) the set $PL^2(M, \mathrm{d}x) \cap \mathcal{S}$ is dense in $PL^2(M, \mathrm{d}x)$. Further, any $f \in PL^2(M, \mathrm{d}x) \cap \mathcal{S}$ is an almost-everywhere gauge invariant function that is smooth. Define $g = f - Pf$. Then g vanishes almost everywhere and is smooth. Hence g is zero everywhere, and we get $f \in \mathcal{S}_G$ and $PL^2(M, \mathrm{d}x) \cap \mathcal{S} \subseteq \mathcal{S}_G$. The opposite inclusion, $\mathcal{S}_G \subseteq PL^2(M, \mathrm{d}x) \cap \mathcal{S}$, is obvious since any $f \in \mathcal{S}_G$ is square integrable and $\mathcal{S}_G \subseteq \mathcal{S}$ by lemma (8). □

A.5 Weyl algebra with closure constraints

Lemma 11. *For any $x \in M$ the maps α_x and β_x from $\mathfrak{A}(\mathcal{S})$ to $\mathfrak{A}(\mathcal{S})$ defined such that for any $f \in \mathcal{S}$*

$$\alpha_x\left(W_{(f)}\right) = W_{(L_x^\star f)}, \qquad \beta_x\left(W_{(f)}\right) = W_{(R_x^\star f)}, \tag{A.26}$$

and extended to the whole $\mathfrak{A}(\mathcal{S})$ by linearity are $\star$-automorphisms.

Proof. By definition α_x and β_x are linear. Further let $f, g \in \mathcal{S}$, then by lemma (3.2.1)

$$\begin{aligned}
\alpha_x\left(W_{(f)}W_{(g)}\right) &= \alpha_x\left(W_{(f+g)}e^{-\frac{i}{s}\Im(f,g)}\right) \\
&= e^{-\frac{i}{2}\Im(f,g)_{L^2}}W_{(L_x^\star f + L_x^\star g)} \\
&= e^{-\frac{i}{2}\Im(L_x^\star f, L_x^\star g)_{L^2}}W_{(L_x^\star f + L_x^\star g)} \\
&= W_{(L^\star f)}W_{(L^\star g)} \\
&= \alpha_x\left(W_{(f)}\right)\alpha_x\left(W_{(g)}\right).
\end{aligned}$$

Also

$$\begin{aligned}
\alpha_x\left(W_{(f)}^\star\right) &= \alpha_x\left(W_{(-f)}\right) \\
&= W_{(-L_x^\star f)} \\
&= \left[\alpha_x\left(W_{(L_x^\star f)}\right)\right]^\star.
\end{aligned}$$

And similar for β_x. □

Restricting $\mathcal{S}$ to $\mathcal{S}_G$ we obtain a subset $\mathfrak{A}_G$ defined as

$$\mathfrak{A}_G = \overline{\operatorname{span}\left\{W_{(f)} \in \mathfrak{A}(\mathcal{S}) \,|\, f \in \mathcal{S}_G\right\}}^{\|\cdot\|_{\mathfrak{A}(\mathcal{S})}}, \tag{A.27}$$

where $\overline{\circ}^{\|\cdot\|_{\mathfrak{A}(\mathcal{S})}}$ denotes the closure in the $\mathfrak{A}(\mathcal{S})$-$C^\star$-algebra norm.

Theorem 12. *$\mathfrak{A}_G$ is a maximal $C^\star$-sub-algebra of $\mathfrak{A}(\mathcal{S})$ that satisfies $\forall A \in \mathfrak{A}_G$, $\beta_{D_h}(A) = A$ for any $h \in G$.*

Proof. $\mathfrak{A}_G$ is spanned by Weyl elements of the form $W_{(f)}$ with $f \in \mathcal{S}_G \subset \mathcal{S}$, hence, $\mathfrak{A}_G \subset \mathfrak{A}(\mathcal{S})$. Since $\mathcal{S}_G$ is closed under addition, and multiplication by real numbers, $\mathfrak{A}_G$ is closed under multiplication and involution,

$$W_{(f)}W_{(g)} = W_{(f+g)}e^{-\frac{i}{2}\Im(f,g)} \in \mathfrak{A}_G,$$
$$W^{\star}_{(f)} = W_{(-f)} \in \mathfrak{A}_G.$$

To show that $\mathfrak{A}_G$ is invariant under β_{D_h} for any $h \in G$ let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathfrak{A}_G$ such that

$$A_n = \sum_{i=0}^{n} c_i W_{(f_i)} \qquad \text{with} \quad c_i \in \mathbb{C}, \quad f_i \in \mathcal{S}_G$$

and that converges to $A \in \mathfrak{A}_G$. Choose $h \in G$. Then by lemma 11 β_{D_h} is a $\star$-automorphism on $\mathfrak{A}(\mathcal{S})$ and the sequence $\left(\beta_{D_h}(A_n)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{A}(\mathcal{S})$ that converges to $\beta_{D_h}(A) \in \mathfrak{A}(\mathcal{S})$. However, if $f_i \in \mathcal{S}_G$ then $\beta_{D_h}\left(W_{(f_i)}\right) = W_{\left(R^{\star}_{D_h} f_i\right)} = W_{(f_i)}$ and the two sequences are identical in $\mathfrak{A}_G$. Thus, the limit points have to be equal and we get, $\beta_{D_h}(A) = A$. The fact that $\mathfrak{A}_G$ is maximal follows from proposition 3.2.1 and the fact that we can decompose, $\mathcal{S} = \mathcal{S}_G + \mathcal{S}_{NG}$ with $\mathcal{S}_G \cap \mathcal{S}_{NG} = \{0\}$. □

Corollary 13. *The Weyl algebra over $\mathcal{S}_G$, denoted $\mathfrak{A}(\mathcal{S}_G)$, is a maximal $C^{\star}$-sub-algebra of $\mathfrak{A}(\mathcal{S})$ that is invariant under β_{D_h} for any $h \in G$.*

Proof. This follows from the fact that $\eta : \mathfrak{A}(\mathcal{S}) \to \mathfrak{A}(\mathcal{S}_G)$ defined on Weyl elements by

$$\eta\left(W_{(f)}\right) = W_{(Pf)}, \tag{A.28}$$

and extended to $\mathfrak{A}(\mathcal{S})$ by linearity is an invertible $\star$-homomorphism from $\mathfrak{A}_G$ to $\mathfrak{A}(\mathcal{S}_G)$. The later is obvious since on $\mathfrak{A}_G$, η acts as an identity. □

A.6 Proof of the equality between the correlation functions

We show that the following equality holds for any $f_i \in \mathcal{S}_{\mathbb{R}}$,

$$\begin{aligned}&\partial_{t_1}\cdots\partial_{t_n}\omega_{\varphi}\left(W_{(t_1 f_1)}\cdots W_{(t_n f_n)}\right)|_{t=0}\\ &= \int dx^1\, f_1\left(x^1\right)\delta_{J\left(x^1\right)}\cdots\int dx^1\, f_n\left(x^n\right)\delta_{J\left(x^n\right)}Z[J]\,|_{J=0},\end{aligned}$$

where $x^i \in M$ for $i \in \{1, \cdots, n\}$.

We proceed by induction: assuming that the assertion holds true for some arbitrary but fixed $n \geqslant 2$ we have

$$\begin{aligned}
&\partial_{t_1} \cdots \partial_{t_{n-1}} \omega_\varphi \left(W_{(t_1 f_1)} \cdots W_{(t_{n-1} f_{n-1})} \right) |_{t=0} \\
&= \int \mathrm{d}x^1 \, f_1 \left(x^1 \right) \delta_{J(x^1)} \cdots \int \mathrm{d}x^{n-1} \, f_{n-1} \left(x^{n-1} \right) \delta_{J(x^{n-1})} Z \left[J \right] |_{J=0},
\end{aligned}$$

We begin with the induction step

$$\left(\imath \partial_{t_n} \right) \imath^{n-1} \partial_{t_1} \cdots \partial_{t_{n-1}} \omega_\varphi \left(W_{(t_1 f_1)} \cdots W_{(t_{n-1} f_{n-1})} W_{(t_n f_n)} \right) |_{t=0}. \tag{A.29}$$

Since $f_i \in \mathcal{S}_\mathbb{R}$ we have by the product of the Weyl algebra

$$W_{(t_1 f_1)} \cdots W_{(t_{n-1} f_{n-1})} W_{(t_n f_n)} = W_{(t_1 f_1 + \cdots + t_n f_n)}. \tag{A.30}$$

By the definition of the state ω_φ we obtain

$$\begin{aligned}
\text{(A.29)} &= \imath^n \partial_{t_n} \partial_{t_1} \cdots \partial_{t_{n-1}} e^{-\frac{\| t_1 f_1 + \cdots + t_n f_n \|_\varphi^2}{4}} e^{\imath \Re(\varphi(t_1 f_1 + \cdots + t_n f_n))} |_{t=0} \\
&= \imath^{n-1} \partial_{t_1} \cdots \partial_{t_{n-1}} \left(e^{-\frac{\| t_1 f_1 + \cdots + t_{n-1} f_{n-1} \|_\varphi^2}{4}} e^{\imath \Re(\varphi(t_1 f_1 + \cdots + t_{n-1} f_{n-1}))} \imath \partial_{t_n} \left(e^{-\frac{(t_n f_n, C(t_1 f_1 + \cdots + t_n f_n))}{2}} e^{\imath \Re(\varphi(t_n f_n))} \right) \right) |_{t=0},
\end{aligned}$$

where in the last equality we used the definition of the norm $\| \cdot \|_\varphi$ and the linearity of $\varphi(\cdot)$ to separate the f_n dependent part. Hence we obtain

$$\begin{aligned}
&\imath^{n-1} \partial_{t_1} \cdots \partial_{t_{n-1}} \left(e^{-\frac{\| t_1 f_1 + \cdots + t_{n-1} f_{n-1} \|_\varphi^2}{4}} e^{\imath \Re(\varphi(t_1 f_1 + \cdots + t_{n-1} f_{n-1}))} \right) \imath \partial_{t_n} \left(e^{-\frac{(t_n f_n, C(t_1 f_1 + \cdots + t_n f_n))}{2}} e^{\imath \Re(\varphi(t_n f_n))} \right) |_{t=0} \\
&+ \imath^{n-1} \sum_j \partial_{t_1} \cdots \hat{\partial}_{t_j} \cdots \partial_{t_{n-1}} \left(e^{-\frac{\| t_1 f_1 + \cdots + t_{n-1} f_{n-1} \|_\varphi^2}{4}} e^{\imath \Re(\varphi(t_1 f_1 + \cdots + t_{n-1} f_{n-1}))} \right) \imath \partial_{t_n} \partial_{t_j} \left(e^{-\frac{(t_n f_n, C(t_1 f_1 + \cdots + t_n f_n))}{2}} e^{\imath \Re(\varphi(t_n f_n))} \right) |_{t=0},
\end{aligned}$$

where $\hat{\partial}_{t_j}$ denotes that the missing derivative with respect to the parameter t_j. By the inductive assumption we have

$$\begin{aligned}
&\int \mathrm{d}x^1 \, f_1 \left(x^1 \right) \delta_{J(x^1)} \cdots \int \mathrm{d}x^{n-1} \, f_{n-1} \left(x^{n-1} \right) \delta_{J(x^{n-1})} Z \left[J \right] |_{J=0} \cdot \Re \left(\varphi \left(f_n \right) \right) \\
&+ \sum_j \int \mathrm{d}x^1 \, f_1 \left(x^1 \right) \delta_{J(x^1)} \cdots \int \mathrm{d}h^j \, f_j \left(x^j \right) \hat{\delta}_{J(x^j)} \cdots \int \mathrm{d}x^{n-1} \, f_{n-1} \left(x^{n-1} \right) \delta_{J(x^{n-1})} Z \left[J \right] |_{J=0} \cdot \frac{\left(f_n, f_j \right)_C}{2}.
\end{aligned}$$

Since $f_n \in \mathcal{S}_\mathbb{R}$ we have $\Re \left(\varphi \left(f_n \right) \right) = \varphi \left(f_n \right) = \int \mathrm{d}x^n \, f \left(x^n \right) \delta_{J(x^n)} e^{\int J \varphi} |_{J=0}$. And similarly

$$\frac{\left(f_n, f_j \right)_C}{2} = \int \mathrm{d}x^n \, f \left(x^n \right) \, \delta_{J(x^n)} \int \mathrm{d}x^j \, f \left(x^j \right) \, \delta_{J(x^j)} e^{\frac{1}{2} \int JCJ} |_{J=0}.$$

Due to this replacements the above equation becomes

$$\int \mathrm{d}x^1 \, f_1 \left(x^1 \right) \delta_{J(x^1)} \cdots \int \mathrm{d}x^n \, f_n \left(x^n \right) \delta_{J(x^n)} Z \left[J \right] |_{J=0}.$$

Equation (3.56) verifies the inductive assumption for $n = 2$, which completes the proof.

4 Symmetry analysis of group field theory

In the last chapter we discussed how to obtain an operator formulation of GFT from the action S — the action for the definition of the covariant formulation. We have seen that the local minima of S play a central role in this construction. Moreover, we have shown that these extrema need to be from the space of tempered micro-functions in order to lead to Fock inequivalent phases of GFT. As we have shown in chapter 2, the action of GFT is multi-local and the corresponding equations of motion involve integrals alongside differential operators, called *integro-differential equations*. Hence, for a perturbative definition of an operator theory for GFT we need to solve integro-differential equations on the space of (tempered) distributions — this, however, is in general a difficult task.

For partial differential equations, symmetries often provide the necessary tools for a characterization of solutions [150] and the symmetry analysis for local theories is well developed. For integro-differential equations the symmetry analysis is understood only for some specific cases [151, 152] but general algorithms to find symmetry groups of integro-differential equations are not known.

On the other hand, one of the open issues of GFT is a characterization of the theory space. Here again, the usual characterization is done in terms of symmetries. It is not clear if the theory space of GFT should be also done in terms of symmetries, or rather in terms of combinatorics or some other principles, but it is worth performing the symmetry analysis, to clarify the situation. For this reason, a symmetry analysis for GFT models is important for conceptual as well as technical reasons [112].

In this chapter we will develop and perform a symmetry analysis for GFT actions using a very basic notion of point symmetries. It will turn out that the definition of point symmetries is too restrictive, but it will also allow us to classify all existing point symmetries of the models.

The discussion of this chapter is purely classical since our goal is the symmetry analysis of a classical action S. To address this problem we will use the geometrical formulations of field theories. We will begin this chapter by reviewing the geometrical formulation of local field theories and then extend it to multi-local actions as used in GFT. In the second part of this chapter we apply the developed framework to specific GFT models and derive their symmetry groups.

4.1 Geometrical construction of local field theories

For the definition of symmetries and the symmetry analysis of GFT models we will use the geometrical formulation of field theory. In this formulation the Lagrangian is defined as a differentiable function on a *jet bundle* and physical degrees of freedom — the fields — are *sections in the underling vector bundle*. A precise definition of vector bundles, jet bundles, sections and other necessary ingredients for classical field theory, is presented in any standard text book on this subject. For this reason we will provide only the basic intuition for the geometric ingredients needed but we refrain from stating a comprehensive and precise definition of all concepts to avoid a creation of copies of known text books. For a rigorous definition of vector bundles and jet bundles we suggest [153, 154] and for their application in field theory we refer to [150, 155].

In the following we will need three ingredients from the differential geometry and the theory of Lie groups: the concept of vector bundles, the concept of jet bundles and the concept of a local group of transformations.

4.1.1 Vector bundles

A vector bundle is a generalization of the direct product between a differentiable manifold and a (finite dimensional) vector space,

$$E = M \times V. \tag{4.1}$$

In fact the simplest vector bundles — called *trivial* — are exactly of this form. The usual nomenclature is as follows: the vector bundle is denoted by the tuple (E, π, M, V) sometimes simply denoted by π, where E is a differential manifold called *the total space*, M is a differential manifold called *the base space*, V is a vector space called *the fiber* and $\pi : E \to M$ is a submersion[1] called the *projection*. In general, a vector bundle can not be written as a direct product, but locally — in a small open neighborhood of each point — every vector bundle is diffeomorphic to a trivial one; the diffeomorphism is called *a local trivialization*, $loc_{\Omega}^{-1} : E \supset \Omega \to U \times V$, where $U = \pi(\Omega)$. Typically, a local trivialization is not unique, and hence, there is no canonical way of decomposing the total space into a Cartesian product. However, the change between

[1] *A differential map π between to differentiable manifolds E and M is called a submersion at point $p \in E$ if its differential at p*

$$D\pi|_p : T_pE \to T_{\pi(p)}M \tag{4.2}$$

is surjective. The map π is called a submersion if it is a submersion at each point of E. In this chapter we use the symbol π as the submersion for the vector bundle and not as the symbol for the representation as we did this in the previous chapter. This should not lead to any confusion, since in this chapter we will not deal with any representations of the algebra and π will exclusively refer to the submersion.

two local trivializations on the same region of E is a group element of the so *called structure group of* π. This allows us to cover the whole E with local trivializations switching from one to the other on the intersections of their domains. Hence, if we restrict our statements to local regions on E we can always work with Cartesian products.

Physical degrees of freedom are described by differentiable functions $\phi : M \to V$ from the base space to the fiber. Going slightly against the usual convention we call these functions *fields*. If the vector bundle is trivial we can define a graph of a field ϕ in E as a set $gr(\phi) = \{(x, \phi(x)) \,|\, x \in M\}$. On a generic vector bundle the notion of a field and its graph, depends on the local trivialization. A *section in the vector bundle* π is a generalization of the concept of a graph, that is independent of the local trivialization. Locally, however, a (smooth) section can always be represented as a graph of some (smooth) field ϕ and conversely, given a local trivialization, every (smooth) field ϕ defines a (smooth) section by its graph $gr(\phi)$. In the following we will consider only smooth sections and therefore drop this specification. We will denote the space of all sections of a vector bundle π by $\Gamma(\pi)$ and use the same notation for the space of smooth fields assuming a local trivialization. For the space of local sections and fields — those that are defined only in a local region on M — we use the notation $\Gamma_{loc}(\pi)$.

Choosing coordinates on M, a basis on V and suitably adjusting the domains of the chart and the local trivialization we can define coordinates on E such that

$$E \ni p \overset{loc_p^{-1}}{\mapsto} (x, u) \overset{\text{chart}}{\mapsto} \left(x^\mu, u^i\right) \in \mathbb{R}^m \times \mathbb{R}^n, \tag{4.3}$$

where $m = \dim(M)$ and $n = \dim(V)$. As in the previous chapter, we will reserve the letters x and y for the elements of the base space. However, the same characters with superscript will refer to coordinates on M. Similarly with the fiber components for which we use the letter $u \in V$ and after choosing a coordinate basis denote them as u^i. Due to local trivialization, when working with vector bundles we can think about the Cartesian products of local regions in $M \times V$ or equivalently in local charts on $\mathbb{R}^m \times \mathbb{R}^n$. This is what we will do in the rest of this chapter.

4.1.2 Jet bundles

A vector bundle, π, provides an appropriate space for the definition of fields in a geometrical way as section of π. The next step is the construction of the appropriate geometrical space for the description of derivatives of functions. This space is called *the (first) jet bundle*.

A (first) jet bundle is a generalization of the tangent space of a manifold. It is a vector bundle, and hence consists of the total space that we denote JE, the base manifold M, the fiber J and the projection $\pi_J : JE \to M$. Locally, it takes a structure of the Cartesian product

$$JE = M \times V \times J \tag{4.4}$$

where J is the fiber of the jet bundle called the *(first) jet space*.

The jet space is an equivalence class of local sections on E, just as a tangent space is an equivalence space of local curves on the manifold. Elements of the jet space, can be understood as differentials of local fields $\phi : M \to V$; that is at some point $x \in M$, $D\phi|_x$ is an element of the jet space at x. Locally, we can therefore write an element of the jet bundle as

$$(x, \phi(x), D\phi|_x), \tag{4.5}$$

and if we do not want to refer to sections we write $(x, u, u_x) \in JE$, where u_x denote the points of J. Generalization to higher order differentials define then higher order jet spaces, and corresponding higher order jet bundles. In the following we will use only first jet bundles and refer to them simply as jet bundles.

Local coordinates of the vector bundle π can be extended to local coordinates on the jet bundle π_J such that

$$JE \ni j \overset{loc_j^{-1}}{\mapsto} (x, u, u_x) \overset{chart}{\mapsto} \left(x^\mu, u^i, u^i_\mu\right) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m \cdot n}. \tag{4.6}$$

These coordinates are constructed in the following way: the coordinates on M induce a coordinate basis on TM, whereas coordinates on V define a basis in the fiber, hence the linear map, $D\phi|_p : T_pM \to V$, becomes a matrix with components u^i_μ. Hence, in a chart the jet space can be understood as the space of matrices $Mat\,(n \times m, \mathbb{R})$.

The jet bundle allows us to define the Lagrangian as a smooth, real valued function,

$$L : JE \to \mathbb{R}. \tag{4.7}$$

It is a function on a finite dimensional space of points of the form $\left(x^\mu, u^i, u^i_\mu\right)$, but in this geometrical formulation it is not a function on $\Gamma(\pi)$, the infinite dimensional space of fields. This makes it possible to use the usual differential calculus for variation of the Lagrangian. In the following we will distinguish between four different derivatives:

PARTIAL DERIVATIVE: is a derivative of the Lagrangian with respect to the base space. In a chart $\left(x^\mu, u^i, u^i_\mu\right)$ this derivative is given by the derivative with respect to the coordinates x^μ. We write for $I \subset \mathbb{R}$ an interval containing zero, $t \in I$, and $\{e^\mu\}$ the canonical basis in $\mathbb{R}^m$,

$$\partial_\nu L|_{(x,u,u_x)} \doteq \frac{d}{dt} L\left(x^\mu + te^\nu, u^i, u^i_\mu\right)|_{t=0}. \tag{4.8}$$

FIBER DERIVATIVE: is a derivative of the Lagrangian with respect to the fiber coordinate. In a chart this derivative is given by the partial derivative with respect to the coordinates u^i. For $I \subset \mathbb{R}$ an interval containing zero, $t \in I$, and $\{e^j\}$ the canonical basis in $\mathbb{R}^n$,

$$\partial_{u^j} L|_{(x,u,u_x)} \doteq \frac{d}{dt} L\left(x^\mu, u^i + te^j, u^i_\mu\right)|_{t=0}. \tag{4.9}$$

JET DERIVATIVE: is a derivative of the Lagrangian with respect to the jet space. In a chart this derivative is given by the partial derivate with respect to the coordinate u^i_μ. For $I \subset \mathbb{R}$ an interval containing zero, $t \in I$, $\{e^\mu\}$ the canonical basis in $\mathbb{R}^m$ and $\{e^i\}$ the canonical basis in $\mathbb{R}^n$,

$$\partial_{u^j_\nu} L_{(x,u,u_x)} \doteq \frac{d}{dt} L\left(x^\mu, u^i, u^i_\mu + t\left(e^\nu \otimes e^j\right)\right)|_{t=0}. \tag{4.10}$$

TOTAL DERIVATIVE: is a derivative of a function on JE, in the case when the points of the jet bundle are given by some smooth section $gr(\phi) = (x^\mu, \phi(x^\mu), D\phi|_{x^\mu})$ and we have to take into account the dependence of ϕ on x^μ. We write

$$D_\mu L|_{(x,u,u_x)} \doteq \partial_\mu L + \sum_{i=1}^{n} \partial_{u^i} L \cdot \partial_{x^\mu}\phi^i + \sum_{i=1}^{n} \partial_{u^i_\mu} L \cdot \partial_{x^\mu} D\phi^i, \tag{4.11}$$

where the right hand side is evaluated at the point (x, u, u_x).

The last necessary piece in the formulation of the field theory is the action. An action S_Ω is a functional on $\Gamma(\pi)$ given by an integral over a local region $\Omega \subset M$ of a Lagrangian such that in coordinates we get

$$S[\phi] = \int_\Omega \mathrm{d}x^\mu \sqrt{g}\, L\left(x^\mu, u^i, u^i_\mu\right), \tag{4.12}$$

where g refers to the determinant of the metric in the coordinate chart x^μ and the points u and u_x are implicitly assumed to be given by the smooth section $gr(\phi) = (x, \phi(x), D\phi|_x)$.

4.1.3 Continuous symmetry and the local group of transformations

In order to discuss the notion of symmetries of the action S_Ω in the geometrical langue we need to introduce the *local group of transformations* [150].

A local group of transformations is a Lie group G_T that acts on the vector bundle π. That means, for $g \in G_T$ and any $x \in M$, $u \in V$ we get new points $\tilde{x} \in M$ and $\tilde{u} \in V$ such that

$$g \cdot (x, u) = (\tilde{x}, \tilde{u}) = (C(x, u), Q(x, u)). \tag{4.13}$$

where the functions $C : E \to M$ and $Q : E \to V$ specify the group action: C prescribes the new point of the base manifold and Q gives the new point of the fiber. Both functions are not invertible in general, but if g is sufficiently close to the identity the inverse of $C \circ (\mathbb{1} \times \phi)$ exists due to the inverse function theorem [150].

If the point in the bundle is given by a (local) section $(x, \phi(x))$ and g is close to identity, $\mathbb{1}$, the transformed point $(\tilde{x}, \tilde{u})$ will be also given by some section such that $(\tilde{x}, \tilde{u}) = \left(\tilde{x}, \tilde{\phi}(\tilde{x})\right)$. It is a well known result [150] that the transformed field $\tilde{\phi}$ is given by

$$\tilde{\phi}(\tilde{x}) = Q \circ (\mathbb{1} \times \phi) \circ [C \circ (\mathbb{1} \times \phi)]^{-1}(\tilde{x}), \tag{4.14}$$

The transformation of ϕ under the group G_T induces a transformation of $D\phi$, which can be calculated in a straight forward way and the action of the group G_T *prolongs* to the jet bundle, such that

$$g \cdot (x, u, u_x) = g \cdot (x, \phi(x), D\phi|_x) = \left(\tilde{x}, \tilde{\phi}(\tilde{x}), D\tilde{\phi}|_{\tilde{x}}\right) = (\tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}). \tag{4.15}$$

The algebra of G_T is associated with vector fields on π, such that for each g close to $\mathbb{1}$ there exists a vector field W on TE and $\epsilon \in \mathbb{R}$ sufficiently close to zero such that $g = \exp(\epsilon W) \doteq g_\epsilon$. The vector field W can be split in its components tangent to the base manifold and parallel to the fiber as follows,

$$W = X_M(x, u)\,\partial_x + X_V(x, u)\,\partial_u, \tag{4.16}$$

and we refer to X_M as the generator of C and X_V as the generator of Q — even though in the strict sense neither C nor Q are elements of G_T.

A symmetry of the action S_Ω is a local group of transformations such that for any $U \subset \Omega$ and any $\phi \in \Gamma(\pi)$.

$$\begin{aligned} S_U[\phi] &= \int_U \mathrm{d}x^\mu \sqrt{g}\, L\left(x^\mu, u^i, u^i_\mu\right) \\ &= \int_{\tilde{U}} \mathrm{d}\tilde{x}^\mu \sqrt{\tilde{g}}\, L\left(\tilde{x}^\mu, \tilde{u}^i, \tilde{u}^i_\mu\right) = \tilde{S}_{\tilde{U}}\left[\tilde{\phi}\right], \end{aligned} \tag{4.17}$$

where, $\tilde{U} = [C \circ (\mathbb{1} \times \phi)](U)$, is the transformed domain of integration, that in general depends not only on $g \in G_T$ but also on the whole field configuration ϕ in U. These transformations are called *Lie point symmetries* or *"geometrical"* symmetries, because they admit a geometrical interpretation of a flow, being generated by vector fields of the vector bundle π.

The requirement that the symmetry does not change the action for any sub-domain of Ω is essential in order to derive a point-wise criteria for the symmetry. Such point-wise statements are much easier to treat than equation (4.17) that involves integrals.

It is a well known result, based on properties of Lie groups, that C and Q define a symmetry of the action if and only if the infinitesimal generators X_M and X_V satisfy the following equation

$$\sum_{\mu=1}^{m}\sum_{i=1}^{n} D_\mu \partial_{u^i_\mu} L \cdot X^i_Q + \sum_{\mu=1}^{m} D_\mu\left(L X^\mu_M\right) + \sum_{i=1}^{n} E^i_L \cdot X^i_Q = 0 \quad . \tag{4.18}$$

Where X_Q is the characteristic of the symmetry defined in local coordinates as $X_Q^i(x,u) = X_V^i(x,u) - \sum_{\mu=1}^{m} X_M^{\mu}(x,u)\,\partial_\mu \phi^i(x)$, and the equations of motion for the Lagrangian L are denoted by E_L^i. For better readability we summarize equation (4.18) as

$$\mathrm{Div}\left(\partial_u L \cdot X_Q + L X_M\right) + E_L\left(X_Q\right) = 0. \tag{4.19}$$

The vector fields X_M and X_V generate a symmetry of the action S if and only if they satisfy the above equation [150]. Due to the "only if" part of this relation, equation (4.19) can be used as a defining equation for the symmetry group. We will show explicitly how this is used in the symmetry analysis in the second part of this chapter.

When the equations of motion are satisfied, $E_L^i = 0$ for all $i \in \{1, \cdots, n\}$, the equation (4.19) defines a conservation law,

$$\mathrm{Div}\left(\partial_u L \cdot X_Q + L X_M\right) = 0.$$

Indeed, equation (4.19) is a special case of a more general Noether theorem that relates the symmetries of the action and conservation laws [156].

4.2 *Multi-local action and its symmetry group*

We now turn to the multi-local case and modify the above formalism, to apply it in the context of GFT, or more generally, a theory with multi-local interactions. We begin by formulating GFT in the above language.

4.2.1 *Vector bundle for GFT*

The vector bundle of GFT is a trivial bundle (E, π, M, V) with the base space $M = G^{\times d}$, for a Lie group G of dimension $r = \dim(G)$. As before we will denote the elements of the base space by x and by slight abuse of notation denote the coordinates by x^μ, where $\mu \in \{1, \cdots, m\}$, $m = \dim(M) = r \cdot d$. The components of x will be denoted by subscripts such that

$$M \ni x = (x_1, \cdots, x_d). \tag{4.20}$$

with $x_j \in G$ for $j \in \{1, \cdots, d\}$. If we need to refer to the coordinates of a single component we denote them by x_j^α, where $\alpha \in \{1, \cdots, r\}$. The measure $\mathrm{d}x$ will refer to the Haar measure on M. In coordinates the Haar measure will be denoted $\mathrm{d}x^\alpha \sqrt{g}$, with the metric determinant g in the chart x^α and where the metric is given by the Killing form. Notably, since the base space is a Cartesian product of differential manifolds each of which can be equipped with the metric, we have the following relation between the Haar measure on M and the Haar measure on G,

$$\mathrm{d}x = \mathrm{d}x^\alpha \sqrt{g} = \prod_{j=1}^{d} \mathrm{d}x_j^\alpha \sqrt{g_j} = \prod_{j=1}^{d} \mathrm{d}x_j, \tag{4.21}$$

where g_j refers to the determinant of the metric on the jth component of M in the chart x_j^α and $\mathrm{d}x_j$ is the Haar measure on it.

In the following we will chose the fiber to be either, $V = \mathbb{R}^{\times c}$, or $V = \mathbb{C}^{\times c}$ with some $c \in \mathbb{N}$ and denote the field components by ϕ^i for $i \in \{1, \cdots c\}$. In the GFT literature the components of the field are called *colors*, and when we define a model with $c > 1$ such that the interaction of the model is linear in each component we speak about a *colored GFT model*.

To save space in rather lengthy equations we will use a short notation for a field at a point x such that

$$\phi_{1,2,\cdots,d} \doteq \phi(x_1, \cdots, x_d) = \phi(x). \tag{4.22}$$

We will denote the gradient on M by ∇. In local coordinates the gradient for a smooth function f on M is given by

$$\nabla f \doteq \sum_{\mu,\nu=1}^{m} \partial_\mu f\, g^{\mu\nu}\, \partial_\nu. \tag{4.23}$$

The divergence on M is denoted div and defined on a tangent vector $X \in TM$ in local coordinates as

$$\mathrm{div}(X)(x) = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, X^\mu\right)|_x. \tag{4.24}$$

If X is a tangent vector and L is a function on the jet bundle, we will distinguish between div and Div such that

$$\mathrm{div}(L \cdot X) = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, L \cdot X^\mu\right), \tag{4.25}$$

$$\mathrm{Div}(L \cdot X) = \frac{1}{\sqrt{g}} D_\mu \left(\sqrt{g}\, L \cdot X^\mu\right), \tag{4.26}$$

where div refers to the partial derivatives and Div implies the total derivative on L. The Laplace-Beltrami operator on M is defined as $\Delta = \mathrm{Div} \circ \nabla$.

For better presentation we will discuss the general procedure on a specific example, but it should be clear that the construction can be applied to a large class of multi-local actions. The action we choose for our discussion is the Boulatov action augmented by a Laplacian term that we introduced in chapter 2, that we rewrite such that the combinatorics of the model becomes explicit,

$$\begin{aligned} S_M[\phi] = &\int_M \mathrm{d}x\, \phi(x) \cdot (-\Delta + m)\, \phi(x) \\ &+ \lambda \int_{M^{\times 4}} \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}w\, \phi(x)\ \phi(y)\ \mathrm{tet}(x,y,z,w)\ \phi(z)\ \phi(w), \end{aligned} \tag{4.27}$$

where

$$\text{tet}(x,y,z,w) \doteq \delta\left(x_1 y_1^{-1}\right)\delta\left(x_2 z_2^{-1}\right)\delta\left(x_3 w_3^{-1}\right) \times \delta\left(y_2 w_2^{-1}\right)\delta\left(y_3 z_3^{-1}\right)\delta\left(z_1 w_1^{-1}\right), \tag{4.28}$$

denotes the combinatorics of the tetrahedron. In the following we restrict our discussion to real valued scalar fields. The equation of motion for the Boulatov action reads then

$$(-\Delta + m)\,\phi(x) + 4\lambda \int_{M^{\times 3}} \mathrm{d}y\mathrm{d}z\mathrm{d}w \quad \text{tet}(x,y,z,w)\,\phi(y)\,\phi(z)\,\phi(w) = 0. \tag{4.29}$$

As we already pointed out this equation is an integro-differential equation, because the second term will contain integrals even after the explicit use of the delta distributions. Because of this, the action can not be written as an integral over a Lagrangian defined over some jet bundle. Nevertheless, we can split the action in two separate parts as

$$S_M[\phi] = S_M^0[\phi] + S_{M^{\times 4}}^1[\phi]. \tag{4.30}$$

The first part is the local action, induced by the local Lagrangian

$$S_M^0[\phi] = \int_M \mathrm{d}x\,\phi(x)\,(-\Delta + m)\,\phi(x),$$

or after integration by parts and in coordinates

$$S_M^0[\phi] = \int_M \mathrm{d}x^\mu\,\sqrt{g}\,L^0\left(x^\mu, u, u_\mu\right) \tag{4.31}$$

$$L^0\left(x^\mu, u, u_\mu\right) = \sum_{\mu=1}^{m} u_\mu u_\mu + m\,uu, \tag{4.32}$$

where the Lagrangian is defined on the first jet bundle over the vector bundle $\pi : M \times \mathbb{R} \to M$.

The second part of eq.(4.30) is given by

$$S_{M^{\times 4}}^1[\phi] = \lambda \int_{M^{\times 4}} \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}w\;\phi(x)\,\phi(y)\,\phi(z)\,\phi(w)\;\text{tet}(x,y,z,w),$$

and as such can not be written as an integral of a differentiable function defined on the vector bundle π. However, we can construct a new vector bundle with the base space $\mathcal{M}$ such that the action becomes of the form

$$S_{\mathcal{M}}^1[\phi] = \int_{\mathcal{M}} \mathrm{d}\mathcal{X}^\gamma\,L^1\left(\mathcal{X}^\gamma, \mathcal{U}^\delta\right) \quad L^1\left(\mathcal{X}^\gamma, \mathcal{U}^\delta\right) = \mathcal{U}^1\mathcal{U}^2\mathcal{U}^3\mathcal{U}^4, \tag{4.33}$$

where $\mathcal{X}_j^\mu$ are coordinates of $\mathcal{M}$ and $\mathcal{U}_j^i$ are coordinates on the suitable fiber. In the following section we detail the construction of the suitable vector bundle for the multi-local interaction.

4.2.2 *Vector bundle for the multi-local interaction*

We begin with the vector bundle of the local part, π, and define the 4 fold direct product of π, such that the total space is $E^{\times 4}$, the base space is $M^{\times 4}$, the fiber is $V^{\times 4}$ and the projection is $\pi^4 = \pi \times \pi \times \pi \times \pi$. This product $\left(E^4, \pi^4, M^4, V^4\right)$ is again a vector bundle [154] that we denote π^4. On M^4 we have canonical projections $pr_j : M^4 \to M$ for $j \in \{1, 2, 3, 4\}$ such that

$$pr_1(x, y, z, w) = x \qquad pr_2(x, y, z, w) = y$$
$$pr_3(x, y, z, w) = z \qquad pr_4(x, y, z, w) = w.$$

Sections in π^4 give smooth fields $\varphi : M^4 \to \mathbb{R}^4$, with components $\varphi^i : M \to \mathbb{R}$ for $i \in \{1, 2, 3, 4\}$.

Next we define a map f_{tet} from $G^{\times 6} \to M^4$ as follows,

$$f_{\text{tet}} : (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto ((x_1, x_2, x_3), (x_1, x_4, x_5), (x_6, x_2, x_5), (x_6, x_4, x_3)) \tag{4.34}$$

This map encodes the combinatorics of the tetrahedron and is a combination of diagonal maps, $(x) \mapsto (x, x)$, and permutation maps $(x_1, x_2) \mapsto (x_2, x_1)$ and hence is differentiable. More precisely, f_{tet} is an embedding from $G^{\times 6}$ in M^4. Therefore the pull back of π^4 by f_{tet} is again a vector bundle [154] — *the pull back bundle of* π^4 that we denote $\left(\mathcal{E}, f^\star_{\text{tet}}\pi^4, \mathcal{M}\right)$ or simply $f^\star_{\text{tet}}\pi^4$, with the total space $\mathcal{E}$ and the base space $\mathcal{M} = G^{\times 6}$.

Smooth sections of π^4 are mapped to smooth section of $f^\star_{\text{tet}}\pi^4$ by the pull back with f_{tet}, such that for a smooth field $\varphi : M^4 \to \mathbb{R}^4$, its pull back $\psi : \mathcal{M} \to \mathbb{R}^4$ is given by

$$\psi(x_1, x_2, x_3, x_4, x_5, x_6) = (\varphi \circ f)(x_1, x_2, x_3, x_4, x_5, x_6). \tag{4.35}$$

In local trivialization we denote the points of the product bundle $f^\star_{\text{tet}}\pi^4$ by $(\mathcal{X}, \mathcal{U})$ and in a chart we write $\left(\mathcal{X}^\alpha_i, \mathcal{U}_j\right)$, where $\alpha \in \{1, \cdots, r\}$, $j \in \{1, \cdots, 4\}$ and $i \in \{1, \cdots, 6\}$. Since the base manifold of $f^\star_{\text{tet}}\pi^4$ is a direct product of the group G we get

$$\mathcal{X} = (x_1, \cdots, x_6). \tag{4.36}$$

For a fixed α and j we get in local coordinates

$$\mathcal{X}^\alpha_j : \mathcal{M} \to \mathbb{R} \qquad (x_1, \cdots, x_6) \mapsto x^\alpha_j, \tag{4.37}$$

where x^α_j is the α's chart component of the jth group G in $\mathcal{M}$.

We define the Lagrangian for the interaction of the Boulatov model in local coordinates as

$$L^1\left(\mathcal{X}^\mu_j, \mathcal{U}^i_j\right) = \mathcal{U}_1\mathcal{U}_2\mathcal{U}_3\mathcal{U}_4. \tag{4.38}$$

Again, the Lagrangian is a function on the pull back vector bundle, and not on smooth fields. Moreover, sections in $f_{\text{tet}}^{\star}\pi^4$ are not always given by physical field degrees of freedom, which are sections in π. To ensure a connection between sections in $f_{\text{tet}}^{\star}\pi^4$ and sections in π we choose a subset, $\Gamma_D\left(f_{\text{tet}}^{\star}\pi^4\right)$, of smooth sections on $f_{\text{tet}}^{\star}\pi^4$ such that

$$\Gamma_D\left(f_{\text{tet}}^{\star}\pi^4\right) = \left\{\psi_\phi \in \Gamma\left(f_{\text{tet}}^{\star}\pi^4\right) | \exists\phi \in \Gamma(\pi) \,:\, \psi_\phi = (\phi,\phi,\phi,\phi)\circ f_{\text{tet}}\right\} \tag{4.39}$$

It is a closed subspace of the space of all sections on $f_{\text{tet}}^{\star}\pi^4$ that we call the space of *diagonal sections*.

If $\mathcal{U}$ is given by a diagonal section in $f_{\text{tet}}^{\star}\pi^4$ we obtain for the Lagrangian,

$$\begin{aligned} L^1(\mathcal{X},\mathcal{U}) &= \prod_{j=1}^{4}\left[\left(\psi_\phi\circ f_{\text{tet}}\right)(\mathcal{X})\right]^j \\ &= \phi(x_1,x_2,x_3)\,\phi(x_1,x_4,x_5)\,\phi(x_6,x_2,x_5)\,\phi(x_6,x_4,x_3)\,. \end{aligned}$$

This is precisely the interaction term of our multi-local action where the delta distributions of tet have been integrated out. Hence, on the pull back bundle $f_{\text{tet}}^{\star}\pi^4$ the action $S^1_{M\times 4}$ is given by an integral over a local Lagrangian and we can perform its symmetry analysis by following the standard procedure from local theories. In summary: we can view the multi-local action S_M as a sum of local actions S^0 and S^1 on which we can separately use the local construction for the symmetry analysis.

In order to perform symmetry analysis of S_M we need first to define the local group of transformations on $f_{\text{tet}}^{\star}\pi^4$ in a consistent way.

4.2.3 *The local group of transformations for multi-local actions*

The function f_{tet} encodes the combinatorial structure of the interaction and provides a relation between different vector bundles. We need to take this relation into account when we discuss the local group of transformations, to make sure that $g \in G_T$ applies the same transformation on both bundles.

A local group of transformations on $f_{\text{tet}}^{\star}\pi^4$ is a (local) diffeomorphism that acts in local trivialization as

$$g\cdot(\mathcal{X},\mathcal{U}) = \left(\tilde{\mathcal{X}},\tilde{\mathcal{U}}\right) = \left(\mathcal{C}(\mathcal{X},\mathcal{U}),\mathcal{Q}(\mathcal{X},\mathcal{U})\right). \tag{4.40}$$

for some $\tilde{\mathcal{X}} \in \mathcal{M}$ and $\tilde{\mathcal{U}} \in \mathcal{V}$. However, since $f_{\text{tet}}^{\star}\pi^4$ is related to π we need to ensure that $\mathcal{C}$ and $\mathcal{Q}$ on $f_{\text{tet}}^{\star}\pi^4$ are consistent with the transformations C and Q on π. To do this we examine the action of G_T on sections in $f_{\text{tet}}^{\star}\pi^4$.

Let $gr(\phi) \in \Gamma_{loc}(\pi)$ be a (local) section in π with the corresponding smooth field ϕ. And let $gr\left(\psi_\phi\right) \in \Gamma_D\left(f_{\text{tet}}^{\star}\pi^4\right)$ be a diagonal section in

$f_{\text{tet}}^{\star}\pi^4$ such that the smooth field ψ_ϕ is related to ϕ as

$$\psi_\phi = \phi^{\times 4} \circ f. \tag{4.41}$$

Let $(\mathcal{X},\mathcal{U})$ be given by a section $gr\left(\psi_\phi\right)$, that is $\mathcal{U} = \psi_\phi\left(\mathcal{X}\right)$. By definition of the pull back bundle we can write

$$\mathcal{U} = (\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3,\mathcal{U}_4) = \left(\phi^{\times 4} \circ f\right)(\mathcal{X}). \tag{4.42}$$

For better presentation let us assume that the image of $f\left(\mathcal{X}\right)$ is given by $(x,y,z,w) \in M^{\times 4}$, then $\mathcal{U} = (\phi\left(x\right), \phi\left(y\right), \phi\left(z\right), \phi\left(w\right))$. Since we know the action of G_T on π we know that the points of M will transform under $g \in G_T$ to $\tilde{x} = C\left(x,u^1\right), \tilde{y} = C\left(y,u^2\right), \tilde{z} = C\left(z,u^3\right), \tilde{w} = C\left(w,u^4\right)$. Hence for consistency we need to require that

$$f\left(\mathcal{C}\left(\mathcal{X},\mathcal{U}\right)\right) = \left(C\left(x,u^1\right), C\left(y,u^2\right), C\left(z,u^3\right), C\left(w,u^4\right)\right), \tag{4.43}$$

or more generally

$$f \circ \mathcal{C} \circ \left(\mathbb{1} \times \psi_\phi\right) = \left[C \circ (\mathbb{1} \times \phi)\right]^{\times 4} \circ f. \tag{4.44}$$

Similarly, due to the action of G_T on π we know that the fiber points transform to $\tilde{u} = Q\left(x,u\right)$ and, hence, we need to require that

$$\mathcal{Q} \circ \left(\mathbb{1} \times \psi_\phi\right) = \left[Q \circ (\mathbb{1} \times \phi)\right]^{-1} \circ f. \tag{4.45}$$

From these transformations it follows with the same arguments as in the local case that the fields transform as

$$\psi_\phi \mapsto \left\{\left[Q \circ (\mathbb{1} \times \phi)\right] \circ \left[C \circ (\mathbb{1} \times \phi)\right]^{-1}\right\}^{\times 4} \circ f, \tag{4.46}$$

The above equations provide a relation between the action of the local transformation group G_T on different vector bundles. However, equation (4.44) does not always lead to a well defined $\mathcal{C}$ on $\mathcal{M}$. If $\mathcal{C}$ were well defined, equation (4.44) would imply that, once $\phi \in \Gamma\left(\pi\right)$ is fixed, for any $\mathcal{X} \in \mathcal{M}$, there exists a $\mathcal{Y} \in \mathcal{M}$ such that

$$f\left(\mathcal{Y}\right) = \left[C \circ (\mathbb{1} \times \phi)\right] \circ f\left(\mathcal{X}\right). \tag{4.47}$$

This however, is not always possible. As an example consider a transformation

$$C\left(x_1,x_2,x_3\right) = \left(x_1 \cdot x_2, x_2, x_3\right), \tag{4.48}$$

then

$$\begin{aligned}\left(C^{\times 4} \circ f\right)\left(x_1 \cdots x_6\right) &= C\left(x_1,x_2,x_3\right) C\left(x_3,x_4,x_5\right) C\left(x_5,x_2,x_6\right) C\left(x_6,x_2,x_1\right), \\ &= \left(\left(x_1 \cdot x_2, x_2, x_3\right)\left(x_3 \cdot x_4, x_4, x_5\right)\right. \\ &\quad \left.\left(x_5 \cdot x_2, x_2, x_6\right)\left(x_6 \cdot x_2, x_2, x_1\right)\right)\end{aligned} \tag{4.49}$$

and if $x_4 \neq \mathbb{1}$ the above point is not in the image of f. For that reason the transformation C is not consistent with the relation between the bundles and can not define a local group of transformations for the Boulatov action. We call a local group of transformations *admissible with* f or simply admissible if it satisfies the conditions in eq. (4.44) and eq. (4.45).

4.2.4 *Symmetries of non-local actions*

We define a symmetry group of the multi-local action S_M as an admissible local transformation group G_T such that for each $g \in G_T$ and each field $\phi \in \Gamma(\pi)$ the following relations hold

$$S^0_\Omega[\phi] = \tilde{S}^0_{\tilde{\Omega}}[\tilde{\phi}] \; \forall \Omega \subset M, \quad S^1_\Omega[\psi_\phi] = \tilde{S}^1_{\tilde{\Omega}}\left[\psi_{\tilde{\phi}}\right] \; \forall \Omega \subset \mathcal{M}, \tag{4.50}$$

where $\tilde{\phi}$ is the transformed field and $\tilde{\Omega}$ is the transformed domain. In other words G_T is a symmetry of the multi-local action if it is a symmetry for each individual part of the action.

Since each of the action parts is local on its own jet bundle we can apply the point-wise symmetry condition (4.18). If we call X_M and X_V the generators of C and Q and denote the generators of $\mathcal{C}$ and $\mathcal{Q}$, by $X_{\mathcal{M}}$ and $X_{\mathcal{V}}$ respectively we can state: G_T is a symmetry group of S_M if and only if

$$\mathrm{Div}_M\left(\partial_u L^0 \cdot X_Q + L^0 X_M\right)(x) + E_{L^0}\left(X_Q\right)(x) = 0, \tag{4.51}$$

$$\mathrm{Div}_{\mathcal{M}}\left(\partial_U L^1 \cdot X_{\mathcal{Q}} + L^1 X_{\mathcal{M}}\right)(\mathcal{X}) + E_{L^1}\left(X_{\mathcal{Q}}\right)(\mathcal{X}) = 0, \tag{4.52}$$

where Div_M refers to the divergence on the base space M and $\mathrm{Div}_{\mathcal{M}}$ denote the divergence on the base space $\mathcal{M}$. It is important to stress that, E_{L^1}, is the equation of motion for the action L^1 treated as a local Lagrangian. Hence the operator E_{L^1} does not involve integrals. Since the sections in the pull back bundle are valued in $\mathbb{R}^4$ we have four equations of motion, one for each of the components of the field. On the diagonal sections these equations read

$$E^1_{L^1}(\mathcal{X}) = (1 \times \phi \times \phi \times \phi)(f(\mathcal{X})) \tag{4.53}$$

$$E^2_{L^1}(\mathcal{X}) = (\phi \times 1 \times \phi \times \phi)(f(\mathcal{X})) \tag{4.54}$$

$$E^3_{L^1}(\mathcal{X}) = (\phi \times \phi \times 1 \times \phi)(f(\mathcal{X})) \tag{4.55}$$

$$E^4_{L^1}(\mathcal{X}) = (\phi \times \phi \times \phi \times 1)(f(\mathcal{X})) \tag{4.56}$$

Where $1 \in \Gamma(\pi)$ is a constant one-function on M. These equations are the equations of motion for the action S^1 if the degree of freedom is the field ψ, however, they do not correspond to the physical equation of motion for the physical degree of freedom ϕ, eq. (4.29). The later, however, can be written in terms of E_{L^1}. For $x \in M$, the equation of motion for the field ϕ reads

$$\begin{aligned} E_{L^0}(x) &+ \int \mathrm{d}y\, E^1_{L^1}(x_1, x_2, x_3, y_1, y_2, y_2) \\ &+ \int \mathrm{d}y\, E^2_{L^1}(x_1, y_1, y_2, x_2, x_3, y_3) \\ &+ \int \mathrm{d}y\, E^3_{L^1}(y_1, x_2, y_2, y_2, x_3, x_1) \\ &+ \int \mathrm{d}y\, E^4_{L^1}(y_1, y_2, x_3, x_2, y_3, x_5) = 0. \end{aligned} \tag{4.57}$$

We denote this special class of points in $\mathcal{M}$ by

$$\begin{aligned} \mathcal{P}_1 &= (x_1, x_2, x_3, y_1, y_2, y_2) \\ \mathcal{P}_2 &= (x_1, y_1, y_2, x_2, x_3, y_3) \\ \mathcal{P}_3 &= (y_1, x_2, y_2, y_2, x_3, x_1) \\ \mathcal{P}_4 &= (y_1, y_2, x_3, x_2, y_3, x_5)\,, \end{aligned}$$

and summarize the above equation as

$$E_{L^0}(x) + \int \mathrm{d}y \sum_{j=1}^{4} E^j_{L^1}(P_j) = 0 \tag{4.58}$$

In local field theories, symmetries always correspond to conservation laws by the Noether theorem. To investigate a similar statement in the multi-local case we use these equation of motion in eq. (4.51), evaluate eq. (4.52) at $\mathcal{P}_1$ (or any other of $\mathcal{P}_j$) and integrate it over dy, subsequently summing both equation to obtain

$$\begin{aligned} &\mathrm{Div}_M\left(\partial_u L^0 \cdot X_Q + L^0 X_M\right)(x,u) \\ &+ \int \mathrm{d}y\, \mathrm{Div}_{\mathcal{M}}\left(\partial_{\mathcal{U}} L^1 \cdot X_{\mathcal{Q}} + L^1 X_{\mathcal{M}}\right)(\mathcal{P}_1, \mathcal{U}) \\ &+ \int \mathrm{d}y \sum_{j=1}^{4}\left(E^j_{L^1}(\mathcal{P}_1)\cdot X^j_{\mathcal{Q}}(\mathcal{P}_1,\mathcal{U}) - E^j_{L^1}(P_j)\cdot X_Q(x,u)\right) = 0. \end{aligned} \tag{4.59}$$

The last line does not vanish in general and the conservation law, $\mathrm{Div}(J) = 0$, is not satisfied. As the result a symmetry of a multi-local action does not lead to a conservation law, unlike in the local case. Defining $\Delta(x) = \int \mathrm{d}y \sum_{j=1}^{4}\left(E^j_{L^1}(\mathcal{P}_j)\cdot X_Q(x,u) - E^j_{L^1}(\mathcal{P}_1)\cdot X^j_{\mathcal{Q}}(\mathcal{P}_1,\mathcal{U})\right)$ we can summarize the non-conserved equations a la Noether as,

$$\begin{aligned} &\Delta(x) - EL\left[X_Q\right] \\ &= \mathrm{Div}_M\left(\partial_u L^0 \cdot X_Q + L^0 X_M\right)(x,u) \\ &\quad + \int \mathrm{d}y\, \mathrm{Div}_{\mathcal{M}}\left(\partial_{\mathcal{U}} L^1 \cdot X_{\mathcal{Q}} + L^1 X_{\mathcal{M}}\right)(\mathcal{P}_1,\mathcal{U}) \end{aligned} \tag{4.60}$$

For the local theory we would have, $E^j X^j_Q(\mathcal{P}_j, \mathcal{U}) = E\, X_Q(x,u)$, and hence,$\Delta(x) = 0$. Moreover, $\mathrm{Div}_{\mathcal{M}} = \mathrm{Div}_M$ and the integral over dy vanishes, leading to

$$-EL\left[X_Q\right] = \mathrm{Div}_M\left(\partial_u\left(L^0 + L^1\right) + \left(L^0 + L^1\right) X_M\right), \qquad (4.61)$$

which is the usual Noether conservation law.

This concludes our geometric construction for multi-local GFT actions. It should be clear from the above discussion that we can apply this procedure to a general class of multi-local field theories, whenever the split in local parts is possible. As we have shown in [157] the same construction can be applied to cases in which the interaction part depends also on derivatives of fields. In this case we need to formulate the jet bundle over $f^\star_{\mathrm{tet}}\pi^4$ but apart from some minor technicalities the constructions remains unchanged. In the next section we will present applications of our method to several models of GFT.

4.3 *Overview of the models*

We are now going to apply the analysis developed above to several models in group field theory. Our main goal of this chapter is to use the point-wise formulation of the definition of a symmetry group for the GFT action to classify all symmetries of given models. We begin by introducing the models.

As already discussed the general structure of the GFT actions takes the form,

$$S[\phi] = S^{loc}_\kappa[\phi] + S^I[\phi], \qquad (4.62)$$

with a local and a multi-local part of the action split as in the previous section. We assume that the local, quadratic part of the action is defined as

$$S^{loc}_\kappa[\phi] = \int_M \mathrm{d}x\, \bar{\phi}(x)\left(-\kappa\Delta + m\right)\phi(x). \qquad (4.63)$$

We will treat cases in which κ can be zero, meaning that the model is static and consider the following two types of interactions.

4.3.1 *Simplicial interactions*

We discussed this interaction type in chapter 2. In these models the interaction is constructed using the combinatorial structure of simplexes. In the 3D case this interaction is given by the Boulatov model that we already used several times

$$S^I[\phi] = \lambda \int_{M^{\times 4}} \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}w \quad \mathrm{tet}(x,y,z,w) \quad \phi(x)\,\phi(y)\,\phi(z)\,\phi(w), \qquad (4.64)$$

where as in eq. (4.28)

$$\text{tet}\,(x,y,z,w) = \delta\left(x_1 y_1^{-1}\right)\delta\left(x_2 z_2^{-1}\right)\delta\left(x_3 w_3^{-1}\right)$$
$$\times\,\delta\left(y_2 w_2^{-1}\right)\delta\left(y_3 z_3^{-1}\right)\delta\left(z_1 w_1^{-1}\right).$$

In 4D the interaction type represents the combinatorics of a pentatope[2] and is called the *Ooguri model* given by the interaction

$$S^I[\phi] = \lambda \int_{M^{\times 5}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}q \quad \text{pent}\,(x,y,z,w,q)$$
$$\times\,\phi(x)\,\phi(y)\,\phi(z)\,\phi(w)\,\phi(q)\,,$$

with

$$\text{pent}\,(x,y,z,w,q) = \delta\left(x_1 q_4^{-1}\right)\delta\left(x_2 w_3^{-1}\right)\delta\left(x_3 z_2^{-1}\right)\delta\left(x_4 y_1^{-1}\right)$$
$$\times\,\delta\left(y_2 q_3^{-1}\right)\delta\left(y_3 w_2^{-1}\right)\delta\left(y_4 z_1^{-1}\right)$$
$$\times\,\delta\left(z_3 q_2^{-1}\right)\delta\left(z_4 w_1^{-1}\right)$$
$$\times\,\delta\left(w_4 q_1^{-1}\right).$$

The vector bundle is again a trivial bundle with the total space $E = Spin\,(4)^{\times 4} \times \mathbb{C}$. We will treat this model for $\kappa = 0$.

Explicitely, the simplicial models we treat are:

Boulatov like: with the Laplace operator and the Boulatov interaction and without the closure constraints

$$S[\phi] = \int_M \mathrm{d}x\,\bar{\phi}(x)\,(-\kappa\Delta + m)\,\phi(x) \tag{4.65}$$
$$+\,\lambda \int_{M^{\times 4}} \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}w \quad \text{tet}\,(x,y,z,w)$$
$$\times\,\phi(x)\,\phi(y)\,\phi(z)\,\phi(w) + c.c.,$$

with $M = SU\,(2)^{\times 3}$ and $E = M \times \mathbb{C}$. We choose here complex fields in order to perform the standard symmetry analysis for the most general case.

Boulatov action: the original Boulatov action

$$S[\phi] = m \int_M \mathrm{d}x\,\phi(x)^2 \tag{4.66}$$
$$+\,\lambda \int_{M^{\times 4}} \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}w \quad \text{tet}\,(x,y,z,w)$$
$$\times\,\phi(x)\,\phi(y)\,\phi(z)\,\phi(w)\,,$$

with $M = SU\,(2)^{\times 3}$ and $E = M \times \mathbb{R}$ and the requirement for the fields to satisfy the closure constraint[3].

[2] *The action that we use here represents the combinatorics of the pentatope without paying attention to the orientation of its faces. As we will see below this will result in a slight reduction of its symmetry group. However, this interaction is the one that is most frequently used in the literature and for that reason we use it here.*

[3] *Reminder*

The closure constraint is a requirement on the field ϕ to be invariant under the right multiplication by the diagonal group element. That is for $h \in G$ and $D : G \to M = G^{\times d}$ as in the previous chapter, the fields satisfy

$$\phi \circ R_{D_h} = \phi. \tag{4.67}$$

Ooguri action: the 4D version of the Boulatov action

$$S[\phi] = m\int_M \mathrm{d}x\,\phi(x)^2 + \lambda\int_{M^{\times 5}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}q \times \mathrm{pent}(x,y,z,w,q) \quad \phi(x)\,\phi(y)\,\phi(z)\,\phi(w)\,\phi(q)\,,$$

with $M = Spin(4)^{\times 4}$ and $E = M \times \mathbb{R}$ and the closure constraint on the fields.

4.3.2 Extended Barrett-Crane model

As we discussed in chapter 2, in four dimensions, gravity can be formulated as a BF theory with additional constraints [158], which are labeled *simplicity constraints*. One GFT model that implements the simplicity constrains is the so-called Barrett-Crane model [82], whose detailed treatment in the language of GFT for the euclidean signature was presented in [100]. Here we show just the main construction of the model and refer to the literature for more details.

The starting point for the Barrett-Crane (BC) model is the Ooguri action with $E = Spin(4)^{\times 4} \times \mathbb{C}$ with the closure constraint. This is the GFT action for the BF part of the Plebanski action. To implement the simplicity constraints the base manifold is extended to $M \times S^3$, however, since the 3-sphere S^3 is isomoprhic to $SU(2)$ we deal with the extended base manifold $M \times SU(2)$.With the shorthand notation

$$\phi(x,k) = \phi(x_1,x_2,x_3,x_4,k) =: \phi_{1,2,3,4,k}, \tag{4.68}$$

the interaction of the model can be written as

$$S^I[\phi] = \int_{M_{ext}^{\times 5}} \mathrm{d}x_e\,\mathrm{d}y_e\,\mathrm{d}z_e\,\mathrm{d}w_e\,\mathrm{d}q_e \times \mathrm{pent}(x,y,z,w,q) \times \phi(x,k_1)\,\phi(y,k_2)\,\phi(z,k_3)\,\phi(w,k_4)\,\phi(q,k_5)\,, \tag{4.69}$$

with $M_{ext} = Spin(4)^{\times 4} \times SU(2)$ and $E = M_{ext} \times \mathbb{R}$ and where the index e on the measure denote the extended measure $\mathrm{d}x_e = \mathrm{d}x \times \mathrm{d}k$ with $\mathrm{d}x$ the Haar measure on M and $\mathrm{d}k$ the Haar measure on $SU(2)$. The simplicity constraints are imposed by requiring invariance of the fields

$$\phi \circ S_u = \phi, \tag{4.70}$$

for any $u \in SU(2)^{\times 4}$ where $S_u : M \times SU(2) \to M \times SU(2)$ is given as follows: if we write a $Spin(4)$ element in its selfdual and anti-selfdual $SU(2)$ components as $x_i = (x_{i-}, x_{i+})$ then the action of S_u on the extended base manifold can be written as

$$\begin{aligned}S_u(x,k) = \Big(&x_{1-}\, k u_1 k^{-1}, x_{1+}\, u_1,\\ &x_{2-}\, k u_2 k^{-1}, x_{2+}\, u_2,\\ &x_{3-}\, k u_3 k^{-1}, x_{3+}\, u_3,\\ &x_{4-}\, k u_4 k^{-1}, x_{4+}\, u_4,\\ &k\Big) \in M \times SU(2)\,.\end{aligned} \tag{4.71}$$

We write in short

$$S_u(x,k) = \left(x_- \cdot k\, u\, k^{-1}, x_+ \cdot u,\, k\right), \tag{4.72}$$

where $\cdot$ denotes the component wise multiplication between the components of x and the components of u. In [100] it has been shown that S_u and R_h can be combined to a single transformation $\mathcal{S}_{u,h}$,

$$\mathcal{S}_{u,h}: \quad (x,k) \quad \mapsto \left(\mathbb{1}, h_-^{-1}\right) \cdot (x_-, x_+, k) \cdot \left(\left(k\, u\, k^{-1}, u\right), \mathbb{1}\right) \cdot (h_-, h_+)\,,$$

where $h \in Spin(4)$, h_+ and h_- are its self dual and anti-selfdual components, $u \in SU(2)^{\times 4}$, $x \in M$ and $k \in SU(2)$. In this formulation the fields have to be invariant under the above transformation, such that

$$\phi \circ \mathcal{S}_{u,h} = \phi. \tag{4.73}$$

The explicit Barrett-Crane action for 4D Riemannian gravity is then given by

$$\begin{aligned}S[\phi] =& m \int_{M_{ext}} \mathrm{d}x_e\, \phi(x,k)^2\\ &+ \lambda \int_{M_{ext}^{\times 5}} \mathrm{d}x_e\, \mathrm{d}y_e\, \mathrm{d}z_e\, \mathrm{d}w_e\, \mathrm{d}q_e\\ &\quad \times \mathrm{pent}(x,y,z,w,q) \quad \phi(x,k_1)\,\phi(y,k_2)\,\phi(z,k_3)\,\phi(w,k_4)\,\phi(q,k_5)\,,\end{aligned}$$

with $M_{ext} = Spin(4)^{\times 4} \times SU(2)$ and $E = M_{ext} \times \mathbb{R}$ and where the index e on the measure denote the extended measure $\mathrm{d}x \times \mathrm{d}k$ with $\mathrm{d}x$ the Haar measure on M and $\mathrm{d}k$ the Haar measure on $SU(2)$.

4.4 Applications of the symmetry analysis in GFT

4.4.1 Boulatov like model

We will use the standard Lie group analysis of point symmetries [150], based on point-wise criteria from eq. (4.52). This analysis can be summarized as follows:

i) We assume a most general vector field W on the vector bundle π and insert it in (4.52), ii) we rearrange the resulting equation by different powers in derivatives of fields. Since the coefficients X_M^i and X_V^i do not depend on derivatives of the fields, it is possible to extract all powers explicitly, iii) different powers of derivatives of ϕ are linearly independent since the condition (4.52) has to be satisfied for all fields. For this reason the coefficients in front of each term have to vanish separately. This results in a set of simple differential equations for the components of the vector field W which can then be easily solved.

By partial integration the local part of the action is written as

$$L\left(x^{\mu}, u^{i}, u^{i}_{\mu}\right) = \sum_{i=1}^{c} \left(\sum_{j=1}^{d} \sum_{\alpha=1}^{r} \kappa \bar{u}^{i}_{j\alpha} u^{i}_{j\alpha} + m\, \bar{u}^{i} u^{i} \right), \tag{4.74}$$

where c is the number of colors in the model, d is the number of copies of the group G in the definition of $M = G^{\times d}$ and $r = \dim(G)$ is the dimension of the group G. In our case $c = 1$ but for now, we leave the color unspecified to keep the discussion more general.

The indices in the following equations range over the following domains: $i, t \in \{1, \cdots, c\}$ the color index, $j, k \in \{1, \cdots, d\}$ the copy of G in M, and $\alpha, \beta, \gamma \in \{1, \cdots, r\}$ components of the chart of G. Further we use the following notation for the generators of the symmetry group on E,

$$W = \sum_{j=1}^{d} \sum_{\alpha=1}^{r} X_M^{j\alpha} \partial_{x_j^{\alpha}} + \sum_{i=1}^{c} \left(X_V^i \partial_{u^i} + \bar{X}_V^i \partial_{\bar{u}^i} \right). \tag{4.75}$$

With this notation the symmetry condition from eq. (4.52) implies

$$X_M(L) + L\,\mathrm{Div}(X_M) + 2\kappa \sum_{i=1}^{c} \sum_{j=1}^{d} \sum_{\alpha=1}^{r} \partial_{j\alpha}\phi^i \cdot D_{j\alpha}\left(X_Q^i\right) + 2m \sum_{i=1}^{c} \phi^i \left(X_Q^i\right) = 0, \tag{4.76}$$

where the whole equation is evaluated at the point $(x, \phi(x), D\phi|_x)$ for some $x \in M$. Explicitly sorting the terms by powers of $u^i_{j\alpha} = \partial_{j,\alpha}\phi^i|_x$ we get (in the following equation all indices are summed over)

$$0 = \left[m \left| u^i \right|^2 \operatorname{div}(X_M) + m\bar{u}^i X_{u^i} + m u^i X_{\bar{u}^i} \right] \tag{4.77}$$

$$+\Re\left[u^t_{j\alpha}\right]\left[m \left| u^i \right|^2 \left(\partial_{\bar{u}^t} X^{j\alpha}_M + \partial_{u^t} X^{j\alpha}_M \right) + \kappa g^{\alpha\beta}_j \left(\partial_{j\beta} X^t_V + \partial_{j\beta} \bar{X}^t_V \right) \right] \tag{4.78}$$

$$+\imath\Im\left[u^t_{j\alpha}\right]\left[m \left| u^i \right|^2 \left(\partial_{u^t} X^{j\alpha}_M - \partial_{\bar{u}^t} X^{j\alpha}_M \right) + \kappa g^{\alpha\beta}_j \left(\partial_{j\beta} \bar{X}^t_V - \partial_{j\beta} X^t_V \right) \right] \tag{4.79}$$

$$+\kappa\Re\left[\bar{u}^i_{\alpha j} u^i_{\beta j}\right]\left[\frac{1}{2} X^{k\gamma}_M \partial_{k\gamma} g^{\alpha\beta}_j - 2 g^{\alpha\gamma}_j \partial_{j\gamma} X^{j\beta}_M + g^{\alpha\beta}_j \left\{ \left(\partial_{u^i} X^i_V + \partial_{\bar{u}^i} \bar{X}^i_V \right) + \operatorname{div}(X_M) \right\} \right] \tag{4.80}$$

$$-2\kappa\Re\left[\bar{u}^i_{j\alpha} u^i_{(k\neq j)\beta}\right]\left[g^{\alpha\gamma}_j \partial_{j\gamma} X^{(k\neq j)\beta}_M \right] \tag{4.81}$$

$$+\kappa g^{\alpha\beta}_j \Re\left[\bar{u}^i_{j\alpha} u^{t\neq i}_{j\beta}\right]\left[\partial_{u^{t\neq i}} X^i_V + \partial_{\bar{u}^i} \bar{X}^{t\neq i}_V \right] \tag{4.82}$$

$$+\imath\kappa g^{\alpha\beta}_j \Im\left[\bar{u}^i_{j\alpha} u^{t\neq i}_{j\beta}\right]\left[\partial_{u^{t\neq i}} X^i_V - \partial_{\bar{u}^i} \bar{X}^{t\neq i}_V \right] \tag{4.83}$$

$$+\kappa g^{\alpha\beta}_j \Re\left[\bar{u}^i_{j\alpha} \bar{u}^t_{j\beta}\right]\left[\partial_{\bar{u}^t} X^i_V + \partial_{u^t} \bar{X}^i_V \right] \tag{4.84}$$

$$+\imath\kappa g^{\alpha\beta}_j \Im\left[\bar{u}^i_{j\alpha} \bar{u}^t_{j\beta}\right]\left[\partial_{\bar{u}^t} X^i_V - \partial_{u^t} \bar{X}^i_V \right] \tag{4.85}$$

$$+2\kappa\bar{u}^i_{\alpha j} u^i_{\gamma j} \Re\left[u^t_{j\beta}\right]\left[-2 g^{\alpha\beta}_j \left(\partial_{u^t} X^{j\gamma}_M + \partial_{\bar{u}^t} X^{j\gamma}_M \right) + g^{\alpha\gamma}_j \left(\partial_{u^t} X^{j\beta}_M + \partial_{\bar{u}^t} X^{j\beta}_M \right) \right] \tag{4.86}$$

$$+\imath 2\kappa\bar{u}^i_{j\alpha} u^i_{j\gamma} \Im\left[\bar{u}^t_{j\beta}\right]\left[-2 g^{\alpha\beta}_j \left(\partial_{u^t} X^{j\gamma}_M - \partial_{\bar{u}^t} X^{j\gamma}_M \right) + g^{\alpha\gamma}_j \left(\partial_{u^t} X^{j\beta}_M - \partial_{\bar{u}^t} X^{j\beta}_M \right) \right] \tag{4.87}$$

$$+\bar{u}^i_{j\alpha} u^i_{(k\neq j)\gamma} \Re\left[u^t_{j\beta}\right]\left[-2\kappa g^{\alpha\beta}_j \left(\partial_{u^t} X^{(k\neq j)\gamma}_M + \partial_{\bar{u}^t} X^{(k\neq j)\gamma}_M \right) \right] \tag{4.88}$$

$$+\imath\bar{u}^i_{j\alpha} u^i_{(k\neq j)\gamma} \Im\left[\bar{u}^t_{j\beta}\right]\left[-2\kappa g^{\alpha\beta}_j \left(\partial_{u^t} X^{(k\neq j)\gamma}_M - \partial_{\bar{u}^t} X^{(k\neq j)\gamma}_M \right) \right]. \tag{4.89}$$

The above equation is sorted such that each line is multiplied with a different power and combination of the coordinates u^i and $u^c_{j\alpha}$. Moreover, it has to hold true for arbitrary fields ϕ and hence for arbitrary u^i and $u^i_{j\alpha}$, hence each line has to vanish independently. From this condition we obtain the following relation for the vector fields:

1. Equations (4.82) and (4.83) imply that the vector field component X^i_V depend only on the field colors they transform, that is (without summation)

$$X^i_V = X^i_V\left(x, u^i, \bar{u}^i\right) \qquad \bar{X}^i_V = \bar{X}^i_V\left(x, u^i, \bar{u}^i\right).$$

2. Equations (4.84) and (4.85) additionally imply that the vector fields X^i do not depend on the complex conjugate of the field, that is

$$X^i_V = X^i_V\left(x, u^i\right) \qquad \bar{X}^i_V = \bar{X}^i_V\left(x, \bar{u}^i\right).$$

3. Equations (4.88) and (4.89) tell us that the vector fields that transform the base manifold do not depend on the field values u^i i.e. $X_M = X_M(x)$. From this condition, equations (4.86) and (4.87) are automatically satisfied.

4. Due to the above, equations (4.78) and (4.79) reduce to (without summation)

$$\partial_{j\alpha} X^i_V = 0 = \partial_{j\alpha} \bar{X}^i_V. \tag{4.90}$$

for any fixed i,j and α. That is, the vector fields do not explicitly depend on the points in the base manifold (without summation)

$$X_V^i = X_V^i\left(u^i\right) \qquad \bar{X}_V^i = X_V^i\left(\bar{u}^i\right).$$

5. Equation (4.77), together with the above conclusions, restricts the vector fields to a specific form

$$X_V^i = C\,u^i \qquad \bar{X}_V^i = \bar{C}\,\bar{u}^i, \tag{4.91}$$

where C is an arbitrary constant, $\bar{C}$ its complex conjugate and both satisfy

$$\operatorname{div}\left(X_M\right) = -C - \bar{C}. \tag{4.92}$$

6. The above condition reduces equations (4.80) and (4.81) to

$$\begin{aligned}\sum_{\gamma=1}^{r}\left(\sum_{k=1}^{d} X_M^{k\gamma}\partial_{k\gamma}g_j^{\alpha\beta} - 2\,g_j^{\alpha\gamma}\,\partial_{j\gamma}X_M^{j\beta} - 2\,g_j^{\gamma\beta}\,\partial_{\gamma j}X_M^{j\alpha}\right) &= 0\\ g_j^{\alpha\gamma}\,\partial_{j\gamma}X_M^{(k\neq j)\beta} + g_k^{\alpha\gamma}\,\partial_{j\gamma}X_M^{(j\neq k)\beta} &= 0.\end{aligned}$$

These two equations are the only ones that are not trivial to solve. However, their solution can be found in a straightforward way. The solution in Hopf coordinates for a 3-sphere (η,ξ,χ)[4] reads as (without summation)

[4] *In this coordinates the elements of* $SU(2)$ *can be parametrized as*

$$\begin{pmatrix} e^{\imath\xi}\sin\eta & -e^{-\imath\chi}\cos\eta \\ e^{\imath\chi}\cos\eta & e^{-\imath\xi}\sin\eta \end{pmatrix} \tag{4.93}$$

and the metric on $SU(2)$ *in this coordinates becomes* $g = d\eta^2 + \sin^2\eta\, d\xi^2 + \cos^2\eta\, d\chi^2$.

$$\begin{aligned} X_M^{j\eta} &= C_1\sin\xi_j\sin\chi_j + C_2\cos\xi_j\sin\chi_j \qquad (4.94)\\ &\quad + C_3\sin\xi_j\cos\chi_j + C_4\cos\xi_j\cos\chi_j \\ X_M^{j\xi} &= \frac{\cos\eta_j}{\sin\eta_j}\partial_{j\xi}X_M^{j\eta} + C_5 \qquad (4.95)\\ X_M^{j\chi} &= -\frac{\sin\eta_j}{\cos\eta_j}\partial_{j\chi}X_M^{j\eta} + C_6, \qquad (4.96)\end{aligned}$$

where C_i's are arbitrary constants.

Setting subsequently C_i to one and the rest of the coefficients to zero we obtain, for each $j \in \{1,\cdots,d\}$, six linearly independent vector fields given by

$$v_1 = \begin{pmatrix} \sin(\xi)\sin(\chi) \\ \cot(\eta)\sin(\xi)\cos(\chi) \\ -\tan(\eta)\sin(\xi)\cos(\chi) \end{pmatrix} \quad (4.97)$$

$$v_2 = \begin{pmatrix} \cos(\xi)\sin(\chi) \\ -\cot(\eta)\sin(\xi)\sin(\chi) \\ -\tan(\eta)\cos(\xi)\cos(\chi) \end{pmatrix} \quad (4.98)$$

$$v_3 = \begin{pmatrix} \sin(\xi)\cos(\chi) \\ \cot(\eta)\cos(\xi)\cos(\chi) \\ \tan(\eta)\sin(\xi)\sin(\chi) \end{pmatrix} \quad (4.99)$$

$$v_4 = \begin{pmatrix} \cos(\xi)\cos(\chi) \\ -\cot(\eta)\sin(\xi)\cos(\chi) \\ \tan(\eta)\cos(\xi)\sin(\chi) \end{pmatrix}, \quad (4.100)$$

and

$$v_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4.101)$$

It is a direct calculation to check that these vector fields are divergence free, $\operatorname{div}(v_i) = 0$ for $i \in \{1,2,3,4,5,6\}$. This fact, together with equation (4.92), implies

$$X_V^i = \imath C\, u^i \qquad \bar{X}^i = -\imath C\, \bar{u}^i \quad , \quad (4.102)$$

which generates the usual $U(1)$ symmetry of fields for each color.

In order to find the symmetry group generated by the fields $v_1, \cdots, v_6$ we look at their algebra. The six dimensional Lie algebra of $v_1, \cdots, v_6$ is given in table (4.1)

Table 4.1: Lie algebra of symmetry vector fields

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	v_5	v_6	0	$-v_2$	$-v_3$
v_2	$-v_5$	0	0	v_6	v_1	$-v_4$
v_3	$-v_6$	0	0	v_5	$-v_4$	v_1
v_4	0	$-v_6$	$-v_5$	0	v_3	v_2
v_5	v_2	$-v_1$	v_4	$-v_3$	0	0
v_6	v_3	v_4	$-v_1$	$-v_2$	0	0

We can split this algebra into $\mathfrak{su}(2) \times \mathfrak{su}(2)$ by taking the following linear combinations

$$
\begin{aligned}
l_1 &= \frac{v_5 - v_6}{2} & r_1 &= \frac{v_5 + v_6}{2} \\
l_2 &= \frac{v_3 - v_2}{\sqrt{2}} & r_2 &= \frac{v_3 + v_2}{\sqrt{2}} \\
l_3 &= \frac{v_4 + v_1}{\sqrt{2}} & r_3 &= \frac{v_4 - v_1}{\sqrt{2}} \quad .
\end{aligned}
\tag{4.103}
$$

The commutators for l_i and r_i become

$$[l_1, l_2] = l_3 \qquad [r_1, r_2] = r_3 \tag{4.104}$$

$$[l_1, l_3] = -l_2 \qquad [r_1, r_3] = -r_2 \tag{4.105}$$

$$[l_2, l_3] = 2l_1 \qquad [r_2, r_3] = 2r_1 \tag{4.106}$$

and

$$\left[l_i, r_j\right] = 0 \quad \forall i, j \in \{1, 2, 3\} \tag{4.107}$$

A closer inspection shows that l_i and r_i form a set of left and right invariant vector fields on $SU(2)$, respectively [159].

Since the above algebra was derived for each copy of the group, the whole symmetry group of the local part of the action becomes

$$[SU(2) \times SU(2)]^{\times 3} \times U(1)\,, \tag{4.108}$$

acting on the base manifold by left and right multiplication as

$$L_\alpha R_\beta (x) = \alpha \cdot x \cdot \beta, \tag{4.109}$$

for $\alpha, \beta \in SU(2)^{\times 3}$ and on fields by multiplication with a $U(1)$ phase.

It is straight forward to check that the above symmetry group is an admissible group of transformations for the Boulatov interaction. However, the $U(1)$ symmetry is not respected by the interaction term and the symmetry group for this model becomes

$$G_T = [SU(2) \times SU(2)]^{\times 3}\,,$$

implementing the symmetry under left and right translations on the group.

4.4.2 *Models with closure constraints*

We now turn to the symmetry analysis of GFT models with closure constraints. In principle it is still possible to use the same procedure that we used above, however, the closure constraint on the fields makes it more difficult to identify independent variables. For that reason, contrary to the previous case, we will use the interaction part to classify the symmetry group.

Boulatov model

We begin our analysis by examining the admissible group of transformations. The combinatorics of the Boulatov interaction is encoded in the function f from equation (4.34)

$$f : (\mathcal{X}) \mapsto ((x_1, x_2, x_3), (x_1, x_4, x_5), (x_6, x_2, x_5), (x_6, x_4, x_3)). \quad (4.110)$$

for $\mathcal{X} = (x_1, \cdots, x_6) \in \mathcal{M}$. Admissible transformations of the base manifold are given by those functions $C : E \to M$ that satisfy the relation (4.44). Therefore, for any $\mathcal{X} \in \mathcal{M}$, there should exist a point $\tilde{\mathcal{X}} \in \mathcal{M}$ such that

$$C^{\times 4} \circ f(\mathcal{X}) = f\left(\tilde{\mathcal{X}}\right). \quad (4.111)$$

Writing C in components as

$$C(x, u) = \left(C^1(x, u), C^2(x, u), C^3(x, u)\right) \in M, \quad (4.112)$$

condition (4.111) implies

$$C^1(x_1, x_2, x_3, u_1) = C^1(x_1, x_4, x_5, u_2) \quad (4.113)$$

$$C^2(x_1, x_2, x_3, u_3) = C^2(x_6, x_2, x_5, u_4) \quad (4.114)$$

$$C^3(x_1, x_2, x_3, u_5) = C^3(x_6, x_4, x_3, u_6), \quad (4.115)$$

where u_i are given by ϕ evaluated at the corresponding points of the base manifold. This implies the following decomposition of C,

$$C(x_1, x_2, x_3, u) = C^1(x_1) C^2(x_2) C^3(x_3). \quad (4.116)$$

In this case the diffeomorphism properties of C carry over to the components C^i.

According to equation (4.46), the fields transform as

$$\phi \mapsto \tilde{\phi} = [Q \circ \mathbb{1} \times \phi] \circ C^{-1}. \quad (4.117)$$

The field $\tilde{\phi}$ needs to satisfy the closure constraint as well, otherwise the transformations C, Q would leave the allowed space of fields. The gauge invariance of $\tilde{\phi}$ reads

$$\tilde{\phi} \circ R_h = [Q \circ (\mathbb{1} \times \phi)] \circ C \circ R_{D_h} \stackrel{!}{=} [Q \circ (\mathbb{1} \times \phi)] \circ C = \tilde{\phi} \qquad \forall h \in G. \quad (4.118)$$

Since this has to be true for all gauge invariant fields ϕ, the point $C \circ R_{D_h}(x)$ needs to be in the same orbit (under the multiplication from the right by the diagonal group) as the point $C(x)$. This means that, for any $h \in G$, there should exist an $\tilde{h} \in G$ such that

$$C \circ R_{D_h} = R_{D_{\tilde{h}}} \circ C, \quad (4.119)$$

or point-wise

$$C(xD_h) = C(x) D_{\tilde{h}}. \quad (4.120)$$

As we show in the appendix B.1, this restricts the C, up to discrete transformations, to be of the form

$$C(x) = L_z \cdot Con_{D_h}(x), \tag{4.121}$$

for some $z \in G^{\times 3}$ and $h \in G$ and where Con_{D_h} is the conjugation by D_h such that $Con_{D_h}(x) = D_h^{-1} \cdot x \cdot D_h$ and L is the left multiplication. The transformation group of the interaction part becomes

$$G^{\times 3} \times G, \tag{4.122}$$

where $G^{\times 3}$ acts by left multiplication and the single G acts by conjugation with diagonal elements. It is evident that this group already forms a symmetry group of the Boulatov action, due to the left and right invariance of the Haar measure. We can summarize the role of combinatorial structure and the gauge invariance on the transformation group of the base manifolds as follows

$$\text{Diff}(M) \underset{\text{combinatorics}}{\longrightarrow} \text{Diff}(G)^{\times 3} \underset{\text{clusre constraint}}{\longrightarrow} G^{\times 3} \times G. \tag{4.123}$$

To make sure that the above group is already the whole symmetry group of the Boulatov action we now follow the procedure of the previous section to obtain possible generators for the Q transformation. The infinitesimal symmetry condition for the simplicial interaction in three dimensions takes the form

$$0 = \partial_{\mathcal{U}^i} L \cdot X^i_{\mathcal{Q}} + \text{Div}_{\mathcal{M}}(L\, X_{\mathcal{M}}), \tag{4.124}$$

Using the fact that the only admissible base manifold transformations are generated by divergence free vector fields, the above equation reduces to

$$\mathcal{U}^2\mathcal{U}^3\mathcal{U}^4\, X^1_{\mathcal{V}} + \mathcal{U}^1\mathcal{U}^3\mathcal{U}^4\, X^2_{\mathcal{V}} + \mathcal{U}^1\mathcal{U}^2\mathcal{U}^4\, X^3_{\mathcal{V}} + \mathcal{U}^1\mathcal{U}^2\mathcal{U}^3\, X^4_{\mathcal{V}} = 0 \tag{4.125}$$

Hereby $X^i_{\mathcal{V}}$ is evaluated at the point $(\mathcal{X}, \mathcal{U})$. Equation (4.125) needs to hold true for any ϕ^i and any point of the base manifold, and hence for any value of $\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3, \mathcal{U}^4$. From this it directly follows that

$$X^i_{\mathcal{V}}(\mathcal{X}, \mathcal{U}) = 0. \tag{4.126}$$

Hence, no further symmetries can be added to the above and the overall symmetry group for the Boulatov action with real fields is

$$G_T = SU(2)^{\times 3} \times SU(2), \tag{4.127}$$

acting by left translation and conjugation, respectively, *Symmetries of Ooguri model*

For the Ooguri model with combinatorics

$$f : (x_1, \cdots, x_{10}) \mapsto ((x_1, x_2, x_3, x_4), (x_4, x_5, x_6, x_7), (x_7, x_3, x_8, x_9), \\ \times (x_9, x_6, x_2, x_{10}), (x_{10}, x_8, x_5, x_1)),$$

the above approach results in the transformation group

$$Spin(4)^{\times 2} \times Spin(4), \tag{4.128}$$

where $Spin(4)^{\times 2}$ acts by left multiplication as

$$(G_1, G_2) \cdot x = (G_1 x_1, G_2 x_2, G_2 x_3, G_1 x_4). \tag{4.129}$$

The fact that the symmetry group is not $Spin(4)^{\times 4} \times Spin(4)$ stems from the particular combinatorics that is typically used in the literature. Even though the combinatorial pattern of the pentatope is encoded in the interaction, the orientation of the faces of the pentatope is different. This results in a different combination of the variables of the field ϕ and consequently in a reduction of the symmetry group from $Spin(4)^{\times 4} \times Spin(4)$ down to $Spin(4)^{\times 2} \times Spin(4)$.

Barrett-Crane model

Applying the above analysis to the BC model from equation (4.69) defined by the following combinatorics

$$f : (x_1, \cdots x_{10}; k_1, \cdots, k_5) \mapsto ((x_{1,2,3,4}; k_1), (x_{4,5,6,7}; k_2), (x_{7,3,8,9}; k_3), \\ \times (x_{9,6,2,10}; k_4), (x_{10,8,5,1}; k_5)),$$

we realize that the symmetry group for the gauge invariant BC model without simplicity constrains would be that of an extended Ooguri model from equation (4.128) with an additional $SU(2)$ symmetry for the extended component,

$$\left[Spin(4)^{\times 2} \times Spin(4)\right] \times SU(2), \tag{4.130}$$

where the additional group $SU(2)$ denotes a group of transformations of the $SU(2)$ element k_i.

In order to obtain the symmetry group of the BC model we need to impose simplicity constraints. As we show in appendix B.2, the requirement on the fields to satisfy

$$\phi \circ \mathcal{S}_{u,h} = \phi, \tag{4.131}$$

for $u \in SU(2)^{\times 4}$ and $h \in SU(2)$ reduce the symmetry group of the Ooguri model (for the chosen combinatorics) down to

$$Spin(4)^{\times 2} \times SU(2), \tag{4.132}$$

that acts on the elements of the local base manifold of the BC model $M \times SU(2)$ by conjugation such that in the selfdual and anti-selfdual components of $Spin(4)$ the action of the $SU(2)$ group is given by

$$(x_-, x_+, k) \overset{SU(2)}{\mapsto} D_h \cdot (x_-, x_+, k) \cdot D_{h^{-1}}.$$

And $Spin(4)^{\times 2}$ acts by the left multiplication as

$$(x, k) \overset{Spin(4)^{\times 2}}{\mapsto} (G_1 x_1, G_2 x_2, G_2 x_3, G_1 x_4, k).$$

4.5 *Summary and Conclusion*

In the first part of this chapter we presented a modification of the symmetry analysis of local theories to theories with multi-local actions. We argued that the multi-local action in GFT can not be written as an integral of a smooth function on a single jet bundle. However, we have shown that it is possible to construct a vector bundle that encodes the combinatorial structure of the interaction and on which the interaction Lagrangian can be defined as a smooth function. The multi-local action can then be split in a sum of actions each of which is an ordinary local action on a suitable vector bundle.

We applied our approach to explicit GFT models with and without closure constraints. For models without closure constraints we used the standard symmetry analysis for the local part of the action and reduced the resulting group of symmetries using the interaction part. For models with closure constraints our approach went the other way around: we started with the interaction part and the condition for admissible transformation groups. This allowed us to derive the whole group of point symmetries for actions with closure constraints even in cases when the local part of the action did not contain derivative operators. This is especially interesting because the ordinary approach would fail in this case.

The definition of point symmetries for multi-local actions seems, however, too strong, since the resulting symmetry groups merely represent the invariance of the Haar measure. For that reason we believe that the notion of point symmetry for multi-local actions is inappropriate. The equations of motion of a multi-local theory depend on integrals over the entire domain of definition and therefore depend on the whole field configuration in the domain. The Lagrangian of this theory is, hence, more appropriately written as a functional on the space of fields, rather than a function of a jet bundle. The geometrical construction of vector bundles and jet bundles should, therefore, be replaced by infinite dimensional manifolds [160]. A group of transformations would act directly on fields, but possibly will not have any interpretation in terms of vector fields on the vector bundle. For this reason the compatibility conditions will not be necessary and will not restrict the possible group of transformation. Such symmetry groups are already present in local field theories and are called the Lie-Baecklund symmetries [150, 161]. However, whereas the Lie-Baecklund symmetries are still local (in a curtain sense), there is no reason to require the same for the multi-local action. Under these considerations the resulting symmetry group is expected to grow significantly.

In fact, the symmetry group of some studied equations in plasma physics (for example Vlasow and Landau equations [162]) and in hydrodynamics (for example the Boltzmann equation [163]) have been shown to be infinite dimensional. The fact that the symmetry group of integro-differential equation is larger than that of partial differential equations is intuitive, since an integration procedure could make even those transformations to symmetries which fail to conserve the action in local subregions of the domain.

In conclusion, our analysis provides a structured method for the derivation of point symmetries for multi-local actions but it also shows that a point definition of symmetries is too restrictive for multi-local models. A symmetry analysis of GFT action needs to be performed using the framework of infinite dimensional manifolds and we hope that this issue will be addressed in future works.

Appendix B

B.1 Reduction of transformations due to gauge invariance

From equation (4.120) the requirement on the transformation reads

$$C(x \cdot D_h) = C(x)\, D_{\tilde{h}}. \tag{B.1}$$

Writing out this equation in components we get

$$\begin{aligned} C^1(x_1 h) &= C^1(x_1)\,\tilde{h} \\ C^2(x_2 h) &= C^2(x_2)\,\tilde{h} \\ C^3(x_3 h) &= C^3(x_3)\,\tilde{h}, \end{aligned} \tag{B.2}$$

with C^i being a diffeomorphism on the group G. At this point we employ the known relation

$$\mathrm{Diff}(G) \simeq G \times \mathrm{Diff}_{\mathbb{1}}(G), \tag{B.3}$$

that states that the group of diffeomorphisms on G is diffeomorphic (as a manifold) to the group G itself (that acts by left multiplication) times a group of diffeomorphisms that stabilizes the identity of G, denoted $\mathrm{Diff}_{\mathbb{1}}(G)$. This implies that every C^i can be written by some $c^i \in G$ and $\mathcal{D}^i \in \mathrm{Diff}_{\mathbb{1}}(G)$ such that $C^i(x) = c^i\, \mathcal{D}^i(x)$ with $\mathcal{D}^i(\mathbb{1}) = \mathbb{1}$. Inserting this relation in equations B.2 and evaluating it at the point $x_1 = x_2 = x_3 = \mathbb{1}$ we observe

$$c^1 \cdot \mathcal{D}^1(h) = c^1 \cdot \tilde{h} \tag{B.4}$$

$$c^2 \cdot \mathcal{D}^2(h) = c^2 \cdot \tilde{h} \tag{B.5}$$

$$c^1 \cdot \mathcal{D}^3(h) = c^1 \cdot \tilde{h}, \tag{B.6}$$

which, in tern, implies

$$\mathcal{D}^1(h) = \mathcal{D}^2(h) = \mathcal{D}^3(h) = \tilde{h} =: \mathcal{D}(h). \tag{B.7}$$

Inserting this relation again in (B.2) at an arbitrary point x we get for $\mathcal{D}$

$$\mathcal{D}(x_i h) = \mathcal{D}(x_i)\, \mathcal{D}(h). \tag{B.8}$$

In other words $\mathcal{D}$ is an homomorphism and therefore an automorphism. On G however, the group of automorphisms splits in the inner automorphisms which are given by a conjugation with a fixed group element and outer automorphisms, which relate to the discreet symmetries. Focusing on continues transformations we get

$$\mathcal{D}(g) = d \cdot g \cdot d^{-1} \tag{B.9}$$

for some fixed $d \in SU(2)$.

B.2 Barrett-Crane model

In this section we are going to show what are the admissible transformation in the Barrett-Crane model.

In the following we will denote a group element of $Spin(4)$ by its two copies of $SU(2)$ as

$$Spin(4) \ni g = (g_-, g_+) \in SU(2) \times SU(2),$$

a tuple of four elements is referred to by the vector notation

$$\vec{g} = (\vec{g}_-, \vec{g}_+).$$

For points of the base manifold $M = Spin(4)^{\times 4}$ we use x and sometimes write $x_{1,2,3,4}$ referring to (x_1, x_2, x_3, x_4).

A base manifold transformation of the model is denoted by $C : Spin(4)^{\times 4} \times SU(2) \to Spin(4)^{\times 4} \times SU(2)$. We denote the components of this transformation as

$$C(\vec{g}, k) = \left(\left(C_1^-, C_1^+\right), \cdots, \left(C_4^-, C_4^+\right), C_k\right).$$

Here all the component functions $C_i^{\pm}$ are functions on the base manifold and therefore depend on points of the form $(\vec{g}, k)$. However, the combinatorial structure of the BC model dictates the following conditions on the components

$$\begin{aligned} C_1(x_{1,2,3,4}, k_1) &= C_4(x_{10,8,5,1}, k_5) \\ C_2(x_{1,2,3,4}, k_1) &= C_3(x_{9,6,2,10}, k_4). \end{aligned}$$

From the above relations we see that the components of the transformation have the following dependences

$$C(x_{1,2,3,4}, k) = (C_1(x_1), C_2(x_2), C_3(x_3), C_4(x_4), C_k(k)).$$

A priori we do not have any additional constrains on the component C_k. However, since C is a diffeomorphism and C_i are diffeomorphisms, the transformation of the normal has to be a diffeomorphism as well [1]. Again invoking the diffeomorphism of manifolds $\text{Diff}(Spin(4)) \simeq$

[1] *Notice, that it would not be true if we didn't have restriction on C_i, since then C_i would not be a diffeomorphism and hence neither needs to be C_k.*

$Spin\,(4) \times \mathrm{Diff}_{\mathbb{1}}$ we denote the components of C that belong to $\mathrm{Diff}_{\mathbb{1}}$ by the lower case c.

To implement simplicity constraints we need to require the invariance of fields such that

$$\phi \circ \mathcal{S}_{u,h} = \phi.$$

for any $u \in SU\,(2)^{\times 4}$ and $h \in Spin\,(4)$. Since the fields are transformed under C as $\phi \mapsto \phi \circ C^{-1}$, we again get the following relations for the transformation C

$$\phi \circ C_{D_h} \circ S_u = \phi \circ C,$$

or equivalently for each $h \in Spin\,(4)$, $u \in SU\,(2)^{\times 4}$ and $x \in M$ there exist $\tilde{u} \in SU\,(2)^{\times 4}$ and $\tilde{h} \in Spin\,(4)$ and $\tilde{k} \in SU\,(2)$ such that

$$C_i\,(x \cdot u_k \cdot h) \quad = \quad C_i\,(x) \cdot \tilde{u}_{C_k} \cdot \tilde{h} \tag{B.10}$$

$$C_k\left(h_-^{-1} k h_+\right) \quad = \quad \tilde{h}_-^{-1} C_k\,(k)\,\tilde{h}_+, \tag{B.11}$$

where we write $u_k = \left(kuk^{-1}, u\right)$. The left multiplication by $Spin\,(4)$ is untouched by this transformation, however this is not true for the normal component C_k. We first focus on the transformations C_i and treat the normal component C_k afterwards.

From the form of u_k we notice that for $u = \mathbb{1}$ the left hand side does not depend on k and so the right hand side also should not. It follows that for $u = \mathbb{1}$ we have $\tilde{u} = \mathbb{1}$. Equation (B.10) then reads for the $\mathrm{Diff}_{\mathbb{1}}$ part,

$$c_i\,(g \cdot h) = c_i\,(g) \cdot \tilde{h}.$$

It follows that c_i is a homomorphism on $Spin\,(4)$ and therefore is either conjugation by a fixed element of $Spin\,(4)$ or a flip of the $SU\,(2)$ parts, which is a discrete transformation. Hence, if c_i is continuous it can be written as

$$c_i\,(g) = s \cdot g \cdot s^{-1},$$

where $g, s \in Spin\,(4)$. This implies

$$\tilde{h} = s \cdot h \cdot s^{-1}.$$

Inserting this relation now in equation (B.11) we obtain

$$C_k\left(h_-^{-1} k h_+\right) = \left(s_- h_-^{-1} s_-^{-1}\right)\, C_k\,(k)\, \left(s_+ h_+ s_+^{-1}\right).$$

Splitting C_k in the left multiplication by $SU\,(2)$ and $\mathrm{Diff}_{\mathbb{1}}$ we get for some fixed $w \in SU\,(2)$

$$w\, c_k\left(h_-^{-1} k h_+\right) = \left(s_- h_-^{-1} s_-^{-1}\right) w\, c_k\,(k)\, \left(s_+ h_+ s_+^{-1}\right). \tag{B.12}$$

Choosing $h_- = h_+$ and setting $k = \mathbb{1}$ we get

$$w = \left(s_- h_-^{-1} s_-^{-1}\right) w\, \left(s_+ h_- s_+^{-1}\right),$$

which can only be satisfied if $w = \mathbb{1}$.

Inserting equation (B.12) in (B.10) and using the fact that c_i is a homomorphism yields

$$\begin{aligned} c_i\left(u_k\right) &= c_i\left(k, \mathbb{1}\right) \cdot c_i\left(u, u\right) \cdot c_i\left(k^{-1}, \mathbb{1}\right) \\ &= c_k\left(k\right) c_i\left(u, u\right) c_k\left(k^{-1}\right). \end{aligned}$$

Hence, $c_i\left(a, b\right) = \left(c_k\left(a\right), c_k\left(b\right)\right)$ and c_k is a homomorphisms itself. Therefor

$$c_i\left(g\right) = \left(s, s\right) \cdot g \cdot \left(s, s\right)^{-1},$$

and $c_k\left(k\right) = sks^{-1}$.

These are the only admissible transformations that preserve the combinatorial structure of the theory and respect the simplicity constraints together with gauge invariance. Notice that $\mathcal{S}$ itself would fail the requirement of admissible transformations and therefor can not be seen as a symmetry.

5

Conclusion and future work

5.1 Summary of the thesis

In this chapter we recapitulate the work presented in this thesis and provide an outlook for future work in the development of group field theory.

In this thesis we discussed the formalism of group field theory in applications to quantum gravity — a framework that suggests an emergence of space-time from more fundamental degrees of freedom. In chapter 1 we introduced a general problem of quantum gravity and presented some direct complications that appear when we try to combine the principles of general relativity and quantum field theory. In chapter 2, we motivated and discussed the framework of group field theory in its functional and operator formulation. The final result of this chapter was the conceptual idea of the operator framework, followed by a realization that an explicit relation between functional and operator GFT does not exist. Nevertheless, this relation is highly desirable for the following reason:

The motivation and explicit form of GFT dynamics is conceptually justified only in the functional approach, but the cosmological calculations and the many body interpretation of GFT is performed only in the functional formulation. Without an explicit dictionary between the ingredients, the dynamics of the operator theory is not justified and could cast doubts on cosmological calculations. On the other hand, an interpretation of geometrically meaningful observables seems to be more easily done in the operator formulation using the intuition from many body physics. For that reason, our final conclusion of chapter 2 was to pose the question for an explicit relation between the functional and the operator approach.

In chapter 3 we provided a perturbative answer to this question and explicitly constructed a correspondence between the formalisms. Our construction uses the framework of algebraic non-relativistic quantum field theory. Our main result was to show that we need to modify the commutator between the field operators in order to include the dynamical relations of the functional GFT. By choosing a coherent state on the resulting algebra and restricting the space of smearing functions to a sub-space of real valued functions we obtain a one-to-one correspondence between the expectation values of field operators and correlation functions of the functional integral. Moreover, due to the corollary (4 on page 53) these coherent states correspond to the Fock representation of the algebra whenever they are translation invariant. We have also shown, that local minima of the action may lead to different operator algebras. We characterized the space of Fock-inequivalent coherent states by the space of tempered micro-functions. At the end of the chapter 3 we suggested an interpretation of Fock-inequivalent representations as phases of the GFT and the modified algebras as the space of effective observables in corresponding phase. We will discuss the concept of phases below in more details, after we recap the results of chapter 4.

A necessary ingredient of our construction is the classification of minima of the action of GFT. To simplify this task we suggested to use symmetry analysis. For this reason we developed the local symmetry analysis for multi-local actions in the first part of chapter 4 and derived the full local point-symmetry group for different GFT models in the second part of this chapter. Our analysis has shown, however, that the concept of point symmetry is too restrictive in the case of multi-local theories. And the (almost) only symmetries we obtained were captured by the left and right invariance of the Haar measure. Nevertheless, all models in question were symmetric under left translations. Even though this result is fairly trivial, its combination with the suggested definition of phases in GFT leads to an interesting consequence that we discuss below.

5.2 Broken symmetry phases of GFT

Due to the corollary 4 in chapter 3 we know that coherent states leading to an infinite particle states can not be symmetric under left translations. At the same time, however, we observed the left translation invariance of all discussed actions in functional GFT. This suggests the classification of phases of GFT in terms of symmetry breaking in the following way.

Assume that we have a tempered micro-function that corresponds to a minimum of the Boulatov action; for example the one that we have used at the end of chapter 3. In this case the solution is not invariant under left translation (even though the action is). For that reason the coherent state, ω^δ, that corresponds to that minimum, will not be translation invariant as well. Using the left translation automorphism α_y on the algebra of observables we can define a family of states $\{\omega^y = \omega^\delta \circ \alpha_y \,|\, y \in M\}$. It can be shown that each of this states leads to an inequivalent representation of the algebra [141], despite the fact that they are related by a symmetry of the action. In QFT's on space-time this process is called spontaneous symmetry breaking, and it is often used for characterization of phases [118, 164]. There, the phenomenon is usually described as an invariance of the Hamiltonian, but non-invariance of the ground state of the model under the symmetry transformation. Even though, in our case the dynamical operator is not directly introduced in the operator formalism, the symmetry breaking is still present. This allows us to make the following classification of phases in GFT (fig. 5.1): the Fock-phase — the one with finitely many particles — is the left translation invariant phase; the non-Fock phases — those with infinitely many particles — correspond to phases that break left translation invariance.

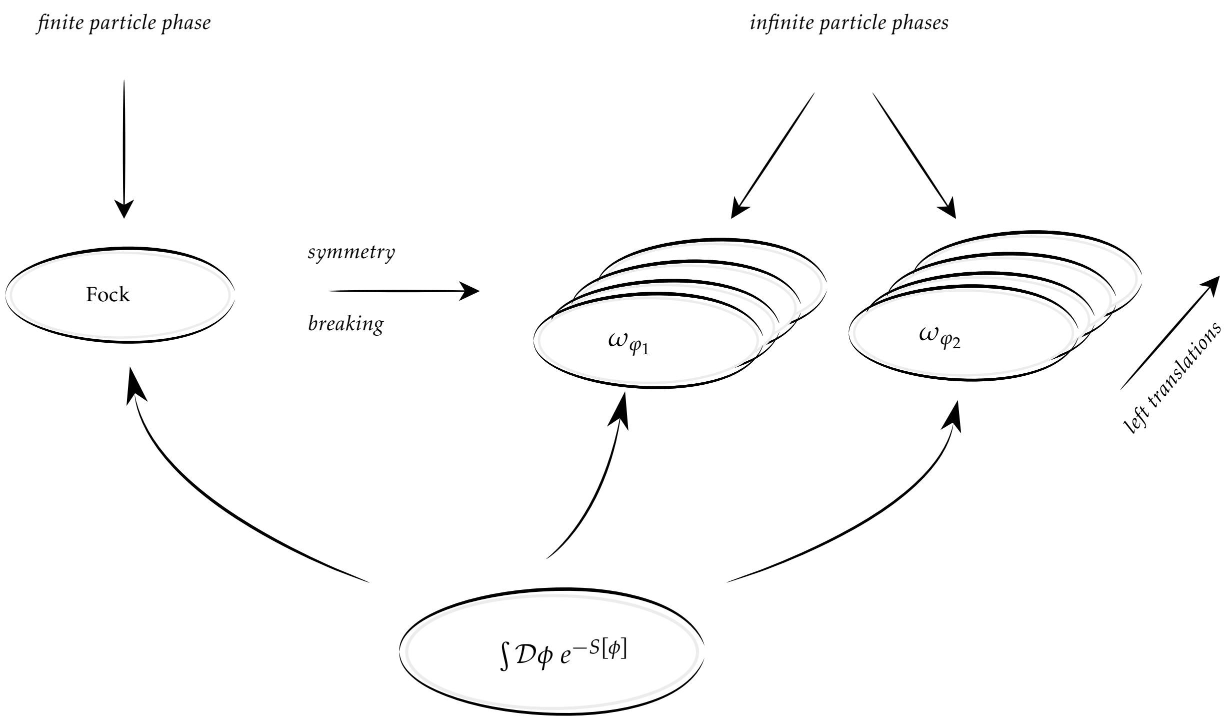


Figure 5.1: Perturbative phase diagram of GFT. Hereby, the whole family of states that appear from left translations of ω_{φ_i} define a single phase, just as in the case of statistical physics [118, 131]. Also phases that correspond to tempered micro-functions break translation invariance.

As we discussed above at least some of the phases with infinitely many particles are expected to contain states that describe smooth geometries. This intuition combined with the above symmetry breaking suggests the following question: can translation invariance of the GFT base manifold be related to discrete geometric information, whereas its breaking signals the phase transition to smooth geometries? We suggest to pursuit this question in future works.

5.3 Further research directions in GFT

We want to conclude this theses with a list of research directions in GFT that, from our point of view, are important to make significant progress in the field. The list is not meant to be complete nor comprehensive. Instead we restrict ourselves only to those directions that directly relate to topics that we discussed in this thesis:

COSMOLOGY FROM GFT PHASES: Starting with the geometric GFT models — such as Barrett-Crane model that we mentioned in chapter 4 — and following the procedure in [142, 148], we should investigate resulting cosmological equations of GFT. In contrast to existing work the dynamics of the model will directly stem from functional GFT and, hence, could be directly related and justified from spin foam models. This will provide a solid basis for cosmological calculations. Moreover, unlike in [58, 97, 142, 144, 147, 148], the included dynamics will not be truncated to the kinetic part of the action but will include the linearized version of the multi-local interactions of the model.

NON-LOCAL SYMMETRIES OF GFT: The concept of point symmetries as we introduced them in our work is too strong for the multi-local models of GFT. For that reason we suggest to extend the definition of symmetries in two ways:

Lie-Baecklund symmetries: This type of symmetries is already present in the analysis of partial differential equations and generalizes the concept of point symmetries. The main idea is to use the generators of the symmetry transformation such that

$$X_M^{LB}\left(x,\phi\left(x\right),\partial_x\phi\left(x\right),\cdots\right)\qquad X_V^{LB}\left(x,\phi\left(x\right),\partial_x\phi\left(x\right),\cdots\right).$$

These generators depend on an arbitrary but finite order of field derivatives and generalize the concept of point symmetries that are generated by $X_M\left(x,\phi\left(x\right)\right)$ and $X_V\left(x,\phi\left(x\right)\right)$. However, this generalization also prohibits us to give a geometrical interpretation to the new generators X_M^{LB} and X_V^{LB}, since they can no longer be seen as vector fields on the vector bundle of the model. Even in the local case of partial differential equations this generalization leads to a large number of new symmetries. In GFT this may enlarge the symmetry group leading to non-trivial symmetries that could be further used for classification of solutions.

Infinite dimensional calculus: Another approach to symmetries could take on a different lead: a muli-local action S can be rewritten as an integral over a Lagrangian L such that

$$S\left[\phi\right]=\int_M \mathrm{d}x\, L\left[x,\phi\right]. \tag{5.1}$$

However, here L itself would be a functional that depends on the whole field configuration ϕ on M rather than on local points of the form $(x, \phi(x), D\phi|_x)$. In this case a formulation of the theory on vector and jet bundles does not seem to be appropriate. The formalism of infinitely dimensional manifolds [160] can, however, be more suitable for this case. Using this formulation a symmetry of S should be formulated purely in terms of field transformations, $\phi \mapsto \tilde{\phi}$, and will not need to satisfy the compatibility conditions eq. (4.44) and eq. (4.45). We expect the resulting symmetry to be very large and general enough to capture the truly multi-local features of GFT.

REGULARIZATION OF THE BOTTOMLESS ACTION: The bottomless structure of GFT action prevents us from a non-perturbative definition of functional and operator GFT. Moreover, for renormalization analysis even using non-perturbative methods such as the functional renormalization group flow [104, 165], the bound from below is highly desired. In the literature a procedure of regularizing a bottomless actions exists [166]. The construction suggests to exchange the original action of the model by one that is bounded from below such that the perturbative expansion of the quantum theory remains untouched. To our knowledge this formulation has not yet been applied in GFT. We believe it is an important direction of development in GFT that should be investigated in future work if a non-perturbative formulation of GFT is desirable.

FULL CLASSIFICATION OF MINIMA FOR GFT MODELS: This is one of the most important technical aspects for our construction. Even if the current symmetry analysis does not provide much insight in the characterization of the minima the problem can be addressed in a straight forward way using Fourier analysis on Lie groups [132]. The characterization of local minima for the Boulatov and the Boulatov-like action with the Laplace-Beltrami operator is the current work in progress by the author and collaborators [**Ben-Geloun:aa**].

FORMULATION OF THE CONSTRAINT OPERATORS IN GFT: This problem is important for better understanding of the operator formulation of GFT. A construction of constraint operators directly on the level of the algebra may lead to better intuition of the dynamics of GFT in the language of many body physics. A step in this direction could be achieved by taking the direct sum over all algebras for each minimum of the action S. A constraint operator would, then, act as a projector on the suitable sub-algebra. The problem with this naive formulation is that the direct-sum-algebra will not be primitive, and hence does not admit faithful irreducible representations. The later, however, are of fundamental importance for the formulation of algebraic QFT [42].

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Index