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Analysis of Teukolsky equations on slowly rotating Kerr spacetimes

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Abstract

In this thesis, we treat the extreme Newman-Penrose components of both the Maxwell field ($s = \pm 1$) and the linearized gravitational perturbations (or "linearized gravity" for short) ($s = \pm 2$) in the exterior of a slowly rotating Kerr black hole. Upon different rescalings, we can obtain spin s components which satisfy the separable Teukolsky master equation (TME). For each of these spin s components defined in Kinnersley tetrad, the resulting equations by performing some first-order differential operator on it once and twice (twice only for $s = \pm 2$), together with the TME, are in the form of an "inhomogeneous spin-weighted wave equation" (ISWWE) with different potentials and constitute a linear spin-weighted wave system. We then prove energy and integrated local energy decay (Morawetz) estimates for this type of ISWWE, and utilize them to achieve both a uniform bound of a positive definite energy and a Morawetz estimate for the regular extreme Newman-Penrose components defined in the regular Hawking-Hartle tetrad.

We also present some brief discussions on mode stability for TME for the case of real frequencies. This says that in a fixed subextremal Kerr spacetime, there is no nontrivial separated mode solutions to TME which are purely ingoing at horizon and purely outgoing at infinity. This yields a representation formula for solutions to inhomogeneous Teukolsky equations, and will play a crucial role in generalizing the above energy and Morawetz estimates results to the full subextremal Kerr case.

Contents

1. Introduction and overview

The stability conjecture of Kerr black holes says that metrics of the subextremal Kerr family of spacetimes $(\mathcal{M}, g = g_{M,a})$ $(|a| < M)$ are (expected to be) stable against small perturbations of initial data as solutions to the vacuum Einstein equations (VEE)

$$\mathbf{Ric}[g]_{\mu\nu} = 0, \tag{1.1}$$

$\mathbf{Ric}[g]_{\mu\nu}$ being the Ricci curvature tensor of the metric. Before approaching this fully nonlinear problem, the null geodesic equations, the scalar wave equation, the Maxwell equations and then some proper linearization of VEE are a sequence of models with increasing accuracy for the nonlinear dynamics.

In the main part of this thesis, we consider the Maxwell equations for a real two-form $\mathbf{F}_{\alpha\beta}$:

$$\nabla^{\alpha}\mathbf{F}_{\alpha\beta} = 0 \qquad\qquad \nabla_{[\gamma}\mathbf{F}_{\alpha\beta]} = 0 \tag{1.2}$$

and solutions to some proper linearization of VEE–linearized gravitational perturbations (linearized gravity)–on a slowly rotating Kerr background. Following the Newman-Penrose (N-P) formalim Newman and Penrose (1962, 1963) to perform tetrad perturbations, Teukolsky Teukolsky (1972) showed that some components of Maxwell field and linearized gravity–the gauge invariant extreme spin components–satisfy a single master equation, the *Teukolsky master equation* (TME). Each extreme spin component and the quantities constructed by applying some differential operators up to certain times on it satisfy an inhomogeneous, linear wave system. By treating these linear wave systems, we follow the author's own works Ma (2017a,b) to prove in this work the uniform boundedness of a positive definite energy and an integrated local energy decay (Morawetz) estimate for each extreme spin component. The pointwise decay estimates, which enjoy their own interests, can be obtained in a standard way from these results. Moreover, from the work of Dafermos et al. (2014), it is likely that we can generalize it to the full subextremal Kerr case with our mode stability result Andersson et al. (2017b) for general spin fields. On the other hand, these estimates for gauge invariant extreme spin components are also crucial for linear stability of Kerr spacetimes. The aforementioned mode stability result for general spin fields on full subextremal Kerr spacetimes is reviewed in the last part in this thesis.

We here give an overview of this thesis. Chapter 2 is devoted to introductory materials about the Kerr spacetimes, the extreme spin components and their governing inhomogeneous, linear wave systems, the main results and a short summary of the relevant results. We prove some red-shift estimates near horizon and Morawetz estimates near infinity for different extreme spin components in Chapter 3, and give a proof of the main results Theorem 2.4.2 in Chapter 4 under assumptions of some estimates. These assumed estimates are then proved in Chapters 5 and 7 for Schwarzschild case and slowly rotating Kerr case respectively, based on a version of energy and Morawetz estimates (i.e. Theorem 2.4.1) for the wave systems proved in Chapter 5 and Chapter 6 in Schwarzschild and slowly rotating Kerr spacetimes. In Chapter 8, we briefly review our mode stability result for general spin fields on subextremal Kerr backgrounds.

2. Preliminaries and main results

Contents

2.1. Kerr metric

The subextremal Kerr Kerr (1963) family of spacetimes $(\mathcal{M}, g_{M,a})$ $(|a| < M)$, in Boyer-Lindquist (B-L) coordinates (t, r, θ, ϕ) Boyer and Lindquist (1967), has the metric

$$\begin{aligned} g_{M,a} = &- \left(1 - \tfrac{2Mr}{\Sigma}\right) dt^2 - \tfrac{2Mar\sin^2\theta}{\Sigma}(dtd\phi + d\phi dt) \\ &+ \tfrac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \tfrac{\sin^2\theta}{\Sigma}\left[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2, \end{aligned} \tag{2.1}$$

where

$$\Delta(r) = r^2 - 2Mr + a^2 \qquad \text{and} \qquad \Sigma(r,\theta) = r^2 + a^2\cos^2\theta. \tag{2.2}$$

A Kerr spacetime is parameterized by its mass M and angular momentum per mass a, and describes a rotating, stationary (with ∂_t Killing), axisymmetric (with ∂_ϕ Killing), asymptotically flat vacuum black hole. Setting $a = 0$ recovers the spherically symmetric Schwarzschild metric Schwarzschild (1916). The function $\Delta(r)$ has two zeros

$$r_+ = M + \sqrt{M^2 - a^2} \qquad \text{and} \qquad r_- = M - \sqrt{M^2 - a^2}, \tag{2.3}$$

which correspond to the locations of event horizon $\mathcal{H}$ and Cauchy horizon, respectively. We will constrain our considerations in the closure of the exterior region of a Kerr black hole, or in another way, in the domain of outer communication (DOC) $\mathcal{D}$. As mentioned below in Section 2.4, we will focus only on the future development by symmetry, hence only the future part $\mathcal{H}^+$ of the event horizon will be of interest for us. In what follows, whenever we say "in a slowly rotating Kerr spacetime", it should always be understood as the DOC of a Kerr spacetime $(\mathcal{M}, g_{M,a})$ with $|a|/M \leq a_0/M \ll 1$ sufficiently small.

Introduce the tortoise coordinate system (t, r^*, θ, ϕ)[1], with r^* defined by:

$$\tfrac{dr^*}{dr} = \tfrac{r^2+a^2}{\Delta}, \qquad r^*(3M) = 0. \tag{2.4}$$

The B-L coordinate system fails to extend across the future event horizon $\mathcal{H}^+$ due to the singularity in the metric coefficients in (2.1). Instead, we define an ingoing Eddington-Finkelstein (E-F) coordinate system $(v, r, \theta, \tilde{\phi})$ which is regular on $\mathcal{H}^+$:

$$\begin{cases} dv = dt + dr^*, \\ d\tilde{\phi} = d\phi + a(r^2+a^2)^{-1} dr^*. \end{cases} \tag{2.5}$$

We finally define a global Kerr coordinate system (t^*, r, θ, ϕ^*), via gluing the coordinate system $(\vartheta = v - r, r, \theta, \tilde{\phi})$ near horizon with the B-L coordinate system (t, r, θ, ϕ) away from horizon smoothly, by

$$\begin{cases} t^* = t + \chi_1(r)\left(r^*(r) - r - r^*(r_0) + r_0\right), \\ \phi^* = \phi + \chi_1(r)\acute{\phi}(r) \mod 2\pi, \quad d\acute{\phi}/dr = a/\Delta. \end{cases} \tag{2.6}$$

Here, the smooth cutoff function $\chi_1(r)$, which equals to 1 in $[r_+, M + r_0/2]$ and identically vanishes for $r \geq r_0$ with $r_0(M)$ fixed in Chapter 3, is chosen such that the initial hypersurface[2]

$$\Sigma_0 = \{(t^*, r, \theta, \phi^*) | t^* = 0\} \cap \mathcal{D} \tag{2.7}$$

[1] This is called as Regge-Wheeler Regge and Wheeler (1957) coordinate system when on Schwarzschild.

[2] Here the initial hypersurface could be taken as $\{t^* = D\}$ hypersurface for any real value D, but for convenience, we take it as in (2.7).

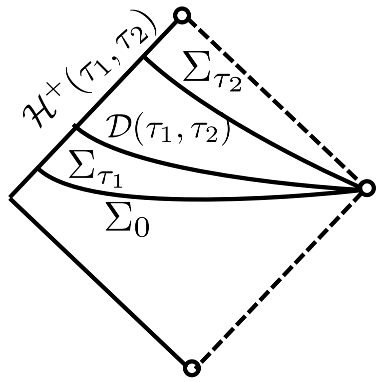

Figure 2.1.: Penrose diagram

is a spacelike hypersurface with

$$c(M) \leq -g(\nabla t^*, \nabla t^*)|_{\Sigma_0} \leq C(M), \tag{2.8}$$

$c(M)$ and $C(M)$ being two universal positive constants. We notice that

$$\partial_{t^*} = \partial_t \triangleq T \quad \text{and} \quad \partial_{\phi^*} = \partial_{\tilde{\phi}} = \partial_\phi, \tag{2.9}$$

and denote φ_τ to be the 1-parameter family of diffeomorphisms generated by T. Define constant time hypersurfaces

$$\Sigma_\tau = \varphi_\tau(\Sigma_0) = \{(t^*, r, \theta, \phi^*)|t^* = \tau\} \cap \mathcal{D}. \tag{2.10}$$

They are spacelike hypersurfaces satisfying (2.8), and in particular, for $r \leq M + r_0/2$, we have

$$1 \leq -g(\nabla t^*, \nabla t^*)|_{\Sigma_\tau} = 1 + 2Mr/\Sigma \leq 3. \tag{2.11}$$

For any $0 \leq \tau_1 < \tau_2$, we use the notations

$$\mathcal{D}(\tau_1,\tau_2) = \bigcup_{\tau\in[\tau_1,\tau_2]} \Sigma_\tau, \qquad \mathcal{H}^+(\tau_1,\tau_2) = \partial\mathcal{D}(\tau_1,\tau_2) \cap \mathcal{H}^+. \tag{2.12}$$

The reader may find the Penrose diagram Figure 2.1 useful.

The volume form of the spacetime manifold is

$$d\text{Vol}_{\mathcal{M}} = \begin{cases} \Sigma dt dr \sin\theta d\theta d\phi & \text{in B-L coordinates,} \\ \Sigma dt^* dr \sin\theta d\theta d\phi^* & \text{in global Kerr coordinates,} \end{cases} \tag{2.13}$$

and the volume form of the hypersurface $\Sigma_\tau(\tau \geq 0)$ is

$$d\text{Vol}_{\Sigma_\tau} = \Sigma dr \sin\theta d\theta d\phi^* \quad \text{in global Kerr coordinates.} \tag{2.14}$$

Unless otherwise specified, we will always suppress these volume forms in this paper when the integral is over a spacetime region or a 3-dimensional submanifold of Σ_τ.

2.2. Teukolsky master equation for spin s components

Following the Newman-Penrose (N-P) formalim Newman and Penrose (1962, 1963), we obtain N-P components of electromagnetic field

$$\Psi_0 = \mathbf{F}_{\mu\nu} l^\mu m^\nu, \;\; \Psi_1 = \tfrac{1}{2}\mathbf{F}_{\mu\nu}(l^\mu n^\nu + \overline{m}^\mu m^\nu), \;\; \Psi_2 = \mathbf{F}_{\mu\nu}\overline{m}^\mu n^\nu, \tag{2.15}$$

and the complete five N-P components of Weyl tensor

$$\begin{aligned} \Phi_0 = -\,\mathbf{W}_{lmlm}, \qquad \Phi_1 = -\mathbf{W}_{lnlm}, \qquad \Phi_2 = -\mathbf{W}_{lm\overline{m}n}, \\ \Phi_3 = -\mathbf{W}_{ln\overline{m}n}, \qquad \Phi_4 = -\mathbf{W}_{n\overline{m}n\overline{m}}, \end{aligned} \tag{2.16}$$

by projecting the Maxwell tensor $\mathbf{F}_{\alpha\beta}$ and the Weyl tensor $\mathbf{W}_{\alpha\beta\gamma\delta}$ onto the Kinnersley null tetrad $(l, n, m, \overline{m})$ Kinnersley (1969):

$$\begin{aligned} l^\mu &= \tfrac{1}{\Delta}(r^2 + a^2, \Delta, 0, a), \\ n^\nu &= \tfrac{1}{2\Sigma}(r^2 + a^2, -\Delta, 0, a), \\ m^\mu &= \tfrac{1}{\sqrt{2}\bar{\rho}}\left(ia\sin\theta, 0, 1, \tfrac{i}{\sin\theta}\right), \end{aligned} \tag{2.17}$$

and $\overline{m}^\mu$ and $\bar{\rho}$ being the complex conjugate of m^μ and $\rho = r - ia\cos\theta$ respectively. The Maxwell equations, and the full set of N-P equations comprising the commutation relations, the Ricci identities, the eliminant relations and the Bianchi identities in (Chandrasekhar, 1998, Chapter 1.8), are both coupled first–order differential systems, with the later one linking the tetrad, the spin coefficients and these five N-P components of Weyl tensor.

The background N-P components on Kerr for Weyl tensor are

$$\Phi_0^B = \Phi_1^B = \Phi_3^B = \Phi_4^B = 0, \qquad\qquad \Phi_2^B = -M\bar{\rho}^{-3}. \tag{2.18}$$

We perturb in the N-P equations the tetrad, the spin coefficients and the five N-P components by $l^T = l^B + l^P$, $\kappa^T = \kappa^B + \kappa^P$,[3] $\Phi_0^T = \Phi_0^B + \Phi_0^P$, etc, and the complete set of equations for linearized gravity is then obtained from the N-P equations by keeping the perturbation terms (with superscript P) only to first order. The total parts of the perturbed extreme N-P components Φ_0^T and Φ_4^T (equal to the perturbation parts Φ_0^P and Φ_4^P) for linearized gravity are the "ingoing and outgoing radiative parts", and are invariant under gauge transformations and infinitesimal tetrad rotations. From now on, we will drop the superscript T and still denote these perturbed extreme N-P components as Φ_0 and Φ_4. Similarly, we can also

[3] κ^B is one of the background spin coefficients used in (Chandrasekhar, 1998, Chapter 1.8).

perturb the N-P components of Maxwell field, and define the perturbed, gauge invariant extreme N-P components Ψ_0^T and Ψ_2^T, or simply Ψ_0 and Ψ_2.

Teukolsky Teukolsky (1972) derived the decoupled equations on Kerr backgrounds for the spin s components $\psi_{[s]}$ ($s = \pm 2$ for linearized gravity and $s = \pm 1$ for Maxwell field)

$$\psi_{[+1]} = \Delta \Psi_0, \qquad \psi_{[-1]} = \Delta^{-1} \rho^2 \Psi_2, \tag{2.19a}$$
$$\psi_{[+2]} = \Delta^2 \Phi_0, \qquad \psi_{[-2]} = \Delta^{-2} \rho^4 \Phi_4, \tag{2.19b}$$

and showed that these decoupled equations are in fact *separable* and governed by a single master equation–the celebrated *Teukolsky Master Equation* (TME)–given in B-L coordinates by[4]

$$\begin{aligned} &- \left[\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\theta\right] \frac{\partial^2 \psi_{[s]}}{\partial t^2} - \frac{4Mar}{\Delta} \frac{\partial^2 \psi_{[s]}}{\partial t \partial \phi} - \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right] \frac{\partial^2 \psi_{[s]}}{\partial \phi^2} \\ &+ \Delta^s \frac{\partial}{\partial r}\left(\Delta^{-s+1} \frac{\partial \psi_{[s]}}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi_{[s]}}{\partial \theta}\right) + 2s \left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right] \frac{\partial \psi_{[s]}}{\partial \phi} \\ &+ 2s \left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right] \frac{\partial \psi_{[s]}}{\partial t} - (s^2 \cot^2\theta + s)\psi_{[s]} = 0. \end{aligned} \tag{2.20}$$

The Kinnersley tetrad is, however, singular on $\mathcal{H}^+$ in ingoing E-F coordinates, suggesting that the perturbed N-P components are not all regular there. We perform a null rotation by

$$\begin{cases} l \to \tilde{l} = \Delta/(2\Sigma) \cdot l, \\ n \to \tilde{n} = (2\Sigma)/\Delta \cdot n, \\ m \to m, \end{cases} \tag{2.21}$$

and the resulting tetrad $(\tilde{l}, \tilde{n}, m, \overline{m})$, namely the Hawking-Hartle (H-H) tetrad Hawking and Hartle (1972), is in fact regular up to and on $\mathcal{H}^+$ in global Kerr coordinates. The regular extreme N-P components of Maxwell field and linearized gravity in regular H-H tetrad are then

$$\widetilde{\Psi_0}(\mathbf{F}) = \mathbf{F}_{\mu\nu} \tilde{l}^\mu m^\nu = \frac{1}{2\Sigma} \psi_{[+1]}, \qquad \widetilde{\Psi_2}(\mathbf{F}) = \mathbf{F}_{\mu\nu} \overline{m}^\mu \tilde{n}^\nu = \frac{2\Sigma}{\rho^2} \psi_{[-1]}, \tag{2.22}$$
$$\widetilde{\Phi_0}(\mathbf{W}) = -\mathbf{W}_{\tilde{l}m\tilde{l}m} = \frac{1}{4\Sigma^2} \psi_{[+2]}, \qquad \widetilde{\Phi_4}(\mathbf{W}) = -\mathbf{W}_{\tilde{n}\overline{m}\tilde{n}\overline{m}} = \frac{4\Sigma^2}{\rho^4} \psi_{[-2]}, \tag{2.23}$$

respectively. The results in this paper will be with respect to complex scalars $\widetilde{\Phi_0}$ and $\widetilde{\Phi_4}$, and $\widetilde{\Psi_0}$ and $\widetilde{\Psi_2}$.

[4]These scalars differ with the ones used in Teukolsky (1972) by a rescaling of Δ^s.

2.3. Coupled systems

Denote the future-directed ingoing and outgoing principal null vector fields in B-L coordinates

$$Y = \tfrac{(r^2+a^2)\partial_t + a\partial_\phi}{\Delta} - \partial_r, \qquad V = \tfrac{(r^2+a^2)\partial_t + a\partial_\phi}{\Delta} + \partial_r. \tag{2.24a}$$

From TME (2.20), the equation for $\psi_{[s]}$ is

$$\left(\Sigma\Box_g + 2is\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - (s^2\cot^2\theta + s)\right)\psi_{[s]} = -2sZ\psi_{[s]}, \tag{2.25}$$

with $\Box_g$ being the scalar wave operator for the metric g, and

$$Z = (r-M)Y - 2r\partial_t. \tag{2.26}$$

Construct the quantities

$$\phi^0_{+1} = \psi_{[+1]}/r^2, \qquad \phi^1_{+1} = rY(r\phi^0_{+1}), \tag{2.27a}$$
$$\phi^0_{-1} = \Delta/r^2\psi_{[-1]}, \qquad \phi^1_{-1} = rV(r\phi^0_{-1}). \tag{2.27b}$$

for spin ±1 components, and

$$\phi^0_{+2} = \psi_{[+2]}/r^4, \qquad \phi^1_{+2} = rY(r\phi^0_{+2}), \qquad \phi^2_{+2} = rY(r\phi^1_{+2}), \tag{2.28a}$$
$$\phi^0_{-2} = \Delta^2/r^4\psi_{[-2]}, \qquad \phi^1_{-2} = -rV(r\phi^0_{-2}), \qquad \phi^2_{-2} = -rV(r\phi^1_{-2}), \tag{2.28b}$$

for spin ±2 components. The upper index here denotes the number of times the differential operator rYr or $-(rVr)$ is performed. We should notice that though it is $\Delta/(r^2+a^2)V$ but not V which is a regular vector field on $\mathcal{H}^+$, the variables $\{\phi^i_s\}$ are indeed smooth up to and on future horizon if the regular N-P scalars $\widetilde{\Psi}_2$ and $\widetilde{\Phi}_4$ are. In global Kerr coordinates, the regular vector field Y equals to $-\partial_r + \partial_{t^*}$ in $[r_+, M + r_0/2]$ and is $\frac{r^2+a^2}{\Delta}\partial_{t^*} + \frac{a}{\Delta}\partial_{\phi^*} - \partial_r$ for $r \geq r_0$.

The governing equations for these quantities are[5]

$$\mathbf{L}^1_{+1}\phi^0_{+1} = F^0_{+1} = \tfrac{2(r^2-3Mr+2a^2)}{r^3}\phi^1_{+1} - \tfrac{4(a^2\partial_t + a\partial_\phi)\phi^0_{+1}}{r}, \tag{2.29a}$$
$$\mathbf{L}^1_{+1}\phi^1_{+1} = F^1_{+1} = -2(a^2\partial_t + a\partial_\phi)\phi^0_{+1}, \tag{2.29b}$$

$$\mathbf{L}^1_{-1}\phi^0_{-1} = F^0_{-1} = -\tfrac{2(r^2-3Mr+2a^2)}{r^3}\phi^1_{-1} + \tfrac{4(a^2\partial_t + a\partial_\phi)\phi^0_{-1}}{r}, \tag{2.30a}$$

[5] There exists a different set for variables of which the governing wave systems contain no ∂_t derivative terms on the right hand side (RHS). See Appendix A.

$$\mathbf{L}^1_{-1}\phi^1_{-1} = F^1_{-1} = -2(a^2\partial_t + a\partial_\phi)\phi^0_{-1}, \tag{2.30b}$$

$$\begin{aligned}
\mathbf{L}^0_{+2}\phi^0_{+2} &= F^0_{+2} = \tfrac{4(r^2-3Mr+2a^2)}{r^3}\phi^1_{+2} - \tfrac{8(a^2\partial_t+a\partial_\phi)\phi^0_{+2}}{r}, & (2.31a)\\
\mathbf{L}^1_{+2}\phi^1_{+2} &= F^1_{+2} = \tfrac{2(r^2-3Mr+2a^2)}{r^3}\phi^2_{+2} + \tfrac{6Mr-12a^2}{r}\phi^0_{+2}\\
&\qquad - \tfrac{4(a^2\partial_t+a\partial_\phi)\phi^1_{+2}}{r} - 6(a^2\partial_t + a\partial_\phi)\phi^0_{+2}, & (2.31b)\\
\mathbf{L}^1_{+2}\phi^2_{+2} &= F^2_{+2} = -8(a^2\partial_t + a\partial_\phi)\phi^1_{+2} - 12a^2\phi^0_{+2}, & (2.31c)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{L}^0_{-2}\phi^0_{-2} &= F^0_{-2} = \tfrac{4(r^2-3Mr+2a^2)}{r^3}\phi^1_{-2} + \tfrac{8(a^2\partial_t+a\partial_\phi)\phi^0_{-2}}{r}, & (2.32a)\\
\mathbf{L}^1_{-2}\phi^1_{-2} &= F^1_{-2} = \tfrac{2(r^2-3Mr+2a^2)}{r^3}\phi^2_{-2} + \tfrac{6Mr-12a^2}{r}\phi^0_{-2}\\
&\qquad + \tfrac{4(a^2\partial_t+a\partial_\phi)\phi^1_{-2}}{r} + 6(a^2\partial_t + a\partial_\phi)\phi^0_{-2}, & (2.32b)\\
\mathbf{L}^1_{-2}\phi^2_{-2} &= F^2_{-2} = 8(a^2\partial_t + a\partial_\phi)\phi^1_{-2} - 12a^2\phi^0_{-2}, & (2.32c)
\end{aligned}$$

respectively. The subscript here indicates the spin weight $s = \pm 1$, ± 2, and the operators $\mathbf{L}^0_s$ and $\mathbf{L}^1_s$, given by

$$\begin{aligned}
\mathbf{L}^0_s &= \Sigma\Box_g + 2is\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - s^2\left(\cot^2\theta + \tfrac{r^2+2Mr-2a^2}{2r^2}\right), & (2.33a)\\
\mathbf{L}^1_s &= \Sigma\Box_g + 2is\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - s^2\left(\cot^2\theta + \tfrac{r^2-2Mr+2a^2}{r^2}\right), & (2.33b)
\end{aligned}$$

are both called as *"spin-weighted wave operators"*, but with different potentials.

Remark 2.3.1. The underlying reason for applying $|s|$ times the first-order differential operators to the spin s components is to make the nonzero boost weight vanishing. This is closely related to *Chandrasekhar transformation* Chandrasekhar (1975) on Schwarzschild as well.

The equations (2.29)–(2.32) for ϕ^i_s are in either of the following forms:

$$\begin{aligned}
\mathbf{L}^0_s\psi &= F; & (2.34a)\\
\mathbf{L}^1_s\psi &= F, & (2.34b)
\end{aligned}$$

which are called as "*inhomogeneous spin-weighted wave equations*" (ISWWE) in this thesis. When there is no confusion of which spin component we are treating, we may suppress the subscript of ϕ^i_s and simply write as ϕ^i.

Remark 2.3.2. After making the substitutions $\partial_t \leftrightarrow -i\omega$, $\partial_\phi \leftrightarrow im$, and separating the operators $\mathbf{L}_s^k$ $(k = 0, 1)$, the angular parts are the spin-weighted spheroidal harmonic operator of angular Teukolsky equation. The radial operator of $\mathbf{L}_s^1$ is the sum of the radial part of the rescaled scalar wave operator $\Sigma\Box_g$ and a potential $s^2(r^2 - \Delta - a^2)/r^2$, and reduces to the radial operator for Fackerell-Ipser (F-I) equation Fackerell and Ipser (1972) or Regge-Wheeler equation Regge and Wheeler (1957) when on Schwarzschild background $(a = 0)$, while the one of $\mathbf{L}_s^0$ is the sum of the radial part of $\Sigma\Box_g$ and another potential $s^2(\Delta + a^2)/(2r^2)$. See more details in Section 5.2 for Schwarzschild case and Section 6.2 for Kerr case.

We also note that in the non-static Kerr case $(a \neq 0)$, the classical F-I operator in Fackerell and Ipser (1972) for Maxwell field has an imaginary zeroth order term in the potential, thus being quite different from the operator $\mathbf{L}_s^1$ here in which the imaginary coefficients are accompanied by first order ∂_t and ∂_ϕ derivatives. This is the main difference which enables us in this paper to not introduce fractional derivative operators as in Andersson and Blue (2015a) which treats the classical F-I equation.

2.4. Main theorems

The TME admits a symmetry that $\Delta^s\psi_{[-s]}(-t, r, \theta, -\phi)$ and $\psi_{[s]}(t, r, \theta, \phi)$ satisfy the same equation, hence we focus only on the future time development in this paper, and one could easily obtain the analogous estimates in the past time direction.

For any complex-valued smooth function $\psi : \mathcal{M} \to \mathbb{C}$ with spin weight s, we define in global Kerr coordinates for any $\tau \geq 0$ that

$$|\partial\psi(t^*, r, \theta, \phi^*)|^2 = |\partial_{t^*}\psi|^2 + |\partial_r\psi|^2 + |\nabla\!\!\!\!/\,\psi|^2, \tag{2.35}$$

$$E_\tau(\psi) = \int_{\Sigma_\tau} |\partial\psi|^2, \tag{2.36}$$

and in ingoing E-F coordinates for any $\tau_2 > \tau_1 \geq 0$ that

$$E_{\mathcal{H}^+(\tau_1,\tau_2)}(\psi) = \int_{\mathcal{H}^+(\tau_1,\tau_2)} (|\partial_v\psi|^2 + |\nabla\!\!\!\!/\,\psi|^2) r^2 dv \sin\theta d\theta d\tilde{\phi}. \tag{2.37}$$

The $\nabla\!\!\!\!/$ used here are not the standard covariant angular derivatives $\check{\nabla\!\!\!\!/}$ on sphere $\mathbb{S}^2(t^*, r)$, but the spin-weighted version of them, and we take $\nabla\!\!\!\!/$ to be any one of

$\not\nabla_j$ $(j = 1, 2, 3)$ defined by

$$\begin{cases} r\not\nabla_1 &= r\check{\not\nabla}_1 - \frac{is\cos\phi}{\sin\theta} = (-\sin\phi\partial_\theta - \frac{\cos\phi}{\sin\theta}\cos\theta\partial_{\phi^*}) - \frac{is\cos\phi}{\sin\theta}, \\ r\not\nabla_2 &= r\check{\not\nabla}_2 - \frac{is\sin\phi}{\sin\theta} = (\cos\phi\partial_\theta - \frac{\sin\phi}{\sin\theta}\cos\theta\partial_{\phi^*}) - \frac{is\sin\phi}{\sin\theta}, \\ r\not\nabla_3 &= r\check{\not\nabla}_3 = \partial_{\phi^*}. \end{cases} \tag{2.38}$$

In global Kerr coordinates, we can express the modulus square of $\not\nabla\psi$ as

$$\begin{aligned} |\not\nabla\psi|^2 = \sum_{i=1,2,3} |\not\nabla\psi|^2 &= \tfrac{1}{r^2}\Big(|\partial_\theta\psi|^2 + \Big|\tfrac{\cos\theta\partial_{\phi^*}\psi + is\psi}{\sin\theta}\Big|^2 + |\partial_{\phi^*}\psi|^2\Big) \\ &= \tfrac{1}{r^2}\Big(|\partial_\theta\psi|^2 + \Big|\tfrac{\partial_{\phi^*}\psi + is\cos\theta\psi}{\sin\theta}\Big|^2 + s^2|\psi|^2\Big). \end{aligned} \tag{2.39}$$

In particular, note from (2.39) that $|\not\nabla\psi|^2$, and thus $|\partial\psi|^2$, already have control over $r^{-2}|\psi|^2$. The same expressions (2.38) and (2.39) hold true in B-L coordinates and ingoing E-F coordinates due to (2.9). For convenience of calculations, we may always refer to these expressions with ∂_ϕ in place of ∂_{ϕ^*} without confusion.

For any smooth function ψ with spin weight s, we define for any multi-index $i = (i_1, i_2, i_3, i_4, i_5)$ with $i_k \geq 0$ $(k = 1, 2, 3, 4, 5)$

$$\partial^i\psi = \partial_{t^*}^{i_1}\partial_r^{i_2}\not\nabla_1^{i_3}\not\nabla_2^{i_4}\not\nabla_3^{i_5}\psi. \tag{2.40}$$

Denote a few Morawetz densities by[6]

$$\mathbb{M}_{\text{deg}}(\psi) = r^{-1-\delta}|\partial_r\psi|^2 + \chi_{\text{trap}}(r)(r^{-1-\delta}|\partial_{t^*}\psi|^2 + r^{-1}|\not\nabla\psi|^2), \tag{2.41a}$$

$$\mathbb{M}(\psi) = r^{-1-\delta}(|\partial_r\psi|^2 + |\partial_{t^*}\psi|^2) + r^{-1}|\not\nabla\psi|^2, \tag{2.41b}$$

$$\widetilde{\mathbb{M}}_{\text{deg}}(\psi) = r^{-1}|\partial_r\psi|^2 + \chi_{\text{trap}}(r)r^{-1}(|\partial_{t^*}\psi|^2 + |\not\nabla\psi|^2), \tag{2.41c}$$

$$\widetilde{\mathbb{M}}(\psi) = r^{-1}|\partial\psi|^2. \tag{2.41d}$$

Here, $\chi_{\text{trap}}(r) = (1 - 3M/r)^2(1 - \eta_{[r^-_{\text{trap}}, r^+_{\text{trap}}]}(r))$, $\eta_{[r^-_{\text{trap}}, r^+_{\text{trap}}]}(r)$ is the indicator function in the radius region bounded by minimal and maximal trapped radii $r^{\pm}_{\text{trap}}(\varepsilon_0, M)$ with ε_0 chosen in Theorem 2.4.1 below, and $\delta \in (0, 1/2)$ is an arbitrary constant. Note that when $\varepsilon_0 \to 0$, $r^{\pm}_{\text{trap}}(\varepsilon_0, M) \to r_{\text{trap}}(0, M) = 3M$.

Theorem 2.4.1. (Energy and Morawetz estimate for the inhomogeneous spin-weighted wave systems) In the DOC of a slowly rotating Kerr spacetime $(\mathcal{M}, g = g_{M,a})$, given any regular Maxwell field[7] $\mathbf{F}_{\alpha\beta}$ to the Maxwell equations (1.2) or

[6] We should distinguish among these different notations that one with a tilde means there is no extra $r^{-\delta}$ power in the coefficients of r- or t^*- derivative terms and one with the subscript deg means there is the trapping degeneracy in the trapped region, and vice versa.

[7] "regular Maxwell field" is defined in Definition 2.6.1.

any smooth[8] regular extreme N-P components as in Section 2.2 which vanish near spatial infinity, $\psi = \phi_s^j$ defined as in (2.27) and (2.28) satisfies an "inhomogeneous spin-weighted wave equation" (ISWWE)

$$\mathbf{L}_s^i \psi = F, \tag{2.42}$$

which takes the form of the corresponding subequation in the linear wave system (2.29)–(2.32) with $s = \pm 1$ or ± 2 being the spin weight and $\mathbf{L}_s^i$ defined as in (2.33), and let φ_s^i be any one of

$$\left\{ r^{4-\delta}\phi_{+2}^0, r^{2-\delta}\phi_{+2}^1, \phi_{+2}^2, \widetilde{\phi_{-2}^0}, \widetilde{\phi_{-2}^1}, \phi_{-2}^2 \right\} \tag{2.43}$$

with the same upper and lower indexes. Then, for any $0 < \delta < 1/2$, there exist universal constants $\varepsilon_0 = \varepsilon_0(M)$ and $C = C(M, \delta, \Sigma_0) = C(M, \delta, \Sigma_\tau)$ such that for all $|a|/M \leq a_0/M \leq \varepsilon_0$ and any $\tau \geq 0$, the following estimates hold true:

- for $|s| = 1$,

$$E_\tau(\psi) + E_{\mathcal{H}^+(0,\tau)}(\psi) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\psi) \leq C \sum_{j=0}^{|s|} E_0(\phi_s^j) + C\mathcal{E}(F). \tag{2.44}$$

  Here, the error term $\mathcal{E}(F)$ coming from the source F takes the form of

$$\mathcal{E}(F) = \left| \int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma} \Re \left(F \partial_t \bar{\psi} \right) \right| + \int_{\mathcal{D}(0,\tau)} r^{-3+\delta} |F|^2 \tag{2.45}$$

  with $\Re(\cdot)$ denoting the real part;

- for $(s, i) = (+2, 0)$ or $(+2, 1)$,

$$E_\tau(\varphi_s^i) + E_{\mathcal{H}^+(0,\tau)}(\varphi_s^i) + \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}_{\text{deg}}(\varphi_s^i) \leq C \left(E_0^{\text{total}}(s) + \mathcal{E}_s^i \right); \tag{2.46a}$$

- for other combinations of (s, i) with $|s| = 2$,

$$E_\tau(\varphi_s^i) + E_{\mathcal{H}^+(0,\tau)}(\varphi_s^i) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\varphi_s^i) \leq C \left(E_0^{\text{total}}(s) + \mathcal{E}_s^i \right). \tag{2.46b}$$

[8] In fact, the N-P components should be viewed as sections of a complex line bundle. Therefore, "smooth" means that these components and their derivatives to any order with respect to $(\partial_{t^*}, \partial_r, \nabla_1, \nabla_2, \nabla_3)$ are continous.

The expressions of $E_0^{\text{total}}(s)$ are given in (6.25), and the error terms here are

$$\mathcal{E}_s^i = \mathcal{E}_{\text{main},s}^i + \mathcal{E}_{\text{ex},s}^i, \tag{2.47}$$

with

$$\begin{aligned}\mathcal{E}_{\text{main},+2}^i = &\left| \int_{\mathcal{D}(0,\tau)} \Sigma^{-1} \Re \left(F_{+2}^i \partial_t \overline{\phi_{+2}^i} \right) \right| \\ &+ \tfrac{|a|}{M} \int_{\mathcal{D}(0,\tau)} \left(\widetilde{\mathbb{M}}(r^{4-\delta}\phi_{+2}^0) + \widetilde{\mathbb{M}}(r^{2-\delta}\phi_{+2}^1) + \mathbb{M}_{\text{deg}}(\phi_{+2}^2) \right), \end{aligned}\tag{2.48a}$$

$$\begin{aligned}\mathcal{E}_{\text{main},-2}^i = &\left| \int_{\mathcal{D}(0,\tau)} \Sigma^{-1} \Re \left(F_{-2}^i \partial_t \overline{\phi_{-2}^i} \right) \right| \\ &+ \tfrac{|a|}{M} \int_{\mathcal{D}(0,\tau)} \left(\mathbb{M}(\widetilde{\phi^0}) + \mathbb{M}(\widetilde{\phi^1}) + \mathbb{M}_{\text{deg}}(\phi_{-2}^2) + |\nabla\!\!\!/\, \widetilde{\phi^0}|^2 + |\nabla\!\!\!/\, \widetilde{\phi^1}|^2 \right), \end{aligned}\tag{2.48b}$$

and

$$\mathcal{E}_{\text{ex},+2}^0 = \epsilon_0 \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(r^{4-\delta}\phi_{+2}^0) + \tfrac{1}{\epsilon_0} \int_{\mathcal{D}(0,\tau)} r^{-3}|\phi_{+2}^1|^2, \tag{2.49a}$$

$$\mathcal{E}_{\text{ex},+2}^1 = \epsilon_1 \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(r^{2-\delta}\phi_{+2}^1) + \tfrac{1}{\epsilon_1} \int_{\mathcal{D}(0,\tau)} \left(\widetilde{\mathbb{M}}_{\text{deg}}(r^{4-\delta}\phi_{+2}^0) + \tfrac{|\phi_{+2}^2|^2}{r^3} \right), \tag{2.49b}$$

$$\mathcal{E}_{\text{ex},+2}^2 = 0, \tag{2.49c}$$

$$\mathcal{E}_{\text{ex},-2}^0 = \epsilon_0 \int_{\mathcal{D}(0,\tau)} \mathbb{M}(\widetilde{\phi^0}) + \tfrac{1}{\epsilon_0} \int_{\mathcal{D}(0,\tau)} r^{-3}|\widetilde{\phi_{-2}^1}|^2, \tag{2.49d}$$

$$\mathcal{E}_{\text{ex},-2}^1 = \epsilon_1 \int_{\mathcal{D}(0,\tau)} \mathbb{M}(\widetilde{\phi^1}) + \tfrac{1}{\epsilon_1} \int_{\mathcal{D}(0,\tau)} \left(r^{-3}|\phi_{-2}^2|^2 + r^{-2}|\widetilde{\phi_{-2}^0}|^2 \right), \tag{2.49e}$$

$$\mathcal{E}_{\text{ex},-2}^2 = 0. \tag{2.49f}$$

Theorem 2.4.2. Under the same assumptions of Theorem 2.4.1, for any $0 < \delta < 1/2$ and nonnegative integer n, there exist universal constants $\varepsilon_0 = \varepsilon_0(M)$ and $C = C(M, \delta, \Sigma_0, n) = C(M, \delta, \Sigma_\tau, n)$ such that for all $|a|/M \le a_0/M \le \varepsilon_0$ and any $\tau \ge 0$, it holds true for regular extreme N-P components:

$$\begin{aligned}&\sum_{|k|\le n} \int_{\mathcal{D}(0,\tau)} \left(\widetilde{\mathbb{M}}(\partial^k \Phi_0^{(0)}) + \widetilde{\mathbb{M}}(\partial^k \Phi_0^{(1)}) + \mathbb{M}_{\text{deg}}(\partial^k \Phi_0^{(2)}) \right) \\ &+ \sum_{|k|\le n} \sum_{i=0}^{2} \left(E_\tau(\partial^k \Phi_0^{(i)}) + E_{\mathcal{H}^+(0,\tau)}(\partial^k \Phi_0^{(i)}) \right) \le C \sum_{|k|\le n} \sum_{i=0}^{2} E_0(\partial^k \Phi_0^{(i)}), \end{aligned}\tag{2.50a}$$

$$\sum_{|k|\le n} \int_{\mathcal{D}(0,\tau)} \left(\mathbb{M}(\partial^k \Phi_4^{(0)}) + \mathbb{M}(\partial^k \Phi_4^{(1)}) + \mathbb{M}_{\text{deg}}(\partial^k \Phi_4^{(2)}) \right)$$

$$+\sum_{|k|\le n}\sum_{i=0}^{2}\Big(E_\tau(\partial^k\Phi_4^{(i)})+E_{\mathcal{H}^+(0,\tau)}(\partial^k\Phi_4^{(i)})\Big)\le C\sum_{|k|\le n}\sum_{i=0}^{2}E_0(\partial^k\Phi_4^{(i)}),\qquad(2.50\text{b})$$

and

$$\sum_{|i|\le n}\int_{\mathcal{D}(0,\tau)}\Big(\widetilde{\mathbb{M}}(\partial^i\Psi_0^{(0)})+\mathbb{M}_{\deg}(\partial^i\Psi_0^{(1)})\Big)$$
$$+\sum_{|i|\le n}\sum_{k=0,1}\Big(E_\tau(\partial^i\Psi_0^{(k)})+E_{\mathcal{H}^+(0,\tau)}(\partial^i\Psi_0^{(k)})\Big)\le C\sum_{|i|\le n}\sum_{k=0,1}E_0(\partial^i\Psi_0^{(k)}),\qquad(2.50\text{c})$$
$$\sum_{|i|\le n}\int_{\mathcal{D}(0,\tau)}\Big(\mathbb{M}(\partial^i\Psi_2^{(0)})+\mathbb{M}_{\deg}(\partial^i\Psi_2^{(1)})\Big)$$
$$+\sum_{|i|\le n}\sum_{k=0,1}\Big(E_\tau(\partial^i\Psi_2^{(k)})+E_{\mathcal{H}^+(0,\tau)}(\partial^i\Psi_2^{(k)})\Big)\le C\sum_{|i|\le n}\sum_{k=0,1}E_0(\partial^i\Psi_2^{(k)}).\qquad(2.50\text{d})$$

Here,

$$\Phi_0^{(0)}=r^{4-\delta}\widetilde{\Phi_0},\qquad\Phi_0^{(1)}=r^{4-\delta}Y\widetilde{\Phi_0},\qquad\Phi_0^{(2)}=r^4YY\widetilde{\Phi_0};\qquad(2.51\text{a})$$
$$\Phi_4^{(0)}=\widetilde{\Phi_4},\qquad\Phi_4^{(1)}=\tfrac{r\Delta}{r^2+a^2}V(r\Phi_4^{(0)}),\qquad\Phi_4^{(2)}=\tfrac{r\Delta}{r^2+a^2}V(r\Phi_4^{(1)}),\qquad(2.51\text{b})$$

and

$$\Psi_0^{(0)}=r^{2-\delta}\widetilde{\Psi_0},\qquad\Psi_0^{(1)}=r^2Y\widetilde{\Psi_0},\qquad(2.51\text{c})$$
$$\Psi_2^{(0)}=\widetilde{\Psi_2},\qquad\Psi_2^{(1)}=\tfrac{r\Delta}{r^2+a^2}V(r\widetilde{\Psi_2}).\qquad(2.51\text{d})$$

Let us give a few remarks before going further.

Remark 2.4.1. The trapping degeneracy for the Morawetz densities $\mathbb{M}_{\deg}(\partial^k\Phi_0^{(2)})$, $\mathbb{M}_{\deg}(\partial^k\Phi_4^{(2)})$, $\mathbb{M}_{\deg}(\partial^k\Psi_0^{(1)})$ and $\mathbb{M}_{\deg}(\partial^k\Psi_2^{(1)})$ with $|k|\le n-1$ can be manifestly removed. We shall only focus on the $n=0$ case, since as shown in Section 4.1, the $n\ge 1$ cases follow straightforwardly from the $n=0$ case.

Remark 2.4.2. The energy and Morawetz estimate (2.50a) or (2.50b) for the spin-2 case is obtained by treating the system (2.31) or (2.32) for ϕ_s^i, and is a single estimate at three levels of regularity for each extreme spin component, since ϕ_s^2 involves at most second-order derivatives of ϕ_s^0. Therefore, in spite of the well-known trapping phenomenon, we prove Morawetz estimates which bound spacetime integrals of nondegenerate Morawetz densities of ϕ_s^0 and ϕ_s^1 in the trapped region. However, the three levels of regularity must be treated simultaneously. One one hand, to estimate the inhomogeneous terms on the RHS of (2.31) and (2.32), it is necessary to eliminate the trapping degeneracy in the Morawetz estimates for ϕ_s^0 and ϕ_s^1 by considering one more order of derivative; on the other hand, it is possible to close the three estimates simultaneously, because the RHS of (2.31) and (2.32) are at two levels of regularity at most, involving no derivatives of ϕ_s^2 and at most one of ϕ_s^0 and ϕ_s^1. Similar phenomenon holds for the spin-1 case.

Remark 2.4.3. In the spin-1 case, we find that the equation (2.29b) and (2.30b) for ϕ_s^1 is coupled to ϕ_s^0. When $|a|/M \leq a_0/M \ll 1$ is sufficiently small, however, the coupling effect with ϕ_s^0 in (2.29b) and (2.30b) is weak. This is the main observation suggesting that one may be able to run through the approach from Schwarzschild case to slowly rotating Kerr case. In the spin-2 case, note that the systems (2.31) and (2.32) are, however, neither weakly coupled, a fact caused by the presence of the ϕ_s^1 term in (2.31a) and (2.32a), or the ϕ_s^0 term in (2.31b) and (2.32b). This is an essential difference compared to the Maxwell ($s = \pm 1$) case. Take the system (2.31) for $s = +2$ for example. Our approach here relies on an estimate bounding ϕ_{+2}^1 from ϕ_{+2}^2 by employing the differential relation (2.28a) between them, which facilitates the treatment for the system in a rough (but accurate in the Schwarzschild case) sense that the error term in the Morawetz estimate for (2.31a) arising from the inhomogeneous term can be controlled by adding a large amount of Morawetz estimate of (2.31c) to the Morawetz estimate of (2.31a); see Chapter 4.

Remark 2.4.4. There remain some difficulties when estimating the error terms $\mathcal{E}(F)$ and $\mathcal{E}_s^i$ in the trapped region and in the large radius region. We also take the system (2.31) for $s = +2$ as an example to illustrate these difficulties. It is obvious that the term $\left|\int_{\mathcal{D}(0,\tau)} \frac{1}{\Sigma}\Re\left(F_{+2}^2 \partial_t \overline{\phi_{+2}^2}\right)\right|$ in $\mathcal{E}_{+2}^2$ can not be estimated directly because of the trapping degeneracy present in the t^*- and angular derivatives terms in the Morawetz density $\mathbb{M}_{\text{deg}}(\phi_{+2}^2)$. In the large radius region, since there is an $r^{-\delta}$ power loss in $\mathbb{M}_{\text{deg}}(\phi_{+2}^1)$ and $\mathbb{M}_{\text{deg}}(\phi_{+2}^0)$, the error term $\int_{\mathcal{D}(0,\tau)} r^{-3+\delta}|F_{+2}^2|^2$ would not be bounded by $C(\mathbb{M}_{\text{deg}}(\phi_{+2}^1) + \mathbb{M}_{\text{deg}}(\phi_{+2}^0))$. There are other error terms where these two difficulties in different regions are present as well, and we will show how to treat them in the main proof Chapters 5 and 7.

2.5. Relevant results

The scalar wave equation in the DOC of vacuum black holes has been studied extensively in the last 15 years. On a Schwarzschild background, uniform boundedness of scalar wave was first obtained in Kay and Wald (1987), while a Morawetz Morawetz (1968) type multiplier, which is first introduced to black hole background in Blue and Soffer (2003), has been utilized in many works, such as Blue and Soffer (2009), Dafermos and Rodnianski (2009), to achieve Morawetz estimate. In non-static ($a \neq 0$) Kerr spacetimes, the fact that the Killing vector field ∂_t fails to be globally timelike as in Schwarzschild case raises a difficulty in constructing a uniformly bounded positive energy or a nonnegative conserved energy. Moreover, the location where the null geodesics can be trapped is enlarged from $r = 3M$ in Schwarzschild to a radius region in Kerr case. Both of the two facts that there is a lack of Killing vector fields (the ∂_t and ∂_ϕ) to commute with the scalar wave operator and these two Killing vector fields do not span a globally timelike direction in the DOC render one to obtain the uniform boundedness or decay estimates for the field itself. However, three independent, different approaches Tataru and Tohaneanu (2011), Andersson and Blue (2015b), Dafermos and Rodnianski (2010) have been developed on slowly rotating Kerr background to achieve uniform bound of a positive definite energy (albeit not conserved) and Morawetz estimate. Different pointwise decay estimates are also proved there. In particular, the separability of the wave equation or the complete integrability of the geodesic flow first found in Carter (1968) is a point of crucial importance in these works.

Decay behaviours for Maxwell field have been proved in Blue (2008) outside a Schwarzschild black hole, and on some spherically symmetric backgrounds in Sterbenz and Tataru (2015). The works above focus on estimating the middle component which satisfies a decoupled, separable Fackerell-Ipser equation Fackerell and Ipser (1972) in a form similar to scalar wave equation, and then make use of these estimates to achieve Morawetz estimates and decay estimates for the extreme components. In contrast with these works, Pasqualotto (2016) treats the extreme components satisfying the TME by applying some first-order differential operators to the extreme components, which then also satisfy the Fackerell-Ipser equation, while a superenergy tensor is constructed in Andersson et al. (2016) to yield a conserved energy current when contracted with ∂_t. In particular, the constructed superenergy tensor vanishes for the non-radiating Coulomb field. A decay estimate is also obtained in Ghanem (2014) under the assumption of a Morawetz estimate. We refer to the recent paper Andersson et al. (2016) for a more complete description of the literature in Maxwell equations on Schwarzschild background.

The method of linearizing VEE subject to metric perturbations was carried out for the Schwarzschild metric in Regge and Wheeler (1957), Vishveshwara (1970), Zerilli (1970). In these papers, the time and angular dependence can be easily separated out from the equations due to the metric being static and spherically symmetric. The resulting radial equations can be reduced to Regge-Wheeler equation governing the odd-parity perturbations and Zerilli equation governing the even-parity perturbations. In particular, these equations were derived later in

Moncrief (1974) without assuming any gauge conditions. The linear stability of Schwarzschild metric has been resolved recently in Dafermos et al. (2016), Hung et al. (2017), with the former one starting from a Regge-Wheeler type equation satisfied by some scalar constructed from *Chandrasekhar transformation Chandrasekhar (1975)* by applying some second order differential operator to the extreme component and the later one treating Regge-Wheeler-Zerilli-Moncrief system. The energy and Morawetz estimates, as well as decay estimates, for this system are also obtained in Andersson et al. (2017a).

For nonzero integer spin fields in the DOC of a Kerr spacetime, only a few results about stability issue can be found in the literature. The only result for Maxwell field we are aware of is given in Andersson and Blue (2015a) on slowly rotating Kerr background, in which energy and Morawetz estimates for both the full Maxwell equations and the Fackerel-Ipser equation for the middle component are proved by introducing fractional derivative operators to treat the presence of an imaginary potential term in Fackerel-Ipser equation. The estimates therein enable the authors to prove a uniform bound of a positive definite energy and the convergence property of the Maxwell field to a stationary Coulomb field. Turning to the extreme components, as mentioned already, they satisfy decoupled, separable TME (2.20). Differential relations between the radial parts of the modes with opposite extreme spin weights, as well as similar relations between the angular parts, are derived in Starobinsky and Churilov (1973), Teukolsky and Press (1974), known as "*Teukolsky-Starobinsky Identities*". In Whiting (1989), it is shown that by assuming some proper boundary conditions the TME admits no modes with frequency having positive imaginary part, or in another way, no exponentially growing mode solutions exist. This mode stability result is recently generalized to the case of real frequencies in Shlapentokh-Rothman (2015) for scalar field and in our paper Andersson et al. (2017b) for general spin fields. We mention here the papers Finster and Smoller (2016) which discusses the stability problem for each azimuthal mode solution to TME, Klainerman and Szeftel (2017) which proves the nonlinear stability of Schwarzschild spacetimes under axially symmetric polarized perturbations and Dafermos et al. (2017) in which part of this thesis, that is, the energy and Morawetz estimates for spin ± 2 components are also proved but with different techniques.

2.6. Further Preliminaries and Notations

2.6.1. Well-posedness theorem

We state here a well-posedness (WP) theorem for a general system of linear wave equations, cf. (Bär et al., 2007, Chapter 3.2). Due to the fact that smooth initial data which vanish near spatial infinity can be approached by smooth, compactly supported data, we restrict our considerations on initial data which are smooth and of compact support on initial hypersurface Σ_0.

Proposition 2.6.1. For any $1 \leq n \in \mathbb{N}^+$ and $0 \leq |a| < M$, let Σ_0 be an initial spacelike hypersurface defined as in (2.10) in the DOC $\mathcal{D}$ of a Kerr spacetime $(\mathcal{M}, g_{M,a})$, and let φ_0^i, φ_1^i be compactly supported smooth sections in a vector bundle $\mathbf{E}$ over the manifold $\mathcal{D}$, $i = 1, 2, \cdots, n$. Then there exists a unique solution $\varphi = (\varphi^i)_{i=1,2,\cdots,n}$, with $\varphi^i \in C^\infty(D^+(\Sigma_0) \cap \mathcal{D}, \mathbf{E})$, to the system of linear wave equations

$$\begin{cases} \mathbf{L}_\varphi \varphi = 0 \\ \varphi^i|_{\Sigma_0} = \varphi_0^i, \quad n_{\Sigma_0}^\mu \partial_\mu \varphi^i|_{\Sigma_0} = \varphi_1^i, \quad \forall i = 1, 2, \cdots, n. \end{cases} \tag{2.52}$$

Here, $D^+(\Sigma_0)$ is the future domain of dependence of Σ_0, $\mathbf{L}_\varphi$ is a linear wave operator for φ and $n_{\Sigma_0} = n_{\Sigma_0}^\mu \partial_\mu$ is the future-directed unit normal vector field of initial hypersurface Σ_0. Moreover, φ is continuously dependent on the initial data (φ_0, φ_1) and C^∞-dependent on the parameter a, i.e. the map

$$(\varphi_0, \varphi_1) \times a \mapsto \varphi \tag{2.53}$$

is a $C^0 \times C^\infty$ map. By finite speed of propagation, the solution φ will be smooth and compactly supported on each Σ_τ for $\tau \geq 0$.

We apply this WP theorem to the linear wave systems (2.29)–(2.32) of $\varphi = (\phi_s^0, \phi_s^1, \cdots, \phi_s^{|s|})$, and ensure the existence and uniqueness of the solution for any given compactly supported smooth initial data.

2.6.2. regular and integrable

Definition 2.6.1.
- A two-form $\mathbf{F}_{\alpha\beta}$ to the Maxwell equations (1.2), with all components in global Kerr coordinates being smooth in the exterior region

of the Kerr spacetime, admitting a smooth extension to the closure of the exterior region in the maximal analytic extension and vanishing near spatial infinity up to a charged stationary Coulomb part, is called a **regular Maxwell field**. In particular, Coulomb part is supported in the middle N-P component Ψ_1, cf. Andersson and Blue (2015a).

- Let $\mathbf{F}_{\alpha\beta}$ be a regular Maxwell field and the smooth regular extreme N-P components of the linearized gravity be vanishing near spatial infinity. A solution $\psi = \phi_s^i$ defined as in (2.27) or (2.28) is called an **integrable** solution to the ISWWE (2.42), if for every integer $n \geq 0$, every multi-index $0 \leq |i| \leq n$ and any $\check{r}_0 > r_+$, we have

$$\sum_{0\leq|i|\leq n} \int_{\mathcal{D}(-\infty,\infty)\cap\{r=\check{r}_0\}} \left(\left|\partial^i\psi\right|^2 + \left|\partial^i F\right|^2\right) < \infty. \tag{2.54}$$

Here, we recall in (2.40) the definition of $\partial^i\psi$ and $\partial_i F$..

2.6.3. Generic constants and general rules

Constants C and c, depending only on a_0, M, δ and Σ_0, are always understood as large constants and small constants respectively, and may change line to line throughout this thesis based on the algebraic rules: $C+C=C$, $CC=C$, $Cc=C$, etc. When there is no confusion, the dependence on M, a_0, δ and Σ_0 may always be suppressed. Once the constants $\varepsilon_0(M)$ and $0<\delta<1/2$ in Theorems 2.4.1 and 2.4.2 are chosen and the choice of function $\chi_1(r)$ in (2.6) defining the global Kerr coordinates is made, these constants can be made to be only dependent on M.

For any two functions F and G, $F \lesssim G$ means that there exists a constant C such that $F \leq CG$ holds everywhere. $F \sim G$ indicates that $F \lesssim G$ and $G \lesssim F$, and we say that F *is equivalent to* G.

The standard Laplacian on unit 2-sphere is denoted as $\triangle_{\mathbb{S}^2}$, and the volume form $d\sigma_{\mathbb{S}^2}$ on unit 2-sphere is $\sin\theta d\theta d\phi^*$ or $\sin\theta d\theta d\phi$ depending on which coordinate system is used.

Some cutoff functions will be used in this paper. Denote $\chi_R(r)$ to be a smooth cutoff function utilized in Section 3.1 which is 1 for $r \geq R$ and vanishes identically for $r \leq R-1$, and $\chi_0(r)$ a smooth cutoff function which equals to 1 for $r \leq r_0$ and is identically zero for $r \geq r_1$; see Chapter 3 for the choices of r_0 and r_1. The function χ is a smooth cutoff both to the future time and to the past time, which will be applied to the solution in the proof of Theorem 2.4.1.

An overline or a bar will always denote the complex conjugate, $\Re(\cdot)$ denotes the real part, and "left hand side(s)" and "right hand side(s)" are short for "LHS" and "RHS" respectively.

Throughout this paper, whenever we talk about "choosing some multiplier for some equation", it should always be understood as multiplying the equation by the multiplier, performing integration by parts, taking the real part and finally integrating in the spacetime region $\mathcal{D}(0,\tau)$ (or $\mathcal{D}(\tau_1,\tau_2)$) in global Kerr coordinate system with respect to the measure $\Sigma \sin\theta dt^* dr d\theta d\phi^*$.

3. Estimates near horizon and near infinity

Contents

Morawetz estimates in large radius r region and red-shift estimates near horizon for different quantities are proved in this chapter. We emphasize that all the R_0 in the estimates in this whole chapter can be *a priori* different, so do all the r_0 and the r_1, but we will take the minimal r_0, the maximal r_1 and the maximal R_0 among them such that the estimates hold true uniformly with the constants C do not depend on them, and still denote them as r_0, r_1 and R_0.

3.1. Morawetz estimates near infinity

We put the equations (2.29), (2.30), (2.31b), (2.31c), (2.32b) and (2.32c) into the general form (2.34b), or equivalently, in an expanded form

$$\begin{aligned}\Sigma\widetilde{\Box}_g\psi \triangleq &\left\{\partial_r(\Delta\partial_r) - \tfrac{\left((r^2+a^2)\partial_t + a\partial_\phi\right)^2}{\Delta}\right.\\ &\left.+\tfrac{1}{\sin\theta}\tfrac{d}{d\theta}\left(\sin\theta\tfrac{d}{d\theta}\right) + \left(\tfrac{\partial_\phi + is\cos\theta}{\sin\theta} + a\sin\theta\partial_t\right)^2\right\}\psi\\ =&\left(4ias\cos\theta\partial_t + s^2\tfrac{\Delta+a^2}{r^2}\right)\psi + F, \end{aligned} \tag{3.1}$$

such that $\Sigma\widetilde{\Box}_g$ is the same as the rescaled scalar wave operator $\Sigma\Box_g$ except for $(\frac{\partial_\phi+is\cos\theta}{\sin\theta} + a\sin\theta\partial_t)^2$ in place of the operator $(\frac{\partial_\phi}{\sin\theta} + a\sin\theta\partial_t)^2$ in the expansion of $\Sigma\Box_g$. Analogously, the equations (2.31a) and (2.32a) can be put into the form of (2.34a), or

$$\begin{aligned}\Sigma\widetilde{\Box}_g\psi \triangleq \Big\{&\partial_r(\Delta\partial_r) - \tfrac{\left((r^2+a^2)\partial_t+a\partial_\phi\right)^2}{\Delta}\\ &+\tfrac{1}{\sin\theta}\tfrac{d}{d\theta}\left(\sin\theta\tfrac{d}{d\theta}\right) + \left(\tfrac{\partial_\phi+is\cos\theta}{\sin\theta}+a\sin\theta\partial_t\right)^2\Big\}\psi\\ =&\left(4ias\cos\theta\partial_t + s^2\tfrac{r^2+2Mr-2a^2}{2r^2}\right)\psi + F.\end{aligned} \tag{3.2}$$

Recall (see e.g. Dafermos and Rodnianski (2010)) that for each $0<\delta<1/2$ there exist constants $R_0 \gg 4M$ and $C=C(\delta)$ such that for all $R\geq R_0$, one can choose a multiplier

$$X_w\bar{\psi} = -\tfrac{1}{\Sigma}\left(f(r)\partial_{r^*}+\tfrac{1}{4}w(r)\right)\bar{\psi} \tag{3.3}$$

for the rescaled inhomogeneous scalar wave equation

$$\Sigma\Box_g\psi = G \tag{3.4}$$

on any subextremal Kerr background, and achieve the following Morawetz estimate in large r region for any $\tau_2>\tau_1$:

$$\begin{aligned}&\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\geq R\}}\left\{\frac{|\partial_{r^*}\psi|^2}{r^{1+\delta}}+\frac{|\partial_t\psi|^2}{r^{1+\delta}}+\frac{|\breve{\nabla}\psi|^2}{r}+\frac{|\psi|^2}{r^{3+\delta}}\right\}\\ &\lesssim \breve{E}_{\tau_1}(\psi)+\breve{E}_{\tau_2}(\psi)+\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{R-1\leq r\leq R\}}\left(|\breve{\partial}\psi|^2+|\psi|^2\right)\\ &\quad+\int_{\mathcal{D}(\tau_1,\tau_2)}\Re\left(G\cdot X_w\bar{\psi}\right).\end{aligned} \tag{3.5}$$

Here,

$$f = \chi_R(r)\cdot(1-r^{-\delta}), \tag{3.6a}$$

$$w = 2\partial_{r^*}f + 4\tfrac{1-2M/r}{r}f - 2\delta\tfrac{1-2M/r}{r^{1+\delta}}f, \tag{3.6b}$$

$\breve{\nabla}$ are the standard covariant angular derivatives on sphere $\mathbb{S}^2(t,r)$ as in (2.38), and

$$\breve{E}_\tau(\psi) = \int_{\Sigma_\tau}|\breve{\partial}\psi|^2 = \int_{\Sigma_\tau}(|\partial_{t^*}\psi|^2+|\partial_r\psi|^2+|\breve{\nabla}\psi|^2). \tag{3.7}$$

Since the difference between the operator $(\frac{1}{\sin\theta}\partial_\phi + is\cot\theta + a\sin\theta\partial_t)^2$ in $\Sigma\widetilde{\Box}_g$ and $(\frac{1}{\sin\theta}\partial_\phi + a\sin\theta\partial_t)^2$ in the expansion of $\Sigma\Box_g$ is terms with coefficients independent of

r, and the terms containing ϕ-derivative in $\Delta^{-1}((r^2+a^2)\partial_t+a\partial_\phi)^2$ have coefficients which are of lower order in r, we will achieve the same type of Morawetz estimate in large r region by utilizing the same multiplier $X_w\bar{\psi}$, with $|\nabla\!\!\!/\psi|^2$ and $|\partial\psi|^2 - |s||\psi|^2/r^2$ in place of $|\check{\nabla\!\!\!/}\psi|^2$ and $|\check{\partial}\psi|^2$, $E(\psi)$ replacing $\check{E}(\psi)$, and a substitution of

$$G = \left(4ias\cos\theta\partial_t + s^2\tfrac{\Delta+a^2}{r^2}\right)\psi + F \tag{3.8a}$$

for (3.1) or

$$G = \left(4ias\cos\theta\partial_t + s^2\tfrac{r^2+2Mr-2a^2}{2r^2}\right)\psi + F \tag{3.8b}$$

for (3.2) respectively in (3.5). The bulk term coming from the source term (3.8) in (3.5) is then

$$C\int_{\mathcal{D}(\tau_1,\tau_2)} -\tfrac{1}{\Sigma}\Re\left(\left(\left(4ias\cos\theta\partial_t + s^2\tfrac{\Delta+a^2}{r^2}\right)\psi + F\right)(f\partial_{r^*}+\tfrac{1}{4}w)\bar{\psi}\right) \tag{3.9}$$

or

$$C\int_{\mathcal{D}(\tau_1,\tau_2)} -\tfrac{1}{\Sigma}\Re\left(\left(\left(4ias\cos\theta\partial_t + s^2\tfrac{r^2+2Mr-2a^2}{2r^2}\right)\psi + F\right)(f\partial_{r^*}+\tfrac{1}{4}w)\bar{\psi}\right), \tag{3.10}$$

which is bounded for R large enough in both cases by

$$\begin{aligned}\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)} -\tfrac{C|\psi|^2}{r^3} + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,R]}|\partial\psi|^2\\ + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,\infty)}\left\{\tfrac{C|a|}{r^2}|\partial\psi|^2 + C\Re\left(FX_w\bar{\psi}\right)\right\}.\end{aligned} \tag{3.11}$$

Therefore, for any fixed $0<\delta<\frac{1}{2}$, we choose the same multiplier $X_w\bar{\psi} = -\Sigma^{-1}\left(f(r)\partial_{r^*}+\frac{1}{4}w(r)\right)\bar{\psi}$ with $f(r)$ and $w(r)$ defined in (3.6) for both (3.1) and (3.2), and easily obtain the following result.

Proposition 3.1.1. In a subextremal Kerr spacetime $(\mathcal{M}, g_{M,a})$ ($|a|\le a_0<M$), for any fixed $0<\delta<\frac{1}{2}$, and for any solution ψ solving any subequation in the linear systems (2.29)–(2.32), there exists constant $R_0(M)$ and universal constant C such that for all $R\ge R_0$, the following estimate holds for any $\tau_2>\tau_1$:

$$\begin{aligned}\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\ge R\}}\mathbb{M}_{\deg}(\psi) \le & C\left(E_{\tau_1}(\psi)+E_{\tau_2}(\psi)+\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{R-1\le r\le R\}}|\partial\psi|^2\right)\\ &+C\left|\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\ge R-1\}}\Re\left(FX_w\bar{\psi}\right)\right|.\end{aligned} \tag{3.12}$$

Remark 3.1.1. Recall in (2.41) the definition of the Morawetz density $\mathbb{M}_{\deg}(\psi)$.

In fact, we can obtain an improved Morawetz estimate in the large radius region for spin $+1$ component for Maxwell field and spin $+2$ component for linearized gravity due to the damping ∂_t term on RHS of (2.25) for large r.

Proposition 3.1.2. In a subextremal Kerr spacetime $(\mathcal{M}, g_{M,a})$ ($|a| \leq a_0 < M$), let $0 < \delta < 1/2$ be given. Then there exists constant $R_0(M)$ and universal constant C such that for all $R \geq R_0$ and any $\tau_2 > \tau_1$, it holds for ϕ^0_{+1} that

$$\begin{aligned}&\int_{\Sigma_{\tau_2}\cap[R,+\infty)} \left|\partial(r^{2-\delta}\phi^0_{+1})\right|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)} r^{-1}\left|\partial(r^{2-\delta}\phi^0_{+1})\right|^2 \\ &\lesssim \int_{\Sigma_{\tau_2}\cap[R-1,R)} \left|\partial(r^{2-\delta}\phi^0_{+1})\right|^2 + \int_{\Sigma_{\tau_1}\cap[R-1,+\infty)} \left|\partial(r^{2-\delta}\phi^0_{+1})\right|^2, \end{aligned} \tag{3.13}$$

and the following estimates hold for ϕ^0_{+2} and ϕ^1_{+2} respectively:

$$\begin{aligned}&\int_{\Sigma_{\tau_2}\cap[R,+\infty)} \left|\partial\left(r^{4-\delta}\phi^0_{+2}\right)\right|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)} r^{-1}\left|\partial\left(r^{4-\delta}\phi^0_{+2}\right)\right|^2 \\ &\lesssim \int_{\Sigma_{\tau_2}\cap[R-1,R)} \left|\partial\left(r^{4-\delta}\phi^0_{+2}\right)\right|^2 + \int_{\Sigma_{\tau_1}\cap[R-1,+\infty)} \left|\partial\left(r^{4-\delta}\phi^0_{+2}\right)\right|^2, \end{aligned} \tag{3.14a}$$

$$\begin{aligned}&\int_{\Sigma_{\tau_2}\cap[R,+\infty)} \left|\partial\left(r^{2-\delta}\phi^1_{+2}\right)\right|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)} r^{-1}\left|\partial\left(r^{2-\delta}\phi^1_{+2}\right)\right|^2 \\ &\lesssim \int_{\Sigma_{\tau_2}\cap[R-1,R)} \left|\partial\left(r^{2-\delta}\phi^1_{+2}\right)\right|^2 + \int_{\Sigma_{\tau_1}\cap[R-1,+\infty)} \left|\partial\left(r^{2-\delta}\phi^1_{+2}\right)\right|^2 \\ &\quad + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,\infty)} \frac{\left|\partial\left(r^{4-\delta}\phi^0_{+2}\right)\right|^2}{r^2}. \end{aligned} \tag{3.14b}$$

Proof. We start with the spin $+1$ component. The equation for $\grave{\phi}^0_{+1} = \frac{r^2(r^2+a^2)^{1-\delta/2}}{\Delta}\phi^0_{+1}$ is

$$\begin{aligned}&\left(\Sigma\Box_g + \tfrac{2i\cos\theta}{\sin^2\theta}\partial_\phi - \cot^2\theta + (1 - \tfrac{3\delta}{2} + \tfrac{\delta^2}{4})\right)\grave{\phi}^0_{+1} \\ &= \tfrac{(r^3-3Mr^2+a^2r+a^2M)}{r^2+a^2}\left(\tfrac{(2-\delta)V(\sqrt{r^2+a^2}\grave{\phi}^0_{+1})}{\sqrt{r^2+a^2}} + \delta\left(\tfrac{r^2+a^2}{\Delta}\partial_t + \tfrac{a}{\Delta}\partial_\phi\right)\grave{\phi}^0_{+1}\right) \\ &\quad + \tfrac{\underline{P}_5(r)}{\Delta(r^2+a^2)^2}\grave{\phi}^0_{+1} + \left(2ia\cos\theta\partial_t - \tfrac{4ar}{r^2+a^2}\partial_\phi\right)\grave{\phi}^0_{+1}. \end{aligned} \tag{3.15}$$

$\underline{P}_5(r)$ here is a polynomial in r with powers no larger than 5 and coefficients depending only on a, M and δ. The coefficients can be calculated explicitly. We make use of the following expansion for any smooth complex scalar ψ of spin weight s

$$\left(\Sigma\Box_g + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\cot^2\theta + |s| - \tfrac{3\delta}{2} + \tfrac{\delta^2}{4})\right)\psi$$

$$= \left(\tfrac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \tfrac{1}{\sin^2\theta}\partial^2_{\phi\phi} + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\cot^2\theta + |s| - \tfrac{3\delta}{2} + \tfrac{\delta^2}{4}\right)\psi$$
$$- \sqrt{r^2+a^2}Y\left(\tfrac{\Delta}{r^2+a^2}V\left(\sqrt{r^2+a^2}\psi\right)\right) + \tfrac{2ar}{r^2+a^2}\partial_\phi\psi$$
$$+ \left(2a\partial^2_{t\phi} + a^2\sin^2\theta\partial^2_{tt}\right)\psi - \tfrac{2Mr^3+a^2r^2-4a^2Mr+a^4}{(r^2+a^2)^2}\psi, \tag{3.16}$$

and notice that the eigenvalues of the operator in the first line on RHS of (3.16) are not larger than $\delta^2/4-3\delta/2$ which is negative. Hence if we choose the multiplier

$$- \tfrac{1}{\Sigma}\chi_R X_0\overline{\grave{\phi}^0_{+1}}$$
$$\triangleq - \tfrac{1}{\Sigma}\chi_R\tfrac{\Delta}{r^2+a^2}\left(\tfrac{(2-\delta)V(\sqrt{r^2+a^2}\overline{\grave{\phi}^0_{+1}})}{\sqrt{r^2+a^2}} + \delta\left(\tfrac{r^2+a^2}{\Delta}\partial_t + \tfrac{a}{\Delta}\partial_\phi\right)\overline{\grave{\phi}^0_{+1}}\right) \tag{3.17}$$

for the equation (3.15), we have

$$\int_{\Sigma_{\tau_2}\cap[R,+\infty)}|\partial\grave{\phi}^0_{+1}|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)} r^{-1}\left(|X_0\grave{\phi}^0_{+1}|^2 + |\nabla\!\!\!\!/\,\grave{\phi}^0_{+1}|^2\right)$$
$$\lesssim\Big(\int_{\Sigma_{\tau_2}\cap[R-1,R)} + \int_{\Sigma_{\tau_1}\cap[R-1,+\infty)}\Big)|\partial\grave{\phi}^0_{+1}|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,\infty)}\tfrac{|\partial\grave{\phi}^0_{+1}|^2}{r^2}. \tag{3.18}$$

Moreover, we choose the multiplier $-\chi_R r^{-3}(1-2M/r)\overline{\grave{\phi}^0_{+1}}$ for (3.15) and arrive at

$$\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)} r^{-1}\left(|\partial_r\grave{\phi}^0_{+1}|^2 + |\nabla\!\!\!\!/\,\grave{\phi}^0_{+1}|^2\right)$$
$$\lesssim\int_{\Sigma_{\tau_2}\cap[R-1,+\infty)}|\partial\grave{\phi}^0_{+1}|^2 + \int_{\Sigma_{\tau_1}\cap[R-1,+\infty)}|\partial\grave{\phi}^0_{+1}|^2$$
$$+ \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,\infty)}\left(r^{-1}|\partial_{t^*}\grave{\phi}^0_{+1}|^2 + r^{-2}|\partial\grave{\phi}^0_{+1}|^2\right). \tag{3.19}$$

The estimate (3.13) follows from adding a sufficiently large multiple of (3.18) to (3.19) and taking R sufficiently large.

For the spin +2 component, we define the variables

$$\phi^{0,4-\delta}_{+2} = \left(\tfrac{r^2+a^2}{\sqrt{\Delta}}\right)^{4-\delta}\cdot\left(\psi_{[+2]}/(r^2+a^2)^2\right), \tag{3.20a}$$
$$\phi^{1,2-\delta}_{+2} = \left(\tfrac{r^2+a^2}{\sqrt{\Delta}}\right)^{2-\delta}\cdot\left(\sqrt{r^2+a^2}Y\left(\psi_{[+2]}/(r^2+a^2)^{3/2}\right)\right), \tag{3.20b}$$

and derive the governing equations of them as follows

$$\left(\Sigma\Box_g + \tfrac{4i\cos\theta}{\sin^2\theta}\partial_\phi - 4\cot^2\theta + (2+\delta^2-5\delta)\right)\phi^{0,4-\delta}_{+2}$$

$$
\begin{aligned}
&= \tfrac{(r^3-3Mr^2+a^2r+a^2M)}{r^2+a^2}\Big(\tfrac{(4-2\delta)V(\sqrt{r^2+a^2}\phi_{+2}^{0,4-\delta})}{\sqrt{r^2+a^2}} + 2\delta\left(\tfrac{r^2+a^2}{\Delta}\partial_t + \tfrac{a}{\Delta}\partial_\phi\right)\phi_{+2}^{0,4-\delta}\Big)\\
&\quad + \left(4ia\cos\theta\partial_t - \tfrac{8ar}{r^2+a^2}\partial_\phi\right)\phi_{+2}^{0,4-\delta} + \tfrac{P_5(r)}{\Delta(r^2+a^2)^2}\phi_{+2}^{0,4-\delta}, \qquad (3.21a)\\
&\quad \left(\Sigma\Box_g + \tfrac{4i\cos\theta}{\sin^2\theta}\partial_\phi - 4\cot^2\theta + (2+\delta^2-5\delta)\right)\phi_{+2}^{1,2-\delta}\\
&= \tfrac{(r^3-3Mr^2+a^2r+a^2M)}{r^2+a^2}\Big(\tfrac{(2-2\delta)V(\sqrt{r^2+a^2}\phi_{+2}^{1,2-\delta})}{\sqrt{r^2+a^2}} + 2\delta\left(\tfrac{r^2+a^2}{\Delta}\partial_t + \tfrac{a}{\Delta}\partial_\phi\right)\phi_{+2}^{1,2-\delta}\Big)\\
&\quad + \tfrac{\underline{P}_5(r)}{\Delta(r^2+a^2)^2}\phi_{+2}^{1,2-\delta} + \left(4ia\cos\theta\partial_t - \tfrac{4ar}{r^2+a^2}\partial_\phi\right)\phi_{+2}^{1,2-\delta}\\
&\quad + \tfrac{6a\Delta(a^2-r^2)}{(r^2+a^2)^3}\partial_\phi\phi_{+2}^{0,4-\delta} + \tfrac{6r\Delta(Mr^3-a^2r^2-3Ma^2r-a^4)}{(r^2+a^2)^4}\phi_{+2}^{0,4-\delta}. \qquad (3.21b)
\end{aligned}
$$

Here, $P_5(r)$ and $\underline{P}_5(r)$ are both polynomials in r with powers no larger than 5, and the coefficients of these two polynomials depend only on a, M and δ and can be calculated explicitly. Similar to (3.16), we have for any smooth complex scalar ψ of spin weight s that

$$
\begin{aligned}
&\left(\Sigma\Box_g + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\cot^2\theta + |s| + \delta^2 - 5\delta\right)\psi\\
&= \left(\tfrac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \tfrac{1}{\sin^2\theta}\partial^2_{\phi\phi} + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\cot^2\theta + |s| + \delta^2 - 5\delta\right)\psi\\
&\quad - \sqrt{r^2+a^2}Y\left(\tfrac{\Delta}{r^2+a^2}V\left(\sqrt{r^2+a^2}\psi\right)\right) + \tfrac{2ar}{r^2+a^2}\partial_\phi\psi\\
&\quad + \left(2a\partial^2_{t\phi} + a^2\sin^2\theta\partial^2_{tt}\right)\psi - \tfrac{2Mr^3+a^2r^2-4a^2Mr+a^4}{(r^2+a^2)^2}\psi. \qquad (3.22)
\end{aligned}
$$

Notice that the eigenvalues of the operator in the first line on RHS of (3.22) are not greater than $\delta^2 - 5\delta$ which is negative, hence if we choose the multiplier

$$
\begin{aligned}
&- \tfrac{1}{\Sigma}\chi_R X_0\overline{\phi_{+2}^{0,4-\delta}}\\
\triangleq\ & - \tfrac{1}{\Sigma}\chi_R\tfrac{\Delta}{r^2+a^2}\Big(\tfrac{(4-2\delta)V(\sqrt{r^2+a^2}\overline{\phi_{+2}^{0,4-\delta}})}{\sqrt{r^2+a^2}} + 2\delta\left(\tfrac{r^2+a^2}{\Delta}\partial_t + \tfrac{a}{\Delta}\partial_\phi\right)\overline{\phi_{+2}^{0,4-\delta}}\Big) \qquad (3.23)
\end{aligned}
$$

for (3.21a), it then follows

$$
\begin{aligned}
&\int_{\Sigma_{\tau_2}\cap[R,+\infty)}|\partial\phi_{+2}^{0,4-\delta}|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)} r^{-1}\left(|X_0\phi_{+2}^{0,4-\delta}|^2 + |\nabla\!\!\!\!/\,\phi_{+2}^{0,4-\delta}|^2\right)\\
\lesssim\ &\left(\int_{\Sigma_{\tau_2}\cap[R-1,R)} + \int_{\Sigma_{\tau_1}\cap[R-1,+\infty)}\right)|\partial\phi_{+2}^{0,4-\delta}|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,\infty)}\tfrac{|\partial\phi_{+2}^{0,4-\delta}|^2}{r^2}. \qquad (3.24)
\end{aligned}
$$

Moreover, by choosing the multiplier $-\chi_R r^{-3}(1-2M/r)\overline{\phi_{+2}^{0,4-\delta}}$ for (3.21a), we arrive at

$$
\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)} r^{-1}\left(|\partial_r\phi_{+2}^{0,4-\delta}|^2 + |\nabla\!\!\!\!/\,\phi_{+2}^{0,4-\delta}|^2\right)
$$

$$\lesssim \int_{\Sigma_{\tau_2}\cap[R-1,+\infty)} |\partial\phi_{+2}^{0,4-\delta}|^2 + \int_{\Sigma_{\tau_1}\cap[R-1,+\infty)} |\partial\phi_{+2}^{0,4-\delta}|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,\infty)} \left(r^{-1}|\partial_{t^*}\phi_{+2}^{0,4-\delta}|^2 + r^{-2}|\partial\phi_{+2}^{0,4-\delta}|^2\right). \tag{3.25}$$

Adding a sufficiently large multiple of (3.24) to (3.25) and taking R sufficiently large, we conclude the inequality (3.14a). The estimate (3.14b) follows in the same way by treating (3.21b). □

3.2. Red-shift estimates near horizon

3.2.1. Red-shift estimates for spin-1 case

The following red-shift estimate near $\mathcal{H}^+$ for rescaled inhomogeneous scalar wave equation (3.4) is taken from (Dafermos and Rodnianski, 2011, Sect.5.2) and Dafermos and Rodnianski (2010).

Lemma 3.2.1. In a slowly rotating Kerr spacetime $(\mathcal{M}, g_{M,a})$, there exist constants $\varepsilon_0(M)$, $r_+ \leq 2M < r_0(M) < r_1(M) < (1+\sqrt{2})M$ and $C = C(\Sigma_{\tau_1}, M) = C(\Sigma_{\tau_2}, M)$, two smooth real functions $y_1(r)$ and $y_2(r)$ on $[r_+,\infty)$ with $y_1(r)\to 1$ and $y_2(r)\to 0$ as $r\to r_+$, and a φ_τ-invariant timelike vector field

$$N = T + \chi_0(r)\left(y_1(r)Y + y_2(r)T\right) \tag{3.26}$$

such that for all $|a|/M \leq a_0/M \leq \varepsilon_0$, by choosing a multiplier

$$-N_{\chi_0}\bar{\psi} = -\chi_0(r)\Sigma^{-1}N\bar{\psi}, \tag{3.27}$$

the following estimate holds for any solution ψ to the rescaled inhomogeneous scalar wave equation (3.4) for any $\tau_2 > \tau_1$:

$$\begin{aligned}&\int_{\Sigma_{\tau_2}\cap\{r\leq r_0\}} \left|\check{\partial}\psi\right|^2 + \check{E}_{\mathcal{H}^+(\tau_1,\tau_2)}(\psi) \\ &+ \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}} \left(\left|\check{\partial}\psi\right|^2 + |\log(r-r_+)|^{-2}|r-r_+|^{-1}\psi^2\right) \\ \leq{}& C\int_{\Sigma_{\tau_1}\cap\{r\leq r_1\}} \left|\check{\partial}\psi\right|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}} \left|\check{\partial}\psi\right|^2 \\ &+ C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_1\}} \Re\left(G\cdot N_{\chi_0}\bar{\psi}\right).\end{aligned} \tag{3.28}$$

Here,

$$\check{E}_{\mathcal{H}^+(\tau_1,\tau_2)}(\psi) = \int_{\mathcal{H}^+(\tau_1,\tau_2)} (|\partial_v\psi|^2 + |\check{\nabla}\psi|^2) r^2 dv \sin\theta d\theta d\tilde{\phi} \tag{3.29}$$

in ingoing E-F coordinates.

As in the last section, we refer to the rewritten form (3.1) of the ISWWE (2.42). The difference between the operator $(\frac{\partial_\phi}{\sin\theta} + is\cot\theta + a\sin\theta\partial_t)^2$ in $\Sigma\widetilde{\Box}_g$ and $(\frac{\partial_\phi}{\sin\theta} + a\sin\theta\partial_t)^2$ in the expansion of $\Sigma\Box_g$ involves only terms with coefficients independent of t, ϕ and r, and the term $a\partial\phi$ in $(r^2+a^2)\partial_t + a\partial_\phi$ has coefficient proportional to a, therefore we could use the same multiplier $-N_{\chi_0}$ to achieve the same estimate for sufficient small $|a|/M$ with the same replacements as in the last section. On RHS, we are left with

$$C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_1\}} -\Re\left(\left(\left(4ias\cos\theta\partial_t + \tfrac{\Delta+a^2}{r^2}\right)\psi + F\right)N_{\chi_0}\bar{\psi}\right), \tag{3.30}$$

which in turn is bounded by

$$\begin{aligned} &\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}} \left(-CN\left(\tfrac{\Delta+a^2}{r^2}|\psi|^2\right) - C\tfrac{r_+-r_-}{r^2}|\psi|^2\right) \\ &+ C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_1\}} \left(|a||\partial\psi|^2 + |F|^2\right) + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}} |\partial\psi|^2. \end{aligned} \tag{3.31}$$

In conclusion, we have the following red-shift estimate for the ISWWE (2.42).

Proposition 3.2.1. In a slowly rotating Kerr spacetime $(\mathcal{M}, g_{M,a})$, there exist constants $\varepsilon_0(M)$, $r_+ \leq 2M < r_0(M) < r_1(M) < (1+\sqrt{2})M$ and a universal constant C, and a φ_τ-invariant vector field N defined as in (3.27) such that for all $|a|/M \leq a_0/M \leq \varepsilon_0$, the following estimate holds for any solution ψ to the ISWWE (2.42) for any $\tau_2 > \tau_1$:

$$\begin{aligned} &E_{\mathcal{H}^+(\tau_1,\tau_2)}(\psi) + \int_{\Sigma_{\tau_2}\cap\{r\leq r_0\}} |\partial\psi|^2 \\ &+ \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}} \left(|\partial\psi|^2 + |\log(r-r_+)|^{-2}|r-r_+|^{-1}\psi^2\right) \\ \leq{}& C\int_{\Sigma_{\tau_1}\cap\{r\leq r_1\}} |\partial\psi|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}} |\partial\psi|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]} |F|^2. \end{aligned} \tag{3.32}$$

Moreover, we can obtain a red-shift estimate near horizon for $\psi_{[-1]}$.

Proposition 3.2.2. Under the same assumptions in Proposition 3.2.1, we have

$$
\begin{aligned}
&E_{\mathcal{H}^+(\tau_1,\tau_2)}(\psi_{[-1]}) + \int_{\Sigma_{\tau_2}\cap\{r\leq r_0\}} \left|\partial\psi_{[-1]}\right|^2 + \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}} \left|\partial\psi_{[-1]}\right|^2 \\
\leq{}& C\int_{\Sigma_{\tau_1}\cap\{r\leq r_1\}} \left|\partial\psi_{[-1]}\right|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}} \left|\partial\psi_{[-1]}\right|^2 \\
&+ C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_+\leq r\leq r_1\}} \left|\phi^1_{-1}\right|^2 .
\end{aligned}
\tag{3.33}
$$

Proof. We have from (2.27b) that

$$
\begin{aligned}
\phi^1_{-1} ={}& rV\left(r^{-1}\Delta\psi_{[-1]}\right)\\
={}& -\Delta Y\psi_{[-1]} + 2\left[(r^2+a^2)\partial_t + a\partial_\phi\right]\psi_{[-1]} + \tfrac{r^2-a^2}{r}\psi_{[-1]}.
\end{aligned}
\tag{3.34}
$$

Hence, from (2.25), the equation for $\psi_{[-1]}$ can be rewritten as

$$
\begin{aligned}
&\left(\Sigma\Box_g - 2i\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - \cot^2\theta\right)\psi_{[-1]}\\
={}& \tfrac{3r^2-5a^2}{2r^2}\psi_{[-1]} + \left(\tfrac{4(r-M)r-5\Delta}{2r}Y + r\partial_t\right)\psi_{[-1]} + \tfrac{5}{r}\left(a^2\partial_t + a\partial_\phi\right)\psi_{[-1]} - \tfrac{5}{2r}\phi^1_{-1}.
\end{aligned}
\tag{3.35}
$$

For small enough $|a|/M$, the coefficient of $\psi_{[-1]}$ term on the RHS is positive near horizon and its derivative with respect to r is nonnegative, and the term $((4(r-M)r-5\Delta)/(2r)Y + r\partial_t)\psi_{[-1]}$ is close to a positive multiple of $N_{\chi_0}\psi_{[-1]}$ when r is sufficiently close to r_+ from the choice of N_{χ_0} in Proposition 3.2.1. Therefore, arguing the same as in the proof of Proposition 3.2.1, there exists a radius constant r_0 close to r_+ such that the red-shift estimate (3.33) near $\mathcal{H}^+$ holds for $\psi_{[-1]}$ holds. □

3.2.2. Red-shift estimates for spin-2 case

Proposition 3.2.3. In a slowly rotating Kerr spacetime $(\mathcal{M}, g_{M,a})$, there exist constants $\varepsilon_0(M)$, $r_+ < 2M < r_0(M) < r_1(M) < (1+\sqrt{2})M$ and $C = C(\Sigma_{\tau_1}, M) = C(\Sigma_{\tau_2}, M)$, two smooth functions $y_1(r)$ and $y_2(r)$ on $[r_+,\infty)$ with $y_1(r) \to 1$, $y_2(r) \to 0$ as $r \to r_+$, and a φ_τ-invariant timelike vector field N as defined in (3.26) with $\chi_0(r)$ a smooth cutoff function which equals to 1 for $r \leq r_0$ and is identically zero for $r \geq r_1$, such that for all $|a|/M \leq a_0/M \leq \varepsilon_0$,

- for $\psi \in \{\phi^1_{+2}, \phi^2_{+2}, \phi^2_{-2}\}$ whose governing equations (2.31b), (2.31c) and (2.32c) can be put into the form of (2.34b) with the relevant inhomogeneous term F, the following estimate holds for any $\tau_2 > \tau_1$:

$$\begin{aligned}
&E_{\mathcal{H}^+(\tau_1,\tau_2)}(\psi) + \int_{\Sigma_{\tau_2}\cap\{r\leq r_0\}} |\partial\psi|^2 \\
&+ \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}} \left(|\partial\psi|^2 + |\log(r-r_+)|^{-2}|r-r_+|^{-1}\psi^2\right) \\
\leq& C\int_{\Sigma_{\tau_1}\cap\{r\leq r_1\}} |\partial\psi|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}} |\partial\psi|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]} |F|^2;
\end{aligned} \tag{3.36}$$

- for the equation (2.31a) of ϕ^0_{+2}, the following estimate near horizon holds for any $\tau_2 > \tau_1$:

$$\begin{aligned}
&E_{\mathcal{H}^+(\tau_1,\tau_2)}(\phi^0_{+2}) + \int_{\Sigma_{\tau_2}\cap\{r\leq r_0\}} |\partial\phi^0_{+2}|^2 \\
&+ \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}} \left(|\partial\phi^0_{+2}|^2 + |\log(r-r_+)|^{-2}|r-r_+|^{-1}|\phi^0_{+2}|^2\right) \\
\leq& C\int_{\Sigma_{\tau_1}\cap\{r\leq r_1\}} |\partial\phi^0_{+2}|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}} |\partial\phi^0_{+2}|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]} |\phi^1_{+2}|^2.
\end{aligned} \tag{3.37}$$

Proof. Following the discussions in Section 3.2.1, the estimate (3.36) manifestly holds true.

For ϕ^0_{+2}, we also make use of the following equivalent form of equation (2.31a):

$$\begin{aligned}
\Sigma\widetilde{\Box}_g(\phi^0_{+2}) =& \tfrac{2(r^2+2Mr-2a^2)}{r} Y\phi^0_{+2} + \tfrac{2r^2-8Mr+12a^2}{r^3}\phi^1_{+2} \\
&- \tfrac{8(a^2\partial_t + a\partial_\phi)\phi^0_{+2}}{r} + 8ia\cos\theta\partial_t(\phi^0_{+2}).
\end{aligned} \tag{3.38}$$

Then the estimate (3.37) follows easily. □

It is convenient to introduce the variables which are not degenerate at $\mathcal{H}^+$

$$\widetilde{\phi^0_{-2}} = \Delta^{-2}r^4\phi^0_{-2}, \qquad \widetilde{\phi^1_{-2}} = \Delta^{-1}r^2\phi^1_{-2}, \tag{3.39}$$

and we may suppress the subindex and simply write as $\widetilde{\phi}^0$ and $\widetilde{\phi}^1$. The equation for $\widetilde{\phi}^0 = \psi_{[-2]}$ reads

$$\Sigma\widetilde{\Box}_g\widetilde{\phi}^0 = \tfrac{8r^2-10a^2}{r^2}\widetilde{\phi}^0 + \left(\tfrac{4(r-M)r-5\Delta}{r}Y + 2r\partial_t\right)\widetilde{\phi}^0 - \tfrac{5\Delta}{r^2}\widetilde{\phi}^0$$

$$+\tfrac{10}{r}\left(a^2\partial_t + a\partial_\phi\right)\widetilde{\phi^0} + \tfrac{5}{r}\widetilde{\phi^1} - 8ia\cos\theta\partial_t\widetilde{\phi^0}, \tag{3.40}$$

and the governing equation for $r^2\widetilde{\phi^1}$ is

$$\begin{aligned}\Sigma\widetilde{\Box}_g(r^2\widetilde{\phi^1}) =& \tfrac{7r^2-3a^2}{2r^2}(r^2\widetilde{\phi^1}) + \left(\tfrac{4(r-M)r-9\Delta}{2r}Y + r\partial_t\right)(r^2\widetilde{\phi^1}) + \tfrac{r}{2}\phi^2\\ &+ \tfrac{6\Delta}{r}\left((Mr-2a^2)\widetilde{\phi^0} + r\left(a^2\partial_t + a\partial_\phi\right)\widetilde{\phi^0}\right)\\ &+ \tfrac{5}{r}\left(a^2\partial_t + a\partial_\phi\right)(r^2\widetilde{\phi^1}) - 8ia\cos\theta\partial_t(r^2\widetilde{\phi^1}).\end{aligned} \tag{3.41}$$

One could easily adapt the proof in Section 3.2.1 to obtain:

Proposition 3.2.4. In a slowly rotating Kerr spacetime $(\mathcal{M}, g_{M,a})$, there exist constants $\varepsilon_0(M)$, $r_+ < 2M < r_0(M) < r_1(M) < (1+\sqrt{2})M$ and $C = C(\Sigma_{\tau_1}, M) = C(\Sigma_{\tau_2}, M)$, and a φ_τ-invariant timelike vector field N defined as in (3.26) for two smooth functions $y_1(r)$ and $y_2(r)$ on $[r_+,\infty)$ with $y_1(r)\to 1$, $y_2(r)\to 0$ as $r\to r_+$, such that for all $|a|/M \le a_0/M \le \varepsilon_0$, the following red-shift estimates hold for $\widetilde{\phi^0_{-2}}$ and $\widetilde{\phi^1_{-2}}$ for any $\tau_2 > \tau_1$:

$$\begin{aligned}&E_{\mathcal{H}^+(\tau_1,\tau_2)}(\widetilde{\phi^0}) + \int_{\Sigma_{\tau_2}\cap\{r\le r_0\}} |\partial\widetilde{\phi^0}|^2\\ &+ \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\le r_0\}} \left(|\partial\widetilde{\phi^0}|^2 + |\log(r-r_+)|^{-2}|r-r_+|^{-1}|\widetilde{\phi^0}|^2\right)\\ \le& C\int_{\Sigma_{\tau_1}\cap\{r\le r_1\}} |\partial\widetilde{\phi^0}|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\le r\le r_1\}} |\partial\widetilde{\phi^0}|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]} |\widetilde{\phi^1}|^2,\end{aligned} \tag{3.42}$$

$$\begin{aligned}&E_{\mathcal{H}^+(\tau_1,\tau_2)}(\widetilde{\phi^1}) + \int_{\Sigma_{\tau_2}\cap\{r\le r_0\}} |\partial\widetilde{\phi^1}|^2\\ &+ \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\le r_0\}} \left(|\partial\widetilde{\phi^1}|^2 + |\log(r-r_+)|^{-2}|r-r_+|^{-1}|\widetilde{\phi^1}|^2\right)\\ \le& C\int_{\Sigma_{\tau_1}\cap\{r\le r_1\}} |\partial\widetilde{\phi^1}|^2 + C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\le r\le r_1\}} |\partial\widetilde{\phi^1}|^2\\ &+ C\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]} \left(|\phi^2_{-2}|^2 + \tfrac{|a|}{M}|\partial\widetilde{\phi^0}|^2 + |\widetilde{\phi^0}|^2\right).\end{aligned} \tag{3.43}$$

4. Outline of Proof

Contents

4.1. Proof of Theorem 2.4.2 for $n \geq 1$ based on $n = 0$ estimates

Assume the $n = 0$ case of Theorem 2.4.2 is true. To prove the inequality (2.50) with integer $n \geq 1$, one just needs to consider the case $n = 1$ by induction. In the spin-2 case, we commute $\chi_0 Y$ with (2.31b), (2.31c), (2.32c), (3.38), (3.40) and (3.41), then it follows easily from the red-shift commutation property (Dafermos and Rodnianski, 2010, Prop.5.4.1), elliptic estimates and the fact that T and ∂_{ϕ^*} are Killing vector fields that the estimate (2.50) for $n = 1$ is valid. Similarly, for the spin-1 case, by commuting the Killing vector field T with (2.42), $\chi_0 Y$ with (2.42) for spin +1 component and (3.35) for spin −1 component, it can be analogously argued that Theorem 2.4.2 holds for $n \geq 1$.

4.2. Estimates for spacetime integrals of ϕ_s^0 and ϕ_s^1

We derive in this section some estimates for ϕ_s^0 and ϕ_s^1 which are used in Section 4.3.

4.2.1. Spin-1

We state a lemma controlling $\left|\nabla\phi_{-1}^0\right|$ by $\left|\nabla\phi_{-1}^1\right|$.

Lemma 4.2.1. In a fixed subextremal Kerr spacetime $(\mathcal{M}, g_{M,a})$ $(|a| \leq a_0 < M)$, the following estimate holds for spin -1 component:

$$\begin{aligned}&\int_{\mathcal{D}(0,\tau)\cap[R,\infty)} |\nabla\phi^0|^2 + \int_{\Sigma_\tau\cap[R,\infty)} r|\nabla\phi^0|^2 \\ \lesssim &\int_{\mathcal{D}(0,\tau)\cap[R-1,\infty)} \frac{|\nabla\phi^1|^2}{r} + \int_{\Sigma_0\cap[R-1,\infty)} r|\nabla\phi^0|^2 + \int_{\mathcal{D}(0,\tau)\cap[R-1,R)} \frac{|\nabla\phi^0|^2}{r}. \end{aligned} \tag{4.1}$$

Proof. We start with an identity that for the cutoff function $\chi_R(r)$, any real value β and ∇_i $(i = 1, 2, 3)$ as defined in (2.38):

$$\begin{aligned}&V\left(\chi_R r^\beta |r^2\nabla_i\phi^0|^2\right) - \beta\chi_R r^{\beta-1}|r^2\nabla_i\phi^0|^2 - \partial_r\chi_R r^\beta|r^2\nabla_i\phi^0|^2 \\ = {}&2\chi_R r^{2+\beta}\Re\left(\nabla_i\phi^0\overline{\nabla_i\phi^1}\right).\end{aligned} \tag{4.2}$$

Integrating (4.2) over $\mathcal{D}(0,\tau)$ with the measure

$$d\check{V} = r^{-2}dV = drdt^*\sin\theta d\theta d\phi^* \tag{4.3}$$

for $\beta = -1$, and applying Cauchy-Schwarz to the last term, it is manifest that the estimate (4.1) follows from summing over $i = 1, 2, 3$. □

4.2.2. Spin-2

4.2.2.1. Spin +2 component

Proposition 4.2.1. In a fixed subextremal Kerr spacetime $(\mathcal{M}, g_{M,a})$ $(|a| \leq a_0 < M)$, the following estimate holds for ϕ_{+2}^1 defined as in (2.28a) from the spin $+2$ component:

$$\int_{\mathcal{D}(0,\tau)} \tfrac{|\phi^1|^2}{r^2} \lesssim \hat{\epsilon}_1 \int_{\mathcal{D}(0,\tau)} \tfrac{|r\phi^1|^2}{r^3} + \hat{\epsilon}_1^{-1} \int_{\mathcal{D}(0,\tau)} \tfrac{|\phi^2|^2}{r^3} + \int_{\Sigma_0} \tfrac{|r\phi^1|^2}{r^2}. \tag{4.4}$$

Proof. We start with a simple identity for any smooth real function $f_{+2}(r)$ and any real value α:

$$Y\left(f_{+2}r^{\alpha}|r\phi^1|^2\right) + f_{+2}\alpha r^{\alpha-1}|r\phi^1|^2 - Y(f_{+2})r^{\alpha}|r\phi^1|^2 = 2f_{+2}r^{\alpha}\Re(\phi^1\overline{\phi^2}). \tag{4.5}$$

Integrate (4.5) over $\mathcal{D}(0,\tau)$ with the measure $d\check{V}$ as in (4.3) for $\alpha = 0$ and $f_{+2} = \frac{\Delta}{r^2+a^2}$. Then, since

$$-Y(f_{+2}) = \partial_r f_{+2} = \tfrac{2M(r^2-a^2)}{(r^2+a^2)^2} \geq \tfrac{c}{r^2}, \tag{4.6}$$

an application of Cauchy-Schwarz inequality to the term $\int_{\mathcal{D}(0,\tau)} f_{+2}\Re(\phi^1\overline{\phi^2})d\check{V}$ proves the estimate (4.4). □

4.2.2.2. Spin -2 component

Proposition 4.2.2. In a fixed subextremal Kerr spacetime $(\mathcal{M}, g_{M,a})$ ($|a| \leq a_0 < M$), it holds for ϕ_{-2}^0 and ϕ_{-2}^1 defined as in (2.28b) from the spin -2 component that

$$\int_{\mathcal{D}(0,\tau)} \tfrac{|\widetilde{\phi^0}|^2}{r^2} \lesssim \int_{\mathcal{D}(0,\tau)} \tfrac{|\phi^2|^2}{r^3} + \int_{\Sigma_0}\left(\tfrac{|\widetilde{\phi^0}|^2}{r} + \tfrac{|\widetilde{\phi^1}|^2}{r}\right), \tag{4.7a}$$

$$\int_{\mathcal{D}(0,\tau)} \tfrac{|\widetilde{\phi^1}|^2}{r^2} \lesssim \int_{\mathcal{D}(0,\tau)} \tfrac{|\phi^2|^2}{r^3} + \int_{\Sigma_0} \tfrac{|\widetilde{\phi^1}|^2}{r}. \tag{4.7b}$$

Moreover, for the angular derivatives of them, we have

$$\begin{aligned}&\int_{\mathcal{D}(0,\tau)\cap[6M,\infty)} |\nabla\widetilde{\phi^0}|^2 + \int_{\Sigma_\tau\cap[6M,\infty)} r|\nabla\widetilde{\phi^0}|^2 \\ \lesssim\ & \int_{\mathcal{D}(0,\tau)\cap[5M,\infty)} \frac{|\nabla\widetilde{\phi^1}|^2}{r} + \int_{\Sigma_0\cap[5M,\infty)} r|\nabla\widetilde{\phi^0}|^2 + \int_{\mathcal{D}(0,\tau)\cap[5M,6M]} \frac{|\nabla\widetilde{\phi^0}|^2}{r},\end{aligned} \tag{4.8a}$$

$$\begin{aligned}&\int_{\mathcal{D}(0,\tau)\cap[6M,\infty)} |\nabla\widetilde{\phi^1}|^2 + \int_{\Sigma_\tau\cap[6M,\infty)} r|\nabla\widetilde{\phi^1}|^2 \\ \lesssim\ & \int_{\mathcal{D}(0,\tau)\cap[5M,\infty)} \frac{|\nabla\phi^2|^2}{r} + \int_{\Sigma_0\cap[5M,\infty)} r|\nabla\widetilde{\phi^1}|^2 + \int_{\mathcal{D}(0,\tau)\cap[5M,6M]} \frac{|\nabla\widetilde{\phi^1}|^2}{r}\end{aligned} \tag{4.8b}$$

Proof. We derive for any real function $f_{-2}(r)$ and any real value β that

$$V(f_{-2}r^{\beta}|r\phi^1|^2) - f_{-2}\beta r^{\beta-1}|r\phi^1|^2 - \partial_r f_{-2}r^{\beta}|r\phi^1|^2 = -2r^{\beta}f_{-2}\Re(\phi^1\overline{\phi^2}). \tag{4.9}$$

By choosing $\beta = -1$ and $f_{-2} = \frac{r^2+a^2}{\Delta}$, since $\partial_r f_{-2} = \frac{-2M(r^2-a^2)}{\Delta^2}$, the estimate (4.7b) then follows from integrating (4.9) over $\mathcal{D}(0,\tau)$ with the measure $d\check{V}$ in (4.3) and applying Cauchy-Schwarz to the integral of the RHS of (4.9).

Similarly, for ϕ^0, we have

$$\int_{\mathcal{D}(0,\tau)} \frac{|\widetilde{\phi^0}|^2}{r^2} \lesssim \int_{\mathcal{D}(0,\tau)} \frac{|\widetilde{\phi^1}|^2}{r^3} + \int_{\Sigma_0} \frac{|\widetilde{\phi^0}|^2}{r}. \tag{4.10}$$

Combining (4.7b) with (4.10) proves the estimate (4.7a).

We prove the inequality (4.8a) below, the proof for (4.8b) being analogous. For a smooth cutoff function $\chi_2(r)$ which is equal to 1 in $[6M,\infty)$ and vanishes in $[r_+,5M]$, any real value β and ∇_j $(j=1,2,3)$ as defined in (2.38), it holds

$$\begin{aligned} &V(f_{-2}\chi_2 r^\beta |r^2\nabla_j\phi^0|^2) - \chi_2\partial_r f_{-2} r^\beta |r^2\nabla_j\phi^0|^2 \\ &-(\beta\chi_2 f_{-2} + \partial_r\chi_2 f_{-2} r)\, r^{\beta-1}|r^2\nabla_j\phi^0|^2 = -2\chi_2 f_{-2} r^{2+\beta}\Re(\nabla_j\phi^0\overline{\nabla_j\phi^1}). \end{aligned} \tag{4.11}$$

Choosing $\beta = -1$ and $f_{-2} = \frac{(r^2+a^2)^3}{\Delta^3}$, integrating (4.11) over $\mathcal{D}(0,\tau)$ with the measure $d\check{V}$ in (4.3), and applying Cauchy-Schwarz to the last term, the estimate (4.8a) for $i=0$ follows manifestly from summing over $j=1,2,3$. □

4.3. Proof of $n = 0$ case

Define two quantities for spin ±1 components respectively

$$\begin{aligned} \Xi_{+1}(0,\tau) =& E_0(r^{2-\delta}\phi^0_{+1}) + E_0(\phi^1_{+1}) + \tfrac{|a|}{M}\left(E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^0_{+1}) + E_{\mathcal{H}^+(0,\tau)}(\phi^1_{+1})\right) \\ &+ \tfrac{|a|}{M}\left[E_\tau(r^{2-\delta}\phi^0_{+1}) + E_\tau(\phi^1_{+1}) + \int_{\mathcal{D}(0,\tau)} (\widetilde{\mathbb{M}}(r^{2-\delta}\phi^0_{+1}) + \mathbb{M}_{\text{deg}}(\phi^1_{+1}))\right], \end{aligned} \tag{4.12a}$$

$$\begin{aligned} \Xi_{-1}(0,\tau) =& E_0(\phi^0) + E_0(\phi^1) + \int_{\Sigma_0} r|\nabla\phi^0|^2 \\ &+ \tfrac{|a|}{M}\left[\sum_{i=0,1}\left(E_\tau(\phi^i) + E_{\mathcal{H}^+(0,\tau)}(\phi^i)\right) + \int_{\mathcal{D}(0,\tau)}\left(\mathbb{M}_{\text{deg}}(\phi^1_{-1}) + \mathbb{M}(\phi^0)\right)\right]. \end{aligned} \tag{4.12b}$$

Moreover, recalling the definition in (3.39), we define two quantities for spin ±2 components respectively that

$$\Xi_{+2}(0,\tau) = E_0(r^{4-\delta}\phi^0_{+2}) + E_0(r^{2-\delta}\phi^1_{+2}) + E_0(\phi^2_{+2})$$

$$
\begin{aligned}
&+ \tfrac{|a|}{M} \left(E_\tau(r^{4-\delta}\phi^0_{+2}) + E_\tau(r^{2-\delta}\phi^1_{+2}) + E_\tau(\phi^2_{+2})\right) \\
&+ \tfrac{|a|}{M} \left(E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0_{+2}) + E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^1_{+2}) + E_{\mathcal{H}^+(0,\tau)}(\phi^2_{+2})\right) \\
&+ \tfrac{|a|}{M} \int_{\mathcal{D}(0,\tau)} \left(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0_{+2}) + \widetilde{\mathbb{M}}(r^{2-\delta}\phi^1_{+2}) + \mathbb{M}_{\text{deg}}(\phi^2_{+2})\right), \qquad (4.13a)
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{-2}(0,\tau) =& E_0(\widetilde{\phi^0}) + E_0(\widetilde{\phi^1}) + E_0(\phi^2_{-2}) + \int_{\Sigma_0} r\left(|\nabla\widetilde{\phi^0}|^2 + |\nabla\widetilde{\phi^1}|^2\right) \\
&+ \tfrac{|a|}{M}\left(\sum_{i=0}^{1}\left(E_\tau(\widetilde{\phi^i}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^i})\right) + E_\tau(\phi^2_{-2}) + E_{\mathcal{H}^+(0,\tau)}(\phi^2_{-2})\right) \\
&+ \tfrac{|a|}{M} \int_{\mathcal{D}(0,\tau)} \left(\mathbb{M}_{\text{deg}}(\phi^2_{-2}) + \mathbb{M}(\widetilde{\phi^1}) + \mathbb{M}(\widetilde{\phi^0})\right). \qquad (4.13b)
\end{aligned}
$$

We say $F_1 \lesssim_a F_2$ for two functions in the region $\mathcal{D}(0,\tau)$ if there exists a universal constant $C = C(a_0, M, \delta, \Sigma_0)$ such that

$$
F_1 \leq CF_2 + C\Xi_s(0,\tau) \qquad (4.14)
$$

depending on which spin component we are considering. We now give the outline of the proof of the estimates (2.50) for different spin components separately.

4.3.1. Spin $+1$ component

We will first show in Chapters 5–7 that

$$
\begin{aligned}
&E_\tau(r^{2-\delta}\phi^0) + E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^0) + \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}_{\text{deg}}(r^{2-\delta}\phi^0) \\
\lesssim_a &\int_{\mathcal{D}(0,\tau)} \left(\epsilon_0\widetilde{\mathbb{M}}(r^{2-\delta}\phi^0) + \epsilon_0^{-1}\tfrac{|\phi^1|^2}{r^3}\right), \qquad (4.15a) \\
&E_\tau(\phi^1) + E_{\mathcal{H}^+(0,\tau)}(\phi^1) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\phi^1) \lesssim_a 0. \qquad (4.15b)
\end{aligned}
$$

By adding an A_0 multiple of estimate (4.15b) to (4.15a) and from the fact that

$$
\int_{\mathcal{D}(0,\tau)} \left(\widetilde{\mathbb{M}}_{\text{deg}}(r^{2-\delta}\phi^0) + \mathbb{M}_{\text{deg}}(\phi^1)\right) \sim \int_{\mathcal{D}(0,\tau)} \left(\widetilde{\mathbb{M}}(r^{2-\delta}\phi^0) + \mathbb{M}_{\text{deg}}(\phi^1)\right), \qquad (4.16)
$$

we can choose ϵ_0 sufficiently small and a sufficiently large constant A_0 such that the RHS of the added inequality can be absorbed by the LHS for sufficiently small $|a|/M \leq a_0/M$. This completes the proof of Theorem 2.4.2 for spin $+1$ component for $n = 0$.

4.3.2. Spin -1 component

For spin -1 component, the following estimates will be justified in Chapters 5–7

$$E_\tau(\phi^0) + E_{\mathcal{H}^+(0,\tau)}(\phi^0) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\phi^0) \lesssim_a \int_{\mathcal{D}(0,\tau)} \left(\epsilon_0 \mathbb{M}(\phi^0) + \tfrac{1}{\epsilon_0}\tfrac{|\phi^1|^2}{r^3}\right), \quad (4.17a)$$

$$E_\tau(\phi^1) + E_{\mathcal{H}^+(0,\tau)}(\phi^1) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\phi^1) \lesssim_a \tfrac{|a|}{M} \int_{\mathcal{D}(0,\tau)} |\nabla\!\!\!/ \phi^0|^2. \quad (4.17b)$$

We add an A_0 multiple of (4.17b) to (4.17a), and similar to the discussions for spin $+1$ component it holds that

$$\int_{\mathcal{D}(0,\tau)} \left(\mathbb{M}_{\text{deg}}(\phi^0) + \mathbb{M}_{\text{deg}}(\phi^1)\right) \sim \int_{\mathcal{D}(0,\tau)} \left(\mathbb{M}(\phi^0) + \mathbb{M}_{\text{deg}}(\phi^1)\right). \quad (4.18)$$

Then we can use the estimate (4.1) to bound the RHS of the gained inequality, and by choosing ϵ_0 sufficiently small and A_0 sufficiently large, Theorem 2.4.2 for $n = 0$ case is proved by taking into account the red-shift estimate (3.33) and the following fact.

Proposition 4.3.1. For the spin -1 component, it holds for any $\tau \geq 0$ that

$$\int_{\Sigma_0} r|\nabla\!\!\!/ \phi^0|^2 \lesssim E_0(\phi^0) + E_0(\phi^1). \quad (4.19)$$

Proof. Notice from the equation (2.30a) of ϕ^0_{-1} that

$$\begin{aligned}\tfrac{\Delta}{r^2} Y\phi^1 &= \triangle_{\mathbb{S}^2}\phi^0 - 2i\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right)\phi^0 - \tfrac{1}{\sin^2\theta}\phi^0 + \tfrac{\Delta}{r^3}\phi^1 \\ &\quad + a^2\cos^2\theta\partial^2_{tt}\phi^0 + 2a\partial^2_{t\phi}\phi^0 + \tfrac{6(a^2\partial_t + a\partial_\phi)}{r}\phi^0 + \tfrac{2ar}{r^2+a^2}\partial_\phi\phi^0. \end{aligned} \quad (4.20)$$

By multiplying $r^{-1}\overline{\phi^0}$ on both sides, taking the real part and integrating over $\Sigma_\tau \cap \{r \geq R_4\}$ ($\tau \geq 0$) with large R_4 to be fixed, it follows

$$\int_{\Sigma_\tau} r\left|\nabla\!\!\!/ \phi^0\right|^2 \lesssim E_\tau(\phi^0) + E_\tau(\phi^1) + a^2\left|\int_{\Sigma_\tau\cap\{r\geq R_4\}} r^{-1}\Re(\partial^2_{tt}\phi^0\overline{\phi^0})\right|. \quad (4.21)$$

We substitute into the last integral the following relation

$$\partial^2_{tt} = \left(\tfrac{\Delta}{r^2+a^2}V - \tfrac{a}{r^2+a^2}\partial_\phi - \partial_{r^*}\right)\left(\tfrac{\Delta}{r^2+a^2}V - \tfrac{a}{r^2+a^2}\partial_\phi - \partial_{r^*}\right), \quad (4.22)$$

use the replacement $V\phi^0 = r^{-2}\phi^1 - r^{-1}\phi^0$, and perform integration by parts, finally ending with

$$\left|\int_{\Sigma_\tau\cap\{r\geq R_4\}} \tfrac{1}{r}\Re(\partial^2_{tt}\phi^0\overline{\phi^0})\right| \lesssim E_\tau(\phi^0) + E_\tau(\phi^1) + \int_{\Sigma_\tau\cap\{r=R_4\}} \left|\partial\phi^0\right|^2. \quad (4.23)$$

We can appropriately choose R_4 such that the last term is bounded by $CE_\tau(\phi^0)$. □

4.3.3. Spin $+2$ component

We will first obtain in Chapters 5–7 the following energy and Morawetz estimates for ϕ^0, ϕ^1 and ϕ^2 defined from the spin $+2$ component:

$$\begin{aligned} &E_\tau(r^{4-\delta}\phi^0) + E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0) + \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0) \\ \lesssim_a &\int_{\mathcal{D}(0,\tau)} \Big(\epsilon_0 \widetilde{\mathbb{M}}(r^{4-\delta}\phi^0) + \epsilon_0^{-1}\tfrac{|\phi^1|^2}{r^3}\Big), \end{aligned} \tag{4.24a}$$

$$\begin{aligned} &E_\tau(r^{2-\delta}\phi^1) + E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^1) + \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1) \\ \lesssim_a &\int_{\mathcal{D}(0,\tau)} \Big(\epsilon_1 \widetilde{\mathbb{M}}(r^{2-\delta}\phi^1) + \epsilon_1^{-1}\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0) + \epsilon_1^{-1}\mathbb{M}_{\deg}(\phi^2)\Big), \end{aligned} \tag{4.24b}$$

$$E_\tau(\phi^2) + E_{\mathcal{H}^+(0,\tau)}(\phi^2) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\phi^2) \lesssim_a 0. \tag{4.24c}$$

In addition, the estimate (4.4) for ϕ^1 in Section 4.2 can be used to bound the last term in (4.24a). The parameters ϵ_0 and ϵ_1 in (4.24), and $\hat{\epsilon}_1$ in (4.4), are small constants to be fixed. Substituting (4.4) into (4.24a) gives

$$\begin{aligned} &E_\tau(r^{4-\delta}\phi^0) + E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0) + \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0) \\ \lesssim_a &\epsilon_0 \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(r^{4-\delta}\phi^0) + \epsilon_0^{-1}\hat{\epsilon}_1 \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(r\phi^1) + \epsilon_0^{-1}\hat{\epsilon}_1^{-1} \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\phi^2). \end{aligned} \tag{4.25}$$

We add A_0 multiple of estimate (4.25) and A_1 multiple of (4.24c) to the estimate (4.24b), and fix the parameters one by one to satisfy

$$\epsilon_1 \ll 1, A_0 \gg \epsilon_1^{-1}, \epsilon_0 \ll A_0^{-1}, \hat{\epsilon}_1 \ll A_0^{-1}\epsilon_0, A_1 \gg A_0\left(\epsilon_0\hat{\epsilon}_1\right)^{-1} + \epsilon_1^{-1}, \tag{4.26}$$

then for sufficiently small $|a|/M \leq a_0/M$ all the spacetime integrals on the RHS of the gained estimate can be absorbed by the LHS, arriving at:

$$\begin{aligned} &E_\tau(r^{4-\delta}\phi^0_{+2}) + E_\tau(r^{2-\delta}\phi^1_{+2}) + E_\tau(\phi^2) \\ &+ \big(E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0_{+2}) + E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^1_{+2}) + E_{\mathcal{H}^+(0,\tau)}(\phi^2)\big) \\ &+ \int_{\mathcal{D}(0,\tau)} \Big(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0_{+2}) + \widetilde{\mathbb{M}}(r^{2-\delta}\phi^1_{+2}) + \mathbb{M}_{\deg}(\phi^2_{+2})\Big) \\ \lesssim\ &E_0(r^{4-\delta}\phi^0_{+2}) + E_0(r^{2-\delta}\phi^1_{+2}) + E_0(\phi^2_{+2}). \end{aligned} \tag{4.27}$$

Here, we have utilized the facts that

$$\int_{\mathcal{D}(0,\tau)} \Big(\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0) + \widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1)\Big)$$

$$\sim \int_{\mathcal{D}(0,\tau)} \left(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0) + \widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1) \right), \tag{4.28a}$$

$$\int_{\mathcal{D}(0,\tau)} \left(\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0) + \widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1) + \mathbb{M}_{\deg}(\phi^2) \right)$$
$$\sim \int_{\mathcal{D}(0,\tau)} \left(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0) + \widetilde{\mathbb{M}}(r^{2-\delta}\phi^1) + \mathbb{M}_{\deg}(\phi^2) \right). \tag{4.28b}$$

In the trapped region, $\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0) + \widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1)$ bounds over $|Y\phi^0|^2$, $|\partial_{r^*}\phi^0|^2$ and $|\phi^0|^2$ and then over $|\phi^0|^2$ and $|H\phi^0|^2$, $H = \partial_t + a/(r^2+a^2)\partial_\phi$ being a globally timelike vector field in the interior of $\mathcal{D}$ with $-g(H,H) = \Delta\Sigma/(r^2+a^2)^2$. Hence, (4.28a) follows from elliptic estimates. The inequality (4.28b) can be similarly justified. The estimate (2.50a) with $n=0$ then follows from (4.27).

4.3.4. Spin -2 component

Similarly as above, ϵ_0 and ϵ_1 are small constants to be fixed and we will prove in Chapters 5–7 the following energy and Morawetz estimates for $\widetilde{\phi^0}$, $\widetilde{\phi^1}$ and ϕ^2 constructed from the spin -2 component:

$$E_\tau(\widetilde{\phi^0}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^0}) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\widetilde{\phi^0}) \lesssim_a \int_{\mathcal{D}(0,\tau)} \left(\epsilon_0 \mathbb{M}(\widetilde{\phi^0}) + \tfrac{1}{\epsilon_0}\tfrac{|\widetilde{\phi^1}|^2}{r^3} \right), \tag{4.29a}$$

$$E_\tau(\widetilde{\phi^1}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^1}) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\widetilde{\phi^1})$$
$$\lesssim_a \int_{\mathcal{D}(0,\tau)} \left(\epsilon_1 \mathbb{M}(\widetilde{\phi^1}) + \tfrac{1}{\epsilon_1}\left(\mathbb{M}_{\deg}(\phi^2) + \tfrac{|a|}{M}|\nabla\widetilde{\phi^0}|^2 + \tfrac{|\widetilde{\phi^0}|^2}{r^2} \right) \right), \tag{4.29b}$$

$$E_\tau(\phi^2) + E_{\mathcal{H}^+(0,\tau)}(\phi^2) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\phi^2) \lesssim_a \tfrac{|a|}{M} \int_{\mathcal{D}(0,\tau)} \left(|\nabla\widetilde{\phi^1}|^2 + \tfrac{|\widetilde{\phi^0}|^2}{r^2} \right). \tag{4.29c}$$

By substituting (4.7b) into (4.29a), (4.7a) and (4.8a) into (4.29b), (4.7a) and (4.8b) into (4.29c), respectively, it follows that

$$E_\tau(\widetilde{\phi^0}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^0}) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\widetilde{\phi^0})$$
$$\lesssim_a \epsilon_0 \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(\widetilde{\phi^0}) + \tfrac{1}{\epsilon_0} \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\phi^2), \tag{4.30}$$
$$E_\tau(\widetilde{\phi^1}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^1}) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\widetilde{\phi^1})$$

$$\lesssim_a \int_{\mathcal{D}(0,\tau)} \left(\epsilon_1 + \tfrac{|a|}{\epsilon_1 M}\right) \mathbb{M}(\widetilde{\phi^1}) + \tfrac{1}{\epsilon_1}\left(\mathbb{M}_{\deg}(\phi^2) + \tfrac{|a|}{M}\mathbb{M}(\phi^0)\right), \tag{4.31}$$

$$E_\tau(\phi^2) + E_{\mathcal{H}^+(0,\tau)}(\phi^2) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\phi^2) \lesssim_a 0. \tag{4.32}$$

We add an A_0 multiple of estimate (4.30) and an A_1 multiple of (4.32) to the estimate (4.31), and fix the parameters in an order such that

$$A_0 \gg 1, \epsilon_1 \ll 1, \epsilon_0 \ll A_0^{-1}, A_1 \gg A_0\epsilon_0^{-1} + \epsilon_1^{-1}, \tag{4.33}$$

then for sufficiently small $|a|/M \leq a_0/M$, all the spacetime integrals on RHS can be absorbed by the LHS, and it holds true that:

$$\begin{aligned}
&\sum_{i=0,1}\left(E_\tau(\widetilde{\phi^i}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^i})\right) + \left(E_\tau(\phi^2) + E_{\mathcal{H}^+(0,\tau)}(\phi^2)\right)\\
&+ \int_{\mathcal{D}(0,\tau)}\left(\widetilde{\mathbb{M}}(\widetilde{\phi^0}) + \widetilde{\mathbb{M}}(\widetilde{\phi^1}) + \mathbb{M}_{\deg}(\phi^2)\right)\\
\lesssim& \sum_{i=0,1} E_0(\widetilde{\phi^i}) + E_0\left(\phi^2\right) + \int_{\Sigma_0} r\left(|\nabla\widetilde{\phi^0}|^2 + |\nabla\widetilde{\phi^1}|^2\right)\\
\lesssim& \sum_{i=0,1} E_0(\widetilde{\phi^i}) + E_0\left(\phi^2\right). \qquad (4.34)
\end{aligned}$$

The inference is as follows. It can be argued in the same way as in the relations (4.28) for the spin +2 component that the trapping degeneracy in the terms $\mathbb{M}_{\deg}(\phi^0)$ and $\mathbb{M}_{\deg}(\phi^1)$ can be removed, and in the last step we have used the the following Proposition.

Proposition 4.3.2. For the spin -2 component, it holds for any $\tau \geq 0$ that

$$\int_{\Sigma_0} r(|\nabla\widetilde{\phi^0}|^2 + |\nabla\widetilde{\phi^1}|^2) \lesssim E_0(\widetilde{\phi^0}) + E_0(\widetilde{\phi^1}) + E_0(\phi^2_{-2}). \tag{4.35}$$

Proof. Rewrite the equations (2.32a) and (2.32b) as

$$\begin{aligned}
0 =& \tfrac{\Delta}{r^2}Y\phi^1 + \triangle_{\mathbb{S}^2}\phi^0 - 4i\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right)\phi^0 - \tfrac{4}{\sin^2\theta}\phi^0 + \tfrac{2r^2-6Mr+6a^2}{r^2}\phi^0\\
&+ a^2\cos^2\theta\partial^2_{tt}\phi^0 + \tfrac{2(a^2\partial_t + a\partial_\phi)}{r}\phi^0 + 2a\partial^2_{t\phi}\phi^0 + \tfrac{2ar}{r^2+a^2}\partial_\phi\phi^0 - \tfrac{3\Delta+a^2}{r^3}\phi^1, \qquad (4.36a)\\
0 =& \tfrac{\Delta}{r^2}Y\phi^2 + \triangle_{\mathbb{S}^2}\phi^1 - 4i\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right)\phi^1 - \tfrac{4}{\sin^2\theta}\phi^1\\
&+ a^2\cos^2\theta\partial^2_{tt}\phi^1 + 2a\partial^2_{t\phi}\phi^1 + \tfrac{6(a^2\partial_t + a\partial_\phi)}{r}\phi^1 + \tfrac{2ar}{r^2+a^2}\partial_\phi\phi^1\\
&- 6(a^2\partial_t + a\partial_\phi)\phi^0 - \tfrac{\Delta}{r^3}\phi^2 + \tfrac{6Mr-6a^2}{r^2}\phi^1 + \tfrac{12a^2-6Mr}{r}\phi^0. \qquad (4.36b)
\end{aligned}$$

By multiplying $r^{-1}\overline{\phi^0}$ on both sides of (4.36a), taking the real part and integrating over $\Sigma_\tau \cap \{r \geq R_3\}$ ($\tau \geq 0$) with $R_3 \geq 5M$ to be fixed, it follows

$$\int_{\Sigma_\tau} r|\nabla\widetilde{\phi^0}|^2 \lesssim E_\tau(\widetilde{\phi^0}) + E_\tau(\widetilde{\phi^1}) + a^2 \left|\int_{\Sigma_\tau\cap\{r\geq R_3\}} r^{-1}\Re(\partial_{tt}^2\phi^0\overline{\phi^0})\right|. \tag{4.37}$$

We substitute into the last integral the relation

$$\partial_{tt}^2 = \left(\tfrac{\Delta}{r^2+a^2}V - \tfrac{a}{r^2+a^2}\partial_\phi - \partial_{r^*}\right)\left(\tfrac{\Delta}{r^2+a^2}V - \tfrac{a}{r^2+a^2}\partial_\phi - \partial_{r^*}\right), \tag{4.38}$$

make the replacement $V\phi^0 = -r^{-2}\phi^1 - r^{-1}\phi^0$, and perform integration by parts, arriving at

$$\left|\int_{\Sigma_\tau\cap\{r\geq R_3\}} \tfrac{1}{r}\Re(\partial_{tt}^2\phi^0\overline{\phi^0})\right| \lesssim E_\tau(\phi^0) + E_\tau(\phi^1) + \int_{\Sigma_\tau\cap\{r=R_3\}} \left|\partial\phi^0\right|^2. \tag{4.39}$$

We can appropriately choose R_3 such that the last term is bounded by $CE_\tau(\phi^0)$, and conclude

$$\int_{\Sigma_\tau} r|\nabla\widetilde{\phi^0}|^2 \lesssim E_\tau(\widetilde{\phi^0}) + E_\tau(\widetilde{\phi^1}). \tag{4.40}$$

Similarly, we can obtain from (4.36b) that

$$\int_{\Sigma_\tau} r|\nabla\widetilde{\phi^1}|^2 \lesssim E_\tau(\widetilde{\phi^0}) + E_\tau(\widetilde{\phi^1}) + E_\tau(\widetilde{\phi^2}) + \int_{\Sigma_\tau} r|\nabla\widetilde{\phi^0}|^2. \tag{4.41}$$

The inequality (4.35) then follows from (4.40) and (4.41). □

From the estimate (4.34), the estimate (2.50) is proved for the other regular N-P component $\widetilde{\Phi_4}$ for $n = 0$.

5. Proof of Theorems 2.4.1 and 2.4.2 on Schwarzschild

Contents

We prove the Theorem 2.4.1 and derive the estimates (4.15), (4.17), (4.24) and (4.29) on Schwarzschild backgrounds, thus finishing the proof of Theorem 2.4.2 on Schwarzschild for $n = 0$ from the discussions in Chapter 4. The $n \geq 1$ cases follow from Section 4.1.

5.1. Coupled system on Schwarzschild

In Schwarzschild spacetime, the governing equations in the systems (2.29) and (2.30) for ϕ_s^i with $s = \pm 1$ are

$$\mathbf{L}_s^1 \phi_s^0 = F_s^0 = \tfrac{2s(r-3M)}{r^2}\phi_s^1, \tag{5.1a}$$
$$\mathbf{L}_s^1 \phi_s^1 = F_s^1 = 0, \tag{5.1b}$$

while for $s = \pm 2$, the subequations in systems (2.31) and (2.32) can be written in a unified form:

$$\mathbf{L}_s^0 \phi_s^0 = F_s^0 = \tfrac{4(r-3M)}{r^2}\phi_s^1, \tag{5.2a}$$
$$\mathbf{L}_s^1 \phi_s^1 = F_s^1 = \tfrac{2(r-3M)}{r^2}\phi_s^2 + 6M\phi_s^0, \tag{5.2b}$$

$$\mathbf{L}_s^1\phi_s^2 = F_s^2 = 0, \tag{5.2c}$$

with the operators simplified to

$$\mathbf{L}_s^0 = \Sigma\square_g + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\left(\cot^2\theta + \tfrac{r+2M}{2r}\right), \tag{5.3a}$$
$$\mathbf{L}_s^1 = \Sigma\square_g + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\left(\cot^2\theta + \tfrac{r-2M}{r}\right). \tag{5.3b}$$

5.2. Decomposition

The equations (5.1), (5.2b) and (5.2c) are all in the form of an ISWWE

$$\mathbf{L}_s^1\varphi^{(1)} = \Sigma\square_g\varphi^{(1)} + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi\varphi^{(1)} - s^2\left(\cot^2\theta + \tfrac{r-2M}{r}\right)\varphi^{(1)} = G^{(1)}. \tag{5.4}$$

Decompose the solution $\varphi^{(1)}$ and the inhomogeneous term $G^{(1)}$ into

$$\varphi^{(1)} = \sum_{m,\ell}\varphi_{m\ell}^{(1)}(t,r)Y_{m\ell}^s(\cos\theta)e^{im\phi}, m\in\mathbb{Z}, \tag{5.5}$$

$$G^{(1)} = \sum_{m,\ell}G_{m\ell}^{(1)}(t,r)Y_{m\ell}^s(\cos\theta)e^{im\phi}, m\in\mathbb{Z}. \tag{5.6}$$

Here, for each m, $\{Y_{m\ell}^s(\cos\theta)\}_\ell$ with $\min\{\ell\} = \max(|m|,|s|) \geq |s|$ are the eigenfunctions of the self-adjoint operator

$$\mathbf{S}_m = \tfrac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta - \tfrac{m^2+2ms\cos\theta+s^2}{\sin^2\theta} \tag{5.7}$$

on $L^2(\sin\theta d\theta)$. These eigenfunctions, called as "*spin-weighted spherical harmonics*", form a complete orthonormal basis on $L^2(\sin\theta d\theta)$ and have eigenvalues $-\Lambda_{m\ell} = -\ell(\ell+1)$ defined by

$$\mathbf{S}_m Y_{m\ell}^s(\cos\theta) = -\Lambda_{m\ell}Y_{m\ell}^s(\cos\theta). \tag{5.8}$$

An integration by parts, together with a usage of Plancherel lemma and the orthonormality property of the basis $\left\{Y_{m\ell}^s(\cos\theta)e^{im\phi}\right\}_{m\ell}$, gives

$$\sum_{m,\ell}\ell(\ell+1)\left|\varphi_{m\ell}^{(1)}(t,r)\right|^2 = \int_0^\pi\int_0^{2\pi}\left|\nabla\!\!\!/\varphi^{(1)}(t,r)\right|^2 r^2\sin\theta d\phi d\theta. \tag{5.9}$$

The equation for $\varphi_{m\ell}^{(1)}$ is now

$$r^4\Delta^{-1}\partial_{tt}^2\varphi_{m\ell}^{(1)} - \partial_r(\Delta\partial_r)\varphi_{m\ell}^{(1)} + \ell(\ell+1)\varphi_{m\ell}^{(1)} - 2s^2M/r\varphi_{m\ell}^{(1)} + G_{m\ell}^{(1)} = 0. \tag{5.10}$$

In the case that the inhomogeneous term $G^{(1)} = 0$, this is exactly the equation one obtains after decomposing into spherical harmonics the solution to the classical Regge-Wheeler equation Regge and Wheeler (1957) or Fackerell-Ipser equation Fackerell and Ipser (1972) on Schwarzschild.

The equation (5.2a), while, is in a form of an ISWWE with another potential:

$$\mathbf{L}_s^0 \varphi^{(0)} = \Sigma \Box_g \varphi^{(0)} + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi \varphi^{(0)} - 4\left(\cot^2\theta + \tfrac{r+2M}{2r}\right)\varphi^{(0)} = G^{(0)}. \tag{5.11}$$

After the decomposition into spin-weighted spherical harmonics as above, the equation for $\varphi^{(0)}_{m\ell}$ reads

$$r^4\Delta^{-1}\partial^2_{tt}\varphi^{(0)}_{m\ell} - \partial_r(\Delta\partial_r)\varphi^{(0)}_{m\ell} + \ell(\ell+1)\varphi^{(0)}_{m\ell} - (2-4M/r)\varphi^{(0)}_{m\ell} + G^{(0)}_{m\ell} = 0, \tag{5.12}$$

with $\min\{\ell\} = \max(|m|,|s|) \geq 2$. The identity (5.9) holds for $\varphi^{(0)}$ as well.

We now consider the general form of the equations (5.10) and (5.12):

$$r^4\Delta^{-1}\partial^2_{tt}\varphi - \partial_r(\Delta\partial_r)\varphi + \ell(\ell+1)\varphi + V(r)\varphi + G = 0, \tag{5.13}$$

with the potential

$$V(r) = \begin{cases} -2s^2M/r & \text{for (5.10)}, \\ -2+4M/r & \text{for (5.12)}. \end{cases} \tag{5.14}$$

5.3. Energy estimate

Multiplying (5.13) by $T\overline{\varphi} = \partial_t\overline{\varphi}$ and taking the real part, we arrive at an identity:

$$\begin{aligned} &\tfrac{1}{2}\partial_t\left(\tfrac{r^4}{\Delta}|\partial_t\varphi|^2 + \Delta|\partial_r\varphi|^2 + \ell(\ell+1)|\varphi|^2 + V|\varphi|^2\right) - \partial_r\left(\Re(\Delta\partial_r\varphi\partial_t\overline{\varphi})\right) \\ = & -\Re(G\partial_t\overline{\varphi}). \end{aligned} \tag{5.15}$$

Since $\ell \geq |s|$, the inequality

$$\ell(\ell+1) + V(r) \geq \tfrac{1}{3}\ell(\ell+1) \tag{5.16}$$

holds for both potentials in (5.14). Summing over m and ℓ, applying the identity (5.9) for $\varphi^{(1)}$ and $\varphi^{(0)}$, and finally integrating with respect to the measure dt^*dr over $\{(t^*,r)|0 \leq t^* \leq \tau, 2M \leq r < \infty\}$, we have the following energy estimate for ψ^i $(i = 0,1)$:

$$E^T_\tau(\varphi^{(i)}) \leq C\left(E^T_0(\varphi^{(i)}) + \int_{\mathcal{D}(0,\tau)} \tfrac{1}{r^2}\left|\Re(G^{(i)}\partial_t\overline{\varphi^{(i)}})\right|\right). \tag{5.17}$$

In global Kerr coordinates, for any $\tau \geq 0$,

$$E^T_\tau(\varphi^{(i)}) \sim \int_{\Sigma_\tau}\left(|\partial_{t^*}\varphi^{(i)}|^2 + |\nabla\!\!\!/\varphi^{(i)}|^2 + \tfrac{\Delta}{r^2}|\partial_r\varphi^{(i)}|^2\right). \tag{5.18}$$

5.4. Morawetz estimate

In this section, taking the choices of the multipliers in Andersson and Blue (2015b), Andersson et al. (2017a), we prove the Morawetz estimate for the separated equations (5.10) and (5.12), which are both in the form of (5.13) with potential as in (5.14), and then derive the Morawetz estimate for (5.4) and (5.11).

We multiply (5.13) by

$$X(\bar{\varphi}) = \hat{f}\partial_r\bar{\varphi} + \hat{q}\bar{\varphi} = \tfrac{2(r-2M)(r-3M)}{r^2}\partial_r\bar{\varphi} + \tfrac{(2r-3M)\Delta}{r^4}\bar{\varphi}, \tag{5.19}$$

take the real part and arrive at

$$\begin{aligned}&\partial_t\left(\Re\left(\tfrac{r^4}{\Delta}X(\varphi)\partial_t\bar{\varphi}\right)\right) + \tfrac{1}{2}\partial_r\left(\hat{f}\left[\ell(\ell+1)|\varphi|^2 - \tfrac{r^4}{\Delta}|\partial_t\varphi|^2 - \Delta|\partial_r\varphi|^2 + V|\varphi|^2\right]\right)\\ &\quad + \tfrac{1}{2}\partial_r\left(\Re(\partial_r(\Delta\hat{q})|\varphi|^2 - 2\Delta\hat{q}\bar{\varphi}\partial_r\varphi - 2\hat{q}(r-M)|\varphi|^2 - r^{-1}B^r(r)|\varphi|^2)\right) + B(\varphi)\\ &= -\Re(X(\varphi)\overline{G}).\end{aligned} \tag{5.20}$$

Here, the bulk term

$$B(\varphi) = B^t(r)|\partial_t\varphi|^2 + r^{-2}B^r(r)|\partial_r(r\varphi)|^2 + B^\ell(r)\left(\ell(\ell+1)|\varphi|^2\right) + B^0(r)|\varphi|^2, \tag{5.21}$$

with

$$\begin{aligned}B^t(r) &= \frac{1}{2}\partial_r\left(\frac{r^4}{\Delta}\hat{f}\right) - \hat{q}\frac{r^4}{\Delta}\\ B^r(r) &= \frac{1}{2}\partial_r\left(\Delta\hat{f}\right) - 2\hat{f}(r-M) + \Delta\hat{q}\\ B^\ell(r) &= -\frac{1}{2}\partial_r(\hat{f}) + \hat{q}\\ B^0(r) &= \partial_r\left(\hat{q}(r-M)\right) - \frac{1}{2}\partial^2_{rr}(\Delta\hat{q}) + V\hat{q} - \frac{1}{2}\partial_r\left(V\hat{f}\right)\\ &\quad + r^2(\partial_r(r^{-3}B^r(r)) + r^{-4}B^r(r)).\end{aligned}$$

With the choices of $\hat{f}$ and $\hat{q}$ as in (5.19),

$$B^t(r) = 0, \quad B^r(r) = \tfrac{6M\Delta^2}{r^4}, \quad B^\ell(r) = \tfrac{2(r-3M)^2}{r^3}, \tag{5.22}$$

and

$$B^0(r) = \begin{cases} -27Mr^{-2} + 162M^2r^{-3} - 234M^3r^{-4} & \text{for (5.10) and } |s|=2,\\ -9Mr^{-2} + 60M^2r^{-3} - 90M^3r^{-4} & \text{for (5.10) and } |s|=1,\\ -4r^{-1} + 33Mr^{-2} - 78M^2r^{-3} + 54M^3r^{-4} & \text{for (5.12).}\end{cases} \tag{5.23}$$

We first treat (5.10) by calculating $\underline{V}^1_{|s|}(r) = B^0(r) + |s|(|s|+1)B^\ell(r)$ that

$$\underline{V}^1_{|s|=2}(r) = 3r^{-4}(4r^3 - 33Mr^2 + 90M^2r - 78M^3), \tag{5.24a}$$
$$\underline{V}^1_{|s|=1}(r) = r^{-4}(4r^3 - 33Mr^2 + 96M^2r - 90M^3). \tag{5.24b}$$

Clearly, it holds that

$$B(\varphi) \geq r^{-2}B^r(r)|\partial_r(r\varphi)|^2 + \underline{V}^1_{|s|}(r)|\varphi|^2 + B^\ell(r)\left(\ell(\ell+1) - |s|(|s|+1)\right)|\varphi|^2. \tag{5.25}$$

Using Mathematica to calculate the roots of the the third order polynomial $\frac{1}{3}r^4\underline{V}^1_{|s|}(r)$, we find there exists only one real root for this polynomial in both $|s| = 1$ and $|s| = 2$, and the roots in both cases are less that $2M$. Hence, in both $|s| = 1$ and $|s| = 2$ cases, there exists a universal constant $c > 0$ such that for any $r \geq 2M$,

$$B(\varphi) \geq c\left(\tfrac{\Delta^2}{r^4}|\partial_r\varphi|^2 + \tfrac{1}{r}|\varphi|^2 + \tfrac{(r-3M)^2}{r^3}\ell(\ell+1)|\varphi|^2\right). \tag{5.26}$$

Instead, if multiplying (5.13) by $h\bar{\varphi}$ with

$$h = -\tfrac{\Delta(r-3M)^2}{r^7}, \tag{5.27}$$

and take the real part, we arrive at

$$\begin{aligned}
&\tfrac{1}{2}\partial_r\left(\Re(\partial_r(\Delta h)|\varphi|^2 - 2\Delta h\bar{\varphi}\partial_r\varphi - 2h(r-M)|\varphi|^2)\right) + h\left(\ell(\ell+1)|\varphi|^2\right)\\
&+ \partial_t\left(\Re\left(\tfrac{r^4}{\Delta}h\varphi\partial_t\bar{\varphi}\right)\right) - h\tfrac{r^4}{2\Delta}|\partial_t\varphi|^2 + \Delta h|\partial_r\varphi|^2\\
&+ \left(\partial_r\left(h(r-M)\right) - \tfrac{1}{2}\partial^2_{rr}(\Delta h) + hV\right)|\varphi|^2\\
=&-\Re(h\varphi\overline{G}).
\end{aligned} \tag{5.28}$$

After integration, this allows us to control the bulk integral of $|\partial_t\varphi|^2$ part by the bulk integral of the RHS of (5.26). We sum over m and ℓ for (5.20) and (5.28) with $\varphi = \varphi^{(1)}$ and $G = G^{(1)}$, apply the identity (5.9), integrate with respect to the measure dt^*dr over $\{(t^*, r)|0 \leq t^* \leq \tau, 2M \leq r < \infty\}$ and take (5.26) into account, then we obtain a Morawetz estimate for (5.4) in global Kerr coordinates:

$$\begin{aligned}
&\int_{\mathcal{D}(0,\tau)}\left(\tfrac{\Delta^2}{r^6}|\partial_r\varphi^{(1)}|^2 + \tfrac{1}{r^4}|\varphi^{(1)}|^2 + \tfrac{(r-3M)^2}{r^2}\left(\tfrac{1}{r^3}|\partial_{t^*}\varphi^{(1)}|^2 + \tfrac{1}{r}|\nabla\!\!\!/\varphi^{(1)}|^2\right)\right)\\
\lesssim\ & E^T_\tau(\varphi^{(1)}) + E^T_0(\varphi^{(1)}) + \int_{\mathcal{D}(0,\tau)}\tfrac{1}{r^2}\left(\left|\Re\left(X(\varphi^{(1)})\overline{G^{(1)}}\right)\right| + \left|\Re\left(h\varphi^{(1)}\overline{G^{(1)}}\right)\right|\right).
\end{aligned} \tag{5.29}$$

Turning now to (5.12), similarly as above, we calculate

$$\begin{aligned}\underline{V}^0(r) &= B^0(r) + 6B^\ell(r)\\ &= r^{-4}(8r^3 - 39Mr^2 + 30M^2r + 54M^3).\end{aligned} \tag{5.30}$$

One can check the roots of this third order polynomial $r^4\underline{V}^0(r)$ by Mathematica and find the only one real root is negative, hence $r^4\underline{V}^0(r)$ is positive for $r \geq 2M$, which yields

$$\begin{aligned}B(\varphi) &\geq r^{-2}B^r(r)|\partial_r(r\varphi)|^2 + \underline{V}^0(r)|\varphi|^2 + B^\ell(r)\left(\ell(\ell+1) - 6\right)|\varphi|^2\\ &\geq c\left(\tfrac{\Delta^2}{r^4}|\partial_r\varphi|^2 + \tfrac{1}{r}|\varphi|^2 + \tfrac{(r-3M)^2}{r^3}\ell(\ell+1)|\varphi|^2\right).\end{aligned} \tag{5.31}$$

Following the argument above for (5.10), it is straightforward to obtain the following Morawetz estimate for equation (5.11) in global Kerr coordinates:

$$\begin{aligned}&\int_{\mathcal{D}(0,\tau)}\left(\tfrac{\Delta^2}{r^6}|\partial_r\varphi^{(0)}|^2 + \tfrac{1}{r^4}|\varphi^{(0)}|^2 + \tfrac{(r-3M)^2}{r^2}\left(\tfrac{1}{r^3}|\partial_{t^*}\varphi^{(0)}|^2 + \tfrac{1}{r}|\nabla\!\!\!/\varphi^{(0)}|^2\right)\right)\\ &\lesssim E^T_\tau(\varphi^{(0)}) + E^T_0(\varphi^{(0)}) + \int_{\mathcal{D}(0,\tau)}\tfrac{1}{r^2}\left(\left|\Re\left(X(\varphi^{(0)})\overline{G^{(0)}}\right)\right| + \left|\Re\left(h\varphi^{(0)}\overline{G^{(0)}}\right)\right|\right).\end{aligned} \tag{5.32}$$

5.5. Proof of Theorems 2.4.1 and 2.4.2 on Schwarzschild

5.5.1. Spin ± 1 components

From the red-shift estimate Proposition 3.2.3, the Morawetz estimates in the large radius region (3.13) and the estimate (5.29) applied to each individual equation in the system (5.1), Theorem 2.4.1 for $s = \pm 1$ is proved. The inequalities (4.15) and (4.17) are obviously valid from the Proposition 3.1.1 and Theorem 2.4.1 for spin ± 1 components.

5.5.2. Spin ± 2 components

We prove the Theorem 2.4.1, as well as the estimates (4.15), (4.17), (4.24) and (4.29) for spin ± 2 components separately.

5.5.2.1. Spin $+2$ component

Applying the Morawetz estimates (5.29) to (5.2b) and (5.2c), and (5.32) to (5.2a), then together with the Morawetz estimates in large r region for $r^{4-\delta}\phi^0$ and $r^{2-\delta}\phi^1$ in Proposition 3.1.2 and red-shift estimates near horizon in Section 3.2, it holds for ϕ^i $(i = 0, 1, 2)$ that

$$\begin{aligned}&E_\tau(r^{4-\delta}\phi^0) + E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0) + \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}_{\text{deg}}(r^{4-\delta}\phi^0)\\ \lesssim\ & E_0(r^{4-\delta}\phi^0) + \mathcal{E}_{\text{schw}}(\phi^0_{+2}),\end{aligned} \tag{5.33}$$

$$\begin{aligned}&E_\tau(r^{2-\delta}\phi^1) + E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^1) + \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}_{\text{deg}}(r^{2-\delta}\phi^1)\\ \lesssim\ & E_0(r^{2-\delta}\phi^1) + \mathcal{E}_{\text{schw}}(\phi^1_{+2}) + \int_{\mathcal{D}(0,\tau)\cap[R-1,\infty)} \frac{\left|\partial\left(r^{4-\delta}\phi^0_{+2}\right)\right|^2}{r^2},\end{aligned} \tag{5.34}$$

$$E_\tau(\phi^2) + E_{\mathcal{H}^+(0,\tau)}(\phi^2) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\phi^2) \lesssim E_0(\phi^2). \tag{5.35}$$

The error term $\mathcal{E}_{\text{schw}}(\phi^0_{+2})$ is bounded by

$$\begin{aligned}&\int_{\mathcal{D}(0,\tau)} \tfrac{1}{r^2}\left(\left|\Re\left(X(\phi^0)\overline{F^0_{+2}}\right)\right| + \left|\Re\left(h\phi^0\overline{F^0_{+2}}\right)\right|\right) + \int_{\mathcal{D}(0,\tau)} \tfrac{1}{r^2}|F^0_{+2}||\partial_t\phi^0|\\ &+ \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\geq R-1\}} \left(\left|\Re\left(F^0_{+2}X_w\overline{\phi^0}\right)\right| + \tfrac{|\phi^1|^2}{r^3}\right)\\ \lesssim\ & \epsilon_0 \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(r\phi^0) + \epsilon_0^{-1}\int_{\mathcal{D}(0,\tau)} r^{-3}|\phi^1|^2,\end{aligned} \tag{5.36}$$

and $\mathcal{E}_{\text{schw}}(\phi^1_{+2})$ is easily controlled from Cauchy-Schwarz inequality by

$$C\epsilon_1 \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(r\phi^1) + C\epsilon_1^{-1}\int_{\mathcal{D}(0,\tau)} \left(\tfrac{|\phi^2|^2}{r^3} + \tfrac{|r\phi^0|^2}{r^3}\right). \tag{5.37}$$

Hence, this completes the proof of Theorem 2.4.1 and (4.24).

5.5.2.2. Spin -2 component

The Morawetz estimate (5.29) applied to (5.2b) and (5.2c), estimate (5.32) applied to (5.2a), the Morawetz estimates in large r region for $\{\phi^i_{-2}\}|_{i=0,1,2}$ in Proposition 3.1.1 and red-shift estimates near horizon in Section 3.2 together imply

$$E_\tau(\widetilde{\phi}^0) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi}^0) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\widetilde{\phi}^0) \lesssim E_0(\widetilde{\phi}^0) + \mathcal{E}_{\text{schw}}(\widetilde{\phi}^0), \tag{5.38}$$

$$E_\tau(\widetilde{\phi^1}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^1}) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\widetilde{\phi^1}) \lesssim E_0(\widetilde{\phi^1}) + \mathcal{E}_{\text{schw}}(\widetilde{\phi^1}), \tag{5.39}$$

$$E_\tau(\phi^2) + E_{\mathcal{H}^+(0,\tau)}(\phi^2) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\deg}(\phi^2) \lesssim E_0(\phi^2). \tag{5.40}$$

Easy to see the Theorem 2.4.1 and the estimates (4.29) hold from the inequality that

$$\begin{aligned}
\mathcal{E}_{\text{schw}}(\widetilde{\phi^0}) \lesssim& \int_{\mathcal{D}(0,\tau)} \tfrac{1}{r^2}\left(\left|\Re\left(X(\phi^0)\overline{F^0_{-2}}\right)\right| + \left|\Re\left(h\phi^0\overline{F^0_{-2}}\right)\right| + |F^0_{-2}||\partial_t\phi^0|\right) \\
&+ \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\geq R-1\}} |F^0_{-2}||X_w\overline{\phi^0}| + \int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]} |\widetilde{\phi^1}|^2 \\
\lesssim& \,\epsilon_0 \int_{\mathcal{D}(0,\tau)} \mathbb{M}(\widetilde{\phi^0}) + \epsilon_0^{-1}\int_{\mathcal{D}(0,\tau)} r^{-3}|\widetilde{\phi^1}|^2
\end{aligned} \tag{5.41}$$

and the following estimate obtained analogously

$$\mathcal{E}_{\text{schw}}(\widetilde{\phi^1}) \lesssim \epsilon_1 \int_{\mathcal{D}(0,\tau)} \mathbb{M}(\widetilde{\phi^1}) + \epsilon_1^{-1}\int_{\mathcal{D}(0,\tau)} \left(\tfrac{|\phi^2|^2}{r^3} + \tfrac{|\widetilde{\phi^0}|^2}{r^2}\right). \tag{5.42}$$

6. Proof of Theorem 2.4.1 on slowly rotating Kerr

Contents

6.1. Energy estimate

We start by choosing a multiplier $-2\Sigma^{-1}\partial_t\bar{\psi}$ for (2.34b), which gives an identity for any $\tau_2 > \tau_1 \geq 0$ that

$$\int_{\Sigma_{\tau_2}} e^1_{\tau_2}(\psi) = \int_{\Sigma_{\tau_1}} e^1_{\tau_1}(\psi) - \int_{\mathcal{D}(\tau_1,\tau_2)} \Re\left(\tfrac{2F}{\Sigma}\partial_t\bar{\psi}\right). \tag{6.1}$$

Here, the energy density in $r \geq r_0$ equals to

$$e^1_\tau(\psi) = \tfrac{1}{\Sigma}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\right|^2 - \tfrac{a^2}{\Delta}|\partial_\phi\psi|^2 + \tfrac{s^2(\Delta+a^2)}{r^2}|\psi|^2\right)$$

$$+\tfrac{(r^2+a^2)^2-a^2\sin^2\theta\Delta}{\Delta\Sigma}|\partial_t\psi|^2+\tfrac{(r^2+a^2)^2}{\Delta\Sigma}|\partial_{r^*}\psi|^2. \tag{6.2}$$

From (5.9), we have for $r\geq r_0$ that

$$\begin{aligned}&\int_{\mathbb{S}^2}\left(|\partial_\theta\psi|^2+\left|\tfrac{\partial_\phi\psi+is\cos\theta\psi}{\sin\theta}\right|^2+s^2|\psi|^2\right)d\sigma_{\mathbb{S}^2}\\ \geq&\int_0^\pi\sum_{m\in\mathbb{Z}}\left(\max\{s^2+|s|,m^2+|m|\}|\psi_m|^2\right)\sin\theta d\theta,\end{aligned} \tag{6.3}$$

with

$$\psi_m(t,r,\theta)=\tfrac{1}{\sqrt{2\pi}}\int_0^{2\pi}e^{-im\phi}\psi(t,r,\theta,\phi)d\phi. \tag{6.4}$$

It follows then that

$$\begin{aligned}&\int_{\mathbb{S}^2}\left(|\partial_\theta\psi|^2+\left|\tfrac{\partial_\phi\psi+is\cos\theta\psi}{\sin\theta}\right|^2-\tfrac{a^2}{\Delta}|\partial_\phi\psi|^2+\tfrac{s^2(\Delta+a^2)}{r^2}|\psi|^2\right)d\sigma_{\mathbb{S}^2}\\ \geq&\int_0^\pi\sum_{m\in\mathbb{Z}}\left(\max\{s^2+|s|,m^2+|m|\}-\tfrac{a^2m^2}{\Delta}+s^2\tfrac{\Delta+a^2-r^2}{r^2}\right)|\psi_m|^2\sin\theta d\theta.\end{aligned} \tag{6.5}$$

Denote

$$A^1_{m,s}=\max\{s^2+|s|,m^2+|m|\}-\tfrac{a^2m^2}{\Delta}+s^2\tfrac{\Delta+a^2-r^2}{r^2}. \tag{6.6}$$

In the case that $|s|=2$, if $|m|=0$ or 1, then clearly

$$A^1_{m,s}\geq 2-\tfrac{a^2m^2}{\Delta}+\tfrac{4(\Delta+a^2)}{r^2}, \tag{6.7}$$

which is nonnegative when $r\geq 2M$. If $|m|\geq 4$, then

$$A^1_{m,s}\geq m^2\left(1-\tfrac{a^2}{\Delta}\right)+\tfrac{4(\Delta+a^2)}{r^2}, \tag{6.8}$$

which is again nonnegative when $r\geq 2M$. For $|m|=2$ (or 3),

$$A^1_{m,s}\geq\tfrac{2\Delta-4a^2}{\Delta}+\tfrac{4(\Delta+a^2)}{r^2}\ \left(\text{or }\tfrac{8\Delta-9a^2}{\Delta}+\tfrac{4(\Delta+a^2)}{r^2}\right), \tag{6.9}$$

with the RHS being nonnegative when $r^2-2Mr-a^2\geq 0$, i.e. $r\geq M+\sqrt{M^2+a^2}$. While in the case that $|s|=1$, obviously we have

$$A^1_{m,s}\geq m^2-\tfrac{a^2m^2}{\Delta}+\tfrac{\Delta+a^2}{r^2}, \tag{6.10}$$

which is positive for $r>2M$.

One can similarly choose the multiplier $-2\Sigma^{-1}\partial_t\bar{\psi}$ for (2.34a) satisfied by ϕ_s^0 ($s = \pm 2$), and arrive at an energy identity for any $\tau_2 > \tau_1 \geq 0$:

$$\int_{\Sigma_{\tau_2}} e_{\tau_2}^0(\psi) = \int_{\Sigma_{\tau_1}} e_{\tau_1}^0(\psi) - \int_{\mathcal{D}(\tau_1,\tau_2)} \Re\left(\tfrac{2F}{\Sigma}\cdot\partial_t\bar{\psi}\right). \tag{6.11}$$

Here, the energy density in $r \geq r_0$ is

$$\begin{aligned} e_\tau^0(\psi) = {}& \tfrac{1}{\Sigma}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\right|^2 - \tfrac{a^2}{\Delta}|\partial_\phi\psi|^2 + \tfrac{s^2(r^2+2Mr-2a^2)}{2r^2}|\psi|^2\right) \\ & + \tfrac{(r^2+a^2)^2 - a^2\sin^2\theta\Delta}{\Delta\Sigma}|\partial_t\psi|^2 + \tfrac{(r^2+a^2)^2}{\Delta\Sigma}|\partial_{r^*}\psi|^2. \end{aligned} \tag{6.12}$$

It follows from (6.3) that for $r \geq r_0$,

$$\begin{aligned} & \int_{\mathbb{S}^2}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\right|^2 - \tfrac{a^2}{\Delta}|\partial_\phi\psi|^2 + \tfrac{s^2(r^2+2Mr-2a^2)}{2r^2}|\psi|^2\right)d\sigma_{S^2} \\ \geq {}& \int_0^\pi \sum_{m\in\mathbb{Z}}\left(\max\{s^2+|s|, m^2+|m|\} - \tfrac{a^2m^2}{\Delta} - s^2\tfrac{\Delta+a^2}{2r^2}\right)|\psi_m|^2\sin\theta d\theta. \end{aligned} \tag{6.13}$$

Denote

$$A_{m,s}^0 = \max\{|s|(|s|+1), |m|(|m|+1)\} - \tfrac{a^2m^2}{\Delta} - s^2\tfrac{\Delta+a^2}{2r^2}. \tag{6.14}$$

Note that $|s| = 2$ here. If $|m| = 0$ or 1,

$$A_{m,s}^0 \geq 2 - \tfrac{a^2m^2}{\Delta} + \tfrac{2(r^2+2Mr-2a^2)}{r^2}, \tag{6.15}$$

and when $|m| \geq 4$,

$$A_{m,s}^0 \geq m^2\left(1 - \tfrac{a^2}{\Delta}\right) + \tfrac{2(r^2+2Mr-2a^2)}{r^2}. \tag{6.16}$$

The RHS of these inequalities are clearly nonnegative when $r \geq 2M$. For the remaining case that $|m| = 2$ (or 3),

$$A_{m,s}^0 \geq \tfrac{2\Delta-4a^2}{\Delta} + \tfrac{2(r^2+2Mr-2a^2)}{r^2}\ \left(\text{or } \tfrac{8\Delta-9a^2}{\Delta} + \tfrac{2(r^2+2Mr-2a^2)}{r^2}\right), \tag{6.17}$$

which is nonnegative when $r^2 - 2Mr - a^2 \geq 0$, i.e., when $r \geq M + \sqrt{M^2+a^2}$.

Hence, we arrive at the conclusion that for $|a|/M$ sufficiently small and $r \geq r_0$, the energy densities $e_\tau^k(\psi)$ ($k = 0, 1$) above for both (2.34b) and (2.34a) are strictly positive and satisfy $e_\tau^k(\psi) \geq c|\partial\psi|^2$.

Since the energy densities $e_\tau^k(\psi)$ are both nonnegative in Schwarzschild case ($a = 0$), it holds true in $[r_+, r_0]$ that for sufficiently small $|a|/M \leq a_0/M \ll 1$ and any $\tau \geq 0$,

$$-e_\tau^k(\psi) \leq \tfrac{Ca^2}{M^2}|\partial\psi|^2. \tag{6.18}$$

Therefore, the above discussions imply the following energy estimate for both (2.34a) and (2.34b):

$$\int_{\Sigma_{\tau_2}\cap[r_0,\infty)} |\partial\psi|^2 \lesssim \int_{\Sigma_{\tau_1}} e^k_{\tau_1}(\psi) + \tfrac{a^2}{M^2}\int_{\Sigma_{\tau_2}\cap[r_+,r_0]} |\partial\psi|^2 + \left|\int_{\mathcal{D}(\tau_1,\tau_2)} \Re\left(\tfrac{F}{\Sigma}T\bar{\psi}\right)\right|. \tag{6.19}$$

From now on, we will suppress the superscript k in the energy density and simply write it as $e_{\tau_1}(\psi)$.

Clearly, there exists an $\varepsilon_0 = \varepsilon_0(M) \geq 0$ and a nonnegative differential function $e_0(\varepsilon_0)$ with $e_0(0) = 0$ such that for all $|a|/M \leq \varepsilon_0$ and any $\tilde{e} > e_0$, by adding to this energy estimate $\tilde{e}$ times the redshift estimate in Proposition 3.2.3 for $\psi \in \{\phi^0_{+2}, \phi^1_{+2}, \phi^2_{+2}, \phi^2_{-2}\}$ and in Proposition 3.2.4 for $\widetilde{\phi^0}$ and $\widetilde{\phi^1}$, we obtain the following result analogous to (Dafermos and Rodnianski, 2010, Prop.5.3.1) for sufficiently small $|a|/M \leq a_0/M$.

Proposition 6.1.1. For $\psi = \phi^i_s$ $(i = 0, 1, \cdots, |s|)$, and $F = F^i_s$ in (2.29) and (2.30) for $s = \pm 1$ and (2.31) and (2.32) for $s = \pm 2$ with the same superscript and subscript as $\psi = \phi^i_s$, define

$$\tilde{\psi} = \begin{cases} \widetilde{\phi^j_{-2}}, & \text{if } \psi = \phi^j_{-2}\ (j = 0, 1); \\ \psi, & \text{if } \psi = \phi^0_{+2}, \phi^1_{+2}, \phi^2_{+2} \text{ or } \phi^2_{-2}. \\ \psi, & \text{if } \psi = \phi^i_s \text{ with } |s| = 1. \end{cases} \tag{6.20}$$

It then holds that

$$\begin{aligned} &\int_{\Sigma_{\tau_2}} |e_{\tau_2}(\tilde{\psi})| + \tilde{e}E_{\tau_2}(\tilde{\psi}) \\ \lesssim &\int_{\Sigma_{\tau_1}} |e_{\tau_1}(\tilde{\psi})| + \tilde{e}E_{\tau_1}(\tilde{\psi}) + \tilde{e}\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}} |\partial\tilde{\psi}|^2 \\ &+ \left(\tilde{e}\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]} \mathcal{B}(\tilde{\psi}, F) + \left|\int_{\mathcal{D}(\tau_1,\tau_2)} \Re\left(\tfrac{F}{\Sigma}T\bar{\psi}\right)\right|\right). \end{aligned} \tag{6.21}$$

Here,

$$\mathcal{B}(\tilde{\psi}, F) = \begin{cases} |F|^2, & \text{for } \tilde{\psi} = \phi^1_{+2}, \phi^2_{+2} \text{ or } \phi^2_{-2}; \\ |\phi^1_{+2}|^2, & \text{for } \tilde{\psi} = \phi^0_{+2}; \\ |\widetilde{\phi^1}|^2, & \text{for } \tilde{\psi} = \widetilde{\phi^0_{-2}}; \\ |\phi^2_{-2}|^2 + |\widetilde{\phi^0}|^2 + \frac{|a|}{M}|\partial\widetilde{\phi^0}|^2, & \text{for } \tilde{\psi} = \widetilde{\phi^1_{-2}}; \\ |F|^2, & \text{for } \tilde{\psi} = \phi^i_s \text{ with } |s| = 1. \end{cases} \tag{6.22}$$

We here state a finite in time energy estimate for the inhomogeneous SWFIE (2.34a) and (2.34b) based on the above discussions, which is an analogue of (Dafermos and Rodnianski, 2010, Prop.5.3.2).

Proposition 6.1.2. (Finite in time energy estimate) Given an arbitrary $\epsilon > 0$, there exists an $a_0 > 0$ depending on ϵ and a universal constant C such that for $|a| \leq a_0$, $1 \geq \tilde{e} \geq e_0(a)$ and for any $\tau_0 \geq 0$ and all $0 \leq \tau \leq \epsilon^{-1}$, the following results hold true: For $\psi = \phi_s^i$ $(i = 0, 1, \cdots, |s|)$, $\tilde{\psi}$ in (6.20) and the corresponding inhomogeneous function $F = F_s^i$ in (2.29) and (2.30) for $s = \pm 1$ and (2.31) and (2.32) for $s = \pm 2$, we have

$$\begin{aligned}
&\int_{\Sigma_{\tau_0+\tau}} |e_{\tau_0+\tau}(\tilde{\psi})| + \tilde{e} E_{\tau_0+\tau}(\tilde{\psi}) \\
&\lesssim (1 + C\tilde{e}) \left(\int_{\Sigma_{\tau_0}} |e_{\tau_0}(\tilde{\psi})| + \tilde{e} E^{\text{total}}_{\tau_0+\tau}(s) \right) \\
&\quad + C \left(\tilde{e} \int_{\mathcal{D}(\tau_0,\tau_0+\tau)\cap[r_+,r_1]} \mathcal{B}(\tilde{\psi}, F) + \left| \int_{\mathcal{D}(\tau_0,\tau_0+\tau)} \Re\left(\tfrac{F}{\Sigma} \cdot T\bar{\tilde{\psi}}\right) \right| \right), \qquad (6.23)
\end{aligned}$$

and, depending on the spin weight s,

$$\int_{\mathcal{D}(\tau_0,\tau_0+\tau)\cap[r_0,r_1]} |\partial\tilde{\psi}|^2 \leq C E^{\text{total}}_{\tau}(s). \qquad (6.24)$$

Here, $\mathcal{B}(\tilde{\psi}, F)$ is already defined in (6.22) and, for any $\tau \geq 0$,

$$E^{\text{total}}_{\tau}(s) = \begin{cases} E_\tau(r^{4-\delta}\phi^0_{+2}) + E_\tau(r^{2-\delta}\phi^1_{+2}) + E_\tau(\phi^2_{+2}), & \text{for } s = +2; \\ E_\tau(\widetilde{\phi^0_{-2}}) + E_\tau(\widetilde{\phi^0_{-2}}) + E_\tau(\phi^2_{-2}), & \text{for } s = -2; \\ E_\tau(\phi^0_{+1}) + E_\tau(\phi^1_{+1}), & \text{for } s = +1; \\ E_\tau(\phi^0_{-1}) + E_\tau(\phi^1_{-1}), & \text{for } s = -1. \end{cases} \qquad (6.25)$$

Proof. The first estimate follows easily from the previous proposition together with the second estimate, while the second estimate follows from the fact that it holds for Schwarzschild case for all ϵ from the discussions in Chapters 5 and 4 and the well-posedness property in Section 2.6.1 applied to the linear wave systems of $\{\phi^i_{+1}\}_{i=0,1}$, $\{\phi^i_{-1}\}_{i=0,1}$, $\{\phi^i_{+2}\}_{i=0,1,2}$ and $\{\widetilde{\phi^0_{-2}}, \widetilde{\phi^1_{-2}}, \phi^2_{-2}\}$. □

6.2. Separated angular and radial equations

In the exterior of a subextremal Kerr black hole, if the solution ψ to the equation (2.34b) is *integrable*, it then holds in $L^2(dt)$ that

$$\psi = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \psi_\omega(r,\theta,\phi) d\omega, \tag{6.26}$$

where ψ_ω is defined as the Fourier transform of ψ:

$$\psi_\omega = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \psi(t,r,\theta,\phi) dt. \tag{6.27}$$

We further decompose ψ_ω in $L^2(\sin\theta d\theta d\phi)$ into

$$\psi_\omega = \sum_{m,\ell} \psi_{m\ell}^{(a\omega)}(r) Y_{m\ell}^s(a\omega, \cos\theta) e^{im\phi}, \quad m \in \mathbb{Z}. \tag{6.28}$$

Here, for each m, $\{Y_{m\ell}^s(a\omega,\cos\theta)\}_\ell$, with $\min\{\ell\} = \max\{|m|,|s|\}$, are the eigenfunctions of the self-adjoint operator

$$\mathbf{S}_m = \tfrac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta - \tfrac{m^2+2ms\cos\theta+s^2}{\sin^2\theta} + a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta \tag{6.29}$$

on $L^2(\sin\theta d\theta)$. These eigenfunctions, called as "*spin-weighted spheroidal harmonics*", form a complete orthonormal basis on $L^2(\sin\theta d\theta)$, and have eigenvalues $-\Lambda_{m\ell,s}^{(a\omega)}$ defined by

$$\mathbf{S}_m Y_{m\ell}^s(a\omega,\cos\theta) = -\Lambda_{m\ell,s}^{(a\omega)} Y_{m\ell}^s(a\omega,\cos\theta). \tag{6.30}$$

One could similarly define F_ω and $F_{m\ell}^{(a\omega)}$.

An integration by parts, together with a usage of Plancherel lemma and the orthonormality property of the basis $\{Y_{m\ell}^s(a\omega,\cos\theta)e^{im\phi}\}_{m\ell}$, gives

$$\begin{aligned}&\int_{-\infty}^{+\infty} \sum_{m,\ell} \Lambda_{m\ell,s}^{(a\omega)} |\psi_{m\ell}^{(a\omega)}|^2 d\omega \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{S}^2} d\sigma_{\mathbb{S}^2} dt \Big\{ |\partial_\theta\psi|^2 + \Big|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\Big|^2 - |a\cos\theta\partial_t\psi + is\psi|^2 + 2s^2|\psi|^2\Big\}.\end{aligned} \tag{6.31a}$$

The radial equation for $\psi_{m\ell}^{(a\omega)}$ is then

$$\Big\{\partial_r(\Delta\partial_r) + (V_1)_{m\ell,s}^{(a\omega)}(r)\Big\}\psi_{m\ell}^{(a\omega)} = F_{m\ell}^{(a\omega)}, \tag{6.32}$$

with the potential

$$(V_1)^{(a\omega)}_{m\ell,s}(r) = \frac{(r^2+a^2)^2\omega^2+a^2m^2-4aMrm\omega}{\Delta} - \left(\lambda^{(a\omega)}_{m\ell,s,1}(r) + a^2\omega^2\right). \tag{6.33}$$

We utilized here a substitution of

$$\lambda^{(a\omega)}_{m\ell,s,1}(r) = \Lambda^{(a\omega)}_{m\ell,s} - \frac{s^2(2Mr-2a^2)}{r^2}, \tag{6.34}$$

by which the above radial equation (6.32) is the same as the radial equation (Dafermos and Rodnianski, 2010, Eq.(33))[1] for the scalar field.

One could obtain for (2.34a) the same angular equation and the following radial equation for $\psi^{(a\omega)}_{m\ell}$ after decomposition:

$$\left\{\partial_r(\Delta\partial_r) + (V_0)^{(a\omega)}_{m\ell,s}(r)\right\}\psi^{(a\omega)}_{m\ell} = F^{(a\omega)}_{m\ell}, \tag{6.35}$$

with the potential

$$(V_0)^{(a\omega)}_{m\ell,s}(r) = \frac{(r^2+a^2)^2\omega^2+a^2m^2-4aMrm\omega}{\Delta} - \left(\lambda^{(a\omega)}_{m\ell,s,0}(r) + a^2\omega^2\right), \tag{6.36}$$

and a substitution of

$$\lambda^{(a\omega)}_{m\ell,s,0}(r) = \Lambda^{(a\omega)}_{m\ell,s} - \frac{s^2(\Delta+a^2)}{2r^2}. \tag{6.37}$$

We state here some basic identities for any $r > r_+$ from properties of Fourier transform and Plancherel lemma:

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\pi} |\psi(t,r,\theta,\phi)|^2 \sin\theta d\theta d\phi dt = \int_{-\infty}^{\infty}\sum_{m,\ell}\left|\psi^{(a\omega)}_{m\ell}(r)\right|^2 d\omega,$$

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\pi} |\partial_r\psi(t,r,\theta,\phi)|^2 \sin\theta d\theta d\phi dt = \int_{-\infty}^{\infty}\sum_{m,\ell}\left|\partial_r\psi^{(a\omega)}_{m\ell}(r)\right|^2 d\omega,$$

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\pi} |\partial_t\psi(t,r,\theta,\phi)|^2 \sin\theta d\theta d\phi dt = \int_{-\infty}^{\infty}\sum_{m,\ell}\omega^2\left|\psi^{(a\omega)}_{m\ell}(r)\right|^2 d\omega.$$

[1] The authors in Dafermos and Rodnianski (2010) missed one term $-4aMrm\omega/\Delta$ in the Equation (33), but what is used thereafter is the Schrödinger equation (34) in Section 9 which is correct. Therefore, the validity of the proof will not be influenced by the missing term.

6.3. Proof of Theorem 2.4.1 for spin-1 case

6.3.1. Cutoff in time

To justify the separation procedures in Section 6.2, one would need the assumption that the solution $\psi(r)$ is *integrable*[2], which is *a priori* unknown. Therefore, we apply cutoff to the solution both to the future and to the past, and then do separation for the wave equation which the gained function satisfies.

Let $\chi_2(x)$ be a smooth cutoff function which equals to 0 for $x \leq 0$ and is identically 1 when $x \geq 1$. Choosing $\varepsilon > 0$ and a fixed $\tau' \geq 2\varepsilon^{-1}$, we define

$$\chi = \chi_{\tau',\varepsilon}(t^*) = \chi_2(\varepsilon t^*)\chi_2(\varepsilon(\tau' - t^*)) \tag{6.38}$$

and

$$\psi_\chi = \chi\psi \tag{6.39}$$

in coordinate system (t^*, r, θ, ϕ^*). The cutoff function ψ_χ is now a smooth function supported in $0 \leq t^* \leq \tau'$, and $\psi_\chi = \psi$ in $\varepsilon^{-1} \leq t^* \leq \tau' - \varepsilon^{-1}$. Moreover, it satisfies the following inhomogeneous equation

$$\begin{aligned}\mathbf{L}_s^k \psi_\chi &= F_\chi \\ &= \chi F + \Sigma\left(2\nabla^\mu\chi\nabla_\mu\psi + (\Box_g\chi)\,\psi\right) - 2isa\cos\theta\partial_t\chi\psi.\end{aligned} \tag{6.40}$$

$k = 0$ or 1 depends on the equation (2.34a) or (2.34b) we are treating. The fact that the aforedefined χ is ϕ^*-independent is utilized here.

Note the fact that the functions ψ_χ and F_χ are compactly supported in $0 \leq t^* \leq \tau'$ at each fixed $r > r_+$, and the assumption that ψ is a compactly supported smooth section solving one subequation of a linear wave system, hence ψ_χ is an integrable solution to (6.40) from Proposition 2.6.1. In the following discussions, we apply the mode decompositions in Section 6.2 to ψ_χ and F_χ, and separate the wave equation (6.40) into the angular equation (6.30) and radial equation (6.32) (or (6.35)), with the radial parts $R_{m\ell}^{(a\omega)} = (\psi_\chi)_{m\ell}^{(a\omega)}$ and $(F_\chi)_{m\ell}^{(a\omega)}$ of ψ_χ and F_χ in place of $\psi_{m\ell}^{(a\omega)}$ and $F_{m\ell}^{(a\omega)}$ respectively.

Before introducing the microlocal currents, we give some estimates for the inhomogeneous term F_χ here. Due to the fact that $\nabla\chi$ and $\Box_g\chi$ are supported in

$$\left\{0 \leq t^* \leq \varepsilon^{-1}\right\} \cup \left\{\tau' - \varepsilon^{-1} \leq t^* \leq \tau'\right\}, \tag{6.41}$$

[2] Recall it in Definition 2.6.1

it holds in the coordinate system (t^*, r, θ, ϕ^*) that

$$|\partial_{t^*}\chi| \leq C\varepsilon, \quad |\Box_g \chi| \leq C\varepsilon^2, \tag{6.42a}$$

$$|\nabla^\mu \chi \nabla_\mu \psi|^2 + \left|\tfrac{ias\cos\theta\partial_t\chi\psi}{\Sigma}\right|^2 \leq C\varepsilon^2 \left(|\partial\psi|^2 + \tfrac{a^2}{M^2}\left|\tfrac{\psi}{r}\right|^2\right). \tag{6.42b}$$

6.3.2. Currents in phase space

In what follows, we will suppress the dependence on a, ω, m, ℓ and s of the functions $R^{(a\omega)}_{m\ell}(r)$, $F^{(a\omega)}_{m\ell}(r)$, $\Lambda^{(a\omega)}_{m\ell,s}$, $\lambda^{(a\omega)}_{m\ell,s,k}(r)$, $(V_k)^{(a\omega)}_{m\ell,s}(r)$ and other functions defined by them, $k = 0, 1$. When there is no confusion, the dependence on r may always be implicit (except for the radial part $R(r)$ to avoid misunderstanding with the radius parameter R). Moreover, in the spin-1 case, k always takes the value 1. Thus we may drop the subscript k as well, and write simply as $R(r)$, F, Λ, λ and V.

We transform the radial equation (6.32) and (6.35) into a Schrödinger form, which will be of great use to define the microlocal currents below, by setting

$$u(r) = \sqrt{r^2 + a^2}R(r), \qquad H(r) = \tfrac{\Delta F_\chi(r)}{(r^2+a^2)^{3/2}}. \tag{6.43}$$

The Schrödinger equation for $u(r)$ reads after some calculations

$$u''(r) + \left(\omega^2 - V(r)\right) u(r) = H(r), \tag{6.44}$$

where

$$\begin{aligned} V =& \omega^2 - \tfrac{\Delta}{(r^2+a^2)^2} V + \tfrac{1}{(r^2+a^2)} \tfrac{d^2}{dr^{*2}} (r^2 + a^2)^{1/2} \\ =& \tfrac{4Mram\omega - a^2m^2 + \Delta\left(\lambda + a^2\omega^2\right)}{(r^2+a^2)^2} + \tfrac{\Delta}{(r^2+a^2)^4}\left(a^2\Delta + 2Mr(r^2 - a^2)\right), \end{aligned} \tag{6.45}$$

and a prime $'$ denotes a partial derivative with respect to r^* in tortoise coordinates.

Given any real, smooth functions y, h and f, define the microlocal currents

$$Q^y = y\left(|u'|^2 + \left(\omega^2 - V\right)|u|^2\right), \tag{6.46a}$$

$$Q^h = h\Re\left(u'\overline{u}\right) - \tfrac{1}{2}h'|u|^2, \tag{6.46b}$$

$$Q^f = Q^{h=f'} + Q^{y=f} = f'\Re\left(u'\overline{u}\right) - \left(\tfrac{1}{2}f'' - f\left(\omega^2 - V\right)\right)|u|^2 + f\left|u'\right|^2. \tag{6.46c}$$

The currents Q^y and Q^h are constructed via multiplying the equation (6.44) by $y\bar{u}'/2$ and $h\bar{u}$ respectively. We calculate the derivatives of the above currents as follows

$$(Q^y)' = y'\left(|u'|^2 + \left(\omega^2 - V\right)|u|^2\right) - yV'|u|^2 + 2y\Re\left(u'\overline{H}\right), \tag{6.47a}$$

$$\left(Q^h\right)' = h\left(|u'|^2 + \left(V - \omega^2\right)|u|^2\right) - \tfrac{1}{2}h''|u|^2 + h\Re\left(u\overline{H}\right), \tag{6.47b}$$

$$\left(Q^f\right)' = 2f'\left|u'\right|^2 - fV'|u|^2 + \Re\left(2f\overline{H}u' + f'\overline{H}u\right) - \tfrac{1}{2}f'''|u|^2. \tag{6.47c}$$

6.3.3. Frequency regimes

Let us start to define the separated frequency regimes, in which we will obtain a phase-space version of Morawetz estimate by choosing different functions y, h and f separately. Let ω_3, λ_3 be (potentially large) parameters and λ_2 be a (potentially small) parameter, all to be determined in the proof below. The frequency space is divided into

- $\mathcal{F}_T = \{(\omega, m, \ell) : |\omega| \geq \omega_3, \lambda < \lambda_2\omega^2\}$;
- $\mathcal{F}_{Tr} = \{(\omega, m, \ell) : |\omega| \geq \omega_3, \lambda \geq \lambda_2\omega^2\}$;
- $\mathcal{F}_A = \{(\omega, m, \ell) : |\omega| \leq \omega_3, \Lambda > \lambda_3\}$;
- $\mathcal{F}_B = \{(\omega, m, \ell) : |\omega| \leq \omega_3, \Lambda \leq \lambda_3\}$.

We fix an arbitrary $2M < r_c < r_0$, with r_0 fixed in Proposition 3.2.3.

Remark 6.3.1. We note here a fact that for all $|a| < M$ and all frequency triplets (ω, m, ℓ), $V'(r) < 0$ for $r \geq R_5$, with $R_5 \geq R$ sufficiently large.

6.3.4. $\mathcal{F}_T$ regime (time-dominated frequency regime)

We here follow the proof in (Dafermos and Rodnianski, 2010, Sect.9.6).

For $|a| \leq a_0 \ll M$, by choosing small enough λ_2 and large enough ω_3, there exists a constant $c < 1$ such that we have in $\mathcal{F}_T$ that

$$\omega^2 - V \geq \tfrac{1-c}{2}\omega^2 \text{ in } [r_+, \infty). \tag{6.48}$$

As to the potential V, apart from the fact in Remark 6.3.1, it holds true that for all r^*,

$$|V'| \leq C\Delta/r^5\left(\left(\lambda + a_0^2\omega^2\right) + 1\right). \tag{6.49}$$

We choose function y to satisfy the following properties:

1. $y \geq 0$, $y' \geq c\Delta/r^4$ in $(r_+, R_5]$,
2. $y \geq 0$, $y' \geq 0$ in $[R_5, R_5 + 1]$,
3. $y = 1$ in $[R_5 + 1, \infty)$.

Then, we have

Lemma 6.3.1. Fix a small constant λ_2 as above. Then for large enough ω_3, for arbitrary $r^*_\infty > (R_5 + 1)^* > R^*$ and $r^*_{-\infty} < r^*_c$, we have in $\mathcal{F}_T$ frequency regime the following estimate

$$
\begin{aligned}
& c\int_{r_c^*}^{R^*} \tfrac{\Delta}{r^4}\Big(|u'|^2 + \big(\omega^2 + (\lambda + a^2\omega^2) + 1\big) |u|^2 \Big) \\
\leq & \int_{r^*_{-\infty}}^{r^*_\infty} 2y\Re\left(u'\overline{H}\right) + Q^y\left(r^*_\infty\right) - Q^y\left(r^*_{-\infty}\right). \qquad (6.50)
\end{aligned}
$$

6.3.5. $\mathcal{F}_{Tr}$ regime (trapped frequency regime)

Here, we have fixed λ_2 as in Section 6.3.4, and will fix ω_3. This is the only frequency regime where trapping phenomenon could happen. We remark without proof that the potential V here shares the same properties as in (Dafermos and Rodnianski, 2010, Sect.9.5). In particular, for ω_3 sufficiently large and $|a|/M \leq a_0/M$ sufficiently small, $V'(r)$ has a unique zero point $r^{(a\omega)}_{m\ell}$ depending smoothly on the frequency triplets (ω, m, ℓ) and parameter a for any (ω, m, ℓ) in $\mathcal{F}_{Tr}$. Choose a function f associated with Q^f current to satisfy the following properties:

1. $f' \geq 0$ for all r^*, and $f' \geq c\Delta/r^4$ for $r_c \leq r \leq R$,
2. f changes sign from negative to positive at $r = r^{(a\omega)}_{m\ell}$, $\lim\limits_{r^* \to -\infty} f = -1$, and $f = 1$ for some large R_4,
3. $-fV' - \frac{1}{2}f''' \geq c(\omega_1)((\lambda + a^2\omega^2) + \omega^2)(r - r^{(a\omega)}_{m\ell})^2\Delta/r^7$ for all r^*.

Therefore, we arrive at the following conclusion.

Lemma 6.3.2. Choosing ω_3 sufficiently large and $|a|/M$ sufficiently small, for arbitrary $r^*_\infty > R^*_4 \geq R^*$ and $r^*_{-\infty} < r^*_c$, we have in $\mathcal{F}_{Tr}$ frequency regime the following estimate

$$
\begin{aligned}
& c\int_{r_c^*}^{R^*} \left(\Delta/r^4 \left(|u'|^2 + |u|^2\right) + \Delta/r^5(1 - r^{-1}r^{(a\omega)}_{m\ell})^2\left(\omega^2 + (\lambda + a^2\omega^2)\right)|u|^2\right) \\
\leq & \int_{r^*_{-\infty}}^{r^*_\infty} \left(2f\Re\left(u'\overline{H}\right) + f'\Re\left(u\overline{H}\right)\right) + Q^f\left(r^*_\infty\right) - Q^f\left(r^*_{-\infty}\right). \qquad (6.51)
\end{aligned}
$$

6.3.6. $\mathcal{F}_A$ regime (angular-dominated frequency regime)

Here, we fix ω_3 as in Section 6.3.5, and will choose λ_3 to be sufficiently large. This regime is contained in $\{(\omega, m, \ell) : |\omega| \leq \omega_3, \lambda \geq \lambda_3 - 1\}$. In this regime, for sufficiently small $|a|/M$, the zero points of $V'(r)$ in $[r_+, \infty)$ are located in a small neighborhood of $r = 3M$. The Q^f current is utilized to achieve the positivity of the bulk term outside this small neighborhood, while $hV|u|^2$ in $(Q^h)'$ is used to compensate the potentially negative bulk term in $(Q^f)'$ with $h(r)$ being a positive constant in this small neighborhood. We will constrain ourselves here not to give the explicit constructions of the functions f and h, but refer to (Dafermos and Rodnianski, 2010, Sect.9.4). We restate the conclusion here.

Lemma 6.3.3. Fix ω_3 as in Section 6.3.5, and choose λ_3 to be sufficiently large and $|a|/M$ sufficiently small, then for arbitrary $r^*_\infty > R^*$ and $r^*_{-\infty} < r^*_c$, we have in $\mathcal{F}_A$ frequency regime the following estimate

$$
\begin{aligned}
&c \int_{r_c^*}^{R^*} \left(|u'|^2 + \Delta/r^5 \left(\omega^2 + (\lambda + a^2\omega^2) + 1\right) |u|^2 \right) \\
&\leq \int_{r^*_{-\infty}}^{r^*_\infty} \left(2f\Re\left(u'\overline{H}\right) + (f' + h)\Re\left(u\overline{H}\right)\right) + \left(Q^f + Q^h\right)(r^*_\infty) - Q^f\left(r^*_{-\infty}\right). \quad (6.52)
\end{aligned}
$$

6.3.7. $\mathcal{F}_B$ regime (bounded frequency regime)

We fix ω_3 and λ_3 as above. This bounded frequency regime is contained in $\{(\omega, m, \ell) : |\omega| \leq \omega_3, \lambda \leq \lambda_3\}$. In this regime, a key fact is that the minimum value of eigenvalues Λ for the separated angular equation (6.30) is close to $\max\{s^2 + |s|, m^2 + |m|\}$ due to smallness of $|a\omega|$. The function λ then satisfies

$$
\begin{aligned}
\lambda &\geq \max\{s^2 + |s|, m^2 + |m|\} - (2Mr - 2a^2)/r^2 - \left(Ca^2\omega_3^2 + cs^2\right) \\
&\geq 3/4 - Ca^2\omega_3^2. \quad (6.53)
\end{aligned}
$$

Therefore, there exists a sufficiently small $\varepsilon_0 = \varepsilon_0(\omega_3) > 0$ and $B_0 = B_0(\varepsilon_0) \geq 1/2$ such that for all $|a|/M \leq \varepsilon_0$,

$$
\lambda + a^2\omega^2 \geq B_0 \left(1 + (m^2 + |m|) + \varepsilon_0^2\omega_3^2\right). \quad (6.54)
$$

It is easy to check that the estimates (Dafermos and Rodnianski, 2010, Eq.(41)–(44)) also hold true in this regime.

We split furthermore this regime into two sub-regimes, depending on the magnitude of $|\omega|$ compared to a suitably small parameter ω_4 to be chosen.

1. *Sub-regime $|\omega| \leq \omega_4$ (near-stationary case).* We will fix a suitably small $\omega_4 > 0$ in this case. One could follow the proof in (Dafermos and Rodnianski, 2010, Sect.9.3.1) and obtain the following result.
   Lemma 6.3.4. Fix a suitably small $\omega_4 > 0$. For arbitrary $r^*_\infty > R^*$ and $r^*_{-\infty} < r^*_c$, we have in the sub-regime $|\omega| \leq \omega_4$ of $\mathcal{F}_B$ frequency regime the following estimate
$$
\begin{aligned}
&c\int_{r^*_c}^{R^*} \left(\Delta/r^2\,|u'|^2 + \Delta/r^5\left(\omega^2 + (\lambda + a^2\omega^2) + 1\right)|u|^2\right)\\
&\leq \int_{r^*_{-\infty}}^{r^*_c} q|r^*|^{-2}|u|^2dr^* + \int_{r^*_{-\infty}}^{r^*_\infty}\left(2y\Re\left(u'\overline{H}\right) + h\Re\left(u\overline{H}\right)\right)\\
&\quad + Q^y\left(r^*_\infty\right) - \left(Q^y + Q^h\right)\left(r^*_{-\infty}\right). \qquad (6.55)
\end{aligned}
$$
   Here, $q > 0$ is an arbitrarily constant, which will be chosen to be sufficiently small in Section 6.3.8.3. In particular, ω_4 is already chosen in the proof of this lemma.

2. *Sub-regime $|\omega| \geq \omega_4$ (non-stationary case).* Fix an ω_4 as above. One could argue in the same way as in (Dafermos and Rodnianski, 2010, Sect.9.3.2) to establish the following conclusion.
   Lemma 6.3.5. Fix an ω_4 as in Lemma 6.3.4. For arbitrary $r^*_\infty > R^*$ and $r^*_{-\infty} < r^*_c$, we have in the sub-regime $|\omega| \geq \omega_4$ of $\mathcal{F}_B$ frequency regime the following estimate
$$
\begin{aligned}
&c\int_{r^*_c}^{R^*} \left(\Delta/r^2\,|u'|^2 + \Delta/r^5\left(\omega^2 + (\lambda + a^2\omega^2) + 1\right)|u|^2\right)\\
&\leq \int_{r^*_{-\infty}}^{r^*_c} Ba_0(\Delta/r^2)|u|^2dr^* + \int_{R^*_2}^{(\epsilon_2^{-1}R_2)^*} B\left(\epsilon_2 r^{-2} + r^{-3}\right)\omega^2|u|^2\\
&\quad + \int_{r^*_{-\infty}}^{r^*_\infty}\left(2y\Re\left(u'\overline{H}\right) + h\Re\left(u\overline{H}\right)\right) + Q^y\left(r^*_\infty\right) - Q^y\left(r^*_{-\infty}\right). \qquad (6.56)
\end{aligned}
$$
   Here again, $\varepsilon_2 > 0$ (suitably small) and $R_2 \geq R$ are arbitrary constants, which will be chosen to be sufficiently small in Section 6.3.8.3.

6.3.8. Summing up

We apply Cauchy-Schwarz inequality to the term $|a\cos\theta\partial_t\psi_\chi + is\psi_\chi|^2$ in (6.31a), it then easily follows from the estimate (6.3) that for any $r > r_c$

$$
\int_{-\infty}^{\infty}\sum_{m,\ell}\lambda(r)|u(r)|^2d\omega
$$

$$\geq \int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\left(r^2+a^2\right)\left(c(r_c)r^2\left|\nabla\psi_\chi\right|^2 - C(r_c)a^2\left|\partial_t\psi_\chi\right|^2\right)d\sigma_{\mathbb{S}^2}dt. \tag{6.57}$$

6.3.8.1. Error terms

The error terms, compared to the ones in Dafermos and Rodnianski (2010), have two additional terms coming from the source term and from the cutoff. We consider first the term arising from the cutoff

$$\int_{-\infty}^{\infty}\int_{r^*_{-\infty}}^{r^*_{\infty}}\sum_{m,\ell}\tfrac{-2as\Delta}{(r^2+a^2)^{3/2}}\Re\left(\left(i\cos\theta\partial_t\chi\psi\right)^{(a\omega)}_{m\ell}\left(c(r)\bar{u}(r)+d(r)\bar{u}'(r)\right)\right)dr^*d\omega, \tag{6.58}$$

and split it into two parts integrated over $r^*_{-\infty}\leq r^*\leq R^*_7$ and $R^*_7\leq r^*\leq r^*_\infty$ respectively. Here, $R_7\geq R$ is fixed such that the functions $y=f=1$ in the above chosen currents are satisfied for all frequencies in $r\geq R_7$. The integral over $r^*_{-\infty}\leq r\leq R^*_7$, after applying Cauchy-Schwarz and Plancherel lemma, is dominated by

$$\begin{aligned}
&C\varepsilon_3^{-1}a^2\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{r^*_{-\infty}}^{R^*_7}\left|\partial_t\chi\cdot(\psi/r)\right|^2\Delta dr^*d\sigma_{\mathbb{S}^2}dt^*\\
&+C\varepsilon_3\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{r^*_{-\infty}}^{R^*_7}r^{-1}\left(\left|\partial_{r^*}\psi_\chi\right|^2+\left|\psi_\chi/r\right|^2\right)\Delta dr^*d\sigma_{\mathbb{S}^2}dt^*\\
\leq& C\left(\varepsilon_3^{-1}a^2+\varepsilon_3\right)\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{r^*_{-\infty}}^{R^*_7}r^{-1}\left(\left|\partial_{r^*}\psi\right|^2+\left|\psi/r\right|^2\right)\Delta dr^*d\sigma_{\mathbb{S}^2}dt^*.
\end{aligned} \tag{6.59}$$

In view of the fact that $f(r)=1$ (and $y(r)=1$) in $R_7\leq r\leq r_\infty$ is independent of the frequency parameters (ω,m,ℓ) , the integral over this radius region equals to

$$\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{R_7}^{r_\infty}\Re\left(\partial_r(\sqrt{r^2+a^2}\chi\bar{\psi})\tfrac{-2as\Delta}{(r^2+a^2)^{3/2}}i\cos\theta\partial_t\chi\psi\right)drd\sigma_{\mathbb{S}^2}dt^* \tag{6.60}$$

by Plancherel lemma. Note that $\partial_{r^*}\chi=0$ for sufficiently large r and this integral is supported in $\{0\leq t^*\leq\tau'-\varepsilon^{-1}\}\cup\{\tau'-\varepsilon^{-1}\leq t^*\leq\tau'\}$, then the integral (6.60) is

$$\begin{aligned}
&\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{R_7}^{r_\infty}\tfrac{-2as\cos\theta\Delta}{r^2+a^2}\left(\Re\left(i\partial_r(\chi\bar{\psi})\partial_t(\chi\psi)\right)-\chi^2\Re\left(i\partial_r\bar{\psi}\partial_t\psi\right)\right)drd\sigma_{\mathbb{S}^2}dt^*\\
\leq& C|a|\left(\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{R_7}^{r_\infty}\tfrac{|\partial\psi|^2}{r^2}r^2drd\sigma_{\mathbb{S}^2}dt^*+\int_{\mathcal{D}(0,\tau')\cap\{r=R_7\}}|\partial\psi|^2\right),
\end{aligned} \tag{6.61}$$

with the second integral in the second line arising from estimating the second term in the first line of (6.61). This is further controlled, via an average of integration, by

$$C|a|\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{R_7-1}^{r_\infty}\tfrac{|\partial\psi|^2}{r^2}r^2drd\sigma_{\mathbb{S}^2}dt^*. \tag{6.62}$$

The additional term coming from the source term F is

$$\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{r^*_{-\infty}}^{r^*_\infty}\sum_{m,\ell}\tfrac{\Delta}{(r^2+a^2)^{3/2}}\Re\left((\chi F)^{(a\omega)}_{m,\ell}\left(c(r)\bar{u}(r)+d(r)\bar{u}'(r)\right)\right)dr^*d\sigma_{\mathbb{S}^2}dt^*. \tag{6.63}$$

As discussed above, we consider this integral over $r^*_{-\infty}\leq r\leq R^*_7$ and $R^*_7\leq r\leq r^*_\infty$ separately. The integral over $r^*_{-\infty}\leq r\leq R^*_7$ can be treated in the same way as above and thus be estimated by

$$\begin{aligned}
&C\varepsilon_3^{-1}\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{r^*_{-\infty}}^{R^*_7}|\chi F|^2\,\Delta dr^*d\sigma_{\mathbb{S}^2}dt^*\\
&\quad+C\varepsilon_3\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{r^*_{-\infty}}^{R^*_7}r^{-1}\left(|\partial_{r^*}\psi_\chi|^2+|\psi_\chi/r|^2\right)\Delta dr^*d\sigma_{\mathbb{S}^2}dt^*\\
\leq&C\varepsilon_3^{-1}\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{r^*_{-\infty}}^{R^*_7}|F|^2\,\Delta dr^*d\sigma_{\mathbb{S}^2}dt^*\\
&\quad+C\varepsilon_3\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{r^*_{-\infty}}^{R^*_7}r^{-1}\left(|\partial_{r^*}\psi|^2+|\psi/r|^2\right)\Delta dr^*d\sigma_{\mathbb{S}^2}dt^*,
\end{aligned} \tag{6.64}$$

while the integral over $R^*_7\leq r\leq r^*_\infty$ equals to

$$\begin{aligned}
&\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{R_7}^{r_\infty}\Re\left(\partial_r(\sqrt{r^2+a^2}\chi\bar{\psi})\tfrac{\Delta}{(r^2+a^2)^{3/2}}\chi F\right)drd\sigma_{\mathbb{S}^2}dt^*\\
=&\int_0^{\tau'}\int_{\mathbb{S}^2}\int_{R_7}^{r_\infty}\chi^2\Re\left(\partial_r(\sqrt{r^2+a^2}\bar{\psi})\tfrac{\Delta}{(r^2+a^2)^{3/2}}F\right)drd\sigma_{\mathbb{S}^2}dt^*
\end{aligned} \tag{6.65}$$

and is furthermore bounded by

$$C\int_0^{\tau'}\int_{\Sigma_{t^*}}\left(\varepsilon^{-1}r^{1+\delta}|F/\Sigma|^2+\varepsilon r^{-1-\delta}(|\partial_{r^*}\psi|^2+|\psi/r|^2)\right)d\mathrm{Vol}_{\Sigma_{t^*}}dt^*. \tag{6.66}$$

The other error terms, being the same as in (Dafermos and Rodnianski, 2010, Sect.10.2-10.3), can be treated in the same way there.

6.3.8.2. Boundary terms

Turning to the boundary terms, they are controlled in the same way as in (Dafermos and Rodnianski, 2010, Sect.10.4) and we will omit the discussions here. In particular, the boundary terms at r_∞ vanish for sufficiently large r_∞ from the reduction in Section 2.6.1.

6.3.8.3. Summing and finishing the proof

Given the above estimates, we need to make some replacements to finish the proof along the line in (Dafermos and Rodnianski, 2010, Sect.10.5). The energy estimate associated with the multiplier $\partial_t\bar{\psi}$, Morawetz estimate in large r region, red-shift estimate, propositions 5.3.1 and 5.3.2 in Dafermos and Rodnianski (2010) for the scalar wave equation should be replaced by the estimates (6.19), (3.12), (3.32), Propositions 6.1.1 and 6.1.2 in this paper, respectively. Upon these replacements, the spin-1 case in the Theorem 2.4.1 is proved.

6.4. Proof of Theorem 2.4.1 for spin-2 case

Following the procedures in Sections 6.3.1 and 6.3.2, we choose $\varepsilon > 0$ and any fixed $\tau' \geq 2\varepsilon^{-1}$, and apply in global Kerr coordinate system the cutoff

$$\chi = \chi_{\tau',\varepsilon}(t^*) = \chi_2(\varepsilon t^*)\chi_2(\varepsilon(\tau' - t^*)) \tag{6.67}$$

to the solution ψ:

$$\phi^i_{s,\chi} = \chi\phi^i_s, \tag{6.68}$$

with $\chi_2(x)$ being a smooth cutoff function which equals to 0 for $x \leq 0$ and is identically 1 for $x \geq 1$. Moreover, it satisfies the following inhomogeneous equation

$$\begin{aligned}\mathbf{L}^k_s\phi^i_{s,\chi} &= F^i_{s,\chi}\\ &= \chi F^i_s + \Sigma\left(2\nabla^\mu\chi\nabla_\mu\phi^i_s + (\Box_g\chi)\,\phi^i_s\right) - 2isa\cos\theta\partial_t\chi\cdot\phi^i_s.\end{aligned} \tag{6.69}$$

$k = 0$ or 1 depends on the equation (2.34a) or (2.34b) we are treating.

From the assumptions in Theorem 2.4.2 and the reduction in Section 2.6.1, $\widetilde{\Phi_j}(j = 0, 4)$, and hence Φ_j, ϕ^i_s and F^i_s, are smooth and compactly supported. As a result, we can apply the mode decompositions in Section 6.2 to $\psi = \phi^i_{s,\chi}$ and $F = F^i_{s,\chi}$, and separate the wave equation (6.69) into the angular equation (6.30) and radial equation (6.32) or radial equation (6.35), with $(R^i_s)^{(a\omega)}_{m\ell} \triangleq (\phi^i_{s,\chi})^{(a\omega)}_{m\ell}$ and $(F^i_{s,\chi})^{(a\omega)}_{m\ell}$ in place of $\psi^{(a\omega)}_{m\ell}$ and $F^{(a\omega)}_{m\ell}$ respectively.

We suppress the dependence on a, ω, m and ℓ in the functions $(R^i_s)^{(a\omega)}_{m\ell}(r)$, $(F^i_{s,\chi})^{(a\omega)}_{m\ell}(r)$, $\Lambda^{(a\omega)}_{m\ell,s,k}$, $\lambda^{(a\omega)}_{m\ell,s,k}(r)$, $(V_k)^{(a\omega)}_{m\ell,s}(r)$ and other functions defined by them, and when there is no confusion, the dependence on r may always be implicit. Define

$$u^i_s(r) = \sqrt{r^2+a^2}R^i_s(r), \qquad H^i_s(r) = \tfrac{\Delta F^i_{s,\chi}(r)}{(r^2+a^2)^{3/2}} \tag{6.70}$$

to transform the radial equation into an equation of Schrödinger form, which is the same as (6.44) with the same potential. One could adapt easily the proof in Section 6.3 to obtain frequency localised Morawetz estimates and sum up these estimates, with corresponding replacements of spin 1 statements in Propositions 3.1.1, 3.1.2, 6.1.1 and 6.1.2 by the spin 2 statements, the red-shift estimate Proposition 3.2.3 for spin 1 case in Section 3.2.1 by the red-shift estimates in Section 3.2.2.

Then we arrive at the estimates (2.46) with all the error terms divided into three categories:

1. error terms by choosing the multipliers $\partial_t\overline{\phi^i_s}$ to obtain energy estimate, $X_w\overline{\phi^i_s}$ to obtain Morawetz estimates for ϕ^i_s in large r region, and N to obtain redshift estimates for ϕ^i_{+2} $(i=0,1,2)$ and ϕ^2_{-2};
2. error terms arising in the currents estimates;
3. extra error terms arising from Morawetz estimates in large radius region in Proposition 3.1.2 for $r^{4-\delta}\phi^0_{+2}$ and $r^{2-\delta}\phi^1_{+2}$ and red-shift estimates in Proposition 3.2.4 for $\widetilde{\phi^0_{-2}}$ and $\widetilde{\phi^1_{-2}}$.

It is obvious from the application of Cauchy-Schwarz inequality that all these three categories are bounded by $C\mathcal{E}^i_s$.

7. Proof of Main Theorem 2.4.2

Contents

7.1. Maxwell field

7.1.1. Spin $+1$ component

The estimate (2.44) applied to $\psi = \phi^0_{+1}$ and $F = F^0_{+1}$ and the estimate (3.13) together imply

$$\begin{aligned} &E_\tau(r^{2-\delta}\phi^0) + E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^0) + \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}_{\text{deg}}(r^{2-\delta}\phi^0) \\ &\lesssim E_0(r^{2-\delta}\phi^0) + E_0(\phi^1) + \mathcal{E}(F^0_{+1}). \end{aligned} \tag{7.1}$$

Instead, if we apply the estimate (2.44) for $\psi = \phi^1_{+1}$ and $F = F^1_{+1}$, then we arrive at

$$E_\tau(\phi^1) + E_{\mathcal{H}^+(0,\tau)}(\phi^1) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\phi^1) \lesssim E_0(\phi^0) + E_0(\phi^1) + \mathcal{E}(F^1_{+1}). \tag{7.2}$$

It is manifest that

$$\mathcal{E}(F^0_{+1}) \lesssim \varepsilon_0 \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\phi^1) + \varepsilon_0^{-1} \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(r^{2-\delta}\phi^0), \tag{7.3}$$

and all terms in $\mathcal{E}(F^1_{+1})$ are bounded by $\frac{Ca}{M}\int_{\mathcal{D}(0,\tau)}(\widetilde{\mathbb{M}}(r^{2-\delta}\phi^0)+\mathbb{M}_{\deg}(\phi^1))$ except for the term

$$\left|\int_{\mathcal{D}(0,\tau)}\Re\left(\Sigma^{-1}F^1_{+1}\partial_{t^*}\overline{\phi^1}\right)\right| \tag{7.4}$$

because of the trapping degeneracy in $\mathbb{M}_{\deg}(\phi^1)$. We have for (7.4) that

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)}\tfrac{1}{\Sigma}\Re\left(F^1_{+1}\partial_t\overline{\phi^1}\right)\right|\\ &\leq\left|\int_{\mathcal{D}(0,\tau)}\tfrac{2a^2}{\Sigma}\Re\left(\partial_t\phi^0\partial_t\overline{\phi^1}\right)\right|+\left|\int_{\mathcal{D}(0,\tau)}\tfrac{2a}{\Sigma}\Re\left(\partial_\phi\phi^0\partial_t\overline{\phi^1}\right)\right|,\end{aligned} \tag{7.5}$$

and

$$\left|\int_{\mathcal{D}(0,\tau)}\tfrac{2a^2}{\Sigma}\Re\left(\partial_t\phi^0\partial_t\overline{\phi^1}\right)\right|=\left|\int_{\mathcal{D}(0,\tau)}\tfrac{a^2}{\Sigma}Y\left(r^2|\partial_t\phi^0|^2\right)\right|\lesssim_a 0. \tag{7.6}$$

As to the other term $\left|\int_{\mathcal{D}(0,\tau)}\frac{2a}{\Sigma}\Re(\partial_\phi\phi^0\partial_t\overline{\phi^1})\right|$, we split it into three sub-integrals with $r_+<\check{r}_2<r^-_{\text{trap}}\leq r^+_{\text{trap}}<R_1<\infty$ to be chosen:

$$\left|\left(\int_{\mathcal{D}(0,\tau)\cap[r_+,\check{r}_2]}+\int_{\mathcal{D}(0,\tau)\cap[R_1,\infty)}+\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,R_1]}\right)\tfrac{2a}{\Sigma}\Re\left(\partial_\phi\phi^0\partial_t\overline{\phi^1}\right)\right|,$$

with the first two sub-integrals controlled by $\frac{Ca}{M}\int_{\mathcal{D}(0,\tau)}(\widetilde{\mathbb{M}}(r^{2-\delta}\phi^0)+\mathbb{M}_{\deg}(\phi^1))$ directly. We substitute the expression

$$\partial_t\phi^1=(r^2+a^2)^{-1}\left(\Delta Y\phi^1-a\partial_\phi\phi^1+\Delta\partial_r\phi^1\right), \tag{7.7}$$

and find the third sub-integral is bounded by

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,R_1]}\left(\tfrac{2a\Delta}{r\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{r\phi^0})Y\phi^1\right)-\tfrac{a^2}{\Sigma(r^2+a^2)}Y\left(\left|\partial_\phi(r\phi^0)\right|^2\right)\right)\right|\\ &+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,R_1]}\tfrac{2a\Delta}{\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{\phi^0})\partial_r\phi^1\right)\right|\lesssim_a 0.\end{aligned} \tag{7.8}$$

In the last step, we applied integration by parts to the first line and controlled the boundary terms at R_1 and $\check{r}_2$ by appropriately choosing these two radius parameters such that these boundary terms are bounded via an average of integration by $\frac{Ca}{M}\int_{\mathcal{D}(0,\tau)}(\widetilde{\mathbb{M}}(r^{2-\delta}\phi^0)+\mathbb{M}_{\deg}(\phi^1))$. These then imply the estimates (4.15).

7.1.2. Spin -1 component

Apply the estimate (2.44) to equation (2.30a) of ϕ^0 and equation (2.30b) of ϕ^1. Manifestly, we have

$$\mathcal{E}(F^0_{-1}) \lesssim_a \int_{\mathcal{D}(0,\tau)} \left(\epsilon_0 \mathbb{M}(\phi^0) + \tfrac{1}{\epsilon_0}\tfrac{|\phi^1|^2}{r^3}\right), \tag{7.9}$$

and for the term $\int_{\mathcal{D}(0,\tau)} r^{-3+\delta}|F^1_{-1}|^2$ in $\mathcal{E}(F^1_{-1})$,

$$\int_{\mathcal{D}(0,\tau)} r^{-3+\delta}|F^1_{-1}|^2 \lesssim \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)} \left(\mathbb{M}(\phi^0) + |\nabla\!\!\!\!/\,\phi^0|^2\right). \tag{7.10}$$

For the remaining term $\left|\int_{\mathcal{D}(0,\tau)} \frac{1}{\Sigma}\Re\left(F^1_{-1}\partial_t\overline{\phi^1}\right)\right|$ in $\mathcal{E}(F^1_{-1})$, we have

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^1_{-1}\partial_t\overline{\phi^1}\right)\right|\\ \le{}& \left|\int_{\mathcal{D}(0,\tau)} \tfrac{2a^2}{\Sigma}\Re\left(\partial_t\phi^0\partial_t\overline{\phi^1}\right)\right| + \left|\int_{\mathcal{D}(0,\tau)} \tfrac{2a}{\Sigma}\Re\left(\partial_\phi\phi^0\partial_t\overline{\phi^1}\right)\right|.\end{aligned} \tag{7.11}$$

We first split the first integral on RHS into two sub-integrals over $[r_+, \check{r}_1]$ and $[\check{r}_1, \infty)$, with $\check{r}_1 \in (r_1, r^-_{\text{trap}})$ to be fixed, and it follows

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)} \tfrac{2a^2}{\Sigma}\Re\left\{\partial_t\phi^0\partial_t\overline{\phi^1}\right\}\right|\\ \le{}& \left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_1,\infty)} \tfrac{a^2}{\Sigma}V\left(r^2|\partial_t\phi^0|^2\right)\right| + \left|\int_{\mathcal{D}(0,\tau)\cap[r_+,\check{r}_1]} \tfrac{2a^2}{\Sigma}\Re\left(\partial_t\phi^0\partial_t\overline{\phi^1}\right)\right|\\ \lesssim_a{}& \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)\cap\{r=\check{r}_1\}} \left|\partial\phi^0\right|^2.\end{aligned} \tag{7.12}$$

We can choose a $\check{r}_1$ such that the last term in (7.12) can be bounded, via an average of integration, by

$$\int_{\mathcal{D}(0,\tau)\cap\{r=\check{r}_1\}} \left|\partial\phi^0\right|^2 \le C\int_{\mathcal{D}(0,\tau)} \mathbb{M}(\phi^0). \tag{7.13}$$

As to the second integral term $\left|\int_{\mathcal{D}(0,\tau)} 2a\Sigma^{-1}\Re(\partial_\phi\phi^0\partial_t\overline{\phi^1})\right|$, we split it into two sub-integrals with $r_+ < \check{r}_3 < r^-_{\text{trap}}$ to be chosen:

$$\left|\left(\int_{\mathcal{D}(0,\tau)\cap[r_+,\check{r}_3]} + \int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty)}\right)\tfrac{2a}{\Sigma}\Re\left(\partial_\phi\phi^0\partial_t\overline{\phi^1}\right)\right|, \tag{7.14}$$

where the first sub-integral is clearly bounded by $\frac{C|a|}{M}\int_{\mathcal{D}(0,\tau)}(\mathbb{M}(\phi^0)+\mathbb{M}_{\text{deg}}(\phi^1))$. We substitute the expression

$$\partial_t\phi^1 = (r^2+a^2)^{-1}\left(\Delta V\phi^1 - a\partial_\phi\phi^1 - \Delta\partial_r\phi^1\right) \tag{7.15}$$

into the second sub-integral and find it is bounded by

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty]}\tfrac{2a\Delta}{r\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{r\phi^0})V\phi^1\right)\right|\\ &+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty]}\tfrac{a^2}{\Sigma(r^2+a^2)}V\left(\left|\partial_\phi(r\phi^0)\right|^2\right)\right|\\ &+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty]}\tfrac{2a\Delta}{\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{\phi^0})\partial_r\phi^1\right)\right|.\end{aligned} \tag{7.16}$$

Integrating by parts for the first two lines then shows that for sufficiently small $|a|/M$

$$\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,\infty)}2a\Sigma^{-1}\Re\left(\partial_\phi\phi^0\partial_t\overline{\phi^1}\right)\right|\lesssim_a \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)}|\nabla\!\!\!/\phi^0|^2. \tag{7.17}$$

Then the estimates (4.17) are valid.

7.2. Linearized gravity

The estimates (4.24) for spin $+2$ component and (4.29) for spin -2 component are proved on slowly rotating Kerr background in this section.

7.2.1. Spin $+2$ component

Let us treat the error terms $\mathcal{E}^i_{+2}$ in the energy and Morawetz estimate (2.46). Manifestly,

$$\left|\int_{\mathcal{D}(0,\tau)}\tfrac{1}{\Sigma}\Re\left(F^0_{+2}\partial_t\overline{\phi^0}\right)\right|\lesssim_a \epsilon_0\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0)+\tfrac{1}{\epsilon_0}\int_{\mathcal{D}(0,\tau)}r^{-3}\left|\phi^1_{+2}\right|^2, \tag{7.18}$$

$$\left|\int_{\mathcal{D}(0,\tau)}\tfrac{1}{\Sigma}\Re\left(F^1_{+2}\partial_t\overline{\phi^1}\right)\right|\lesssim_a \int_{\mathcal{D}(0,\tau)}\left(\epsilon_1\widetilde{\mathbb{M}}(r^{2-\delta}\phi^1)+\tfrac{1}{\epsilon_1}\left(\tfrac{|\phi^0_{+2}|^2}{r^2}+\tfrac{|\phi^2_{+2}|^2}{r^3}\right)\right). \tag{7.19}$$

For the term $\left|\int_{\mathcal{D}(0,\tau)} \frac{1}{\Sigma}\Re(F_{+2}^2\partial_t\overline{\phi^2})\right|$, which *a prior* can not be controlled in the trapped region due to the trapping degeneracy, we control it by

$$\begin{aligned}\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F_{+2}^2\partial_t\overline{\phi^2}\right)\right| \lesssim{}& \left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\partial_t\phi^1\partial_t\overline{\phi^2}\right)\right| \\ &+\left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\phi^0\partial_t\overline{\phi^2}\right)\right| + \left|\int_{\mathcal{D}(0,\tau)} \tfrac{a}{\Sigma}\Re\left(\partial_\phi\phi^1\partial_t\overline{\phi^2}\right)\right|. \end{aligned} \tag{7.20}$$

The sum of the first two integrals on RHS is

$$\left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{2\Sigma}Y\left(r^2|\partial_t\phi^1|^2\right)\right| + \left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\left(\partial_{t^*}\left(\Re(\phi^0\overline{\phi^2})\right) - \Re(\partial_t\phi^0\overline{\phi^2})\right)\right| \lesssim_a 0. \tag{7.21}$$

As to the third integral term, we choose $\check{r}_1 \in (r_0, r_{\text{trap}}^-)$ and $\check{R}_1 > r_{\text{trap}}^+$, and split the integral in radius into three sub-integrals over $[r_+, \check{r}_1]$, $[\check{r}_1, \check{R}_1]$ and $[\check{R}_1, \infty)$, respectively. The sum of the sub-integrals over $[r_+, \check{r}_1]$ and $[\check{R}_1, \infty)$ is manifestly bounded by $C\Xi_{+2}(0,\tau)$. For the left sub-integral over $[\check{r}_1, \check{R}_1]$, we utilize the expression

$$\partial_t\phi^2 = (r^2+a^2)^{-1}\left(\Delta Y\phi^2 - a\partial_\phi\phi^2 + \Delta\partial_r\phi^2\right), \tag{7.22}$$

and find this left sub-integral is bounded by

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,R_1]} \left(\tfrac{2a\Delta}{r\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{r\phi^1})Y\phi^2\right) - \tfrac{a^2}{\Sigma(r^2+a^2)}Y\left(\left|\partial_\phi(r\phi^1)\right|^2\right)\right)\right| \\ &+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,R_1]} \tfrac{2a\Delta}{\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{\phi^1})\partial_r\phi^2\right)\right| \lesssim_a 0. \end{aligned} \tag{7.23}$$

In the last step, integration by parts is applied to the first line and two radius parameters $\check{r}_1$ and $\check{R}_1$ are appropriately chosen such that the boundary terms at $\check{r}_1$ and $\check{R}_1$ are bounded via an average of integration by $\frac{C|a|}{M}\int_{\mathcal{D}(0,\tau)}(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0) + \widetilde{\mathbb{M}}(r^{2-\delta}\phi^1) + \mathbb{M}_{\text{deg}}(\phi^2))$. Therefore, it holds that

$$\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_1,\check{R}_1]} \tfrac{a}{\Sigma}\Re\left(\partial_\phi\phi^1\partial_t\overline{\phi^2}\right)\right| \lesssim_a 0, \tag{7.24}$$

which further implies together with the above discussions that

$$\left|\int_{\mathcal{D}(0,\tau)} \Sigma^{-1}\Re\left(F_{+2}^2\partial_t\overline{\phi^2}\right)\right| \lesssim_a 0. \tag{7.25}$$

The estimates (4.24) are then proved.

7.2.2. Spin -2 component

We shall now bound the error terms $\mathcal{E}^i_{-2}$ in the energy and Morawetz estimate (2.46). Notice that

$$\begin{aligned}
&\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^0_{-2}\partial_t\overline{\phi^0}\right)\right| \\
\lesssim_a\ & \epsilon_0\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\phi^0)+\epsilon_0^{-1}\int_{\mathcal{D}(0,\tau)} r^{-3}|\phi^1|^2, \qquad (7.26)\\
&\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^1_{-2}\partial_t\overline{\phi^1}\right)\right| \\
\lesssim_a\ & \int_{\mathcal{D}(0,\tau)}\left(\epsilon_1\mathbb{M}(\phi^1)+\tfrac{1}{\epsilon_1}\left(\tfrac{|\widetilde{\phi^0}|^2}{r^2}+\tfrac{|\phi^2|^2}{r^3}+\tfrac{|a|}{M}|\nabla\!\!\!\!/\,\widetilde{\phi^0}|^2\right)\right). \qquad (7.27)
\end{aligned}$$

For the term $\left|\int_{\mathcal{D}(0,\tau)}\frac{1}{\Sigma}\Re\left(F^2_{-2}\partial_t\overline{\phi^2}\right)\right|$, we have

$$\begin{aligned}
\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^2_{-2}\partial_t\overline{\phi^2}\right)\right| \lesssim{}& \left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\partial_t\phi^1\partial_t\overline{\phi^2}\right)\right| \\
&+\left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\phi^0\partial_t\overline{\phi^2}\right)\right|+\left|\int_{\mathcal{D}(0,\tau)} \tfrac{a}{\Sigma}\Re\left(\partial_\phi\phi^1\partial_t\overline{\phi^2}\right)\right|. \qquad (7.28)
\end{aligned}$$

We split the first integral into two sub-integrals over $[r_+,\check{r}_2]$ and $[\check{r}_2,\infty)$, with $\check{r}_2\in(r_1,r^-_{\text{trap}})$ to be fixed, and obtain

$$\begin{aligned}
&\left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\partial_t\phi^1\partial_t\overline{\phi^2}\right)\right| \\
\leq{}& \left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,\infty)} \tfrac{a^2}{2\Sigma}V\left(r^2|\partial_t\phi^1|^2\right)\right|+\left|\int_{\mathcal{D}(0,\tau)\cap[r_+,\check{r}_2]} \tfrac{a^2}{\Sigma}\Re\left(\partial_t\phi^1\partial_t\overline{\phi^2}\right)\right| \\
\lesssim_a{}& \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)\cap\{r=\check{r}_2\}}\left|\partial\phi^1\right|^2. \qquad (7.29)
\end{aligned}$$

We can choose a $\check{r}_2$ such that the last term in (7.29) can be bounded, via an average of integration, by

$$\tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)\cap\{r=\check{r}_2\}}\left|\partial\phi^1\right|^2\lesssim\tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\phi^1)\lesssim\tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\widetilde{\phi^1}). \qquad (7.30)$$

We split the integral region of the second line of (7.28) into two subregions $[r_+,\check{r}_3]$ and $(\check{r}_3,\infty)$ with $\check{r}_3\in(r_0,r^-_{\text{trap}})$ to be fixed. The terms integrated over $[r_+,\check{r}_3]$ are clearly bounded by $C\Xi_{-2}(0,\tau)$. While, for the integrals over $(\check{r}_3, \infty)$, we use the substitution

$$\partial_t \phi^2 = (r^2+a^2)^{-1}\left(\Delta V\phi^2 - a\partial_\phi\phi^2 - \Delta\partial_r\phi^2\right), \tag{7.31}$$

and find they are dominated by

$$\begin{aligned}
&\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty)} \left(\tfrac{a\Delta}{r\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{r\phi^1})V\phi^2\right) - \tfrac{a^2}{2\Sigma(r^2+a^2)}V\left(\left|\partial_\phi(r\phi^1)\right|^2\right)\right)\right| \\
&+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty)} \tfrac{a^2\Delta}{r\Sigma(r^2+a^2)}\Re\left((\overline{r\phi^0})V\phi^2\right)\right| \\
&+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty)} \tfrac{\Delta}{\Sigma(r^2+a^2)}\left(\Re\left(a\partial_\phi\overline{\phi^1}\partial_r\phi^2\right) + \Re\left(a^2\overline{\phi^0}\partial_r\phi^2\right)\right)\right| \\
&+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty)} \tfrac{a^2}{\Sigma(r^2+a^2)}\left(\partial_\phi\left(\Re\left(\phi^0\overline{\phi^2}\right)\right) - \Re\left(\partial_\phi\phi^0\overline{\phi^2}\right)\right)\right| \\
\lesssim_a &\int_{\mathcal{D}(0,\tau)} \tfrac{|a|}{M}\left(|\nabla\widetilde{\phi}^1|^2 + \tfrac{|\widetilde{\phi^0}|^2}{r^2}\right).
\end{aligned} \tag{7.32}$$

Here, we applied integration by parts to the first two lines and utilized the definition (2.28b) and similar estimates as (7.13) to control the boundary terms at $\check{R}_3$ and $\check{r}_3$ by appropriately choosing these two radius parameters. In summary, we have

$$\left|\int_{\mathcal{D}(0,\tau)} \Sigma^{-1}\Re\left(F^2_{-2}\partial_t\overline{\phi^2}\right)\right| \lesssim_a \int_{\mathcal{D}(0,\tau)} \tfrac{|a|}{M}\left(|\nabla\widetilde{\phi}^1|^2 + \tfrac{|\widetilde{\phi^0}|^2}{r^2}\right). \tag{7.33}$$

It is manifest from the estimates (2.46), (7.26), (7.27) and (7.33) that the estimates (4.29) hold true.

8. Brief overview of mode stability result

We briefly review our mode stability result Andersson et al. (2017b), which is a joint work with Lars Andersson, Claudio Paganini and Bernard F Whiting.

In Boyer-Lindquist coordinates (t, r, θ, ϕ), let

$$\begin{aligned}\mathbf{L} = \partial_r \Delta \partial_r - \frac{1}{\Delta}\left\{(r^2+a^2)\partial_t + a\partial_\phi - (r-M)s\right\}^2 \\ - 4s(r + ia\cos\theta)\partial_t + \frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta \\ + \frac{1}{\sin^2\theta}\left\{a\sin^2\theta\partial_t + \partial_\phi + is\cos\theta\right\}^2 .\end{aligned} \tag{8.1}$$

Then the spin s component Φ solves a separable, spin-weighted wave equation–TME Teukolsky (1973), given by

$$\mathbf{L}\Phi = 0 \tag{8.2}$$

We should note here that this TME is different from the TME (8.2) since the field $\Phi = \Delta^{-s/2}\psi_{[s]}$.

The TME admits separated solutions (or modes for simplicity) of the form

$$\Phi = e^{-i\omega t} e^{im\phi} S(\theta) R(r), \tag{8.3}$$

where ω, m are the frequencies corresponding to the Killing vector fields $(\partial_t)^a$, $(\partial_\phi)^a$. Let

$$K = (r^2 + a^2)\omega - am. \tag{8.4}$$

Then with

$$\mathbf{R} = \partial_r \Delta \partial_r + \frac{K^2 - 2iK(r-M)s - (r-M)^2 s^2}{\Delta} + 4sir\omega - \Lambda \tag{8.5}$$

$$\mathbf{S} = \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta - \frac{m^2}{\sin^2\theta} + a^2\cos\theta^2\omega^2 - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta$$

$$+\Lambda + 2a\omega m - a^2\omega^2, \tag{8.6}$$

where Λ is a separation constant, which can be assumed to be real for real ω, we have after making the substitutions $\partial_t \leftrightarrow -i\omega$, $\partial_\phi \leftrightarrow im$,

$$\mathbf{L} = \mathbf{R} + \mathbf{S}, \tag{8.7}$$

and

$$[\mathbf{R}, \mathbf{S}] = 0. \tag{8.8}$$

In particular, $\mathbf{R}, \mathbf{S}$ are commuting symmetry operators for $\mathbf{L}$. It follows from the above that for modes of the form (8.3), (8.2) is equivalent to the equations $\mathbf{R}R = 0$, $\mathbf{S}S = 0$. We shall refer to the equations

$$\mathbf{R}R = 0 \tag{8.9a}$$

$$\mathbf{S}S = 0 \tag{8.9b}$$

as the radial and angular Teukolsky equations, respectively. As for the treatment of the real frequency case by Shlapentokh-Rothman (2015), we shall not be concerned with the analysis of the angular Teukolsky equation here, but point out that $\mathbf{S}$ is formally self-adjoint on $[0, \pi]$ with respect to $\sin\theta d\theta$. Imposing the condition that the solutions correspond to regular spin-weighted functions fixes the boundary conditions at $\theta = 0, \pi$ and equation (8.9b) becomes a Sturm-Liouville problem which has a discrete, real spectrum; see Leaver (1986) for more details. The separation constant used here is related to that used in Teukolsky and Press (1974) by $\Lambda + 2a\omega m - a^2\omega^2 = E - s^2$, and to the one used in Whiting (1989) and Teukolsky (1972) by $\Lambda + 2a\omega m - a^2\omega^2 = A + s$.

For fields of non-zero spin, the TME does not admit a real action, and hence standard arguments do not yield energy conservation and dispersive estimates. This is an obstacle to proving stability for the test fields with non-zero spin on the Kerr exterior spacetime.

A proof of mode stability is given in Whiting (1989). It shows that the TME has no modes which are such that the frequency ω has positive imaginary part, and which have no incoming radiation in the sense that the wave is outgoing at infinity, and ingoing at the horizon. To be concise, the main result of Whiting (1989) states that the TME admits no exponentially growing solutions without incoming radiation. In the case of $\Im\omega > 0$, the condition of no incoming radiation can be restated as saying that the solution has support only on the future horizon and null infinity. On the other hand, there do exist mode solutions with no incoming radiation for certain frequencies with negative imaginary part. This case corresponds to quasi-normal modes (Kokkotas and Schmidt, 1999), which are exponentially decaying in time.

It is known that exponentially growing modes must arise by quasi-normal frequencies passing from the lower half plane through the real axis into the upper half plane as a is changed from zero. This was argued heuristically by Press and Teukolsky (Press and Teukolsky, 1973, p. 651) and later shown by Hartle and Wilkins Hartle and Wilkins (1974), see also (Teukolsky and Press, 1974, p. 452). For this reason, the mode stability problem can be reduced to considering the case of real frequencies.

Recently, the mode stability argument has been revisited for the case of real frequencies, restricting to the spin-0 case (Shlapentokh-Rothman, 2015). In the case of real frequencies, the mode stability result states that restricting to modes with no incoming radiation in the above sense, the radial Teukolsky equation has no non-trivial solutions. This has the consequence that there are linearly independent solutions $R_{\text{hor}}, R_{\text{out}}$ which are purely ingoing at the horizon and outgoing at infinity, respectively, a fact which plays a central role in the proof of boundedness and decay for scalar waves on subextremal Kerr exterior spacetimes (Dafermos et al., 2014), in particular it is used to treat the superradiant range of frequencies.

Motivated by the relevance of the TME for the black hole stability problem we give a proof of mode stability on the real axis for fields with arbitrary spin. Our main result in Andersson et al. (2017b) is the following.

Theorem 8.0.1 (Mode stability on the real axis). Let Φ be a mode to the TME with $\omega \in \mathbb{R}$ for the subextremal Kerr black hole. Assume that Φ has purely ingoing radiation at the horizon and purely outgoing radiation at infinity. Then $\Phi = 0$.

Remark 8.0.1. A classical scattering argument can be used to show mode stability on the real axis for half-integer spins, or for frequences outside of the superradiant range $\omega(\omega - am/(2Mr_+)) < 0$. The proof of mode stability on the real axis presented in this paper is independent of that scattering argument.

The fact that there are no modes to the TME with no incoming radiation has the important consequence that the radial Teukolsky equation has two fundamental solutions R_{hor} and R_{out} which are ingoing at the horizon, and outgoing at infinity, respectively, and are linearly independent, with non-vanishing Wronskian. This implies that one can construct solutions of the inhomogenous Teukolsky equation using the method of variation of the parameter. The properties of the solutions R_{hor} and R_{out} can be used to estimate the solution of the inhomogenous Teukolsky equation, and will be crucial in generalizing the results in the previous chapters from slowly rotating Kerr backgrounds to full subextremal Kerr backgrounds.

A. Another set of variables

We here introduce another set of variables $\hat{\phi}_s^i$ $(i = 0, 1, \cdots, |s|)$ constructed from the spin s components, with their governing equations containing no ∂_t derivative terms on the RHS. This fact is in particular useful when we consider the case of axisymmetric perturbations where the ∂_ϕ derivative terms are identically zero, since the systems (A.4), (A.5), (A.7) and (A.8) will see no derivative terms on the RHS.

Define two first order differential operators

$$\mathcal{Y}(\cdot) = \sqrt{r^2+a^2}Y(\sqrt{r^2+a^2}\cdot), \qquad \mathcal{V}(\cdot) = -\sqrt{r^2+a^2}V(\sqrt{r^2+a^2}\cdot), \tag{A.1}$$

and the wave operators $\hat{\mathbf{L}}_s^0$ and $\hat{\mathbf{L}}_s^1$

$$\hat{\mathbf{L}}_s^1 = \Sigma\Box_g + 2is\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - s^2\left(\cot^2\theta + \tfrac{r^4-2Mr^3+6a^2Mr-a^4}{(r^2+a^2)^2}\right), \tag{A.2a}$$

$$\hat{\mathbf{L}}_s^0 = \Sigma\Box_g + 2is\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - s^2\left(\cot^2\theta + \tfrac{r^2+2Mr-a^2}{2(r^2+a^2)}\right). \tag{A.2b}$$

A.1. Spin-1

Define

$$\hat{\phi}_{+1}^0 = \psi_{[+1]}/(r^2+a^2), \qquad \hat{\phi}_{+1}^1 = \mathcal{Y}(\hat{\phi}_{+1}^0), \tag{A.3a}$$

$$\hat{\phi}_{-1}^0 = \Delta\psi_{[-1]}/(r^2+a^2), \qquad \hat{\phi}_{-1}^1 = \mathcal{V}(\hat{\phi}_{-1}^0). \tag{A.3b}$$

These variables will satisfy the following equations

$$\hat{\mathbf{L}}_{+1}^1\hat{\phi}_{+1}^0 = \tfrac{2(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}_{+1}^1 - \tfrac{4ar}{r^2+a^2}\partial_\phi\hat{\phi}_{+1}^0, \tag{A.4a}$$

$$\hat{\mathbf{L}}_{+1}^1\hat{\phi}_{+1}^1 = \tfrac{-2a^2(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}_{+1}^0 + \tfrac{2a(a^4-r^4)}{(r^2+a^2)^2}\partial_\phi\hat{\phi}_{+1}^0, \tag{A.4b}$$

and

$$\hat{\mathbf{L}}_{-1}^1\hat{\phi}_{-1}^0 = \tfrac{2(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}_{-1}^1 + \tfrac{4ar}{r^2+a^2}\partial_\phi\hat{\phi}_{-1}^0, \tag{A.5a}$$

$$\hat{\mathbf{L}}_{-1}^1\hat{\phi}_{-1}^1 = \tfrac{-2a^2(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}_{-1}^0 - \tfrac{2a(a^4-r^4)}{(r^2+a^2)^2}\partial_\phi\hat{\phi}_{-1}^0, \tag{A.5b}$$

respectively.

A.2. Spin-2

Define the variables

$$\hat{\phi}^0_{+2} = \psi_{[+2]}/(r^2+a^2)^2, \qquad \hat{\phi}^1_{+2} = \mathcal{Y}(\hat{\phi}^0_{+2}), \qquad \hat{\phi}^2_{+2} = \mathcal{Y}(\hat{\phi}^1_{+2}), \tag{A.6a}$$
$$\hat{\phi}^0_{-2} = \Delta^2\psi_{[-2]}/(r^2+a^2)^2, \qquad \hat{\phi}^1_{-2} = \mathcal{V}(\hat{\phi}^0_{-2}), \qquad \hat{\phi}^2_{-2} = \mathcal{V}(\hat{\phi}^1_{-2}). \tag{A.6b}$$

These variables will satisfy the following equations

$$\hat{\mathbf{L}}^0_{+2}\hat{\phi}^0_{+2} = \hat{F}^0_{+2} = \tfrac{4(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}^1_{+2} - \tfrac{8ar}{r^2+a^2}\partial_\phi\hat{\phi}^0_{+2}, \tag{A.7a}$$
$$\begin{aligned}\hat{\mathbf{L}}^1_{+2}\hat{\phi}^1_{+2} = \hat{F}^1_{+2} = {} & \tfrac{2(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}^2_{+2} - \tfrac{4ar}{r^2+a^2}\partial_\phi\hat{\phi}^1_{+2} \\ & + \tfrac{6r(Mr^3-a^2r^2-3Ma^2r-a^4)}{(r^2+a^2)^2}\hat{\phi}^0_{+2} + \tfrac{6a(a^2-r^2)}{r^2+a^2}\partial_\phi\hat{\phi}^0_{+2},\end{aligned} \tag{A.7b}$$
$$\begin{aligned}\hat{\mathbf{L}}^1_{+2}\hat{\phi}^2_{+2} = \hat{F}^2_{+2} = {} & -\tfrac{20a^2(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}^1_{+2} + \tfrac{8a(a^2-r^2)}{r^2+a^2}\partial_\phi\hat{\phi}^1_{+2} \\ & + \tfrac{6a^2(a^4+6a^2Mr-10Mr^3-r^4)}{(r^2+a^2)^2}\hat{\phi}^0_{+2} + \tfrac{24a^3r}{r^2+a^2}\partial_\phi\hat{\phi}^0_{+2},\end{aligned} \tag{A.7c}$$

and

$$\hat{\mathbf{L}}^0_{-2}\hat{\phi}^0_{-2} = \hat{F}^0_{-2} = \tfrac{4(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}^1_{-2} + \tfrac{8ar}{r^2+a^2}\partial_\phi\hat{\phi}^0_{-2}, \tag{A.8a}$$
$$\begin{aligned}\hat{\mathbf{L}}^1_{-2}\hat{\phi}^1_{-2} = \hat{F}^1_{-2} = {} & \tfrac{2(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}^2_{-2} + \tfrac{4ar}{r^2+a^2}\partial_\phi\hat{\phi}^1_{-2} \\ & + \tfrac{6r(Mr^3-a^2r^2-3a^2Mr-a^4)}{(r^2+a^2)^2}\hat{\phi}^0_{-2} - \tfrac{6a(a^2-r^2)}{r^2+a^2}\partial_\phi\hat{\phi}^0_{-2},\end{aligned} \tag{A.8b}$$
$$\begin{aligned}\hat{\mathbf{L}}^1_{-2}\hat{\phi}^2_{-2} = \hat{F}^2_{-2} = {} & -\tfrac{20a^2(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\hat{\phi}^1_{-2} - \tfrac{8a(a^2-r^2)}{r^2+a^2}\partial_\phi\hat{\phi}^1_{-2} \\ & + \tfrac{6a^2(a^4+6a^2Mr-10Mr^3-r^4)}{(r^2+a^2)^2}\hat{\phi}^0_{-2} - \tfrac{24a^3r}{r^2+a^2}\partial_\phi\hat{\phi}^0_{-2},\end{aligned} \tag{A.8c}$$

respectively.

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