The Navier-Stokes Equations for Elliptic Quasicomplexes

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Die selbständige und eigenhändige Anfertigung versichere ich an Eides statt.

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Contents

Introduction

The main equations of mathematical physics describe the dynamics of processes whose steady state is modeled on elliptic equations. So are for example the classical heat and wave equations. On the other hand, elliptic equations are nowadays interpreted mostly within the framework of elliptic complexes which are subjects of homological algebra, a part of geometry. For this reason, they make still sense in the global context of analysis on manifolds. As but two examples of great importance for real and complex (analytic) geometry we mention the de Rham and Dolbeault complexes. To any of these complexes one is able to assign a so-called symbol complex of finite-dimensional vector spaces and their linear mappings. Then, by the ellipticity is meant that the corresponding complex of symbols is exact. Unfortunately, the complexes do not survive even under those perturbations of the differential which do not affect the symbol complex. In that sense it is natural to pass from complexes to more stable subjects called quasicomplexes. They are characterized by the property that their curvature, i.e., the differential applied twice, is small in some reasonable sense, e.g., is compact or belongs to some operator ideal. A brief survey of quasicomplexes is presented in Chapter 1.

To illustrate this, consider the de Rham complex on a compact smooth manifold $\mathcal{X}$ (possibly, with boundary)

$$0 \to \Omega^0(\mathcal{X}) \stackrel{d}{\to} \Omega^1(\mathcal{X}) \stackrel{d}{\to} \ldots \stackrel{d}{\to} \Omega^n(\mathcal{X}) \to 0$$

where $\Omega^i(\mathcal{X}) := C^\infty(\mathcal{X}, \Lambda^i)$ is the space of all differential forms of degree i with smooth coefficients on $\mathcal{X}$, d is the exterior derivative and n stands for the dimension of $\mathcal{X}$. Fix a differential form $a \in \Omega^1(\mathcal{X})$ and consider the operator $d + a$ which is defined by $(d + a)u = du + a \wedge u$ for $u \in \Omega^i(\mathcal{X})$. It maps $\Omega^i(\mathcal{X})$ continuously into $\Omega^{i+1}(\mathcal{X})$ for every $i = 0, 1, \ldots$ and satisfies

$$\begin{aligned} (d+a)^2 u &= (d+a)(du + a \wedge u) \\ &= d^2 u + da \wedge u - a \wedge du + a \wedge du + a^2 \wedge u \\ &= da \wedge u \end{aligned}$$

for all $u \in \Omega^i(\mathcal{X})$, since $a \wedge a = 0$. Hence, we obtain a new complex on $\mathcal{X}$ provided that the differential form a is closed, i.e., $da = 0$. Otherwise we get a sequence

$$0 \longrightarrow \Omega^0(\mathcal{X}) \xrightarrow{d+a} \Omega^1(\mathcal{X}) \xrightarrow{d+a} \dots \xrightarrow{d+a} \Omega^n(\mathcal{X}) \longrightarrow 0 \tag{0.0.1}$$

whose curvature is given by the differential operator $u \mapsto da \wedge u$ of order zero.

When evaluated in appropriate Sobolev space, the curvature is a compact operator, this means, $d + a$ is a "small" perturbation of the exterior derivative. The sequence of symbols corresponding to (0.0.1) is actually a complex, for it coincides with the symbol sequence of the de Rham complex. Hence, the notion of ellipticity still applies to sequence (0.0.1). In this way we obtain what is referred to as quasicomplexes. They bear a rich structure reflecting on geometry of $\mathcal{X}$. The Laplacian of quasicomplex (0.0.1) reduces to the Laplacian of the de Rham complex up to lower order terms. This recovers once again the fact that quasicomplex (0.0.1) is elliptic. Our example motivates rather strikingly the study of classical boundary value problems for the Laplace operators related to elliptic quasicomplexes. The basic stationary boundary value problem is the so-called Neumann problem after Spencer which appears when one tries to extend the Hoge theory for compact closed manifolds to compact manifolds with boundary. The corresponding direct sum decomposition is a far-reaching development of the classical Helmholtz decomposition which is a necessary tool to eliminate the pressure from the Navier-Stokes equations. These latter constitute the main problem of the present thesis. The Navier-Stokes equations present an evolution equation for the Laplace operator of the de Rham complex at step 1. This is a parabolic system containing an additional nonlinear term of very involved geometric structure. An efficient study of the Navier-Stokes equations should include the study of their linearisations which prove to be parabolic systems whose coefficients have restricted smoothness. To be more prepared for the study of such problems, we first examine the techniques of quasicomplexes on considering the first mixed problem for the generalised Lamé system related to an elliptic quasicomplex. It should be noted that the theory developed in this thesis is new, for the paper [MT14] deals with complexes.

In his work on a systematic dynamical theory of elasticity Gabriel Lamé in mid 1881 derived from Newtonian mechanics his basic equations which are also the conditions for equilibrium. From those he went on to derive what are now known as nonstationary Lamé equations in elastodynamics

$$\rho u''_{tt} = -\mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u + f, \tag{0.0.2}$$

where $u : \mathcal{X} \times (0,T) \to \mathbb{R}^3$ is a search-for displacement vector, ρ is the mass density, λ and μ are physical characteristics of the body under consideration called Lamé constants, $\Delta u = -u''_{x_1x_1} - u''_{x_2x_2} - u''_{x_3x_3}$ is the nonnegative Laplace operator in $\mathbb{R}^3$, and f is the density vector of outer forces, see [ES75], [KGBB76], [LL70], [TS82], and elsewhere. Here, $\mathcal{X}$ stands for a bounded domain in $\mathbb{R}^3$ whose boundary is assumed to be smooth enough. Hence, to specify a particular solution of nonstationary Lamé equations, we consider the first mixed problem for (0.0.2) in the cylinder $\mathcal{X} \times (0,T)$ by posing the initial conditions

$$\begin{array}{rcll} u(x,0) & = & u_0(x), & \text{for} \quad x \in \mathcal{X}, \\ u'_t(x,0) & = & u_1(x), & \text{for} \quad x \in \mathcal{X}, \end{array} \tag{0.0.3}$$

on the lower basis of the cylinder and a Dirichlet condition

$$u(x,t) \;=\; u_l(x,t), \quad \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T), \tag{0.0.4}$$

on the lateral surface.

When working in adequate function spaces surviving under restriction to the lateral boundary, one can assume without loss of generality that $u_l \equiv 0$, for if not, one first solves the Dirichlet problem with data on $\partial\mathcal{X} \times [0,T]$ in the class of smooth functions.

To a certain extent the theory of mixed problems for hyperbolic partial differential equations with variable coefficients is a completion of the classical area studying the Cauchy problem and mixed problem for the wave equation. The fundamental idea of J. Leray in the early 1950s is that the energy form corresponding to a hyperbolic operator with simple real characteristics is an elliptic form with parameter, which allows one to obtain estimates in the case of variable coefficients. For a recent account of the theory we refer the reader to Chapter 3 in [GV96]. The energy method for hyperbolic equations takes a considerable part in [GV96]. This method automatically extends to $2b$-parabolic differential equations with variable coefficients. Within the framework of energy method the theories of hyperbolic and parabolic equations can be combined into one theory of operators with dominating principal quasihomogeneous part.

In the second chapter we apply the theory to the first mixed problem for a generalised Lamé system. While the classical Lamé system of (0.0.2) stems from dynamical theory of elasticity, the generalised Lamé system is well motivated by its origin in homological algebra. The chapter is intended to bring together two areas of mathematics, one of these being applied and the other area purely theoretical. This is a part of our program to specify the main equations of applied mathematics within the framework of differential geometry. Although the theory of mixed problems for equations with dominating principal quasihomogeneous part is well understood, see [GV96], the focus of the present work is mainly on the study of a very particular and well motivated class of hyperbolic equations to which the general theory applies successfully.

Our approach makes no appeal to the theory of [GV96] and it is much more delicate than that of [GV96]. Using the geometric structure of the generalised Lamé system, we develop the Galerkin method which enables us to construct an approximate solution of the mixed problem. We also prove the existence of a classical solution.

Chapters 3, 4 and 5 are devoted to the generalised Navier-Stokes equations for elliptic complexes. The problem of describing the dynamics of incompressible viscous fluid is of great importance in applications. In 2006 the Clay Mathematics Institute announced it as the sixth prize millennium problem, see [Fef00]. The dynamics is described by the Navier-Stokes equations and the problem consists in finding a classical solution to the equations. By classical we would mean here a solution of a class which is good motivated by applications and for which a uniqueness theorem is available. Essential contributions are published in the research articles [Ler34a, Ler34b], [Kol42], [Hop51], [LS60], [Sol03] as well as surveys and books [Lad70, Lad03]), [Lio61, Lio69], [Tem79], [FV80], etc.

In physics by the Navier-Stokes equations is meant the impulse equation for the flow. In the computational fluid dynamics the impulse equation is enlarged by the continuity and energy equations.

The impulse equation of dynamics of (compressible) viscous fluid was formulated in differential form independently by Claude Navier (1827) and George Stokes (1845). This is

$$\rho(u'_t + u'_x u) = -\mu \varDelta u + (\lambda+\mu)\,\nabla\,\mathrm{div}\,u - \nabla p + f, \qquad (0.0.5)$$

where $u : \mathcal{X} \times (0,T) \to \mathbb{R}^3$ and $p : \mathcal{X} \times (0,T) \to \mathbb{R}$ are the search-for velocity vector field and pressure of a particle in the flow, respectively, ρ is the mass density, λ and μ are the first Lamé constant and the dynamical viscosity of the fluid under consideration, respectively, by u'_x is meant the Jacobi matrix of u in the spatial variables, $\varDelta u = -u''_{x_1x_1} - u''_{x_2x_2} - u''_{x_3x_3}$ is the nonnegative Laplace operator in $\mathbb{R}^3$, and f is the density vector of outer forces, such as gravitation and so on, see [Tem79] and elsewhere. Here, $\mathcal{X}$ stands for a bounded domain in $\mathbb{R}^3$ whose boundary is assumed to be smooth enough. Hence, to specify a particular solution of (0.0.5), we consider the first mixed problem in the cylinder $\mathcal{X} \times (0,T)$ by posing the initial conditions on the lower basis of the cylinder and a Dirichlet condition on the lateral surface.

To wit,

$$\begin{array}{rcll} u(x,0) & = & u_0(x), & \text{for} \quad x \in \mathcal{X}, \\ u(x,t) & = & u_l(x,t), & \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T). \end{array} \tag{0.0.6}$$

It is worth pointing out that the pressure p is determined solely from the impulse equation up to an additive constant. To fix this constant it suffices to put a moment condition on p.

If the density ρ does not change along the trajectories of particles, the flow is said to be incompressible. It is the assumption that is most often used in applications. For incompressible fluid the continuity equation takes the especially simple form $\operatorname{div} u = 0$ in $\mathcal{X} \times (0,T)$, i.e., the vector field u should be divergence free (solenoidal). In many practical problems the flow is not only incompressible but it has even a constant density. In this case one can divide by ρ in (0.0.5) which reduces the impulse equation to

$$\begin{array}{rcl} u'_t + u'_x u & = & -\nu \Delta u - \nabla p + f, \\ \operatorname{div} u & = & 0 \end{array} \tag{0.0.7}$$

in $\mathcal{X} \times (0,T)$, where $\nu = \mu/\rho$ is the so-called kinematic viscosity and we use the same letters p and f to designate p/ρ and f/ρ. In this way we obtain what is referred to as but the Navier-Stokes equations.

Using manipulations of the nonlinear term $u'_x u$ Hopf [Hop51] proved that equations (0.0.7) under homogeneous data (0.0.6) have a weak solution satisfying the estimate

$$\begin{array}{rl} & \|u(\cdot,t)\|^2_{L^2(\mathcal{X},\mathbb{R}^3)} + \displaystyle\int_0^t \|u'(\cdot,t')\|^2_{L^2(\mathcal{X},\mathbb{R}^{3\times 3})} dt' \\ \leq & \|u_0\|^2_{L^2(\mathcal{X},\mathbb{R}^3)} + \displaystyle\int_0^t \|f(\cdot,t')\|^2_{L^2(\mathcal{X},\mathbb{R}^3)} dt' \end{array}$$

for all $t < T$.

However, in this full generality no uniqueness theorem for a weak solution has been known. On the other hand, under stronger conditions on the solution, it is unique, cf. [Lad70, Lad03]. In contrast to [Fef00], we believe that the main problem concerning the Navier-Stokes equations consists in removal of this gap, i.e., in specifying adequate function spaces in which both existence and uniqueness theorems are valid. From the viewpoint of pure mathematics this would initiate new problems similar to that the complex Neumann problem gave rise to the study of subelliptic operators, and even greater ones, let alone phenomena evoked by nonlinear perturbations.

In this work we are aimed at elaborating another insight into the classical Navier-Stokes equations. In consists in specifying this problem within the framework of global analysis of elliptic complexes on manifolds. In this manner we obtain what is referred to as the generalised Navier-Stokes equations on $\mathcal{X}$.

Chapters 6 and 7 deal with the Neumann problem after Spencer for elliptic quasicomplexes. This theory has been used to eliminate the pressure from the generalised Navier-Stokes equations. In the theory of elliptic linear partial differential equations the terms coercive is used to describe a certain class of boundary value problems for elliptic systems $Lu = f$, in which, for functions u satisfying the boundary conditions, it is possible to estimate in relevant norm all the derivatives of u of order equal to the order m of L in terms of the norm of Lu and in terms of suitable norms for the given boundary data. That is, there is no loss in derivatives - in going from Lu to u we gain precisely m derivatives.

In connection with the study of inhomogeneous overdetermined systems of partial differential equations, Spencer [Spe63] proposed a method which leads in some cases to well determined elliptic boundary value problems which fail however to be coercive operators in the proper L^2 Sobolev spaces. In the case where the system consists of the inhomogeneous Cauchy-Riemann equations for differential forms, the resulting boundary value problem is called the $\bar{\partial}$-Neumann problem. Extending a basic inequality of [Mor63] this problem was solved in [Koh63] for forms on strongly pseudoconvex domains on a complex manifold. The elliptic operator L in the $\bar{\partial}$-Neumann problem is of second order, and in going from Lu to u, in a pseudoconvex domain, one gains only one derivative instead of two. This makes the problem more difficult than a coercive one, the main difficulty occurring in the proof of regularity at the boundary. The regularity proof in [Koh63] is rather complicated. A simpler proof was found in [Mor63]. In [KN65] is also presented a simpler proof which yields a rather general theorem for elliptic equations, Theorem 7.1.1. The result for the $\bar{\partial}$-Neumann problem is a very special case of this theorem.

In [KN65], the results are expressed in a fairly general form which may eventually prove useful in carrying out Spencer's attack on overdetermined equations. For functions u and v with values in $\mathbb{C}^{k_i}$ or in a smooth vector bundle F^i over a compact manifold with boundary $\mathcal{X}$ one considers a sesquilinear form $Q(u, v)$ which is an integral over $\mathcal{X}$ of an expression involving derivatives of u and v. For functions u, v lying in a linear space $\mathcal{D}$ determined by certain boundary conditions one is looking for a solution $u \in \mathcal{D}$ of $Q(u, v) = (f, v)$ for all $v \in \mathcal{D}$, where f is a given function with values in F^i and $(\cdot, \cdot)$ denotes the L^2 scalar product of sections in $\mathcal{X}$. The form Q is primarily assumed to be almost Hermitean and that $\Re\, Q(u, u) \geq 0$ for $u \in \mathcal{D}$. The paper [KN65] is aimed at obtaining solutions that are regular in $\mathcal{X}$ up to the boundary. The solutions then lie in $\mathcal{D}$ and satisfy also "free" or "natural" boundary conditions.

It was Sweeney, a PhD student of Spencer, who developed the approach of [KN65] within the framework of overdetermined systems, see [Swe69], [Swe71], [Swe72], [Swe76], [HS86]. A differential operator A^0 is said to be overdetermined if there is a differential operator $A^1 \neq 0$ with the property that $A^1 A^0 \equiv 0$. Then, for the local solvability of the inhomogeneous equation $A^0 u = f$ it is necessary that the right-hand side satisfies $A^1 f = 0$. The above papers deal with sesquilinear forms $Q(f,g) = (A^1 f, A^1 g) + (A^{0*} f, A^{0*} g) + (f,g)$ called the Dirichlet forms. We consider the Neumann problem after Spencer for more general Dirichlet forms which correspond to quasicomplexes of differential operators.

Assume that $\mathcal{X}$ is a compact manifold with boundary. For each nonnegative integer i let F^i be a vector bundle over $\mathcal{X}$, and let A^i be a first order differential operator which maps C^∞ sections of F^i to C^∞ sections of F^{i+1}. Suppose that the compositions $A^i A^{i-1}$ are all of order not exceeding 1 so that the operators A^i form a sequence

$$0 \longrightarrow C^\infty(\mathcal{X}, F^0) \xrightarrow{A^0} C^\infty(\mathcal{X}, F^1) \xrightarrow{A^1} \dots \xrightarrow{A^{N-1}} C^\infty(\mathcal{X}, F^N) \longrightarrow 0 \quad (0.0.8)$$

whose curvature $A^i A^{i-1}$ evaluated in appropriate Sobolev spaces is compact at each step. The assumption that all of A^i have order 1 simplifies the notation essentially. This will usually not be the case in practice. However, this assumptions is fulfilled for classical complexes of differential operators which arise in differential geometry, see [Wel73], [Tar95, Ch. 1] and elsewhere. The purpose of this work is to show how one obtains existence and regularity theorems for the Neumann problem after Spencer, see (6.4.1), if an estimate of the form

$$\|f\|_{1/2}^2 \leq c \left(\|A^i f\|^2 + \|A^{i-1*} f\|^2 + \|f\|^2 \right)$$

holds for all smooth f satisfying certain boundary conditions. In the case of zero curvature, i.e., $A^i A^{i-1} \equiv 0$, basic results are contained in [KN65]. However, if $A^i A^{i-1} \not\equiv 0$, the theorems of [KN65] do not apply. Our contribution rests on a detailed study of the boundary conditions which settles the matter of "free boundary conditions."

A major part of Chapter 7 is concerned with solving equations of the type $Q(u,v) = (f,v)$ for all $v \in \mathcal{D}$. The form $Q(u,v)$ is an integral of a sum of squares. In [KN65] also more general forms are considered, admitting a mild non-Hermitean part. Since the problem is not assumed to be coercive, one must be rather careful in handling the error terms which usually arise from derivatives of the coefficients, when deriving estimates. On assuming that $Q(u,u) \geq \|u\|^2$ for u in a subspace $\mathcal{D}$ (after adding (u,v) to Q), and that $Q(u,u)^{1/2}$ is compact with respect to the L^2 norm, i.e., that any sequence

(u_ν) with $Q(u_\nu, u_\nu)$ bounded has a convergent subsequence in L^2, one shows that the equation can be solved, the space of solutions of the homogeneous equation is finite dimensional, and that the solution operator is compact. On assuming a gain of derivatives we readily present a regularity theorem for solutions.

Similarly to [KN65], our results are not complete in themselves, but are meant as a technical aid in obtaining more definitive results. For no indication is given when a priori estimates hold. Indeed it seems to be rather difficult to say in general when they can be established.

It is very easy to prove the existence of a Hilbert space solution of the equation $Q(u, v) = (f, v)$ for all $v \in \mathcal{D}$. But we are interested in those solutions which are smooth in $\mathcal{X}$. To do this we derive a priori estimates for the L^2 norms of derivatives of u. Near the boundary we first estimate derivatives in directions tangential to the boundary by essentially setting v equal to tangential derivatives of u. To this end, we assume that the boundary conditions are, in some sense, invariant with respect to translation along the boundary. Then, assuming the boundary to be noncharacteristic, we estimate also the normal derivatives. Then we are faced with the standard problem of going from a priori estimates of derivatives to the proof of their existence.

There is, as yet, no general theorem which states that whenever one has a priori estimates for derivatives of a function then, in fact, these derivatives exist. In each individual case one has to prove this separately, and this is often the most tedious and technical aspect of existence theorems. One way which is often used is to apply a smoothing operator to the solution. In order to apply the a priori estimates to the resulting functions it is necessary to handle the term arising from the commutator of the differential operator and the smoothing operator. This is sometimes rather complicated. This method is used extensively in the book [Hoe63], where a number of special lemmas concerned with the commutators of differential and smoothing operators are given.

In [KN65] another method of smoothing is used. It is more closely related to differential operators, and has proved useful in a wide class of problems. It consists in adding ε times an elliptic operator so that the resulting equation becomes elliptic and coercive under the given boundary conditions for $\varepsilon > 0$, even if the original equation is not elliptic. Thus we rely on the fact that the differentiability theorems are well known for such problems and we wish to reduce the differentiability theorems to those for coercive elliptic problems. The new equation, being coercive elliptic, has a smooth solution u_ε in $\mathcal{X}$ and, if the elliptic term has been added in a suitable way, the method of obtaining a priori estimates applies as well to the new equation as to the original one, and yields estimates for the derivatives of u_ε which are independent of ε. Letting $\varepsilon \to 0$ through a sequence ε_ν, it follows that a subsequence of the u_{ε_ν}, together with derivatives, converges to a smooth solution of the original problem.

This method, therefore, does not show that a generalised solution u is smooth, but constructs a smooth solution. If there is uniqueness among generalised solutions, then one may also infer that u is smooth. The situation is very similar to that with Hopf's weak solution to the Navier-Stokes equations.

Chapter 1

Elliptic quasicomplexes

1.1 Complexes

Complexes of operators are generalizations of single operators. If $A\colon V \to W$ is a linear map between vector spaces, then A defines the so-called short complex
$$0 \to V \stackrel{A}{\to} W \to 0.$$

By a (cochain) complex $V^\cdot$ is meant a sequence of linear maps between vector spaces
$$V^\cdot\colon\ 0 \to V^0 \stackrel{A^0}{\to} V^1 \stackrel{A^1}{\to} \ldots \stackrel{A^{N-1}}{\to} V^N \to 0$$
with $A^{i+1}A^i = 0$ for all $i = 0, 1, \ldots, N-1$. For such a complex $V^\cdot$, we set $V^i = 0$ for $i \in \mathbb{Z} \setminus \{0, ..., N\}$ and $A^i = 0$ for $i \in \mathbb{Z} \setminus \{0, ..., N-1\}$. To each complex the differential A is associated by $Av = A^i v$ for $v \in V^i$. Since $A^2 = 0$ the differential is nilpotent. We will write $(V^\cdot, A)$ instead of $V^\cdot$, if we want to emphasize which differential is used.

For any $v \in V^{i-1}$,
$$A^i(A^{i-1}v) = (A^i A^{i-1})v = 0$$
whence $\operatorname{im} A^{i-1} \subset \ker A^i$, i.e., the image of A^{i-1} is a subspace of the kernel of A^i. The quotient space
$$H^i(V^\cdot) := \ker A^i / \operatorname{im} A^{i-1}$$
is called the cohomology of the complex at step i. A complex is said to be exact at step i if $H^i(V^\cdot) = 0$.

Remark 1.1.1. *For a short complex we have*
$$\begin{array}{rclclcl} H^0(V^\cdot) &=& \ker A^0/\operatorname{im} A^{-1} &=& \ker A/\{0\} &\cong& \ker A, \\ H^1(V^\cdot) &=& \ker A^1/\operatorname{im} A^0 &=& W/\operatorname{im} A &=:& \operatorname{coker} A. \end{array}$$

Assume that $V^\cdot$ is a complex with finite dimensional cohomology. The Euler characteristic of $V^\cdot$ is defined by

$$\chi(V^\cdot) := \sum_i (-1)^i \dim H^i(V^\cdot).$$

Example 1.1.2. If $V^\cdot$ is a short complex, then from Remark 1.1.1 it follows that

$$\chi(V^\cdot) = \dim H^0(V^\cdot) - \dim H^1(V^\cdot) = \dim \ker A - \dim \operatorname{coker} A,$$

which is the index of the operator A.

Let $\{V^\cdot, A\}$ and $\{W^\cdot, B\}$ be two complexes. Without restriction of generality we can take complexes of the same length N. A homomorphism of the complexes $V^\cdot$ and $W^\cdot$ is a sequence of linear maps $L^i \colon V^i \to W^i$ which makes the diagram

$$\begin{array}{ccccccccc}
0 & \to & V^0 & \stackrel{A^0}{\to} & V^1 & \stackrel{A^1}{\to} & \dots & \stackrel{A^{N-1}}{\to} & V^N & \to & 0 \\
 & & \downarrow L^0 & & \downarrow L^1 & & & & \downarrow L^N & & \\
0 & \to & W^0 & \stackrel{B^0}{\to} & W^1 & \stackrel{B^1}{\to} & \dots & \stackrel{B^{N-1}}{\to} & W^N & \to & 0
\end{array}$$

commutative, i.e. $L^{i+1}A^i = B^i L^i$ for all $i = 0, 1, \dots, N-1$. Each homomorphism $\{L^i\}$ induces a sequence of homomorphisms $HL^i \colon H^i(V^\cdot) \to H^i(W^\cdot)$ of the cohomology by $HL^i[v] := [L^i v]$ for $[v] \in H^i(V^\cdot)$. It is easy to see that this is well defined. The homomorphisms of $V^\cdot$ to $V^\cdot$ itself are called endomorphisms of this complex.

Suppose $V^\cdot$ is a complex with finite dimensional cohomology and $\{E^i\}$ an endomorphism of the complex. Then HE^i is an endomorphism of the finite dimensional space $H^i(V^\cdot)$, and so the trace $\operatorname{tr} HE^i$ is well defined. The alternating sum

$$L(E) := \sum_i (-1)^i \operatorname{tr} HE^i$$

is called the Lefschetz number of the endomorphism. If $E^i = I_{V^i}$ are the identity maps, then the trace $\operatorname{tr} HE^i$ just amounts to the dimension of $H^i(V^\cdot)$ whence $L(I_{V^\cdot}) = \chi(V^\cdot)$.

We have established complexes as sequences of linear maps between vector spaces which is adequate for algebraic analysis. If we want to use methods of calculus, we have to include continuous linear maps between topological vector spaces. For the rest of the chapter we will understand complexes in this topological sense. If we use homomorphisms $\{L^i\}$, we will suppose the maps L^i to be continuous, too.

Example 1.1.3. Let $\mathcal{X}$ be a smooth (i.e. C^∞) manifold of dimension n. Denote by $\Omega^q(\mathcal{X})$ the space of differential forms of degree q with smooth coefficients on $\mathcal{X}$ and $d\colon \Omega^q(\mathcal{X}) \to \Omega^{q+1}(\mathcal{X})$ the exterior derivative. Locally any $\omega \in \Omega^q(\mathcal{X})$ looks like

$$\omega(x) = \sum_{\substack{J=(j_1,\ldots,j_q)\\ 1\le j_1<\ldots<j_q\le n}} \omega_J(x)\, dx^J$$

for $x = (x^1, \ldots, x^n)$ in a coordinate patch U of $\mathcal{X}$, where $dx^J = dx^{j_1} \wedge \ldots \wedge dx^{j_q}$ and $\omega_J \in C^\infty(U, \mathbb{R})$. The derivative is given by

$$d\omega(x) = \sum_{\substack{J=(j_1,\ldots,j_q)\\ 1\le j_1<\ldots<j_q\le n}} d\omega_J(x) \wedge dx^J$$

for $x \in U$. It is linear and satisfies $d^2 = 0$. Hence

$$\Omega^{\cdot}(\mathcal{X})\colon\ 0 \to \Omega^0(\mathcal{X}) \xrightarrow{d} \Omega^1(\mathcal{X}) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(\mathcal{X}) \to 0$$

is a complex. This complex is referred to as the de Rham complex of $\mathcal{X}$ and its cohomology $H^i_{dR}(\mathcal{X}) := H^i(\Omega^{\cdot}(\mathcal{X}))$ are called the de Rham cohomology of $\mathcal{X}$.

The de Rham complex is a classical example of complexes. The numbers $\dim H^i_{dR}(\mathcal{X})$ are called Betti numbers of the underlying manifold $\mathcal{X}$. They depend on certain topological properties of $\mathcal{X}$.

1.2 Quasicomplexes

It is well known that compact perturbation of Fredholm operators are Fredholm. However, compact perturbations of Fredholm complexes lead beyond these complexes, for a perturbation of a complex need not be a complex. This is a motivation for considering sequences of vector spaces and their linear mappings whose compositions are "small" in some reasonable sense. For example, these can be compact mappings or mappings from some operator ideal.

We focus on studying such sequences of Hilbert spaces, for we deal with Sobolev spaces $H^s = W^{s,2}$ in the sequel. Consider a sequence

$$0 \longrightarrow V^0 \xrightarrow{A^0} V^1 \xrightarrow{A^1} \ldots \xrightarrow{A^{N-1}} V^N \longrightarrow 0 \tag{1.2.1}$$

where $V^0, V^1, \ldots, V^N$ are Hilbert spaces and $A^i : V^i \to V^{i+1}$ bounded linear operators for $i = 0, 1, \ldots, N-1$. As above, introduce a graded operator

A, acting from V^i to V^{i+1} for each i, by setting $Au := A^i u$ for $u \in V^i$. Using the geometric language, one thinks of the composition $A^2 := A^{i+1} \circ A^i$ as the curvature of sequence (1.2.1). The sequence (1.2.1) is said to be a quasicomplex if its curvature is a compact operator at each step i. This condition is automatically fulfilled for $i \geq N-1$. Any compact perturbation of a quasicomplex is a quasicomplex, for if K^i is a compact mapping from V^i to V^{i+1}, then

$$\begin{aligned} (A+K)^2 &= (A^{i+1}+K^{i+1})(A^i+K^i) \\ &= A^{i+1}A^i + A^{i+1}K^i + K^{i+1}A^i + K^{i+1}K^i \\ &= 0 \end{aligned}$$

modulo compact operators. Using the concept of Calkin algebra it is possible to introduce a symbol complex for any quasicomplex (1.2.1) and define Fredholm quasicomplexes in this way, see [Tar07]. However, this goes beyond the framework of this work. Mention that the Fredholm property can be characterised within any algebra with symbol structure by the invertibility of relevant symbols, which is usually called the ellipticity. The symbol structure is especially simple for algebras of pseudifferential operators on a smooth manifold $\mathcal{X}$.

Let

$$0 \longrightarrow C^\infty(\mathcal{X}, F^0) \xrightarrow{A^0} C^\infty(\mathcal{X}, F^1) \xrightarrow{A^1} \dots \xrightarrow{A^{N-1}} C^\infty(\mathcal{X}, F^N) \longrightarrow 0 \quad (1.2.2)$$

be a sequence of first order differential operators between sections of smooth vector bundles F^i over $\mathcal{X}$. As usual, we write $\sigma^1(A^i)$ for the principal symbol of A^i. This is a function on the cotangent bundle $T^*\mathcal{X}$ of $\mathcal{X}$ with values in the bundle homomorphisms, i.e., $\sigma^1(A^i)(x,\xi)$ is a linear mapping of F^i_x to F^{i+1}_x for each point $(x,\xi) \in T^*\mathcal{X}$ (see Section 1.4). We do not assume that $A^{i+1}A^i$ is zero but "small" for all i. Since each continuous linear operator from $C^\infty(\mathcal{X}, F^i)$ to $C^\infty(\mathcal{X}, F^{i+1})$ is compact, we need another class of operators in order to characterise the "smallness" of the curvature. As but one natural way to do this we mention appropriate extensions of sequence (1.2.2) to Sobolev space of sections on $\mathcal{X}$. A necessary condition for the compactness $A^2 : H^s(\mathcal{X}, F^i) \to H^{s-2}(\mathcal{X}, F^{i+2})$ is $\sigma^2(A^2) = 0$. This condition is also sufficient if $\mathcal{X}$ is compact and closed. From $\sigma^2(A^2) = 0$ it follows that the curvature A^2 has order ≤ 1 at each step i. For this reason, what we will mean by quasicomplexes in this work, are sequences (1.2.2) whose curvature has order at most 1. This corresponds to the calculus of pseudodifferential operators in the interior of $\mathcal{X}$.

Passing to symbol mappings in quasicomplex (1.2.2) yields a family of sequences

$$0 \longrightarrow F_x^0 \stackrel{\sigma^1(A^0)(x,\xi)}{\longrightarrow} F_x^1 \stackrel{\sigma^1(A^1)(x,\xi)}{\longrightarrow} \dots \stackrel{\sigma^1(A^{N-1})(x,\xi)}{\longrightarrow} F_x^N \longrightarrow 0 \qquad (1.2.3)$$

of linear mappings of finite-dimensional vector spaces F_x^i parametrised by the points (x, ξ) of $T^*\mathcal{X}$. Since the order of A^2 is less than 2, it follows readily that

$$\begin{aligned} \sigma^1(A^{i+1})(x,\xi)\sigma^1(A^i)(x,\xi) &= \sigma^2(A^{i+1}A^i)(x,\xi) \\ &= 0 \end{aligned}$$

for all $(x, \xi) \in T^*\mathcal{X}$. Therefore, the symbol sequence (1.2.3) has curvature zero, i.e., (1.2.3) is a complex. Quasicomplex (1.2.2) is said to be elliptic at step i if the cohomology of (1.2.3) is trivial at step i. We call (1.2.2) elliptic if it is elliptic at each step. Thus, we are in a position to introduce interior ellipticity for quasicomplexes of differential operators like (1.2.2). The operators $\varDelta^i = A^{i*}A^i + A^{i-1}A^{i-1*}$ are called the Laplace operators (or Laplacians) of quasicomplex (1.2.2). These are second order differential operators between sections of the vector bundle F^i over $\mathcal{X}$. Using the properties of the principal symbol mapping we obtain

$$\sigma^2(\varDelta^i) = \sigma^1(A^i)^*\sigma^1(A^i) + \sigma^1(A^{i-1})\sigma^1(A^{i-1})^*,$$

and so the principal symbol of $\varDelta^i$ coincides with the Laplacian of symbol complex (1.2.3). A familiar argument of linear algebra now shows that the ellipticity of quasicomplex (1.2.2) at step i just amounts to saying that the symbol $\sigma^2(\varDelta^i)(x, \xi)$ is invertible for all $x \in \mathcal{X}$ and $\xi \in T_x^*\mathcal{X} \setminus \{0\}$. This latter means that $\varDelta^i$ is an elliptic operator. On summarising we conclude that the ellipticity of quasicomplex (1.2.2) at step i is equivalent to the ellipticity of its Laplace operator $\varDelta^i$ in the calculus of pseudodifferential operators in the interior of $\mathcal{X}$. By the very definition of elliptic quasicomplexes, they survive under "small" perturbations.

We complete this section by considering certain "small" perturbations of the Dolbeault complex.

Example 1.2.1. Assume that $\mathcal{X}$ is a complex (analytic) manifold of dimension n. As usual, we denote by $\varOmega^{0,q}(\mathcal{X})$ the space of all differential forms of bidegree $(0, q)$ with C^∞coefficients on $\mathcal{X}$, where $0 \le q \le n$. Locally such a form can be written as

$$f(z) = \sum_{\substack{J=(j_1,\dots,j_q) \\ 1\le j_1<\dots<j_q\le n}} f_J(z) d\bar{z}^J,$$

where $z = (z^1, \ldots, z^n)$ are local coordinates, $d\bar{z}^J = d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}$ and f_I are C^∞ functions of z with complex values. Analogously to the exterior derivative d one defines the Cauchy-Riemann operator $\bar{\partial}$ which maps the differential forms of bidegree $(0, q)$ to differential forms of bidegree $(0, q+1)$ on $\mathcal{X}$, see for instance [Wel73]. Moreover, $\bar{\partial}^2 = 0$, i.e., the spaces $\Omega^{0,q}(\mathcal{X})$ are gathered together to constitute a complex of first order differential operators on $\mathcal{X}$ called the Dolbeault complex. This complex is proved to be elliptic in (the interior of) $\mathcal{X}$. Choose any differential form a of bidegree $(0, 1)$ with smooth coefficients on $\mathcal{X}$ and consider the sequence

$$0 \longrightarrow \Omega^{0,0}(\mathcal{X}) \xrightarrow{\bar{\partial}+a} \Omega^{0,1}(\mathcal{X}) \xrightarrow{\bar{\partial}+a} \ldots \xrightarrow{\bar{\partial}+a} \Omega^{0,n}(\mathcal{X}) \longrightarrow 0 \qquad (1.2.4)$$

which is equipped with differential $\bar{\partial} + a$ given by $(\bar{\partial} + a)u = \bar{\partial}u + a \wedge u$ for $u \in \Omega^{0,q}$. Since

$$\begin{aligned}
(\bar{\partial} + a)^2 u &= (\bar{\partial} + a)(\bar{\partial}u + a \wedge u) \\
&= \bar{\partial}^2 u + \bar{\partial}a \wedge u - a \wedge \bar{\partial}u + a \wedge \bar{\partial}u + a \wedge a \wedge u \\
&= \bar{\partial}a \wedge u,
\end{aligned}$$

the curvature of sequence (1.2.4) is equal to $\bar{\partial}a$. It follows that (1.2.4) is a quasicomplex. Moreover, it is a complex if a is $\bar{\partial}$-closed. The symbol sequence of (1.2.4) coincides with that of the Dolbeault complex, and so the quasicomplex is elliptic in $\mathcal{X}$.

1.3 Sobolev spaces

Set $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$ for $\xi \in \mathbb{R}^n$. For any $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ is defined to consist of all temperate distributions $f \in S'(\mathbb{R}^n)$ with the property that $\langle \xi \rangle^s \hat{f} \in L^2(\mathbb{R}^n)$, where $\hat{f}$ stands for the Fourier transorm of f. When endowed with scalar product

$$(f, g) = (\langle \xi \rangle^s \hat{f}, \langle \xi \rangle^s \hat{g})_{L^2(\mathbb{R}^n)},$$

the space $H^s(\mathbb{R}^n)$ is a Hilbert space. From the definition it readily follows that the dual of $H^s(\mathbb{R}^n)$ is isomorphic to $H^{-s}(\mathbb{R}^n)$.

Theorem 1.3.1. *If $s > [n/2] + k$ for some integer $k \geq 0$, then the space $H^s(\mathbb{R}^n)$ is continuously embedded into $C^k(\mathbb{R}^n)$.*

The commonly used case of this theorem corresponds to $k = 0$ when the classes of $H^s(\mathbb{R}^n)$ gain unique continuous representatives.

Example 1.3.2. Consider the delta function δ on $\mathbb{R}^n$. It is easy to see that $\delta \in (C(\mathbb{R}^n))'$. By Theorem 1.3.1 we deduce that $H^s(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$ for $s > n/2$. Hence,

$$\delta \in (H^s(\mathbb{R}^n))' \cong H^{-s}(\mathbb{R}^n)$$

for all $s > n/2$.

Denote by $\mathbb{H}^n$ the closed half-space of $\mathbb{R}^n$ consisting of all $x = (x^1, \ldots, x^n)$ which satisfy $x^n \geq 0$. We give $\mathbb{H}^n$ the topology induced from $\mathbb{R}^n$, i.e., by open sets in the half-space are meant the intersections of open sets in $\mathbb{R}^n$ with $\mathbb{H}^n$. For any $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{H}^n)$ is defined to be the restriction of $H^s(\mathbb{R}^n)$ into $\mathbb{H}^n$. To specify $H^s(\mathbb{H}^n)$ within the framework of Hilbert spaces, one invokes the construction of a quotient space. Namely,

$$H^s(\mathbb{H}^n) := H^s(\mathbb{R}^n)/H^s_{\overline{\mathbb{R}^n \setminus \mathbb{H}^n}}(\mathbb{R}^n),$$

the denominator being the subspace of $H^s(\mathbb{R}^n)$ consisting of all functions with support in $\overline{\mathbb{R}^n \setminus \mathbb{H}^n}$.

Our next objective is to introduce Sobolev spaces of sections of a vector bundle over a compact manifold with or without boundary. In the case of compact closed manifolds we just refer to [Wel73]. Let $\mathcal{X}$ be a compact smooth manifold with boundary and F a Hermitean vector bundle over $\mathcal{X}$. Choose a finite open covering $\{U_\nu\}$ of $\mathcal{X}$ by coordinate patches, such that F is trivial over each U_ν. Thus, $F\restriction_{U_\nu} \cong U_\nu \times \mathbb{C}^k$ and we fix any trivialisation of F over U_ν. The Sobolev spaces in question will not depend on the particular choices of the covering, local coordinates, trivialisations, etc. up to unitary structure. Pick a smooth partition $\{\chi_\nu\}$ of unity on $\mathcal{X}$ subordinate to the covering $\{U_\nu\}$. Any section f of F can be written as $f = \sum(\chi_\nu f)$, where $\chi_\nu f$ vanishes away from a compact subset of U_ν in $\mathcal{X}$. Therefore, if $x = h_\nu(p)$ are local coordinates in U_ν, then the pullback of $\chi_\nu f$ under the inverse mapping of h_ν

$$(h_\nu^{-1})^*(\chi_\nu f)(x) = \begin{cases} (\chi_\nu f)(h_\nu^{-1}(x)), & \text{if } x \in h_\nu(U_\nu), \\ 0, & \text{if } x \in \mathcal{X} \setminus h_\nu(U_\nu), \end{cases}$$

is specified within functions on $\mathbb{H}^n$ with values in $\mathbb{C}^k$ whose support is a compact subset of $h_\nu(U_\nu)$. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathcal{X}, F)$ is defined to consist of all sections f of F over $\mathcal{X}$ with the property that $(h_\nu^{-1})^*(\chi_\nu f)$ belongs to $H^s(\mathbb{H}^n, \mathbb{C}^k)$ for every ν. Obviously, $H^s(\mathcal{X}, F)$ is Hilbert under the scalar product

$$(f, g) = \sum_\nu ((h_\nu^{-1})^*(\chi_\nu f), (h_\nu^{-1})^*(\chi_\nu g))_{H^s(\mathbb{H}^n, \mathbb{C}^k)}$$

for $f, g \in H^s(\mathcal{X}, F)$.

The union of all spaces $H^s(\mathcal{X}, F)$ is what is meant by distribution sections of F on $\mathcal{X}$. The Sobolev embedding theorem still holds in this full generality.

Theorem 1.3.3. *Suppose that $\mathcal{X}$ is a compact manifold with or without boundary and F is a smooth vector bundle over $\mathcal{X}$. If $s > [n/2] + k$ for some integer $k \geq 0$, then the space $H^s(\mathcal{X}, F)$ is continuously embedded into $C^k(\mathcal{X}, F)$.*

Theorem 1.3.4. *Let $\mathcal{X}$ be a compact manifold with or without boundary and F a smooth vector bundle over $\mathcal{X}$. If $s > t$, then the natural embedding*

$$H^s(\mathcal{X}, F) \hookrightarrow H^t(\mathcal{X}, F)$$

is compact.

1.4 Pseudodifferential operators

Let U be an open set in $\mathbb{R}^n$ and m a real number. Denote by $\mathcal{S}^m(U \times \mathbb{R}^n)$ the space of all smooth functions a on $U \times \mathbb{R}^n$ with the property that, for each $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ and compact set $K \subset U$, there exists a constant $c_{\alpha,\beta,K} > 0$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq c_{\alpha,\beta,K} \langle \xi \rangle^{m-|\beta|}$$

for all $(x,\xi) \in K \times \mathbb{R}^n$. The elements of $\mathcal{S}^m(U \times \mathbb{R}^n)$ are called symbols and those of

$$\mathcal{S}^{-\infty}(U \times \mathbb{R}^n) = \bigcap_m \mathcal{S}^m(U \times \mathbb{R}^n)$$

smoothing symbols.

To any symbol $a \in \mathcal{S}^m(U \times \mathbb{R}^n)$ we assign the canonical pseudodifferential operator $A = a(x, D)$ by

$$Au\,(x) = \mathcal{F}^{-1}_{\xi \mapsto x} a(x,\xi) \mathcal{F}_{x \mapsto \xi} u$$

for $u \in C^\infty_{\mathrm{comp}}(U)$, where $\mathcal{F}u$ is the Fourier transform of u. Note that A maps $C^\infty_{\mathrm{comp}}(U)$ continuously into $C^\infty(U)$. The function $\sigma(A) := a$ is called the symbol of A.

We now want to consider classical pseudodifferential operators. They form an important subclass of canonical pseudodifferential operators which is closed under basic operations. Classical pseudodifferential operators were introduced in 1965 by J. J. Kohn and L. Nirenberg who reinforced the theory of S. G. Michlin, A. P. Calderon, etc. The main property of this class is the existence of a principal symbol. More precisely, a symbol $a \in \mathcal{S}^m(U \times \mathbb{R}^n)$ is said to be classical (or multihomogeneous) if there is a sequence $\{a_{m-j}\}_{j=0,1,\ldots}$ of functions $a_{m-j} \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$ positively homogeneous of degree $m-j$ in ξ, such that

$$a - \chi \sum_{j=0}^{N} a_{m-j} \in \mathcal{S}^{m-N-1}(U \times \mathbb{R}^n)$$

for all $N = 0, 1, \ldots$, where $\chi \in C^\infty(\mathbb{R}^n)$ is a cut-off function with respect to $\xi = 0$. Obviously, all the components a_{m-j} are uniquely determined by a. A canonical pseudodifferential operator A on U is called classical if its symbol $\sigma(A)$ is classical. The set of all classical pseudodifferential operators of degree m on U is denoted by $\Psi^m_{\mathrm{cl}}(U)$. The component $\sigma^m(A) := a_m$ is called the principal symbol of A.

Example 1.4.1. Any (scalar) linear partial differential operator A of order m on U has the form

$$A(x, \partial) := \sum_{|\alpha| \leq m} A_\alpha(x) \partial^\alpha,$$

where $A_\alpha \in C^\infty(U)$. This is a classical pseudodifferential operator with symbol $\sigma(A)(x,\xi) = A(x, \imath\xi)$. The principal symbol of A is

$$\sigma^m(A)(x,\xi) = \sum_{|\alpha| = m} A_\alpha(x)(\imath\xi)^\alpha.$$

Canonical pseudodifferential operators on open sets of $\mathbb{R}^n$ glue together to give rise to pseudodifferential operators on sections of vector bundles over a smooth manifold $\mathcal{X}$ of dimension n. Locally sections of a vector bundle F of rank k are functions with values in $\mathbb{C}^k$. Canonical pseudodifferential operators mapping functions with values in $\mathbb{C}^k$ to functions with values in $\mathbb{C}^l$ are simply $(l \times k)$-matrices

$$\Big(a_{ij}(x, \partial)\Big)_{\substack{i=1,\ldots,l \\ j=1,\ldots,k}} \tag{1.4.1}$$

of canonical pseudodifferential operators on scalar-valued functions. The notions of multihomogeneity and principal symbol are extended to (1.4.1) in entry wise unless a sophisticated approach is elaborated. Given vector bundles F and G of ranks k and l over $\mathcal{X}$, by a pseudodifferential operator mapping sections of F to those of G is meant any map $A : C^\infty_{\mathrm{comp}}(\mathcal{X}, F) \to C^\infty(\mathcal{X}, G)$ which has form (1.4.1) in any coordinate patch U in $\mathcal{X}$ over which both F

and G are trivial, for any choice of local coordinates and trivialisations. The space of all classical pseudodifferential operators of order m on $\mathcal{X}$ mapping sections of F to those of G is denoted by $\Psi^m_{\mathrm{cl}}(\mathcal{X}; F, G)$. For $A \in \Psi^m_{\mathrm{cl}}(\mathcal{X}; F, G)$, the principal symbol $\sigma^m(A)$ proves to be a well-defined homomorphism of induced bundles $\pi^* F \to \pi^* G$ over $T^*\mathcal{X} \setminus \{0\}$, where $\pi\colon (T^*\mathcal{X} \setminus \{0\}) \to \mathcal{X}$ is the natural projection. Locally the principal symbol is a family of linear maps $\sigma^m(A)(x,\xi)\colon F_x \to G_x$ parametrised by $\xi \in \mathbb{R}^n \setminus \{0\}$, where F_x and G_x stand for the fibers of F and G over a point $x \in \mathcal{X}$. It is positively homogeneous of order m in ξ.

Since pseudodifferential operators are not local they can not be composed with each other in general and so they do not form any operator algebra on $\mathcal{X}$. Moreover, one encounters severe difficulties related to the behaviour of pseudodifferential operators near the boundary of $\mathcal{X}$. To save proper action of pseudodifferential operators in Sobolev spaces one imposes a restriction on operators under consideration called the transmission property with respect to the boundary of $\mathcal{X}$. However, this topic exceeds the scope of this work and we refer the reader to [RS82] for a thorough treatment. From now on we make the standing assumptions that the pseudodifferential operators under study bear the transmission property with respect to the boundary. Such operators form an algebra containing in particular all differential operators on $\mathcal{X}$. Note that the parametrices of elliptic differential operators on a manifold slightly "greater" than $\mathcal{X}$ also bear the transmission property with respect to the hypersurface $\partial\mathcal{X}$.

Theorem 1.4.2. *Suppose that $A \in \Psi^m_{\mathrm{cl}}(\mathcal{X}; F, G)$ and $B \in \Psi^n_{\mathrm{cl}}(\mathcal{X}; G, H)$ are pseudodifferential operators with transmission property with respect to the boundary. Then $BA \in \Psi^{m+n}_{\mathrm{cl}}(\mathcal{X}; F, H)$ bears this property, too, and the equality*

$$\sigma^{m+n}(BA) = \sigma^n(B)\sigma^m(A)$$

holds.

An operator $A \in \Psi^m_{\mathrm{cl}}(\mathcal{X}; F, G)$ is said to be elliptic in the interior of $\mathcal{X}$ if $\sigma^m(A)(x,\xi)\colon F_x \to G_x$ is invertible for all $x \in \mathcal{X}$ and $\xi \in T^*_x\mathcal{X} \setminus \{0\}$. In the following theorem by $\mathcal{X}'$ is meant a smooth manifold slightly larger than $\mathcal{X}$, so that $\mathcal{X}$ is compactly embedded into $\mathcal{X}'$. The operators are considered on $\mathcal{X}'$.

Theorem 1.4.3. *For each elliptic operator $A \in \Psi^m_{\mathrm{cl}}(\mathcal{X}'; F, G)$ there is a properly supported operator $P \in \Psi^{-m}_{\mathrm{cl}}(\mathcal{X}'; G, F)$, such that*

$$\begin{array}{rcl} I - PA & \in & \Psi^{-\infty}(\mathcal{X}'; F), \\ I - AP & \in & \Psi^{-\infty}(\mathcal{X}'; G). \end{array} \tag{1.4.2}$$

Any operator $P \in \Psi^{-m}_{\mathrm{cl}}(X; F, E)$ satisfying equalities (1.4.2) is said to be a parametrix of A in the interior of $\mathcal{X}'$. The eqialities are satisfied for all distribution sections of F and G' over $\mathcal{X}'$ with compact support in $\mathcal{X}$, i.e., we get

$$\begin{array}{rcl} P(Au) & = & u - (I - PA)u, \\ A(Pf) & = & f - (I - AP)f \end{array}$$

for all $u \in \mathcal{D}'_{\mathcal{X}}(\mathcal{X}', F)$ and $f \in \mathcal{D}'_{\mathcal{X}}(\mathcal{X}', G)$. However, these formulas do not lead to any parametrix of A in the sence of Banach spaces unless $\mathcal{X}$ is a compact closed manifold.

Theorem 1.4.4. *For each $s \in \mathbb{R}$, any operator $A \in \Psi^{m}_{\mathrm{cl}}(\mathcal{X}; F, G)$ with transmission property with respect to the boundary maps $H^s(\mathcal{X}, F)$ continuously into $H^{s-m}(\mathcal{X}, G)$.*

Chapter 2

The first mixed problem for the Lamé system

We find an adequate interpretation of the stationary Lamé operator within the framework of elliptic quasicomplexes and study the first mixed problem for the nonstationary Lamé system.

2.1 Generalised Lamé system

The stationary Lamé equations are easily specified within the framework of quasicomplexes on the underlying manifold $\mathcal{X}$. On introducing the de Rham complex of $\mathcal{X}$

$$0 \longrightarrow \Omega^0(\mathcal{X}) \xrightarrow{d} \Omega^1(\mathcal{X}) \xrightarrow{d} \Omega^2(\mathcal{X}) \xrightarrow{d} \Omega^3(\mathcal{X}) \longrightarrow 0$$

we can rewrite system (0.0.2) in the invariant form

$$\rho u''_{tt} = -\mu \Delta u - (\lambda + \mu)\, dd^*\, u + f \qquad (2.1.1)$$

in the semicylinder $\mathcal{X} \times [0, \infty)$, where $\Delta = d^*d + dd^*$ is the Laplacian of the de Rham complex.

Example 2.1.1. When restricted to functions, i.e., differential forms of degree $i = 0$, equation (2.1.1) reads

$$\rho u''_{tt} = -\mu \Delta u + f,$$

which is precisely the wave equation in the cylinder $\mathcal{X} \times (0, T)$.

More generally, let $\mathcal{X}$ be a C^∞ compact manifold with boundary of dimension n. Consider a quasicomplex of first order differential operators acting in sections of vector bundles over $\mathcal{X}$,

$$0 \to C^\infty(\mathcal{X}, F^0) \stackrel{A^0}{\to} C^\infty(\mathcal{X}, F^1) \stackrel{A^1}{\to} \ldots \stackrel{A^{N-1}}{\to} C^\infty(\mathcal{X}, F^N) \to 0, \qquad (2.1.2)$$

where $A^i \in \mathrm{Diff}^1(\mathcal{X}; F^i, F^{i-1})$ satisfy $A^{i+1}A^i = 0$ up to first order terms for all $i = 0, 1, \ldots, N-2$. Our basic assumption is that (2.1.2) is elliptic, i.e., the corresponding complex of principal symbols is exact away from the zero section of the cotangent bundle $T^*\mathcal{X}$. We endow the manifold $\mathcal{X}$ and the vector bundles F^i by Riemannian metrics.

Set

$$F = \bigoplus_{i=0}^{N} F^i$$

and consider two first order differential operators A and A^* in $C^\infty(\mathcal{X}, F)$ given by the $((N+1) \times (N+1))$-matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ A^0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & A^1 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & A^{N-1} & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 0 & A^{0*} & 0 & \ldots & 0 & 0 \\ 0 & 0 & A^{1*} & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & A^{N-1*} \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

where $A^i \in \mathrm{Diff}^1(\mathcal{X}; F^{i+1}, F^i)$ stands for the formal adjoint of A^i. It is easily verified that $A \circ A = 0$ and $A^* \circ A^* = 0$ up to first order terms and

$$\Delta := A^*A + AA^* = \begin{pmatrix} \Delta^0 & 0 & \ldots & 0 \\ 0 & \Delta^1 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & \Delta^N \end{pmatrix}, \qquad (2.1.3)$$

where $\Delta^i = A^{i*}A^i + A^{i-1}A^{i-1*}$ for $i = 0, 1, \ldots, N$ are the so-called Laplacians of complex (2.1.2). The ellipticity of quasicomplex (2.1.2) just amounts to that of its Laplacians $\Delta^0, \Delta^1, \ldots, \Delta^N$.

Lemma 2.1.2. *Let r, s be real or complex numbers. Then $rA + sA^* \in \mathrm{Diff}^1(\mathcal{X}; F)$ is elliptic if and only if $rs \neq 0$.*

Proof. Necessity. If at least one of the scalars r and s vanishes then the operator $rA + sA^*$ reduces to a scalar multiple of A or A^*, which operators can not be elliptic because of their nilpotency.

Sufficiency. If both r and s are different from zero then a trivial verification gives

$$\begin{array}{rcl}(s^{-1}A + r^{-1}A^*)(rA + sA^*) & = & AA^* + A^*A, \\ (rA + sA^*)(s^{-1}A + r^{-1}A^*) & = & AA^* + A^*A\end{array}$$

up to first order terms, showing the ellipticity of $rA + sA^*$. □

By generalised stationary Lamé operators related to quasicomplex (2.1.2) are meant the products of two operators of the form $rA+sA^*$, where $rs \neq 0$. These are precisely operators $L \in \mathrm{Diff}^2(\mathcal{X}; F)$ of the form $L = rA^*A+sAA^*$, where $rs \neq 0$. They are elliptic and preserve the grading of quasicomplex (2.1.2) in the sense that if u is a section of F^i, then so is Lu.

Consider the Dirichlet problem for the elliptic operator

$$\varDelta^2 = (A^*A)^2 + (AA^*)^2$$

on $\mathcal{X}$ with data

$$\begin{array}{rcl} u & = & 0 \quad \text{at} \quad \partial\mathcal{X}, \\ (A + A^*)u & = & 0 \quad \text{at} \quad \partial\mathcal{X}. \end{array} \tag{2.1.4}$$

This boundary value problem is elliptic and formally selfadjoint. As usual, it can be treated within the framework of densely defined unbounded operators in the Hilbert space $L^2(\mathcal{X}, F)$, cf. [ST03]. In particular, there is a bounded operator $G : L^2(\mathcal{X}, F) \to H^4(\mathcal{X}, F)$ called the Green operator, such that $u = Gf$ satisfies (2.1.4) and

$$f = Hf + \varDelta^2(Gf) \tag{2.1.5}$$

for all $f \in L^2(\mathcal{X}, F)$, where H is the orthogonal projection of $L^2(\mathcal{X}, F)$ onto the finite-dimensional subspace of $L^2(\mathcal{X}, F)$ consisting of all $h \in C^\infty(\mathcal{X}, F)$ which satisfy $(A + A^*)h = 0$ in $\mathcal{X}$ and $h = 0$ at $\partial\mathcal{X}$. The Green operator G is actually known to be a pseudodifferential operator of order -4 in Boutet de Monvel's algebra on $\mathcal{X}$, see [BdM71].

If $A + A^*$ has the uniqueness property for the global Cauchy problem on $\mathcal{X}$ then $H = 0$. By the uniqueness property is meant that if h is any solution to $(A + A^*)h = 0$ in a connected open set U in $\mathcal{X}$ and h vanishes in a nonempty open subset of U then h is identically zero in U.

Lemma 2.1.3. *Suppose that $L = rA^*A+sAA^*$ is a stationary Lamé operator on $\mathcal{X}'$, where $rs \neq 0$. Then $P = (\varDelta^2/L)G$, with $\varDelta^2/L = r^{-1}A^*A + s^{-1}AA^*$, is a parametrix of L.*

Proof. By the above, we get

$$\begin{aligned} LP &= L(\varDelta^2/L)G \\ &= \varDelta^2 G \\ &= I - H \end{aligned}$$

where $H \in \Psi^{-\infty}(\mathcal{X}; F)$. Hence, P is a left parametrix of L. Since L is elliptic, P is also a right parametrix of L in the interior of $\mathcal{X}$. □

Write

$$\begin{aligned} L &= r\varDelta + (s-r)AA^* \\ &= -\mu\varDelta - (\lambda+\mu)AA^*, \end{aligned}$$

where $r = -\mu$ and $s = -\lambda - 2\mu$. Then for the ellipticity of L it is necessary and sufficient that $\mu \neq 0$ and $\lambda + 2\mu \neq 0$.

2.2 Wave equation

In the open cylinder $\mathcal{C}_T = \overset{\circ}{\mathcal{X}} \times (0,T)$ for some $T > 0$ we consider the hyperbolic system

$$\rho u''_{tt} = -\mu\varDelta u - (\lambda+\mu)AA^*u + f \tag{2.2.1}$$

for a section u of the bundle $(x,t) \mapsto F^i_x$ over $\mathcal{X} \times [0,T]$, which we write F^i for short, cf. Fig. 2.1. Assume $\rho = 1$ and $\mu > 0$.

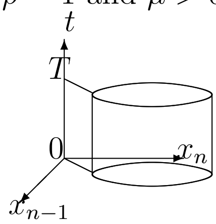

Fig. 2.1: *A cylinder* $\mathcal{C}_T$

A function $u \in C^2(\mathcal{C}_T, F^i) \cap C^1(\mathcal{X} \times [0,T), F^i)$ satisfying equation (2.2.1) in $\mathcal{C}_T$, the initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad \text{for} \quad x \in \overset{\circ}{\mathcal{X}}, \\ u'_t(x,0) &= u_1(x), \quad \text{for} \quad x \in \overset{\circ}{\mathcal{X}}, \end{aligned} \tag{2.2.2}$$

on the lower basis of the cylinder and a Dirichlet condition

$$u(x,t) = u_l(x,t), \quad \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T), \tag{2.2.3}$$

on the lateral surface is said to be a classical solution of the first mixed problem for the generalised Lamé equations. Since the case of inhomogeneous boundary conditions reduces easily to the case of homogeneous ones, we will assume $u_l \equiv 0$ in the sequel.

Let u be a classical solution of the first mixed problem for the generalised Lamé equations with $f \in L^2(\mathcal{C}_T, F^i)$. Given any $\varepsilon > 0$, we multiply both sides of (2.2.1) with g^*, where g is an arbitrary smooth function in the closure of $\mathcal{C}_{T-\varepsilon}$ vanishing at the lateral surface and the head of this cylinder, and integrate the resulting equality over $\mathcal{C}_{T-\varepsilon}$. We will write the inner product of the values of f and g at any point $(x,t) \in \mathcal{C}_{T-\varepsilon}$ simply (f,g) when no confusion can arise. Using the Stokes theorem, we get

$$\begin{aligned}\int_{\mathcal{C}_{T-\varepsilon}} (f,g)\, dxdt &= \int_{\mathcal{C}_{T-\varepsilon}} (u''_{tt} + \mu\Delta u + (\lambda+\mu)AA^*u, g)\, dxdt \\ = -\int_{\mathcal{X}} (u_1,g)dx &+ \int_{\mathcal{C}_{T-\varepsilon}} (-(u'_t, g'_t) + \mu(Au, Ag) + (\lambda+2\mu)(A^*u, A^*g))\, dxdt.\end{aligned}$$

We exploit this identity to introduce the concept of weak solution of the first mixed problem for the generalised Lamé system. We assume that $f \in L^2(\mathcal{C}_T, F^i)$ and $u_1 \in L^2(\mathcal{X}, F^i)$.

A function $u \in H^1(\mathcal{C}_T, F^i)$ is called a weak solution of the first mixed problem for (2.2.1) in $\mathcal{C}_T$, if u satisfies

$$\begin{array}{rcll} u(x,0) &=& u_0(x), & \text{for} \quad x \in \overset{\circ}{\mathcal{X}}, \\ u(x,t) &=& 0, & \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T), \end{array}$$

and

$$\begin{aligned} &\int_{\mathcal{C}_T} (-(u'_t, g'_t) + \mu(Au, Ag) + (\lambda + 2\mu)(A^*u, A^*g))\, dxdt \\ &= \int_{\mathcal{X}} (u_1, g)dx + \int_{\mathcal{C}_T} (f,g)\, dxdt \end{aligned} \tag{2.2.4}$$

for all $g \in H^1(\mathcal{C}_T, F^i)$, such that

$$\begin{array}{rcll} g(x,T) &=& 0, & \text{for} \quad x \in \overset{\circ}{\mathcal{X}}, \\ g(x,t) &=& 0, & \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T). \end{array} \tag{2.2.5}$$

Just as classical solution, if u is a weak solution of the first mixed problem for the generalised Lamé system in $\mathcal{C}_T$, then u is a weak solution of the corresponding problem also in the cylinder $\mathcal{C}_{T'}$ with any $T' < T$. Indeed, u belongs to $H^1(\mathcal{C}_{T'}, F^i)$ for all $T' < T$ and it vanishes on the lateral boundary of $\mathcal{C}_{T'}$. Moreover, the identity (2.2.4) is fulfilled for all $g \in H^1(\mathcal{C}_T, F^i)$ with property (2.2.5). It is immediately verified that if a function g belongs to $H^1(\mathcal{C}_{T'}, F^i)$, the trace of g at the cross-section $\{t = T'\}$ is zero and $g = 0$ in $\mathcal{C}_T \setminus \mathcal{C}_{T'}$, then $g \in H^1(\mathcal{C}_T)$ and $g(x, T) = 0$ for all x in the interior of $\mathcal{X}$. If moreover $g = 0$ at $\partial\mathcal{X} \times (0, T')$, then g vanishes at the lateral boundary of $\mathcal{C}_T$. Hence it follows that the function u satisfies the integral identity by means of which one defines the weak solution of the corresponding mixed problem in $\mathcal{C}_{T'}$.

Note that we introduced the concept of weak solution of the first mixed problem as natural generalisation of the concept of classical solution (with $f \in L^2(\mathcal{C}_T, F^i)$). We have actually proved that the classical solution of the first mixed problem in $\mathcal{C}_T$ with $f \in L^2(\mathcal{C}_T, F^i)$ is a weak solution of this problem in the smaller cylinder $\mathcal{C}_{T-\varepsilon}$ for any $\varepsilon \in (0, T)$.

Along with classical and weak solutions of the first mixed problem one can introduce the notion of 'almost everywhere' solution. A function u is said to be an 'almost everywhere' solution of the first mixed problem if $u \in H^2(\mathcal{C}_T, F^i)$ satisfies equation (2.2.1) for almost all $(x, t) \in \mathcal{C}_T$, initial conditions (2.2.2) for almost all x in the interior of $\mathcal{X}$ and the trace of u on the lateral surface vanishes almost everywhere. From the definition it follows immediately that if the classical solution of the first mixed problem belongs to $H^2(\mathcal{C}_T, F^i)$ then it is also an 'almost everywhere' solution. Moreover, if an 'almost everywhere' solution u of the first mixed problem belongs to the class $C^2(\mathcal{C}_T, F^i) \cap C^1(\mathcal{X} \times [0, T), F^i)$ then u is obviously a classical solution, too.

Every 'almost everywhere' solution of the first mixed problem in $\mathcal{C}_T$ is a weak solution of this problem in $\mathcal{C}_T$. The converse assertion is also true.

Lemma 2.2.1. *If a weak solution of the first mixed problem belongs to the space $H^2(\mathcal{C}_T, F^i)$ then it is an 'almost everywhere' solution of this problem. If a weak solution of the first mixed problem belongs to $C^2(\mathcal{C}_T, F^i) \cap C^1(\mathcal{X} \times [0, T), F^i)$ then it is a classical solution of this problem.*

Proof. This is a standard fact on functions with generalised derivatives, cf. Lemma 1 in [Mik76, p. 287]. □

We are now in a position to prove a uniqueness theorem for the weak solution of the first mixed problem.

Theorem 2.2.2. *Suppose $\mu \geq 0$ and $\lambda + 2\mu \geq 0$. Then the first mixed problem for the generalised Lamé system has at most one weak solution.*

Proof. Let $u \in H^1(\mathcal{C}_T, F^i)$ be a weak solution of the first mixed problem with $f = 0$ in $\mathcal{C}_T$ and $u_0 = u_1 = 0$ in the interior of $\mathcal{X}$.

Pick an arbitrary $s \in (0, T)$ and consider the function

$$g(x,t) = \begin{cases} \displaystyle\int_t^s u(x,\theta)d\theta, & \text{if } \ 0 < t < s, \\ 0, & \text{if } \ s < t < T, \end{cases}$$

defined in $\mathcal{C}_T$. It is immediately verified that the function g has generalised derivatives

$$g'_{x^j}(x,t) = \begin{cases} \displaystyle\int_t^s u'_{x^j}(x,\theta)d\theta, & \text{if } \ 0 < t < s, \\ 0, & \text{if } \ s < t < T, \end{cases}$$

and

$$g'_t(x,t) = \begin{cases} -u(x,t), & \text{if } \ 0 < t < s, \\ 0, & \text{if } \ s < t < T, \end{cases}$$

in $\mathcal{C}_T$. Therefore, we get $g \in H^1(\mathcal{C}_T, F^i)$. Moreover, g vanishes at the lateral boundary and the head of the cylinder $\mathcal{C}_T$.

Substituting the function g into identity (2.2.4) yields

$$\int_{\mathcal{C}_s}\!\Big((u'_t, u) + \mu(Au, \int_t^s Au(\cdot,\theta)d\theta) + (\lambda{+}2\mu)(A^*u, \int_t^s A^*u(\cdot,\theta)d\theta) \Big)\, dxdt = 0$$

for all $s \in (0, T)$. It is obvious that

$$\Re \int_{\mathcal{C}_s} (u'_t, u)\, dxdt = \frac{1}{2} \int_{\mathcal{X}} |u(x,s)|^2 dx.$$

Since

$$\begin{aligned} \int_{\mathcal{C}_s} (Au(x,t), \int_t^s Au(x,\theta)d\theta)\, dxdt &= \int_{\mathcal{X}} \int_0^s (Au(x,t), \int_t^s Au(x,\theta)d\theta)\, dxdt \\ &= \int_{\mathcal{X}} \int_0^s (\int_0^\theta Au(x,t)dt, Au(x,\theta))\, dxd\theta \end{aligned}$$

which transforms to

$$\begin{aligned} &\int_{\mathcal{X}} (\int_0^s Au(x,t)dt, \int_0^s Au(x,\theta)d\theta)\, dx - \int_{\mathcal{X}} \int_0^s (\int_\theta^s Au(x,t)dt, Au(x,\theta))\, dxd\theta \\ &\quad = \int_{\mathcal{X}} |\int_0^s Au(x,t)dt|^2\, dx - \int_{\mathcal{C}_s} (\int_\theta^s Au(x,t)dt, Au(x,\theta))\, dxd\theta, \end{aligned}$$

we get

$$\Re \int_{\mathcal{C}_s} (Au(x,t), \int_t^s Au(x,\theta)d\theta)\, dxdt = \frac{1}{2} \int_{\mathcal{X}} | \int_0^s Au(x,t)dt|^2\, dx.$$

Similarly we obtain

$$\Re \int_{\mathcal{C}_s} (A^* u(x,t), \int_t^s A^* u(x,\theta)d\theta)\, dxdt = \frac{1}{2} \int_{\mathcal{X}} | \int_0^s A^* u(x,t)dt|^2\, dx$$

whence

$$\int_{\mathcal{X}} |u(x,s)|^2 dx + \mu \int_{\mathcal{X}} |\int_0^s Au(x,t)dt|^2\, dx + (\lambda{+}2\mu) \int_{\mathcal{X}} |\int_0^s A^* u(x,t)dt|^2 dx = 0 \tag{2.2.6}$$

for all $s \in (0,T)$.

Since $\mu \geq 0$ and $\mu + 2\lambda \geq 0$, we conclude from (2.2.6) that

$$\int_{\mathcal{X}} |u(x,s)|^2 dx = 0$$

for all $s \in (0,T)$, and so $u = 0$ in $\mathcal{C}_T$, as desired. □

As mentioned, a classical solution of the first mixed problem is also a weak solution of this problem in $\mathcal{C}_{T-\varepsilon}$ for each $\varepsilon \in (0,T)$. Hence, Theorem 2.2.2 implies the uniqueness of classical solution as well. Furthermore, since almost everywhere solutions are weak solutions, we also deduce that, if $\mu \geq 0$ and $\mu{+}2\lambda \geq 0$, then the first mixed problem for the generalised Lamé system has at most one almost everywhere solution.

2.3 Existence of a weak solution

We now turn to showing the existence of solutions of the first mixed problem for the generalised Lamé system. To this end we use the Fourier method which consists in looking the solution of the mixed problem in the form of series over eigenfunctions of the corresponding elliptic boundary value problem.

Let v be a weak eigenfunction of the first boundary value problem for the generalised Lamé system

$$\begin{array}{rcll} -\mu \Delta v - (\lambda{+}\mu) AA^* v & = & \varkappa v & \text{in} \ \ \overset{\circ}{\mathcal{X}}, \\ v & = & 0 & \text{at} \ \ \partial \mathcal{X}, \end{array} \tag{2.3.1}$$

where $\varkappa$ is a corresponding eigenvalue. This just amounts to saying that

$$\int_{\mathcal{X}} (-\mu(Av, Ag)_x - (\lambda+2\mu)(A^*v, A^*g)_x)\, dx - \varkappa \int_{\mathcal{X}} (v, g)_x dx = 0 \qquad (2.3.2)$$

for all $g \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$.

Consider the orthonormal system $(v_k)_{k=1,2,\ldots}$ in $L^2(\mathcal{X}, F^i)$ consisting of all weak eigenfunctions of problem (2.3.1). Let $(\varkappa_k)_{k=1,2,\ldots}$ be the sequence of corresponding eigenvalues. As usual we think of this sequence as nonincreasing sequence with $\varkappa_1 < 0$ and each eigenvalue repeats itself in accord with its multiplicity. The system $(v_k)_{k=1,2,\ldots}$ is known to be an orthonormal basis in $L^2(\mathcal{X}, F^i)$ and $\varkappa_k \to -\infty$ when $k \to \infty$. Moreover, the first eigenvalue $\varkappa_1$ is strongly negative, if $\mu > 0$ and $\lambda + 2\mu > 0$.

Suppose that the initial data u_0 and u_1 in (2.2.2) belong to $L^2(\mathcal{X}, F^i)$, and f belongs to $L^2(\mathcal{C}_T, F^i)$. By the Fubini theorem, $f(\cdot, t) \in L^2(\mathcal{X}, F^i)$ holds for almost all $t \in (0, T)$. We represent the functions u_0 and u_1 and the function $f(\cdot, t)$ for almost all $t \in (0, T)$ as Fourier series over the system $(v_k)_{k=1,2,\ldots}$ of eigenfunction of problem (2.3.1). To wit,

$$u_0(x) = \sum_{k=1}^{\infty} u_{0,k} v_k(x), \;\; u_1(x) = \sum_{k=1}^{\infty} u_{1,k} v_k(x),$$

where $u_{0,k} = (u_0, v_k)_{L^2(\mathcal{X}, F^i)}$ and $u_{1,k} = (u_1, v_k)_{L^2(\mathcal{X}, F^i)}$ for $k = 1, 2, \ldots$. By the Parseval equality, we get

$$\begin{aligned} \sum_{k=1}^{\infty} |u_{0,k}|^2 &= \|u_0\|^2_{L^2(\mathcal{X}, F^i)}, \\ \sum_{k=1}^{\infty} |u_{1,k}|^2 &= \|u_1\|^2_{L^2(\mathcal{X}, F^i)}. \end{aligned} \qquad (2.3.3)$$

Similarly we get

$$f(x, t) = \sum_{k=1}^{\infty} f_k(t) v_k(x),$$

where $f_k(t) = \displaystyle\int_{\mathcal{X}} (f(\cdot, t), v_k)_x dx$ for $k = 1, 2, \ldots$. Since

$$|f_k(t)|^2 \le \int_{\mathcal{X}} |f(\cdot, t)|^2 dx \int_{\mathcal{X}} |v_k|^2 dx = \int_{\mathcal{X}} |f(\cdot, t)|^2 dx,$$

it follows that $f_k \in L^2(0, T)$ for all $k = 1, 2, \ldots$. Moreover,

$$\sum_{k=1}^{\infty} |f_k(t)|^2 = \int_{\mathcal{X}} |f(\cdot, t)|^2 dx$$

holds for almost all $t \in (0,T)$, which is due to the Parseval equality. This yields readily

$$\sum_{k=1}^{\infty} \int_0^T |f_k(t)|^2 dt = \int_{\mathcal{C}_T} |f(x,t)|^2 dx dt. \tag{2.3.4}$$

Take first the k th harmonics $u_{0,k} v_k$ and $u_{1,k} v_k$ as initial data in (2.2.2), and the function $f_k(t) v_k(x)$ as function in the right-hand of (2.2.1), where $k = 1, 2, \ldots$. Consider the function

$$u_k(x,t) = w_k(t) v_k(x), \tag{2.3.5}$$

where

$$w_k(t) = u_{0,k} \cos \sqrt{-\varkappa_k} t + u_{1,k} \frac{\sin \sqrt{-\varkappa_k} t}{\sqrt{-\varkappa_k}} + \int_0^t f_k(t') \frac{\sin \sqrt{-\varkappa_k}(t-t')}{\sqrt{-\varkappa_k}} dt'.$$

Note that this formula still makes sense if $\varkappa_k = 0$, for the limit of the right-hand side exists as $\varkappa_k \to 0$. The function w_k belongs obviously to $H^2(0,T)$, satisfies the initial conditions $w_k(0) = u_{0,k}$ and $w'_k(0) = u_{1,k}$ and is a solution of the ordinary differential equation

$$w''_k - \varkappa_k w_k = f_k \tag{2.3.6}$$

for almost all $t \in (0,T)$.

Our next objective is to show that if v_k is an eigenfunction of problem (2.3.1) corresponding to the eigenvalue $\varkappa_k$ then $u_k(x,t)$ is a weak solution of the first mixed problem for the equation

$$u''_{tt}(x,t) = -\mu \varDelta u(x,t) - (\lambda+\mu) A A^* u(x,t) + f_k(t) v_k(x)$$

in $\mathcal{C}_T$ with initial data

$$\begin{array}{rcll} u(x,0) & = & u_{0,k} v_k(x), & \text{for} \quad x \in \overset{\circ}{\mathcal{X}}, \\ u'_t(x,0) & = & u_{1,k} v_k(x), & \text{for} \quad x \in \overset{\circ}{\mathcal{X}}. \end{array}$$

Indeed, the function u_k given by (2.3.5) belongs to $H^1(\mathcal{C}_T, F^i)$, satisfies the initial conditions and vanishes at the lateral boundary of the cylinder. It remains to show that

$$\begin{aligned} & \int_{\mathcal{C}_T} (-((u_k)'_t, g'_t) + \mu(Au_k, Ag) + (\lambda+2\mu)(A^* u_k, A^* g))\, dx dt \\ & \qquad = \int_{\mathcal{X}} u_{1,k} (v_k, g) dx + \int_{\mathcal{C}_T} f_k(t)(v_k, g)\, dx dt \end{aligned}$$

for all $g \in H^1(\mathcal{C}_T, F^i)$ satisfying (2.2.5). It is sufficient to establish the above identity only for functions $g \in C^1(\overline{\mathcal{C}}_T, F^i)$ satisfying (2.2.5).

By (2.3.5) and integration by parts,

$$\begin{aligned}\int_{\mathcal{C}_T} ((u_k)'_t, g'_t)\, dxdt &= \int_{\mathcal{X}} \Big(v_k, \int_0^T w'_k(t) g'_t dt\Big)_x\, dx \\ &= \int_{\mathcal{X}} \Big(v_k, -u_{1,k} g(x,0) - \int_0^T w''_k(t) g dt\Big)_x\, dx\end{aligned}$$

which reduces, by (2.3.6), to

$$-\int_{\mathcal{X}} u_{1,k} (v_k, g(x,0))_x\, dx - \varkappa_k \int_{\mathcal{C}_T} (u_k, g)\, dxdt - \int_{\mathcal{C}_T} f_k(t)(v_k, g)\, dxdt.$$

Hence, the desired identity follows from (2.3.2), for

$$\begin{aligned}&\int_{\mathcal{C}_T} (\mu(Au_k, Ag) + (\lambda+2\mu)(A^* u_k, A^* g))\, dxdt \\ &= \int_0^T w_k(t) \Big(\int_{\mathcal{X}} (\mu(Av_k, Ag)_x + (\lambda+2\mu)(A^* v_k, A^* g)_x)\, dx \Big) dt \\ &= -\int_0^T w_k(t) \Big(\varkappa_k \int_{\mathcal{X}} (v_k, g)_x dx \Big) dt,\end{aligned}$$

as desired.

If one takes the partial sums

$$\sum_{k=1}^N u_{0,k} v_k(x), \quad \sum_{k=1}^N u_{1,k} v_k(x)$$

of the Fourier series for the functions u_0 and u_1, respectively, as initial data and the partial sum

$$\sum_{k=1}^N f_k(t) v_k(x)$$

of the Fourier series for f as the right-hand side of the equation, then the weak solution of the first mixed problem is

$$s_N(x,t) = \sum_{k=1}^N u_k(x,t) = \sum_{k=1}^N w_k(t) v_k(x).$$

In particular, the function s_N satisfies the identity

$$\begin{aligned}
& \int_{\mathcal{C}_T} (-((s_N)'_t, g'_t) + \mu(As_N, Ag) + (\lambda{+}2\mu)(A^* s_N, A^* g))\, dxdt \\
= & \int_{\mathcal{X}} \Big(\sum_{k=1}^{N} u_{1,k} v_k, g \Big)\, dx + \int_{\mathcal{C}_T} \Big(\sum_{k=1}^{N} f_k(t) v_k, g \Big)\, dxdt
\end{aligned} \tag{2.3.7}$$

for all $g \in H^1(\mathcal{C}_T, F^i)$ satisfying (2.2.5).

Thus it is to be expected that under certain assumptions on u_0, u_1 and f the solution of the first mixed problem for the generalised Lamé system can be represented as series

$$u(x,t) = \sum_{k=0}^{\infty} w_k(t) v_k(x), \tag{2.3.8}$$

where $(v_k)_{k=1,2,\ldots}$ are weak eigenfunctions of problem (2.3.1).

Theorem 2.3.1. *Let $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$, $u_1 \in L^2(\mathcal{X}, F^i)$ and $f \in L^2(\mathcal{C}_T, F^i)$. Then the first mixed problem possesses a weak solution given by series (2.3.8) which converges in $H^1(\mathcal{C}_T, F^i)$. Moreover,*

$$\|u\|_{H^1(\mathcal{C}_T, F^i)} \le C \Big(\|f\|_{L^2(\mathcal{C}_T, F^i)} + \|u_0\|_{H^1(\mathcal{X}, F^i)} + \|u_1\|_{L^2(\mathcal{X}, F^i)} \Big) \tag{2.3.9}$$

with C a constant independent of u_0, u_1 and f.

Proof. From the formula for w_k it follows that

$$|w_k(t)| \le |u_{0,k}| + \frac{1}{\sqrt{|\varkappa_k|}} |u_{1,k}| + \frac{1}{\sqrt{|\varkappa_k|}} \int_0^T |f_k(t')|\, dt'$$

for all $t \in [0,T]$ and $k = 1, 2, \ldots$. Hence,

$$\begin{aligned}
|w_k(t)|^2 & \le 3\, |u_{0,k}|^2 + \frac{3}{|\varkappa_k|} |u_{1,k}|^2 + \frac{3}{|\varkappa_k|} \Big(\int_0^T |f_k(t')|\, dt' \Big)^2 \\
& \le c(T) \Big(|u_{0,k}|^2 + |\varkappa_k|^{-1} |u_{1,k}|^2 + |\varkappa_k|^{-1} \int_0^T |f_k(t')|^2\, dt' \Big).
\end{aligned} \tag{2.3.10}$$

Furthermore, since

$$|w'_k(t)| \le \sqrt{|\varkappa_k|}\, |u_{0,k}| + |u_{1,k}| + \int_0^T |f_k(t')|\, dt'$$

for all $t \in [0, T]$, we get

$$|w_k'(t)|^2 \le c(T) \Big(|\varkappa_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2\, dt' \Big). \tag{2.3.11}$$

Since the function u_0 belongs to $\overset{\circ}{H}{}^1(\mathcal{X}, F^i)$, its Fourier series over the orthonormal system $(v_k)_{k=1,2,\ldots}$ converges to u_0 actually in the $H^1(\mathcal{X}, F^i)$-norm, see Theorem 3 in [Mik76, p. 181] and elsewhere. Moreover, there is a constant $c > 0$ with the property that

$$\sum_{k=1}^{\infty} |\varkappa_k|\, |u_{0,k}|^2 \le c\, \|u_0\|^2_{H^1(\mathcal{X}, F^i)} \tag{2.3.12}$$

for all $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$.

Consider the partial sum $s_N(x, t)$ of Fourier series (2.3.8). Since both w_k and w_k' are continuous on $[0, T]$, for each fixed $t \in [0, T]$, the function s_N and its derivative in t belong to

$$\overset{\circ}{H}{}^1(\mathcal{X}, F^i).$$

To study the values of $t \mapsto s_N(\cdot, t)$ in $\overset{\circ}{H}{}^1(\mathcal{X}, F^i)$, it is convenient to endow this space with the so-called Dirichlet scalar product

$$D(v, g) = \int_{\mathcal{X}} \Big(\mu (Av, Ag)_x + (\lambda{+}2\mu)(A^* v, A^* g)_x \Big) dx$$

and the Dirichlet norm $D(v) := \sqrt{D(v, v)}$. The system

$$\Big(\frac{v_k}{\sqrt{-\varkappa_k}} \Big)_{k=1,2,\ldots}$$

is obviously orthonormal with respect to the Dirichlet scalar product. By (2.3.10), if $1 \le M < N$, then

$$\begin{aligned}
D(s_N(\cdot, t) - s_M(\cdot, t))^2 &= D\Big(\sum_{k=M+1}^{N} w_k(t) v_k \Big)^2 \\
&= \sum_{k=M+1}^{N} |w_k(t)|^2\, |\varkappa_k| \\
&\le c(T) \sum_{k=M+1}^{N} \Big(|\varkappa_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2\, dt' \Big)
\end{aligned}$$

for all $t \in [0,T]$. Similarly, using (2.3.11), we get

$$\begin{aligned}
& \|(s_N)'_t(\cdot,t) - (s_M)'_t(\cdot,t)\|^2_{L^2(\mathcal{X},F^i)} \\
= & \ \| \sum_{k=M+1}^{N} w'_k(t) v_k \|^2_{L^2(\mathcal{X},F^i)} \\
= & \sum_{k=M+1}^{N} |w'_k(t)|^2 \\
\leq & \ c(T) \sum_{k=M+1}^{N} \Big(|\varkappa_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2\, dt' \Big)
\end{aligned}$$

for $t \in [0,T]$. Here, $c(T)$ stands for a constant which depends on T but not on M and N, and which can be different in diverse applications.

On integrating these two inequalities in $t \in [0,T]$ and summing up them we obtain immediately

$$\|s_N - s_M\|^2_{H^1(\mathcal{C}_T,F^i)} \leq c(T) \sum_{k=M+1}^{N} \Big(|\varkappa_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2\, dt' \Big) \tag{2.3.13}$$

for all $1 \leq M < N$. Combining (2.3.13) with (2.3.3), (2.3.4) and (2.3.12) we conclude that $(s_N)_{N=1,2,\ldots}$ is a Cauchy sequence in $H^1(\mathcal{C}_T, F^i)$. Therefore, series (2.3.8) converges in this space to a function $u(x,t)$ in $H^1(\mathcal{C}_T, F^i)$. Obviously, u satisfies the initial conditions (2.2.2) and vanishes at the lateral boundary of $\mathcal{C}_T$. Letting $N \to \infty$ in (2.3.7) we deduce that u is a weak solution of the first mixed problem for the generalised Lamé system.

In much the same way we derive inequalities

$$\begin{aligned}
D(s_N(\cdot,t))^2 & = D\Big(\sum_{k=1}^{N} w_k(t) v_k \Big)^2 \\
& = \sum_{k=1}^{N} |w_k(t)|^2\, |\varkappa_k| \\
& \leq c(T) \sum_{k=1}^{N} \Big(|\varkappa_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2\, dt' \Big)
\end{aligned}$$

and

$$\begin{aligned}
\|(s_N)'_t(\cdot,t)\|^2_{L^2(\mathcal{X},F^i)} &= \|\sum_{k=1}^{N} w'_k(t)v_k\|^2_{L^2(\mathcal{X},F^i)} \\
&= \sum_{k=1}^{N} |w'_k(t)|^2 \\
&\leq c(T)\sum_{k=1}^{N}\Big(|\varkappa_k|\,|u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2\,dt'\Big)
\end{aligned}$$

for all $t \in [0,T]$ and $N \geq 1$. Integrating these inequalities in $t \in [0,T]$, summing up them and using (2.3.3), (2.3.4) and (2.3.12) we establish estimate (2.3.9), thus completing the proof. □

2.4 Galerkin method

There are also other proofs of the existence of weak solutions to mixed problems which do not exploit eigenfunctions. In this section we present the so-called Galerkin method which allows one to also construct an approximate solution of the mixed problem. In contrast to the Fourier method, the Galerkin method applies also in the case where the coefficients of A depend not only on the space variables but also on the time t.

As before, we assume $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X},F^i)$, $u_1 \in L^2(\mathcal{X},F^i)$ and $f \in L^2(\mathcal{C}_T,F^i)$. Pick an arbitrary system $(v_k)_{k=1,2,\ldots}$ in $C^2(\mathcal{X},F^i)$ which satisfies $v_k = 0$ at $\partial\mathcal{X}$ and is complete in

$$\overset{\circ}{H}{}^1(\mathcal{X},F^i).$$

Given any integer $N \geq 1$, we solve problem (2.2.1), (2.2.2) and (2.2.3) with $u_l = 0$ in the finite-dimensional subspace V_N of $L^2(\mathcal{X},F^i)$ spanned by functions $v_1,\ldots,v_N$. More precisely, we look for a function u_N in $H^2(\mathcal{C}_T,F^i)$, such that $u_N(\cdot,t)$ belongs to the subspace V_N for any fixed $t \in [0,T]$, u_N satisfies conditions (2.2.2) with initial data

$$\begin{aligned}
u_{0,N}(x) &= \sum_{k=1}^{N} u_{0,k}v_k(x), \\
u_{1,N}(x) &= \sum_{k=1}^{N} u_{1,k}v_k(x)
\end{aligned}$$

being orthogonal projections of u_0 and u_1 onto V_N, respectively, and the orthogonal projections of $(u_N)''_{tt} + \mu\Delta u_N + (\lambda+\mu)AA^*u_N$ and f onto V_N

coincide for almost all $t \in [0,T]$. (Note that the orthogonality refers here to the inner product of $L^2(\mathcal{X}, F^i)$.)

We thus search for functions $w_1(t), \ldots, w_N(t)$ in $H^2(0,T)$ which satisfy $w_k(0) = u_{0,k}$ and $w'_k(0) = u_{1,k}$ for all $k = 1, \ldots, N$, and such that

$$u_N(x,t) = \sum_{k=1}^{N} w_k(t) v_k(x)$$

fulfills

$$\int_{\mathcal{X}} \left((u_N)''_{tt} + \mu \Delta u_N + (\lambda+\mu) A A^* u_N, v_k \right)_x dx = \int_{\mathcal{X}} (f, v_k)_x \, dx \qquad (2.4.1)$$

for almost all $t \in [0,T]$ (for which $f(\cdot,t) \in L^2(\mathcal{X}, F^i)$), where $k = 1, \ldots, N$. The Galerkin method consists in approximating the solution u of mixed problem (2.2.1), (2.2.2) and (2.2.3) with $u_l = 0$ by solutions u_N of the projected problems. To substantiate this method one ought to show that each projected problem has a unique solution u_N and the sequence $(u_N)_{N=1,2,\ldots}$ converges in some sense (weakly in $H^1(\mathcal{C}_T, F^i)$) to u.

For simplicity, we restrict ourselves to the case of homogeneous initial conditions $u_0 = 0$ and $u_1 = 0$. Then the coefficients $u_{0,k}$ and $u_{1,k}$ vanish and we are lead to the system

$$\begin{array}{rcl} w_k(0) & = & 0, \\ w'_k(0) & = & 0 \end{array} \qquad (2.4.2)$$

for all $k = 1, \ldots, N$.

Equations (2.4.1) constitute a quadratic system of second order linear ordinary differential equations with constant coefficients for unknown functions $w_1(t), \ldots, w_N(t)$. To wit,

$$\sum_{j=1}^{N} \left(w''_j(t) \, (v_j, v_k)_{L^2(\mathcal{X}, F^i)} + w_j(t) \, D(v_j, v_k) \right) = f_k(t) \qquad (2.4.3)$$

for $k = 1, \ldots, N$, where

$$f_k(t) = \int_{\mathcal{X}} (f(\cdot,t), v_k)_x dx$$

belongs to $L^2(\mathcal{X}, F^i)$.

Our task is to prove that system (2.4.3) has a unique solution $w_1, \ldots, w_N$ with components in $H^1(0,T)$ satisfying initial conditions (2.4.2). Since the system $v_1, \ldots, v_N$ is linearly independent for all integer $N \geq 1$, the (Gram-Schmidt) determinant of the $(N \times N)$-matrix with entries $(v_j, v_k)_{L^2(\mathcal{X}, F^i)}$ is different from zero. Hence, system (2.4.3) can be resolved with respect to the higher order derivatives. It follows that problem (2.4.3), (2.4.2) reduces to the initial problem of canonical form on $[0,T]$, namely

$$\begin{array}{rcl} W'(t) & = & AW(t)+F(t), \quad \text{if} \quad t\in(0,T), \\ W(0) & = & 0, \end{array} \tag{2.4.4}$$

where $W=(w',w)^T$ and

$$A=-\begin{pmatrix} 0 & \left((v_j,v_k)_{L^2(\mathcal{X},F^i)}\right)^{-1}(D(v_j,v_k)) \\ E_N & 0 \end{pmatrix}.$$

The components of the $2N$-column $F(t)$ belong to $L^2(0,T)$. We look for a solution W of problem (2.4.4) in $H^1((0,T),\mathbb{C}^{2N})$. As usual, we replace this problem by the equivalent system of integral equations

$$W(t)=\int_0^t AW(t')dt'+\int_0^t F(t')dt', \tag{2.4.5}$$

the free term on the right-hand side belonging to $H^1((0,T),\mathbb{C}^{2N})$ and so being continuous on $[0,T]$. If $W\in H^1((0,T),\mathbb{C}^{2N})$ is a solution of (2.4.4), then it is continuous on $[0,T]$ and satisfies equation (2.4.5). Conversely, if $W:[0,T]\to\mathbb{C}^{2N}$ is a continuous solution of equation (2.4.5), then it is actually of class $H^1((0,T),\mathbb{C}^{2N})$ and satisfies (2.4.4). And the existence and uniqueness of a continuous solution to equation (2.4.4) is a direct consequence of the Banach fixed point theorem. We have thus proved that system (2.4.3) has a unique solution $w_1,\dots,w_N$ in $H^1(0,T)$ satisfying (2.4.2).

Multiply equality (2.4.1) by $w_k'(t)$, integrate over $t\in(0,t')$, where t' is an arbitrary number of $[0,T]$, and sum up for $k=1,\dots,N$. Then we get

$$\int_{\mathcal{C}_{t'}}((u_N)''_{tt}+\mu\Delta u_N+(\lambda+\mu)AA^*u_N,(u_N)'_t)_x\,dxdt=\int_{\mathcal{C}_{t'}}(f,(u_N)'_t)_x\,dxdt. \tag{2.4.6}$$

Using the Stokes formula one transforms the real part of the left-hand side of this equality to

$$\frac{1}{2}\int_{\mathcal{X}}\left(|(u_N)'_t(x,t')|^2+\mu\,|Au_N(x,t')|^2+(\lambda+2\mu)\,|A^*u_N(x,t')|^2\right)dx$$

for all $t'\in[0,T]$. On the subspace $H^1_b(\mathcal{C}_T,F^i)$ of $H^1(\mathcal{C}_T,F^i)$ consisting of those functions which vanish on the lateral boundary of $\mathcal{C}_T$ and its base, the norm can be equivalently given by

$$\|u\|^2_{H^1_b(\mathcal{C}_T,F^i)}=\int_{\mathcal{C}_T}|u'_t|^2dxdt+\int_0^T D(u(\cdot,t))^2dt,$$

where $D(v)$ is the Dirichlet norm of $v \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$. Hence,

$$\Re \int_0^T dt' \int_{\mathcal{C}_{t'}} ((u_N)''_{tt} + \mu \Delta u_N + (\lambda{+}\mu) A A^* u_N, (u_N)'_t)_x \, dxdt = \frac{1}{2} \, \|u_N\|^2_{H^1_b(\mathcal{C}_T, F^i)}$$

and equality (2.4.6) yields

$$\begin{aligned} \|u_N\|^2_{H^1_b(\mathcal{C}_T, F^i)} &= 2 \, \Re \int_0^T dt' \int_{\mathcal{C}_{t'}} (f, (u_N)'_t)_x \, dxdt \\ &= 2 \, \Re \int_{\mathcal{C}_T} (T - t) \, (f, (u_N)'_t)_x \, dxdt \\ &\leq 2T \, \|f\|_{L^2(\mathcal{C}_T, F^i)} \|u_N\|_{H^1_b(\mathcal{C}_T, F^i)} \end{aligned}$$

whence

$$\|u_N\|_{H^1_b(\mathcal{C}_T, F^i)} \leq 2T \, \|f\|_{L^2(\mathcal{C}_T, F^i)}.$$

We have thus proved that the set of functions u_N, where $N = 1, 2, \ldots$, is bounded in the Hilbert space $H^1_b(\mathcal{C}_T, F^i)$. Therefore, this set is weakly compact in $H^1_b(\mathcal{C}_T, F^i)$, i.e., it has a subsequence which converges weakly in $H^1_b(\mathcal{C}_T, F^i)$ to a function $u \in H^1_b(\mathcal{C}_T, F^i)$. By abuse of notation, we continue to write u_N for this subsequence.

We claim that u is the desired weak solution of the first mixed problem for the generalised Lamé system. To show this it is sufficient to verify that the integral identity

$$\int_{\mathcal{C}_T} (-(u'_t, g'_t) + \mu (Au, Ag) + (\lambda{+}2\mu)(A^* u, A^* g)) \, dxdt = \int_{\mathcal{C}_T} (f, g) \, dxdt \tag{2.4.7}$$

holds for all $g \in H^1(\mathcal{C}_T, F^i)$ which vanish at the lateral boundary of $\mathcal{C}_T$ and the cylinder head, cf. (2.2.4) with $u_1 = 0$. Let us introduce the temporary notation $H^1_c(\mathcal{C}_T, F^i)$ for the (obviously, closed) subspace of $H^1(\mathcal{C}_T, F^i)$ consisting of all such g. It is actually sufficient to establish (2.4.7) for all g in a complete subset Σ of $H^1_c(\mathcal{C}_T, F^i)$.

As Σ we take the set of all functions of the form $z(t) v_k(x)$, where $k \geq 1$ is an integer and $z(t)$ a smooth function on $[0, T]$ satisfying $z(T) = 0$. We first show that equality (2.4.7) is true for each function $g(x, t) = z(t) v_k(x)$ and then that the linear combinations of such functions are dense in $H^1_c(\mathcal{C}_T, F^i)$. To this end, we multiply equality (2.4.1) by $z(t)$, integrate it over $t \in (0, T)$ and apply the Stokes formula, obtaining

$$\int_{\mathcal{C}_T} (-((u_N)'_t, g'_t) + \mu (A u_N, Ag) + (\lambda{+}2\mu)(A^* u_N, A^* g))_x \, dxdt = \int_{\mathcal{C}_T} (f, g)_x dxdt$$

for all $N \geq k$, where $g = zv_k$. This implies readily (2.4.7), for $u_N \to u$ weakly in $H^1(\mathcal{C}_T, F^i)$.

Our next goal is to show that the linear hull of Σ is dense in $H^1_c(\mathcal{C}_T, F^i)$. To do this it is sufficient to prove that each function $g \in C^2(\overline{\mathcal{C}}_T, F^i)$ vanishing at the lateral boundary of the cylinder and its head (the set of such functions is dense in $H^1_c(\mathcal{C}_T, F^i)$) can be approximated in the $H^1(\mathcal{C}_T, F^i)$-norm by linear combinations of functions in Σ. This last assertion is actually well known within the framework of theory of Sobolev spaces. For a proof, we refer the reader to [Mik76, p. 302] and elsewhere.

Remark 2.4.1. *Since the weak solution of the first mixed problem exists and is unique, not only a subsequence but also the sequence $(u_N)_{N=1,2,\ldots}$ itself converges weakly in $H^1(\mathcal{C}_T, F^i)$ to u.*

2.5 Regularity of weak solutions

Assume that the boundary $\partial\mathcal{X}$ of $\mathcal{X}$ is of class C^s for some integer $s \geq 1$. Then the eigenfunctions $(v_k)_{k=1,2,\ldots}$ of problem (2.3.1) belong to $H^s(\mathcal{X}, F^i)$ and satisfy the boundary conditions

$$L^i v_k \;=\; 0 \quad \text{at} \quad \partial\mathcal{X} \tag{2.5.1}$$

for $i = 0, 1, \ldots, \left[\dfrac{s-1}{2}\right]$.

Let $H^s_{\mathcal{D}}(\mathcal{X}, F^i)$ stand for the subspace of $H^s(\mathcal{X}, F^i)$ consisting of all functions v satisfying (2.5.1). We put additional restrictions on the data of the problem to attain to a classical solution. More precisely, we require that $u_0 \in H^s_{\mathcal{D}}(\mathcal{X}, F^i)$, $u_1 \in H^{s-1}_{\mathcal{D}}(\mathcal{X}, F^i)$ and f belongs to the subspace of $H^{s-1}(\mathcal{C}_T, F^i)$ consisting of all functions satisfying

$$L^i f \;=\; 0 \quad \text{at} \quad \partial\mathcal{X} \times (0, T) \tag{2.5.2}$$

for $i = 0, 1, \ldots, \left[\dfrac{s}{2}\right] - 1$.

For $s = 1$, the latter equations are empty and we arrive at $f \in L^2(\mathcal{X}, F^i)$, as above.

Theorem 2.5.1. *Under the above hypotheses, series (2.3.8) converges to the weak solution $u(x,t)$ in $H^s(\mathcal{X}, F^i)$ uniformly in $t \in [0,T]$. Given any $j = 1, \ldots, s$, the series obtained from (2.3.8) by the j-fold termwise differentiation in t converges in $H^{s-j}(\mathcal{X}, F^i)$ uniformly in $t \in [0,T]$. Moreover,* *there is a constant $c > 0$ independent of t, such that*

$$\begin{aligned}\sum_{j=0}^{s} \| \sum_{k=1}^{\infty} w_k^{(j)}(t) v_k \|^2_{H^{s-j}(\mathcal{X},F^i)} \\ \leq \; c \left(\|u_0\|^2_{H^s(\mathcal{X},F^i)} + \|u_1\|^2_{H^{s-1}(\mathcal{X},F^i)} + \|f\|^2_{H^{s-1}(\mathcal{C}_T,F^i)} \right)\end{aligned} \tag{2.5.3}$$

for all $t \in [0, T]$.

Proof. The proof of this theorem runs similarly to the proof of Theorem 3 of [Mik76, p. 305], if one exploits the techniques developed earlier in Sections 2.2 and 2.3. □

By (2.5.3), if $1 \leq M < N$, then

$$\sup_{t \in [0,T]} \Big\| \sum_{k=M+1}^{N} w_k^{(j)}(t) v_k \Big\|^2_{H^{s-j}(\mathcal{X},F^i)} \to 0$$

as $M \to \infty$. Hence, the partial sums of series (2.3.8) converge in $H^s(\mathcal{C}_T, F^i)$ and from (2.5.3) it follows that

$$\|u\|_{H^s(\mathcal{C}_T,F^i)} \leq c' \left(\|u_0\|_{H^s(\mathcal{X},F^i)} + \|u_1\|_{H^{s-1}(\mathcal{X},F^i)} + \|f\|_{H^{s-1}(\mathcal{C}_T,F^i)} \right). \tag{2.5.4}$$

Corollary 2.5.2. *Under the above hypotheses, the weak solution of the first mixed problem for the generalised Lamé system belongs to $H^s(\mathcal{C}_T, F^i)$. Moreover, series (2.3.8) converges to the weak solution in the $H^s(\mathcal{C}_T, F^i)$ -norm and inequality (2.5.4) holds true.*

From Corollary 2.5.2 with $s = 2$ it follows that the weak solution of the first mixed problem belongs to $H^2(\mathcal{C}_T, F^i)$, and so it is a solution almost everywhere. If moreover $s > n/2 + 2$, then the weak solution u belongs to the space $C^2(\overline{\mathcal{C}}_T, F^i)$, which is due to the Sobolev embedding theorem, and so u is a classical solution of the problem.

Note that along with the smoothness of u_0, u_1 and f Theorem 2.5.1 assumes that u_0 satisfies (2.5.1), u_1 satisfies (2.5.1) with s replaced by $s - 1$, and f satisfies (2.5.2). The conditions are actually necessary. To show this, suppose $s \geq 2$. Since $u_0(x) = u(x, 0)$ is represented by series (2.3.8) which converges in $H^s(\mathcal{X}, F^i)$, and $u_1(x) = u'_t(x, 0)$ is represented by series (2.3.8) which is differentiated termwise in t and converges in $H^{s-1}(\mathcal{X}, F^i)$, we conclude readily that u_0 satisfies (2.5.1) and u_1 satisfies (2.5.1) with s replaced by $s - 1$. Furthermore, since series (2.3.8) converges to u in $H^s(\mathcal{C}_T, F^i)$, the series obtained from (2.3.8) by termwise applying the operators L and the second derivative in t converge in $H^{s-2}(\mathcal{C}_T, F^i)$ to Lu and u''_{tt}, respectively. Hence, if $s \geq 3$, then $f = u''_{tt} - Lu$ satisfies equalities (2.5.2) with s replaced by $s-1$. In case s is even, the last condition of (2.5.2) is superfluous indeed, see Corollary 2 in [Mik76, p. 311].

However, if one wants to prove the smoothness of the weak solution of the first mixed problem rather than the convergence of the Fourier series in the corresponding spaces, then conditions (2.5.1) and (2.5.2) can be essentially relaxed, see Theorem 3' in [Mik76, p. 323].

Chapter 3

The Navier-Stokes equations for elliptic complexes

We continue our study of invariant forms of the classical equations of mathematical physics, such as the Maxwell equations or the Lamé system, on manifold with boundary. To this end we interpret them in terms of the de Rham complex at a certain step. On using the structure of the complex we get an insight to predict a degeneracy deeply encoded in the equations. In the present paper we develop an invariant approach to the classical Navier-Stokes equations.

3.1 Generalised Navier-Stokes equations

Let $\mathcal{X}$ be a compact differentiable manifold of dimension n with or without boundary. Consider the de Rham complex

$$0 \to \Omega^0(\mathcal{X}) \stackrel{d}{\to} \Omega^1(\mathcal{X}) \stackrel{d}{\to} \ldots \stackrel{d}{\to} \Omega^n(\mathcal{X}) \to 0$$

on $\mathcal{X}$, where $\Omega^i(\mathcal{X})$ are the spaces of differential forms of degree i with C^∞ coefficients on $\mathcal{X}$. The impulse equation can be immediately rewritten in terms of one-forms as

$$\rho u'_t + \mu \, \Delta u + (\lambda + \mu) dd^* u + dp + (u')^* u = f$$

in $\mathcal{X} \times (0, T)$, where d^* is the formal adjoint of d, $\Delta = d^*d + dd^*$ the Laplace operator of Hodge, and $(u')^*$ the dual of the tangential mapping $u'(x) : T_x\mathcal{X} \to T_x\mathcal{X}$ Every term in the equation makes still sense for differential forms u of arbitrary degree $0 \leq i \leq n$, except for the nonlinear perturbation $(u')^* u$ which is defined solely for one-forms. On the other hand, the specific form $(u')^* u$ does not survive under simple transforms like a shift $u \mapsto u + v$ which are needed to reduce nonzero initial or boundary data to the zero ones. Hence, to specify the nonlinearity we write it in a more abstract form $N(u)$, where N^i is an unbounded nonlinear operator in the space of differential forms of degree i with square integrable coefficients on $\mathcal{X}$ and, as usual, we set $Nu = N^i u$ for $u \in \Omega^i(\mathcal{X})$. Later on we impose an additional condition on N^i which implies an energy estimate.

Hence, the impulse equation generalises to arbitrary step i of the de Rham complex in the form

$$\rho u'_t + \mu\, \Delta u + (\lambda + \mu) d d^* u + dp + N(u) = f$$

while the continuity equation for incompressible fluid reads $d^* u = 0$.

As usual one investigates these evolution equations in the open cylinder $\mathcal{C}_T := \overset{\circ}{\mathcal{X}} \times (0, T)$ whose base is the interior of $\mathcal{X}$. Up to the pressure p the linear part of the impulse equation looks like the generalised Lamé system, cf. Chapter 2. The crucial difference lies in the fact that the impulse equation of hydrodynamics is parabolic while the Lamé system is of hyperbolic type. This is clarified within the framework elasticity theory which proceeds from the assumption that the displacement u befalls along the optical fibres similarly to waves.

We now assume that

$$0 \to C^\infty(\mathcal{X}, F^0) \overset{A}{\to} C^\infty(\mathcal{X}, F^1) \overset{A}{\to} \ldots \overset{A}{\to} C^\infty(\mathcal{X}, F^N) \to 0 \tag{3.1.1}$$

is an arbitrary elliptic complex of first order differential operators between sections of vector bundles F^i over $\mathcal{X}$. The differential A of this complex is given by a sequence $A^i \in \mathrm{Diff}^1(\mathcal{X}; F^i, F^{i+1})$ satisfying $A^{i+1} A^i = 0$, where $A^i \equiv 0$ unless $i = 0, 1, \ldots, N-1$. We introduce the generalised Navier-Stokes equations by

$$\begin{array}{rcl} u'_t + \nu\, \Delta u + Ap + N(u) & = & f, \\ A^* u & = & 0 \end{array} \tag{3.1.2}$$

for unknown sections u and p of the (induced) vector bundles F^i and F^{i-1} over $\mathcal{C}_T$, respectively, where $\Delta = A^* A + A A^*$ is the Laplacian of complex (3.1.1) and N a graded operator corresponding to a sequence $\{N^i\}$ of unbounded nonlinear operators in the spaces $L^2(\mathcal{X}, F^i)$ of square integrable sections of the vector bundles F^i. By the above, we set $\nu = \nu/\rho$.

Example 3.1.1. For $i = 0$ equations (3.1.2) reduce obviously to

$$u'_t + \nu\, \Delta u + N(u) = f$$

in C_T because of $A^{-1} \equiv 0$. This equation can be thought of as a far-reaching generalisation of the well-known Burgers equation in one spatial variable, see [Bur40], [Hop50].

When posing initial and boundary conditions for a solution (u, p) of (3.1.2), we observe that the "pressure" p is no longer determined by u up to a finite-dimensional subspace of $L^2(\mathcal{X}, F^{i-1})$, for the null-space of A^{i-1} need not be of finite dimension. Hence, we have to subject p to certain boundary conditions. Since we are going to project the first equation of (3.1.2) onto the space of solutions to $A^*u = 0$, we look for a suitable boundary condition within the framework of the Neumann problem after Spencer, see [Tar95]. Given any $v \in L^2(\mathcal{X}, F^i)$, it consists in finding a section $g \in L^2(\mathcal{X}, F^i)$ satisfying $\Delta g = v$ in $\mathcal{X}$ and $n(g) = n(Ag) = 0$ at $\partial\mathcal{X}$ in a weak sense. Here, by $n(g)$ is meant the so-called normal part of g at the boundary which bears the Cauchy data of g with respect to A^*. As already mentioned, the study of this problem stimulated to essential development of analysis and geometry in the 1960s.

Lemma 3.1.2. *Suppose that the Neumann problem is solvable at step i for the complex (3.1.1) and H and G are the corresponding harmonic projection and the Green operator. Then the operator $Pg := Hg + A^*A\,Gg$ is an orthogonal projection in $L^2(\mathcal{X}, F^i)$.*

Proof. Under the assumption of the lemma, the space of all $g \in L^2(\mathcal{X}, F^i)$ satisfying $\Delta g = 0$ in $\mathcal{X}$ and $n(g) = n(Ag) = 0$ at $\partial\mathcal{X}$ is finite dimensional. The elements of this space are called harmonic sections and they actually satisfy $Ag = A^*g = 0$ in $\mathcal{X}$. The harmonic sections prove to be C^∞ sections of F^i over $\mathcal{X}$, and so the orthogonal projection H onto the space is a smoothing operator. Moreover, there is a compact selfadjoint operator G in $L^2(\mathcal{X}, F^i)$, such that $n(Gg) = n(AGg) = 0$ at $\partial\mathcal{X}$ for all $g \in L^2(\mathcal{X}, F^i)$ and the identity operator in $L^2(\mathcal{X}, F^i)$ splits into $H + A^*AG + AA^*G$. The Green operator G is of pseudodifferential nature. The decomposition $g = Hg + A^*AGg + AA^*Gg$ valid for all $g \in L^2(\mathcal{X}, F^i)$ is usually referred to as the generalised Hodge decomposition. Since $A^2 = 0$, the summands are pairwise orthogonal, and so both A^*AG and AA^*G are orthogonal projections, too. For a thorough discussion of the Neumann problem we refer the reader to [Tar95, Ch. 4]. □

The projector P is an analogue of the Helmholtz projector onto vector fields which are divergence free. A slightly different approach to this decomposition is presented in [Lad70].

Lemma 3.1.3. *In order that $Pg = g$ be valid it is necessary and sufficient that $A^*g = 0$ in $\mathcal{X}$ and $n(g) = 0$ at $\partial\mathcal{X}$.*

Proof. Suppose that $Pg = g$. Then $A^*g = A^*(Hg + A^*A\,Gg)$ vanishes in $\mathcal{X}$ and $n(g) = n(Hg + A^*A\,Gg)$ vanishes at $\partial\mathcal{X}$, for $n(AGg) = 0$ implies immediately $n(A^*\,AGg) = 0$, as is easy to check. On the other hand, if $A^*g = 0$ in $\mathcal{X}$ and $n(g) = 0$ at $\partial\mathcal{X}$, then an easy calculation shows that $A^*(Gg) = 0$ whence $Pg = g$, as desired. $\square$

From Lemma 3.1.3 it follows that P vanishes on sections of the form Ap with $p \in L^2(\mathcal{X}, F^{i-1})$ and $Ap \in L^2(\mathcal{X}, F^i)$. Indeed, to prove this it suffices to show that Ap is orthogonal to all sections g satisfying $A^*g = 0$ in $\mathcal{X}$ and $n(g) = 0$ at the boundary. For such a section g we get

$$\begin{array}{rcl} (Ap, g)_{L^2(\mathcal{X},F^i)} & = & (p, A^*g)_{L^2(\mathcal{X},F^{i-1})} \\ & = & 0, \end{array}$$

as desired.

Since equations (3.1.2) do not contain any derivative of p in t, we formulate an initial condition

$$u(x,0) = u_0(x), \tag{3.1.3}$$

for each $x \in \overset{\circ}{\mathcal{X}}$, on the base of the cylinder $\mathcal{C}_T$, and a boundary condition

$$u(x,t) = u_l(x,t) \tag{3.1.4}$$

for all $(x,t) \in \partial\mathcal{X} \times (0,T)$ on the lateral surface of $\mathcal{C}_T$. If u is affixed to u_0 at the initial moment $t = 0$ strong enough, then u_0 should inherit from u the condition $A^*u_0 = 0$ in some weak sense in $\mathcal{X}$. Moreover, the normal component of the "velocity" u at the lateral surface should vanish, hence $n(u_l) = 0$ at $\partial\mathcal{X} \times (0,T)$. We thus get

$$\begin{array}{rcll} A^*u_0 & = & 0 & \text{in} \ \ \mathcal{X}, \\ n(u_l) & = & 0 & \text{at} \ \ \partial\mathcal{X} \times (0,T), \end{array} \tag{3.1.5}$$

a compatibility condition completing the physical interpretation of (3.1.2) as generalised Navier-Stokes equations.

If the smoothness of u allows one to control the values of u at $\partial\mathcal{X}$ up to $t = 0$, then (3.1.5) implies, in particular, that u_0 is a solution of the Cauchy problem for the formal adjoint of A^{i-1} with zero data in $\mathcal{X}$. For $i = N$ the differential operator $A^{i-1\,*}$ is (possibly, overdetermined) elliptic, and so u_0 is specified within a subspace of $C^\infty(\mathcal{X}, F^N)$ of finite dimension. Note that for $i = 0$ both equations of (3.1.5) are empty.

Neither of equations (3.1.3) and (3.1.4) puts any restriction on the "pressure" p, and so p remains still undetermined. If looking for a $p \in L^2(\mathcal{C}_T, F^{i-1})$ within the framework of the Neumann problem, one can determine p uniquely from the condition that p is orthogonal to the subspace of $L^2(\mathcal{X}, F^{i-1})$ consisting of all solutions v to the homogeneous equation $Av = 0$ in $\mathcal{X}$. This solution is called canonical (it amounts to $A^* G(Ap)$).

3.2 Energy estimates

As is known, one of the main relations for incompressible viscous fluid in a bounded domain $\mathcal{X} \subset \mathbb{R}^n$ with smooth boundary is the so-called energy balance relation

$$\frac{1}{2} \int_{\mathcal{X}} |u|^2 dx \,\Big|_{t'}^{t''} + \nu \int_{t'}^{t''} \!\! \int_{\mathcal{X}} |u'_x|^2 dx dt = \int_{t'}^{t''} \!\! \int_{\mathcal{X}} (f, u)\, dx dt$$

for all $t', t'' \in (0, T)$. It is valid for all sufficiently smooth nonstationary vector fields $u(x,t)$ on the cylinder $\mathcal{C}_T$ over $\mathcal{X}$, satisfying (0.0.7) under the homogeneous boundary condition $u = 0$ at $\partial\mathcal{X}$. The proof is based on a lemma which provides an insight into the nonlinearity.

Lemma 3.2.1. *For each $u \in C^1(\overline{\mathcal{X}}, \mathbb{R}^n)$ it follows that*

$$\int_{\mathcal{X}} (u'_x u, u)\, dx = - \int_{\mathcal{X}} \frac{1}{2} |u|^2 \operatorname{div} u \, dx + \int_{\partial\mathcal{X}} \frac{1}{2} |u|^2 \, (u, \nu)\, ds,$$

where ds is the area form of the hypersurface $\partial\mathcal{X}$ and $\nu(x)$ the outward unit normal vector at a point $x \in \partial\mathcal{X}$.

Proof. Using the Stokes formula, we get

$$\begin{aligned} \int_{\mathcal{X}} (u'_x u, u)\, dx &= \int_{\mathcal{X}} \sum_{j=1}^{n} \sum_{k=1}^{n} (\partial_k u_j)\, u_k u_j dx \\ &= \int_{\mathcal{X}} \sum_{k=1}^{n} \partial_k \Big(\frac{1}{2} \sum_{j=1}^{n} u_j^2 \Big) u_k \, dx \\ &= - \int_{\mathcal{X}} \frac{1}{2} |u|^2 \operatorname{div} u \, dx + \int_{\partial\mathcal{X}} \frac{1}{2} |u|^2 \, (u, \nu)\, ds, \end{aligned}$$

as desired. □

To guarantee an energy estimate for generalised Navier-Stokes equations (3.1.2) we impose a special restriction on the nonlinear term $N(u)$. In the sequel we assume that

$$(N(u), u)_{L^2(\mathcal{X}, F^i)} = 0 \qquad (3.2.1)$$

for all $u \in L^2(\mathcal{X}, F^i)$ in the domain of N satisfying $A^*u = 0$ in $\mathcal{X}$ and $n(u) = 0$ at $\partial\mathcal{X}$. Equality (3.2.1) is fulfilled, in particular, if

$$(N(u), v)_x = (u, AB(v, u))_x$$

pointwise for all $u, v \in C^\infty(\mathcal{X}, F^i)$ up to a term vanishing for $u = v$, where B is a smooth sesquilinear form on $F^i \times F^i$ with values in F^{i-1}. In the classical case we have

$$B(u, v) = \frac{1}{2}\,(u, v)_x.$$

Theorem 3.2.2. *Let u be a bounded section of Slobodetskii space $H^{2,1}(\mathcal{C}_T, F^i)$ satisfying equations (3.1.2) in $\mathcal{C}_T$ and vanishing at the lateral surface of the cylinder. Then,*

$$\frac{1}{2}\int_{\mathcal{X}} |u|^2 dx\,\Big|_{t'}^{t''} + \nu \int_{t'}^{t''}\!\!\int_{\mathcal{X}} |Au|^2\, dxdt = \Re \int_{t'}^{t''}\!\!\int_{\mathcal{X}} (f, u)\, dxdt \qquad (3.2.2)$$

for all $t', t'' \in (0, T)$.

Proof. Since u is bounded, we can take the pointwise scalar product of both sides of the impulse equation of (3.1.2) with u and integrate it over the cylinder $\mathcal{X} \times (t', t'')$. This gives

$$\int_{t'}^{t''}\!\!\int_{\mathcal{X}} (u'_t + \nu\varDelta u + Ap + N(u), u)_x\, dxdt = \int_{t'}^{t''}\!\!\int_{\mathcal{X}} (f, u)_x\, dxdt$$

for all $t', t'' \in (0, T)$.

It is easily seen that

$$\Re\,(u'_t, u)_x = \frac{1}{2}\frac{\partial}{\partial t}(u, u)_x$$

whence

$$\Re \int_{t'}^{t''}\!\!\int_{\mathcal{X}} (u'_t, u)_x dxdt = \frac{1}{2}\int_{\mathcal{X}} |u|^2 dx\,\Big|_{t'}^{t''}$$

by the Newton-Leibniz formula.

Furthermore, using the "continuity equation" $A^*u = 0$ in $\mathcal{X}$ and integration by parts we obtain

$$\int_{\mathcal{X}} (\varDelta u, u)_x\, dx \;=\; \int_{\mathcal{X}} |Au|_x^2\, dx + \int_{\partial\mathcal{X}} ((\sigma^i)^* Au, u)_x\, ds,$$

where σ^i is ($\sqrt{-1}$ times) the principal symbol of the differential operator A^i evaluated at the point $(x, \nu) \in T^*\mathcal{X}$. The integral over $\partial\mathcal{X}$ on the right-hand side vanishes, for u has zero Cauchy data with respect to the differential operator A^i at the boundary.

Since $p \in L^2(\mathcal{X}, F^{i-1})$ and $Ap \in L^2(\mathcal{X}, F^i)$, it follows from what is said in Section 3.1 that

$$\int_{\mathcal{X}} (Ap, u)_x \, dx = 0$$

for all $t \in (0, T)$.

Finally, we take into consideration the structure of nonlinearity $N(u)$ described in (3.2.1) to deduce that

$$\begin{aligned}\int_{\mathcal{X}} (N(u), u)_x \, dx &= \int_{\mathcal{X}} (u, AB(u,u))_x \, dx \\ &= \int_{\mathcal{X}} (A^* u, B(u,u))_x \, dx - \int_{\partial \mathcal{X}} ((\sigma^{i-1})^* u, B(u,u))_x \, ds \\ &= 0\end{aligned}$$

in much the same way as in the proof of Lemma 3.2.1. Summarising we arrive at equality (3.2.2), as desired. □

From Theorem 3.2.2 it follows readily that for solutions of the generalised Navier-Stokes equations, which vanish at the lateral surface of $\mathcal{C}_T$, one can estimate the energy norm

$$\|u\|_{\mathrm{EN}} := \sup_{0 \le t \le T} \|u\|_{L^2(\mathcal{X}, F^i)} + \|Au\|_{L^2(\mathcal{C}_T, F^{i+1})} \tag{3.2.3}$$

only through the norms $\|f\|_{L^{2,1}(\mathcal{C}_T, F^i)}$ and $\|u_0\|_{L^2(\mathcal{X}, F^i)}$, where

$$\|f\|_{L^{q,r}(\mathcal{C}_T, F^i)} := \Big(\int_0^T \Big(\int_{\mathcal{X}} |f(x,t)|^q dx \Big)^{r/q} dt \Big)^{1/r},$$

cf. [Lad70].

The set of sections $u(x,t)$ having finite energy norm (3.2.3) forms a Banach space. Its elements need not be continuous in t in the $L^2(\mathcal{X}, F^i)$-norm. By analogy with other studied problems one might believe that this class is fairly natural for the Navier-Stokes equations. Such a class was first introduced in [Hop51]. However, the class has proved to be too large for the classical Navier-Stokes equations in $\mathbb{R}^3$, for the uniqueness theorem of the first mixed problem is violated in this class. Under finite energy norm, the uniqueness property for the classical Navier-Stokes equations takes place first in $L^{q,r}(\mathcal{C}_T, \mathbb{R}^n)$ with $n/2q + 1/r \le 1/2$, see [Lad70] and [Lad03].

It is worth pointing out that the structure of nonlinearity specified by (3.2.1) is still too general to introduce weak solutions of class $L^2(\mathcal{C}_T, F^i)$ to the generalised Navier-Stokes equations.

3.3 First steps towards the solution

On applying the Helmholtz projector to the generalised impulse equation one obtains

$$(Pu)'_t + \nu\, P(\Delta u) + P(Ap) + PN(u) = Pf$$

while the continuity equation means that $Pu = u$ in $\mathcal{C}_T$. Since $P(Ap) = 0$, this allows one to eliminate the "pressure" from the impulse equation, thus obtaining an equivalent form

$$\begin{array}{rcl} (Pu)'_t + \nu\, P(\Delta Pu) + P(Ap) + PN(Pu) & = & Pf, \\ ((I-P)Pu)'_t + \nu\,(I-P)(\Delta Pu) + (I-P)Ap + (I-P)N(Pu) & = & (I-P)f \end{array}$$

of the Navier-Stokes equations, for $(I-P)P = 0$ and

$$\begin{array}{rcl} (I-P)\Delta P & = & (I - H - A^*AG)\,\Delta\,(H + A^*AG) \\ & = & AA^*G\,A^*A\,\Delta G \\ & = & 0, \end{array}$$

the last equality being due to the fact that A^*GA^* vanishes on sections of zero Cauchy data with respect to A^* at $\partial\mathcal{X}$.

In other words, we separate the generalised Navier-Stokes equations into two single problems

$$(Pu)'_t + \nu\, P(\Delta Pu) + P(Ap) + PN(Pu) = Pf$$

in $\mathcal{C}_T$ under the initial and boundary conditions

$$\begin{array}{rcll} Pu\,(x,0) & = & u_0(x), & \text{for} \quad x \in \mathcal{X}, \\ Pu\,(x,t) & = & u_l(x,t), & \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T), \end{array}$$

and

$$\begin{array}{rcll} Ap & = & (I-P)\,(f - N(Pu)) & \text{in} \quad \mathcal{C}_T, \\ (p,v)_{L^2(\mathcal{X},F^{i-1})} & = & 0 & \text{for} \quad v \in \ker A. \end{array} \tag{3.3.1}$$

As already mentioned in Section 3.1, if the Neumann problem of Spencer is solvable at step i of elliptic complex (3.1.1), then the only solution of problem (3.3.1) is given by

$$p = A^*G(I-P)\,(f - N(Pu)).$$

The operator $P\Delta$ is sometimes called the Stokes operator. It is of pseudodifferential nature.

Lemma 3.3.1. *Suppose that $u \in H^{2,1}(\mathcal{C}_T, F^i)$ is a bounded solution to the first mixed problem*

$$\begin{array}{rcll} u'_t + \nu\, P\Delta u + PN(u) & = & Pf & \textit{in} \;\; \mathcal{C}_T, \\ u & = & u_0 & \textit{at} \;\; \overset{\circ}{\mathcal{X}} \times \{0\}, \\ u & = & u_l & \textit{at} \;\; \partial\mathcal{X} \times (0,T) \end{array} \tag{3.3.2}$$

in the cylinder. Then $A^* u = 0$ in $\mathcal{C}_T$.

Proof. Indeed, from the differential equation of (3.3.2) it follows that

$$\frac{\partial}{\partial t} A^* u = 0$$

in $\mathcal{C}_T$. Since $A^* u = A^* u_0 = 0$ for $t = 0$, we deduce readily that $A^* u = 0$ for all $t \in (0,T)$, as desired. □

Summarising we choose the following way of solving the generalised Navier-Stokes equations. We first construct a solution u of mixed problem (3.3.2). According to Lemma 3.3.1, u satisfies $A^* u = 0$ in $\mathcal{C}_T$, and so $Pu = u$. Substitute this section into equation (3.3.1) for p. From this equation the "pressure" p is determined uniquely and bears the appropriate regularity of the canonical solution of the Neumann problem for complex (3.1.1) at step i. Finally, on combining the equations (3.3.2) and (3.3.1) we conclude that the pair (u,p) is a solution of (3.1.2) under conditions (3.1.3) and (3.1.4).

In the sequel we focus on the study of operator equation (3.3.2) by Hilbert space methods.

3.4 A WKB solution

To handle the nonlinear term $N(u)$ in the generalised Navier-Stokes equations it might be useful to gain a small parameter ε multiplying $N(u)$. To this end we restrict our attention to those N which are of the form $N(u) = N(u,u)$, where $N(u,v)$ is a first order bidifferential operator between sections of F^i on $\mathcal{X}$, as in the classical case. Pick an arbitrary $\varepsilon \neq 0$ and change the dependent variable by $u = \varepsilon \tilde{u}$. Substituting u into (3.3.2) and using the specific form of nonlinearity to divide both sides by ε, we get

$$\begin{array}{rcll} \tilde{u}'_t + \nu\, P\Delta \tilde{u} + \varepsilon\, PN(\tilde{u}) & = & P\tilde{f} & \text{in} \;\; \mathcal{C}_T, \\ \tilde{u} & = & \tilde{u}_0 & \text{at} \;\; \overset{\circ}{\mathcal{X}} \times \{0\}, \\ \tilde{u} & = & \tilde{u}_l & \text{at} \;\; \partial\mathcal{X} \times (0,T), \end{array} \tag{3.4.1}$$

where $\tilde{f} = f/\varepsilon$, $\tilde{u}_0 = u_0/\varepsilon$ and $\tilde{u}_l = u_l/\varepsilon$ are as arbitrary as f, u_0 and u_l if the domains for f, u_0 and u_l are invariant under stretching. We have thus arrived at the same mixed problem for the Navier-Stokes equations in the cylinder $\mathcal{C}_T$ but the problem now contains a small parameter ε multiplying the nonlinear term. By abuse of notation we omit the sign "tilde" and write u, f, u_0 and u_l for the new variables.

By experience with other studied mixed problems for parabolic equations, the problem (3.4.1) for $\varepsilon = 0$ has a unique solution in $H^{2,1}(\mathcal{C}_T, F^i)$ which depends continuously on the data f, u_0 and u_l. Therefore, we may try to exploit a WKB approximation

$$u(x,t) = \sum_{k=1}^{\infty} c_k(x,t)\, \varepsilon^k$$

to construct a solution for nonlinear problem (3.4.1), the series being asymptotic for $\varepsilon \to 0$. On substituting this expansion into (3.4.1) and equating the coefficients of the same powers of ε we get a linear mixed problem for determining the initial approximation c_0

$$\begin{array}{rcll} \dfrac{\partial}{\partial t} c_0 + \nu\, P \varDelta c_0 & = & P f & \text{in} \ \ \mathcal{C}_T, \\ c_0 & = & u_0 & \text{at} \ \ \mathring{\mathcal{X}} \times \{0\}, \\ c_0 & = & u_l & \text{at} \ \ \partial\mathcal{X} \times (0,T) \end{array} \tag{3.4.2}$$

and a system of recurrent equations

$$\begin{array}{rcll} \dfrac{\partial}{\partial t} c_k + \nu\, P \varDelta c_k & = & - \displaystyle\sum_{i+j=k-1} P N(c_i, c_j) & \text{in} \ \ \mathcal{C}_T, \\ c_k & = & 0 & \text{at} \ \ \mathring{\mathcal{X}} \times \{0\}, \\ c_k & = & 0 & \text{at} \ \ \partial\mathcal{X} \times (0,T) \end{array}$$

for $k = 1, 2, \ldots$.

The recurrent equations display once again the main problem in solving the Navier-Stokes equations. We start with data $f \in L^2(\mathcal{C}_T, F^i)$ and u_0, u_l of relevant regularity. The initial approximation c_0 will belong to the Slobodetskii space $H^{2,1}(\mathcal{C}_T, F^i)$. The mixed problem for c_1 has the right-hand side $-N(c_0, c_0)$ and zero initial and boundary data. To evaluate the right-hand side one uses the so-called multiplicative inequalities, see for instance [Lad70]. However, one can see from the very beginning that $N(c_0, c_0)$ fails to belong to $L^2(\mathcal{C}_T, F^i)$ and so no iteration is possible to determine c_1, etc. within the L^2-approach. If c_0 is additionally bounded then $N(c_0, c_0) \in L^2(\mathcal{X}, F^i)$ and one can find c_1 in $H^{2,1}(\mathcal{C}_T, F^i)$, etc. However, $H^{2,1}(\mathcal{C}_T, F^i)$ is embedded into $L^\infty(\mathcal{C}_T, F^i)$ only for $n = 1$ and $n = 2$ while no other criteria for the existence of a bounded solution have been known. Thus, the WKB approximation gives an evidence to the lack of smoothness controlled by L^2-scales.

On the other hand, if we look for a solution $u \in W^{(2,1),q}(\mathcal{X}, F^i)$ of the Navier-Stokes equations with q large enough, so that $W^{(2,1),q}(\mathcal{X}, F^i)$ is embedded continuously into $C(\mathcal{X}, F^i)$, then the construction of a WKB solution goes through. This motivates the study of the linearised Navier-Stokes equations in Banach spaces $W^{(2,1),q}(\mathcal{X}, F^i)$, where q is sufficiently large. (By the Sobolev embedding theorem, $q > n$ is sufficient.)

Chapter 4

Particular cases

4.1 Analysis in the case of closed manifolds

In this section we consider in detail the case where $\mathcal{X}$ is a smooth compact closed manifold of dimension n. Recall that the classical Hodge theory extends to elliptic complexes on compact closed manifolds without any essential changes, see for instance [Wel73], [Tar95] and elsewhere. As but one byproduct of this theory we mention the fact that the projector P is a classical pseudodifferential operator of order 0 between sections of the vector bundle F^i on $\mathcal{X}$.

Given an arbitrary section $f \in L^2(\mathcal{C}_T, F^i)$, consider the pseudodifferential equation

$$u'_t + \nu\, P\Delta u + \varepsilon\, PN(u) = Pf \tag{4.1.1}$$

in $\mathcal{C}_T$ for an unknown section $u \in H^{2,1}(\mathcal{C}_T, F^i)$, where ε is a small parameter. We tacitly assume that the term $N(u)(\cdot, t)$ belongs to $L^2(\mathcal{X}, F^i)$ for almost all $t \in (0, T)$.

To treat (4.1.1) within the framework of ordinary differential equations with operator-valued coefficients, we should give the operators proper domains. The closure of Δ in $L^2(\mathcal{X}, F^i)$ has domain $H^2(\mathcal{X}, F^i)$ and is nonnegative in this domain. The projector P is obviously nonnegative, hence we make $\nu\, P\Delta$ into a positive operator by adding λI with any $\lambda > 0$. To this end change the dependent variable by $u = e^{\lambda t}\tilde{u}$. Substituting this into (4.1.1), dividing by $e^{\lambda t}$ and writing $\tilde{u}$ and $\tilde{f} = e^{-\lambda t} f$ simply u and f we get

$$u'_t + Lu + \varepsilon\, e^{\lambda t} PN(u) = Pf$$

where

$$Lu = P(\nu\, \Delta + \lambda I)u.$$

It is precisely the abstract form in which we study the Navier-Stokes equations in $\mathcal{C}_T$, cf. [Lad56, Lad58].

Remark 4.1.1. *Combining Lemma 3.1.3 and Lemma 3.3.1 makes it reasonable to restrict the domains of operators to the subspace of* $L^2(\mathcal{X}, F^*)$ *consisting of weak solutions to* $A^*u = 0$ *in* $\mathcal{X}$.

If $\varepsilon = 0$, then the unique solution to (4.1.1) under the initial condition $u(\cdot, 0) = u_0$ is

$$u(\cdot, t) = e^{-tL} u_0 + \int_0^t e^{-(t-t')L}\, Pf(\cdot, t')\, dt'$$

for $t \in (0, T)$, which we denote by $c_0(\cdot, t)$. On using this formula one reduces (4.1.1) to a nonlinear integral equation of the Fredholm type $u = c_0 + \varepsilon K(u)$ in $(0, T)$, where

$$K(u)(\cdot, t) = -\int_0^t e^{-(t-t')L}\, e^{\lambda t'} PN(u)(\cdot, t')\, dt'.$$

Since

$$K(u)(\cdot, t) - K(v)(\cdot, t) = -\int_0^t e^{-(t-t')L}\, e^{\lambda t'} P(N(u)(\cdot, t') - N(v)(\cdot, t'))\, dt',$$

the small parameter multiplying $K(u)$ may be useful only for Lipschitz nonlinearities $N(u)$, which is not the case for $N(u)$ on the whole space. This gives an evidence to the fact that the contraction mapping principle does not apply on the whole space.

Lemma 4.1.2. *Suppose that* K *is a compact operator in a Hilbert space* H. *If all solutions of the equation* $u = c_0 + \varepsilon' K(u)$ *with* $\varepsilon' \in (0, \varepsilon]$ *lie in a ball* $B(c_0, R)$ *of finite radius, then the equation* $u = c_0 + \varepsilon K(u)$ *has at least one solution in the closure of the ball.*

Proof. The lemma follows immediately from the mapping degree theory of Leray-Schauder. Indeed, on increasing R, if necessary, one can assume that the mapping family $h_\vartheta(u) = u - u_0 - \vartheta \varepsilon K(u)$ for $\vartheta \in [0, 1]$ does not vanish at the boundary of $B(c_0, R)$. Hence, the mapping degree $\deg(h_\vartheta, B(c_0, R))$ is independent of ϑ. For $\vartheta = 0$, the degree just amounts to 1 by the normalisation property. It follows that $\deg(h_1, B(c_0, R)) = 1$, and so $h_1(u) = 0$ has at least one solution in $B(c_0, R)$, as desired. □

In order to prove that all solutions of the equation $u = c_0 + \varepsilon' K(u)$ with $\varepsilon' \in (0, \varepsilon]$ lie in a ball $B(c_0, R)$ with a finite R, one uses the so-called a priori estimates for the solutions.

For a study of the abstract initial value problem $u = c_0 + \varepsilon K(u)$ within the theory of operator semigroups we refer the reader to [FK64, Kat84], etc. It exploits fractional powers of the positive selfadjoint operator L in $L^2(\mathcal{X}, F^i)$ and enables one to prove existence and uniqueness theorems for small intervals $(0, T)$ or for small initial data.

4.2 Potential equations

Assume that the right-hand side $f \in L^2(\mathcal{C}_T, F^i)$ of the generalised impulse equation

$$u'_t + \nu\, \Delta u + Ap + N(u) = f$$

is potential, i.e., $f = A\varpi$ for some section $\varpi \in L^2(\mathcal{C}_T, F^{i-1})$. Then it is to be expected that the equation possesses a potential solution $u = A\wp$ in the cylinder with $\wp \in L^2(\mathcal{C}_T, F^{i-1})$. On substituting both f and u in the impulse equation we get the equation

$$A\wp'_t + A\,\nu \Delta\wp + Ap + N(A\wp) = A\varpi$$

for the unknown potential $\wp$. If this equation possesses a solution then $N(A\wp)$ is a potential again.

Hence it follows that the structure condition $NA = AN$ on the nonlinearity is well motivated by applications in natural sciences. On the other hand, this condition is well understood within the framework of homological algebra, for it specifies the so-called cochain mappings (endomorphisms) of complexes. This condition is fulfilled for the classical Navier-Stokes equations if the nonlinear term at step $i = 0$ is defined by

$$N^0(\wp) := \frac{1}{2} \sum_{k=1}^{n} (\partial_k \wp)^2 .$$

Lemma 4.2.1. *For any vector field u of the form $u = \wp'$ in a domain $\mathcal{X} \subset \mathbb{R}^n$, where $\wp \in C^2(\mathcal{X})$, we have*

$$u'_x u = (N^0(\wp))'_x,$$

i.e., $N^1 d = dN^0$.

Proof. Since

$$u = \begin{pmatrix} \partial_1 \wp \\ \cdots \\ \partial_n \wp \end{pmatrix},$$

it follows that

$$u'_x u = \begin{pmatrix} \displaystyle\sum_{k=1}^{n} (\partial_k \partial_1 \wp)\, \partial_k \wp \\ \cdots \\ \displaystyle\sum_{k=1}^{n} (\partial_k \partial_n \wp)\, \partial_k \wp \end{pmatrix} = \begin{pmatrix} \displaystyle\partial_1 \sum_{k=1}^{n} \frac{(\partial_k \wp)^2}{2} \\ \cdots \\ \displaystyle\partial_n \sum_{k=1}^{n} \frac{(\partial_k \wp)^2}{2} \end{pmatrix} = \big(N^0(\wp)\big)'_x,$$

as desired. □

Our viewpoint sheds some new light on the generalised Navier-Stokes equations (3.1.2). More precisely, the structure of the classical Navier-Stokes equations actually specifies the nonlinear term $N(u)$ at each step i through the commutative relations $N^i d = dN^{i-1}$. Since the Neumann problem after Spencer is solvable for the de Rham complex at each step i, the space $L^2(\mathcal{X}, \Lambda^i T^* \mathcal{X})$ splits into the range of P and the range of $I - P$. On the range of P the nonlinearity structure is specified by (3.2.1). And on the range of $I - P$ which coincides with the range of A the nonlinear term $N(u)$ is uniquely determined by the commutative relations $N^i d = dN^{i-1}$ and by the explicit formula for N^1. For arbitrary elliptic complexes (3.1.1) we may argue in much the same way if the Neumann problem after Spencer is solvable at each step $i > 0$ for (3.1.1).

To wit, by a cochain mapping of the complex $C^\infty(\mathcal{X}, F^\cdot)$ is meant any sequence of (possibly, nonlinear) self-mappings N^i of $C^\infty(\mathcal{X}, F^i)$ with the property that the diagram

$$\begin{array}{ccccccccc} 0 & \to & C^\infty(\mathcal{X}, F^0) & \stackrel{A}{\to} & C^\infty(\mathcal{X}, F^1) & \stackrel{A}{\to} & \ldots \stackrel{A}{\to} & C^\infty(\mathcal{X}, F^N) & \to & 0 \\ & & \downarrow N & & \downarrow N & & & \downarrow N & & \\ 0 & \to & C^\infty(\mathcal{X}, F^0) & \stackrel{A}{\to} & C^\infty(\mathcal{X}, F^1) & \stackrel{A}{\to} & \ldots \stackrel{A}{\to} & C^\infty(\mathcal{X}, F^N) & \to & 0 \end{array}$$

commutes. Our standing assumption on the nonlinear terms N^i of the generalised Navier-Stokes equations will be that they constitute a cochain mapping N of complex (3.1.1), i.e.,

$$N^i A^{i-1} = A^{i-1} N^{i-1} \tag{4.2.1}$$

for all $i = 1, \ldots, N$.

The interest of the class of Navier-Stokes equations is that it is closed under building potential equations. Namely, for each $i = 1, \ldots, N$, the Navier-Stokes equations at step $i-1$ are potential equations for those at step i, as is easy to check.

Example 4.2.2. By Lemma 4.2.1, the Navier-Stokes equations for the de Rham complex at step $i = 0$ read

$$u'_t + \frac{1}{2}\,|u'_x|^2 = \nu \mathit{\Delta} u + f(x,t), \tag{4.2.2}$$

u being an unknown function in the cylinder $\mathcal{C}_T$.

Equation (4.2.2) has been frequently studied as a nonlinear model for the motion of an interface under deposition, when the forcing potential f is random, delta-correlated in both space and time, see [KPZ86].

4.3 The homogeneous Burgers equation

The Cole-Hopf transformation was discovered independently by Hopf [Hop50] and Cole [Col51] around 1950. It changes Burgers' equation $u'_t + uu'_x = u''_{xx}$ into the heat equation $v'_t = v''_{xx}$. To derive the transform, we let $u = \wp'_x$. Then Burgers' equation can be integrated yielding $\wp'_t + (\wp'_x)^2/2 = \wp''_{xx}$ up to a function depending on t only. Let $\wp = -2\log v$. Thus, $u = -2v'_x/v$. Applying some algebra to this we get $v'_t = v''_{xx}$.

More generally, the n-dimensional impulse equation

$$u'_t + u'_x u = \nu \mathit{\Delta} u + f'_x(x,t)$$

for a vector field $u = \wp'_x$, which describes the dynamics of a stirred, pressureless and vorticity-free fluid, has found interesting applications in a wide range of non-equilibrium statistical physics problems, see [BK03]. The associated Hamilton-Jacobi equation, satisfied by the velocity potential $\wp$, just amounts to equation (4.2.2) of Example 4.2.2.

Starting with this example, we now consider a quasilinear partial differential equation

$$\wp'_t = \mathit{\Delta}\wp - a(\wp)\,|\wp'_x|^2 \tag{4.3.1}$$

in $\mathbb{R}^{n+1}$, where a is a continuous real-valued function on the real axis. Choose a strictly monotone decreasing C^2 function $v = \mathcal{H}(\wp)$ on $\mathbb{R}$, such that

$$-a(\wp) = \frac{\mathcal{H}''(\wp)}{\mathcal{H}'(\wp)}$$

for all $\wp \in \mathbb{R}$. The general solution of this ordinary differential equation satisfying the initial condition $\mathcal{H}'(0) = \mathcal{H}_1 < 0$ is

$$\mathcal{H}'(\wp) = \exp\Big(- \int_0^{\wp} a(\vartheta) d\vartheta \Big) \mathcal{H}_1,$$

which is a smooth function on $\mathbb{R}$ with positive values. The function $v = \mathcal{H}(\wp)$ may be found by integration. In this way we recover what is referred to as the Cole-Hopf transformation.

A simple computation shows that the change of variables $v = \mathcal{H}(\wp)$ reduces (4.3.1) to the heat equation

$$v'_t = \Delta v \tag{4.3.2}$$

for the new unknown function v. Hence, the general solution to (4.3.1) is $\wp = \mathcal{H}^{-1}(v)$, with v satisfying (4.3.2).

Example 4.3.1. Let a be constant. Then

$$\begin{aligned} \mathcal{H}(\wp) &= \mathcal{H}_0 + \mathcal{H}_1 \frac{1 - \exp(-a\wp)}{a}, \\ \mathcal{H}^{-1}(v) &= \frac{-1}{a} \log \Big(1 - a \frac{v - \mathcal{H}_0}{\mathcal{H}_1} \Big). \end{aligned}$$

Using the function $\mathcal{H}$ allows one to endow the set of solutions to equation (4.3.1) with the symmetry $\wp_1 \circ \wp_2 := \mathcal{H}^{-1}(\mathcal{H}(\wp_1) + \mathcal{H}(\wp_2))$.

In [Hop50], the transformation $\mathcal{H}$ is applied to study the Cauchy problem for the homogeneous Burgers equation $u'_t + \nu \Delta u + u'_x u = 0$, cf. Example 3.1.1. In the last decades, mathematicians become increasingly interested in problems related to the behaviour of solutions to a partial differential equation in which the highest order terms occur linearly with small coefficients. These problems originate from physical applications, mainly from modern fluid dynamics (compressible fluids of small kinematic viscosity $\nu > 0$ and of small heat conductivity λ). Research in these fields has led to some general mathematical observations, such as the following two. The solution of the initial value problem for equations of fluid flow tends for "most" values to a limit function as both ν and λ tend to zero. The limit function is, in general, discontinuous and is pieced together by solutions of the equations in which those highest order coefficients vanish (ideal fluid with contact and shock discontinuities). These observations are perhaps valid in a much wider range of partial differential equations. The second observation is restricted to nonlinear equations, but it seems to point out a typical occurrence in the general case. Exact formulation and rigorous proof of these observations are still tasks for the future. As is noted in [Hop50], a careful study of special problems is still a commendable way towards greater insight into the matter. Among the partial differential equations studied in this direction one meets rarely those in which the totality of solutions is rigorously determined and in which the passage to the limit can thus be studied in detail. On using the Cole-Hopf transformation one obtains a complete solution for the Burgers equation. It was first introduced in [Bur40] as a simple model for the differential equations of fluid flow. Although the Burgers equation is a too simple model to fully illustrate the statistics of free turbulence, a theory of this equation serves as an instructive introduction into some mathematical problems involved. There is a close analogy between the Burgers equation and the Navier-Stokes equations. However, no additional dependent variables such as pressure, density or temperature appear in the Burgers equation. Nevertheless [Bur40] observed certain analogy between some solutions of the Burgers equation and one-dimensional flows of a compressible fluid. According to [Hop50], Burgers had an intuitive picture of the limit case $\nu \to 0$ in the solutions and determined the origin and the law of propagation of discontinuity. Like [Bur40] the paper [Hop50] studies the boundary-free initial value problem, to wit, given u for all x and $t = 0$, one wants u for all x and $t > 0$. The solution is achieved by an exact integration of the Burgers equation. Both problems, the behaviour of the solution as $t \to \infty$ while ν is constant, and its behaviour as $\nu \to 0$ while the initial data are kept fixed, are treated.

4.4 Linearised Navier-Stokes equations

The study of a nonlinear equation begins with the study of its linearisation. We need only to consider the linearisation of the impulse equation. The nonlinear term is $N(u) := N(u, u)$, where N is a first order bidifferential operator of the type $F^i \times F^i \to F^i$ on $\mathcal{X}$. Hence it follows that the linearisation at a fixed section u_0 of F^i is

$$\begin{aligned} N(u) &= N(u_0, u_0) + N(u_0, u - u_0) + N(u - u_0, u) \\ &= -N(u_0) + N(u_0, u) + N(u, u_0) + o(\|u - u_0\|) \end{aligned}$$

for u close to u_0. The question still open is of how to choose a domain for the solution.

Note that both partial differential operators $N(u_0, u)$ and $N(u, u_0)$ have discontinuous coefficients unless u_0 bear excess smoothness. Therefore, the study of relevant linearisations of the Navier-Stokes equations requires a fairly delicate analysis. The best general reference here is [Lad70]. Instead of this we consider the linear mixed problem which underlies the construction of a WKB solution to the Navier-Stokes equations, see Section 3.4. The mixed problem (3.4.2) is an evolution equation for the Toeplitz operator $P\Delta$, where P is the Helmholtz projector. To provide an insight into the problem we neglect P and look for a solution of the mixed problem

$$\begin{array}{rcll} u'_t + \nu\,\Delta u & = & f & \text{in } \ \mathcal{C}_T, \\ u & = & u_0 & \text{at } \ \overset{\circ}{\mathcal{X}} \times \{0\}, \\ u & = & u_l & \text{at } \ \partial\mathcal{X} \times (0,T), \end{array} \qquad (4.4.1)$$

the right-hand side f being a given section of the bundle F^i over the cylinder $\mathcal{C}_T$ and the initial data u_0 and boundary data u_l being prescribed sections of F^i over the lower basis $\mathcal{X} \times \{0\}$ and the lateral boundary $\partial\mathcal{X} \times (0,T)$ of $\mathcal{C}_T$, respectively. As is usual for evolution equations, no condition is posed on the upper basis of the cylinder.

For a recent account of the theory of mixed problems we refer the reader to Chapter 3 of [GV96]. The energy method for hyperbolic equations takes a considerable part in [GV96]. This method automatically extends to $2b$-parabolic differential equations with variable coefficients. In this section we specify the general theory for the parabolic equation $u'_t - \nu\,\Delta u = f$ including the Laplacian Δ of elliptic complex (3.1.1).

By a classical solution of problem (4.4.1) is meant any section $u \in C^{2,1}_{\mathrm{loc}}(\mathcal{C}_T, F^i)$ which is continuous up to $\mathcal{X} \times \{0\}$ and $\partial\mathcal{X} \times (0,T)$ and satisfies pointwise the equations of (4.4.1).

Since the case of inhomogeneous boundary conditions reduces in a familiar manner to the case of homogeneous boundary conditions, we will assume in the sequel that $u_l = 0$.

Lemma 4.4.1. *Suppose $f \in L^2(\mathcal{C}_T, F^i)$. If $u \in C^{1,0}(\mathcal{X} \times (0,T), F^i)$ is a classical solution of problem (4.4.1), then $u \in H^{1,0}(\mathcal{C}_T, F^i)$.*

Proof. We pick arbitrary $\varepsilon, t' \in (0,T)$ satisfying $\varepsilon < t'$, multiply the differential equation in (4.4.1) by u^* and integrate the equality over the cylinder $\mathcal{X} \times (\varepsilon, t')$. Since

$$\Re\,(u'_t, u)_x = \frac{1}{2}\,\frac{\partial}{\partial t}(u,u)_x$$

for all $t \in (0,T)$, the Stokes formula implies

$$\frac{1}{2}\int\limits_{\mathcal{X}} \left(|u(\cdot,t')|^2 - |u(\cdot,\varepsilon)|^2\right) dx + \nu \iint\limits_{\mathcal{X}\ \varepsilon}^{\quad t'} \left(|Au|^2 + |A^*u|^2\right) dx dt = \Re \iint\limits_{\mathcal{X}\ \varepsilon}^{\quad t'} (f,u)_x \, dx dt.$$

Hence it follows that

$$\begin{aligned}
& \frac{1}{2}\int_{\mathcal{X}} |u(\cdot,t')|^2\,dx + \nu \int_{\mathcal{X}}\int_{\varepsilon}^{t'} \left(|Au|^2 + |A^*u|^2\right) dxdt \\
& \quad \leq \frac{1}{2}\int_{\mathcal{X}} |u(\cdot,\varepsilon)|^2\,dx + \int_{\mathcal{X}}\int_{\varepsilon}^{t'} |f|\,|u|\,dxdt \\
& \quad \leq \frac{1}{2}\int_{\mathcal{X}} |u(\cdot,\varepsilon)|^2\,dx + \|f\|_{L^2(\mathcal{C}_T,F^i)}\,\|u\|_{L^2(\mathcal{X}\times(\varepsilon,t'),F^i)},
\end{aligned}$$

and so on passing in this inequality to the limit as $\varepsilon \to 0$ we get

$$\begin{aligned}
\frac{1}{2}\int_{\mathcal{X}} |u(\cdot,t')|^2\,dx & \leq \frac{1}{2}\,\|u_0\|^2_{L^2(\mathcal{X},F^i)} + \|f\|_{L^2(\mathcal{C}_T,F^i)}\,\|u\|_{L^2(\mathcal{C}_{t'},F^i)}, \\
\nu\int_{\mathcal{C}_{t'}} \left(|Au|^2 + |A^*u|^2\right) dxdt & \leq \frac{1}{2}\,\|u_0\|^2_{L^2(\mathcal{X},F^i)} + \|f\|_{L^2(\mathcal{C}_T,F^i)}\,\|u\|_{L^2(\mathcal{C}_{t'},F^i)}.
\end{aligned} \tag{4.4.2}$$

Choose an arbitrary $t \in (0,T)$ and integrate the first inequality of (4.4.2) in $t' \in (0,t)$. This yields

$$\begin{aligned}
\int_0^t \Big(\int_{\mathcal{X}} |u(\cdot,t')|^2\,dx\Big)dt' & \leq T\,\|u_0\|^2_{L^2(\mathcal{X},F^i)} + 2T\,\|f\|_{L^2(\mathcal{C}_T,F^i)}\,\|u\|_{L^2(\mathcal{C}_t,F^i)} \\
& \leq T\,\|u_0\|^2_{L^2(\mathcal{X},F^i)} + 2T^2\,\|f\|^2_{L^2(\mathcal{C}_T,F^i)} + \frac{1}{2}\,\|u\|^2_{L^2(\mathcal{C}_t,F^i)}
\end{aligned}$$

whence

$$\|u\|^2_{L^2(\mathcal{C}_t,F^i)} \leq 2T\,\|u_0\|^2_{L^2(\mathcal{X},F^i)} + 4T^2\,\|f\|^2_{L^2(\mathcal{C}_T,F^i)} =: C$$

for all $t \in (0,T)$, with C a nonnegative constant independent of t. We have thus proved that $\|u\|_{L^2(\mathcal{C}_T,F^i)} \leq \sqrt{C}$, and so the second inequality of (4.4.2) shows readily that

$$\|Au\|^2_{L^2(\mathcal{C}_{t'},F^{i+1})} + \|A^*u\|^2_{L^2(\mathcal{C}_{t'},F^i)} \leq \frac{1}{2\nu}\,\|u_0\|^2_{L^2(\mathcal{X},F^i)} + \frac{\sqrt{C}}{\nu}\,\|f\|_{L^2(\mathcal{C}_T,F^i)}$$

for all $t' \in (0,T)$. On using a familiar argument with the Dirichlet scalar product on $\overset{\circ}{H}{}^1(\mathcal{X},F^i)$ we now conclude that $|u'_x|$ is square integrable on $\mathcal{C}_T$, which establishes the lemma. □

Remark 4.4.2. *On combining the first inequality of (4.4.2) and the estimate* $\|u\|_{L^2(\mathcal{C}_T,F^i)} \leq \sqrt{C}$ *one sees that the classical solution* $u \in C^{1,0}(\mathcal{X}\times(0,T),F^i)$ *of problem (4.4.1) satisfies* $\|u(\cdot,t')\|_{L^2(\mathcal{X},F^i)} \leq C'$ *for all* $t' \in (0,T)$*, where the constant* C' *depends only on* T *and* $\|u_0\|_{L^2(\mathcal{X},F^i)}$*,* $\|f\|_{L^2(\mathcal{C}_T,F^i)}$*.*

Let $f \in L^2(\mathcal{C}_T, F^i)$ and let $u \in C^{1,0}(\mathcal{X} \times (0,T), F^i)$ be a classical solution of mixed problem (4.4.1). We multiply the differential equation of (4.4.1) by v^*, where v is a C^1 section of F^i over the closure of $\mathcal{C}_T$ satisfying $v(x,T) = 0$ for all $x \in \mathcal{X}$ and $v = 0$ at $\partial\mathcal{X} \times (0,T)$, and integrate the resulting equality over the cylinder $\mathcal{X} \times (\varepsilon, t')$, where $0 < \varepsilon < t' < T$. On applying the Stokes formula we arrive at the equality

$$\int_{\mathcal{X}} (u(\cdot,t'), v(\cdot,t'))_x dx + \int_{\mathcal{X}} \int_{\varepsilon}^{t'} (-(u, v'_t)_x + \nu((Au, Av)_x + (A^*u, A^*v)_x))\, dxdt$$
$$= \int_{\mathcal{X}} (u(\cdot,\varepsilon), v(\cdot,\varepsilon))_x dx + \int_{\mathcal{X}} \int_{\varepsilon}^{t'} (f, v)_x \, dxdt.$$

By Lemma 4.4.1, $u \in H^{1,0}(\mathcal{C}_T, F^i)$, and so the restriction of u to the lateral surface belongs to $L^2(\partial\mathcal{X} \times (0,T), F^i)$. Using Remark 4.4.2, we pass in the last equality to the limit as $\varepsilon \to 0$ and $t' \to T$, thus obtaining

$$\int\limits_{\mathcal{C}_T} (-(u, v'_t)_x + \nu((Au, Av)_x + (A^*u, A^*v)_x))\, dxdt$$
$$= \int\limits_{\mathcal{X}} (u_0, v(\cdot,0))_x dx + \int\limits_{\mathcal{C}_T} (f, v)_x dxdt \tag{4.4.3}$$

for all sections $v \in C^1(\overline{\mathcal{C}}_T, F^i)$ vanishing both on $\mathcal{X} \times \{T\}$ and $\partial\mathcal{X} \times (0,T)$. By continuity, (4.4.3) still holds for all $v \in H^1(\mathcal{C}_T, F^i)$ satisfying $v = 0$ at $\mathcal{X} \times \{T\}$ and $\partial\mathcal{X} \times (0,T)$.

We use the identity (4.4.3) to introduce weak solutions to the mixed problem (4.4.1). In the sequel we assume that $f \in L^2(\mathcal{C}_T, F^i)$ and $u_0 \in L^2(\mathcal{X}, F^i)$. A section $u \in H^{1,0}(\mathcal{C}_T, F^i)$ is said to be a weak solution of problem (4.4.1), if $u = 0$ on $\partial\mathcal{X} \times (0,T)$ and the identity (4.4.3) is fulfilled for all $v \in H^1(\mathcal{C}_T, F^i)$ vanishing on the cylinder top $\mathcal{X} \times \{T\}$ and on the lateral surface $\partial\mathcal{X} \times (0,T)$. Along with classical and weak solutions of the first mixed problem one can introduce the concept of 'almost everywhere' solution. A section u is said to be an 'almost everywhere' solution of the mixed problem if it belongs to the space $H^{2,1}(\mathcal{C}_T, F^i)$ and satisfies the differential equation of (4.4.1) for almost all $(x,t) \in \mathcal{C}_T$, the initial condition for almost all $x \in \mathcal{X}$ and the trace of u on the lateral surface vanishes almost everywhere.

Lemma 4.4.1 shows that any classical solution of problem (4.4.1) which belongs to $C^{1,0}(\partial X \times (0,T), F^i)$ is also a weak solution of the first mixed problem. Similarly one proves that any 'almost everywhere' solution of the first mixed problem is a weak solution. It is easily seen that if a weak solution of problem (4.4.1) belongs to $H^{2,1}(\mathcal{C}_T, F^i)$ then it is an 'almost everywhere' solution. And if a weak solution of the first mixed problem belongs to $C^{2,1}(\mathcal{C}_T, F^i)$ and is continuous up to the lower basis and the lateral surface of the cylinder $\mathcal{C}_T$, then it is a classical solution. For proofs of the corresponding assertions for solutions of the first mixed problem for the Lamé system we refer the reader to [MT14]. It is worth pointing out that any of the classical, weak or 'almost everywhere' solutions bears the following property: If $u(x,t)$ is a solution of (4.4.1) in the cylinder $\mathcal{C}_T$, then it is a solution in any cylinder $\mathcal{C}_{t'}$ with $0 < t' < T$.

Chapter 5

Linear Navier-Stokes equations

The assumption that (3.1.1) is a complex is unnecessarily restrictive for what we discuss in this chapter. Similarly to Chapter 2 we relax this condition and consider arbitrary elliptic quasicomplexes (3.1.1) of first order differential operators on $\mathcal{X}$.

5.1 Uniqueness of a weak solution

Our next objective is to establish a uniqueness theorem for solutions of the first mixed problem.

Theorem 5.1.1. *As defined above, the first mixed problem (4.4.1) has at most one weak solution.*

Proof. The proof is analogous to that of Theorem 3.2 in [MT14]. Let $u_1(x,t)$ and $u_2(x,t)$ be two weak solutions of (4.4.1). Then the difference $u = u_1 - u_2$ is a weak solution of the corresponding homogeneous problem with $f = 0$ and $u_0 = 0$. We have to show that $u = 0$ in $\mathcal{C}_T$.

Let $u \in H^{1,0}(\mathcal{C}_T, F^i)$ be a weak solution of the first mixed problem with $f = 0$ in $\mathcal{C}_T$ and $u_0 = 0$ in $\mathcal{X}$. Consider the function

$$v(x,t) = \int_t^T u(x,\theta)d\theta$$

defined in $\mathcal{C}_T$. It is immediately verified that the function v has generalised derivatives

$$v'_{x^j}(x,t) = \int_t^T u'_{x^j}(x,\theta)d\theta,$$

for $j = 1, \ldots, n$, and

$$v'_t(x,t) = -u(x,t)$$

in $\mathcal{C}_T$. Since v and v'_{x^j}, v'_t belong to $L^2(\mathcal{C}_T, F^i)$, we deduce that $v \in H^1(\mathcal{C}_T, F^i)$. Moreover, this section vanishes at the lateral boundary and on the top of the cylinder $\mathcal{C}_T$.

Substituting the function v into identity (4.4.3) yields

$$\int_{\mathcal{C}_T} \Big(|u|^2 + \nu (Au, \int_t^T Au(\cdot,\theta) d\theta)_x + \nu (A^* u, \int_t^T A^* u(\cdot,\theta) d\theta)_x \Big) dx dt = 0.$$

Since

$$\begin{aligned} \int\limits_{\mathcal{C}_T} (Au(x,t), \int\limits_t^T Au(x,\theta) d\theta)_x dx dt &= \int\limits_{\mathcal{X}} \int\limits_0^T (Au(x,t), \int\limits_t^T Au(x,\theta) d\theta)_x dx dt \\ &= \int\limits_{\mathcal{X}} \int\limits_0^T (\int\limits_0^\theta Au(x,t) dt, Au(x,\theta))_x dx d\theta \end{aligned}$$

which transforms to

$$\begin{aligned} &\int\limits_{\mathcal{X}} (\int\limits_0^T Au(x,t) dt, \int\limits_0^T Au(x,\theta) d\theta)_x dx - \int\limits_{\mathcal{X}} \int\limits_0^T (\int\limits_\theta^T Au(x,t) dt, Au(x,\theta))_x dx d\theta \\ &\qquad = \int\limits_{\mathcal{X}} |\int\limits_0^T Au(x,t) dt|^2 \, dx - \int\limits_{\mathcal{C}_T} (\int\limits_\theta^T Au(x,t) dt, Au(x,\theta))_x dx d\theta, \end{aligned}$$

we get

$$\Re \int_{\mathcal{C}_T} (Au(x,t), \int_t^T Au(x,\theta) d\theta)_x dx dt = \frac{1}{2} \int_{\mathcal{X}} |\int_0^T Au(x,t) dt|^2 \, dx.$$

Similarly we obtain

$$\Re \int_{\mathcal{C}_T} (A^* u(x,t), \int_t^T A^* u(x,\theta) d\theta)_x dx dt = \frac{1}{2} \int_{\mathcal{X}} |\int_0^T A^* u(x,t) dt|^2 \, dx$$

whence

$$\int_{\mathcal{C}_T} |u(x,t)|^2 dx dt + \frac{\nu}{2} \int_{\mathcal{X}} |\int_0^T Au(x,t) dt|^2 \, dx + \frac{\nu}{2} \int_{\mathcal{X}} |\int_0^T A^* u(x,t) dt|^2 \, dx = 0. \tag{5.1.1}$$

Since $\nu \geq 0$, we conclude from (5.1.1) that

$$\int_{\mathcal{X}} |u(x,t)|^2 dx = 0$$

for all $t \in (0,T)$, and so $u = 0$ in $\mathcal{C}_T$, as desired. □

Since an 'almost everywhere' solution of problem (4.4.1) is actually a weak solution to this problem, Theorem 5.1.1 implies

Corollary 5.1.2. *As defined above, problem (4.4.1) has at most one 'almost everywhere' solution.*

On combining Theorem 5.1.1 and Lemma 4.4.1 we also deduce that the first mixed problem has at most one classical solution belonging to the space $C^{1,0}(\mathcal{X} \times (0,T), F^i)$.

5.2 Existence of a weak solution

We now turn to the proof of the existence of solutions to problem (4.4.1). To this end we use the Fourier method in the same way as for hyperbolic equations, see [MT14].

Let v be a weak eigenfunction of the first boundary value problem for the $-\nu$ multiple of the Laplace operator

$$\begin{array}{rcll} -\nu\, \Delta v & = & \varkappa v & \text{in} \quad \overset{\circ}{\mathcal{X}}, \\ v & = & 0 & \text{at} \quad \partial \mathcal{X}, \end{array} \tag{5.2.1}$$

where $\varkappa$ is the corresponding eigenvalue. Thus, v belongs to $\overset{\circ}{H}{}^1(\mathcal{X}, F^i)$ and satisfies the integral identity

$$\nu \int_{\mathcal{X}} ((Av, Ag)_x + (A^* v, A^* g)_x)\, dx + \varkappa \int_{\mathcal{X}} (v, g)_x dx = 0$$

for all $g \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$.

Consider the orthonormal system $(v_k)_{k=1,2,\ldots}$ in $L^2(\mathcal{X}, F^i)$ consisting of all weak eigenfunction of problem (5.2.1). Let

$$(\varkappa_k)_{k=1,2,\ldots}$$

be the sequence of corresponding eigenvalues. As usual we think of this sequence as nonincreasing sequence with $\varkappa_1 \leq 0$ and each eigenvalue repeats itself in accord with its multiplicity. The system $(v_k)_{k=1,2,\ldots}$ is known to be an orthonormal basis in $L^2(\mathcal{X}, F^i)$ and $\varkappa_k \to -\infty$ when $k \to \infty$. Moreover, the first eigenvalue $\varkappa_1$ is obviously strongly negative, if $\nu > 0$ holds.

Assume that $f \in L^2(\mathcal{C}_T, F^i)$ and $u_0 \in L^2(\mathcal{X}, F^i)$. By the Fubini theorem we deduce that $f(\cdot, t) \in L^2(\mathcal{X}, F^i)$ for almost all $t \in (0,T)$. We expand the sections $f(\cdot,t)$ and u_0 as Fourier series over the system of eigenfunctions $(v_k)_{k=1,2,\ldots}$ in $\mathcal{X}$, namely

$$\begin{array}{rcl} f(x,t) & = & \displaystyle\sum_{k=1}^{\infty} f_k(t) v_k(x), \\ u_0(x) & = & \displaystyle\sum_{k=1}^{\infty} u_{0,k} v_k(x), \end{array}$$

where $f_k(t) = (f(\cdot,t), v_k)_{L^2(\mathcal{X},F^i)}$ and $u_{0,k} = (u_0, v_k)_{L^2(\mathcal{X},F^i)}$ for $k = 1,2,\ldots$, the functions f_k belonging to $L^2(0,T)$. By the Parseval equality, we get

$$\begin{array}{rcl} \displaystyle\sum_{k=1}^{\infty} |f_k(t)|^2 & = & \|f(\cdot,t)\|^2_{L^2(\mathcal{X},F^i)}, \\ \displaystyle\sum_{k=1}^{\infty} |u_{0,k}|^2 & = & \|u_0\|^2_{L^2(\mathcal{X},F^i)}, \end{array} \tag{5.2.2}$$

where the first equality is valid for almost all $t \in (0,T)$. On integrating both sides of the first equality of (5.2.2) in $t \in (0,T)$ and using the theorem of Beppo Levi we see that

$$\sum_{k=1}^{\infty} \int_0^T |f_k(t)|^2 dt = \int_{\mathcal{C}_T} |f(x,t)|^2 dx dt. \tag{5.2.3}$$

For any $k = 1,2,\ldots$, we introduce the function

$$w_k(t) = u_{0,k} \exp(\varkappa_k t) + \int_0^t f_k(t') \exp\left(\varkappa_k (t-t')\right) dt',$$

which obviously belongs to $H^1(0,T)$ and satisfies the initial value problem

$$\begin{array}{rcl} w_k' - \varkappa_k w_k & = & f_k \quad \text{a.e. on} \quad (0,T), \\ w_k(0) & = & u_{0,k}, \end{array} \tag{5.2.4}$$

the initial condition is well defined, for $H^1(0,T) \hookrightarrow C[0,T]$ by the Sobolev embedding theorem. In much the same way as in [MT14] one verifies that the section $u_k(x,t) = w_k(t)v_k(x)$ is a weak solution of problem (4.4.1) with the right-hand side $f(x,t) = f_k(t)v_k(x)$ and initial data $u_0(x) = u_{0,k}v_k(x)$. Hence it follows by linearity that the partial sums

$$s_N(x,t) = \sum_{k=1}^{N} w_k(t) v_k(x)$$

are weak solutions of problem (4.4.1) whose right-hand side and initial data are given by the corresponding partial sums of the Fourier series for f and u_0, respectively. To wit,

$$\begin{aligned}\int_{\mathcal{C}_T} & (-(s_N, v'_t)_x + \nu((As_N, Av)_x + (A^* s_N, A^* v)_x))\, dx dt \\ = & \int_{\mathcal{X}} \sum_{k=1}^{N} u_{0,k}\, (v_k, v(\cdot, 0))_x dx + \int_{\mathcal{C}_T} \sum_{k=1}^{N} f_k(t)\, (v_k, v(\cdot, t))_x dx dt\end{aligned} \tag{5.2.5}$$

for all sections $v \in H^1(\mathcal{C}_T, F^i)$ vanishing both on $\mathcal{X} \times \{T\}$ and $\partial\mathcal{X} \times (0, T)$, cf. (4.4.3).

Our next objective is to show that the series

$$u(x, t) = \sum_{k=1}^{\infty} w_k(t) v_k(x) \tag{5.2.6}$$

converges in $H^{1,0}(\mathcal{C}_T, F^i)$ and its sum gives a weak solution to the first mixed problem (4.4.1).

Theorem 5.2.1. *If $f \in L^2(\mathcal{C}_T, F^i)$ and $u_0 \in L^2(\mathcal{X}, F^i)$, then problem (4.4.1) has a weak solution u. The solution is represented by series (5.2.6) which converges in $H^{1,0}(\mathcal{C}_T, F^i)$. Moreover, there is a constant $C > 0$ independent of f and u_0, such that*

$$\|u\|_{H^{1,0}(\mathcal{C}_T, F^i)} \leq C \left(\|f\|_{L^2(\mathcal{C}_T, F^i)} + \|u_0\|_{L^2(\mathcal{X}, F^i)} \right). \tag{5.2.7}$$

Proof. From the formula for $w_k(t)$ it follows readily by the Cauchy-Schwarz inequality that

$$\begin{aligned} |w_k(t)| & \leq |u_{0,k}|\, \exp(\varkappa_k t) + \int_0^t |f_k(t')|\, \exp\left(\varkappa_k (t - t')\right) dt' \\ & \leq |u_{0,k}|\, \exp(\varkappa_k t) + \|f_k\|_{L^2(0,T)}\, \frac{1}{\sqrt{2|\varkappa_k|}} \end{aligned}$$

whenever $t \in [0, T]$. Therefore,

$$|w_k(t)|^2 \leq 2 \exp(2\varkappa_k t)\, |u_{0,k}|^2 + \frac{1}{|\varkappa_k|}\, \|f_k\|^2_{L^2(0,T)} \tag{5.2.8}$$

for all $t \in [0, T]$.

Consider a partial sum $s_N(x,t)$ of series (5.2.6). For any fixed $t \in [0,T]$ it belongs to the space

$$\overset{\circ}{H}{}^1(\mathcal{X}, F^i).$$

It is convenient to endow this space with the so-called Dirichlet scalar product

$$D(v,g) = \int_{\mathcal{X}} \nu \Big((Av, Ag)_x + (A^* v, A^* g)_x \Big) dx$$

and the Dirichlet norm $D(v) := \sqrt{D(v,v)}$. Since the system

$$\Big\{ \frac{v_k}{\sqrt{-\varkappa_k}} \Big\}_{k=1,2,\ldots}$$

is obviously orthonormal with respect to the scalar product $D(v,g)$, we obtain by (5.2.8)

$$\begin{aligned}
\|s_N(\cdot,t) - s_M(\cdot,t)\|^2_{H^1(\mathcal{X},F^i)} &= \| \sum_{k=M+1}^{N} w_k(t) v_k \|^2_{H^1(\mathcal{X},F^i)} \\
&\le C \sum_{k=M+1}^{N} |w_k(t)|^2 \, |\varkappa_k| \\
&\le C \sum_{k=M+1}^{N} \Big(2|\varkappa_k| \exp(2\varkappa_k t) |u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)} \Big)
\end{aligned}$$

for all M and N satisfying $1 \le M < N$, and all $t \in [0,T]$, with C a constant independent of M, N and t. Along with this inequality we obtain in the same manner

$$\|s_N(\cdot,t)\|^2_{H^1(\mathcal{X},F^i)} \le C \sum_{k=1}^{N} \Big(2|\varkappa_k| \exp(2\varkappa_k t) \, |u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)} \Big)$$

for all $N = 1, 2, \ldots$ and $t \in [0,T]$. On integrating the last two inequalities in $t \in [0,T]$ we obtain

$$\begin{aligned}
\|s_N - s_M\|^2_{H^{1,0}(\mathcal{C}_T,F^i)} &\le C' \sum_{k=M+1}^{N} \Big(|u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)} \Big), \\
\|s_N\|^2_{H^{1,0}(\mathcal{C}_T,F^i)} &\le C' \sum_{k=1}^{N} \Big(|u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)} \Big),
\end{aligned} \qquad (5.2.9)$$

where the constant C' is independent of N and M.

By (5.2.2) the series with general term $|u_{0,k}|^2 + \|f_k\|^2_{L^2(0,T)}$ converges. Hence, from the first estimate of (5.2.9) we deduce that series (5.2.6) converges in $H^{1,0}(\mathcal{C}_T, F^i)$, and so its sum $u(x,t)$ belongs to $H^{1,0}(\mathcal{C}_T, F^i)$ and satisfies $u = 0$ on the lateral boundary $\partial\mathcal{X} \times (0,T)$ of the cylinder. Letting $N \to \infty$ in identity (5.2.5) we see that the section u is a weak solution of problem (4.4.1). Estimate (5.2.7) follows immediately from the second inequality of (5.2.9), if we let $N \to \infty$ in (5.2.9) and use equalities (5.2.2). □

Note that similarly to the hyperbolic case [MT14] one can prove the existence of a weak solutions to problem (4.4.1) by means of the Galerkin method.

5.3 Regularity of weak solutions

We now discuss briefly the regularity of weak solutions. Assume that the boundary $\partial\mathcal{X}$ of $\mathcal{X}$ is of class C^{2s} for some integer $s \geq 1$. Then the eigenfunctions $(v_k)_{k=1,2,\ldots}$ of problem (5.2.1) belong to $H^{2s}(\mathcal{X}, F^i)$ and satisfy the boundary conditions

$$(-\nu\varDelta)^i v_k \;\; = \;\; 0 \quad \text{on} \quad \partial\mathcal{X} \tag{5.3.1}$$

for $i = 0, 1, \ldots, s-1$.

Let $H^{2s}_{\mathcal{D}}(\mathcal{X}, F^i)$ stand for the subspace of $H^{2s}(\mathcal{X}, F^i)$ consisting of all functions v satisfying (5.3.1). We put additional restrictions on the data of the problem to attain to a classical solution. More precisely, we require that $u_0 \in H^{2s-1}_{\mathcal{D}}(\mathcal{X}, F^i)$ and f belongs to the subspace of $H^{2(s-1),s-1}(\mathcal{C}_T, F^i)$ that consists of all functions satisfying

$$(-\nu\varDelta)^i f \;\; = \;\; 0 \quad \text{at} \quad \partial\mathcal{X} \times (0,T) \tag{5.3.2}$$

for $i = 0, 1, \ldots, s-2$.

For $s = 1$, the latter equations are empty and we arrive at $f \in L^2(\mathcal{X}, F^i)$, as above.

Theorem 5.3.1. *Under the above hypotheses, series (5.2.6) converges to the weak solution u of problem (4.4.1) in $H^{2s,s}(\mathcal{C}_T, F^i)$. Moreover, there is a constant $C > 0$ independent of f and u_0, such that*

$$\|u\|_{H^{2s,s}(\mathcal{C}_T,F^i)} \leq C \left(\|f\|_{H^{2(s-1),s-1}(\mathcal{C}_T,F^i)} + \|u_0\|_{H^{2s-1}(\mathcal{X},F^i)} \right). \tag{5.3.3}$$

Proof. The proof of this theorem runs similarly to the proof of Theorem 4 of [Mik76, p. 372], if one exploits the techniques developed above. □

Since a weak solution of problem (4.4.1) which belongs to $H^{2,1}(\mathcal{C}_T, F^i)$ is an 'almost everywhere' solution, Theorem 5.3.1 for $s = 1$ implies

Corollary 5.3.2. *Suppose $\partial\mathcal{X} \in C^2$ and $f \in L^2(\mathcal{C}_T, F^i)$, $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$. Then series (5.2.6) converges in $H^{2,1}(\mathcal{C}_T, F^i)$ and its sum is an 'almost everywhere' solution of problem (4.4.1). Moreover, there is a constant C independent of f and u_0, such that*

$$\|u\|_{H^{2,1}(\mathcal{C}_T, F^i)} \le C \left(\|f\|_{L^2(\mathcal{C}_T, F^i)} + \|u_0\|_{H^1(\mathcal{X}, F^i)} \right).$$

If the boundary of $\mathcal{X}$ is of class $C^{[n/2]+3}$, then the eigenfunctions $v_k(x)$ of problem (5.2.1) belong to the space $H^{[n/2]+3}(\mathcal{X}, F^i)$, and so to the space $C^2(\mathcal{X}, F^i)$, which is due to the Sobolev embedding theorem. Therefore, the partial sums s_N of series (5.2.6 are in $C^{2,1}(\overline{\mathcal{C}}_T, F^i)$

Corollary 5.3.3. *Assume that $\partial\mathcal{X} \in C^{2s+1}$, where $2s + 1 \ge [n/2] + 3$, and $f \in H^{2s,s}_{\mathcal{D}}(\mathcal{C}_T, F^i)$, $u_0 \in H^{2s+1}_{\mathcal{D}}(\mathcal{X}, F^i)$. Then series (5.2.6) converges in $C^{2,1}(\overline{\mathcal{C}}_T, F^i)$ and its sum u is a classical solution of problem (4.4.1). Moreover, there is a constant $C > 0$ independent of f and u_0, with the property that*

$$\|u\|_{C(\overline{\mathcal{C}}_T, F^i)} \le C \left(\|f\|_{H^{2(s-1),s-1}(\mathcal{C}_T, F^i)} + \|u_0\|_{H^{2s-1}(\mathcal{X}, F^i)} \right).$$

Proof. The proof of this corollary runs in much the same way as the proof of Theorem 5 of [Mik76, p. 381]. □

5.4 Generalised Navier-Stokes equations revisited

The arguments of this chapter still apply if we replace the Laplacian Δ by the composition $P\Delta$, where P is the Helmholtz projector introduced in Lemma 3.1.2. The only difference is in substituting Pf for f, i.e., in the choice of data $f(t, \cdot)$ and u_0 in the subspace of $L^2(\mathcal{X}, F^i)$ consisting of those sections which belong to the kernel of the adjoint operator for A^i in the sense of Hilbert spaces. More precisely, assume that the boundary of $\mathcal{X}$ is of class C^2 and $f \in L^2(\mathcal{C}_T, F^i)$, $u_0 \in H^1(\mathcal{X}, F^i)$ satisfies $A^* u_0 = 0$ in $\mathcal{X}$ and $u_0 = 0$ at $\partial\mathcal{X}$. Then there is a unique section $u \in H^{2,1}(\mathcal{C}_T, F^i)$ such that

$$\begin{array}{rcll} u'_t + \nu P\Delta u & = & Pf & \text{in } \ \mathcal{C}_T, \\ u & = & u_0 & \text{at } \ \mathcal{X} \times \{0\}, \\ u & = & 0 & \text{at } \ \partial\mathcal{X} \times (0, T) \end{array} \qquad (5.4.1)$$

in a weak sense or, what is equivalent, almost everywhere on the corresponding strata. Moreover,

$$\|u\|_{H^{2,1}(\mathcal{C}_T,F^i)} \leq C \left(\|Pf\|_{L^2(\mathcal{C}_T,F^i)} + \|u_0\|_{H^1(\mathcal{X},F^i)}\right)$$

with $C > 0$ a constant independent of f and u_0.

We complete the work by describing those nonlinear perturbations $N(u)$ of the equation $u'_t + \nu\, \Delta u = f$ which can be handled within the Leray-Schauder continuation method. Without restriction of generality we can assume that the initial data u_0 is zero.

Denote by U the subspace of $H^{2,1}(\mathcal{C}_T, F^i)$ consisting of those sections u which satisfy $Pu = u$ and vanish on the basis $\mathcal{X} \times \{0\}$ and on the lateral surface $\partial\mathcal{X} \times (0,T)$ of $\mathcal{C}_T$. When endowed with the scalar product induced from $H^{2,1}(\mathcal{C}_T, F^i)$, the space U is Hilbert. By the above, the mapping

$$Lu = u'_t + \nu\, P\Delta u$$

is an isomorphism of U onto the range of the projector P in $L^2(\mathcal{C}_T, F^i)$. Let N be a compact continuous mapping of $H^{2,1}(\mathcal{C}_T, F^i)$ into $L^2(\mathcal{C}_T, F^i)$. Given any $f \in L^2(\mathcal{C}_T, F^i)$, we look for $u \in U$ satisfying

$$Lu + PN(u) = Pf \tag{5.4.2}$$

in $\mathcal{C}_T$.

On applying L^{-1} to both sides of this equation we transform it to the form $u = c_0 + K(u)$, where $c_0 = L^{-1}Pf$ and

$$K(u) := -L^{-1}PN(u)$$

for $u \in U$. Since both L^{-1} and P are bounded linear operators, we conclude readily that K is a compact continuous self-mapping of U. If $u \in U$ is a solution of the equation

$$u = c_0 + \vartheta K(u),$$

for some $\vartheta \in [0,1]$, then

$$\begin{array}{rcl} \|u - c_0\|_U & = & \vartheta \|K(u)\|_U \\ & \leq & \|L^{-1}\|\, \|N(u)\|_{L^2(\mathcal{C}_T,F^i)} \end{array}$$

for all $u \in U$.

Assume that

$$\|N(u)\|_{L^2(\mathcal{C}_T,F^i)} = o\left(\|u\|_U\right) \tag{5.4.3}$$

for $\|u\|_U \to \infty$. Then from the above inequality it follows that there is a number $R > 0$ independent of u, such that $\|u - c_0\|_U < R$. In other words, any solution to the equation $u = c_0 + \vartheta K(u)$ with some $\vartheta \in [0,1]$ belongs to the ball $B(c_0, R)$ in U. On applying Lemma 4.1.2 we see that the equation $u = c_0 + K(u)$ possesses at least one solution in U. We can now return to the perturbed equation (5.4.2) and conclude that under condition (5.4.3) it has at least one solution $u \in U$ for each right-hand side $f \in L^2(\mathcal{C}_T, F^i)$.

Condition (5.4.3) gives rise to a broad class of nonlinear parabolic equations for which the first mixed problem is solvable in the space $H^{2,1}(\mathcal{C}_T, F^i)$. Still, as is mentioned in Section 4.1, the nonlinearity in the classical Navier-Stokes equations does not satisfy (5.4.3).

Chapter 6

The Neumann problem after Spencer for quasicomplexes

When trying to extend the Hodge theory for elliptic complexes on compact closed manifolds to the case of compact manifolds with boundary one is led to a boundary value problem for the Laplacian of the complex which is usually referred to as Neumann problem. We study the Neumann problem for a larger class of sequences of differential operators on a compact manifold with boundary. These are sequences of small curvature, i.e., bearing the property that the composition of any two neighbouring operators has order less than two.

6.1 Preliminaries

Corresponding to each point $x \in \mathcal{X}$ and cotangent vector $\xi \in T^*_x\mathcal{X}$ there is associated with (0.0.8) a sequence of linear mappings

$$0 \longrightarrow F^0_x \stackrel{\sigma^1(A^0)(x,\xi)}{\longrightarrow} F^1_x \stackrel{\sigma^1(A^1)(x,\xi)}{\longrightarrow} \ldots \stackrel{\sigma^1(A^{N-1})(x,\xi)}{\longrightarrow} F^N_x \longrightarrow 0, \qquad (6.1.1)$$

where F^i_x is the fibre of the bundle F^i over x and $\sigma^1(A^i)(x,\xi)$ the principal homogeneous symbol of A^i at (x,ξ). Since A^iA^{i-1} is of order ≤ 1, it follows that $0 \equiv \sigma^2(A^iA^{i-1}) = \sigma^1(A^i)\sigma^1(A^{i-1}) \equiv 0$, i.e., the symbol sequence (6.1.1) constitutes a complex. A cotangent vector $\xi \in T^*_x\mathcal{X}$ is said to be noncharacteristic for the quasicomplex (0.0.8) if the symbol complex is exact.

In what follows, functional methods are used to study quasicomplex (0.0.8), and it will be necessary to have L^2 norms defined for sections of the vector bundles F^i. Accordingly, we shall always consider $\mathcal{X}$ to have a Riemannian structure with volume element dv, and we shall assume that each F^i has a C^∞ Hermitean inner product $(\cdot,\cdot)_x$ defined along its fibres. For arbitrary sections $f, g \in C^\infty(\mathcal{X}, F^i)$, we define

$$(f,g) = \int_{\mathcal{X}} (f(x), g(x))_x dv$$

and $\|f\| = \sqrt{(f,f)}$. Then $L^2(\mathcal{X}, F^i)$ can be defined as the completion of $C^\infty(\mathcal{X}, F^i)$ in the norm $\|\cdot\|$.

In a similar way, we use the induced area element ds on the boundary $\mathcal{S}$ of $\mathcal{X}$ to introduce the space $L^2(\mathcal{S}, F^i)$ with scalar product $(\cdot,\cdot)_{\mathcal{S}}$ and norm $\|\cdot\|_{\mathcal{S}}$.

As usual, we write A^{i-1*} for the formal adjoint of A^{i-1} as determined by the inner products in the spaces $L^2(\mathcal{X}, F^{i-1})$ and $L^2(\mathcal{X}, F^i)$. Thus A^{i-1*} is the unique differential operator from sections of F^i to sections of F^{i-1} of order 1, such that $(A^{i-1}u, g) = (u, A^{i-1*}g)$ whenever $u \in C^\infty(\mathcal{X}, F^{i-1})$ and $g \in C^\infty(\mathcal{X}, F^i)$ have support in the interior of $\mathcal{X}$.

We will also use the Sobolev norms $\|\cdot\|_s$ defined for sections of F^i, where s is a real number. Remark that if $\mathcal{X}$ the closure of an open set in $\mathbb{R}^n$, $F^i = \mathcal{X} \times \mathbb{C}^{k_i}$ and s is a nonnegative integer, then the norm $\|\cdot\|_s$ on $C^\infty(\mathcal{X}, F^i)$ is equivalent to the norm

$$f \mapsto \Big(\sum_{|\alpha| \le s} \|\partial^\alpha f\|^2 \Big)^{1/2},$$

where $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_1}$.

The construction of Sobolev spaces on the compact closed manifold $\mathcal{S}$ is more direct. We write $\|\cdot\|_{\mathcal{S},s}$ for the Sobolev norm on $C^\infty(\mathcal{S}, F^i)$ and $H^s(\mathcal{S}, F^i)$ for the corresponding function space.

6.2 A boundary decomposition

The operators $\Delta^i = A^{i*}A^i + A^{i-1}A^{i-1*}$ are called the Laplacians of (0.0.8). The unit normal vector $\nu(x)$ of the boundary $\partial\mathcal{X}$ is noncharacteristic for the quasicomplex at step i if and only if $\partial\mathcal{X}$ is noncharacteristic for the Laplacian $\Delta^i \in \mathrm{Diff}^2(\mathcal{X}; F^i)$ at x. Throughout this chapter and the next we make the standing assumption that the conormal bundle of the boundary is noncharacteristic for quasicomplex (0.0.8) at steps $i-1$ and i.

We can assume without loss of generality that $\mathcal{X}$ is embedded into a larger smooth manifold $\mathcal{X}'$ without boundary. Choose a smooth function ϱ in a neighbourhood U of $\partial\mathcal{X}$ in $\mathcal{X}'$ which is negative in $U \cap (\mathcal{X} \setminus \partial\mathcal{X})$, positive in $U \cap (\mathcal{X}' \setminus \mathcal{X})$ and whose differential does not vanish on $\partial\mathcal{X}$. By shrinking

U if necessary, we may actually assume that $|d\varrho(x)| = 1$ holds for all $x \in \partial\mathcal{X}$, for if not, we replace ϱ by $\varrho/|d\varrho|$.

Lemma 6.2.1. *For $x \in \partial\mathcal{X}$, the cotangent vector $d\varrho(x) \in T_x^*\mathcal{X}$ is independent of the particular choice of ϱ.*

Proof. Let ϱ_1 and ϱ_2 be two functions with the properties described above. For each $x \in \partial\mathcal{X}$ there is a neighbourhood U_x of this point in $\mathcal{X}'$, such that $\varrho_2 = f\varrho_1$ in U_x with some smooth function f in U_x. It is clear that f is positive in $U_x \setminus \partial\mathcal{X}$. Furthermore, we get $d\varrho_2 = f d\varrho_1$ on $U_x \cap \partial\mathcal{X}$ whence $f \equiv 1$ on $U_x \cap \partial\mathcal{X}$, as desired. □

Write $\sigma^i(x)$ for the principal homogeneous symbol of A^i evaluated at the point $(x, d\varrho(x))$ of $T^*\mathcal{X}$. This is a smooth section of the bundle $\mathrm{Hom}(F^i, F^{i+1})$ whose restriction to the surface $\partial\mathcal{X}$ does not depend on the particular choice of ϱ, the latter being due to Lemma 6.2.1. The principal homogeneous symbol of Δ^i evaluated at $(x, d\varrho(x))$ is $\sigma^i(x)^*\sigma^i(x) + \sigma^{i-1}(x)\sigma^{i-1}(x)^*$, which we denote by $\ell^i(x)$ for short. Since the boundary is noncharacteristic for Δ^i, the map $\ell^i(x) \in \mathrm{Hom}(F_x^i)$ is invertible for all x in some neighbourhood of $\partial\mathcal{X}$ in $\mathcal{X}'$, and similarly for the symbol $\ell^{i-1}(x)$.

Theorem 6.2.2. *The restriction of the bundle F^i to the surface $\partial\mathcal{X}$ splits into the direct sum*

$$F^i \restriction_{\partial\mathcal{X}} = F_t^i \oplus \sigma^{i-1} F_t^{i-1},$$

where $F_t^i = \sigma^{i*}\sigma^i(\ell^i)^{-1} F^i \restriction_{\partial\mathcal{X}}$ is a smooth subbundle of $F^i \restriction_{\partial\mathcal{X}}$.

Proof. For each $x \in \mathcal{X}$ close to the boundary, any $f \in F_x^i$ can be written in the form

$$f = t(f) + \sigma^{i-1}(x) n(f), \tag{6.2.1}$$

where

$$\begin{aligned} t(f) &= \sigma^i(x)^*\sigma^i(x)(\ell^i(x))^{-1} f, \\ n(f) &= \sigma^{i-1}(x)^*(\ell^i(x))^{-1} f \end{aligned}$$

prove to satisfy $t \circ t = t$, $t \circ n = n$, $n \circ t = 0$ and $n \circ n = 0$. This establishes the theorem. □

Note that if $F^i = \Lambda^i T^*\mathcal{X}$ is the bundle of exterior forms of degree i over $\mathcal{X}$ then $F_t^i = \iota^* F^i$ is the pullback of F^i under the embedding $\partial\mathcal{X} \hookrightarrow \mathcal{X}$. It follows that $F_t^i = \Lambda^i T^*(\partial\mathcal{X})$.

6.3 Green formula

To describe natural boundary value problems for solutions of $\Delta^i u = f$ in $\mathcal{X}$, one uses a Green formula related to the Laplacian Δ^i. Such formulas are well understood in general, see for instance Lemma 3.2.10 in [Tar95]. In this section we just compute explicitly the terms included into this formula, to get it in the form we need.

Theorem 6.3.1 (Green formula)**.** *For all smooth sections u and v of F^i over $\mathcal{X}$ it follows that*

$$\int_{\partial\mathcal{X}} \Big((t(u), \imath\ell n(Av))_x - (\imath\ell n(u), t(A^*v))_x + (t(A^*u), \imath\ell n(v))_x - (\imath\ell n(Au), t(v))_x \Big) ds$$
$$= \int_{\mathcal{X}} \Big((\Delta u, v)_x - (u, \Delta v)_x \Big) dv,$$

where $\imath = \sqrt{-1}$.

Proof. Let $G_A(*g, u)$ stand for the Green operator for a differential operator $A = A^i$, see § 2.4.2 of [Tar95]. Here, $* : F^{i+1} \to F^{i+1\prime}$ is the fibrewise Hodge star operator determined by

$$\langle *g, f \rangle = (f, g)_x$$

for all $f \in F^{i+1}_x$. An easy computation shows that the pullbacks of differential forms $G_A(*g, u)$ and $G_{A^*}(*u, g)$ under the inclusion $\partial\mathcal{X} \hookrightarrow \mathcal{X}$ amount to

$$\begin{array}{rcl} \iota^*\, G_A(*g, u) &=& (t(u), \imath\ell\, n(g))_x ds, \\ \iota^*\, G_{A^*}(*u, g) &=& -(\imath\ell\, n(g), t(u))_x ds \end{array}$$

on $\partial\mathcal{X}$ for all smooth sections g and u of F^{i+1} and F^i, respectively, cf. § 3.2.2 *ibid.* Applying Corollary 2.5.14 of [Tar95] establishes the formula. □

Theorem 6.3.1 shows immediately that the quadrupel $t(u)$, $n(u)$, $t(A^*u)$ and $n(Au)$ gives a representation of the Cauchy data of u on the surface $\partial\mathcal{X}$ relative to the Laplacian Δ. The tangential part of the Cauchy data, $(t(u), t(A^*u))$, is usually referred to as the Dirichlet data, and the normal part of the Cauchy data, $(n(u), n(Au))$, is referred to as the Neumann data. This designation is due rather to the whimsical development of mathematics than to well-motivated choice, for, at the last step of the quasicomplex, the data $(t(u), t(A^*u))$ reduce to $t(A^*u)$, which is the classical Neumann data, and $(n(u), n(Au))$ reduce to $n(u)$, which is the classical Dirichlet data.

6.4 The Neumann problem

In his paper [Spe63], Spencer proposed a method of studying the cohomology of an elliptic complex similar to (0.0.8) at step i. The main step involves the boundary value problem

$$\begin{array}{rcll} \Delta^i u & = & f & \text{in} \quad \mathcal{X}, \\ n(u) & = & 0 & \text{on} \quad \partial\mathcal{X}, \\ n(Au) & = & 0 & \text{on} \quad \partial\mathcal{X}, \end{array} \tag{6.4.1}$$

where f is a given section of F^i over $\mathcal{X}$.

Example 6.4.1. In the special case of the de Rham complex and $i = 0$ problem (6.4.1) reduces to the classical Neumann problem. For $n(du)$ amounts to the normal derivative of u at $\partial\mathcal{X}$.

Even in the classical case, (6.4.1) is not solvable unless f satisfies additional conditions. Since (6.4.1) is a boundary value problem symmetric with respect to the Green formula, it is solvable only if f is orthogonal in the L^2 sense to the space $\mathcal{H}^i(\mathcal{X})$ of all $h \in C^\infty(\mathcal{X}, F^i)$ satisfying the corresponding homogeneous problem, i.e., $\Delta^i h = 0$ in $\mathcal{X}$ and $n(h) = 0$, $n(Ah) = 0$ on $\partial\mathcal{X}$. The sections of $\mathcal{H}^i(\mathcal{X})$ are called harmonic.

Lemma 6.4.2. *A section $h \in C^\infty(\mathcal{X}, F^i)$ is harmonic if and only if $Ah = 0$, $A^*h = 0$ in $\mathcal{X}$ and $n(h) = 0$ on $\partial\mathcal{X}$.*

Proof. The point here is that the boundary conditions of (6.4.1) allow us to integrate by parts without introducing integrals on the boundary. The sufficiency is obvious. To show the necessity, pick a section $h \in \mathcal{H}^i(\mathcal{X})$. On integrating by parts we readily obtain

$$0 = (\Delta^i h, h) = \|A^i h\|^2 + \|A^{i-1*} h\|^2,$$

and the lemma follows. $\square$

The main step in the approach of [Spe63] is to establish that $\mathcal{H}^i(\mathcal{X})$ is finite-dimensional and if $f \in C^\infty(\mathcal{X}, F^i)$ is orthogonal to $\mathcal{H}^i(\mathcal{X})$ then (6.4.1) can be solved for $u \in C^\infty(\mathcal{X}, F^i)$. Suppose that these solvability properties for problem (6.4.1) have been established. We introduce the subspace $\mathcal{N}^i(\mathcal{X})$ of $C^\infty(\mathcal{X}, F^i)$ consisting of those sections u which satisfy the boundary conditions in (6.4.1), i.e., $n(u) = 0$ and $n(Au) = 0$ on $\partial\mathcal{X}$. Given any $f \in C^\infty(\mathcal{X}, F^i)$, we denote by $H^i f$ the orthogonal projection of f into $\mathcal{H}^i(\mathcal{X})$. The difference $f - H^i f$ still belongs to $C^\infty(\mathcal{X}, F^i)$ and is orthogonal to $\mathcal{H}^i(\mathcal{X})$, hence there is a section $u \in \mathcal{N}^i(\mathcal{X})$ such that $\Delta^i u = f - H^i f$ in $\mathcal{X}$. Set $N^i f := u - H^i u$, thus obtaining a linear operator from $C^\infty(\mathcal{X}, F^i)$ to $\mathcal{N}^i(\mathcal{X})$. This operator is well defined, for from $u_1, u_2 \in \mathcal{N}^i(\mathcal{X})$ and $\Delta^i u_1 = \Delta^i u_2$ it follows that $u_1 - H^i u_1 = u_2 - H^i u_2$. We see that any section $f \in C^\infty(\mathcal{X}, F^i)$ can be written as

$$f = H^i f + A^{i*} A^i \, N^i f + A^{i-1} A^{i-1*} \, N^i f \tag{6.4.2}$$

in $\mathcal{X}$.

If the curvature of (0.0.8) vanishes at step i, i.e., $A^i A^{i-1} \equiv 0$, then the terms on the right-hand side of (6.4.2) are mutually orthogonal, as is easy to check. In this case formula (6.4.2) furnishes an isomorphism between the cohomology of (0.0.8) at step i and the space $\mathcal{H}^i(\mathcal{X})$ of harmonic sections, see [Wel73], [Tar95, 4.1] for more details.

Chapter 7

Subelliptic estimates

7.1 The main theorem

Quasicomplex (0.0.8) is said to be elliptic at step i if the symbol complex (6.1.1) is exact at step i for each $x \in \mathcal{X}$ and for each cotangent vector $\xi \in T_x^*\mathcal{X}$ different from zero. This is equivalent to the fact that the Laplacian Δ^i is a second order elliptic operator on $\mathcal{X}$.

Theorem 7.1.1. *Suppose (0.0.8) is elliptic at steps $i-1$ and i and there is a constant c such that*

$$\|u\|_{1/2}^2 \leq c\left(\|A^i u\|^2 + \|A^{i-1*}u\|^2 + \|u\|^2\right) \tag{7.1.1}$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying $n(u) = 0$. Then $\mathcal{H}^i(\mathcal{X})$ is finite-dimensional, and if $f \in C^\infty(\mathcal{X}, F^i)$ is orthogonal to $\mathcal{H}^i(\mathcal{X})$ then there exists $u \in C^\infty(\mathcal{X}, F^i)$ satisfying (6.4.1).

As is mentioned in the introductory remarks, this theorem is contained in [KN65] if the curvature of (0.0.8) vanishes at step i.

The first step in proving the theorem is to extend the Laplacian Δ^i to a closed operator L^i on the Hilbert space $L^2(\mathcal{X}, F^i)$. To this end we apply a classical method of (Kurt) Friedrichs, cf. [AS80]. In functional analysis, by the Friedrichs extension is meant a canonical self-adjoint extension of a nonnegative densely defined symmetric operator. This extension is particularly useful in situations where an operator may fail to be essentially self-adjoint or whose essential self-adjointness is difficult to show. The definition of the Friedrichs extension is based on the theory of closed positive forms on Hilbert spaces. If T is a nonnegative operator in a Hilbert space H, then $Q(u,v) = (u, Tv) + (u,v)$ is a sesquilinear form on $\operatorname{Dom} T$ and $Q(u,u) \geq \|u\|^2$. Thus Q defines an inner product on $\operatorname{Dom} T$. Let H_1 be the completion of $\operatorname{Dom} T$ with respect to Q. This is an abstractly defined space. For instance its elements can be represented as equivalence classes of Cauchy sequences of elements of $\operatorname{Dom} T$. It is not obvious that all elements in H_1 can be identified with elements of H. However, the canonical inclusion $\operatorname{Dom} T \hookrightarrow H$ extends to an injective continuous map $H_1 \hookrightarrow H$. We regard H_1 as a subspace of H. Define an operator T_1 in H whose domain consists of all $u \in H_1$ such that $v \mapsto Q(u, v)$ is a bounded conjugate-linear functional on H_1. Here, bounded is relative to the topology of H_1 inherited from H. Pick $u \in \operatorname{Dom} T_1$. By the Riesz representation theorem applied to the linear functional $v \mapsto \overline{Q(u, v)}$ extended to all of H, there is a unique $f \in H$ such that $Q(u, v) = (f, v)$ for all $v \in H_1$. Set $T_1 u := f$. Then T_1 is a nonnegative self-adjoint operator in H, such that $T_1 - I$ extends T. The operator $T_1 - I$ is called the Friedrichs extension of T.

The operator Δ^i in $L^2(\mathcal{X}, F^i)$ with domain $\mathcal{N}^i(\mathcal{X})$ is nonnegative, densely defined and symmetric. The sesquilinear form $Q(u, v) = (u, \Delta^i v) + (u, v)$ on $\mathcal{N}^i(\mathcal{X})$ reduces readily to

$$D(u, v) := (A^i u, A^i v) + (A^{i-1*} u, A^{i-1*} v) + (u, v),$$

which is known as the Dirichlet scalar product on $C^\infty(\mathcal{X}, F^i)$. When completing $\mathcal{N}^i(\mathcal{X})$ in the norm $D(u) := \sqrt{D(u, u)}$, one can scarcely retain the boundary condition $n(Au) = 0$ at $\partial\mathcal{X}$. Hence, one disregards this condition from the very beginning and considers the Dirichlet inner product on the subspace of $C^\infty(\mathcal{X}, F^i)$ which consists of all u satisfying $n(u) = 0$ on $\partial\mathcal{X}$. We write $\mathcal{D}^i$ for its completion to a Hilbert space. It is not difficult to see that $\mathcal{D}^i$ can be thought of as a subspace of $L^2(\mathcal{X}, F^i)$. We now define $L^i + I$ to be the operator whose domain consists of all $u \in \mathcal{D}^i$ such that $v \mapsto D(v, u)$ extends to a bounded linear functional on $L^2(\mathcal{X}, F^i)$ and whose rule of correspondence is given by $D(u, v) = ((L^i + I)u, v)$, for all sections $v \in \mathcal{D}^i$. Then $L^i + I$ is a self-adjoint operator on $L^2(\mathcal{X}, F^i)$, and $(L^i + I)u = (\Delta^i + I)u$ if $u \in \mathcal{N}^i(\mathcal{X})$. Also, $L^i + I$ is surjective, and $(L^i + I)^{-1}$ is bounded as an operator from $L^2(\mathcal{X}, F^i)$ to $\mathcal{D}^i$. It follows by (7.1.1) and Rellich's theorem that $(L^i + I)^{-1}$ is a compact operator from $L^2(\mathcal{X}, F^i)$ to itself, and hence $L^i = (L^i + I) - I$ must have closed range and finite-dimensional null space. Since L^i is self-adjoint, its null space is the orthogonal complement of the range of L^i. Hence, any $f \in L^2(\mathcal{X}, F^i)$ can be written in the form $f = h + L^i u$, where h belongs to the null space of L^i and u is in the domain of L^i. The proof of Theorem 7.1.1 will now be complete when we establish two facts. The first of the two is that if u lies in the domain of L^i and if $L^i u$ is C^∞, then u is C^∞. The second fact is that every smooth section u in the domain of L^i must satisfy the boundary conditions $n(u) = 0$ and $n(Au) = 0$ on $\partial\mathcal{X}$. If f is C^∞, then the first statement will imply that the sections h and u in

$f = h + L^i u$ are C^∞. Its proof will occupy the next three sections. The second statement will then imply that h is in $\mathcal{H}^i(\mathcal{X})$, that u is in $\mathcal{N}^i(\mathcal{X})$, and that $L^i u = \Delta^i u$. We turn to the proof of the second statement right now.

Lemma 7.1.2. *Every C^∞ section u in $\mathcal{D}^i$ satisfies the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$.*

Proof. Since $u \in \mathcal{D}^i$, there exists a sequence $\{u_j\}$ in $C^\infty(\mathcal{X}, F^i)$ such that $n(u_j) = 0$ on $\partial\mathcal{X}$ and $D(u - u_j) \to 0$ as $j \to \infty$. Since $n(u_j) = 0$ on $\partial\mathcal{X}$, integration by parts yields the equality $(A^* u_j, \varphi) = (u_j, A\varphi)$ for every $\varphi \in C^\infty(\mathcal{X}, F^{i-1})$. Since $D(u - u_j) \to 0$, we may pass to the limit in the equality to obtain $(A^* u, \varphi) = (u, A\varphi)$ for every φ. In view of the integration-by-parts formula (see the proof of Theorem 6.3.1), this means that

$$\int_{\partial\mathcal{X}} (\imath\ell n(u), t(\varphi))_x ds = 0$$

for all $\varphi \in C^\infty(\mathcal{X}, F^{i-1})$. Hence the lemma holds. $\square$

Lemma 7.1.3. *Suppose the boundary is noncharacteristic for quasicomplex (0.0.8) at step $i-1$. Then every $u \in C^\infty(\mathcal{X}, F^i)$ which belongs to the domain of L^i satisfies $n(Au) = 0$ on $\partial\mathcal{X}$.*

Proof. If $u \in C^\infty(\mathcal{X}, F^i)$ belongs to the domain of L^i, then for every C^∞ section v in $\mathcal{D}^i$ we get

$$\begin{aligned} 0 &= D(u,v) - ((L^i + I)u, v) \\ &= ((A^i u, A^i v) - (A^{i*} A^i u, v)) + ((A^{i-1*} u, A^{i-1*} v) - (A^{i-1} A^{i-1*} u, v)) \\ &= \int_{\partial\mathcal{X}} (\imath\ell n(Au), t(v))_x \, ds - \int_{\partial\mathcal{X}} (t(A^* u), \imath\ell n(v))_x \, ds, \end{aligned}$$

the last equality being due to the integration-by-parts-formula. As $n(v) = 0$ on the surface $\partial\mathcal{X}$, the second term on the right-hand side vanishes, which readily gives

$$\int_{\partial\mathcal{X}} (\imath\ell n(Au), t(v))_x \, ds = 0$$

for all $v \in C^\infty(\mathcal{X}, F^i)$ satisfying $n(v) = 0$ on $\partial\mathcal{X}$. On applying Theorem 6.2.2 we conclude that $n(Au) = 0$ on $\partial\mathcal{X}$, as desired. $\square$

7.2 A priori estimates

To complete the proof of Theorem 7.1.1 we must prove that u is C^∞, whenever $L^i u$ is. In this section we derive certain a priori estimates which help establish this result. In what follows, c will denote a generic constant.

We shall need the norms $\|f\|_{(r,s)}$ when f is a C^∞ function with compact support in the closed half-space $\mathbb{R}^n_{\geq 0}$ consisting of all $x \in \mathbb{R}^n$ with $x^n \geq 0$. For the definition of these norms in terms of Fourier transform we refer to Section 2.5 of [Hoe63]. We only remark that if r and s are nonnegative integers, then $\|\cdot\|_{(r,s)}$ is equivalent to the norm

$$f \mapsto \Big(\sum_{\substack{|\alpha| \leq r+s \\ \alpha_n \leq r}} \int_{\mathbb{R}^n_{\geq 0}} |\partial^\alpha f(x)|^2 dv \Big)^{1/2}.$$

So $\|f\|_{(r,s)}$ controls the L^2 norms of those partial derivatives of f which are of total order $\leq r+s$ and are of order $\leq r$ in the normal derivative $\partial/\partial x^n$. We list the main properties of the norms $\|\cdot\|_{(r,s)}$ in

Lemma 7.2.1. *As defined above, the scale $\|\cdot\|_{(r,s)}$ bears the following properties:*

1) $\|f\|_{(r,0)} = \|f\|_r$, the Sobolev r-norm on $\mathbb{R}^n_{\geq 0}$;

2) $\|f\|_{(r,s)} \leq \|f\|_{(r',s')}$ if $r \leq r'$ and $r+s \leq r'+s'$;

3) $\|Pf\|_{(r,s)} \leq c\,\|f\|_{(r+m,s)}$ holds with some constant c independent of f, if P is a differential operator of order m;

4) $\|f\|_{(r,s)} \leq c\,(\|Pf\|_{(r-m,s)} + \|f\|_{(r',s')})$ holds with a constant c independent of f, if P is an elliptic differential operator of order m and $r+s = r'+s'$;

5) $\|f\|_{S,s} \leq \|f\|_{(1,s-1)}$, where $\|\cdot\|_{S,s}$ is the Sobolev s-norm on $\{x^n = 0\}$;

6) $2\,\Re\,(f,g) \leq \|f\|_{((0,s)}\|g\|_{((0,-s)}$ for any s.

Proof. Assertion 4) is Lemma 2.1.1 in [Hoe66]. The rest of the lemma is contained in Section 2.5 of [Hoe63]. □

Let U be a coordinate neighbourhood in $\mathcal{X}$ such that the bundles F^{i-1}, F^i, and F^{i+1} are trivial over U. Assume that the coordinate $x = (x^1, \ldots, x^n)$ on U maps U into the closed half-space $\mathbb{R}^n_{\geq 0}$. Then any C^∞ function with support in U can be considered as a function on $\mathbb{R}^n_{\geq 0}$, and hence the norms $\|f\|_{(r,s)}$ are defined for $f \in C^\infty_{\mathrm{comp}}(U)$. Now fix a frame in $F^i|_U$, that is, choose sections $e_1, \ldots, e_{k^i}$ in $C^\infty(U, F^i)$ with the property that for each $x \in U$ the elements $e_1(x), \ldots, e_{k^i}(x)$ form a basis for the fibre over x. Then each $u \in C^\infty_{\mathrm{comp}}(U, F^i)$ has component functions defined by

$$u = u^1 e_1 + \ldots + u^{k^i} e_{k^i},$$

and we may define

$$\|u\|_{(r,s)} = \Big(\sum_{j=1}^{k^i} \|u^j\|^2_{(r,s)} \Big)^{1/2}.$$

It is easy to check that the assertions in Lemma 7.2.1 continue to hold for these norms.

Let $D' = (D_1, \ldots, D_{n-1})$, where $D_j = \dfrac{1}{\sqrt{-1}}\dfrac{\partial}{\partial x^j}$. Consider the pseudodifferential operator

$$\Lambda^s = \chi(D')\left(1 + |D'|^2\right)^{s/2}$$

on $\mathbb{R}^{n-1}$, where $\chi \in C^\infty(\mathbb{R}^{n-1})$ is 0 on a neighbourhood of the origin and 1 outside a slightly larger set. On letting Λ^s act along the first $n-1$ coordinate directions we define $\Lambda^s f$ when f is a C^∞ function on $\mathcal{X}$ with compact support in U. And with a fixed choice of frame in F^i over U we can define $\Lambda^s u$ for $u \in C^\infty_{\mathrm{comp}}(U, F^i)$ by letting Λ^s act on the component functions of u as determined by the frame. If $\varphi \in C^\infty_{\mathrm{comp}}(U)$ and

$$T^s = \varphi\Lambda^s\varphi. \tag{7.2.1}$$

then T^s is an operator which acts on $C^\infty(\mathcal{X})$ and also, with a choice of local frame, an arbitrary smooth sections of F^{i-1}, F^i, or F^{i+1}.

If an appropriate frame is used to define T^s on sections of F^i, then T^s becomes a formally self-adjoint operator. In fact, let $e'_1, \ldots, e'_{k^i} \in C^\infty(\mathcal{X}, F^i)$ be such that for each $x \in U$ the elements $e'_1(x), \ldots, e'_{k^i}(x)$ form an orthonormal basis for the fibre, and let the volume element be given by $dv = v(x)dx$ in the coordinate x on U. Then define $e_j = e'_j/\sqrt{v}$, for $j = 1, \ldots, k^i$, so that if $u = u^j e_j$ and $v = v^j e_j$ have support in U, then

$$(u, v) = \int_U \sum_{j=1}^{k^i} u^i(x)\overline{v^i(x)}dx.$$

If we define

$$T^s u = (T^s u^j)e_j$$

for $u = u^j e_j \in C^\infty(U, F^i)$, then $(T^s u, v) = (u, T^s v)$ for all C^∞ sections u and v. When letting T^s operate on sections of a bundle, we shall assume that the frame being used makes T^s self-adjoint.

Lemma 7.2.2. *Suppose φ, ψ, ω are C^∞ functions with compact support in U and $\varphi = 1$ on the support of ω, $\psi = 1$ on the support of φ. Let T^s be the operator defined by (7.2.1). Then,*

1) for each r, t there is a constant c such that $\|T^s f\|_{(r,t)} \le c\,\|\psi f\|_{(r,t+s)}$;

2) if moreover P is a differential operator of order m, then for each r, t there exists a constant c such that

$$\begin{aligned} \|[P, T^s]f\|_{(r,t)} &\le c\,\|\psi f\|_{(r+m,t+s-1)}, \\ \|[[P, T^s], T^s]f\|_{(r,t)} &\le c\,\|\psi f\|_{(r+m,t+2s-2)}; \end{aligned}$$

3) for each t there is a constant c such that

$$\|\omega f\|_{(0,t+s)} \leq c \left(\|T^s f\|_{(0,t)} + \|f\|_{t+s-1} \right).$$

As usual, the bracket $[P, Q]$ of two operators denotes their commutator $PQ - QP$.

Proof. Assertions 1) and 2) are well-known properties of classical pseudodifferential operators. 3) holds because T^s is tangentially elliptic on the support of ω, see Theorem 4.7 in [Hoe65]. $\square$

Lemma 7.2.3. *Assume that (0.0.8) is elliptic at F^i and let $u \in C^\infty(\mathcal{X}, F^i)$ satisfy $n(u) = 0$ on $\partial\mathcal{X}$. Then there exist $v, u', u'' \in C^\infty(U, F^i)$ with support in* $\operatorname{supp}\varphi$ *such that*

1) $T^s T^s u = v + T^s u' + u''$;

2) $n(v) = 0$ on $\partial\mathcal{X}$;

3) for each t there is a constant c such that

$$\begin{array}{rcl} \|u'\|_{(1,t)} & \leq & c\,\|\psi u\|_{(1,t+s-1)}, \\ \|u''\|_{(1,t)} & \leq & c\,\|\psi u\|_{(1,t+2s-2)}. \end{array}$$

Proof. We follow the proof of Lemma 4 in [Swe71]. Theorem 6.2.2 shows immediately that the homotopy formula $\sigma\, n(u) + n(\sigma u) = u$ holds for all $u \in C^\infty(\partial\mathcal{X}, F^i)$, where $n^2 = 0$. Hence, the results of [Swe71] apply with $\mathcal{A} = n$, $\mathcal{B} = n$ and $\mathcal{R} = \sigma(x)$. Consider

$$\begin{array}{rcl} w & = & \sigma(x)\, n(T^s T^s u) \\ & = & \sigma(x)\, T^s [n, T^s] u + \sigma(x)\, [n, T^s] T^s u \\ & = & T^s w' + w'', \end{array}$$

where $w' = 2\sigma(x)\, [n, T^s] u$ and

$$\begin{array}{rcl} w'' & = & [\sigma(x)[n, T^s], T^s] u + [\sigma(x), T^s]\, [n, T^s] u \\ & = & \sigma(x)\, [[n, T^s], T^s] u + 2\, [\sigma(x), T^s]\, [n, T^s] u. \end{array}$$

Using Lemmata 7.2.1, 7.2.2 and inequality $\|\sigma(x) u\|_{\mathcal{S},s} \leq c\,\|u\|_{\mathcal{S},s}$ with c a constant independent of u, we infer

$$\begin{array}{rcl} \|w'\|_{\mathcal{S},t+1/2} & \leq & c\,\|[n, T^s] u\|_{\mathcal{S},t+1/2} \\ & \leq & c\,\|\psi u\|_{\mathcal{S},t+s-1/2} \\ & \leq & c\,\|\psi u\|_{(1,t+s-1)} \end{array}$$

and

$$\begin{aligned}
\|w''\|_{\mathcal{S},t+1/2} &\leq c\,(\|[[n,T^s],T^s]u\|_{\mathcal{S},t+1/2} + \|[n,T^s]u\|_{\mathcal{S},t+s-1/2}) \\
&\leq c\,(\|\psi u\|_{\mathcal{S},t+2s-3/2} + \|\psi u\|_{\mathcal{S},t+2s-3/2}) \\
&\leq c\,\|\psi u\|_{(1,t+2s-2)}.
\end{aligned}$$

By Theorem 2.5.7 in [Hoe63] we can choose $u', u'' \in C^\infty_{\mathrm{comp}}(U, F^i)$ such that

$$\begin{aligned}
u' &= w', \\
u'' &= w''
\end{aligned}$$

on the boundary of $\mathcal{X}$ and

$$\begin{aligned}
\|u'\|_{(1,t)} &\leq c\,\|w'\|_{\mathcal{S},t+1/2}, \\
\|u''\|_{(1,t)} &\leq c\,\|w''\|_{\mathcal{S},t+1/2}.
\end{aligned}$$

In view of the estimates for w' and w'' which have already been obtained we get

$$\begin{aligned}
\|u'\|_{(1,t)} &\leq c\,\|\psi u\|_{(1,t+s-1)}, \\
\|u''\|_{(1,t)} &\leq c\,\|\psi u\|_{(1,t+2s-2)},
\end{aligned}$$

as required. Since

$$\begin{aligned}
T^s u' + u'' &= T^s w' + w'' \\
&= \sigma(x)\, n(T^s T^s u)
\end{aligned}$$

on $\partial\mathcal{X}$, we can define $v = T^s T^s u - T^s u' - u''$, and the proof is complete. □

In [KN65] the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$ is assumed to be invariant with respect to action in the directions parallel to the boundary. This means, in particular, that if $n(u) = 0$ on $\partial\mathcal{X}$ then also $n(T^s u) = 0$, in which case Lemma 7.2.3 is trivial. How can the condition $\sigma(x)^* u = 0$ imply $\sigma(x)^* T^s = 0$ on the boundary? This can be achieved only in the case if $n(u) = 0$ just amounts to saying that several components of the section u of F^i vanish on $\partial\mathcal{X}$. Since quasicomplex (0.0.8) is elliptic at the step i, this can certainly be achieved by choosing special local frames for the bundle F^i. The decomposition of Theorem 6.2.2 actually gives such a vector bundle F^i_t which is a direct summand of F^i. Technically this means that all norms under consideration are independent up to equivalent norms of the particular choices of local frames, which is an ungrateful exercise in functional analysis of sections of smooth vector bundles over $\partial\mathcal{X}$.

Lemma 7.2.4. *For all $u \in C^\infty(\mathcal{X}, F^i)$,*

$$D(u, T^s T^s u) = D(T^s u, T^s u) + O\left(\|\psi u\|_{(1,s-1)}\right).$$

Proof. Since T^s is formally self-adjoint, the lemma reduces to Lemma 3.1 in [KN65]. The proof is essentially algebraic, using only self-adjointness and those properties of T^s which are mentioned in Lemma 7.2.2. $\square$

Lemma 7.2.5. *Assume that quasicomplex (0.0.8) is elliptic at steps $i-1$ and i. Let the estimate $\|u\|_{1/2}^2 \leq c\, D(u,u)$ hold for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$. Then for each $s \geq 1/2$ there is a constant c with the property that*

$$\|T^s u\|_{1/2}^2 \leq c\, D(T^s u, T^s u) \leq c\,(\|(\Delta + I)u\|_{s-1/2}^2 + \|u\|_s^2) \qquad (7.2.2)$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$ in the domain of L^i.

Proof. Since u is in the domain of L^i, we have $D(u,v) = ((\Delta + I)u, v)$ for all $v \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(v) = 0$ on $\partial\mathcal{X}$. Hence, this equality holds in particular for the section $v = T^s T^s u - T^s u' - u''$ described in Lemma 7.2.3. Thus,

$$\begin{aligned} & D(u, T^s T^s u) \\ = \ & D(u, T^s u') + D(u, u'') + ((\Delta + I)u, T^s T^s u) + ((\Delta + I)u, T^s u' + u''). \end{aligned} \qquad (7.2.3)$$

We shall treat the terms on the right of (7.2.3) one by one.

To treat the first term we first claim that

$$D(u, T^s u') = D(T^s u, u') + O(\|\psi u\|_{(1,s-1)}^2). \qquad (7.2.4)$$

In fact, to prove this we must majorise two terms like

$$(Au, AT^s u') - (AT^s u, Au') = (Au, [A, T^s]u') + ([T^s, A]u, Au'),$$

and by the preceding lemmata this expression is bounded by

$$\begin{aligned} \|A(\psi u)\|_{(0,s-1)} \|[A, T^s]u'\|_{(0,-s+1)} + \|[T^s, A]u\|\, \|Au'\| & \leq c\, \|\psi u\|_{(1,s-1)} \|u'\|_{(1,0)} \\ & \leq c\, \|\psi u\|_{(1,s-1)}^2. \end{aligned}$$

Therefore, (7.2.4) holds, and since

$$\begin{aligned} |D(T^s u, u')| & \leq \sqrt{D(T^s u, T^s u)} \sqrt{D(u', u')} \\ & \leq \frac{1}{4} D(T^s u, T^s u) + c\, \|u'\|_1^2 \\ & \leq \frac{1}{4} D(T^s u, T^s u) + c\, \|\psi u\|_{(1,s-1)}^2, \end{aligned}$$

we get

$$|D(u, T^s u')| \leq \frac{1}{4} D(T^s u, T^s u) + c\,\|\psi u\|^2_{(1,s-1)}.$$

As for the second term in (7.2.3) we claim that $|D(u, u'')| \leq c\,\|\psi u\|^2_{(1,s-1)}$. In fact, for a typical term, we have

$$\begin{aligned} |(Au, Au'')| &\leq c\,\|A(\psi u)\|_{(0,s-1)} \|u''\|_{(1,-s+1)} \\ &\leq c\,\|\psi u\|^2_{(1,s-1)}, \end{aligned}$$

and hence the above estimate holds.

The third term in (7.2.3) is majorised as

$$\begin{aligned} |((\varDelta + I)u, T^s T^s u)| &= |(T^s(\varDelta + I)u, T^s u)| \\ &\leq \|T^s(\varDelta + I)u\|_{(0,-1/2)} \|T^s u\|_{(0,1/2)} \\ &\leq c\,\|(\varDelta + I)u\|_{s-1/2} \|T^s u\|_{1/2} \\ &\leq c\,(\varepsilon^2 \|T^s u\|^2_{1/2} + \varepsilon^{-2} \|(\varDelta + I)u\|^2_{s-1/2}) \\ &\leq c\,\varepsilon^2 D(T^s u, T^s u) + c\varepsilon^{-2} \|(\varDelta + I)u\|^2_{s-1/2}, \end{aligned}$$

where $\varepsilon > 0$ is taken so small that $c\varepsilon^2 < \dfrac{1}{4}$.

The remaining term in (7.2.3) can now be estimated by

$$\begin{aligned} |((\varDelta + I)u, T^s u' + u'')| &\leq \|\psi(\varDelta + I)u\|_{(0,s-1/2)} \|T^s u' + u''\|_{(0,-s+1/2)} \\ &\leq c\,(\|(\varDelta + I)u\|^2_{s-1/2} + \|\psi u\|^2_{(1,s-1)}), \end{aligned}$$

and thus we have proved that

$$D(u, T^s T^s u) \leq \frac{1}{2}\, D(T^s u, T^s u) + c\,(\|(\varDelta + I)u\|^2_{s-1/2} + \|\psi u\|^2_{(1,s-1)}).$$

On using Lemma 7.2.4 and substracting the term $\dfrac{1}{2}\, D(T^s u, T^s u)$ from both sides we get

$$\frac{1}{2}\, D(T^s u, T^s u) \leq c\,(\|(\varDelta + I)u\|^2_{s-1/2} + \|\psi u\|^2_{(1,s-1)}).$$

To complete the proof it suffices to show that $\|\psi u\|^2_{(1,s-1)}$ is majorised by the right-hand side of (7.3.1). But since quasicomplex (0.0.8) is elliptic at step i, the operator $\varDelta^i + I$ is elliptic, and so, by part 4) of Lemma 7.2.1,

$$\begin{aligned} \|\psi u\|^2_{(1,s-1)} &\leq c\,(\|(\varDelta + I)\psi u\|^2_{(-1,s-1)} + \|u\|^2_s) \\ &\leq c\,(\|(\varDelta + I)\psi u\|^2_{s-3/2} + \|u\|^2_s) \\ &\leq c\,(\|(\varDelta + I)u\|^2_{s-3/2} + \|[\varDelta, \psi]u\|^2_{s-3/2} + \|u\|^2_s) \\ &\leq c\,(\|(\varDelta + I)u\|^2_{s-3/2} + \|u\|^2_s), \end{aligned}$$

as desired. $\square$

Recall that by ω we mean a C^∞ function with compact support in U, such that $\varphi = 1$ on the support of ω.

Lemma 7.2.6. *Suppose the quasicomplex (0.0.8) is elliptic at step i. Then for each $s \geq 1/2$ there is a constant c such that*

$$\|\omega u\|_{s+1/2} \leq c\,(\|T^s u\|_{1/2} + \|(\Delta + I)u\|_{s-3/2} + \|u\|_{s-1/2})$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$.

Proof. Since quasicomplex (0.0.8) is elliptic at step i, the operator $\Delta^i + I$ is elliptic, and part 4) of Lemma 7.2.1 yields

$$\begin{aligned}
\|\omega u\|_{s+1/2} &\leq c\,(\|(\Delta + I)\omega u\|_{s-3/2} + \|\omega u\|_{(0,s+1/2)}) \\
&\leq c\,(\|(\Delta + I)u\|_{s-3/2} + \|[\Delta, \omega]u\|_{s-3/2} + \|\omega u\|_{(0,s+1/2)}) \\
&\leq c\,(\|(\Delta + I)u\|_{s-3/2} + \|u\|_{s-1/2} + \|\omega u\|_{(0,s+1/2)}).
\end{aligned}$$

The desired estimate now follows from part 3) of Lemma 7.2.2. □

Theorem 7.2.7. *Assume that quasicomplex (0.0.8) is elliptic at steps $i-1$ and i. Let the estimate $\|u\|^2_{1/2} \leq c\,D(u,u)$ hold for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$. Then for each $s \geq 1/2$ there is a constant c such that the estimate*

$$\|u\|_{s+1/2} \leq c\,\|(\Delta + I)u\|_{s-1/2} \tag{7.2.5}$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$ in the domain of L^i.

Proof. Choose a finite covering $\{U_\nu\}$ of $\mathcal{X}$ by coordinate neighbourhoods of the form used above. For each ν, let ω_ν, φ_ν, ψ_ν and T^s_ν be as described in Lemma 7.2.2. We can assume that $\{\omega_\nu\}$ forms a partition of unity on $\mathcal{X}$. Then, by Lemmata 7.2.5 and 7.2.6, we get

$$\begin{aligned}
\|u\|_{s+1/2} &\leq c\left(\sum_\nu \|K^s_\nu u\|_{1/2} + \|(\Delta + I)u\|_{s-3/2} + \|u\|_{s-1/2}\right) \\
&\leq c\,(\|(\Delta + I)u\|_{s-1/2} + \|u\|_s)
\end{aligned}$$

for all smooth u in the domain of L^i. Using the interpolation inequality

$$\|u\|_s \leq \varepsilon\|u\|_{s+1/2} + C(\varepsilon)\,\|u\|$$

with $\varepsilon > 0$ sufficiently small, we obtain

$$\|u\|_{s+1/2} \leq c\,(\|(\Delta + I)u\|_{s-1/2} + C(\varepsilon)\,\|u\|) + \frac{1}{2}\,\|u\|_{s+1/2}$$

whence

$$\|u\|_{s+1/2} \leq c\,(\|(\varDelta + I)u\|_{s-1/2} + \|u\|). \tag{7.2.6}$$

Since

$$\|u\|^2 \leq D(u,u) = ((\varDelta + I)u, u) \leq \|(\varDelta + I)u\|\,\|u\|$$

for all u in the domain of L^i, we obtain

$$\|u\| \leq \|(\varDelta + I)u\| \leq \|(\varDelta + I)u\|_{s-1/2}.$$

Estimate (7.3.1) now follows from (7.2.6) and the last inequality, as desired. □

7.3 Elliptic regularisation

Following [KN65], we use the techniques of elliptic regularisation in this section to prove that u is C^∞ whenever $L^i u$ is C^∞. This will complete the proof of Theorem 7.1.1.

Choose a bundle F and a differential operator $\partial : C^\infty(\mathcal{X}, F^i) \to C^\infty(\mathcal{X}, F)$ of order 1 such that $\|\partial u\| \geq \|u\|_1$ for all $u \in C^\infty(\mathcal{X}, F^i)$. Define

$$A^i_\varepsilon = A^i \oplus \varepsilon\partial : C^\infty(\mathcal{X}, F^i) \to C^\infty(\mathcal{X}, F^{i+1}) \oplus C^\infty(\mathcal{X}, F)$$

for $\varepsilon \geq 0$. Except for the fact that the composition $A^i_\varepsilon A^{i-1}$ need not be of order 1 when $\varepsilon > 0$, the operators A^{i-1} and A^i_ε share most of the properties of A^{i-1} and A^i which were used in the last two sections. In particular, we can use the sesquilinear form

$$\begin{aligned} D_\varepsilon(u,v) &= (A^i_\varepsilon u, A^i_\varepsilon v) + (A^{i-1*}u, A^{i-1*}v) + (u,v) \\ &= D(u,v) + \varepsilon^2\,(\partial u, \partial v) \end{aligned}$$

to define a self-adjoint operator L^i_ε on $L^2(\mathcal{X}, F^i)$ such that

$$D_\varepsilon(u,v) = ((L^i_\varepsilon + I)u, v)$$

for all u in the domain of L^i_ε and all C^∞ sections v satisfying $n(v) = 0$ on $\partial\mathcal{X}$.

We still give $D_\varepsilon(u,v)$ the domain that consists of all $u, v \in C^\infty(\mathcal{X}, F^i)$ whose normal parts vanish on $\partial\mathcal{X}$. The only problem is on the additional boundary condition for $A^i_\varepsilon u$ for smooth sections $u \in C^\infty(\mathcal{X}, F^i)$ lying in the domain of L^i_ε. An easy verification using the Green formula shows that this free boundary condition reduces to

$$\ell^i(x)n(Au) + \varepsilon^2(\sigma^1(\partial)(x, d\varrho(x)))^*\partial u = 0$$

on $\partial\mathcal{X}$.

Lemma 7.3.1. *Assume that quasicomplex (0.0.8) is elliptic at steps* $i-1$ *and* i*. Let the estimate* $\|u\|_{1/2}^2 \le c\,D(u,u)$ *hold for all* $u \in C^\infty(\mathcal{X}, F^i)$ *satisfying the boundary condition* $n(u) = 0$ *on* $\partial\mathcal{X}$*. Then for each* $s \ge 1/2$ *there is a constant* c *with the property that*

$$\|u\|_{s+1/2} \le c\,\|(L_\varepsilon^i + I)u\|_{s-1/2} \tag{7.3.1}$$

holds whenever $u \in C^\infty(\mathcal{X}, F^i)$ *is in the domain of* L_ε^i *and* $0 \le \varepsilon \le 1$*.*

Proof. All the arguments used to prove (7.3.1) continue to be valid when A^i is replaced by A_ε^i, and it is easy to see that the constant c in each of the various estimates can be chosen independently of ε. □

The reason for introducing A_ε^i is that when $\varepsilon > 0$ then the coercive estimate

$$\varepsilon^2 \|u\|_1^2 \le D_\varepsilon(u,u) \tag{7.3.2}$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$, and it is fairly easy to obtain a regularity theorem for L_ε^i. In fact, we have

Theorem 7.3.2. *Suppose that quasicomplex (0.0.8) is elliptic at steps* $i-1$ *and* i*. Let the estimate* $\|u\|_{1/2}^2 \le c\,D(u,u)$ *hold for all* $u \in C^\infty(\mathcal{X}, F^i)$ *satisfying the boundary condition* $n(u) = 0$ *on* $\partial\mathcal{X}$ *and let* $0 < \varepsilon \le 1$*. Then for every* $f \in C^\infty(\mathcal{X}, F^i)$ *there is a unique section* $u \in C^\infty(\mathcal{X}, F^i)$ *in the domain of* L_ε^i *such that* $(L_\varepsilon^i + I)u = f$*.*

Proof. The operator L_ε^i was constructed in such a way that $L_\varepsilon^i + I$ automatically maps its domain onto $L^2(\mathcal{X}, F^i)$ in a one-to-one fashion. Hence, to prove the theorem, it will suffice to show that if u is in the domain of L_ε^i and if $(L_\varepsilon^i + I)u$ is C^∞, then u is also C^∞. We shall use the method of difference quotients which occurs, e.g., in [Nir55] and [Agm65].

If f is a function on the closed upper half-space in $\mathbb{R}^n$, if $1 \le j < n$ and $h > 0$, then we write

$$\delta_{h,j} f(x) = \frac{1}{\sqrt{-1}}\, \frac{f(x^1, \ldots, x^j + h, \ldots, x^n) - f(x^1, \ldots, x^j - h, \ldots, x^n)}{2h}$$

and, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_n = 0$, we set

$$\delta_h^\alpha = \delta_{h,1}^{\alpha_1} \ldots \delta_{h,n-1}^{\alpha_{n-1}}.$$

After choosing a coordinate system $x : U \to \mathbb{R}^n$ on $\mathcal{X}$, which maps U into the closed upper half-space, and after choosing a function $\varphi \in C_{\mathrm{comp}}^\infty(U)$ we can use a local orthonormal frame to define

$$T_h^\alpha u = \varphi\, \delta_h^\alpha\,(\varphi u)\,,$$

when u is a section of one of the vector bundles F^i or of F. For details we refer to the discussion just above Lemma 7.2.2.

If, in Lemma 7.2.2, the operator T^s is replaced by the operator T_h^α with $|\alpha| = s$, then statements 1), 2), and 3) continue to hold even if the constants c are required to be independent of h for $0 < h \leq 1$. Consequently, Lemmata 7.2.3 and 7.2.4 also hold for the operators T_h^α, where again the constants can be chosen independent of h. Using (7.3.2) and the arguments in the proof of Lemma 7.2.5, one can show that for each $\varepsilon > 0$ and every integer $s \geq 1$ there is a constant c such that

$$\|T_h^\alpha u\|_1 \leq c\left(\|\psi(L_\varepsilon^i + I)u\|_{(0,s-1)} + \|\psi u\|_{(1,s-1)}\right), \tag{7.3.3}$$

provided $|\alpha| = s$, $0 < h \leq 1$, u belongs to the domain of L_ε^i, $\psi u \in H^{(1,s-1)}(\mathcal{X}, F^i)$ and $(L_\varepsilon^i + I)u \in C^\infty(\mathcal{X}, F^i)$. Now, if α and u satisfy these conditions, then (7.3.3) shows that $(T_h^\alpha u)_{0<h\leq 1}$ is a bounded subset of $H^1(\mathcal{X}, F^i)$. Hence, there is a sequence h_ν converging to zero such that $T_{h_\nu}^\alpha u$ converges weakly to some element f of $H^1(\mathcal{X}, F^i)$. Since $T_{h_\nu}^\alpha u$ converges in the distribution sense to $\varphi D^\alpha(\varphi u)$ as $h \to 0$, we infer that $f = \varphi D^\alpha(\varphi u)$ an hence $\varphi D^\alpha(\varphi u) \in H^1(\mathcal{X}, F^i)$. Thus, if $\varphi = 1$on the support of $\omega \in C_{\mathrm{comp}}^\infty(U)$, we conclude that $\omega u \in H^{(1,s)}(\mathcal{X}, F^i)$.

Now let u be in the domain of L_ε^i, such that $(L_\varepsilon^i + I)u \in C^\infty(\mathcal{X}, F^i)$, and let p be a fixed point of $\partial\mathcal{X}$. Then the argument just given shows that if u is in $H^{(1,s-1)}$ on a neighbourhood U of p, then u is in $H^{(1,s)}$ on a slightly smaller neighbourhood. Thus, for each integer s there exists a function $\omega \in C_{\mathrm{comp}}^\infty(U)$ such that $\omega u \in H^{(1,s)}(U, F^i)$ and hence, by Theorem 2.5.7 in [Hoe63], the restriction of ωu to the boundary belongs to $H^s(\partial\mathcal{X}, F^i)$. It follows that $u \in H^s(\partial\mathcal{X}, F^i)$ for each s, and so $u \restriction_{\partial\mathcal{X}}$ must be C^∞ by Sobolev's lemma. Since both $(L_\varepsilon^i + I)u$ and $u \restriction_{\partial\mathcal{X}}$ are C^∞, the regularity theorem for the Dirichlet problem implies that u is C^∞ also (see for instance Theorem 9.9 in [Agm65]). The proof of the theorem is thus complete. □

Corollary 7.3.3. *Suppose that quasicomplex (0.0.8) is elliptic at steps $i-1$ and i. Let the estimate $\|u\|_{1/2}^2 \leq c\,D(u,u)$ hold for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$ and let u belong to the domain of L^i. Then,*

1) u is C^∞ if $(L^i + I)u$ is C^∞;
2) $u \in H^{s+1}(\mathcal{X}, F^i)$ if $(L^i + I)u \in H^s(\mathcal{X}, F^i)$;
3) $u \in H^{s+1}(\mathcal{X}, F^i)$ if $L^i u \in H^s(\mathcal{X}, F^i)$;
4) u is C^∞ if $L^i u$ is C^∞.

Proof. To prove 1) assume that $(L^i + I)u$ is C^∞ and for each $0 < \varepsilon \leq 1$ let u_ε be the unique C^∞ section satisfying $(L_\varepsilon^i + I)u_\varepsilon = (L^i + I)u$. If $s \geq 1/2$, then (7.3.1) shows that $(u_\varepsilon)_{0<\varepsilon\le 1}$ is bounded in the norm $\| \dots \|_{s+1/2}$, and by Rellich's theorem there is a sequence ε_ν converging to zero, such that u_{ε_ν} converges in the norm $\| \cdot \|_s$ to an element u_0 of $H^s(\mathcal{X}, F^i)$. On passing to the limit in $D_\varepsilon(u_\varepsilon, v) = ((L^i + I)u, v)$ we obtain

$$D(u_0, v) = ((L^i + I)u, v)$$

for all $v \in C^\infty(\mathcal{X}, F^i)$ satisfying $n(v) = 0$ on the boundary. Thus, u_0 is in the domain of L^i and $(L^i + I)u_0 = (L^i + I)u$. Since $L^i + I$ is one-to-one, we conclude that $u_0 = u$ and so $u \in H^s(\mathcal{X}, F^i)$. Since $s \ge 1/2$ can be arbitrarily large, it follows that u is C^∞.

To prove 2), let $s \ge 0$ and assume $(L^i + I)u$ is in $H^s(\mathcal{X}, F^i)$. Choose a sequence f_ν of C^∞ sections which converges to $(L^i + I)u$ in the norm $\| \cdot \|_s$, and let u_ν be the unique C^∞ section satisfying $(L^i + I)u_\nu = f_\nu$. Then, by (7.3.1), the sequence u_ν converges in the norm $\| \cdot \|_{s+1}$ to some element u_0 of $H^{s+1}(\mathcal{X}, F^i)$. Since $L^i + I$ has closed graph, we get $(L^i + I)u_0 = (L^i + I)u$ and hence $u = u_0$. Thus, u belongs to $H^{s+1}(\mathcal{X}, F^i)$, as required.

If $s = 0$, then 3) follows immediately from 2). Let m be a positive integer and assume that 3) holds for all s with $0 \le s \le m - 1$. Let $m - 1 < s \le m$, and assume that $L^i u$ is in $H^s(\mathcal{X}, F^i)$. Then, since $L^i u \in H^{s-1}(\mathcal{X}, F^i)$, we conclude that $u \in H^s(\mathcal{X}, F^i)$ by the inductive hypothesis, and so $(L^i + I)u$ belongs to $H^s(\mathcal{X}, F^i)$. Thus, by 2), we see that u is in $H^{s+1}(\mathcal{X}, F^i)$, as desired.

The assertion 4) follows obviously from 3) by Sobolev's lemma, and the proof is complete. □

7.4 A regularity theorem

In this section we assume that the curvature of quasicomplex (0.0.8) vanishes at step i, i.e., $A^i A^{i-1} \equiv 0$. Then, the inhomogeneous equation $A^{i-1}u = f$ might be locally solvable only for those f which satisfy $A^i f = 0$. This is a starting point of [Spe63].

Let T denote the operator from $L^2(\mathcal{X}, F^{i-1})$ to $L^2(\mathcal{X}, F^i)$ obtained by closing the graph of $A : C^\infty(\mathcal{X}, F^{i-1}) \to C^\infty(\mathcal{X}, F^i)$. Thus, u is in the domain of T and $Tu = f$ if and only only if there is a sequence (u_ν) in $C^\infty(\mathcal{X}, F^{i-1})$ such that $u_\nu \to u$ and $Au_\nu \to f$ in the L^2-norm. Our aim in this section is to prove

Theorem 7.4.1. *Assume that the quasicomplex (0.0.8) is elliptic at steps $i-1$, i and $i+1$, and assume that the estimate $\|f\|^2_{1/2} \le c\, D(f, f)$ holds for all $f \in C^\infty(\mathcal{X}, F^i)$ satisfying $n(f) = 0$ on $\partial\mathcal{X}$. Let u be in the domain of*

T, let u be orthogonal to the kernel of T, and let $Tu \in H^s(\mathcal{X}, F^i)$ for some $s \geq 0$. Then u belongs to $H^{s+1/2}(\mathcal{X}, F^{i-1})$.

Such a theorem has proved useful in studying counterexamples for a priori estimates like $\|f\|^2_{1/2} \leq c\, D(f,f)$, see, e.g., [Swe67].

Lemma 7.4.2. *Under the assumptions of Theorem 7.4.1, for each s there is a constant c such that*

$$\|\omega A u\|_s + \|\omega A^* u\|_s \leq c \left(\|\omega A u\|_{(0,s)} + \|\omega A^* u\|_{(0,s)} + \|\Delta u\|_{s-1} + \|u\|_s \right) \tag{7.4.1}$$

is valid for all $u \in C^\infty(\mathcal{X}, F^i)$.

Proof. Using the ellipticity of the quasicomplex at F^{i-1} and F^{i+1}, one checks readily that

$$(g,h) \mapsto (A^* g, Ag + A^* h, Ah)$$

is an elliptic operator between sections of $F^{i-1} \oplus F^{i+1}$ and $F^{i-2} \oplus F^i \oplus F^{i+2}$. Hence, by Lemma 7.2.1, part 4),

$$\begin{aligned} \|\omega A^* u\|_s + \|\omega A u\|_s \leq c \, (& \|\omega A^* u\|_{(0,s)} + \|\omega A u\|_{(0,s)} \\ + \quad & \|A^*(\omega A^* u)\|_{s-1} + \|A(\omega A^* u) + A^*(\omega A u)\|_{s-1} + \|A(\omega A u)\|_{s-1}), \end{aligned}$$

and since the commutators $[A^*, \omega]$, $[A, \omega]$, etc., have order zero, and the operators $A^* A^*$, AA have order one, we get

$$\begin{array}{rcl} \|A^*(\omega A^* u)\|_{s-1} & \leq & c\, \|u\|_s, \\ \|A(\omega A^* u) + A^*(\omega A u)\|_{s-1} & \leq & c \left(\|\Delta u\|_{s-1} + \|u\|_s \right), \\ \|A(\omega A u)\|_{s-1} & \leq & c\, \|u\|_s. \end{array}$$

Estimate (7.4.2) now follows. □

Lemma 7.4.3. *Under the assumptions of Theorem 7.4.1, for each $s \geq 1/2$ there is a constant c such that*

$$\|A^* u\|_s \leq c\, \|(\Delta + I)u\|_{s-1/2} \tag{7.4.2}$$

holds for each $u \in C^\infty(\mathcal{X}, F^i)$ in the domain of L^i.

Proof. By Lemma 7.2.2 and Lemma 7.4.2 we have

$$\begin{array}{rcl} \|\omega A^* u\|^2_s & \leq & c \left(\|\omega A^* u\|^2_{(0,s)} + \|\omega A u\|^2_{(0,s)} + \|\Delta u\|^2_{s-1} + \|u\|^2_s \right) \\ & \leq & c \left(\|T^s A^* u\|^2 + \|T^s A u\|^2 + \|\Delta u\|^2_{s-1} + \|u\|^2_s \right) \\ & \leq & c \left(D(T^s u, T^s u) + \|\Delta u\|^2_{s-1} + \|u\|^2_s \right) \end{array}$$

for all $u \in C^\infty(\mathcal{X}, F^i)$. If u belongs to the domain of L^i, then by Lemma 7.2.5

$$\begin{aligned} \|\omega A^* u\|_s^2 &\leq c\left(\|(\Delta + I)u\|_{s-1/2}^2 + \|u\|_s^2\right) \\ &\leq c\,\|(\Delta + I)u\|_{s-1/2}^2. \end{aligned}$$

Now cover $\mathcal{X}$ with a finite number of neighbourhoods U_ν of the kind used in Lemma 7.2.2 and choose the corresponding functions ω_ν to form a partition of unity on $\mathcal{X}$. Then

$$\|A^* u\|_s \leq \sum_\nu \|\omega_\nu A^* u\|_s \leq c\,\|(\Delta + I)u\|_{s-1/2}$$

for all $u \in C^\infty(\mathcal{X}, F^i)$ in the domain of L^i, as desired. □

Lemma 7.4.4. *Under the assumptions of Theorem 7.4.1, let $u \in L^2(\mathcal{X}, F^i)$ belong to the domain of L^i, and assume that $L^i u \in H^{s-1/2}$ for some $s \geq 1/2$. Then u is in the domain of T^* and $T^* u$ belongs to $H^s(\mathcal{X}, F^{i-1})$.*

Proof. In view of part 2) of Corollary 7.3.3 we get $u \in H^{s+1/2}(\mathcal{X}, F^i)$ and hence $(L^i + I)u \in H^{s-1/2}(\mathcal{X}, F^i)$. Choose a sequence (f_ν) in $C^\infty(\mathcal{X}, F^i)$ which converges to $(L^i + I)u$ in the norm $\|\cdot\|_{s-1/2}$ and let $u_\nu \in C^\infty(\mathcal{X}, F^i)$ be the unique solution to

$$(L^i + I)u_\nu = f_\nu.$$

Then by (7.3.1) the sequence (u_ν) converges in the norm $\|\cdot\|_{s+1/2}$, and since $L^i + I$ gas closed graph, the limit must be u. Now Lemma 7.1.2 and the Green formula show that each u_ν is in the domain of T^*, and $T^* u_\nu = A^* u_\nu$. The estimate (7.4.2) now implies that $(T^* u_\nu)$ converges in the norm $\|\cdot\|_s$. Since T^* has closed graph, we conclude that $u = \lim u_\nu$ is in the domain of T^* and $T^* u = \lim T^* u_\nu$ is in $H^s(\mathcal{X}, F^{i-1})$. The proof is complete. □

As is remarked in Section 7.1, any $f \in L^2(\mathcal{X}, F^i)$ can be written as $f = h + L^i u$, where h lies in the null space of L^i and u is in the domain of L^i. If we require that u be orthogonal to the null space of L^i, then f determines u uniquely and the correspondence $f \mapsto u$ defines an operator $N^i : L^2(\mathcal{X}, F^i) \to L^2(\mathcal{X}, F^i)$ which, as one easily sees, is self-adjoint and bounded.

Proof of Theorem 7.4.1. Let u be in the domain of T^{i-1}, let u be orthogonal to the kernel of T^{i-1}, and assume that Tu is in $H^s(\mathcal{X}, F^i)$ for some $s \geq 0$. Then, since $Tu = h + L^i(NTu)$, where $h \in \mathcal{H}^i(\mathcal{X})$ is C^∞ on $\mathcal{X}$, Lemma 7.4.4 shows that NTu is in the domain of T^* and T^*NTu belongs to $H^{s+1/2}(\mathcal{X}, F^{i-1})$. To complete the proof we show that

$$u = T^*NTu.$$

In fact, if $v \in C^\infty(\mathcal{X}, F^{i-1})$ is an arbitrary section with support in the interior of $\mathcal{X}$, then $Av = h + \Delta NAv$, where $h \in \mathcal{H}^i(\mathcal{X})$. Hence,

$$Av - AA^*NAv = h + A^*ANAv.$$

Since $A^iA^{i-1} \equiv 0$, the terms on the right-hand side are orthogonal to the terms on the left-hand side. It follows that $A(I - A^*NA)v = 0$ and so $(I - A^*NA)v$ is in the null space of T. Since u is orthogonal to the null space of T, we obtain

$$\begin{aligned} 0 &= (u, (I - A^*NA)v) \\ &= ((I - T^*NT)u, v), \end{aligned}$$

and $u = T^*NTu$ now follows. $\square$

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