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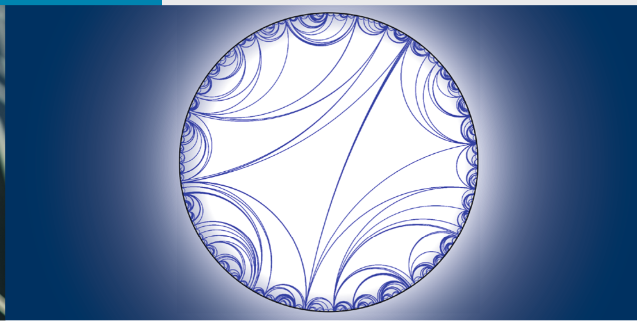

Anatolii V. Zhuchok

Relatively Free Doppelsemigroups

Lectures in Pure and Applied Mathematics | 5

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Preface

The main objective of these lecture notes is the study of variety of doppelsemigroups. A *variety of algebras* is a class of algebras closed under Cartesian products, homomorphic images, and subalgebras. Equivalently, a *variety* is a class of algebras defined by some family of identities. The study of varieties was initiated by Garrett Birkhoff in 1935, while the term "variety" was introduced by Philip Hall in 1949. Many structures in general algebra such as groups, semigroups, rings, Boolean algebras, dimonoids and etc. form a variety. *The free object in a variety over a set X* is an algebra in the variety generated by X and such that every mapping of X into any other algebra in the variety can be extended to a homomorphism of the free object into that algebra. One of the key problems that arise is the word problem for the variety. In order to study this problem, it is often useful to know the structure of the free object in the variety. Free objects have many interesting properties. The most deep results and problems of the variety theory are connected to the investigation of concrete varieties and construction of relatively free algebras.

In general, relatively free objects in any variety of algebras are important in the study of that variety and this has been true, particularly, in the study of doppelsemigroups. A doppelalgebra is an algebra defined on a vector space with two binary linear associative operations. Doppelalgebras play a prominent role in algebraic K-theory. We consider doppelsemigroups, that is, sets with two binary associative operations satisfying the axioms of a doppelalgebra. Doppelsemigroups are a generalization of semigroups and they related to such algebraic structures as duplexes, interassociative semigroups, restrictive bisemigroups, dimonoids, and trioids.

This book is devoted to the study of the structure of relatively free doppelsemigroups. The results form the variety theory of algebraic systems, develop the theory of interassociative semigroups and they can be applied to constructing relatively free doppelalgebras. The material is mainly based on the results obtained by the author in [44, 46, 48, 53]. The lecture notes offers promising results for future research in the variety theory of algebraic systems.

The lecture course is mainly oriented on students and Ph.D students, specialized in Algebra, and on specialists in the variety theory of algebras. A prerequisite needed for reading this book is knowledge of basic facts from semigroup theory and universal algebra. For a more extended reading in this direction we recommend [5, 6, 15, 19, 20, 30].

To simplify the understanding, results are given with the proofs. Rather, the exposition is focused on acquaintance with main ideas and approaches of the variety theory of algebraic systems. Standard technical details from universal algebra are used. After the main theorems, I give exercises for readers. At the end of each chapter, I try to give more historical information, more motivations and references for further development. The book will be very useful for persons beginning to study semigroup theory and universal algebra.

The contents of the book is the subject of a mini-course on universal algebra that I delivered at the University of Potsdam to Ph.D students in Fall 2017. I would like to express my sincere gratitude to PD Dr. Jörg Koppitz for helpful remarks which improved the presentation.

Contents

Chapter 1

Examples and the independence of axioms

In this chapter, we give numerous examples of doppelsemigroups and of strong doppelsemigroups, and establish the independence of axioms of a strong doppelsemigroup.

1.1 Examples of doppelsemigroups

In this section, we give examples of doppelsemigroups.

Let us start with a description of a problem we will study.

Let R be a class of universal algebras. It is well known that the free object in R always exists if R is a variety of universal algebras. The problem is to construct the free object for a given variety. Free algebras play an important role in the study of algebras, since every algebra is a homomorphic image of some free algebra. Therefore, we may acquire thorough knowledge of properties of every concrete algebra studying the properties of free algebras.

The main aim of the lecture notes is to present free objects in some varieties of algebras.

We begin with precise definitions.

Definition 1.1.1 ([48])**.** A *doppelsemigroup* is a nonempty set equipped with two binary operations $\dashv$ and $\vdash$ satisfying the axioms

$$(x \dashv y) \vdash z = x \dashv (y \vdash z), \tag{D1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{D2}$$

$$(x \dashv y) \dashv z = x \dashv (y \dashv z), \tag{D4}$$

$$(x \vdash y) \vdash z = x \vdash (y \vdash z). \tag{D5}$$

Definition 1.1.2 ([53])**.** A doppelsemigroup $(D, \dashv, \vdash)$ is called *strong* if it satisfies the axiom

$$x \dashv (y \vdash z) = x \vdash (y \dashv z). \tag{D3}$$

The class of all (strong) doppelsemigroups is a variety. It is natural to raise the problem of constructing doppelsemigroups which are free in the variety V of (strong) doppelsemigroups and in the subvarieties of V. We will solve this problem in the following chapters.

The following example shows relationships between doppelsemigroups and semigroups.

Example 1.1.3. Let $(D, \dashv, \vdash)$ be a doppelsemigroup. If the operations $\dashv$ and $\vdash$ of $(D, \dashv, \vdash)$ coincide, then the doppelsemigroup becomes a semigroup. Thus, every semigroup can be considered as a doppelsemigroup and doppelsemigroups are a generalization of semigroups.

Motivated by the problems of algebraic K-theory, J.-L. Loday introduced the notion of a dimonoid.

Definition 1.1.4 ([24])**.** A *dimonoid* is a nonempty set equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms (D2) and

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \qquad (x \dashv y) \vdash z = x \vdash (y \vdash z).$$

For an extensive information on dimonoids see [38, 40].

Definition 1.1.5 ([37])**.** A dimonoid is called *commutative* if both its operations are commutative.

The following statement establishes relationships between doppelsemigroups and dimonoids.

Proposition 1.1.6. *Every commutative dimonoid is a doppelsemigroup.*

So, the variety of commutative dimonoids is a subclass of the variety of doppelsemigroups. Examples of commutative dimonoids can be found in [37, 40, 41]. The connection between commutative dimonoids and doppelsemigroups was established in [49]. Free dimonoids were constructed in [24] and [42].

Exercise 1.1.7. Prove that every commutative dimonoid is a doppelsemigroup.

Remark 1.1.8. The considerations in this manuscript are restricted to doppelsemigroups; however, it is natural to study the problem of the construction of free objects for the variety of dimonoids (see, e.g., [52]).

For constructing new doppelsemigroups let us consider one class of semigroups.

Let S be a semigroup and $a \in S$. Define a new binary operation $\circ_a$ on S by $x \circ_a y = xay$ for all $x, y \in S$. Then $\circ_a$ is associative and hence $(S, \circ_a)$ is a semigroup [17]. The semigroup $(S, \circ_a)$ is called a *variant* of S, or, alternatively, the *sandwich semigroup of S with respect to the sandwich element a*, or the *semigroup with deformed multiplication*. The operation $\circ_a$ is usually called the *sandwich operation*.

Lemma 1.1.9. *Let S be a semigroup and let $a, b \in S$. Then $(S, \circ_a, \circ_b)$ is a doppelsemigroup.*

Proof. The proof follows by a direct verification. □

Exercise 1.1.10. Prove Lemma 1.1.9.

The notion of a left (right) translation plays an important role for the description of the structure of semigroups. Recall definitions.

Definition 1.1.11 ([6])**.** A transformation $\lambda(\rho)$ of a semigroup S is called a *left (right) translation* if $(xy)\lambda = x\lambda y$ $((xy)\rho = x(y\rho))$ for any $x, y \in S$.

Using one-sided translations, we can construct doppelsemigroups.

Proposition 1.1.12. *Let S be a semigroup and let λ_1, λ_2 be left translations of S, ρ_1, ρ_2 right translations of S and $\lambda_1\rho_1 = \rho_1\lambda_1$, $\lambda_2\rho_2 = \rho_2\lambda_2$, $\lambda_2\rho_1 = \rho_1\lambda_2$, $\lambda_1\rho_2 = \rho_2\lambda_1$. Then S with operations $\dashv$ and $\vdash$, defined by*

$$x \dashv y := x\rho_1(y\lambda_1) \quad \textit{and} \quad x \vdash y := x\rho_2(y\lambda_2)$$

for all $x, y \in S$, is a doppelsemigroup.

Proof. For any $x,y,z \in S$, using conditions of our proposition, we obtain

$$\begin{aligned}(x \dashv y) \dashv z &= (x\rho_1(y\lambda_1)) \dashv z = (x\rho_1(y\lambda_1))\rho_1(z\lambda_1)\\ &= x\rho_1(y\lambda_1\rho_1)(z\lambda_1) = x\rho_1(y\rho_1\lambda_1)(z\lambda_1)\\ &= x\rho_1(y\rho_1(z\lambda_1))\lambda_1 = x \dashv (y\rho_1(z\lambda_1)) = x \dashv (y \dashv z),\end{aligned}$$

$$\begin{aligned}(x \dashv y) \vdash z &= (x\rho_1(y\lambda_1)) \vdash z = (x\rho_1(y\lambda_1))\rho_2(z\lambda_2)\\ &= x\rho_1(y\lambda_1\rho_2)(z\lambda_2) = x\rho_1(y\rho_2\lambda_1)(z\lambda_2)\\ &= x\rho_1(y\rho_2(z\lambda_2))\lambda_1 = x \dashv (y\rho_2(z\lambda_2)) = x \dashv (y \vdash z),\end{aligned}$$

$$\begin{aligned}(x \vdash y) \dashv z &= (x\rho_2(y\lambda_2)) \dashv z = (x\rho_2(y\lambda_2))\rho_1(z\lambda_1)\\ &= x\rho_2(y\lambda_2\rho_1)(z\lambda_1) = x\rho_2(y\rho_1\lambda_2)(z\lambda_1)\\ &= x\rho_2(y\rho_1(z\lambda_1))\lambda_2 = x \vdash (y\rho_1(z\lambda_1)) = x \vdash (y \dashv z)\end{aligned}$$

and

$$\begin{aligned}(x \vdash y) \vdash z &= (x\rho_2(y\lambda_2)) \vdash z = (x\rho_2(y\lambda_2))\rho_2(z\lambda_2)\\ &= x\rho_2(y\lambda_2\rho_2)(z\lambda_2) = x\rho_2(y\rho_2\lambda_2)(z\lambda_2)\\ &= x\rho_2(y\rho_2(z\lambda_2))\lambda_2 = x \vdash (y\rho_2(z\lambda_2)) = x \vdash (y \vdash z).\end{aligned}$$

□

The doppelsemigroup obtained in Proposition 1.1.12 is denoted by $S^{\lambda_1,\lambda_2}_{\rho_1,\rho_2}$.

Let $\overline{D} = (D, \dashv, \vdash)$ be an arbitrary doppelsemigroup and let I, J be arbitrary nonempty sets for which the map

$$p : J \times I \to D : (j,i) \mapsto (j,i)p = p_{ji}$$

is defined. Consider operations $\dashv'$ and $\vdash'$ on $D' := I \times D \times J$ defined by

$$\begin{aligned}(i,a,j) \dashv' (k,b,t) &:= (i, a \dashv p_{jk} \dashv b, t),\\ (i,a,j) \vdash' (k,b,t) &:= (i, a \vdash p_{jk} \vdash b, t)\end{aligned}$$

for all $(i,a,j),(k,b,t) \in D'$. The algebra $(D', \dashv', \vdash')$ is denoted by $\mathrm{Dop}(I, \overline{D}, J; p)$.

Proposition 1.1.13. $\mathrm{Dop}(I,\overline{D},J;p)$ *is a doppelsemigroup.*

Proof. Let (i,a,j), (k,b,t), (l,c,m) be arbitrary elements of $\mathrm{Dop}(I,\overline{D},J;p)$. Then

$$\begin{aligned}((i,a,j)\dashv'(k,b,t))\dashv'(l,c,m) &= (i,a\dashv p_{jk}\dashv b,t)\dashv'(l,c,m)\\ &= (i,(a\dashv p_{jk}\dashv b)\dashv p_{tl}\dashv c,m) = (i,a\dashv p_{jk}\dashv(b\dashv p_{tl}\dashv c),m)\\ &= (i,a,j)\dashv'(k,b\dashv p_{tl}\dashv c,m) = (i,a,j)\dashv'((k,b,t)\dashv'(l,c,m))\end{aligned}$$

by the associativity of the operation $\dashv$. Moreover,

$$\begin{aligned}((i,a,j)\dashv'(k,b,t))\vdash'(l,c,m) &= (i,a\dashv p_{jk}\dashv b,t)\vdash'(l,c,m)\\ &= (i,(a\dashv p_{jk}\dashv b)\vdash p_{tl}\vdash c,m) = (i,((a\dashv p_{jk})\dashv b)\vdash(p_{tl}\vdash c),m)\\ &= (i,(a\dashv p_{jk})\dashv(b\vdash(p_{tl}\vdash c)),m) = (i,a,j)\dashv'(k,b\vdash p_{tl}\vdash c,m)\\ &= (i,a,j)\dashv'((k,b,t)\vdash'(l,c,m))\end{aligned}$$

according to the associativity of operations $\dashv$, $\vdash$ and the axiom (D1). Similarly, the associativity of $\vdash'$ and the axiom (D2) can be checked.

Thus, $\mathrm{Dop}(I,\overline{D},J;p)$ is a doppelsemigroup. □

Exercise 1.1.14. Prove that the associativity of $\vdash'$ and the axiom (D2) are satisfied in $\mathrm{Dop}(I,\overline{D},J;p)$.

Observe that if operations of a doppelsemigroup $\overline{D}$ coincide and it is a group G, then we obtain a Rees semigroup $\mathrm{Dop}(I,G,J;p)$ of the matrix type [6]. So, $\mathrm{Dop}(I,\overline{D},J;p)$ generalizes the semigroup $\mathrm{Dop}(I,G,J;p)$. The doppelsemigroup $\mathrm{Dop}(I,\overline{D},J;p)$ is called a *Rees doppelsemigroup* [48]. The Rees-Sushkevich theorem [6] states that a semigroup is completely 0-simple if and only if it is isomorphic to a Rees semigroup of the matrix type. In connection with this fact the following question naturally appears.

Open Problem 1.1.15. Obtain an analog of the Rees-Sushkevich theorem for semigroups in the class of doppelsemigroups.

Using sandwich operations, we obtain the following example of a doppelsemigroup.

Example 1.1.16. Let $(D,\dashv,\vdash)$ be a doppelsemigroup and $a,b\in D$. Define operations $\dashv_a$ and $\vdash_b$ on D by

$$x\dashv_a y := x\dashv a\dashv y \quad\text{and}\quad x\vdash_b y := x\vdash b\vdash y$$

for all $x,y \in D$. By a direct verification, $(D, \dashv_a, \vdash_b)$ is a doppelsemigroup.

The doppelsemigroup $(D, \dashv_a, \vdash_b)$ is called a *variant* of $(D, \dashv, \vdash)$, or, alternatively, the *sandwich doppelsemigroup of* $(D, \dashv, \vdash)$ *with respect to the sandwich elements a and b*, or the *doppelsemigroup with deformed multiplications* [46].

Example 1.1.17. The direct product $\prod_{i\in I} D_i$ of doppelsemigroups D_i, $i \in I$, is a doppelsemigroup since the class of all doppelsemigroups is a variety.

Definition 1.1.18. An element 0 of a doppelsemigroup $(D, \dashv, \vdash)$ is called *zero* if $x * 0 = 0 = 0 * x$ for all $x \in D$ and $* \in \{\dashv, \vdash\}$.

Now we give one class of doppelsemigroups with zero.

Let $\overline{D} = (D, \dashv, \vdash)$ be an arbitrary doppelsemigroup and let I be an arbitrary nonempty set. Define operations $\dashv'$ and $\vdash'$ on $D' := (I \times D \times I) \cup \{0\}$ by

$$(i,a,j) *' (k,b,t) := \begin{cases} (i, a*b, t) & \text{if } j = k, \\ 0 & \text{if } j \neq k \end{cases}$$

and

$$(i,a,j) *' 0 := 0 *' (i,a,j) := 0 *' 0 := 0$$

for all $(i,a,j), (k,b,t) \in D' \setminus \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra $(D', \dashv', \vdash')$ is denoted by $B(\overline{D}, I)$.

Proposition 1.1.19. *$B(\overline{D}, I)$ is a doppelsemigroup with zero.*

Proof. The proof is similar to the proof of Proposition 1 from [43]. □

Exercise 1.1.20. Prove Proposition 1.1.19.

Observe that if operations of a doppelsemigroup $\overline{D}$ coincide and it is a group G, then any Brandt semigroup [6] is isomorphic to some semigroup $B(G, I)$. So, $B(\overline{D}, I)$ generalizes the semigroup $B(G, I)$. The doppelsemigroup $B(\overline{D}, I)$ is called a *Brandt doppelsemigroup*.

1.2 Examples of strong doppelsemigroups

In this section, we consider examples of strong doppelsemigroups.

Example 1.2.1. Every semigroup can be considered as a strong doppelsemigroup and strong doppelsemigroups are a generalization of semigroups.

A collection of constructions of relatively free dimonoids was given in [52]. In [54], it was shown that any dimonoid is isomorphically embedded into some dimonoid constructed from a semigroup. The following example shows connections between strong doppelsemigroups and commutative dimonoids.

Example 1.2.2. By [37, Lemma 2], every commutative dimonoid is a strong doppelsemigroup. So, the variety of commutative dimonoids is a subclass of the variety of strong doppelsemigroups. Examples of commutative dimonoids can be found in [37, 40, 41].

Let $(D,\dashv,\vdash)$ be a doppelsemigroup and $a,b \in D$. Recall that the doppelsemigroup $(D,\dashv_a,\vdash_b)$ with operations, defined by

$$x \dashv_a y := x \dashv a \dashv y \quad \text{and} \quad x \vdash_b y := x \vdash b \vdash y$$

for all $x,y \in D$, is called a *variant of* $(D,\dashv,\vdash)$ (see Example 1.1.16).

Proposition 1.2.3. *Let $(D,\dashv,\vdash)$ be a commutative strong doppelsemigroup. Then any variant of $(D,\dashv,\vdash)$ is a commutative strong doppelsemigroup.*

Proof. The proof follows by a direct verification. □

Exercise 1.2.4. Prove Proposition 1.2.3.

The proof of the following proposition follows from the proof of Proposition 1.1.12.

Proposition 1.2.5. *Let S be a semigroup and let λ_1,λ_2 be left translations of S, ρ_1,ρ_2 right translations of S and $\lambda_1\rho_1 = \rho_1\lambda_1$, $\lambda_2\rho_2 = \rho_2\lambda_2$, $\lambda_2\rho_1 = \rho_1\lambda_2$, $\lambda_1\rho_2 = \rho_2\lambda_1$. Then the doppelsemigroup $S^{\lambda_1;\lambda_2}_{\rho_1;\rho_2}$ is strong if and only if*

$$x\rho_1(y\rho_2\lambda_1)(z\lambda_2) = x\rho_2(y\rho_1\lambda_2)(z\lambda_1)$$

for all $x,y,z \in S$.

Exercise 1.2.6. Prove Proposition 1.2.5.

As usual, we denote the set of all positive integers by $\mathbb{N}$. The following three lemmas are needed for the sequel, namely, for solving the problem of constructing free objects in the variety of doppelsemigroups.

Lemma 1.2.7. *In a doppelsemigroup* $(D, \dashv, \vdash)$, *for any* $1 < n \in \mathbb{N}$, *and any* $x_i \in D$, $1 \leqslant i \leqslant n+1$, *and* $*_j \in \{\dashv, \vdash\}$, $1 \leqslant j \leqslant n$, *any parenthesizing of*

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1}$$

gives the same element from D.

Proof. The proof follows from the associativity of operations of a doppelsemigroup and from its axioms. □

Exercise 1.2.8. Prove Lemma 1.2.7.

Lemma 1.2.9. *In a strong doppelsemigroup* $(D, \dashv, \vdash)$, *for any* $n \in \mathbb{N}$ *and any* $x_i \in D$ *with* $1 \leqslant i \leqslant n+1$, *and* $*_j \in \{\dashv, \vdash\}$ *with* $1 \leqslant j \leqslant n$,

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1} = x_1 *_{1\pi} x_2 *_{2\pi} \ldots *_{n\pi} x_{n+1},$$

where π *is a permutation of* $\{1, 2, \ldots, n\}$.

Proof. The proof follows from Lemma 1.2.7 and the axiom (D3) of a strong doppelsemigroup. □

Exercise 1.2.10. Prove Lemma 1.2.9.

Lemma 1.2.11. *In a strong doppelsemigroup* $(D, \dashv, \vdash)$, *for any* $k, n \in \mathbb{N}$, *and any* $x_i \in D$ *with* $1 \leqslant i \leqslant k+n$,

$$x_1 \vdash \ldots \vdash x_k \dashv x_{k+1} \dashv \ldots \dashv x_{k+n} = x_1 \dashv \ldots \dashv x_{n+1} \vdash \ldots \vdash x_{n+k}.$$

Proof. The proof follows from Lemma 1.2.9. □

Exercise 1.2.12. Prove Lemma 1.2.11.

It is natural to consider the question when a Rees doppelsemigroup is strong.

Let $\mathrm{Dop}(I,\overline{D},J;p)$ be a Rees doppelsemigroup defined in Section 1.1. If $\overline{D}$ is a strong doppelsemigroup, denote $\mathrm{Dop}(I,\overline{D},J;p)$ by $\mathrm{SDop}(I,\overline{D},J;p)$.

Proposition 1.2.13. $\mathrm{SDop}(I,\overline{D},J;p)$ *is a strong Rees doppelsemigroup.*

Proof. By Proposition 1.1.13, $\mathrm{SDop}(I,\overline{D},J;p)$ is a Rees doppelsemigroup.

Let (i,a,j), (k,b,t), (l,c,m) be arbitrary elements of $\mathrm{SDop}(I,\overline{D},J;p)$. Then

$$\begin{aligned}(i,a,j)\vdash'((k,b,t)\dashv'(l,c,m)) &= (i,a,j)\vdash'(k,b\dashv p_{tl}\dashv c,m)\\ &= (i,a\vdash p_{jk}\vdash(b\dashv p_{tl}\dashv c),m) = (i,a\dashv p_{jk}\dashv(b\vdash p_{tl}\vdash c),m)\\ &= (i,a,j)\dashv'(k,b\vdash p_{tl}\vdash c,m) = (i,a,j)\dashv'((k,b,t)\vdash'(l,c,m))\end{aligned}$$

according to Lemmas 1.2.7 and 1.2.11. Thus, $\mathrm{SDop}(I,\overline{D},J;p)$ is a strong doppelsemigroup. □

The construction of $\mathrm{SDop}(I,\overline{D},J;p)$ generalizes a Rees semigroup of the matrix type [6].

The concept of $\mathscr{P}$-related semigroups was introduced by Hewitt and Zuckerman. Let us recall the definition.

Definition 1.2.14 ([16])**.** Semigroups $(D,\dashv)$ and $(D,\vdash)$ are called *$\mathscr{P}$-related* if

$$x\dashv y\dashv z = x\vdash y\vdash z$$

for all $x,y,z\in D$.

By [37, Lemma 2], the semigroups $(D,\dashv)$ and $(D,\vdash)$ of a commutative dimonoid $(D,\dashv,\vdash)$ are $\mathscr{P}$-related.

Definition 1.2.15. If for a semigroup S, distinct a_1, a_2 do not exist such that $a_1x = a_2x$, $xa_1 = xa_2$ for all $x\in S$, then S is called a *weakly reductive semigroup*.

By [14, Theorem 2], two $\mathscr{P}$-related, weakly reductive semigroups defined on the same set are strongly interassociative, that is, they satisfy the axioms (D1)–(D3). So, obtain

Proposition 1.2.16. *If* $(D,\dashv)$ *and* $(D,\vdash)$ *are $\mathscr{P}$-related, weakly reductive semigroups, then* $(D,\dashv,\vdash)$ *is a strong doppelsemigroup.*

1.3 The independence of axioms of a strong doppelsemigroup

The independence of axioms of the given axiomatic theory plays an important role for constructing the theory. In this section, we prove the independence of axioms of a strong doppelsemigroup.

Definition 1.3.1. A system of axioms Σ is *independent* if any axiom α from Σ can not be deduced from the system of axioms $\Sigma\setminus\{\alpha\}$.

Lemma 1.3.2. *Let* $X := \{a,b,c,d,e,f,g,h\}$. *Define operations* $\dashv$ *and* $\vdash$ *on* X *by*

$$a\vdash a := c,$$
$$a\vdash b := a\dashv c := c\dashv a := d,$$
$$a\vdash c := c\vdash a := f,$$
$$b\vdash a := g, \quad a\dashv a := b,$$
$$a\dashv b := b\dashv a := e,$$
$$u\dashv v := u\vdash v := h \quad \textit{otherwise.}$$

The model $(X,\dashv,\vdash)$ *satisfies the axioms* (D2)–(D5) *but does not satisfy* (D1).

Proof. Let $x,y,z\in X$ with $\{x,y,z\}\neq\{a\}$. Then it is easy to verify that

$$(x\vdash y)\dashv z = x\vdash(y\dashv z) = h,$$
$$(x\dashv y)\dashv z = x\dashv(y\dashv z) = h,$$
$$(x\vdash y)\vdash z = x\vdash(y\vdash z) = h,$$
$$x\dashv(y\vdash z) = x\vdash(y\dashv z) = h.$$

So, it remains to check the cases where only a appears in the terms. We have

$$(a\vdash a)\dashv a = c\dashv a = d = a\vdash b = a\vdash(a\dashv a),$$
$$(a\dashv a)\dashv a = b\dashv a = e = a\dashv b = a\dashv(a\dashv a),$$
$$(a\vdash a)\vdash a = c\vdash a = f = a\vdash c = a\vdash(a\vdash a),$$
$$a\dashv(a\vdash a) = a\dashv c = d = a\vdash b = a\vdash(a\dashv a).$$

This shows that the axioms (D2)–(D5) are satisfied. But the axiom (D1) is not valid in $(X,\dashv,\vdash)$ since we have

$$a\dashv(a\vdash a)=a\dashv c=d\neq g=b\vdash a=(a\dashv a)\vdash a. \qquad \square$$

Lemma 1.3.3. *Let $X:=\{a,b,c,d,e,f,g,h\}$. Define operations $\dashv$ and $\vdash$ on X by*

$$a\vdash a:=c,$$
$$a\vdash b:=b\vdash a:=a\dashv c:=e,$$
$$a\vdash c:=c\vdash a:=f,$$
$$c\dashv a:=g,\quad a\dashv a:=b,$$
$$a\dashv b:=b\dashv a:=d,$$
$$u\dashv v:=u\vdash v:=h\quad \textit{otherwise.}$$

The model $(X,\dashv,\vdash)$ satisfies the axioms (D1), (D3)–(D5) *but does not satisfy* (D2).

Proof. The proof is similar to the proof of Lemma 1.3.2. □

Exercise 1.3.4. Prove Lemma 1.3.3.

Lemma 1.3.5. *Let $\mathbb{N}^0$ be the set of all positive integers with zero, and let*

$$x\dashv y:=2x,\quad z\dashv 0:=0:=0\dashv z\quad \textit{and}\quad z\vdash c:=0$$

for all $x,y\in\mathbb{N}$ and all $z,c\in\mathbb{N}^0$. The model $(\mathbb{N}^0,\dashv,\vdash)$ satisfies the axioms (D1)–(D3), (D5) *but does not satisfy* (D4).

Proof. Indeed, for all $z,c,a\in\mathbb{N}^0$,

$$(z\dashv c)\vdash a=0=z\dashv(c\vdash a),$$
$$(z\vdash c)\dashv a=0=z\vdash(c\dashv a),$$
$$z\dashv(c\vdash a)=0=z\vdash(c\dashv a),$$
$$(z\vdash c)\vdash a=0=z\vdash(c\vdash a).$$

In addition, for all $x, y, b \in \mathbb{N}$ we get

$$(x \dashv y) \dashv b = 2x \dashv b = 4x \neq 2x = x \dashv 2y = x \dashv (y \dashv b). \qquad \square$$

Lemma 1.3.6. *Let $\mathbb{N}^0$ be the set of all positive integers with zero. Put*

$$z \dashv c := 0, \quad x \vdash y := 2y \quad \textit{and} \quad z \vdash 0 := 0 := 0 \vdash z$$

for all $z, c \in \mathbb{N}^0$ and all $x, y \in \mathbb{N}$. The model $(\mathbb{N}^0, \dashv, \vdash)$ satisfies the axioms (D1)–(D4) *but does not satisfy* (D5).

Proof. The proof is similar to the proof of Lemma 1.3.5. $\square$

Exercise 1.3.7. Prove Lemma 1.3.6.

Lemma 1.3.8. *Any doppelsemigroup which is not strong satisfies the axioms* (D1)*,* (D2)*,* (D4)*,* (D5) *but does not satisfy* (D3).

Proof. Let $F[X]$ be the free semigroup on a set X and $a, b \in F[X]$. Consider variants $(F[X], \circ_a)$ and $(F[X], \circ_b)$ of the semigroup $F[X]$. By Lemma 1.1.9, $(F[X], \circ_a, \circ_b)$ is a doppelsemigroup. So, $(F[X], \circ_a, \circ_b)$ satisfies the axioms (D1), (D2), (D4), (D5). Moreover, for all $x, y, z \in F[X]$ we get

$$x \circ_a (y \circ_b z) = x \circ_a (ybz) = xaybz \neq xbyaz = x \circ_b (yaz) = x \circ_b (y \circ_a z).$$

Thus, the axiom (D3) is not valid in $(F[X], \circ_a, \circ_b)$. $\square$

From Lemmas 1.3.2, 1.3.3, 1.3.5, 1.3.6 and 1.3.8 we obtain

Theorem 1.3.9. *The system of axioms* (D1)–(D5) *is independent.*

The independence of axioms of a doppelsemigroup proved in [46] follows from the last theorem.

At the end of the chapter we give an information about interassociativity which is closely related to doppelsemigroups.

The term interassociativity was introduced by Zupnik in [55] for groupoids. Recall that two binary operations $\dashv$ and $\vdash$ on a nonempty set S are *interassociative* if the axiom

(D1) holds. This notion is useful in the investigation of functional equations for algebraic systems. Later, using the Zupnik's concept, Drouzy [7] defined interassociativity for semigroups. Namely, a semigroup $(D,\vdash)$ is called an *interassociate of a semigroup* $(D,\dashv)$ if the axioms (D1) and (D2) hold. So, a semigroup $(D,\vdash)$ is an interassociate of a semigroup $(D,\dashv)$ if and only if the algebra $(D,\dashv,\vdash)$ satisfies the following hyperidentity of associativity [28, 29]: $\Lambda(x,\Phi(y,z)) = \Phi(\Lambda(x,y),z)$. Such semigroups have been extensively studied over the past several decades. Some methods of constructing interassociates for semigroups were developed in [4]. Boyd and Gould [3] discussed the questions about an isomorphism between interassociative semigroups. More recently, attention has turned to the consideration of all interassociates of a monogenic semigroup [13], of the free commutative semigroup [8, 12] and of the free semigroup on the two-element alphabet [11]. Interassociates of bicyclic semigroups were studied in [9]. The construction of a variant of a semigroup was proposed in [23] and studied later on for various classes of semigroups by several authors, see, e.g., [17, 18, 27] and others. Variants of semigroups play a special role for constructing interassociative semigroups. This confirms the fact that two variants of a semigroup are interassociative. Moreover, if a semigroup $(D,\vdash)$ is an interassociate of a monoid $(D,\dashv)$, then $(D,\vdash)$ is a variant of $(D,\dashv)$ (see [28], p. 133; [29]). The main result of Drouzy's paper [7] follows from here: Two interassociative groups are isomorphic.

One of the kind of interassociativity is a strong interassociativity. This concept is of significance in studying $\mathscr{P}$-related semigroups [16]. Strong interassociativity for semigroups was introduced by Gould and Richardson [14]. Recall that a semigroup $(D,\vdash)$ is called a *strong interassociate of a semigroup* $(D,\dashv)$ if $(D,\dashv,\vdash)$ is a strong doppelsemigroup. The class of all strong doppelsemigroups forms a subvariety of the variety of doppelsemigroups. In particular, if $(D,\dashv,\vdash)$ is a commutative doppelsemigroup, that is, a doppelsemigroup with commutative operations, then a semigroup $(D,\vdash)$ is a strong interassociate of a semigroup $(D,\dashv)$ (see [48], Lemma 4.1). The constructions of the free strong doppelsemigroup, of the free n-dinilpotent strong doppelsemigroup, of the free commutative strong doppelsemigroup and of the free n-nilpotent strong doppelsemigroup will be presented in Chapter 3 (see also [53]).

Chapter 2

Structure of free doppelsemigroups

In this chapter, we introduce the construction of a free product in the class of universal algebras and present a free product in the variety of doppelsemigroups. As a consequence, we obtain a free doppelsemigroup. We also construct and study some relatively free doppelsemigroups and characterize the least congruences on a free doppelsemigroup.

2.1 Free products and free doppelsemigroups

In this section, we give the definition of a free product in the class of algebras. Then, using this definition, we construct a free product of doppelsemigroups and, as a consequence, obtain a free doppelsemigroup of an arbitrary rank. This result generalizes the construction of the free doppelsemigroup of rank 1 presented in [31, 32]. We also establish that the semigroups of the constructed free doppelsemigroup are isomorphic and the automorphism group of the free doppelsemigroup is isomorphic to the symmetric group.

Definition 2.1.1. Let R be a class of universal algebras A_β, $\beta \in \Omega$. A *free product in the class R of algebras A_β, $\beta \in \Omega$,* is an algebra A from the class R which contains all A_β as subalgebras and such that any family of homomorphisms of algebras A_β into any algebra B from R can be extended to a homomorphism of the algebra A into B.

It is well known (see, e.g., [21, 36]) that the free product in R always exists if R is a variety of universal algebras, and every free algebra is the free product of one-generated free algebras. The structure of one-generated free doppelsemigroups was described in [31, 32]. So, having the construction of the free product of doppelsemigroups and, using one-generated free doppelsemigroups, we can obtain a free doppelsemigroup of an arbitrary rank.

In order to construct a free product of doppelsemigroups we need to use a free product of semigroups. Recall the construction of a free product of semigroups.

Proposition 2.1.2. *Let $\{S_i\}_{i\in I}$ be a family of arbitrary pairwise disjoint semigroups S_i, $i\in I$, and* FR *the set of all such finite nonempty sequences $a_1a_2\dots a_k$ that if $a_j\in S_{i_j}$, $1\le j\le k$, then $i_j\ne i_{j+1}$, $1\le j\le k-1$. Define the operation $\bullet$ on* FR *by*

$$a_1a_2\dots a_k\bullet b_1b_2\dots b_s$$
$$:=\begin{cases} a_1a_2\dots a_kb_1b_2\dots b_s & \text{if } a_k\in S_i,\, b_1\in S_j,\, i\ne j,\\ a_1a_2\dots a_{k-1}(a_k\cdot b_1)b_2\dots b_s & \text{if } a_k,b_1\in S_i,\ \text{“}\cdot\text{” is the operation on } S_i,\, i\in I. \end{cases}$$

The set FR *with respect to this operation is a semigroup.*

This semigroup is called the *free product of semigroups S_i, $i\in I$.*

Exercise 2.1.3. Prove that $(\mathrm{FR},\bullet)$ is a semigroup.

Now we are ready to solve the problem of constructing a free product of doppelsemigroups.

Let $\mathrm{Fr}[S_i]_{i\in I}$ be the free product of arbitrary semigroups S_i, $i\in I$. For every $w\in \mathrm{Fr}[S_i]_{i\in I}$ denote the first (respectively, last) letter of w by $w^{(0)}$ (respectively, $w^{(1)}$) and the length of w by l_w.

Let $\{(D_i,\dashv_i,\vdash_i)\}_{i\in I}$ be a family of arbitrary pairwise disjoint doppelsemigroups, T the free monoid on the two-element set $\{a,b\}$ and $\theta\in T$ the empty word. The operations on $\mathrm{Fr}[(D_i,\dashv_i)]_{i\in I}$ and $\mathrm{Fr}[(D_i,\vdash_i)]_{i\in I}$ are denoted by $\dashv$ and $\vdash$, respectively. By definition, the length l_θ of θ is equal to 0. Define operations $\dashv'$ and $\vdash'$ on

$$\mathrm{Fr}:=\{(w,u)\in \mathrm{Fr}[(D_i,\dashv_i)]_{i\in I}\times T\mid l_w-l_u=1\}$$

by

$$(w_1,u_1) \dashv' (w_2,u_2) := \begin{cases} (w_1w_2, u_1 \circ_a u_2) & \text{if } w_1^{(1)} \in D_i,\, w_2^{(0)} \in D_j,\, i,j \in I, i \neq j, \\ (w_1 \dashv w_2, u_1u_2) & \text{if } w_1^{(1)}, w_2^{(0)} \in D_i,\, i \in I \end{cases}$$

and

$$(w_1,u_1) \vdash' (w_2,u_2) := \begin{cases} (w_1w_2, u_1 \circ_b u_2) & \text{if } w_1^{(1)} \in D_i,\, w_2^{(0)} \in D_j,\, i,j \in I,\, i \neq j, \\ (w_1 \vdash w_2, u_1u_2) & \text{if } w_1^{(1)}, w_2^{(0)} \in D_i,\, i \in I \end{cases}$$

for all $(w_1,u_1),(w_2,u_2) \in \mathrm{Fr}$. The obtained algebra is denoted by $\mathrm{FrD}(D_i)_{i\in I}$. Since universes of $\mathrm{Fr}[(D_i,\dashv_i)]_{i\in I}$ and $\mathrm{Fr}[(D_i,\vdash_i)]_{i\in I}$ are equal, we can use $\mathrm{Fr}[(D_i,\vdash_i)]_{i\in I}$ instead of $\mathrm{Fr}[(D_i,\dashv_i)]_{i\in I}$ in the definition of Fr.

If $s=1$, we will regard the sequence $y_1y_2\dots y_{s-1} \in T$ as θ.

The main result of this section is the following.

Theorem 2.1.4. $\mathrm{FrD}(D_i)_{i\in I}$ *is the free product of doppelsemigroups* $(D_i,\dashv_i,\vdash_i)$*,* $i \in I$*.*

Proof. Using the associativity of the operation of a free product of semigroups, of a free monoid and of sandwich operations $\circ_a$, $\circ_b$, one can directly check that $\mathrm{FrD}(D_i)_{i\in I}$ is a doppelsemigroup. For each $(D_i,\dashv_i,\vdash_i), i \in I$, we have

$$(D_i,\dashv_i,\vdash_i) \cong \overline{D}_i := \{(w,u) \in \mathrm{FrD}(D_i)_{i\in I} \mid w \in D_i\}$$

and all doppelsemigroups $\overline{D}_i, i \in I$, generate $\mathrm{FrD}(D_i)_{i\in I}$. Moreover, from the definition of $\mathrm{FrD}(D_i)_{i\in I}$ it follows that any its element has a unique representation in the form of the product of a finite number of different elements from $\cup_{i\in I}\overline{D}_i$.

In order to complete the proof we should check the condition of continuability of a homomorphism. For every $i \in I$ let

$$\alpha_i : (D_i,\dashv_i,\vdash_i) \to (K,\dashv'',\vdash'')$$

be a homomorphism of $(D_i,\dashv_i,\vdash_i)$ into an arbitrary doppelsemigroup $(K,\dashv'',\vdash'')$. Take $(z_{m_1}z_{m_2}\dots z_{m_s}, y_1y_2\dots y_{s-1}) \in \mathrm{FrD}(D_i)_{i\in I}$, where $z_{m_k} \in D_{m_k}$, $1 \leqslant k \leqslant s$, $y_p \in \{a,b\}$,

$1 \leqslant p \leqslant s-1$. Define a map

$$\alpha : \mathrm{FrD}(D_i)_{i\in I} \to (K, \dashv'', \vdash'')$$

by

$$\omega\alpha := \begin{cases} z_{m_1}\alpha_{m_1}\widetilde{y}_1 z_{m_2}\alpha_{m_2}\widetilde{y}_2 \ldots \widetilde{y}_{s-1} z_{m_s}\alpha_{m_s} & \text{if } \omega = (z_{m_1}z_{m_2}\ldots z_{m_s}, y_1y_2\ldots y_{s-1}), \\ & \quad s>1, \\ z_{m_1}\alpha_{m_1} & \text{if } \omega = (z_{m_1}, \theta), \end{cases}$$

where

$$\widetilde{y}_p := \begin{cases} \dashv'' & \text{if } y_p = a, \\ \vdash'' & \text{if } y_p = b \end{cases}$$

for all $1 \leqslant p \leqslant s-1$, $s>1$. According to Lemma 1.2.7 α is well-defined.

Using homomorphisms $\alpha_i, i \in I$, one can show that α is a homomorphism continuing $\alpha_i, i \in I$. Thus, $\mathrm{FrD}(D_i)_{i\in I}$ is the free product of doppelsemigroups $(D_i, \dashv_i, \vdash_i), i \in I$. □

Exercise 2.1.5. Show that the map α defined in the proof of the previous theorem is a homomorphism continuing $\alpha_i, i \in I$.

Remark 2.1.6. Consider separately the set $\mathrm{Fr}[(D_i, \dashv_i)]_{i\in I}$ with operations $\prec'$ and $\succ'$ defined by the rules

$$w_1 \prec' w_2 := \begin{cases} w_1w_2 & \text{if } w_1^{(1)} \in D_i,\, w_2^{(0)} \in D_j,\, i,j \in I,\, i \neq j, \\ w_1 \dashv w_2 & \text{if } w_1^{(1)}, w_2^{(0)} \in D_i,\, i \in I \end{cases}$$

and

$$w_1 \succ' w_2 := \begin{cases} w_1w_2 & \text{if } w_1^{(1)} \in D_i,\, w_2^{(0)} \in D_j,\, i,j \in I,\, i \neq j, \\ w_1 \vdash w_2 & \text{if } w_1^{(1)}, w_2^{(0)} \in D_i,\, i \in I \end{cases}$$

for all $w_1, w_2 \in \mathrm{Fr}[(D_i, \dashv_i)]_{i\in I}$. From the proof of Theorem 2.1.4 it follows that $(\mathrm{Fr}[(D_i, \dashv_i)]_{i\in I}, \prec', \succ')$ is a doppelsemigroup. This doppelsemigroup is generated by $\cup_{i\in I} D_i$ but it is not a free product of doppelsemigroups $(D_i, \dashv_i, \vdash_i)$, $i \in I$. Indeed, any element $xy \in \mathrm{Fr}[(D_i, \dashv_i)]_{i\in I}$, where $x \in D_i$, $y \in D_j$, $i \neq j$, can be written as $x \prec' y$ and $x \succ' y$ that contradicts the uniqueness of representation of elements of the free product.

Further we are going to show that the constructions of a free product of doppelsemigroups and of a free product of dimonoids are different. Let us recall the construction of a free product of dimonoids [47].

Let $\mathrm{Fr}[S_i]_{i\in X}$ be the free product of arbitrary semigroups S_i, $i\in X$. Consider the set

$$G(S_i)_{i\in X} := \{(w,m)\in \mathrm{Fr}[S_i]_{i\in X}\times\mathbb{N}\,|\,l_w\geq m\}.$$

For all $(w,m)\in G(S_i)_{i\in X}$ and $u\in \mathrm{Fr}[S_i]_{i\in X}$ let

$$f^u_{(w,m)} := \begin{cases} l_u+m & \text{if } l_{u^{(1)}w^{(0)}}=2,\\ l_u+m-1 & \text{if } l_{u^{(1)}w^{(0)}}=1.\end{cases}$$

For a given relation ρ on a dimonoid $(D,\dashv,\vdash)$ (see Definition 1.1.4), the *congruence generated by* ρ is the least congruence on $(D,\dashv,\vdash)$ containing ρ. It is denoted by $\rho^\star$ and can be characterized as the intersection of all congruences on $(D,\dashv,\vdash)$ containing ρ.

Let $\{(D_i,\dashv_i,\vdash_i)\}_{i\in X}$ be a family of arbitrary pairwise disjoint dimonoids. The operations on $\mathrm{Fr}[(D_i,\dashv_i)]_{i\in X}$ and $\mathrm{Fr}[(D_i,\vdash_i)]_{i\in X}$ are denoted by $\dashv$ and $\vdash$, respectively. For every $i\in X$ consider a relation

$$\theta_i = \{(a\vdash_i b, a\dashv_i b)\mid a,b\in D_i\}$$

on a dimonoid $(D_i,\dashv_i,\vdash_i)$. It is clear that operations of $(D_i,\dashv_i,\vdash_i)/\theta_i^\star$ coincide and it is a semigroup.

Let $\omega_1=(x_1x_2\ldots x_k\ldots x_s,t)$, $\omega_2=(y_1y_2\ldots y_k\ldots y_p,r)\in G((D_i,\dashv_i))_{i\in X}$, where

$$x_1,x_2,\ldots,x_k,\ldots,x_s,y_1,y_2,\ldots,y_k,\ldots,y_p\in\bigcup_{i\in X}D_i.$$

Define a relation $\sim$ on $G((D_i,\dashv_i))_{i\in X}$ by

$$\omega_1\sim\omega_2$$

if and only if

$$s=p,\quad t=r,\quad x_k\theta^\star_{j_k}y_k \text{ for all } 1\leq k\leq s \text{ and some } j_k\in X,\quad\text{and}\quad x_t=y_r.$$

It is not hard to check that $\sim$ is an equivalence relation. Denote the equivalence class of $\sim$ containing an element $(w,m) \in G((D_i, \dashv_i))_{i\in X}$ by $[w,m]$ and the quotient set $G((D_i, \dashv_i))_{i\in X}/\sim$ by $G^{\star}((D_i, \dashv_i))_{i\in X}$.

Define operations $\dashv'$ and $\vdash'$ on $G^{\star}((D_i, \dashv_i))_{i\in X}$ by

$$[w_1,m_1] \dashv' [w_2,m_2] := [w_1 \dashv w_2, m_1],$$
$$[w_1,m_1] \vdash' [w_2,m_2] := [w_1 \vdash w_2, f^{w_1}_{(w_2,m_2)}]$$

for all $[w_1,m_1],[w_2,m_2] \in G^{\star}((D_i, \dashv_i))_{i\in X}$. The algebra $(G^{\star}((D_i, \dashv_i))_{i\in X}, \dashv', \vdash')$ is denoted by $\breve{G}(D_i)_{i\in X}$.

Theorem 2.1.7 ([47], Theorem 2.3). *$\breve{G}(D_i)_{i\in X}$ is the free product of dimonoids $(D_i, \dashv_i, \vdash_i)$, $i \in X$.*

Remark 2.1.8. We constructed the free product of doppelsemigroups using the Cartesian product of the free product of semigroups and the free monoid on the two-element set. In contrast, the free product of dimonoids was constructed from a free product of semigroups and the set of all positive integers, and using some factorization. Since the axioms of a dimonoid and of a doppelsemigroup are different, the details of the constructions are also different.

Our next task is to obtain a free object in the variety of doppelsemigroups. Recall the construction of a free semigroup.

Let X be an arbitrary nonempty set. Let us denote by $F[X]$ the set of all nonempty finite words $a_1a_2\ldots a_m$ in the alphabet X. A binary operation is defined on $F[X]$ by juxtaposition:

$$(a_1a_2\ldots a_m)(b_1b_2\ldots b_n) := a_1a_2\ldots a_mb_1b_2\ldots b_n.$$

With respect to this operation $F[X]$ is a semigroup, called the *free semigroup* on X. The set X is called the *generating set* of $F[X]$.

Let $\mathrm{F}[X]$ be the free semigroup on X. Define operations $\prec$ and $\succ$ on

$$F := \{(w,u) \in \mathrm{F}[X] \times T \mid l_w - l_u = 1\}$$

by

$$(w_1,u_1) \prec (w_2,u_2) := (w_1w_2, u_1 \circ_a u_2), \tag{2.1}$$

$$(w_1,u_1) \succ (w_2,u_2) := (w_1w_2, u_1 \circ_b u_2) \tag{2.2}$$

for all $(w_1,u_1),(w_2,u_2)\in F$. The obtained algebra is denoted by $\mathrm{FDS}(X)$.

Theorem 2.1.9. $\mathrm{FDS}(X)$ *is the free doppelsemigroup.*

Proof. Let $\{(D_i,\dashv_i,\vdash_i)\}_{i\in I}$ be a family of pairwise disjoint one-generated free doppelsemigroups. It is known [31, 32] that for each $i\in I$, $(D_i,\dashv_i,\vdash_i)\cong(T,\circ_a,\circ_b)$. Since any free algebra is the free product of one-generated free algebras, we obtain that the free doppelsemigroup of rank n is the free product $\mathrm{FrD}(D_i)_{i\in I}$ of doppelsemigroups $(D_i,\dashv_i,\vdash_i)$, $i\in I$, where $|I|=n$.

It is immediate to check that $\mathrm{FDS}(X)$ is a doppelsemigroup. Let us show that $\mathrm{FrD}(D_i)_{i\in I}\cong\mathrm{FDS}(X)$ if $|I|=|X|$.

If $X=\{r\}$, one easily proves that the map

$$(T,\circ_a,\circ_b)\to\mathrm{FDS}(X): u\mapsto(r^{l_u+1},u)$$

is an isomorphism. Thus, for each $i\in I$ and $|X|=1$, $(D_i,\dashv_i,\vdash_i)\cong\mathrm{FDS}(X)$. Further for every $x\in X$ assume $\Gamma_x=\mathrm{FDS}(\{x\})$.

Consider the free product of doppelsemigroups $\Gamma_x, x\in X$, and for convenience, denote it by $(\Gamma_x)_{x\in X}$. By the construction of the free product from Theorem 2.1.4, elements of $(\Gamma_x)_{x\in X}$ have the form

$$((x_1^{k_1},u_1)(x_2^{k_2},u_2)\dots(x_s^{k_s},u_s),y_1y_2\dots y_{s-1}),$$

where $x_j\in X$, $k_j\in\mathbb{N}$, $u_j\in T$, $1\leqslant j\leqslant s$, $y_p\in\{a,b\}$, $1\leqslant p\leqslant s-1$ and $x_j\neq x_{j+1}$ for $1\leqslant j\leqslant s-1$. Define a map $\varpi:(\Gamma_x)_{x\in X}\to\mathrm{FDS}(X)$ by the rule

$$((x_1^{k_1},u_1)(x_2^{k_2},u_2)\dots(x_s^{k_s},u_s),y_1y_2\dots y_{s-1})\mapsto(x_1^{k_1}x_2^{k_2}\dots x_s^{k_s},u_1y_1u_2y_2\dots y_{s-1}u_s).$$

An immediate verification shows that ϖ is an isomorphism. So, $\mathrm{FDS}(X)$ is the free doppelsemigroup. □

This result generalizes the construction of the free doppelsemigroup of rank 1 presented in [31, 32].

Exercise 2.1.10. Show that the map ϖ defined in the proof of the previous theorem is an isomorphism.

Remark 2.1.11. Theorem 2.1.9 can be also proved by the following standard method. Firstly, we can prove that $\mathrm{FDS}(X)$ is a doppelsemigroup generated by $X \times \{\theta\}$ and, secondly, that every map of $X \times \{\theta\}$ into any other doppelsemigroup can be uniquely extended to a homomorphism of $\mathrm{FDS}(X)$ into that doppelsemigroup.

The following lemma establishes a relationship between both semigroups of the free doppelsemigroup $\mathrm{FDS}(X)$ (see also [8]).

Lemma 2.1.12. *The semigroups* $(F, \prec)$ *and* $(F, \succ)$ *are isomorphic.*

Exercise 2.1.13. Show that the semigroups $(F, \prec)$ and $(F, \succ)$ are isomorphic.

Denote the symmetric group on X by $\mathfrak{I}[X]$ and the automorphism group of a doppelsemigroup D' by $\mathrm{Aut}\, D'$. Since the set $X \times \{\theta\}$ is generating for $\mathrm{FDS}(X)$, we obtain the following description of the automorphism group of the free doppelsemigroup.

Lemma 2.1.14. $\mathrm{Aut}\,\mathrm{FDS}(X) \cong \mathfrak{I}[X]$.

Exercise 2.1.15. Prove Lemma 2.1.14.

Free doppelsemigroups play an important role in studying interassociativity for semigroups since every doppelsemigroup is a homomorphic image of the free doppelsemigroup. Therefore, we may obtain new pairs of interassociative semigroups via constructing congruences on free doppelsemigroups.

2.2 Free commutative and free n-nilpotent doppelsemigroups

In this section, we consider doppelsemigroups with both commutative operations and study the nilpotency in doppelsemigroups. As a result, we construct a free doppelsemigroup in the variety of commutative (n-nilpotent) doppelsemigroups. We also characterize some least congruences on a free doppelsemigroup, study relationships between the semigroups of the constructed free algebras and describe the automorphism groups of the free algebras.

We start with the definition of the free commutative semigroup.

Let $X = \{w_1, w_2, \ldots, w_n\}$ be a finite set. Let us denote by $\mathrm{F}^*[X]$ the set of all nonempty finite words $w = w_1^{\alpha_1} w_2^{\alpha_2} \ldots w_n^{\alpha_n}$, where α_1, α_2, ..., $\alpha_n \in \mathbb{N} \cup \{0\}$ are not simultaneously equal to zero. Here $w_i^0, 1 \leq i \leq n$, is the empty word and $w^1 = w$ for all $w \in X$. A binary operation is defined on $\mathrm{F}^*[X]$ by the rule

$$(w_1^{\alpha_1} w_2^{\alpha_2} ... w_n^{\alpha_n})(w_1^{\beta_1} w_2^{\beta_2} ... w_n^{\beta_n}) := w_1^{\alpha_1+\beta_1} w_2^{\alpha_2+\beta_2} ... w_n^{\alpha_n+\beta_n}.$$

With respect to this operation $\mathrm{F}^*[X]$ is a semigroup, called the *free commutative semigroup* on X. The set X is called the *generating set* of $\mathrm{F}^*[X]$. Similarly, the free commutative semigroup generated by an infinite set is defined.

The free commutative semigroup will be useful for constructing a free object in the variety of commutative doppelsemigroups.

Definition 2.2.1. A doppelsemigroup $(D, \dashv, \vdash)$ is called *commutative* if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative.

Definition 2.2.2. A doppelsemigroup $(D, \dashv, \vdash)$ with zero 0 is called *nilpotent* if for some $n \in \mathbb{N}$ and any $x_i \in D$, $1 \leqslant i \leqslant n+1$, and $*_j \in \{\dashv, \vdash\}$, $1 \leqslant j \leqslant n$,

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1} = 0.$$

The least such n is called the *nilpotency index* of $(D, \dashv, \vdash)$. For $k \in \mathbb{N}$ a nilpotent doppelsemigroup of nilpotency index $\leqslant k$ is called *k-nilpotent*.

It is obvious that operations of any 1-nilpotent doppelsemigroup coincide and it is a zero semigroup.

Observe that the class of all commutative (n-nilpotent) doppelsemigroups forms a subvariety of the variety of doppelsemigroups.

Definition 2.2.3. A doppelsemigroup which is free in the variety of commutative (n-nilpotent) doppelsemigroups is called a *free commutative (n-nilpotent) doppelsemigroup*.

Our problem is to construct a free commutative (n-nilpotent) doppelsemigroup. The following two lemmas are needed for the sequel.

Lemma 2.2.4. *In a commutative doppelsemigroup* $(D, \dashv, \vdash)$*,*

$$(x \vdash y) \dashv z = x \vdash (y \dashv z) = (x \dashv y) \vdash z = x \dashv (y \vdash z)$$

for all $x, y, z \in D$.

Proof. For all $x, y, z \in D$, we have

$$\begin{aligned}(x \vdash y) \dashv z = z \dashv (x \vdash y) &= (z \dashv x) \vdash y \\ &= (x \dashv z) \vdash y = x \dashv (z \vdash y) = x \dashv (y \vdash z)\end{aligned}$$

by the commutativity of $\dashv$, $\vdash$ and the axiom (D1). □

Lemma 2.2.5. *In a commutative doppelsemigroup* $(D, \dashv, \vdash)$*, for any* $n \in \mathbb{N}$ *and any* $x_i \in D$*,* $1 \leqslant i \leqslant n+1$*, and* $*_j \in \{\dashv, \vdash\}$*,* $1 \leqslant j \leqslant n$*,*

$$\begin{aligned}x_1 *_1 x_2 *_2 \ldots *_n x_{n+1} &= x_{1\pi} *_1 x_{2\pi} *_2 \ldots *_n x_{(n+1)\pi} \\ &= x_1 *_{1\pi'} x_2 *_{2\pi'} \ldots *_{n\pi'} x_{n+1},\end{aligned}$$

where π*,* π' *are permutations of* $\{1, 2, \ldots, n+1\}$ *and* $\{1, 2, \ldots, n\}$*, respectively.*

Exercise 2.2.6. Using Lemmas 1.2.7, 2.2.4 and the commutativity of $\dashv$, $\vdash$, prove Lemma 2.2.5.

In the construction of $\mathrm{FDS}(X)$ (see Section 2.1) instead of the free semigroup $\mathrm{F}[X]$ on X take the free commutative semigroup $\mathrm{F}^*[X]$ on X and instead of the free monoid T on $\{a, b\}$ take the free commutative monoid T^* on $\{a, b\}$ with the empty word θ. In this case, denote by $\mathrm{FDS}^*(X)$ the algebra $(F, \prec, \succ)$ with operations defined by (2.1), (2.2).

We will write the sequence $y_1 y_2 \ldots y_{s-1} \in T^*$ as θ if $s = 1$.

Theorem 2.2.7. $\mathrm{FDS}^*(X)$ *is the free commutative doppelsemigroup.*

Proof. The fact that $\mathrm{FDS}(X)$ is a doppelsemigroup implies that $\mathrm{FDS}^*(X)$ is a doppelsemigroup too. Obviously, $\mathrm{FDS}^*(X)$ is commutative. It is clear from the definition of $\mathrm{FDS}^*(X)$ that it is generated by $X \times \{\theta\}$.

Now let ρ' be a map of $X \times \{\theta\}$ into an arbitrary commutative doppelsemigroup $(K, \dashv, \vdash)$. We wish to show that there exists a unique homomorphism χ extending ρ'.

Consider a map $\rho : X \to K$ such that $x\rho = (x, \theta)\rho'$ for all $x \in X$. Take

$$(x_1 x_2 \ldots x_s, y_1 y_2 \ldots y_{s-1}) \in \mathrm{FDS}^*(X),$$

where $x_j \in X$, $1 \leqslant j \leqslant s$, $y_p \in \{a,b\}$, $1 \leqslant p \leqslant s-1$. We now define a map

$$\chi : \mathrm{FDS}^*(X) \to (K, \dashv, \vdash)$$

by

$$\omega\chi := \begin{cases} x_1\rho\widetilde{y}_1 x_2\rho\widetilde{y}_2\ldots\widetilde{y}_{s-1}x_s\rho & \text{if } \omega = (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}),\ s>1, \\ x_1\rho & \text{if } \omega = (x_1,\theta), \end{cases}$$

where

$$\widetilde{y}_p := \begin{cases} \dashv & \text{if } y_p = a, \\ \vdash & \text{if } y_p = b \end{cases}$$

for all $1 \leqslant p \leqslant s-1$, $s>1$. According to Lemmas 1.2.7 and 2.2.5 χ is well-defined.

One can show that χ is a homomorphism. It is clear that $(x,\theta)\chi = (x,\theta)\rho'$ for all $(x,\theta) \in X \times \{\theta\}$. Since $X \times \{\theta\}$ generates $\mathrm{FDS}^*(X)$, we obtain that χ is unique. Thus, $\mathrm{FDS}^*(X)$ is the free commutative doppelsemigroup. □

Exercise 2.2.8. Show that the map χ defined in the proof of Theorem 2.2.7 is a homomorphism.

Theorem 2.2.7 gives us an idea that one-generated free commutative doppelsemigroups have also other presentation. Thus, from Theorem 2.2.7 we obtain

Corollary 2.2.9. $(T^*, \circ_a, \circ_b)$ *is the free commutative doppelsemigroup of rank* 1.

Further we recall the well-known construction of the free n-nilpotent semigroup.

Fix $n \in \mathbb{N}$. Let X be an arbitrary nonempty set and FN the set of all nonempty finite words $w_1w_2\ldots w_m$, where $w_1, w_2, \ldots, w_m \in X$ and $m \leq n$. A binary operation is defined on $\mathrm{FN}\cup\{0\}$ by the rule

$$(w_1w_2\ldots w_m)(\omega_1\omega_2\ldots\omega_k) := \begin{cases} w_1w_2\ldots w_m\omega_1\omega_2\ldots\omega_k & \text{if } m+k \leqslant n, \\ 0 & \text{if } m+k > n. \end{cases}$$

With respect to this operation $\mathrm{FN}\cup\{0\}$ is a semigroup, called the *free n-nilpotent semigroup* on X. The set X is called the *generating set* of $\mathrm{FN}\cup\{0\}$.

The problem is to extend this construction to the case of doppelsemigroups.

Let $F_n := \{(w,u) \in \mathrm{FDS}(X) \mid l_w \leqslant n\} \cup \{0\}$. Define operations $\dashv$ and $\vdash$ on F_n by

$$(w_1,u_1) \dashv (w_2,u_2) := \begin{cases} (w_1w_2, u_1 \circ_a u_2) & \text{if } l_{w_1w_2} \leqslant n, \\ 0 & \text{if } l_{w_1w_2} > n, \end{cases}$$

$$(w_1,u_1) \vdash (w_2,u_2) := \begin{cases} (w_1w_2, u_1 \circ_b u_2) & \text{if } l_{w_1w_2} \leqslant n, \\ 0 & \text{if } l_{w_1w_2} > n \end{cases}$$

and

$$(w_1,u_1) * 0 := 0 * (w_1,u_1) := 0 * 0 := 0$$

for all $(w_1,u_1),(w_2,u_2) \in F_n \setminus \{0\}$ and $* \in \{\dashv,\vdash\}$. The algebra $(F_n,\dashv,\vdash)$ is denoted by $\mathrm{FNDS}_n(X)$.

Theorem 2.2.10. $\mathrm{FNDS}_n(X)$ *is the free n-nilpotent doppelsemigroup.*

Proof. Similarly to Theorem 1 from [43], the fact that $\mathrm{FNDS}_n(X)$ is an n-nilpotent doppelsemigroup can be proved.

Let us show that $\mathrm{FNDS}_n(X)$ is free in the variety of n-nilpotent doppelsemigroups.

Let $(K,\dashv',\vdash')$ be an arbitrary n-nilpotent doppelsemigroup and $\beta : X \to K$ an arbitrary map. Define a map

$$\mu : \mathrm{FNDS}_n(X) \to (K,\dashv',\vdash') : \omega \mapsto \omega\mu$$

as

$$\omega\mu := \begin{cases} x_1\beta\widetilde{y}_1x_2\beta\widetilde{y}_2\ldots\widetilde{y}_{s-1}x_s\beta & \text{if } \omega = (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}), \\ & \quad x_j \in X, 1 \leqslant j \leqslant s, y_p \in \{a,b\}, \\ & \quad 1 \leqslant p \leqslant s-1, s > 1, \\ x_1\beta & \text{if } \omega = (x_1,\theta), x_1 \in X, \\ 0 & \text{if } \omega = 0, \end{cases}$$

where

$$\widetilde{y}_p := \begin{cases} \dashv' & \text{if } y_p = a, \\ \vdash' & \text{if } y_p = b \end{cases}$$

for all $1 \leqslant p \leqslant s-1$, $s > 1$. According to Lemma 1.2.7 μ is well-defined.

A direct verification shows that μ is a homomorphism. □

Exercise 2.2.11. Show that the map μ defined in the proof of Theorem 2.2.10 is a homomorphism.

The following problem is to find a more elegant version for the free n-nilpotent doppelsemigroup of rank 1. Now we construct a doppelsemigroup which is isomorphic to the free n-nilpotent doppelsemigroup of rank 1.

Fix $n \in \mathbb{N}$ and assume $\overline{T}_n := \{u \in T \mid l_u + 1 \leqslant n\} \cup \{0\}$. Define operations $\dashv$ and $\vdash$ on $\overline{T}_n$ by

$$u_1 \dashv u_2 := \begin{cases} u_1 \circ_a u_2 & \text{if } l_{u_1 u_2} + 2 \leqslant n, \\ 0 & \text{if } l_{u_1 u_2} + 2 > n, \end{cases}$$

$$u_1 \vdash u_2 := \begin{cases} u_1 \circ_b u_2 & \text{if } l_{u_1 u_2} + 2 \leqslant n, \\ 0 & \text{if } l_{u_1 u_2} + 2 > n \end{cases}$$

and

$$u_1 * 0 := 0 * u_1 := 0 * 0 := 0$$

for all $u_1, u_2 \in \overline{T}_n \setminus \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra $(\overline{T}_n, \dashv, \vdash)$ is denoted by T_n. Obviously, T_n is a doppelsemigroup.

Exercise 2.2.12. Prove that T_n is a doppelsemigroup.

Lemma 2.2.13. *If* $|X| = 1$*, then* $T_n \cong \mathrm{FNDS}_n(X)$.

Proof. Let $X = \{r\}$. An easy verification shows that a map

$$\delta : T_n \to \mathrm{FNDS}_n(X),$$

defined by

$$u\delta := \begin{cases} (r^{l_u+1}, u) & \text{if } u \in \overline{T}_n \setminus \{0\}, \\ 0 & \text{if } u = 0, \end{cases}$$

is an isomorphism. □

Exercise 2.2.14. Show that T_n and $\mathrm{FNDS}_n(X)$ are isomorphic if $|X| = 1$.

Let us discuss a relation between semigroups of the constructed free algebras and automorphisms of the free algebras. The following two lemmas could be proved immediately.

Lemma 2.2.15. *The semigroups* $(F,\prec)$ *and* $(F,\succ)$ $($*respectively,* $(F_n,\dashv)$ *and* $(F_n,\vdash))$ *of* $\mathrm{FDS}^*(X)$ $($*respectively,* $\mathrm{FNDS}_n(X))$ *are isomorphic.*

Exercise 2.2.16. Prove that the semigroups $(F,\prec)$ and $(F,\succ)$ as well $(F_n,\dashv)$ and $(F_n,\vdash)$ indicated in the previous lemma are isomorphic.

Lemma 2.2.17. $\mathrm{Aut}\,\mathrm{FDS}^*(X) \cong \mathrm{Aut}\,\mathrm{FNDS}_n(X) \cong \mathfrak{I}[X]$.

Exercise 2.2.18. Show that the groups from Lemma 2.2.17 are isomorphic.

If $f : D_1 \to D_2$ is a homomorphism of doppelsemigroups, the corresponding congruence on D_1 is denoted by Δ_f. If ρ is a congruence on a doppelsemigroup $(D,\dashv,\vdash)$ such that $(D,\dashv,\vdash)/\rho$ is a commutative (n-nilpotent) doppelsemigroup, we say that ρ is a *commutative (n-nilpotent) congruence*. By $\star$ ($\circ$, respectively) denote the operation on $\mathrm{F}^*[X]$ (on T^*, respectively).

The least congruences on a free algebra play an important role for the description of all congruences on this algebra. Now we present the least commutative (n-nilpotent) congruence on a free doppelsemigroup.

Take $(x_1x_2\dots x_s, y_1y_2\dots y_{s-1}), (a_1a_2\dots a_h, b_1b_2\dots b_{h-1}) \in \mathrm{FDS}(X)$, where $x_j \in X$, $1 \leqslant j \leqslant s$, $y_p \in \{a,b\}$, $1 \leqslant p \leqslant s-1$, $a_q \in X$, $1 \leqslant q \leqslant h$, $b_d \in \{a,b\}$, $1 \leqslant d \leqslant h-1$, and define a relation ξ on $\mathrm{FDS}(X)$ by

$$(x_1x_2\dots x_s, y_1y_2\dots y_{s-1})\,\xi\,(a_1a_2\dots a_h, b_1b_2\dots b_{h-1})$$

if and only if

$$(x_1\star x_2\star\dots\star x_s, y_1\circ y_2\circ\dots\circ y_{s-1}) = (a_1\star a_2\star\dots\star a_h, b_1\circ b_2\circ\dots\circ b_{h-1}).$$

Define a relation ζ_n on $\mathrm{FDS}(X)$ by

$$(w_1,u_1)\,\zeta_n\,(w_2,u_2)$$

if and only if

$$(w_1,u_1) = (w_2,u_2) \quad \text{or} \quad l_{w_1} > n,\, l_{w_2} > n.$$

Theorem 2.2.19. *The relation* ξ (ζ_n) *is the least commutative (n-nilpotent) congruence on the free doppelsemigroup* $\mathrm{FDS}(X)$.

Proof. Define a map $\tau : \mathrm{FDS}(X) \to \mathrm{FDS}^*(X)$ by

$$\omega\tau := \begin{cases} (x_1 \star x_2 \star \ldots \star x_s, y_1 \circ y_2 \circ \ldots \circ y_{s-1}) & \text{if } \omega = (x_1 x_2 \ldots x_s, y_1 y_2 \ldots y_{s-1}),\ s > 1, \\ \omega & \text{otherwise.} \end{cases}$$

It is immediate to show that τ is a surjective homomorphism. By Theorem 2.2.7, $\mathrm{FDS}^*(X)$ is the free commutative doppelsemigroup. Then Δ_τ is the least commutative congruence on $\mathrm{FDS}(X)$. From the definition of τ it follows that $\Delta_\tau = \xi$.

Let us prove the second statement of the theorem.

Consider a map $\phi_n : \mathrm{FDS}(X) \to \mathrm{FNDS}_n(X)$ defined by the rule

$$(w,u)\phi_n := \begin{cases} (w,u) & \text{if } l_w \leqslant n, \\ 0 & \text{if } l_w > n, \end{cases}$$

where $(w,u) \in \mathrm{FDS}(X)$. Similarly to the proof of Theorem 4 from [43], the facts that ϕ_n is a surjective homomorphism and $\Delta_{\phi_n} = \zeta_n$ can be proved. According to Theorem 2.2.10 $\mathrm{FNDS}_n(X)$ is the free n-nilpotent doppelsemigroup. So, ζ_n is the least n-nilpotent congruence on $\mathrm{FDS}(X)$. □

Exercise 2.2.20. Show that the map τ defined in the proof of the previous theorem is a surjective homomorphism and $\Delta_\tau = \xi$.

Exercise 2.2.21. Prove that the map ϕ_n defined in the proof of the previous theorem is a surjective homomorphism and $\Delta_{\phi_n} = \zeta_n$.

Let us consider a relation between operations of a doppelsemigroup $(D, \dashv, \vdash)$ in which $(D, \dashv)$ or $(D, \vdash)$ is a rectangular band.

Definition 2.2.22. A semigroup is called a *rectangular band* if it satisfies the identity $xyx = x$.

Note that operations of a doppelsemigroup $(D, \dashv, \vdash)$ with a rectangular band $(D, \dashv)$ or $(D, \vdash)$ coincide (see [3]).

Exercise 2.2.23. Prove that operations of a doppelsemigroup $(D, \dashv, \vdash)$ with a rectangular band $(D, \dashv)$ or $(D, \vdash)$ coincide.

We conclude this section with the description of an important property of commutative doppelsemigroups.

Lemma 2.2.24. *Operations of a commutative doppelsemigroup* $(D, \dashv, \vdash)$ *with idempotent operations* $\dashv$ *and* $\vdash$ *coincide.*

Proof. From Lemma 2.2.4 it follows that $(x \vdash y) \dashv z = (x \dashv y) \vdash z$ for all $x, y, z \in D$. Hence, in the case $x = y$, using the idempotent property of $\dashv$ and $\vdash$, obtain $x \dashv z = x \vdash z$. □

2.3 Free n-dinilpotent doppelsemigroups

In this section, doppelsemigroups with both nilpotent semigroups are considered. We define and study the variety of n-dinilpotent doppelsemigroups. The main purpose of the section is to construct a free n-dinilpotent doppelsemigroup of an arbitrary rank and characterize separately free n-dinilpotent doppelsemigroups of rank 1. Moreover, we study some properties of a free n-dinilpotent doppelsemigroup and discuss one important congruence on a free doppelsemigroup.

Recall the definition of a k-nilpotent semigroup (see also [40, 46, 52]). As usual, $\mathbb{N}$ denotes the set of all positive integers.

Definition 2.3.1. A semigroup S is called *nilpotent* if $S^{n+1} = 0$ for some $n \in \mathbb{N}$. The least such n is called the *nilpotency index of* S. For $k \in \mathbb{N}$ a nilpotent semigroup of nilpotency index $\leqslant k$ is called *k-nilpotent*.

Let us introduce the notion of an n-dinilpotent doppelsemigroup.

Definition 2.3.2. A doppelsemigroup $(D, \dashv, \vdash)$ with zero (see Definition 1.1.18) is called *dinilpotent* if $(D, \dashv)$ and $(D, \vdash)$ are nilpotent semigroups.

Definition 2.3.3. A dinilpotent doppelsemigroup $(D, \dashv, \vdash)$ is called *n-dinilpotent* if $(D, \dashv)$ and $(D, \vdash)$ are n-nilpotent semigroups.

Note that operations of any 1-dinilpotent doppelsemigroup coincide and it is a zero semigroup. The class of all n-dinilpotent doppelsemigroups forms a subvariety of the variety of doppelsemigroups. It is not difficult to check that the variety of n-nilpotent doppelsemigroups (see Section 2.2) is a subvariety of the variety of n-dinilpotent doppelsemigroups.

Definition 2.3.4. A doppelsemigroup which is free in the variety of n-dinilpotent doppelsemigroups is called a *free n-dinilpotent doppelsemigroup*.

The problem is to construct a free n-dinilpotent doppelsemigroup.

As in Section 2.1, let $\mathrm{F}[X]$ be the free semigroup on X, T the free monoid on the two-element set $\{a,b\}$ and $\theta \in T$ the empty word. For $x \in \{a,b\}$ and all $u \in T$, the number of occurrences of an element x in u is denoted by $d_x(u)$. Obviously, $d_x(\theta) = 0$. Fix $n \in \mathbb{N}$ and put

$$M_n := \{(w,u) \in \mathrm{F}[X] \times T \mid l_w - l_u = 1, d_x(u) + 1 \leqslant n, x \in \{a,b\}\} \cup \{0\}.$$

Define operations $\dashv$ and $\vdash$ on M_n by

$$(w_1,u_1) \dashv (w_2,u_2) := \begin{cases} (w_1w_2, u_1au_2) & \text{if } d_x(u_1au_2) + 1 \leqslant n, x \in \{a,b\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(w_1,u_1) \vdash (w_2,u_2) := \begin{cases} (w_1w_2, u_1bu_2) & \text{if } d_x(u_1bu_2) + 1 \leqslant n, x \in \{a,b\}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(w_1,u_1) * 0 := 0 * (w_1,u_1) := 0 * 0 := 0$$

for all $(w_1,u_1), (w_2,u_2) \in M_n \setminus \{0\}$ and $* \in \{\dashv, \vdash\}$. The obtained algebra is denoted by $\mathrm{FDDS}_n(X)$.

Theorem 2.3.5. $\mathrm{FDDS}_n(X)$ *is the free n-dinilpotent doppelsemigroup.*

Proof. First prove that $\mathrm{FDDS}_n(X)$ is a doppelsemigroup. Let (w_1,u_1), (w_2,u_2), $(w_3,u_3) \in M_n \setminus \{0\}$. For $x,y,z \in \{a,b\}$ it is clear that

$$d_x(u_1yu_2zu_3) + 1 \leqslant n$$

implies

$$d_x(u_1yu_2) + 1 \leqslant n, \tag{2.3}$$

$$d_x(u_2zu_3) + 1 \leqslant n. \tag{2.4}$$

Let $d_x(u_1au_2au_3)+1 \leqslant n$ for all $x \in \{a,b\}$. Then, using (2.3), (2.4), we get

$$\begin{aligned}((w_1,u_1)\dashv(w_2,u_2))\dashv(w_3,u_3) &= (w_1w_2,u_1au_2)\dashv(w_3,u_3)\\ &= (w_1w_2w_3,u_1au_2au_3) = (w_1,u_1)\dashv(w_2w_3,u_2au_3)\\ &= (w_1,u_1)\dashv((w_2,u_2)\dashv(w_3,u_3)).\end{aligned}$$

If $d_x(u_1au_2au_3)+1 > n$ for some $x \in \{a,b\}$, then, obviously,

$$((w_1,u_1)\dashv(w_2,u_2))\dashv(w_3,u_3) = 0 = (w_1,u_1)\dashv((w_2,u_2)\dashv(w_3,u_3)).$$

So, the axiom (D3) of a doppelsemigroup holds.

If $d_x(u_1au_2bu_3)+1 \leqslant n$ for all $x \in \{a,b\}$, then, using (2.3), (2.4), obtain

$$\begin{aligned}((w_1,u_1)\dashv(w_2,u_2))\vdash(w_3,u_3) &= (w_1w_2,u_1au_2)\vdash(w_3,u_3)\\ &= (w_1w_2w_3,u_1au_2bu_3) = (w_1,u_1)\dashv(w_2w_3,u_2bu_3)\\ &= (w_1,u_1)\dashv((w_2,u_2)\vdash(w_3,u_3)).\end{aligned}$$

Let $d_x(u_1au_2bu_3)+1 > n$ for some $x \in \{a,b\}$. Then, clearly,

$$((w_1,u_1)\dashv(w_2,u_2))\vdash(w_3,u_3) = 0 = (w_1,u_1)\dashv((w_2,u_2)\vdash(w_3,u_3)).$$

Thus, the axiom (D1) of a doppelsemigroup holds. Similarly, one can check the axioms (D2) and (D4). Thus, $\mathrm{FDDS}_n(X)$ is a doppelsemigroup.

Take arbitrary elements $(w_i,u_i) \in M_n \setminus \{0\}$, $1 \leqslant i \leqslant n+1$. It is clear that

$$d_a(u_1au_2a\ldots au_{n+1})+1 > n.$$

From here

$$(w_1,u_1)\dashv(w_2,u_2)\dashv\ldots\dashv(w_{n+1},u_{n+1}) = 0.$$

At the same time, assuming $y^0 = \theta$ for $y \in \{a,b\}$, for any $(x_i,\theta) \in M_n \setminus \{0\}$, where $x_i \in X$, $1 \leqslant i \leqslant n$, get

$$(x_1,\theta)\dashv(x_2,\theta)\dashv\ldots\dashv(x_n,\theta) = (x_1x_2\ldots x_n,a^{n-1}) \neq 0.$$

From the last arguments we conclude that $(M_n, \dashv)$ is a nilpotent semigroup of nilpotency index n. Analogously, we can prove that $(M_n, \vdash)$ is a nilpotent semigroup of nilpotency index n. So, $\mathrm{FDDS}_n(X)$ is an n-dinilpotent doppelsemigroup.

Let us show that $\mathrm{FDDS}_n(X)$ is free in the variety of n-dinilpotent doppelsemigroups.

Obviously, $\mathrm{FDDS}_n(X)$ is generated by $X \times \{\theta\}$. Let $(K, \dashv', \vdash')$ be an arbitrary n-dinilpotent doppelsemigroup. Let $\beta : X \times \{\theta\} \to K$ be an arbitrary map. Consider a map $\alpha : X \to K$ such that $x\alpha = (x, \theta)\beta$ for all $x \in X$, and define a map

$$\pi : \mathrm{FDDS}_n(X) \to (K, \dashv', \vdash')$$

by

$$\omega\pi := \begin{cases} x_1\alpha\widetilde{y}_1 x_2\alpha\widetilde{y}_2 \ldots \widetilde{y}_{s-1}x_s\alpha & \text{if } \omega = (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}), \\ & \quad x_d \in X,\ 1 \leqslant d \leqslant s,\ y_p \in \{a, b\}, \\ & \quad 1 \leqslant p \leqslant s-1,\ s > 1, \\ x_1\alpha & \text{if } \omega = (x_1, \theta),\ x_1 \in X, \\ 0 & \text{if } \omega = 0, \end{cases}$$

where

$$\widetilde{y}_p := \begin{cases} \dashv' & \text{if } y_p = a, \\ \vdash' & \text{if } y_p = b \end{cases}$$

for all $1 \leqslant p \leqslant s-1$, $s > 1$. According to Lemma 1.2.7 π is well-defined.

To show that π is a homomorphism we will use the axioms of a doppelsemigroup and the identities of an n-dinilpotent doppelsemigroup.

If $s = 1$, we will regard the sequence $y_1y_2\ldots y_{s-1} \in T$ as θ. For arbitrary elements

$$(w_1, u_1) = (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}),$$
$$(w_2, u_2) = (z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1}) \in \mathrm{FDDS}_n(X),$$

where $x_d, z_i \in X$, $1 \leqslant d \leqslant s$, $1 \leqslant i \leqslant k$, $y_p, c_j \in \{a, b\}$, $1 \leqslant p \leqslant s-1$, $1 \leqslant j \leqslant k-1$, in the case $d_x(u_1au_2) + 1 \leqslant n$ for all $x \in \{a, b\}$, we get

$$((x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}) \dashv (z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1}))\pi$$
$$= (x_1\ldots x_sz_1\ldots z_k, y_1\ldots y_{s-1}ac_1\ldots c_{k-1})\pi$$

$$
\begin{aligned}
&= x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha\widetilde{a}z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha \\
&= (x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha)\dashv'(z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha) \\
&= (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1})\pi\dashv'(z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1})\pi.
\end{aligned}
$$

If $d_x(u_1au_2)+1>n$ for some $x\in\{a,b\}$, then

$$((x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1})\dashv(z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1}))\pi = 0\pi = 0.$$

Since $(K,\dashv',\vdash')$ is n-dinilpotent, we have

$$
\begin{aligned}
0 &= x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha\widetilde{a}z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha \\
&= (x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha)\dashv'(z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha) \\
&= (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1})\pi\dashv'(z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1})\pi.
\end{aligned}
$$

So,

$$((w_1,u_1)\dashv(w_2,u_2))\pi = (w_1,u_1)\pi\dashv'(w_2,u_2)\pi$$

for all $(w_1,u_1),(w_2,u_2)\in \mathrm{FDDS}_n(X)$.

Similarly for $\vdash$. So, π is a homomorphism. It is clear that $(x,\theta)\pi=(x,\theta)\beta$ for all $(x,\theta)\in X\times\{\theta\}$. Since $X\times\{\theta\}$ generates $\mathrm{FDDS}_n(X)$, the uniqueness of such homomorphism π is obvious. Thus, $\mathrm{FDDS}_n(X)$ is free in the variety of n-dinilpotent doppelsemigroups. □

Exercise 2.3.6. Check the axioms (D2) and (D4) of $\mathrm{FDDS}_n(X)$.

Exercise 2.3.7. Prove that $(M_n,\vdash)$ is a nilpotent semigroup of nilpotency index n.

Exercise 2.3.8. Consider the map π defined in the proof of Theorem 2.3.5. Show that

$$((w_1,u_1)\vdash(w_2,u_2))\pi = (w_1,u_1)\pi\vdash'(w_2,u_2)\pi$$

for all $(w_1,u_1),(w_2,u_2)\in \mathrm{FDDS}_n(X)$.

Theorem 2.3.5 gives us an idea that one-generated free n-dinilpotent doppelsemigroups have also other presentation. Construct a doppelsemigroup which is isomorphic to the free n-dinilpotent doppelsemigroup of rank 1.

Fix $n \in \mathbb{N}$ and let

$$\overline{\Phi}_n := \{u \in T \mid d_x(u) + 1 \leqslant n, x \in \{a,b\}\} \cup \{0\}.$$

Define operations $\dashv$ and $\vdash$ on $\overline{\Phi}_n$ by

$$u_1 \dashv u_2 := \begin{cases} u_1 a u_2 & \text{if } d_x(u_1 a u_2) + 1 \leqslant n, x \in \{a,b\}, \\ 0 & \text{otherwise}, \end{cases}$$

$$u_1 \vdash u_2 := \begin{cases} u_1 b u_2 & \text{if } d_x(u_1 b u_2) + 1 \leqslant n, x \in \{a,b\}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_1 * 0 := 0 * u_1 := 0 * 0 := 0$$

for all $u_1, u_2 \in \overline{\Phi}_n \setminus \{0\}$ and $* \in \{\dashv, \vdash\}$. The obtained algebra is denoted by Φ_n. Obviously, Φ_n is a doppelsemigroup.

Exercise 2.3.9. Prove that Φ_n is a doppelsemigroup.

Lemma 2.3.10. *If* $|X| = 1$, *then* $\Phi_n \cong \mathrm{FDDS}_n(X)$.

Exercise 2.3.11. Let $X = \{r\}$. Prove that a map $\gamma : \Phi_n \to \mathrm{FDDS}_n(X)$, defined by the rule

$$u\gamma := \begin{cases} (r^{l_u+1}, u) & \text{if } u \in \overline{\Phi}_n \setminus \{0\}, \\ 0 & \text{if } u = 0, \end{cases}$$

is an isomorphism.

The following lemma establishes a relationship between both semigroups of the free n-dinilpotent doppelsemigroup $\mathrm{FDDS}_n(X)$.

Lemma 2.3.12. *The semigroups* $(M_n, \dashv)$ *and* $(M_n, \vdash)$ *are isomorphic.*

Proof. Let $\widehat{a} = b$, $\widehat{b} = a$ and define a map $\sigma : (M_n, \dashv) \to (M_n, \vdash)$ by putting

$$t\sigma := \begin{cases} (w, \widehat{y_1}\widehat{y_2} \ldots \widehat{y_m}) & \text{if } t = (w, y_1 y_2 \ldots y_m) \in M_n \setminus \{0\}, y_p \in \{a,b\}, 1 \leqslant p \leqslant m, \\ t & \text{otherwise}. \end{cases}$$

An immediate verification shows that σ is an isomorphism. $\square$

Exercise 2.3.13. Show that the map σ defined in the proof of Lemma 2.3.12 is an isomorphism.

Since the set $X \times \{\theta\}$ is generating for $\mathrm{FDDS}_n(X)$, we obtain the following description of the automorphism group of the free n-dinilpotent doppelsemigroup.

Lemma 2.3.14. $\mathrm{Aut}\,\mathrm{FDDS}_n(X) \cong \mathfrak{I}[X]$.

Exercise 2.3.15. Prove Lemma 2.3.14.

Definition 2.3.16. If ρ is a congruence on a doppelsemigroup $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is an n-dinilpotent doppelsemigroup, we say that ρ is an *n-dinilpotent congruence*.

At the end of this section we present the least n-dinilpotent congruence on a free doppelsemigroup.

Recall that if $f : D_1 \to D_2$ is a homomorphism of doppelsemigroups, the corresponding congruence on D_1 is denoted by Δ_f.

Let $\mathrm{FDS}(X)$ be the free doppelsemigroup (see Section 2.1) and $n \in \mathbb{N}$. Define a relation $\mu_{(n)}$ on $\mathrm{FDS}(X)$ by

$$(w_1, u_1)\mu_{(n)}(w_2, u_2)$$

if and only if

$$(w_1, u_1) = (w_2, u_2) \quad \text{or} \quad \begin{cases} d_x(u_1) + 1 > n & \text{for some } x \in \{a, b\}, \\ d_y(u_2) + 1 > n & \text{for some } y \in \{a, b\}. \end{cases}$$

Theorem 2.3.17. *The relation $\mu_{(n)}$ is the least n-dinilpotent congruence on the free doppelsemigroup* $\mathrm{FDS}(X)$.

Proof. Define a map $\varphi : \mathrm{FDS}(X) \to \mathrm{FDDS}_n(X)$ by

$$(w, u)\varphi := \begin{cases} (w, u) & \text{if } d_x(u) + 1 \leqslant n \text{ for all } x \in \{a, b\}, \\ 0 & \text{otherwise} \end{cases}$$

$((w, u) \in \mathrm{FDS}(X))$. Show that φ is a homomorphism.

Let $(w_1, u_1), (w_2, u_2) \in \mathrm{FDS}(X)$ and $d_x(u_1 a u_2) + 1 \leqslant n$ for all $x \in \{a, b\}$. From the last inequality it follows that $d_x(u_1) + 1 \leqslant n$ and $d_x(u_2) + 1 \leqslant n$ for all $x \in \{a, b\}$. Then

$$((w_1, u_1) \dashv (w_2, u_2))\varphi = (w_1 w_2, u_1 a u_2)\varphi = (w_1 w_2, u_1 a u_2)$$
$$= (w_1, u_1) \dashv (w_2, u_2) = (w_1, u_1)\varphi \dashv (w_2, u_2)\varphi.$$

If $d_x(u_1 a u_2) + 1 > n$ for some $x \in \{a, b\}$, then

$$((w_1, u_1) \dashv (w_2, u_2))\varphi = (w_1 w_2, u_1 a u_2)\varphi = 0 = (w_1, u_1)\varphi \dashv (w_2, u_2)\varphi.$$

Let further $d_x(u_1 b u_2) + 1 \leqslant n$ for all $x \in \{a, b\}$. Then $d_x(u_1) + 1 \leqslant n$, $d_x(u_2) + 1 \leqslant n$ for all $x \in \{a, b\}$ and

$$((w_1, u_1) \vdash (w_2, u_2))\varphi = (w_1 w_2, u_1 b u_2)\varphi = (w_1 w_2, u_1 b u_2)$$
$$= (w_1, u_1) \vdash (w_2, u_2) = (w_1, u_1)\varphi \vdash (w_2, u_2)\varphi.$$

If $d_x(u_1 b u_2) + 1 > n$ for some $x \in \{a, b\}$, then

$$((w_1, u_1) \vdash (w_2, u_2))\varphi = (w_1 w_2, u_1 b u_2)\varphi = 0 = (w_1, u_1)\varphi \vdash (w_2, u_2)\varphi.$$

Thus, φ is a surjective homomorphism. By Theorem 2.3.5, $\mathrm{FDDS}_n(X)$ is the free n-dinilpotent doppelsemigroup. Then Δ_φ is the least n-dinilpotent congruence on $\mathrm{FDS}(X)$. From the definition of φ it follows that $\Delta_\varphi = \mu_{(n)}$. □

2.4 Free left n-dinilpotent doppelsemigroups

In this section, we introduce left (right) n-dinilpotent doppelsemigroups which are analogs of left (right) nilpotent semigroups of rank n considered by Schein. A free object in the variety of left (right) n-dinilpotent doppelsemigroups is constructed and studied. Using the constructed free algebras, we also characterize some least congruences on a free doppelsemigroup.

Consider the notion of a left (right) nilpotent semigroup of rank n.

Definition 2.4.1 ([33])**.** A semigroup G is called a *left (right) nilpotent semigroup of rank* n if the product of any n elements from this semigroup gives a left (right) zero.

The class of all left nilpotent semigroups of rank n is characterized by the identity

$$x_1x_2\ldots x_nx_{n+1} = x_1x_2\ldots x_n.$$

The identity which characterizes right nilpotent semigroups of rank n is defined dually. It is well-known (see [33], Lemma 1) that a left (right) nilpotent semigroup of rank n is also a left (right) nilpotent semigroup of any rank m greater than n. Right nilpotent semigroups appear in automata theory, namely, such semigroups are semigroups of self-adaptive automata (see [10, 22]).

For doppelsemigroups the question about introducing an analog of a left (right) nilpotent semigroup of rank n is natural.

Definition 2.4.2. A doppelsemigroup $(D,\dashv,\vdash)$ is called *left dinilpotent* if for some $n\in\mathbb{N}$ and any $x_1,\ldots,x_n,\, x\in D$ the following identities hold:

$$(x_1 *_1 \cdots *_{n-1} x_n)\dashv x = x_1 *_1 \cdots *_{n-1} x_n = (x_1 *_1 \cdots *_{n-1} x_n)\vdash x, \tag{2.5}$$

where $*_1,\ldots,*_{n-1}\in\{\dashv,\vdash\}$. The least such n is called the *left dinilpotency index of* $(D,\dashv,\vdash)$. For $k\in\mathbb{N}$ a left dinilpotent doppelsemigroup of left dinilpotency index $\leqslant k$ is called *left k-dinilpotent*.

Definition 2.4.3. A doppelsemigroup $(D,\dashv,\vdash)$ is called *right dinilpotent* if for some $n\in\mathbb{N}$ and any $x_1,\ldots,x_n,\, x\in D$ the following identities hold:

$$x\dashv(x_1 *_1 \cdots *_{n-1} x_n) = x_1 *_1 \cdots *_{n-1} x_n = x\vdash(x_1 *_1 \cdots *_{n-1} x_n),$$

where $*_1,\ldots,*_{n-1}\in\{\dashv,\vdash\}$. The least such n is called the *right dinilpotency index of* $(D,\dashv,\vdash)$. For $k\in\mathbb{N}$ a right dinilpotent doppelsemigroup of right dinilpotency index $\leqslant k$ is called *right k-dinilpotent*.

It is clear that operations of any left (right) 1-dinilpotent doppelsemigroup coincide. In this case, we obtain a left (right) zero semigroup. Moreover, the class of all left (right) n-dinilpotent doppelsemigroups forms a subvariety of the variety of doppelsemigroups.

Definition 2.4.4. A doppelsemigroup which is free in the variety of left (right) n-dinilpotent doppelsemigroups is called a *free left (right) n-dinilpotent doppelsemigroup*.

Further we will solve the problem of constructing a free left (right) n-dinilpotent doppelsemigroup. We use the notations from Section 2.1.

Let $w \in \mathrm{F}[X]$. Fix $n \in \mathbb{N}$. If $l_w \geq n$, by $\overset{n}{\overrightarrow{w}}$ $\left(\overset{n}{\overleftarrow{w}}\right)$ denote the initial (terminal) subword with the length n of w. By definition, $\overset{0}{\overrightarrow{u}} = \theta$ $\left(\overset{0}{\overleftarrow{u}} = \theta\right)$ for all $u \in T \setminus \{\theta\}$. Define operations $\dashv$ and $\vdash$ on

$$L_n := \{(w,u) \in F[X] \times T \mid l_w - l_u = 1, l_w \leqslant n\}$$

by

$$(w_1,u_1) \dashv (w_2,u_2) := \begin{cases} (w_1w_2, u_1au_2) & \text{if } l_{w_1} + l_{w_2} \leqslant n, \\ \left(\overset{n}{\overrightarrow{w_1w_2}}, \overset{n-1}{\overrightarrow{u_1au_2}}\right) & \text{if } l_{w_1} + l_{w_2} > n \end{cases}$$

and

$$(w_1,u_1) \vdash (w_2,u_2) := \begin{cases} (w_1w_2, u_1bu_2) & \text{if } l_{w_1} + l_{w_2} \leqslant n, \\ \left(\overset{n}{\overrightarrow{w_1w_2}}, \overset{n-1}{\overrightarrow{u_1bu_2}}\right) & \text{if } l_{w_1} + l_{w_2} > n \end{cases}$$

for all $(w_1,u_1),(w_2,u_2) \in L_n$. The obtained algebra is denoted by $\mathrm{FDDS}_n^l(X)$.

Lemma 2.4.5. *The operation* $\dashv$ *of* $\mathrm{FDDS}_n^l(X)$ *is associative.*

Proof. Let $(w_1,u_1),(w_2,u_2),(w_3,u_3) \in \mathrm{FDDS}_n^l(X)$ and

$$l_{w_1} + l_{w_2} + l_{w_3} \leqslant n. \tag{2.6}$$

Obviously, (2.6) implies

$$l_{w_1} + l_{w_2} < n, \tag{2.7}$$

$$l_{w_2} + l_{w_3} < n. \tag{2.8}$$

Using (2.6)–(2.8), obtain

$$((w_1,u_1) \dashv (w_2,u_2)) \dashv (w_3,u_3) = (w_1w_2, u_1au_2) \dashv (w_3,u_3)$$

$$= (w_1w_2w_3, u_1au_2au_3) = (w_1, u_1) \dashv (w_2w_3, u_2au_3)$$
$$= (w_1, u_1) \dashv ((w_2, u_2) \dashv (w_3, u_3)).$$

Let further

$$l_{w_1} + l_{w_2} + l_{w_3} > n. \tag{2.9}$$

Divide this case into subcases:

$$l_{w_1} + l_{w_2} \leqslant n, \qquad l_{w_2} + l_{w_3} \leqslant n, \tag{2.10}$$
$$l_{w_1} + l_{w_2} \leqslant n, \qquad l_{w_2} + l_{w_3} > n, \tag{2.11}$$
$$l_{w_1} + l_{w_2} > n, \qquad l_{w_2} + l_{w_3} \leqslant n, \tag{2.12}$$
$$l_{w_1} + l_{w_2} > n, \qquad l_{w_2} + l_{w_3} > n. \tag{2.13}$$

Consider the case (2.11):

$$((w_1, u_1) \dashv (w_2, u_2)) \dashv (w_3, u_3) = (w_1w_2, u_1au_2) \dashv (w_3, u_3)$$
$$= \left(\overset{n}{\overrightarrow{w_1w_2w_3}}, \overset{n-1}{\overrightarrow{u_1au_2au_3}}\right) = \left(\overset{n}{\overrightarrow{w_1\, \overset{n}{\overrightarrow{w_2w_3}}}}, \overset{n-1}{\overrightarrow{u_1a\, \overset{n-1}{\overrightarrow{u_2au_3}}}}\right)$$
$$= (w_1, u_1) \dashv \left(\overset{n}{\overrightarrow{w_2w_3}}, \overset{n-1}{\overrightarrow{u_2au_3}}\right) = (w_1, u_1) \dashv ((w_2, u_2) \dashv (w_3, u_3)).$$

Let us turn to the case (2.12). We have

$$((w_1, u_1) \dashv (w_2, u_2)) \dashv (w_3, u_3) = \left(\overset{n}{\overrightarrow{w_1w_2}}, \overset{n-1}{\overrightarrow{u_1au_2}}\right) \dashv (w_3, u_3)$$
$$= \left(\overset{n}{\overrightarrow{\overset{n}{\overrightarrow{w_1w_2}}\, w_3}}, \overset{n-1}{\overrightarrow{\overset{n-1}{\overrightarrow{u_1au_2}}\, au_3}}\right) = \left(\overset{n}{\overrightarrow{w_1w_2}}, \overset{n-1}{\overrightarrow{u_1au_2}}\right) = \left(\overset{n}{\overrightarrow{w_1w_2w_3}}, \overset{n-1}{\overrightarrow{u_1au_2au_3}}\right)$$
$$= (w_1, u_1) \dashv (w_2w_3, u_2au_3\big) = (w_1, u_1) \dashv ((w_2, u_2) \dashv (w_3, u_3)).$$

The cases (2.10) and (2.13) are considered in a similar way. Thus, the operation $\dashv$ is associative. □

Lemma 2.4.6. *The operation $\vdash$ of $\mathrm{FDDS}_n^l(X)$ is associative.*

Proof. The proof is similar to the proof of Lemma 2.4.5. □

Exercise 2.4.7. Prove Lemma 2.4.6.

Lemma 2.4.8. $\mathrm{FDDS}_n^l(X)$ *satisfies the axiom* (D2) *of a doppelsemigroup.*

Proof. Let $(w_1,u_1),(w_2,u_2),(w_3,u_3) \in \mathrm{FDDS}_n^l(X)$ and (2.6) holds. The condition (2.6) implies (2.7) and (2.8). Using (2.6)–(2.8), get

$$\begin{aligned}((w_1,u_1) \vdash (w_2,u_2)) \dashv (w_3,u_3) &= (w_1w_2,u_1bu_2) \dashv (w_3,u_3)\\ &= (w_1w_2w_3,u_1bu_2au_3) = (w_1,u_1) \vdash (w_2w_3,u_2au_3)\\ &= (w_1,u_1) \vdash ((w_2,u_2) \dashv (w_3,u_3)).\end{aligned}$$

Let now (2.9) holds. Divide this case into subcases (2.10)–(2.13) (see the proof of Lemma 2.4.5).

Consider the case (2.11):

$$\begin{aligned}&((w_1,u_1) \vdash (w_2,u_2)) \dashv (w_3,u_3) = (w_1w_2,u_1bu_2) \dashv (w_3,u_3)\\ &= \left(\overset{n}{\overrightarrow{w_1w_2w_3}},\overset{n-1}{\overrightarrow{u_1bu_2au_3}}\right) = \left(\overset{n}{\overrightarrow{w_1\,\overset{n}{\overrightarrow{w_2w_3}}}},\overset{n-1}{\overrightarrow{u_1b\,\overset{n-1}{\overrightarrow{u_2au_3}}}}\right)\\ &= (w_1,u_1) \vdash \left(\overset{n}{\overrightarrow{w_2w_3}},\overset{n-1}{\overrightarrow{u_2au_3}}\right) = (w_1,u_1) \vdash ((w_2,u_2) \dashv (w_3,u_3)).\end{aligned}$$

If we have (2.13), then

$$\begin{aligned}&((w_1,u_1) \vdash (w_2,u_2)) \dashv (w_3,u_3) = \left(\overset{n}{\overrightarrow{w_1w_2}},\overset{n-1}{\overrightarrow{u_1bu_2}}\right) \dashv (w_3,u_3)\\ &= \left(\overset{n}{\overrightarrow{\overset{n}{\overrightarrow{w_1w_2}}\,w_3}},\overset{n-1}{\overrightarrow{\overset{n-1}{\overrightarrow{u_1bu_2}}\,au_3}}\right) = \left(\overset{n}{\overrightarrow{w_1w_2}},\overset{n-1}{\overrightarrow{u_1bu_2}}\right) = \left(\overset{n}{\overrightarrow{w_1w_2w_3}},\overset{n-1}{\overrightarrow{u_1bu_2au_3}}\right)\\ &= \left(\overset{n}{\overrightarrow{w_1\,\overset{n}{\overrightarrow{w_2w_3}}}},\overset{n-1}{\overrightarrow{u_1b\,\overset{n-1}{\overrightarrow{u_2au_3}}}}\right) = (w_1,u_1) \vdash \left(\overset{n}{\overrightarrow{w_2w_3}},\overset{n-1}{\overrightarrow{u_2au_3}}\right)\\ &= (w_1,u_1) \vdash ((w_2,u_2) \dashv (w_3,u_3)).\end{aligned}$$

The cases (2.10) and (2.12) are considered in a similar way. So, $\mathrm{FDDS}_n^l(X)$ satisfies the axiom (D2) of a doppelsemigroup. □

Lemma 2.4.9. $\mathrm{FDDS}_n^l(X)$ *satisfies the axiom* (D1) *of a doppelsemigroup.*

Proof. The proof is similar to the proof of Lemma 2.4.8. □

Exercise 2.4.10. Prove Lemma 2.4.9.

Lemma 2.4.11. $\mathrm{FDDS}_n^l(X)$ *is a left n-dinilpotent doppelsemigroup.*

Proof. By Lemmas 2.4.5, 2.4.6, 2.4.8 and 2.4.9, $\mathrm{FDDS}_n^l(X)$ is a doppelsemigroup. Show that it is left n-dinilpotent.

Let $(w_1,u_1),\dots,(w_n,u_n) \in \mathrm{FDDS}_n^l(X)$ and $*_1,\dots,*_{n-1} \in \{\dashv,\vdash\}$. Then the expression

$$(t,u) := (w_1,u_1) *_1 \cdots *_{n-1} (w_n,u_n) \in \mathrm{FDDS}_n^l(X).$$

It is clear that $l_t = n$. Then

$$(t,u) \dashv (w,g) = \left(\overrightarrow{tw}^{n}, \overrightarrow{uag}^{n-1}\right) = (t,u)$$

and

$$(t,u) \vdash (w,g) = \left(\overrightarrow{tw}^{n}, \overrightarrow{ubg}^{n-1}\right) = (t,u)$$

for any $(w,g) \in \mathrm{FDDS}_n^l(X)$. So, $\mathrm{FDDS}_n^l(X)$ is a left dinilpotent doppelsemigroup. From the last calculations it follows that left dinilpotency index of $\mathrm{FDDS}_n^l(X)$ is $\leq n$. □

Lemma 2.4.12. $\mathrm{FDDS}_n^l(X)$ *is free in the variety of left n-dinilpotent doppelsemigroups.*

Proof. Let $(S,\dashv',\vdash')$ be an arbitrary left n-dinilpotent doppelsemigroup and $\alpha : X \to S$ an arbitrary map. Take $(x_1 \dots x_s, y_1 \dots y_{s-1}) \in \mathrm{FDDS}_n^l(X)$, where $x_i \in X$, $1 \leqslant i \leqslant s$, $y_j \in \{a,b\}$, $1 \leqslant j \leqslant s-1$. If $s = 1$, we will regard the sequence $y_1 \dots y_{s-1} \in T$ as θ. For $y \in \{a,b\}$ let

$$\widetilde{y} := \begin{cases} \dashv' & \text{if } y = a, \\ \vdash' & \text{if } y = b. \end{cases}$$

Define a map $\psi : \mathrm{FDDS}_n^l(X) \to (S, \dashv', \vdash')$ by the rule

$$\omega\psi := \begin{cases} x_1\alpha\widetilde{y}_1x_2\alpha\widetilde{y}_2\ldots\widetilde{y}_{s-1}x_s\alpha & \text{if } \omega = (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}),\ s > 1, \\ x_1\alpha & \text{if } \omega = (x_1, \theta). \end{cases}$$

By Lemma 1.2.7, ψ is well-defined. Show that ψ is a homomorphism. We will use Lemma 1.2.7 and (2.5).

For arbitrary elements

$$(x_1\ldots x_s, y_1\ldots y_{s-1}), (z_1\ldots z_k, c_1\ldots c_{k-1}) \in \mathrm{FDDS}_n^l(X),$$

where $z_d \in X$, $1 \leqslant d \leqslant k$, $c_q \in \{a, b\}$, $1 \leqslant q \leqslant k-1$, consider two cases:

$$s + k \leqslant n, \tag{2.14}$$

$$s + k > n. \tag{2.15}$$

In the case (2.14) obtain

$$\begin{aligned} &((x_1\ldots x_s, y_1\ldots y_{s-1}) \dashv (z_1\ldots z_k, c_1\ldots c_{k-1}))\psi \\ &\quad = (x_1\ldots x_s z_1\ldots z_k, y_1\ldots y_{s-1}ac_1\ldots c_{k-1})\psi \\ &\quad = x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha\widetilde{a}z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha \\ &\quad = (x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha) \dashv' (z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha) \\ &\quad = (x_1\ldots x_s, y_1\ldots y_{s-1})\psi \dashv' (z_1\ldots z_k, c_1\ldots c_{k-1})\psi. \end{aligned}$$

If (2.15) holds, then assume $s + f = n$ for some $f \in \mathbb{N}^0$ and consider the following three cases:

$$f = 0, \tag{2.16}$$

$$f = 1, \tag{2.17}$$

$$f > 1. \tag{2.18}$$

In the case (2.16), we have

$$((x_1\ldots x_s, y_1\ldots y_{s-1}) \dashv (z_1\ldots z_k, c_1\ldots c_{k-1}))\psi$$

$$= \Big(\overrightarrow{x_1 \dots x_s z_1 \dots z_k}^{n}, \overrightarrow{y_1 \dots y_{s-1} a c_1 \dots c_{k-1}}^{n-1}\Big)\psi$$
$$= (x_1 \dots x_s, y_1 \dots y_{s-1})\psi = x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha$$
$$= (x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha) \dashv' (z_1 \alpha \widetilde{c}_1 \dots \widetilde{c}_{k-1} z_k \alpha)$$
$$= (x_1 \dots x_s, y_1 \dots y_{s-1})\psi \dashv' (z_1 \dots z_k, c_1 \dots c_{k-1})\psi.$$

Suppose that the condition (2.17) is satisfied. Then

$$((x_1 \dots x_s, y_1 \dots y_{s-1}) \dashv (z_1 \dots z_k, c_1 \dots c_{k-1}))\psi$$
$$= \Big(\overrightarrow{x_1 \dots x_s z_1 \dots z_k}^{n}, \overrightarrow{y_1 \dots y_{s-1} a c_1 \dots c_{k-1}}^{n-1}\Big)\psi$$
$$= (x_1 \dots x_s z_1, y_1 \dots y_{s-1} a)\psi = x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha \widetilde{a} z_1 \alpha$$
$$= (x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha \widetilde{a} z_1 \alpha)\widetilde{c}_1 (z_2 \alpha \widetilde{c}_2 \dots \widetilde{c}_{k-1} z_k \alpha)$$
$$= (x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha)\widetilde{a}(z_1 \alpha \widetilde{c}_1 \dots \widetilde{c}_{k-1} z_k \alpha)$$
$$= (x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha) \dashv' (z_1 \alpha \widetilde{c}_1 \dots \widetilde{c}_{k-1} z_k \alpha)$$
$$= (x_1 \dots x_s, y_1 \dots y_{s-1})\psi \dashv' (z_1 \dots z_k, c_1 \dots c_{k-1})\psi.$$

Finally, in the case (2.18), we get

$$((x_1 \dots x_s, y_1 \dots y_{s-1}) \dashv (z_1 \dots z_k, c_1 \dots c_{k-1}))\psi$$
$$= \Big(\overrightarrow{x_1 \dots x_s z_1 \dots z_k}^{n}, \overrightarrow{y_1 \dots y_{s-1} a c_1 \dots c_{k-1}}^{n-1}\Big)\psi$$
$$= (x_1 \dots x_s z_1 \dots z_f, y_1 \dots y_{s-1} a c_1 \dots c_{f-1})\psi$$
$$= x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha \widetilde{a} z_1 \alpha \widetilde{c}_1 \dots \widetilde{c}_{f-1} z_f \alpha$$
$$= (x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha \widetilde{a} z_1 \alpha \widetilde{c}_1 \dots \widetilde{c}_{f-1} z_f \alpha)\widetilde{c}_f (z_{f+1} \alpha \widetilde{c}_{f+1} \dots \widetilde{c}_{k-1} z_k \alpha)$$
$$= (x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha)\widetilde{a}(z_1 \alpha \widetilde{c}_1 \dots \widetilde{c}_{k-1} z_k \alpha)$$
$$= (x_1 \alpha \widetilde{y}_1 \dots \widetilde{y}_{s-1} x_s \alpha) \dashv' (z_1 \alpha \widetilde{c}_1 \dots \widetilde{c}_{k-1} z_k \alpha)$$
$$= (x_1 \dots x_s, y_1 \dots y_{s-1})\psi \dashv' (z_1 \dots z_k, c_1 \dots c_{k-1})\psi.$$

Similarly for $\vdash$. So, ψ is a homomorphism.

Thus, $\mathrm{FDDS}_n^l(X)$ is free in the variety of left n-dinilpotent doppelsemigroups. □

Exercise 2.4.13. Consider the map ψ defined in the proof of Lemma 2.4.12. Show that

$$((x_1 \dots x_s, y_1 \dots y_{s-1}) \vdash (z_1 \dots z_k, c_1 \dots c_{k-1}))\psi$$
$$= (x_1 \dots x_s, y_1 \dots y_{s-1})\psi \vdash' (z_1 \dots z_k, c_1 \dots c_{k-1})\psi$$

for all $(x_1 \dots x_s, y_1 \dots y_{s-1}), (z_1 \dots z_k, c_1 \dots c_{k-1}) \in \mathrm{FDDS}_n^l(X)$.

The main result of this section is the following.

Theorem 2.4.14. $\mathrm{FDDS}_n^l(X)$ *is the free left n-dinilpotent doppelsemigroup.*

Proof. The proof follows from Lemmas 2.4.5, 2.4.6, 2.4.8, 2.4.9, 2.4.11 and 2.4.12. □

The following statement establishes a relationship between both semigroups of the free left n-dinilpotent doppelsemigroup.

Lemma 2.4.15. *The semigroups* $(L_n, \dashv)$ *and* $(L_n, \vdash)$ *of the free left n-dinilpotent doppelsemigroup* $\mathrm{FDDS}_n^l(X)$ *are isomorphic.*

Proof. Let $\widehat{a} = b$, $\widehat{b} = a$ and define a map $\phi : (L_n, \dashv) \to (L_n, \vdash)$ by putting

$$\omega\phi := \begin{cases} (w, \widehat{c_1}\widehat{c_2} \dots \widehat{c_k}) & \text{if } \omega = (w, c_1 c_2 \dots c_k) \in L_n,\ c_q \in \{a, b\},\ 1 \leqslant q \leqslant k, \\ \omega & \text{if } \omega = (w, \theta),\ w \in X. \end{cases}$$

An immediate verification shows that ϕ is an isomorphism. □

Exercise 2.4.16. Show that the map ϕ defined in the proof of Lemma 2.4.15 is an isomorphism.

In the case of rank 1 we have a different construction for the free left n-dinilpotent doppelsemigroup.

Fix $n \in \mathbb{N}$ and define operations $\dashv$ and $\vdash$ on $\overline{\Omega}_n := \{u \in T \mid l_u + 1 \leqslant n\}$ by

$$u_1 \dashv u_2 := \begin{cases} u_1 a u_2 & \text{if } l_{u_1 u_2} + 2 \leqslant n, \\ \overset{n-1}{\overrightarrow{u_1 a u_2}} & \text{if } l_{u_1 u_2} + 2 > n \end{cases}$$

and

$$u_1 \vdash u_2 := \begin{cases} u_1 b u_2 & \text{if } l_{u_1 u_2} + 2 \leqslant n, \\ \overset{n-1}{\overrightarrow{u_1 b u_2}} & \text{if } l_{u_1 u_2} + 2 > n \end{cases}$$

for all $u_1, u_2 \in \overline{\Omega}_n$. The algebra $(\overline{\Omega}_n, \dashv, \vdash)$ is denoted by Ω_n. Obviously, Ω_n is a doppelsemigroup.

Exercise 2.4.17. Prove that Ω_n is a doppelsemigroup.

Lemma 2.4.18. *If* $|X| = 1$*, then a map* $\gamma : \Omega_n \to \mathrm{FDDS}_n^l(X)$*, defined by* $u\gamma := (r^{l_u+1}, u)$*, is an isomorphism.*

Exercise 2.4.19. Prove that $\Omega_n \cong \mathrm{FDDS}_n^l(X)$ if $|X| = 1$.

The free left n-dinilpotent doppelsemigroup $\mathrm{FDDS}_n^l(X)$ is determined uniquely up to isomorphism by cardinality of the set X because the generating set of $\mathrm{FDDS}_n^l(X)$ has the same cardinality as X. Hence, obtain the following description of the automorphism group of the free left n-dinilpotent doppelsemigroup.

Lemma 2.4.20. $\mathrm{Aut}\,\mathrm{FDDS}_n^l(X) \cong \mathfrak{I}[X]$.

Exercise 2.4.21. Prove Lemma 2.4.20.

Definition 2.4.22. If ρ is a congruence on a doppelsemigroup $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is a left (right) n-dinilpotent doppelsemigroup, we say that ρ is a *left (right) n-dinilpotent congruence*.

At the end of this section we characterize the least left n-dinilpotent congruence on a free doppelsemigroup.

As above, if $f : D_1 \to D_2$ is a homomorphism of doppelsemigroups, the corresponding congruence on D_1 is denoted by Δ_f.

Theorem 2.4.23. *A map* $\pi : \mathrm{FDS}(X) \to \mathrm{FDDS}_n^l(X)$*, defined by*

$$(w,u) \mapsto (w,u)\pi := \begin{cases} (w,u) & \text{if} \quad l_w \leqslant n, \\ \left(\overset{n}{\overrightarrow{w}}, \overset{n-1}{\overrightarrow{u}}\right) & \text{if} \quad l_w > n, \end{cases}$$

is an epimorphism and Δ_π *is the least left n-dinilpotent congruence on* $\mathrm{FDS}(X)$.

Proof. Let $(w_1,u_1), (w_2,u_2) \in \mathrm{FDS}(X)$. Assume $l_{w_1} + l_{w_2} \leqslant n$. Then $l_{w_1} < n$, $l_{w_2} < n$ and

$$\begin{aligned} ((w_1,u_1) \dashv (w_2,u_2))\pi &= (w_1w_2, u_1au_2)\pi = (w_1w_2, u_1au_2) \\ &= (w_1,u_1) \dashv (w_2,u_2) = (w_1,u_1)\pi \dashv (w_2,u_2)\pi. \end{aligned}$$

Let $l_{w_1}+l_{w_2}>n$. Divide this case into such subcases:

$$l_{w_1}\leqslant n,\qquad l_{w_2}\leqslant n, \tag{2.19}$$

$$l_{w_1}\leqslant n,\qquad l_{w_2}> n, \tag{2.20}$$

$$l_{w_1}> n,\qquad l_{w_2}\leqslant n, \tag{2.21}$$

$$l_{w_1}> n,\qquad l_{w_2}> n. \tag{2.22}$$

Consider the case (2.19):

$$\begin{aligned}((w_1,u_1)\dashv(w_2,u_2))\pi &= (w_1w_2,u_1au_2)\pi=\left(\overset{n}{\overrightarrow{w_1w_2}},\overset{n-1}{\overrightarrow{u_1au_2}}\right)\\ &=(w_1,u_1)\dashv(w_2,u_2)=(w_1,u_1)\pi\dashv(w_2,u_2)\pi.\end{aligned}$$

If the case (2.20) holds, then

$$\begin{aligned}((w_1,u_1)\dashv(w_2,u_2))\pi &= (w_1w_2,u_1au_2)\pi=\left(\overset{n}{\overrightarrow{w_1w_2}},\overset{n-1}{\overrightarrow{u_1au_2}}\right)\\ &=\left(\overset{n}{\overrightarrow{w_1\overset{n}{\overrightarrow{w_2}}}},\overset{n-1}{\overrightarrow{u_1a\overset{n-1}{\overrightarrow{u_2}}}}\right)=(w_1,u_1)\dashv\left(\overset{n}{\overrightarrow{w_2}},\overset{n-1}{\overrightarrow{u_2}}\right)=(w_1,u_1)\pi\dashv(w_2,u_2)\pi.\end{aligned}$$

In the case (2.21), we get

$$\begin{aligned}((w_1,u_1)\dashv(w_2,u_2))\pi &= (w_1w_2,u_1au_2)\pi=\left(\overset{n}{\overrightarrow{w_1w_2}},\overset{n-1}{\overrightarrow{u_1au_2}}\right)=\left(\overset{n}{\overrightarrow{w_1}},\overset{n-1}{\overrightarrow{u_1}}\right)\\ &=\left(\overset{n}{\overrightarrow{\overset{n}{\overrightarrow{w_1}}w_2}},\overset{n-1}{\overrightarrow{\overset{n-1}{\overrightarrow{u_1}}au_2}}\right)=\left(\overset{n}{\overrightarrow{w_1}},\overset{n-1}{\overrightarrow{u_1}}\right)\dashv(w_2,u_2)=(w_1,u_1)\pi\dashv(w_2,u_2)\pi.\end{aligned}$$

Finally, if (2.22) is true, then

$$\begin{aligned}((w_1,u_1)\dashv(w_2,u_2))\pi &= (w_1w_2,u_1au_2)\pi=\left(\overset{n}{\overrightarrow{w_1w_2}},\overset{n-1}{\overrightarrow{u_1au_2}}\right)=\left(\overset{n}{\overrightarrow{w_1}},\overset{n-1}{\overrightarrow{u_1}}\right)\\ &=\left(\overset{n}{\overrightarrow{\overset{n}{\overrightarrow{w_1}}\overset{n}{\overrightarrow{w_2}}}},\overset{n-1}{\overrightarrow{\overset{n-1}{\overrightarrow{u_1}}a\overset{n-1}{\overrightarrow{u_2}}}}\right)=\left(\overset{n}{\overrightarrow{w_1}},\overset{n-1}{\overrightarrow{u_1}}\right)\dashv\left(\overset{n}{\overrightarrow{w_2}},\overset{n-1}{\overrightarrow{u_2}}\right)=(w_1,u_1)\pi\dashv(w_2,u_2)\pi.\end{aligned}$$

Thus,

$$((w_1,u_1)\dashv(w_2,u_2))\pi=(w_1,u_1)\pi\dashv(w_2,u_2)\pi$$

for all $(w_1,u_1),(w_2,u_2)\in \mathrm{FDS}(X)$.

Similarly for $\vdash$. So, π is a homomorphism. Evidently, π is a surjection. Since by Theorem 2.4.14, $\mathrm{FDDS}_n^l(X)$ is the free left n-dinilpotent doppelsemigroup, Δ_π is the least left n-dinilpotent congruence on $\mathrm{FDS}(X)$. □

Exercise 2.4.24. Consider the map π defined in the proof of Theorem 2.4.23. Show that

$$((w_1,u_1)\vdash(w_2,u_2))\pi=(w_1,u_1)\pi\vdash(w_2,u_2)\pi$$

for all $(w_1,u_1),(w_2,u_2)\in \mathrm{FDS}(X)$.

In order to construct free right n-dinilpotent doppelsemigroups, characterize the least right n-dinilpotent congruence on the free doppelsemigroup and the automorphism group of the free right n-dinilpotent doppelsemigroup we use the duality principle.

Note that doppelsemigroups have a relation with restrictive bisemigroups. Recall the definition of a restrictive bisemigroup [34, 35].

Definition 2.4.25. Let B an arbitrary nonempty set and $\dashv$, $\vdash$ binary operations on B. An ordered triple $(B,\dashv,\vdash)$ is called a *restrictive bisemigroup* if the axioms (D1), (D4), (D5) and

$$x\dashv y\dashv z=y\dashv x\dashv z,\qquad x\vdash y\vdash z=x\vdash z\vdash y,$$
$$x\dashv x=x=x\vdash x$$

hold for all $x,y,z\in B$.

Restrictive bisemigroups have applications in the theory of binary relations.

It is known that operations of a doppelsemigroup $(D,\dashv,\vdash)$ with a rectangular band $(D,\dashv)$ or $(D,\vdash)$ coincide (see Exercise 2.2.23) and operations of a commutative doppelsemigroup $(D,\dashv,\vdash)$ with idempotent operations $\dashv$ and $\vdash$ coincide too (see Lemma 2.2.24). Obviously, operations of a doppelsemigroup $(D,\dashv,\vdash)$ with a left (right) zero semigroup $(D,\dashv)$ or $(D,\vdash)$ coincide. At the same time, there exist restrictive bisemigroups [34] whose binary idempotent operations $\dashv$ and $\vdash$ are distinct.

Now we formulate one open problem.

Open Problem 2.4.26. A doppelsemigroup is called *idempotent* if both its operations are idempotent. A doppelsemigroup which is free in the variety of idempotent doppelsemigroups is called a *free idempotent doppelsemigroup*.

Construct a free idempotent doppelsemigroup.

At the end of the chapter we give more information about doppelalgebras and their relationships with doppelsemigroups.

Richter [32] considered so-called *doppelalgebras*, that is, vector spaces over a field equipped with two binary linear associative operations $\dashv$ and $\vdash$ satisfying the axioms (D1) and (D2), and gave a condition for such algebras to be Lie algebras. In particular, in [32] it was shown that the universal enveloping algebra for Lie algebras has the structure of a doppelalgebra and that there exists a functor from the category of doppelalgebras to the category of associative algebras. Doppelalgebras appear in [1] as algebras over some operads. If in the definition of a doppelalgebra instead of a vector space over a field we take a set and omit the linearity of operations, we obtain the notion of a doppelsemigroup. The term "doppelsemigroup" was first proposed in [48]. From the last definition it follows that a doppelalgebra is just a linear analog of a doppelsemigroup and, therefore, all results obtained for doppelsemigroups can be applied to doppelalgebras. Moreover, a semigroup $(D,\vdash)$ is an interassociate of a semigroup $(D,\dashv)$ if and only if $(D,\dashv,\vdash)$ is a doppelsemigroup. So, the problem of the description of interassociates of a semigroup is reduced to the description of doppelsemigroups. These facts provide one of the main motivations for the study of doppelsemigroups. A principal way in which one can obtain interassociative semigroups is, firstly, to describe relatively free doppelsemigroups and, secondly, congruences on them. This is demonstrated very successfully in this chapter (see also [48, 46, 44]), where the author constructed the free doppelsemigroup, the free commutative doppelsemigroup, the free n-nilpotent doppelsemigroup, the free n-dinilpotent doppelsemigroup and the free left (right) n-dinilpotent doppelsemigroup.

Chapter 3

Structure of free strong doppelsemigroups

In this chapter, we present a free object in the variety of strong doppelsemigroups. We also construct and study some relatively free strong doppelsemigroups, and characterize the least congruences on a free strong doppelsemigroup.

3.1 Free strong doppelsemigroups

Strong interassociativity for semigroups was introduced by Gould and Richardson [14]: Two semigroups defined on the same set are strongly interassociative if the axioms (D1)–(D3) relating operations of these semigroups are satisfied. This section deals with strong doppelsemigroups which are sets with two binary associative operations satisfying axioms of strong interassociativity. So, a semigroup $(D,\vdash)$ is a strong interassociate of a semigroup $(D,\dashv)$ if and only if $(D,\dashv,\vdash)$ is a strong doppelsemigroup, and the problem of the description of strong interassociates of a semigroup is reduced to the description of strong doppelsemigroups. This fact motivates to study strong doppelsemigroups. Commutative dimonoids in the sense of Loday are examples of strong doppelsemigroups and two strongly interassociative semigroups give rise to a strong doppelsemigroup.

In this section, we describe a free strong doppelsemigroup of an arbitrary rank and for this doppelsemigroup we construct an isomorphic one. We also consider separately free strong doppelsemigroups of rank 1.

Recall that a doppelsemigroup $(D,\dashv,\vdash)$ is called *strong* if it satisfies the axiom

$$x \dashv (y \vdash z) = x \vdash (y \dashv z). \tag{D3}$$

In the following, we will use definitions from Sections 2.2 and 2.3.

The class of all n-dinilpotent (respectively, commutative, n-nilpotent) strong doppelsemigroups forms a subvariety of the variety of strong doppelsemigroups. It is not difficult to see that the variety of n-nilpotent strong doppelsemigroups is a subvariety of the variety of n-dinilpotent strong doppelsemigroups.

Definition 3.1.1. A strong doppelsemigroup which is free in the variety of strong doppelsemigroups (respectively, n-dinilpotent strong doppelsemigroups, commutative strong doppelsemigroups, n-nilpotent strong doppelsemigroups) is called a *free strong doppelsemigroup* (respectively, *free n-dinilpotent strong doppelsemigroup*, *free commutative strong doppelsemigroup*, *free n-nilpotent strong doppelsemigroup*).

The main problem of this section is to construct a free strong doppelsemigroup.

Let X be an arbitrary nonempty set, $n \in \mathbb{N}$. We denote the union of n different copies of X^n by Y_n and assume $D(X) := \bigcup_{n\geq 1} Y_n$. Denoting by $x_1 \ldots \breve{x}_i \ldots x_n$ an element in the i-th component of Y_n, define operations $\dashv$ and $\vdash$ on $D(X)$ by

$$(x_1 \ldots \breve{x}_i \ldots x_k) \dashv (x_{k+1} \ldots \breve{x}_j \ldots x_l) := x_1 \ldots \breve{x}_{i+j-k} \ldots x_l,$$
$$(x_1 \ldots \breve{x}_i \ldots x_k) \vdash (x_{k+1} \ldots \breve{x}_j \ldots x_l) := x_1 \ldots \breve{x}_{i+j-k-1} \ldots x_l$$

for all $x_1 \ldots \breve{x}_i \ldots x_k, x_{k+1} \ldots \breve{x}_j \ldots x_l \in D(X)$. The algebra $(D(X),\dashv,\vdash)$ is denoted by $\mathrm{FSD}(X)$.

Theorem 3.1.2. $\mathrm{FSD}(X)$ *is the free strong doppelsemigroup.*

Proof. For all $x_1 \ldots \breve{x}_i \ldots x_k$, $x_{k+1} \ldots \breve{x}_j \ldots x_l$, $x_{l+1} \ldots \breve{x}_s \ldots x_m \in \mathrm{FSD}(X)$, we have

$$\begin{aligned}
&((x_1 \ldots \breve{x}_i \ldots x_k) \dashv (x_{k+1} \ldots \breve{x}_j \ldots x_l)) \dashv (x_{l+1} \ldots \breve{x}_s \ldots x_m) \\
&\quad = (x_1 \ldots \breve{x}_{i+j-k} \ldots x_l) \dashv (x_{l+1} \ldots \breve{x}_s \ldots x_m) \\
&\quad = x_1 \ldots \breve{x}_{i+j-k+s-l} \ldots x_m = (x_1 \ldots \breve{x}_i \ldots x_k) \dashv (x_{k+1} \ldots \breve{x}_{j+s-l} \ldots x_m) \\
&\quad = (x_1 \ldots \breve{x}_i \ldots x_k) \dashv ((x_{k+1} \ldots \breve{x}_j \ldots x_l) \dashv (x_{l+1} \ldots \breve{x}_s \ldots x_m)),
\end{aligned}$$

$$\begin{aligned}
&((x_1\dots\breve{x}_i\dots x_k)\vdash(x_{k+1}\dots\breve{x}_j\dots x_l))\vdash(x_{l+1}\dots\breve{x}_s\dots x_m)\\
&\quad=(x_1\dots\breve{x}_{i+j-k-1}\dots x_l)\vdash(x_{l+1}\dots\breve{x}_s\dots x_m)\\
&\quad=x_1\dots\breve{x}_{i+j-k+s-l-2}\dots x_m=(x_1\dots\breve{x}_i\dots x_k)\vdash(x_{k+1}\dots\breve{x}_{j+s-l-1}\dots x_m)\\
&\quad=(x_1\dots\breve{x}_i\dots x_k)\vdash((x_{k+1}\dots\breve{x}_j\dots x_l)\vdash(x_{l+1}\dots\breve{x}_s\dots x_m)),
\end{aligned}$$

$$\begin{aligned}
&((x_1\dots\breve{x}_i\dots x_k)\dashv(x_{k+1}\dots\breve{x}_j\dots x_l))\vdash(x_{l+1}\dots\breve{x}_s\dots x_m)\\
&\quad=(x_1\dots\breve{x}_{i+j-k}\dots x_l)\vdash(x_{l+1}\dots\breve{x}_s\dots x_m)\\
&\quad=x_1\dots\breve{x}_{i+j-k+s-l-1}\dots x_m=(x_1\dots\breve{x}_i\dots x_k)\dashv(x_{k+1}\dots\breve{x}_{j+s-l-1}\dots x_m)\\
&\quad=(x_1\dots\breve{x}_i\dots x_k)\dashv((x_{k+1}\dots\breve{x}_j\dots x_l)\vdash(x_{l+1}\dots\breve{x}_s\dots x_m))
\end{aligned}$$

and

$$\begin{aligned}
&((x_1\dots\breve{x}_i\dots x_k)\vdash(x_{k+1}\dots\breve{x}_j\dots x_l))\dashv(x_{l+1}\dots\breve{x}_s\dots x_m)\\
&\quad=(x_1\dots\breve{x}_{i+j-k-1}\dots x_l)\dashv(x_{l+1}\dots\breve{x}_s\dots x_m)\\
&\quad=x_1\dots\breve{x}_{i+j-k+s-l-1}\dots x_m=(x_1\dots\breve{x}_i\dots x_k)\vdash(x_{k+1}\dots\breve{x}_{j+s-l}\dots x_m)\\
&\quad=(x_1\dots\breve{x}_i\dots x_k)\vdash((x_{k+1}\dots\breve{x}_j\dots x_l)\dashv(x_{l+1}\dots\breve{x}_s\dots x_m)).
\end{aligned}$$

Thus, $\mathrm{FSD}(X)$ is a strong doppelsemigroup.

Let us show that $\mathrm{FSD}(X)$ is free in the variety of strong doppelsemigroups.

First note that $\mathrm{FSD}(X)$ is generated by $\breve{X}=\{\breve{x}\,|\,x\in X\}$. Indeed, every element $x_1\dots\breve{x}_i\dots x_k\in\mathrm{FSD}(X)$ has the following representation:

$$x_1\dots\breve{x}_i\dots x_k=\breve{x}_1\dashv\dots\dashv\breve{x}_i\vdash\dots\vdash\breve{x}_k.$$

According to Lemma 1.2.9 such representation is unique up to order of symbols $\underbrace{\dashv,\dots,\dashv}_{i-1},\underbrace{\vdash,\dots,\vdash}_{k-i}$. Thus, $\langle\breve{X}\rangle=\mathrm{FSD}(X)$.

Let $(S,\dashv',\vdash')$ be an arbitrary strong doppelsemigroup and $\gamma':\breve{X}\to S$ an arbitrary map. Consider a map $\gamma:X\to S$ such that $x\gamma=\breve{x}\gamma'$ for all $x\in X$, and define a map

$$\phi:\mathrm{FSD}(X)\to(S,\dashv',\vdash')$$

by

$$(x_1\dots\breve{x}_i\dots x_k)\phi:=x_1\gamma\dashv'\dots\dashv' x_i\gamma\vdash'\dots\vdash' x_k\gamma$$

for all $x_1 \ldots \breve{x}_i \ldots x_k \in \mathrm{FSD}(X)$. According to Lemmas 1.2.7 and 1.2.9 ϕ is well-defined.

To show that ϕ is a homomorphism we will use Lemmas 1.2.7 and 1.2.11. For arbitrary elements $x_1 \ldots \breve{x}_i \ldots x_k, x_{k+1} \ldots \breve{x}_j \ldots x_l \in \mathrm{FSD}(X)$ we get

$$
\begin{aligned}
&((x_1 \ldots \breve{x}_i \ldots x_k) \dashv (x_{k+1} \ldots \breve{x}_j \ldots x_l))\phi \\
&\quad = (x_1 \ldots \breve{x}_{i+j-k} \ldots x_l)\phi = x_1\gamma \dashv' \ldots \dashv' x_{i+j-k}\gamma \vdash' \ldots \vdash' x_l\gamma \\
&\quad = x_1\gamma \dashv' \ldots \dashv' (x_i\gamma \dashv' \ldots \dashv' x_{i+j-k}\gamma \vdash' \ldots \vdash' x_j\gamma) \vdash' \ldots \vdash' x_l\gamma \\
&\quad = x_1\gamma \dashv' \ldots \dashv' (x_i\gamma \vdash' \ldots \vdash' x_k\gamma \dashv' x_{k+1}\gamma \dashv' \ldots \dashv' x_j\gamma) \vdash' \ldots \vdash' x_l\gamma \\
&\quad = (x_1\gamma \dashv' \ldots \dashv' x_i\gamma \vdash' \ldots \vdash' x_k\gamma) \dashv' (x_{k+1}\gamma \dashv' \ldots \dashv' x_j\gamma \vdash' \ldots \vdash' x_l\gamma) \\
&\quad = (x_1 \ldots \breve{x}_i \ldots x_k)\phi \dashv' (x_{k+1} \ldots \breve{x}_j \ldots x_l)\phi
\end{aligned}
$$

and

$$
\begin{aligned}
&((x_1 \ldots \breve{x}_i \ldots x_k) \vdash (x_{k+1} \ldots \breve{x}_j \ldots x_l))\phi \\
&\quad = (x_1 \ldots \breve{x}_{i+j-k-1} \ldots x_l)\phi = x_1\gamma \dashv' \ldots \dashv' x_{i+j-k-1}\gamma \vdash' \ldots \vdash x_l\gamma \\
&\quad = x_1\gamma \dashv' \ldots \dashv' (x_i\gamma \dashv' \ldots \dashv' x_{i+j-k-1}\gamma \vdash' \cdots \vdash' x_j\gamma) \vdash' \ldots \vdash' x_l\gamma \\
&\quad = x_1\gamma \dashv' \ldots \dashv' (x_i\gamma \vdash' \ldots \vdash' x_k\gamma \vdash' x_{k+1}\gamma \dashv' \cdots \dashv' x_j\gamma) \vdash' \ldots \vdash x_l\gamma \\
&\quad = (x_1\gamma \dashv' \ldots \dashv' x_i\gamma \vdash' \ldots \vdash' x_k\gamma) \vdash' (x_{k+1}\gamma \dashv' \ldots \dashv' x_j\gamma \vdash' \ldots \vdash x_l\gamma) \\
&\quad = (x_1 \ldots \breve{x}_i \ldots x_k)\phi \vdash' (x_{k+1} \ldots \breve{x}_j \ldots x_l)\phi.
\end{aligned}
$$

So, ϕ is a homomorphism. Clearly, $\breve{x}\phi = \breve{x}\gamma'$ for all $\breve{x} \in \breve{X}$. Since $\breve{X}$ generates $\mathrm{FSD}(X)$, the uniqueness of such homomorphism ϕ is obvious. Thus, $\mathrm{FSD}(X)$ is the free strong doppelsemigroup. □

The question of the description of a more elegant version for the free strong doppelsemigroup is natural. Now we construct a strong doppelsemigroup which is isomorphic to the free strong doppelsemigroup $\mathrm{FSD}(X)$.

We denote $\mathbb{N}$ with zero by $\mathbb{N}^0$. Let $\mathrm{F}[X]$ be the free semigroup in the alphabet X. Define operations $\dashv$ and $\vdash$ on

$$C := \{(w, m) \in \mathrm{F}[X] \times \mathbb{N}^0 \mid l_w > m\}$$

by

$$(w_1, m_1) \dashv (w_2, m_2) := (w_1 w_2, m_1 + m_2 + 1), \tag{3.1}$$

$$(w_1, m_1) \vdash (w_2, m_2) := (w_1 w_2, m_1 + m_2) \tag{3.2}$$

for all (w_1, m_1), $(w_2, m_2) \in C$. The algebra $(C, \dashv, \vdash)$ is denoted by $\widetilde{\mathrm{F}}[X]$.

Lemma 3.1.3. $\widetilde{\mathrm{F}}[X]$ *is a strong doppelsemigroup generated by* $X \times \{0\}$.

Proof. A direct verification shows that $\widetilde{\mathrm{F}}[X]$ is a strong doppelsemigroup. It is clear from the definition of $\widetilde{\mathrm{F}}[X]$ that it is generated by $X \times \{0\}$. □

Exercise 3.1.4. Prove Lemma 3.1.3.

Lemma 3.1.5. $\mathrm{FSD}(X)$ *and* $\widetilde{\mathrm{F}}[X]$ *are isomorphic.*

Proof. Define a map $\mu : \mathrm{FSD}(X) \to \widetilde{\mathrm{F}}[X]$ by

$$(x_1 \ldots \check{x}_i \ldots x_k)\mu := (x_1 \ldots x_i \ldots x_k, i-1), \quad \text{where } x_1 \ldots \check{x}_i \ldots x_k \in \mathrm{FSD}(X).$$

It is clear that μ is a bijection. Show that μ is a homomorphism.

For arbitrary elements $x_1 \ldots \check{x}_i \ldots x_k$, $x_{k+1} \ldots \check{x}_j \ldots x_l \in \mathrm{FSD}(X)$ obtain

$$\begin{aligned}
&((x_1 \ldots \check{x}_i \ldots x_k) \dashv (x_{k+1} \ldots \check{x}_j \ldots x_l))\mu \\
&\quad = (x_1 \ldots \check{x}_{i+j-k} \ldots x_l)\mu = (x_1 \ldots x_{i+j-k} \ldots x_l, i+j-k-1) \\
&\quad = (x_1 \ldots x_i \ldots x_k x_{k+1} \ldots x_j \ldots x_l, i-1+j-k) \\
&\quad = (x_1 \ldots x_i \ldots x_k, i-1) \dashv (x_{k+1} \ldots x_j \ldots x_l, j-k-1) \\
&\quad = (x_1 \ldots \check{x}_i \ldots x_k)\mu \dashv (x_{k+1} \ldots \check{x}_j \ldots x_l)\mu
\end{aligned}$$

and

$$\begin{aligned}
&((x_1 \ldots \check{x}_i \ldots x_k) \vdash (x_{k+1} \ldots \check{x}_j \ldots x_l))\mu \\
&\quad = (x_1 \ldots \check{x}_{i+j-k-1} \ldots x_l)\mu = (x_1 \ldots x_{i+j-k-1} \ldots x_l, i+j-k-2) \\
&\quad = (x_1 \ldots x_i \ldots x_k x_{k+1} \ldots x_j \ldots x_l, i-1+j-k-1) \\
&\quad = (x_1 \ldots x_i \ldots x_k, i-1) \vdash (x_{k+1} \ldots x_j \ldots x_l, j-k-1) \\
&\quad = (x_1 \ldots \check{x}_i \ldots x_k)\mu \vdash (x_{k+1} \ldots \check{x}_j \ldots x_l)\mu.
\end{aligned}$$

Thus, μ is an isomorphism. □

The following natural problem is to present a more elegant version for one-generated free strong doppelsemigroups. Now we construct a strong doppelsemigroup which is isomorphic to the free strong doppelsemigroup of rank 1.

Define operations $\dashv$ and $\vdash$ on $\overline{R} := \{(n,m) \in \mathbb{N} \times \mathbb{N}^0 \mid n > m\}$ by

$$(n_1,m_1) \dashv (n_2,m_2) := (n_1+n_2, m_1+m_2+1),$$
$$(n_1,m_1) \vdash (n_2,m_2) := (n_1+n_2, m_1+m_2)$$

for all (n_1,m_1), $(n_2,m_2) \in \overline{R}$. The algebra $(\overline{R}, \dashv, \vdash)$ is denoted by R. It is not difficult to check that R is a strong doppelsemigroup.

Exercise 3.1.6. Prove that R is a strong doppelsemigroup.

Lemma 3.1.7. *If* $|X| = 1$, *then* $R \cong \widetilde{\mathrm{F}}[X]$.

Proof. Let $X = \{r\}$. One can show that a map $\varpi : R \to \widetilde{\mathrm{F}}[X]$, defined by

$$(n,m)\varpi := (r^n, m),$$

is an isomorphism. □

Exercise 3.1.8. Prove that the map ϖ defined in the proof of Lemma 3.1.7 is an isomorphism.

A principal importance of free strong doppelsemigroups is the fact that we may obtain new pairs of strongly interassociative semigroups via constructing congruences on free strong doppelsemigroups.

3.2 Free n-dinilpotent strong doppelsemigroups

As we have noted, the class of all n-dinilpotent strong doppelsemigroups forms a variety. It is natural to construct a free object in this variety. In this section, we consider a free n-dinilpotent strong doppelsemigroup of an arbitrary rank and discuss separately free n-dinilpotent strong doppelsemigroups of rank 1.

As in Section 2.1, let F$[X]$ be the free semigroup on X. Fix $n \in \mathbb{N}$ and assume

$$D_n := \{(w,m) \in \mathrm{F}[X] \times \mathbb{N}^0 \mid l_w > m, m+1 \leqslant n, l_w - m \leqslant n\} \cup \{0\}.$$

Define operations $\dashv$ and $\vdash$ on D_n by

$$(w_1,m_1) \dashv (w_2,m_2) := \begin{cases} (w_1w_2, m_1+m_2+1) & \text{if } m_1+m_2+2 \leqslant n, \\ & \quad l_{w_1w_2} - m_1 - m_2 - 1 \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

$$(w_1,m_1) \vdash (w_2,m_2) := \begin{cases} (w_1w_2, m_1+m_2) & \text{if } m_1+m_2+1 \leqslant n, \\ & \quad l_{w_1w_2} - m_1 - m_2 \leqslant n, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(w_1,m_1) * 0 := 0 * (w_1,m_1) := 0 * 0 := 0$$

for all $(w_1,m_1),(w_2,m_2) \in D_n \setminus \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra obtained in this way is denoted by $\mathrm{FDSD}_n(X)$.

Theorem 3.2.1. $\mathrm{FDSD}_n(X)$ *is the free n-dinilpotent strong doppelsemigroup.*

Proof. First prove that $\mathrm{FDSD}_n(X)$ is a strong doppelsemigroup. Let

$$(w_1,m_1),(w_2,m_2),(w_3,m_3) \in D_n \setminus \{0\}$$

and let

$$m_1+m_2+m_3+3 \leqslant n \quad \text{and} \quad l_{w_1w_2w_3} - m_1 - m_2 - m_3 - 2 \leqslant n. \tag{3.3}$$

From (3.3) it follows

$$m_1+m_2+2 < n, \qquad m_2+m_3+2 < n, \qquad l_{w_2w_3} - m_2 - m_3 - 1 \leqslant n \tag{3.4}$$

and

$$l_{w_1w_2} - m_1 - m_2 - 1 \leqslant n. \tag{3.5}$$

Using (3.3)–(3.5), we get

$$
\begin{aligned}
&((w_1,m_1)\dashv(w_2,m_2))\dashv(w_3,m_3)\\
&\qquad=(w_1w_2,m_1+m_2+1)\dashv(w_3,m_3)=(w_1w_2w_3,m_1+m_2+m_3+2)\\
&\qquad=(w_1,m_1)\dashv(w_2w_3,m_2+m_3+1)=(w_1,m_1)\dashv((w_2,m_2)\dashv(w_3,m_3)).
\end{aligned}
$$

If $m_1+m_2+m_3+3>n$ or $l_{w_1w_2w_3}-m_1-m_2-m_3-2>n$, then

$$((w_1,m_1)\dashv(w_2,m_2))\dashv(w_3,m_3)=0=(w_1,m_1)\dashv((w_2,m_2)\dashv(w_3,m_3)).$$

So, the axiom (D4) of a strong doppelsemigroup holds.

Let

$$m_1+m_2+m_3+1\leqslant n \quad\text{and}\quad l_{w_1w_2w_3}-m_1-m_2-m_3\leqslant n. \tag{3.6}$$

From (3.6) it follows

$$m_1+m_2+1\leqslant n \tag{3.7}$$

and

$$m_2+m_3+1\leqslant n,\qquad l_{w_1w_2}-m_1-m_2<n,\qquad l_{w_2w_3}-m_2-m_3<n. \tag{3.8}$$

Using (3.6)–(3.8), we obtain

$$
\begin{aligned}
&((w_1,m_1)\vdash(w_2,m_2))\vdash(w_3,m_3)\\
&\qquad=(w_1w_2,m_1+m_2)\vdash(w_3,m_3)=(w_1w_2w_3,m_1+m_2+m_3)\\
&\qquad=(w_1,m_1)\vdash(w_2w_3,m_2+m_3)=(w_1,m_1)\vdash((w_2,m_2)\vdash(w_3,m_3)).
\end{aligned}
$$

If $m_1+m_2+m_3+1>n$ or $l_{w_1w_2w_3}-m_1-m_2-m_3>n$, then

$$((w_1,m_1)\vdash(w_2,m_2))\vdash(w_3,m_3)=0=(w_1,m_1)\vdash((w_2,m_2)\vdash(w_3,m_3)).$$

Thus, the axiom (D5) of a strong doppelsemigroup holds.

Let

$$m_1+m_2+m_3+2\leqslant n \quad\text{and}\quad l_{w_1w_2w_3}-m_1-m_2-m_3-1\leqslant n. \tag{3.9}$$

From (3.9) it follows

$$m_1+m_2+2 \leqslant n, \tag{3.10}$$

$$m_2+m_3+1<n, \qquad l_{w_2w_3}-m_2-m_3 \leqslant n, \tag{3.11}$$

$$l_{w_1w_2}-m_1-m_2 \leqslant n. \tag{3.12}$$

According to (3.9)–(3.12) we get

$$\begin{aligned}
&((w_1,m_1)\dashv(w_2,m_2))\vdash(w_3,m_3)\\
&\quad=(w_1w_2,m_1+m_2+1)\vdash(w_3,m_3)=(w_1w_2w_3,m_1+m_2+m_3+1)\\
&\quad=(w_1,m_1)\dashv(w_2w_3,m_2+m_3)=(w_1,m_1)\dashv((w_2,m_2)\vdash(w_3,m_3))
\end{aligned}$$

and

$$\begin{aligned}
&((w_1,m_1)\vdash(w_2,m_2))\dashv(w_3,m_3)\\
&\quad=(w_1w_2,m_1+m_2)\dashv(w_3,m_3)=(w_1w_2w_3,m_1+m_2+m_3+1)\\
&\quad=(w_1,m_1)\vdash(w_2w_3,m_2+m_3+1)=(w_1,m_1)\vdash((w_2,m_2)\dashv(w_3,m_3)).
\end{aligned}$$

If $m_1+m_2+m_3+2>n$ or $l_{w_1w_2w_3}-m_1-m_2-m_3-1>n$, then

$$\begin{aligned}
&((w_1,m_1)\dashv(w_2,m_2))\vdash(w_3,m_3)=(w_1,m_1)\dashv((w_2,m_2)\vdash(w_3,m_3))\\
&\quad=((w_1,m_1)\vdash(w_2,m_2))\dashv(w_3,m_3)=(w_1,m_1)\vdash((w_2,m_2)\dashv(w_3,m_3))=0.
\end{aligned}$$

Consequently, the axioms (D1), (D2) and (D3) of a strong doppelsemigroup are satisfied.

The proofs of the remaining cases are obvious. Thus, $\mathrm{FDSD}_n(X)$ is a strong doppelsemigroup.

Take arbitrary elements $(w_i,m_i)\in D_n\setminus\{0\}$ with $1\leqslant i\leqslant n+1$. It is clear that

$$m_1+m_2+\cdots+m_{n+1}+n+1>n.$$

From here

$$(w_1,m_1)\dashv(w_2,m_2)\dashv\ldots\dashv(w_{n+1},m_{n+1})=0.$$

At the same time, for any $(x_i,0) \in D_n \setminus \{0\}$, where $x_i \in X$ with $1 \leqslant i \leqslant n$, get

$$(x_1,0) \dashv (x_2,0) \dashv \ldots \dashv (x_n,0) = (x_1x_2\ldots x_n, n-1) \neq 0.$$

From the last arguments we conclude that $(D_n,\dashv)$ is a nilpotent semigroup of nilpotency index n.

Further note that

$$\begin{aligned} &l_{w_1w_2\ldots w_{n+1}} - m_1 - m_2 - \cdots - m_{n+1} \\ &\qquad = (l_{w_1} - m_1) + (l_{w_2} - m_2) + \cdots + (l_{w_{n+1}} - m_{n+1}) > n \end{aligned}$$

as $l_{w_i} - m_i \geq 1$ for all $1 \leqslant i \leqslant n+1$. From the above it follows that

$$(w_1,m_1) \vdash (w_2,m_2) \vdash \ldots \vdash (w_{n+1},m_{n+1}) = 0.$$

Moreover,

$$(x_1,0) \vdash (x_2,0) \vdash \ldots \vdash (x_n,0) = (x_1x_2\ldots x_n,0) \neq 0$$

for any $(x_i,0) \in D_n \setminus \{0\}$, where $x_i \in X$ with $1 \leqslant i \leqslant n$. Thus, $(D_n,\vdash)$ is a nilpotent semigroup of nilpotency index n. So, $\mathrm{FDSD}_n(X)$ is an n-dinilpotent strong doppelsemigroup.

Let us show that $\mathrm{FDSD}_n(X)$ is free in the variety of n-dinilpotent strong doppelsemigroups.

Obviously, $\mathrm{FDSD}_n(X)$ is generated by $X \times \{0\}$. Let $(K,\dashv',\vdash')$ be an arbitrary n-dinilpotent strong doppelsemigroup. Let $\beta' : X \times \{0\} \to K$ be an arbitrary map. Consider a map $\beta : X \to K$ such that $x\beta = (x,0)\beta'$ for all $x \in X$, and define a map

$$\pi : \mathrm{FDSD}_n(X) \to (K,\dashv',\vdash')$$

by

$$\omega\pi := \begin{cases} x_1\beta \dashv' \ldots \dashv' x_{i+1}\beta \vdash' \ldots \vdash' x_k\beta & \text{if } \omega = (x_1\ldots x_k, i), \\ & \quad x_p \in X,\, 1 \leqslant p \leqslant k,\, k > 1, \\ x_1\beta & \text{if } \omega = (x_1,0),\, x_1 \in X, \\ 0 & \text{if } \omega = 0. \end{cases}$$

According to Lemmas 1.2.7 and 1.2.9 π is well-defined.

To show that π is a homomorphism we will use the axioms of an n-dinilpotent strong doppelsemigroup and Lemmas 1.2.7, 1.2.11.

Let $(x_1 \dots x_k, i), (x_{k+1} \dots x_l, j) \in \mathrm{FDSD}_n(X)$, where $x_p \in X$ for $1 \leqslant p \leqslant l$. Assume

$$i + j + 2 \leqslant n \quad \text{and} \quad l - i - j - 1 \leqslant n. \tag{3.13}$$

Then, using (3.13), we get

$$\begin{aligned}
&((x_1 \dots x_k, i) \dashv (x_{k+1} \dots x_l, j))\pi \\
&\quad = (x_1 \dots x_k x_{k+1} \dots x_l, i + j + 1)\pi = x_1\beta \dashv' \dots \dashv' x_{i+j+2}\beta \vdash' \dots \vdash' x_l\beta \\
&\quad = (x_1\beta \dashv' \dots \dashv' x_{i+1}\beta \vdash' \dots \vdash' x_k\beta) \dashv' (x_{k+1}\beta \dashv' \dots \dashv' x_{k+j+1}\beta \vdash' \dots \vdash' x_l\beta) \\
&\quad = (x_1 \dots x_k, i)\pi \dashv' (x_{k+1} \dots x_l, j)\pi.
\end{aligned}$$

If $i + j + 2 > n$ or $l - i - j - 1 > n$, then

$$\begin{aligned}
&((x_1 \dots x_k, i) \dashv (x_{k+1} \dots x_l, j))\pi = 0\pi = 0 = x_1\beta \dashv' \dots \dashv' x_{i+j+2}\beta \vdash' \dots \vdash' x_l\beta \\
&\quad = (x_1\beta \dashv' \dots \dashv' x_{i+1}\beta \vdash' \dots \vdash' x_k\beta) \dashv' (x_{k+1}\beta \dashv' \dots \dashv' x_{k+j+1}\beta \vdash' \dots \vdash' x_l\beta) \\
&\quad = (x_1 \dots x_k, i)\pi \dashv' (x_{k+1} \dots x_l, j)\pi.
\end{aligned}$$

So,

$$((x_1 \dots x_k, i) \dashv (x_{k+1} \dots x_l, j))\pi = (x_1 \dots x_k, i)\pi \dashv' (x_{k+1} \dots x_l, j)\pi$$

for all $(x_1 \dots x_k, i), (x_{k+1} \dots x_l, j) \in \mathrm{FDSD}_n(X)$.

Further assume

$$i + j + 1 \leqslant n \quad \text{and} \quad l - i - j \leqslant n. \tag{3.14}$$

Using (3.14), we obtain

$$\begin{aligned}
&((x_1 \dots x_k, i) \vdash (x_{k+1} \dots x_l, j))\pi \\
&\quad = (x_1 \dots x_k x_{k+1} \dots x_l, i + j)\pi = x_1\beta \dashv' \dots \dashv' x_{i+j+1}\beta \vdash' \dots \vdash' x_l\beta \\
&\quad = (x_1\beta \dashv' \dots \dashv' x_{i+1}\beta \vdash' \dots \vdash' x_k\beta) \vdash' (x_{k+1}\beta \dashv' \dots \dashv' x_{k+j+1}\beta \vdash' \dots \vdash' x_l\beta) \\
&\quad = (x_1 \dots x_k, i)\pi \vdash' (x_{k+1} \dots x_l, j)\pi.
\end{aligned}$$

If $i+j+1>n$ or $l-i-j>n$, then

$$\begin{aligned}((x_1\dots x_k,i)\vdash(x_{k+1}\dots x_l,j))\pi &= 0\pi = 0 = x_1\beta\dashv'\dots\dashv' x_{i+j+1}\beta\vdash'\dots\vdash' x_l\beta\\ &= (x_1\beta\dashv'\dots\dashv' x_{i+1}\beta\vdash'\dots\vdash' x_k\beta)\vdash'(x_{k+1}\beta\dashv'\dots\dashv' x_{k+j+1}\beta\vdash'\dots\vdash' x_l\beta)\\ &= (x_1\dots x_k,i)\pi\vdash'(x_{k+1}\dots x_l,j)\pi.\end{aligned}$$

Thus,

$$((x_1\dots x_k,i)\vdash(x_{k+1}\dots x_l,j))\pi = (x_1\dots x_k,i)\pi\vdash'(x_{k+1}\dots x_l,j)\pi$$

for all $(x_1\dots x_k,i),(x_{k+1}\dots x_l,j)\in \mathrm{FDSD}_n(X)$.

The proofs of the remaining cases are obvious. So, π is a homomorphism. Evidently, $(x,0)\pi=(x,0)\beta'$ for all $(x,0)\in X\times\{0\}$. Since $X\times\{0\}$ generates $\mathrm{FDSD}_n(X)$, we conclude that such homomorphism π is unique. Thus, $\mathrm{FDSD}_n(X)$ is free in the variety of n-dinilpotent strong doppelsemigroups. □

Further, in this section we construct a strong doppelsemigroup which is isomorphic to the free n-dinilpotent strong doppelsemigroup of rank 1. This construction is more elegant as it is obtained only from $\mathbb{N}^0$.

Fix $n\in\mathbb{N}$ and let

$$\overline{\Upsilon}_n := \{(k,m)\in\mathbb{N}\times\mathbb{N}^0 \mid k>m,\, m+1\leqslant n,\, k-m\leqslant n\}\cup\{0\}.$$

Define operations $\dashv$ and $\vdash$ on $\overline{\Upsilon}_n$ by

$$(k_1,m_1)\dashv(k_2,m_2) := \begin{cases}(k_1+k_2,m_1+m_2+1) & \text{if } m_1+m_2+2\leqslant n,\\ & \quad k_1+k_2-m_1-m_2-1\leqslant n,\\ 0 & \text{otherwise},\end{cases}$$

$$(k_1,m_1)\vdash(k_2,m_2) := \begin{cases}(k_1+k_2,m_1+m_2) & \text{if } m_1+m_2+1\leqslant n,\\ & \quad k_1+k_2-m_1-m_2\leqslant n,\\ 0 & \text{otherwise}\end{cases}$$

and

$$(k_1,m_1)*0 := 0*(k_1,m_1) := 0*0 := 0$$

for all $(k_1,m_1),(k_2,m_2) \in \overline{\Upsilon}_n \setminus \{0\}$ and $* \in \{\dashv,\vdash\}$. The algebra obtained in this way is denoted by Υ_n. A direct verification shows that Υ_n is a strong doppelsemigroup.

Exercise 3.2.2. Prove that Υ_n is a strong doppelsemigroup.

Lemma 3.2.3. *If* $|X| = 1$*, then* $\Upsilon_n \cong \mathrm{FDSD}_n(X)$.

Proof. Let $X = \{r\}$. One can show that a map $\tau : \Upsilon_n \to \mathrm{FDSD}_n(X)$, defined by the rule

$$\omega\tau := \begin{cases} (r^k, m) & \text{if } \omega = (k,m) \in \overline{\Upsilon}_n \setminus \{0\}, \\ 0 & \text{if } \omega = 0, \end{cases}$$

is an isomorphism. □

Exercise 3.2.4. Prove that Υ_n and $\mathrm{FDSD}_n(X)$ are isomorphic if $|X| = 1$.

3.3 Free commutative and free n-nilpotent strong doppelsemigroups

As we have seen in the previous chapter commutative (n-nilpotent) doppelsemigroups are defined by the family of identities and so, they form a variety. In this section, for the indicated variety we deal with a subvariety of commutative (n-nilpotent) strong doppelsemigroups. We consider a free commutative (n-nilpotent) strong doppelsemigroup of an arbitrary rank and discuss separately free commutative (n-nilpotent) strong doppelsemigroups of rank 1.

Every commutative strong doppelsemigroup is a commutative doppelsemigroup, and conversely, by Lemma 2.2.4, every commutative doppelsemigroup is strong. So, we obtain the following lemma.

Lemma 3.3.1. *The varieties of commutative strong doppelsemigroups and of commutative doppelsemigroups coincide.*

Using the notation of Section 2.2, from the last lemma we obtain

Corollary 3.3.2. (i) $\mathrm{FDS}^*(X)$ *is the free commutative strong doppelsemigroup.*

(ii) $(T^*, \circ_a, \circ_b)$ *is the free commutative strong doppelsemigroup of rank* 1.

Now we construct a strong doppelsemigroup which is isomorphic to the free commutative strong doppelsemigroup $\mathrm{FDS}^*(X)$.

In the construction of $\widetilde{\mathrm{F}}[X]$ (see Section 3.1) instead of the free semigroup on X take the free commutative semigroup $\mathrm{F}^*[X]$ on X. In this case, denote by $\widetilde{F}^*[X]$ the algebra $(C, \dashv, \vdash)$ with operations defined by (3.1), (3.2).

Lemma 3.3.3. $\mathrm{FDS}^*(X) \cong \widetilde{F}^*[X]$.

Proof. Define the map

$$\delta : \mathrm{FDS}^*(X) \to \widetilde{F}^*[X] : (w,u) \mapsto (w, d_a(u)),$$

where $d_a(u)$ is the number of occurrences of an element a in u. By definition, $d_a(\theta) = 0$. For all $(w_1,u_1),(w_2,u_2) \in \mathrm{FDS}^*(X)$, we have

$$\begin{aligned}((w_1u_1) \prec (w_2,u_2))\delta &= (w_1w_2, u_1au_2)\delta = (w_1w_2, d_a(u_1au_2)) \\ &= (w_1w_2, d_a(u_1)+1+d_a(u_2)) = (w_1, d_a(u_1)) \dashv (w_2, d_a(u_2)) \\ &= (w_1,u_1)\delta \dashv (w_2,u_2)\delta\end{aligned}$$

and

$$\begin{aligned}((w_1,u_1) \succ (w_2,u_2))\delta &= (w_1w_2, u_1bu_2)\delta = (w_1w_2, d_a(u_1bu_2)) \\ &= (w_1w_2, d_a(u_1)+d_a(u_2)) = (w_1, d_a(u_1)) \vdash (w_2, d_a(u_2)) \\ &= (w_1,u_1)\delta \vdash (w_2,u_2)\delta.\end{aligned}$$

Thus, δ is a homomorphism.

The map δ is a surjection. Indeed, assuming $y^0 = \theta$ for $y \in \{a,b\}$, for any element $(w,m) \in \widetilde{F}^*[X]$ there exists $(w, a^m b^{l_w-m-1}) \in \mathrm{FDS}^*(X)$ such that

$$(w, a^m b^{l_w-m-1})\delta = (w, d_a(a^m b^{l_w-m-1})) = (w,m).$$

Let $(w_1,u_1) \neq (w_2,u_2)$. If $w_1 \neq w_2$, then

$$(w_1,u_1)\delta = (w_1, d_a(u_1)) \neq (w_2, d_a(u_2)) = (w_2,u_2)\delta.$$

Assume $u_1 \neq u_2$. If $l_{u_1} \neq l_{u_2}$, then it is obvious that $w_1 \neq w_2$, and from here as above $(w_1,u_1)\delta \neq (w_2,u_2)\delta$. In the case $l_{u_1} = l_{u_2}$, we have $d_a(u_1) \neq d_a(u_2)$ and so, $(w_1,u_1)\delta \neq (w_2,u_2)\delta$ again. Thereby, δ is an injection.

Thus, δ is an isomorphism. □

Further consider separately one-generated free commutative strong doppelsemigroups. Using the notation of Section 3.1, from Lemma 3.3.3 we obtain

Corollary 3.3.4. $(T^*,\circ_a,\circ_b) \cong R$.

Exercise 3.3.5. Prove Corollary 3.3.4.

Let us turn to constructing free n-nilpotent strong doppelsemigroups.

Fix $n \in \mathbb{N}$ and assume $C_n := \{(w,m) \in \widetilde{\mathrm{F}}[X] \mid l_w \leqslant n\} \cup \{0\}$. Define operations $\dashv$ and $\vdash$ on C_n by

$$(w_1,m_1) \dashv (w_2,m_2) := \begin{cases} (w_1w_2, m_1+m_2+1) & \text{if } l_{w_1w_2} \leqslant n, \\ 0 & \text{if } l_{w_1w_2} > n, \end{cases}$$

$$(w_1,m_1) \vdash (w_2,m_2) := \begin{cases} (w_1w_2, m_1+m_2) & \text{if } l_{w_1w_2} \leqslant n, \\ 0 & \text{if } l_{w_1w_2} > n \end{cases}$$

and

$$(w_1,m_1) * 0 := 0 * (w_1,m_1) := 0 * 0 := 0$$

for all $(w_1,m_1),(w_2,m_2) \in C_n \setminus \{0\}$ and $* \in \{\dashv,\vdash\}$. The algebra $(C_n,\dashv,\vdash)$ is denoted by $\mathrm{FNSD}_n(X)$.

Theorem 3.3.6. $\mathrm{FNSD}_n(X)$ *is the free n-nilpotent strong doppelsemigroup.*

Proof. Similarly to Theorem 1 from [43], the fact that $\mathrm{FNSD}_n(X)$ is an n-nilpotent strong doppelsemigroup can be proved.

Let us show that $\mathrm{FNSD}_n(X)$ is free in the variety of n-nilpotent strong doppelsemigroups.

Let $(K,\dashv',\vdash')$ be an arbitrary n-nilpotent strong doppelsemigroup and $\beta : X \to K$ an arbitrary map. Define a map

$$\varphi : \mathrm{FNSD}_n(X) \to (K,\dashv',\vdash') : \omega \mapsto \omega\varphi$$

as

$$\omega\varphi := \begin{cases} x_1\beta \dashv' \ldots \dashv' x_{i+1}\beta \vdash' \ldots \vdash' x_k\beta & \text{if } \omega = (x_1 \ldots x_k, i), \\ & \quad x_p \in X, 1 \leqslant p \leqslant k, k > 1, \\ x_1\beta & \text{if } \omega = (x_1, 0), x_1 \in X, \\ 0 & \text{if } \omega = 0. \end{cases}$$

According to Lemmas 1.2.7 and 1.2.9 φ is well-defined. Using Lemmas 1.2.7 and 1.2.11, one can prove that φ is a homomorphism. □

Exercise 3.3.7. Show that $\mathrm{FNSD}_n(X)$ is an n-nilpotent strong doppelsemigroup.

Exercise 3.3.8. Prove that the map φ defined in the proof of Theorem 3.3.6 is a homomorphism.

Now we present an alternative construction for the free n-nilpotent strong doppelsemigroup of rank 1.

Fix $n \in \mathbb{N}$ and let $\overline{P}_n := \{(k,m) \in R \mid k \leqslant n\} \cup \{0\}$ (see Section 3.1). Define operations $\dashv$ and $\vdash$ on $\overline{P}_n$ by

$$(k_1,m_1) \dashv (k_2,m_2) := \begin{cases} (k_1+k_2, m_1+m_2+1) & \text{if } k_1+k_2 \leqslant n, \\ 0 & \text{if } k_1+k_2 > n, \end{cases}$$

$$(k_1,m_1) \vdash (k_2,m_2) := \begin{cases} (k_1+k_2, m_1+m_2) & \text{if } k_1+k_2 \leqslant n, \\ 0 & \text{if } k_1+k_2 > n \end{cases}$$

and

$$(k_1,m_1) * 0 := 0 * (k_1,m_1) := 0 * 0 := 0$$

for all $(k_1,m_1),(k_2,m_2) \in \overline{P}_n \setminus \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra $(\overline{P}_n, \dashv, \vdash)$ is denoted by P_n. One can check that P_n is a strong doppelsemigroup.

Exercise 3.3.9. Show that P_n is a strong doppelsemigroup.

Lemma 3.3.10. *If* $|X| = 1$*, then* $P_n \cong \mathrm{FNSD}_n(X)$.

Proof. Let $X = \{r\}$. An easy verification shows that a map $\xi : P_n \to \mathrm{FNSD}_n(X)$, defined by

$$\omega\xi := \begin{cases} (r^k, m) & \text{if } \omega = (k,m) \in \overline{P}_n \setminus \{0\}, \\ 0 & \text{if } \omega = 0, \end{cases}$$

is an isomorphism. □

Exercise 3.3.11. Prove that P_n and $\mathrm{FNSD}_n(X)$ are isomorphic if $|X|=1$.

3.4 The least congruences on a free strong doppelsemigroup

In the previous chapter we have given the definitions of an n-dinilpotent congruence, of a commutative congruence and of an n-nilpotent congruence on a doppelsemigroup. The contents of this section is the description of the least n-dinilpotent congruence, of the least commutative congruence and of the least n-nilpotent congruence on a free strong doppelsemigroup. We also establish that the automorphism groups of $\mathrm{FSD}(X)$, $\mathrm{FDSD}_n(X)$ and $\mathrm{FNSD}_n(X)$ are isomorphic to the symmetric group on X.

Let $\widetilde{\mathrm{F}}[X]$ be the free strong doppelsemigroup (see Section 3.1). Fix $n\in\mathbb{N}$ and let

$$Y_n := \{(w,m)\in\widetilde{\mathrm{F}}[X] \mid m+1>n \text{ or } l_w-m>n\}.$$

Define a relation $\varepsilon_{(n)}$ on $\widetilde{\mathrm{F}}[X]$ by

$$(w_1,m_1)\varepsilon_{(n)}(w_2,m_2)$$

if and only if

$$(w_1,m_1)=(w_2,m_2) \quad \text{or} \quad (w_1,m_1),(w_2,m_2)\in Y_n.$$

By $\star$ denote the operation on $\mathrm{F}^*[X]$. Take $(x_1\dots x_k,i),(y_1\dots y_h,j)\in\widetilde{\mathrm{F}}[X]$, where $x_p,y_q\in X$ for $1\leqslant p\leqslant k$, $1\leqslant q\leqslant h$, and define a relation η on $\widetilde{\mathrm{F}}[X]$ by

$$(x_1\dots x_k,i)\eta(y_1\dots y_h,j)$$

if and only if

$$x_1\star\dots\star x_k=y_1\star\dots\star y_h,\ i=j.$$

For every $n\in\mathbb{N}$ define a relation $\wp_{(n)}$ on $\widetilde{\mathrm{F}}[X]$ by

$$(w_1,m_1)\wp_{(n)}(w_2,m_2)$$

if and only if

$$(w_1, m_1) = (w_2, m_2) \quad \text{or} \quad l_{w_1} > n,\, l_{w_2} > n.$$

Theorem 3.4.1. *Let* $\widetilde{\mathrm{F}}[X]$ *be the free strong doppelsemigroup. Then*

(i) $\varepsilon_{(n)}$ *is the least n-dinilpotent congruence on* $\widetilde{\mathrm{F}}[X]$*;*

(ii) η *is the least commutative congruence on* $\widetilde{\mathrm{F}}[X]$*;*

(iii) $\wp_{(n)}$ *is the least n-nilpotent congruence on* $\widetilde{\mathrm{F}}[X]$.

Proof. (i) Define a map $\kappa_n : \widetilde{\mathrm{F}}[X] \to \mathrm{FDSD}_n(X)$ by

$$(w,m)\kappa_n := \begin{cases} (w,m) & \text{if } m+1 \leqslant n,\, l_w - m \leqslant n, \\ 0 & \text{if } m+1 > n \text{ or } l_w - m > n, \end{cases}$$

where $(w,m) \in \widetilde{\mathrm{F}}[X]$. Show that κ_n is a homomorphism.

Take $(w_1,m_1),(w_2,m_2) \in \widetilde{\mathrm{F}}[X]$. Let (3.10) and (3.5) hold. From (3.10) and (3.5) it follows

$$m_1 + 1 < n, \qquad m_2 + 1 < n, \qquad l_{w_1} - m_1 \leqslant n \quad \text{and} \quad l_{w_2} - m_2 \leqslant n. \tag{3.15}$$

Using (3.10), (3.5) and (3.15), we get

$$\begin{aligned} ((w_1,m_1) \dashv (w_2,m_2))\kappa_n &= (w_1w_2, m_1+m_2+1)\kappa_n = (w_1w_2, m_1+m_2+1) \\ &= (w_1,m_1) \dashv (w_2,m_2) = (w_1,m_1)\kappa_n \dashv (w_2,m_2)\kappa_n. \end{aligned}$$

If $m_1 + m_2 + 2 > n$ or $l_{w_1w_2} - m_1 - m_2 - 1 > n$, then

$$((w_1,m_1) \dashv (w_2,m_2))\kappa_n = (w_1w_2, m_1+m_2+1)\kappa_n = 0 = (w_1,m_1)\kappa_n \dashv (w_2,m_2)\kappa_n.$$

Let further (3.7) and (3.12) hold. From (3.7) and (3.12) it follows

$$m_1 + 1 \leqslant n, \qquad m_2 + 1 \leqslant n, \qquad l_{w_1} - m_1 < n \quad \text{and} \quad l_{w_2} - m_2 < n. \tag{3.16}$$

Using (3.7), (3.12) and (3.16), we obtain

$$((w_1,m_1)\vdash(w_2,m_2))\kappa_n = (w_1w_2,m_1+m_2)\kappa_n = (w_1w_2,m_1+m_2)$$
$$= (w_1,m_1)\vdash(w_2,m_2) = (w_1,m_1)\kappa_n\vdash(w_2,m_2)\kappa_n.$$

If $m_1+m_2+1>n$ or $l_{w_1w_2}-m_1-m_2>n$, then

$$((w_1,m_1)\vdash(w_2,m_2))\kappa_n = (w_1w_2,m_1+m_2)\kappa_n = 0 = (w_1,u_1)\kappa_n\vdash(w_2,u_2)\kappa_n.$$

Thus, κ_n is a surjective homomorphism. By Theorem 3.2.1, $\mathrm{FDSD}_n(X)$ is the free n-dinilpotent strong doppelsemigroup. Then Δ_{κ_n} is the least n-dinilpotent congruence on $\widetilde{\mathrm{F}}[X]$. From the definition of κ_n it follows that $\Delta_{\kappa_n} = \varepsilon_{(n)}$.

(ii) Define a map $\zeta : \widetilde{\mathrm{F}}[X] \to \widetilde{\mathrm{F}}^*[X]$ by

$$\omega\zeta := \begin{cases} (x_1\star\ldots\star x_k, i) & \text{if } \omega=(x_1\ldots x_k,i),\, k>1,\\ (x_1,0) & \text{if } \omega=(x_1,0).\end{cases}$$

It is immediate to show that ζ is a surjective homomorphism. By Lemma 3.3.3, $\widetilde{\mathrm{F}}^*[X]$ is the free commutative strong doppelsemigroup. Then Δ_ζ is the least commutative congruence on $\widetilde{\mathrm{F}}[X]$. From the definition of ζ it follows that $\Delta_\zeta = \eta$.

(iii) Consider a map $\rho_n : \widetilde{\mathrm{F}}[X] \to \mathrm{FNSD}_n(X)$ defined by the rule

$$(w,m)\rho_n := \begin{cases} (w,m) & \text{if } l_w \leqslant n,\\ 0 & \text{if } l_w > n,\end{cases}$$

where $(w,m)\in\widetilde{\mathrm{F}}[X]$. Similarly to the proof of Theorem 4 from [43], the fact that ρ_n is a surjective homomorphism can be proved. According to Theorem 3.3.6 $\mathrm{FNSD}_n(X)$ is the free n-nilpotent strong doppelsemigroup. It means that Δ_{ρ_n} is the least n-nilpotent congruence on $\widetilde{\mathrm{F}}[X]$. From the construction of ρ_n it follows that $\Delta_{\rho_n} = \wp_{(n)}$. □

Exercise 3.4.2. Show that the maps ζ and ρ_n defined in the proof of Theorem 3.4.1 are surjective homomorphisms.

Recall that we denote the symmetric group on X by $\mathfrak{I}[X]$ and the automorphism group of a doppelsemigroup D' by $\mathrm{Aut}\,D'$.

It is not difficult to see that the free algebras constructed in Sections 3.1, 3.2 and 3.3 are determined uniquely up to isomorphism by cardinality of the set X. Hence, obtain the following lemma.

Lemma 3.4.3. $\mathrm{Aut}\,\mathrm{FSD}(X) \cong \mathrm{Aut}\,\mathrm{FDSD}_n(X) \cong \mathrm{Aut}\,\mathrm{FNSD}_n(X) \cong \mathfrak{I}[X]$.

Exercise 3.4.4. Prove Lemma 3.4.3.

Note that by Lemma 2.2.17, $\mathrm{Aut}\,\mathrm{FDS}^*(X) \cong \mathfrak{I}[X]$.

Now we formulate one open problem.

Open Problem 3.4.5. Let R be a class of universal algebras. It is well known that the free product in R always exists if R is a variety of universal algebras. The free product of arbitrary dimonoids and the free product of arbitrary doppelsemigroups were constructed in Section 2.1 (see also [47] and [48], respectively). It is natural to raise the following problem.

Construct a free product of strong doppelsemigroups.

At the end of the chapter we show connections between doppelsemigroups and other algebraic structures.

One of the important motivations of studying doppelsemigroups comes from their connections to duplexes, restrictive bisemigroups and trioids causing the greatest interest from the point of view of applications. In [31], the notion of a duplex, that is, a nonempty set equipped with two binary associative operations, was introduced and the free duplex was constructed. Duplexes with operations $\dashv$ and $\vdash$ satisfying the axioms (D1), (D2) were considered in [31], and in this work the free doppelsemigroup of rank 1 was given. The operations of one-generated free doppelsemigroups were used in [25]. Duplexes with two binary idempotent operations $\dashv$ and $\vdash$ satisfying the axiom (D1) (so-called restrictive bisemigroups) were studied in the work of Schein [34]. The axiom (D2) also appears in defining identities of trialgebras and of trioids introduced by Loday and Ronco [26] (see also [51]) in the context of algebraic topology.

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A doppelalgebra is an algebra defined on a vector space with two binary linear associative operations. Doppelalgebras play a prominent role in algebraic *K*-theory. We consider doppelsemigroups, that is, sets with two binary associative operations satisfying the axioms of a doppelalgebra. Doppelsemigroups are a generalization of semigroups and they have relationships with such algebraic structures as interassociative semigroups, restrictive bisemigroups, dimonoids, and trioids. In the lecture notes numerous examples of doppelsemigroups and of strong doppelsemigroups are given. The independence of axioms of a strong doppelsemigroup is established. A free product in the variety of doppelsemigroups is presented. We also construct a free (strong) doppelsemigroup, a free commutative (strong) doppelsemigroup, a free n-nilpotent (strong) doppelsemigroup, a free n-dinilpotent (strong) doppelsemigroup and a free left n-dinilpotent doppelsemigroup. Moreover, the least commutative congruence, the least n-nilpotent congruence, the least n-dinilpotent congruence on a free (strong) doppelsemigroup and the least left n-dinilpotent congruence on a free doppelsemigroup are characterized. The book addresses graduate students, post-graduate students, researchers in algebra and interested readers.

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