Well-Posedness of Einstein-Euler Systems in Asymptotically Flat Spacetimes

Uwe Brauer and Lavi Karp

Departamento Matemática Aplicada
Universidad Complutense Madrid
28040 Madrid, Spain
E-mail: oub@mat.ucm.es

Department of Mathematics
ORT Braude College
P. O. Box 78, 21982 Karmiel, Israel
E-mail: karp@braude.ac.il

Abstract

We prove a local in time existence and uniqueness theorem of classical solutions of the coupled Einstein–Euler system, and therefore establish the well posedness of this system. We use the condition that the energy density might vanish or tends to zero at infinity and that the pressure is a certain function of the energy density, conditions which are used to describe simplified stellar models. In order to achieve our goals we are enforced, by the complexity of the problem, to deal with these equations in a new type of weighted Sobolev spaces of fractional order. Beside their construction, we develop tools for PDEs and techniques for hyperbolic and elliptic equations in these spaces. The well posedness is obtained in these spaces.

1 Introduction

This paper deals with the Cauchy problem for the Einstein-Euler system describing a relativistic self-gravitating perfect fluid, whose density either has compact support or falls off at infinity in an appropriate manner, that is the density belongs to a certain weighted Sobolev space.

The evolution of the gravitational field is described by the Einstein equations:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{1.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor of a spacetime metric $g_{\alpha\beta}$ and $T_{\alpha\beta}$ is the energy momentum tensor. In the case of a perfect fluid the later takes the form $T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}$, here ϵ is energy density, p is the pressure and u^α is a four velocity unit timelike vector.

Since $G_{\alpha\beta}$ is a divergence free tensor, the energy momentum tensor must satisfy the Euler equations

$$\nabla_\alpha T^{\alpha\beta} = 0. \tag{1.2}$$

However, equations (1.1) and (1.2) are not sufficient to determinate the structure uniquely, a functional relation between the pressure p and the energy density ϵ (equation of state) is also necessary. We choose an equation of state which has been used in astrophysical problems. It is the analogue of the well known polytropic equation of state in the non-relativistic theory, given by

$$p = K\epsilon^\gamma, \qquad 0 < K, \quad 1 < \gamma. \tag{1.3}$$

So the present Einstein-Euler system consists of the coupled equations (1.1) and (1.2) with the equation of state (1.3). The unknowns are the gravitational field $g_{\alpha\beta}$, the velocity vector u^α and the energy density ϵ.

The common method to solve the Cauchy problem for the Einstein equations consists usually of two steps. Unlike ordinary initial value problems, initial data must satisfy constraint equations intrinsic to the initial hypersurface. Therefore, the first step is to construct solutions of these constraints. The second step is to solve the evolution equations with these initial data, in the present case these are first order symmetric hyperbolic systems. As we describe later in detail, the complexity of our problems forces us to consider an additional third step, that is, after solving the constraint equations, we have to construct the initial data for the fluid equations.

The nature of this Einstein-Euler system (1.1), (1.2) and (1.3) forces us to treat both the constraint and the evolution equations in the same type of functional spaces. Under the above consideration, we have established the well posedness of this Einstein-Euler system in a weighted Sobolev spaces of fractional order. Oliynyk has recently studied the Newtonian limit of this system in weighted Sobolev spaces of integer order [33].

We will briefly resume the situation in the mathematical theory of self gravitation perfect fluids describing compact bodies, such as stars: For the Euler-Poisson system Makino proved a local existence theorem in the case the density has compact support and it vanishes at the boundary, [28]. Since the Euler equations are singular when the density ρ is zero, Makino had to regularize the system by introducing a new matter variable ($w = M(\rho)$). His solution however, has some disadvantages such as the fact they do not contain static solutions and moreover, the connection between the physical density and the new matter density remains obscure.

Rendall generalized Makino's result to the relativistic case of the Einstein–Euler equations, [34]. His result however suffers from the same disadvantages as Makino's result and moreover it has two essential restrictions: 1. Rendall assumed time symmetry, that means that the extrinsic curvature of the initial manifold is zero and therefore the Einstein's constraint equations are reduced to a single scalar equation; 2. Both the data and solutions are C_0^∞ functions. This regularity condition implies a severe restriction on the equation of state $p = K\epsilon^\gamma$, namely $\gamma \in \mathbb{N}$.

Similarly to Makino and Rendall, we have also used the Makino variable

$$w = M(\epsilon) = \epsilon^{\frac{\gamma-1}{2}}. \tag{1.4}$$

Our approach is motivated by the following observation: As it turns out, the system of evolution equations have the following form

$$A^0\partial_t U + A^k\partial_k U = Q(\epsilon, ..), \tag{1.5}$$

where the unknown U consists of the gravitational field $g_{\alpha\beta}$ the velocity of the fluid u^α and the Makino variable w, and the lower order term Q contains the energy density ϵ. Thus, we need to estimate ϵ by w in the corresponding norm of the function spaces. Combining this estimation with the Makino variable (1.4), it results in an algebraic relation between the order of the functional space k and the coefficient γ of the equation of state (1.3) of the form

$$1 < \gamma \leq \frac{2+k}{k}. \tag{1.6}$$

This relation can be easily derived by considering $\|D^\alpha w\|_{L_2}$, $|\alpha| \leq k$. Moreover, it can be interpreted either as a restriction on γ or on k. Thus, unlike typical hyperbolic systems where often the regularity parameter is bounded from below, here we have both lower and upper bounds for differentiability conditions of the sort $\frac{5}{2} < k \leq \frac{2}{\gamma-1}$. Similar phenomena for Euler-Poisson equations was noticed by Gamblin [17].

We want to interpret (1.6) as an restriction on k rather than on γ. Therefore, instead of imposing conditions on the equation of state and in order to sharpen the regularity conditions for existence theorems, we are lead to the conclusion of considering function spaces of fractional order, and in addition, the Einstein equations consist of quasi linear hyperbolic and elliptic equations. The only function spaces which are known to be useful for existence theorems of the constraint equations in the asymptotically flat case, are the weighted Sobolev spaces $H_{k,\delta}$, $k \in \mathbb{N}$, $\delta \in \mathbb{R}$, which were introduced by Nirenberg and Walker, [32] and Cantor [6], and they are the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\left(\|u\|_{k,\delta}\right)^2 = \sum_{|\alpha|\leq k} \int \left((1+|x|)^{\delta+|\alpha|}|\partial^\alpha u|\right)^2 dx. \tag{1.7}$$

Hence we are forced to consider new function spaces $H_{s,\delta}$, $s \in \mathbb{R}$ which generalize $H_{k,\delta}$ to fractional order. The well posedness of the Einstein-Euler system is obtained in these space and to achieve this, we have to solve both the constraint and the evolution equations in the $H_{s,\delta}$ spaces.

Another difficulty which arises from the non-linear equation of state (1.3) is the compatibility problem of the initial data for the fluid and the gravitational fields. There are three types of initial data for the Einstein-Euler system:

- The gravitational data is a triple (M, h, K), where M is space-like manifold, $h = h_{ab}$ is a proper Riemannian metric on M and $K = K_{ab}$ is a second fundamental form on M (extrinsic curvature). The pair (h, K) must satisfy the constrain equations

$$\begin{cases} R(h) - K_{ab}K^{ab} + (h^{ab}K_{ab})^2 &= 16\pi z, \\ {}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b (h^{bc}K_{bc}) &= -8\pi j^a, \end{cases} \tag{1.8}$$

where $R(h) = h^{ab}R_{ab}$ is the scalar curvature with respect to the metric h.

- The matter variables, consisting of the energy density z and the momentum density j^a, appear in the right hand side of the constraints (1.8).

- The initial data for Makino's variable w and the velocity vector u^α of the perfect fluid.

The projection of the velocity vector u^α, $\bar{u}^\alpha$, on the tangent space of the initial manifold M leads to the following relations

$$\begin{cases} z &= \epsilon + (\epsilon + p)h_{ab}\bar{u}^a\bar{u}^b \\ j^\alpha &= (\epsilon + p)\bar{u}^a\sqrt{1 + h_{ab}\bar{u}^a\bar{u}^b} \end{cases} \tag{1.9}$$

between the matters variable (z, j^a) and $(w, \bar{u}^a)$. We cannot give ϵ, p, $\bar{u}^b$ and solve for z and j^α, since this is incompatible with the conformal scaling, see section 4.1. Therefore we have to give z, j^α and solve for ϵ p, $\bar{u}^b$. Relations (1.9) are by no means trivial, and they enforce us to modify the conformal method for solving the constraint equations (see e. g. [13], [2]). Therefore the free initial data for the Einstein-Euler system will be partially invariant under conformal transformations.

The paper is organized as follows: In the next section we formulate Einstein-Euler system and introduce the Makino's variable. Dealing with these systems requires to transform them into a hyperbolic type of PDEs. Choquet-Bruhat showed that the choice of harmonic coordinates converts the fields equations (1.1) into wave equations and which then can be written as a first order symmetric hyperbolic system [10], [13]. Reducing the Euler equations (1.2) to a first order symmetric hyperbolic system is not a trivial matter. We use a fluid decomposition and present a new reduction of the Euler equations. Beside having a very clear geometric interpretation, we are giving a complete description of the structure of the characteristics conformal cone of the system, namely, it is a union of a three-dimensional hyperplane tangent to the initial manifold and the sound cone.

In Section 3 we define the weighted Sobolev spaces of fractional order $H_{s,\delta}$ and present our main results. These include a solution of the compatibility problem, the construction of initial data and a solution to the evolution equations in the $H_{s,\delta}$ spaces. The announcement of the main results has been published in [5].

Section 4 deals with the constructions of the initial data. The common Lichnerowicz-York [13], [7], [42] scaling method for solving the constraint equations cannot be applied here directly, since it violates the relations (1.9). We need to invert of (1.9) in order construct the initial data and there are two conditions which guarantee it: the dominate energy condition $h_{ab}j^aj^b \leq z^2$, which is invariant under scaling; since $\sqrt{\frac{\partial p}{\partial \epsilon}}$ is the speed of sound, we have the causality condition $\frac{\partial p}{\partial \epsilon} = \frac{\partial}{\partial \epsilon}(K\epsilon^\gamma) < c^2$. Unfortunately the last condition is not invariant under scaling. It is also necessary to restrict the matter variables (z, j^a) to a certain region. We show the inversion of (1.9) exists provided that (z, j^a) belong to a certain region. This fact enables us to construct initial data for evolution equations.

The local existence for first order symmetric hyperbolic systems in $H_{s,\delta}$ is discussed in Section 5. The known existence results in the H^s space [16], [23], [20], [38], [37], [27] cannot be applied to the $H_{s,\delta}$ spaces. The main difficulty here is the establishment of energy estimates for linear hyperbolic systems. In order to achieve it we have defined a specific inner-product in $H_{s,\delta}$ and in addition the Kato-Ponce commutator estimate [24], [38], [37] has an essential role in our approach. Once the energy estimates and other tools have been established in the $H_{s,\delta}$ space, we follow Majda's [27] iteration procedure and show existence, uniqueness and continuity in that norm.

In Section 6 we study elliptic theory in $H_{s,\delta}$ which is essential for the solution of the constraint equations. We will extend earlier results in weighted Sobolev spaces of integer order which were obtained by Cantor [7], Choquet-Bruhat and Christodoulou [11], Choquet-Bruhat, Isenberg and York [12], and Christodoulou and O'Murchadha [14] to the fractional ordered spaces. The central tool is a priori estimate for elliptic systems in the $H_{s,\delta}$ spaces (6.21). Its proof requires first the establishment of analogous a priori estimate in Bessel potential spaces H^s. Our approach is based on the techniques of pseudodifferential operators which have symbols with limited regularity and in order to achieve that we are adopting ideas being presented in Taylor's books [38]and [39]. A different method was derived recently by Maxwell [29] who also showed existence of solutions to Einstein constraint equations in vacuum in $H_{s,\delta}$ with the best possible regularity condition, namely $s > \frac{3}{2}$. The semi-linear elliptic equation is solved by following Cantor's homotopy argument [7] and generalize it in $H_{s,\delta}$ spaces.

Finally, in the appendix we deal with of the construction, properties and tools for PDEs in the weighted Sobolev spaces of fractional order $H_{s,\delta}$. Triebel extended the $H_{k,\delta}$ spaces given by the norm (1.7) to a fractional order [40], [41]. We present three equivalent norms, one of which is a combination of the norm (1.7) and the norm of Lipschitz-Sobolevskij spaces [35]. This definition is essential for the understanding of the relations between the integer and the fractional order spaces (see (A.3)). However the double integral makes it almost impossible to establish any property needed for PDEs. Throughout the effort to solve this problem, we were looking for an equivalent definition of the norm: we let $\{\psi_j\}_{j=0}^\infty$ be a dyadic resolution of unity in $\mathbb{R}^3$ and set

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2^j}\|_{H^s}^2, \tag{1.10}$$

where $(f)_\epsilon(x) = f(\epsilon x)$. When s is an integer, then the norms (1.7) and (1.10) are equivalent. Our guiding philosophy is to apply the known properties of the Bessel potential spaces H^s term-wise to each of the norms in the infinite sum (1.10) and in that way to extend them to the $H_{s,\delta}$ spaces. Of course, this requires a careful treatment and a sound consideration of the additional parameter δ. Among the properties which we have extended to the $H_{s,\delta}$ are algebra, Moser type estimates, fractional power, embedding to the continuous and intermediate estimates.

2 The Initial Value Problem for the Euler-Einstein System

We consider the Einstein-Euler system describing a relativistic self-gravitating perfect fluid. The unknowns in the equations are functions of t and x^a where x^a $(a = 1, 2, 3)$ are Cartesian coordinates on $\mathbb{R}^3$. The alternative notation $x^0 = t$ will also be used and Greek indices will take the values $0, 1, 2, 3$ in the following. The evolution of the gravitational field is described by the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta} \tag{2.1}$$

where $g_{\alpha\beta}$ is a semi Riemannian metric having a signature $(-,+,+,+)$, $R_{\alpha\beta}$ is the Ricci curvature tensor, these are functions of $g_{\alpha\beta}$ and its first and second order partial derivatives and R is the scalar curvature. The right hand side of (2.1) consists of the energy-momentum tensor of the matter, $T_{\alpha\beta}$ and in the case of a perfect fluid the latter takes the form

$$T^{\alpha\beta} = \mu u^\alpha u^\beta + p g^{\alpha\beta}, \qquad \mu = \epsilon + p \tag{2.2}$$

where ϵ is the energy density, p is the pressure and u^α is the four-velocity vector. The vector u^α is a unit timelike vector, which means that it is required to satisfy the normalization condition

$$g_{\alpha\beta} u^\alpha u^\beta = -1. \tag{2.3}$$

The Euler equations describing the evolution of the fluid take the form

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{2.4}$$

where ∇ denotes the covariant derivative associated to the metric $g_{\alpha\beta}$. In order to close the system of equations it is necessary to specify a relation between ϵ and p (equation of state). The choice we make here is one which has been used for astrophysical problems. It is an analogue of the well known polytropic equation of state of the non-relativistic theory given by:

$$p = f(\epsilon) = K\epsilon^\gamma, \qquad K, \gamma \in \mathbb{R}^+, \qquad 1 < \gamma. \tag{2.5}$$

The sound velocity is denoted by

$$\sigma^2 = \frac{\partial p}{\partial \epsilon}. \tag{2.6}$$

The new matter variable $w = M(\epsilon)$ which regularize the Euler equations even for $\epsilon = 0$ is given by the expression

$$w = M(\epsilon) = \epsilon^{\frac{\gamma-1}{2}}. \tag{2.7}$$

2.1 The Euler equations written as a symmetric hyperbolic system

It is not obvious that the Euler equations written in the conservative form $\nabla_\alpha T^{\alpha\beta} = 0$ are symmetric hyperbolic. In fact these equations have to be transformed in order to be expressed in a symmetric hyperbolic form. Rendall presented such a transformation of the equations [34], however it's geometrical meaning is not entirely clear and it might be difficult to generalize it to the non time symmetric case. Hence we will present a different hyperbolic reduction of the Euler equations and discuss it in some details, for we have not seen it anywhere in the literature. The basic idea is to perform the standard *fluid decomposition* and then to modify the equation by adding, in an appropriate manner, the normalization condition (2.3) which will be considered as a constraint equation.

The fluid decomposition method consists of:

1. The equation $\nabla_\nu T^{\nu\beta} = 0$ is once projected orthogonal onto u^α which leads to

$$u_\beta \nabla_\nu T^{\nu\beta} = 0. \tag{2.8}$$

2. The equation $\nabla_\nu T^{\nu\beta} = 0$ is projected into the rest pace $\mathcal{O}$ orthogonal to u^α of a fluid particle gives us:

$$P_{\alpha\beta}\nabla_\nu T^{\nu\beta} = 0 \qquad \text{with} \quad P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \quad P_{\alpha\beta}u^\beta = 0. \tag{2.9}$$

The resulting system is of the form:

$$u^\nu \nabla_\nu \epsilon + \mu \nabla_\nu u^\nu \;=\; 0; \tag{2.10a}$$

$$\mu P_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p \;=\; 0. \tag{2.10b}$$

Note that we have beside the evolution equations (2.10a) and (2.10b) the following constraint equation: $g_{\alpha\beta}u^\alpha u^\beta = -1$. We will show later, in subsection 2.1.1 that this constraint equation is conserved under the evolution equation, that is, if it holds initially at $t = t_0$, then it will hold for $t > t_0$. Note that in most textbooks, the equation (2.10b) is presented as $\mu g_{\alpha\beta}u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0$, which is an equivalent form, since due to the normalization condition (2.3) we have $u_\beta \nabla_\nu u^\beta = 0$.

In order to obtain a symmetric hyperbolic system that we have to modify the system in the following way. The normalization condition (2.3) gives that $u_\beta u^\nu \nabla_\nu u^\beta = 0$, so we add $\mu u_\beta u^\nu \nabla_\nu u^\beta = 0$ to equation (2.10a) and $u_\alpha u_\beta u^\nu \nabla_\nu u^\beta = 0$ to (2.10b), which together with (2.6) results in,

$$u^\nu \nabla_\nu \epsilon + \mu P^\nu{}_\beta \nabla_\nu u^\beta \quad = \quad 0 \tag{2.11a}$$

$$\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + \frac{\sigma^2}{\mu} P^\nu{}_\alpha \nabla_\nu \epsilon \quad = \quad 0, \tag{2.11b}$$

where $\Gamma_{\alpha\beta} = P_{\alpha\beta} + u_\alpha u_\beta = g_{\alpha\beta} + 2u_\alpha u_\beta$. As mentioned above we will introduce a new nonlinear matter variable which is given by (2.7). The idea which is behind this is the following: The system (2.11a) and (2.11b) is almost of symmetric hyperbolic form, it would be symmetric if we multiply the system by appropriate factors, for example, (2.11a) by $\frac{\partial p}{\partial \epsilon}$ and (2.11b) by μ. However, doing so we will be faced with a system in which the coefficients will either tend to zero or to infinity, as $\epsilon \to 0$. Hence, it is impossible to represent this system in a non-degenerate form using these multiplications.

The central point is now to introduce a new variable $w = M(\epsilon)$ which will regularize the equations even for $\epsilon = 0$. We do this by multiplying equation (2.11a) by $\kappa^2 M' = \kappa^2 \frac{\partial M}{\partial \epsilon}$. This results in the following system which we have written in matrix form:

$$\left(\begin{array}{c|c} \kappa^2 u^\nu & \kappa^2 \mu M' P^\nu{}_\beta \\ \hline \frac{\sigma^2}{\mu M'} P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu \end{array}\right) \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.12}$$

In order to obtain symmetry we have to demand

$$M' = \frac{\sigma}{\mu\kappa}, \tag{2.13}$$

where $\kappa \gg 0$ has been introduced in order to simplify the expression for w. We choose κ so that

$$\frac{\sqrt{f'(\epsilon)}}{\mu\kappa} = \frac{2}{\gamma - 1} \frac{\epsilon^{\frac{\gamma-1}{2}}}{\epsilon}, \tag{2.14}$$

which gives the Makino variable (2.7). Taking into account the equation of state (2.5), we see that

$$\kappa = \frac{\gamma - 1}{2} \frac{\sqrt{K\gamma}}{1 + K\epsilon^{\gamma-1}} \gg 0. \tag{2.15}$$

Finally we have obtained the following system

$$\left(\begin{array}{c|c} \kappa^2 u^\nu & \sigma\kappa P^\nu{}_\beta \\ \hline \kappa\sigma P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu \end{array}\right) \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.16}$$

which is both symmetric and non-degenerated. The covariant derivative ∇_ν takes in local coordinates the form $\nabla_\nu = \partial_\nu + \Gamma(g^{\gamma\delta}, \partial g_{\alpha\beta})$ which expresses the fact that the fluid u^α is coupled to equations (2.1) for the gravitational field $g_{\alpha\beta}$. In addition, from the Makino Variable (2.7) we see that $\epsilon^{\gamma-1} = w^2$, so from the expression (2.6), $\sigma = \sqrt{\gamma K} w$ and therefore κ which is given by (2.15) is a C^∞ function of w. Thus the fractional power of the equation of state (2.5) does not appear in the coefficients of the system (2.16), and these coefficients are C^∞ functions of the scalar w, the four vector u^α and the gravitational fields $g_{\alpha\beta}$.

Let us now recall a general definition of symmetric hyperbolic systems.

Definition 2.1 ***(First order symmetric hyperbolic systems)*** *A quasilinear, symmetric hyperbolic system is a system of differential equations of the form*

$$L[U] = \sum_{\alpha=0}^{4} A^\alpha(U;x)\partial_\alpha U + B(U;x) = 0 \tag{2.17}$$

where the matrices A^α are symmetric and for every arbitrary $U \in G$ there exists a covector ξ such that

$$\xi_\alpha A^\alpha(U;x) \tag{2.18}$$

is positive definite. The covectors ξ_α for which (2.18) is positive definite, are spacelike with respect to the equation (2.17). Both matrices A^α, B satisfy certain regularity conditions, which are going to be formulated later.

Usually ξ is chosen to be the vector $(1,0,0,0)$ which implies via the condition (2.18) that the matrix A^0 has to be is positive definite.

Now we want to show that A^0 of our system (2.16) is indeed positive definite. We do this in several steps.

1. Explicit computation of the principle symbol (2.16);
2. We show that $-u_\alpha$ is a space like covector with respect to the equations;
3. Then we apply a deformation argument and show that the covector $t_\alpha := (1,0,0,0)$ is a space like covector with respect to the equation.

For each $\xi_\alpha \in T_x^*V$ the principle symbol is a linear map from $\mathbb{R} \times E_x$ to $\mathbb{R} \times F_x$, where E_x is a fiber in $T_x V$ and F_x is a fiber in the cotangent space $T_x^* V$. Since in local coordinates $\nabla_\nu = \partial_\nu + \Gamma(g^{\gamma\delta}, \partial g_{\alpha\beta})$, the principle symbol of system (2.16) is

$$\xi_\nu A^\nu = \left(\begin{array}{c|c} \kappa^2(u^\nu \xi_\nu) & \sigma\kappa {P^\nu}_\beta \xi_\nu \\ \hline \sigma\kappa {P^\nu}_\alpha \xi_\nu & (u^\nu\xi_\nu)\Gamma_{\alpha\beta} \end{array} \right) \tag{2.19}$$

and the characteristics are the set of covectors for which $(\xi_\nu A^\nu)$ is not an isomorphism. Hence the characteristics are the zeros of $Q(\xi) := \det(\xi_\nu A^\nu)$.

The geometric advantages of the fluid decomposition are the following. The operators in the blocks of the matrix (2.19) are ${P^\nu}_\alpha$, the projection on the rest hyperplane $\mathcal{O}$ and $\Gamma_{\alpha\beta}$, the reflection with respect to the same hyperplane. Therefore, the following relations hold:

$$\Gamma^{\alpha\gamma}\Gamma_{\gamma\beta} = {\delta_\beta}^\alpha, \qquad \Gamma^{\alpha\gamma}{P_\gamma}^\nu = P^{\alpha\nu} \qquad \text{and} \qquad {P_\beta}^\alpha {P_\alpha}^\nu = {P^\nu}_\beta,$$

which yields

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} \end{array}\right)(\xi_\nu A^\nu) = \left(\begin{array}{c|c} \kappa^2(u^\nu\xi_\nu) & \sigma\kappa {P^\nu}_\beta\xi_\nu \\ \hline \sigma\kappa P^{\alpha\nu}\xi_\nu & (u^\nu\xi_\nu)\left(\delta^\alpha_\beta\right) \end{array}\right). \tag{2.20}$$

It is now fairly easy to calculate the determinate of the right hand side of (2.20) and we have

$$\det\left(\begin{array}{c|c} \kappa^2(u^\nu\xi_\nu) & \sigma\kappa {P^\nu}_\beta\xi_\nu \\ \hline \sigma\kappa P^{\alpha\nu}\xi_\nu & (u^\nu\xi_\nu)\left(\delta^\alpha_\beta\right) \end{array}\right) = \kappa^2(u^\nu\xi_\nu)^3\left((u^\nu\xi_\nu)^2 - \sigma^2 P^{\alpha\nu}\xi_\nu P^\nu_\alpha\xi_\nu\right).$$

Since P^α_β is a projection,

$$P^{\alpha\nu}\xi_\nu P^\nu_\alpha\xi_\nu = g^{\nu\beta}\xi_\nu P^\alpha_\beta P^\nu_\alpha\xi_\nu = g^{\nu\beta}\xi_\nu {P^\nu}_\beta\xi_\nu = {P^\nu}_\beta\xi_\nu\xi^\beta \tag{2.21}$$

and since $\Gamma^\gamma_\beta : \mathbb{R}^4 \to \mathbb{R}^4$ is a reflection with respect to a hyperplane,

$$\det\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} \end{array}\right) = \det\left(g^{\alpha\beta}\Gamma^\gamma_\beta\right) = \det\left(g^{\alpha\beta}\right)\det\left(\Gamma^\gamma_\beta\right) = -\left(\det\left(g_{\alpha\beta}\right)\right)^{-1}. \tag{2.22}$$

Consequently,

$$Q(\xi) := \det(\xi_\nu A^\nu) = -\kappa^2\det(g_{\alpha\beta})(u^\nu\xi_\nu)^3\left\{(u^\nu\xi_\nu)^2 - \sigma^2 {P^\alpha}_\beta\xi_\alpha\xi^\beta\right\} \tag{2.23}$$

and therefore the characteristic covectors are given by two simple equations:

$$\xi_\nu u^\nu = 0; \tag{2.24}$$

$$(\xi_\nu u^\nu)^2 - \sigma^2 {P^\alpha}_\beta\xi_\alpha\xi^\beta = 0. \tag{2.25}$$

Remark 2.2 ***(The structure of the characteristics conormal cone of)*** *The characteristics conormal cone is therefore a union of two hypersurfaces in T_x^*V. One of these hypersurfaces is given by the condition (2.24) and it is a three dimensional hyperplane $\mathcal{O}$ with the normal u^α. The other hypersurface is given by the condition (2.25) and forms a three dimensional cone the so called sound cone.*

Remark 2.3 *Equation (2.25) plays an essential role in determining whether the equations form a symmetric hyperbolic system.*

Let us now consider the timelike vector u_ν and the linear combination $-u_\nu A^\nu$, with A^ν from equation (2.16), we then obtain that

$$-u_\nu A^\nu = \left(\begin{array}{c|c} \kappa^2 & 0 \\ \hline 0 & \Gamma_{\alpha\beta} \end{array}\right) \tag{2.26}$$

is positive definite. Indeed, $\Gamma_{\alpha\beta}$ is a reflection with respect to a hyperplane which its normal is a timelike vector. Hence, $-u_\nu$ is for the hydrodynamical equations a spacelike covector in the sense of partial differential equations. Herewith one has showed relatively elegant and elementary that the relativistic hydrodynamical equations are symmetric–hyperbolic.

Now we want however to show that the covector $t_\alpha = (1,0,0,0)$ is spacelike with respect to the system (2.16). Since $P^\alpha{}_\beta u_\alpha = 0$, the covector $-u_\nu$ belongs to the sound cone

$$(\xi_\nu u^\nu)^2 - \sigma^2 P^\alpha{}_\beta \xi_\alpha \xi^\beta > 0. \tag{2.27}$$

Inserting $t_\nu = (1,0,0,0)$ the right hand side of (2.27) yields

$$(u^0)^2(1-\sigma^2) - \sigma^2 g^{00}. \tag{2.28}$$

Since the sound velocity is always less than the light speed, that is $\sigma^2 = \frac{\partial p}{\partial \epsilon} < c^2 = 1$, we conclude from (2.28) that t_ν also belongs to the sound cone (2.27). Hence, the vector $-u_\nu$ can be continuously deformed to t_ν while condition (2.27) holds along the deformation path. Consequently, the determinant of (2.23) remains positive under this process and hence $t_\nu A^\nu = A^0$ is also positive definite.

2.1.1 Conservation of the constraint equation $g_{\alpha\beta}u^\alpha u^\beta = -1$

Now it will be shown that the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$, which acts as a constraint equation for the evolution equation, is conserved along stream lines u^α. Because, if for $t = t_0$ the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$ holds and if it is conserved a long stream lines, then $g_{\alpha\beta}u^\alpha u^\beta = -1$ holds also for $t > t_0$. So let $c(t)$ be a curve such that $c'(t) = u^\alpha$ and set $Z(t) = (u \circ c)_\beta (u \circ c)^\beta$, then we need to establish

$$\frac{d}{dt} Z(t) = 2u_\beta \nabla_{c'(t)} u^\beta = 2u^\nu u_\beta \nabla_\nu u^\beta = 0. \tag{2.29}$$

Multiplying the last four last rows of the Euler system (2.16) by u^α and recalling that $P^\nu{}_\alpha$ is the projection on the rest space $\mathcal{O}$ orthogonal to u^α, we have

$$\begin{aligned} 0 &= u^\alpha \left(\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + \kappa\sigma P^\nu{}_\alpha \nabla_\nu w\right) \\ &= u^\alpha P_{\alpha\beta} u^\nu \nabla_\nu u^\beta - u^\nu u_\beta \nabla_\nu u^\beta + \kappa\sigma u^\alpha P^\nu{}_\alpha \nabla_\nu w \\ &= -u^\nu u_\beta \nabla_\nu u^\beta. \end{aligned}$$

2.2 The reduced Einstein field equations

In this paper we study the fields equations (2.1) with the choice of the harmonic coordinates. This condition take the form

$$g^{\alpha\beta} g^{\gamma\delta}(\partial_\gamma g_{\beta\delta} - \frac{1}{2}\partial_\delta g_{\beta\gamma}) = 0 \tag{2.30}$$

and when it is imposed, then it well known that the Einstein equations (2.1) convert to

$$g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = H_{\alpha\beta}(g, \partial g) - 16\pi T_{\alpha\beta} + 8\pi g^{\mu\nu} T_{\mu\nu} g_{\alpha\beta}, \tag{2.31}$$

see for example [10]. Since (2.31) are quasi linear wave equations, the introducing auxiliary variables

$$h_{\alpha\beta\gamma} = \partial_\gamma g_{\alpha\beta}, \tag{2.32}$$

reduce them into a first order symmetric hyperbolic system:

$$\begin{aligned} \partial_t g_{\alpha\beta} &= h_{\alpha\beta 0} \\ g^{ab}\partial_t h_{\gamma\delta a} &= g^{ab}\partial_a h_{\gamma\delta 0} \\ -g^{00}\partial_t h_{\gamma\delta 0} &= 2g^{0a}\partial_a h_{\gamma\delta 0} + g^{ab}\partial_a h_{\gamma\delta b} \\ &\quad + C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\gamma\delta\alpha\beta\rho\sigma} h_{\epsilon\zeta\eta} h_{\kappa\lambda\mu} g^{\alpha\beta} g^{\rho\sigma} - 16\pi T_{\gamma\delta} + 8\pi g^{\rho\sigma} T_{\rho\sigma} g_{\gamma\delta} \end{aligned} \tag{2.33}$$

The object $C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\gamma\delta\alpha\beta\rho\sigma}$ is a combination of Kronecker deltas with integer coefficients. We therefore conclude:

Conclusion 2.4 ***(The evolution equations in a first order symmetric hyperbolic form)*** *The equations for Einstein gravitational fields (2.1) coupled with the Euler equations (2.4) with the normalization conditions (2.3) and the equation of state (2.5), are equivalent to the system (2.33) and (2.16). The coupled systems (2.33) and (2.16) take the form of a first order symmetric hyperbolic system in accordance with Definition 2.1 and where A^0 is a positive definite matrix.*

3 New Function Spaces and the Principle Results

The principle results concern the solution to Einstein constraint equations (1.8), which lead to elliptic systems and the coupled evolution equations (2.1) and (2.4), which we have showed are equivalent to the hyperbolic system (2.16) and (2.33). The Bessel potential spaces H^s which are the natural choice for the hyperbolic systems are inappropriate for the solutions of the constraint equations in asymptotically flat manifolds. Roughly speaking, because the Laplacian is not invertible in these spaces.

As we explained in the introduction, the Nirenberg-Walker-Cantor weighted Sobolev spaces of integer order $H_{m,\delta}$ [6], [32] are suitable for for the solutions of the constraints in asymptotically flat manifolds. Their norm is given by (1.7).

Our aim is solving the Einstein-Euler systems and therefore it is essential to solve both the evolution equations (2.16) and (2.33), and the constraint equations (1.8) in one type of functions spaces. In addition, since we want also to improve the regularity conditions for the solutions of the Einstein-Euler system, we are lead to consider weighted Sobolev spaces of fractional order.

Triebel [40] presented two equivalent extensions of the integer order norm (1.7) to a fractional order. The first one is analogous to Lipschitz-Sobolevskij norm and it is given by (A.3) in the Appendix. The double integral in (A.3) causes many difficulties which makes it useless as one turns to prove certain properties which are needed for PDEs in these spaces.

The second one is based upon a dyadic resolution of the unity in $\mathbb{R}^3$: Let $K_j = \{x : 2^{j-3} \leq |x| \leq 2^{j+2}\}$, $(j = 1, 2, ...)$ and $K_0 = \{x : |x| \leq 4\}$. Let $\{\psi_j\}_{j=0}^{\infty}$ be a sequence of $C_0^\infty(\mathbb{R}^3)$ such that $\psi_j(x) = 1$ on K_j, $\mathrm{supp}(\psi_j) \subset \cup_{l=j-4}^{j+3} K_l$, for $j \geq 1$ and $\mathrm{supp}(\psi_0) \subset K_0 \cup K_1$.

We denote by H^s the Bessel potential spaces with the norm $(p = 2)$

$$\|u\|_{H^s}^2 = c \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of u. Also, for a function f, $f_\varepsilon(x) = f(\varepsilon x)$.

Definition 3.1 ***(Weighted fractional Sobolev spaces: infinite sum of semi norms)*** *For $s \geq 0$ and $-\infty < \delta < \infty$,*

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{(2^j)}\|_{H^s}^2. \tag{3.1}$$

The space $H_{s,\delta}$ is the set of all temperate distributions with a finite norm given by (3.1).

3.1 The principle results

Our principle results are the compatibility of the initial data for the fluid and the gravitational field, the solution of the constraints equations and the well-posedness of the evolution equations in the $H_{s,\delta}$ spaces.

3.1.1 The compatibility of the initial data for the fluid and the gravitational field

The matter data (non-gravitational) (z, j) which are given by (4.2) and (4.3) arise from external sources and appear in the right hand side of the fields equations (2.1). They are coupled with the initial data for the hyperbolic system (2.16) via the relations

$$\begin{cases} z = w^{\frac{2}{\gamma-1}} \left(1 + (1 + Kw^2)\, h_{ab}\bar{u}^a\bar{u}^b\right) \\ j^a = w^{\frac{2}{\gamma-1}} \,(1 + Kw^2)\, u^a\sqrt{1 + h_{ab}\bar{u}^a\bar{u}^b} \end{cases}, \tag{3.2}$$

here h_{ab} is the given Riemannian metric on the initial manifold and $\bar{u}^\alpha$ is the projection of the velocity vector on the initial manifold. Thus, an indispensable condition for obtaining a solution of the Einstein-Euler system is the inversion of (3.2). This system is not invertible for all $(z, j^a) \in \mathbb{R}_+ \times \mathbb{R}^3$, but the inverse does exist in a certain region.

Theorem 3.2 ***(Reconstruction theorem for the initial data)*** *There is a real function $S : [0, 1) \to \mathbb{R}$ such that if*

$$0 \le z < S(\sqrt{h_{ab}j^aj^b}/z), \tag{3.3}$$

then system (3.2) has a unique inverse. Moreover, the inverse mapping is continuous in $H_{s,\delta}$ norm.

Remark 3.3 *The matter initial data (z, j^a) for the Einstein-Euler system with the the equation of state (2.5) cannot be given freely. They must satisfy condition (3.3). This condition includes the inequality*

$$z^2 \ge h_{ab}j^aj^b, \tag{3.4}$$

which is known as the dominate energy condition.

3.1.2 Solution to the constraint equations

The gravitational data is a triple (M, h, K), where M is a space-like asymptotically flat manifold, $h = h_{ab}$ is a proper Riemannian metric on M, and $K = K_{ab}$ is the second fundamental form on M (extrinsic curvature). The metric h_{ab} and the extrinsic curvature K must satisfy Einstein's constraint equations (4.8) and (4.9). The free initial data is a set $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{z}, \hat{j}^a)$, where h_{ab} is a Riemannian metric, $\bar{A}_{ab}$ is divergence and trace free form, $\hat{z}$ is a scalar function and $\hat{j}^a$ is a vector.

Theorem 3.4 ***(Solution of the constraint equations)***

(i) Given the free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{z}, \hat{j}^a)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{z}, \hat{j}^a) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then there exists two positive functions

α and ϕ such that $(\alpha-1),(\phi-1)\in H_{s,\delta}$, a vector field $W\in H_{s,\delta}$ such that the gravitational data

$$\text{and} \qquad K_{ab}=(\phi\alpha)^{-2}\bar{A}_{ab}+\phi^{-2}\hat{\mathcal{L}}(W)$$

satisfy the constraint equations (4.8) and (4.9) with $z=\phi^{-8}\hat{z}$ and $j^b=\phi^{-10}\hat{j}^b$ as the right hand side, here $\hat{\mathcal{L}}$ is the Killing vector field operator. In addition, the $H_{s,\delta},H_{s-1,\delta+1}$ norms of $(h_{ab}-I,K_{ab})$ depend continuously on the $H_{s,\delta},H_{s-1,\delta+1},H_{s-1,\delta+2}$ norms of $(\bar{h}_{ab}-I,\bar{A}_{ab},\hat{y},\hat{j}^a)$.

(ii) Given the free data $(\bar{h}_{ab},\bar{A}_{ab},\hat{z},\hat{j}^a)$ such that $(\bar{h}_{ab}-I)\in H_{s,\delta}$, $\bar{A}_{ab}\in H_{s-1,\delta+1}$, $(\hat{z},\hat{j}^a)\in H_{s-1,\delta+2}$, $\frac{5}{2}<s<\frac{2}{\gamma-1}+\frac{3}{2}$, $-\frac{3}{2}<\delta<-\frac{1}{2}$ and $((\alpha^4\bar{h}_{ab}),\hat{z},\hat{j}^a)$ satisfies (3.3). Let Ω^{-1} denote the inverse of relations (3.2). Then the data for the four velocity vector and Makino variable are given by: $z=\phi^{-8}\hat{z}$, $j^a=\phi^{-10}\hat{j}^a$,

$$(w,\bar{u}^a):=\Omega^{-1}(z,j^a) \qquad \text{and} \qquad \bar{u}^0=1+h_{ab}\bar{u}^a\bar{u}^b$$

and they satisfy the compatibility conditions (3.2). In addition, the $H_{s-1,\delta+2}$ norms of $(w,\bar{u}^a,u^0-1)$ depend continuously on the $H_{s,\delta},H_{s-1,\delta+2}$ norms of $(\bar{h}_{ab}-I,\hat{y},\hat{j}^a)$.

3.1.3 Solution to the evolution equations

The unkowns of the evolution equations are the gravitaional field $g_{\alpha\beta}$ and its first order partial derivatives $\partial_\alpha g_{\gamma\delta}$, the Makino variable w and the velocity vector u^α. We represent them by the vector $U=(g_{\alpha\beta}-\eta_{\alpha\beta},\partial_a g_{\gamma\delta},\partial_0 g_{\gamma\delta},w,u^a,u^0-1)$, here $\eta_{\alpha\beta}$ denotes the Minkowski metric. The solutions of the constraint equations serve as initial data for the hyperbolic systems (2.33) of the Einstein gravitational fields and (2.16) of the perfect fluid. Applying Theorem 5.18 to the vector U we obtain:

Theorem 3.5 ***(Solutions of the evolution equations (2.33) and (2.16))*** *Let $\frac{7}{2}<s<\frac{2}{\gamma-1}+\frac{3}{2}$ and $-\frac{3}{2}<\delta<-\frac{1}{2}$. Given the solutions of the constraint equations as describe in Theorem 3.4, then there exists a $T>0$, a unique semi-Riemannian metric $g_{\alpha\beta}$ solution to (2.33) and a unique pair (w,u^α) solution to (2.16) such that*

$$(g_{\alpha,\beta}-\eta_{\alpha,\beta}) \in C([0,T],H_{s,\delta})\cap C^1([0,T],H_{s-1,\delta+1}) \tag{3.5}$$
$$(w,u^a,u^0-1) \in C([0,T],H_{s-1,\delta+2})\cap C^1([0,T],H_{s-2,\delta+3}). \tag{3.6}$$

4 The Initial Data

The Cauchy problem for Einstein fields equations (2.1) coupled with the Euler system (2.4) consists of solving the coupled hyperbolic systems (2.33) and (2.16) with given initial data. There are two types of data, gravitational and matter data.

The gravitational data is a triple (M, h, K), where M is a space-like manifold, $h = h_{ab}$ is a proper Riemannian metric on M and $K = K_{ab}$ is the second fundamental form on M (extrinsic curvature). On the space-like manifold M the semi-metric g satisfies the following relations:

$$\begin{cases} g_{ab}|_M = h_{ab}, g_{a0}|_M = 0, g_{00}|_M = -1 \\ \qquad -\frac{1}{2}\partial_0 g_{ab}|_M = K_{ab}. \end{cases} . \tag{4.1}$$

Let n be the unit normal to the hypersurface M, $\delta^\alpha_\beta + n^\alpha n_\beta$ be the projection on M and define

$$z = T_{\alpha\beta} n^\alpha n^\beta, \tag{4.2}$$
$$j^\alpha = (\delta^\alpha_\gamma + n^\alpha n_\gamma) T^{\gamma\beta} n_\beta. \tag{4.3}$$

The scalar z is the energy density and the vecor j^α is the momentum density. These quantities are called matter variables and they appear as sources in the constraint equations (4.8) and (4.9) below.

In conjunction with these we must supply initial data for the velocity vector u^α. So we apply the projection to u^α and set $\bar{u}^\alpha = ({\delta^\alpha}_\beta + n^\alpha n_\beta)u^\beta$. Then from the relation of the perfect fluid (2.2) , (4.2), and (4.3) we see that

$$z = (\epsilon + p)(n_\beta u^\beta)^2 - p, \tag{4.4}$$
$$j^\alpha = (\epsilon + p)\bar{u}^\alpha (n_\beta u^\beta). \tag{4.5}$$

The vectors j^α and $\bar{u}^\alpha$ are tangent to the initial surface and so they can be identified with vectors j^a and $\bar{u}^a$ intrinsic to this surface. Recalling the normalisation condition (2.3) we have $-1 = -(n_\beta u^\beta)^2 + h_{ab}\bar{u}^a\bar{u}^b$. Thus the matter data (z, j^a) can be identify with the initial data for the velocity vector as follow:

$$z = \epsilon + (\epsilon + p) h_{ab}\bar{u}^a\bar{u}^b, \tag{4.6}$$
$$j^\alpha = (\epsilon + p)\bar{u}^a \sqrt{1 + h_{ab}\bar{u}^a\bar{u}^b}. \tag{4.7}$$

These two types of data cannot be given freely, because the hypersurface (M, h) is a submanifold of (V, g) so the Gauss Codazzi equations lead to Einstein constraint equations

$$R(h) - K_{ab}K^{ab} + (h^{ab}K_{ab})^2 = 16\pi z, \tag{4.8}$$
$${}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b (h^{bc}K_{bc}) = -8\pi j^a, \tag{4.9}$$

where $R(h) = h^{ab}R_{ab}$ is the scalar curvature with respect to the metric h.

We turn now to the conformal method which allows us to construct the solutions of the constraint equations (4.8) and (4.9). Before entering into details we have to discuss the relations between the initial data for the system of Einstein gravitational fields (2.33) and the system of the fluid (2.16) which are given by (4.6) and (4.7). As it turns out this relations is by no means trivial, and indeed they will force us to modify the conformal method.

4.1 The compatibility problem of the initial data for the fluid and the gravitational fields

On the one hand, the initial data for the Euler equations (2.16) are $w(\epsilon)$ and u^α. On the other hand $z = F(w(\epsilon), \bar{u}^a)$ and $j^a = H(w(\epsilon), \bar{u}^a)$, which are given by (4.6) and (4.7) respectively, appear as sources in the constraint equations (4.8) and (4.9). There we have the possibility of either to consider w and u^α as the fundamental quantities and construct then z and j^a or, vice verse, to consider z and j^a as the fundamental quantities and construct then w and u^α.

The first possibility does not work because the geometric quantities which occur on the left hand side of the constraint equations are supposed to scale with some power of a scalar function ϕ. So z and j^a, which are the sources in the constraint equations, must also scale with a definite power of ϕ. If ϵ is scaled with a certain power of ϕ, then p would be scaled, according to the equation of state (2.5), to a different power. Hence, by (4.6) z is a sum of different powers. Thus, the power which ϵ and p are scaled would have to be zero and they would be left unchanged by the rescaling. Similarly it can be seen that $\bar{u}^a$ would remain unchanged. So in fact z would be unchanged and this is inconsistent with the scalding used in the conformal method.

Instead of constructing $(w, \bar{u}^a)$ from (z, j^a) it is more useful to introduce some auxiliary quantities. Beside the Makino variable $w = \epsilon^{\frac{\gamma-1}{2}}$, we set

$$y = z^{\frac{\gamma-1}{2}} \qquad \text{and} \qquad v^a = \frac{j^a}{z}. \tag{4.10}$$

Now we consider the following map

$$\Phi \begin{pmatrix} w \\ \bar{u}^a \end{pmatrix} = \begin{pmatrix} w[1 + (1 + Kw^2) h_{ab}\bar{u}^a\bar{u}^b)]^{\frac{\gamma-1}{2}} \\ \frac{(1+Kw^2)\bar{u}^a\sqrt{1+h_{bc}\bar{u}^b\bar{u}^c}}{1+(1+Kw^2)h_{bc}\bar{u}^b\bar{u}^c} \end{pmatrix} = \begin{pmatrix} y \\ v^a \end{pmatrix}, \tag{4.11}$$

which is equivalent to equations (4.6) and (4.7). The initial data $(w, \bar{u}^a)$ for the fluid are reconstructed through the inversion of Φ above.

Theorem 4.1 ***(Reconstruction theorem for the initial data)*** *There is a function $s : [0, 1) \to \mathbb{R}$ such that the map Φ defined by (4.11) is a diffeormophism from $[0, (\sqrt{\gamma K})^{-\frac{1}{2}}) \times \mathbb{R}^3$ to Ω, where*

$$\Omega = \{(y, v^a) : 0 \le y < s\left(\sqrt{h_{ab}v^a v^b}\right), h_{ab}v^a v^b < 1\}. \tag{4.12}$$

Proof (of theorem 4.1) Let $\rho = \sqrt{h_{ab}\bar{u}^a\bar{u}^b}$, $\bar{u}_0$ be a unit vector and $R_{\bar{u}^a}$ be the rotation with respect to the metric h_{ab} such that $\bar{u}^a = \rho R_{\bar{u}^a}\bar{u}_0$. Then

$$\Phi \begin{pmatrix} w \\ \bar{u}^a \end{pmatrix} = \Phi \begin{pmatrix} w \\ \rho R_{\bar{u}^a}\bar{u}_0 \end{pmatrix} = \begin{pmatrix} w[1 + (1 + Kw^2)\rho^2]^{\frac{\gamma-1}{2}} \\ \frac{(1+Kw^2)R_{\bar{u}^a}\bar{u}_0\rho\sqrt{1+\rho^2}}{1+(1+Kw^2)\rho^2} \end{pmatrix}. \tag{4.13}$$

Therefore, we can first invert the two dimensional map

$$\Theta \begin{pmatrix} w \\ \rho \end{pmatrix} := \begin{pmatrix} w[1+(1+Kw^2)\rho^2]^{\frac{\gamma-1}{2}} \\ \frac{(1+Kw^2)\rho\sqrt{1+\rho^2}}{1+(1+Kw^2)\rho^2} \end{pmatrix} \tag{4.14}$$

for $\rho \geq 0$ and then apply again the rotation. For $w > 0$, we decompose Θ of (4.14) as follows:

$$\begin{pmatrix} w \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \epsilon + (\epsilon + p(\epsilon))\rho^2 \\ (\epsilon + p(\epsilon))\rho\sqrt{1+\rho^2} \end{pmatrix} =: \begin{pmatrix} z \\ r \end{pmatrix} \mapsto \begin{pmatrix} z^{\frac{\gamma-1}{2}} \\ \frac{r}{z} \end{pmatrix}. \tag{4.15}$$

In order to show that this is a one to one map, we need to show that the Jacobian of $G(\epsilon, \rho) := (\epsilon + (\epsilon + p(\epsilon))\rho^2, (\epsilon + p(\epsilon))\rho\sqrt{1+\rho^2})$ does not vanish. This computation results with

$$\det \begin{pmatrix} 1+(1+p')\rho^2 & (1+p')\rho\sqrt{1+\rho^2} \\ (\epsilon+p)2\rho & (\epsilon+p)\frac{1+2\rho^2}{\sqrt{1+\rho^2}} \end{pmatrix} = \frac{(\epsilon+p)}{\sqrt{1+\rho^2}}\left(1+\rho^2(1-p')\right). \tag{4.16}$$

Recall that $p' = \frac{\partial p}{\partial \epsilon} = \sigma^2$ is the speed of sound, therefore the causality condition $\sigma^2 < c^2 = 1$ imposes the below restriction of the domain of definition of the map Θ:

$$\sigma^2 = p' = \frac{\partial p}{\partial \epsilon} = \frac{\partial}{\partial \epsilon}(K\epsilon^\gamma) = \gamma K \epsilon^{\gamma-1} = \gamma K w^2 < 1. \tag{4.17}$$

Let S be the strip $\{0 \leq w < (\sqrt{\gamma K})^{-\frac{1}{2}}, 0 \leq \rho < \infty\}$. We now want to show that $\Theta : S \to \Theta(S)$ is a bijection. Clearly, $\Theta(0, \rho) = (0, \frac{\rho}{\sqrt{1+\rho^2}})$ maps $\{0\} \times [0, \infty)$ to $\{0\} \times [0, 1)$ in a one to one manner, and $\Theta(w, 0) = (w, 0)$ is of course a bijection. The line $(\sqrt{\gamma K})^{-\frac{1}{2}}, \rho)$ is mapped to the curve

$$\begin{pmatrix} y(\rho) \\ x(\rho) \end{pmatrix} = \begin{pmatrix} (\sqrt{\gamma K})^{-\frac{1}{2}}(1+2\rho^2)^{\frac{\gamma-1}{2}} \\ \frac{2\rho\sqrt{1+\rho^2}}{1+2\rho^2} \end{pmatrix}. \tag{4.18}$$

Since $\frac{dx}{d\rho} > 0$, there exists a function $s : [0, 1) \to \mathbb{R}$ such that the curve (4.18) is given by the graph of s and the image of Θ is the set below the graph, that is,

$$\Theta(S) = \{(y, x) : y < s(x), 0 \leq x < 1\}. \tag{4.19}$$

By (4.15), (4.16) and (4.17) we conclude that the Jacobian of the map Θ, does not vanish in the interior of S, hence $\Theta : S \to \Theta(S)$ is locally one to one map. It is well known that a locally one to one map between two simply connected sets is a bijective map. ∎

4.2 Cantor's conformal method for solving the constraint equations

In principle there are two possibilities for solving the constraint equation for an asymptotically flat manifold:

- Either to adapt directly the method of York et all, but then one is forced to impose certain relations between $R(\bar{h})$ and the second fundamental form (see Choquet-Bruhat and York [13] for details).
- These undesirable conditions can be substituted by a method developed by Cantor which we will describe in the following. (This method has been discussed in detail in the literature, see for example [2], [13], [7] [14] and reference therein.)

In this method parts of the data are chosen (the so-called free data), and the remaining parts are determined by the constraint equations (4.8) and (4.9). The free initial data are $\left(\bar{h}_{ab}, \bar{A}_{ab}, \bar{z}, \bar{j}\right)$, where A_{ab} is a divergence and trace free 2-tensor. The main idea is to consider two conformal scaling functions, α and ϕ.

1. We start with $\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}$. If α is a positive solution to (4.25), then $R(\hat{h}) = 0$. The Brill-Cantor condition (see Definition 4.5) is necessary and sufficient for the existence of positive solutions. Having solved equation (4.25), we now adjust the given data to the new metric: $\hat{A}^{ab} = \alpha^{-10}\bar{A}^{ab}$, $\hat{z} = \alpha^{-8}\bar{z}$ and $\hat{j}^a = \alpha^{-10}\bar{j}^a$.
2. The second step here is solve the Lichnerowicz Laplacian (4.29) and set
$$\hat{K}^{ab} = \left(\mathcal{L}(W)\right)^{ab} + \alpha^{-10}A^{ab}, \tag{4.20}$$
where $\left(\mathcal{L}(W)\right)^{ab}$ is the Killing operator giving by (4.27).
3. The third step is: If ϕ is a solution to the Lichnerowicz equation (4.30), then it follows from (4.32) that the data $h_{ab} = \phi^4 \hat{h}_{ab}$, $K^{ab} = \phi^{-10}\hat{K}^{ab}$, $z = \phi^{-8}\hat{z}$ and $j^a = \phi^{-10}\hat{j}^a$ satisfy the constraint equations (4.8) and (4.9).

For the Einstein-Euler system with the equation of state (2.5) it is essential that the initial data will satisfy condition (4.12) of Theorem 4.1. Therefore it is necessary to adjust this method in this case.

Here the free initial data are:
$$\left(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b\right). \tag{4.21}$$
where $\bar{A}_{ab}$ is trace and divergence free, that is, $\bar{D}_a\bar{A}^{ab} = \bar{h}_{ab}\bar{A}^{ab} = 0$, where $\bar{D}_a$ is the covariant derivative with respect to the metric $\bar{h}_{ab}$. We require that the matter data $(\hat{y}, \hat{v}^a)$, will satisfy the condition
$$0 \leq \hat{y} < s\left(\sqrt{\hat{h}_{ab}\hat{v}^a\hat{v}^b}\right), \tag{4.22}$$

where $s(\cdot)$ is given by (4.19). The remaining initial data are determined by the constraint equations (4.8) and (4.9), relations (4.10) and Theorem 4.1.

Remark 4.2 *The distinction between the gravitational data $(\bar{h}_{ab}, \bar{A}_{ab})$ and the matter data $(\hat{z}, \hat{j}^b)$ is caused by condition (4.12). For if we make the scaling $\hat{h}_{ab} = \phi^4 \bar{h}_{ab}$, $\hat{z} = \phi^{-8}\bar{z}$, and $\hat{j}^b = \phi^{-10}\bar{j}^b$, then $\hat{v}^b = \phi^{-2}\bar{v}^b$, $\hat{y} = \phi^{-4(\gamma-1)}\bar{y}$ and $\hat{h}_{ab}\hat{v}^a\hat{v}^b = \bar{h}_{ab}\bar{v}^a\bar{v}^b$. Thus, under this conformal transformation, the argument of s in (4.12) is invariant, while the left hand side will be effected. Therefore the free initial data are partially invariant under conformal transformations.*

Now, if we perform the conformal transformation

$$\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}, \tag{4.23}$$

then the scalar curvature with respect to the metric $\hat{h}_{ab}$, $R(\hat{h})$, satisfies

$$-8\Delta_{\bar{h}}\alpha + R(\bar{h})\alpha = R(\hat{h})\alpha^5. \tag{4.24}$$

Therefore, if there exists a nonnegative solution to the equation

$$-\Delta_{\bar{h}}\alpha + \frac{1}{8}R(\bar{h})\alpha = 0, \tag{4.25}$$

then the metric $\hat{h}_{ab}$ given by (4.23) will have zero scalar curvature. We proceed the construction as follow. Let $\hat{A}^{ab} = \alpha^{-10}\bar{A}^{ab}$, $\hat{D}_a$ denotes the covariant derivative with respect to the metric $\hat{h}_{ab}$, since $\hat{D}_a\hat{A}^{ab} = \alpha^{-10}\bar{D}_a\bar{A}^{ab}$, $\hat{A}^{ab}$ is a divergence and trace free 2 tensor.

Assume $\hat{K}$ is a symmetric covariant 2-tensor which satisfies the maximal slice condition, that is $\hat{h}_{ab}\hat{K}^{ab} = 0$. Then we split $\hat{K}$ by writing it for some vector W:

$$\hat{K} = \hat{A} + \hat{\mathcal{L}}(W), \tag{4.26}$$

where $\hat{\mathcal{L}}$ is the Killing field operator

$$\left(\hat{\mathcal{L}}(W)\right)^{ab} = \left(\hat{\pounds}_W \hat{h}\right)^{ab} - \frac{1}{3}\hat{h}^{ab}\mathrm{Tr}\hat{\pounds}_W\hat{h} = \hat{D}_a W^b + \hat{D}_b W^a - \frac{1}{3}\hat{h}^{ab}\mathrm{Tr}\hat{\pounds}_W\hat{h}, \tag{4.27}$$

and $\hat{\pounds}_W\hat{h}$ is the Lie derivative. The vector W must be chosen so that

$$\hat{D}_a\hat{K}^{ab} = \hat{D}_a\left(\hat{\mathcal{L}}(W)\right)^{ab} = \hat{j}^b, \tag{4.28}$$

that is, W is a solution to the Lichnerowicz Laplacian system

$$\left(\Delta_{L_{\hat{h}}}W\right)^b := \hat{D}_a\left(\hat{\mathcal{L}}(W)\right)^{ab} = \Delta_{\hat{h}}W + \frac{1}{3}\hat{D}^b\left(\hat{D}_a W^a\right) + \hat{R}^b_a W^a = \hat{j}^b, \tag{4.29}$$

here $\hat{R}^b_a$ is the Ricci curvature tensor with respect to the metric $\hat{h}_{ab}$.

Having solved the Lichnerowicz Laplacian (4.29) we consider the Lichnerowicz equation

$$-\Delta_{\hat{h}}\phi = 2\pi\hat{z}\phi^{-3} + \frac{1}{8}\hat{K}_a^b\hat{K}_b^a\phi^{-7}. \tag{4.30}$$

Now we put $h_{ab} = \phi^4\hat{h}_{ab}$, $K_{ab} = \phi^{-2}\hat{K}_{ab}$, $z = \phi^{-8}\hat{z}$ and $j^b = \phi^{-10}\hat{j}^b$. Since

$$D_aK^{ab} = \phi^{-10}\hat{D}_a\hat{K}^{ab} = \phi^{-10}\hat{j}^b = j^b \tag{4.31}$$

and

$$-\Delta_{\hat{h}}\phi = \phi^5\frac{1}{8}R(h) = \phi^5\left(2\pi z + \frac{1}{8}K_a^bK_b^a\right) = \phi^5\left(2\pi\hat{z}\phi^{-8} + \frac{1}{8}\hat{K}_a^b\hat{K}_b^a\phi^{-12}\right), \tag{4.32}$$

we see that (h_{ab}, K_{ab}, z, j^b) satisfy the constraint equations (4.8) and (4.9). In order that the matter variables and (z, j) satisfy the compatibility conditions (4.6) and (4.7) it is necessary to check that $y = z^{\frac{\gamma-1}{2}} = (\phi^{-8}\hat{z})^{\frac{\gamma-1}{2}} = \phi^{-4(\gamma-1)}\hat{y}$ and $v^b = \frac{j^b}{z} = \phi^{-2}\hat{v}^b$ satisfy condition (4.12). Indeed,

$$0 \leq y < s\left(\sqrt{h_{ab}v^av^b}\right) \Leftrightarrow 0 \leq \phi^{-4(\gamma-1)}\hat{y} < s\left(\sqrt{\hat{h}_{ab}\hat{v}^a\hat{v}^b}\right), \tag{4.33}$$

but since $\phi \geq 1$, $\phi^{-4(\gamma-1)}\hat{y} \leq \hat{y}$ and thus assumption (4.22) assures condition (4.12).

Theorem 4.3 ***(Construction of the gravitational data)*** *Given the free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{y}, \hat{v}^b) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then the gravitational data:*

$$h_{ab} = (\phi\alpha)^4\bar{h}_{ab} \qquad \text{and} \qquad K_{ab} = (\phi\alpha)^{-2}\bar{A}_{ab} + \phi^{-2}\hat{\mathcal{L}}(W)$$

satisfy the constraint equations (4.8) and (4.9) with $z = \phi^{-8}\hat{z}$ and $j^b = \phi^{-10}\hat{j}^b$ as the right hand side. In addition, $(h_{ab} - I) \in H_{s,\delta}$ and $K_{ab} \in H_{s-1,\delta+1}$ and therefore if $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, then these data have the needed regularity so they can serve as initial data for hyperbolic system (2.33) of Einstein gravitation fields.

Proof (of Theorem 4.3)

- The free data are $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b)$, where $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$ a divergence a trace free 2-tensor and $(\hat{y}, \hat{v}^b) \in H_{s-1,\delta+2}$.
- The function α satisfies equation (4.25), so by Theorem 4.6 $\alpha > 0$ and $(\alpha - 1) \in H_{s,\delta}$ provided that $s \geq 2$ and $\delta > -\frac{3}{2}$. Since α is continuous and $\lim_{|x|\to\infty}\alpha(x) - 1 = 0$, there is a compact set D of $\mathbb{R}^3$ such that $\alpha(x) \geq \frac{1}{2}$ for $x \notin D$ and $\min_D \alpha(x) \geq t_0 > 0$.
- The function $F(t) := \frac{1-t}{t}$ has bounded derivatives in $[\min\{t_0, \frac{1}{2}\}, \infty)$, so by Moser type estimate Theorem B.7 $\alpha^{-1} - 1 = \frac{1-\alpha}{\alpha} \in H_{s,\delta}$.

- Now, by algebra (Proposition B.5), $(\hat{h}_{ab} - I) = (\alpha^4 \bar{h}_{ab} - I) \in H_{s,\delta}$ and $\hat{A}^{ab} = \alpha^{-10} A^{ab} \in H_{s-1,\delta+1}$.

- The matter variables $(\hat{z}, \hat{j}^b)$ are given by $\hat{z} = \hat{y}^{\frac{2}{\gamma-1}}$, $\hat{j}^b = \hat{z}\hat{v}^b$. So by Proposition B.6, $\hat{z} \in H_{s-1,\delta+2}$ provided that $\frac{3}{2} < s-1 < \frac{2}{\gamma-1} + \frac{1}{2}$ and also $j^b \in H_{s-1,\delta+2}$ by the algebra property .

- The vector W is a solution of the Lichnerowicz Laplacian (4.29), thus according to Theorem 4.8 below, $W \in H_{s,\delta}$ if $s \geq 2$. Hence $\hat{K}^{ab}$ given in (4.26) belongs to $H_{s-1,\delta+1}$. Again, by Proposition B.5, $\hat{K}^b_a \hat{K}^a_b \in H_{s-2,\delta+2}$ if $s \geq 2$ and $\delta \geq -\frac{3}{2}$.

- Setting $u = \phi - 1$, then Lichnerowicz equation (4.32) becomes
$$- \Delta_{\hat{h}} u = 2\pi \hat{z}(u+1)^{-3} + \frac{1}{8}\hat{K}^b_a \hat{K}^a_b (u+1)^{-7}. \tag{4.34}$$

- So applying Theorem 6.12 with $s' = s$ and $\delta' = \delta$ results that $(\phi - 1) = u \in H_{s,\delta}$ and $(\phi - 1) = u \geq 0$.

■

Combining our results of Section 4.1 with theorem 4.3 we obtain the following corollary:

Corollary 4.4 ***(Construction of the data for the fluid)*** *Given the free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{y}, \hat{v}^b) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, $-\frac{3}{2} < \delta < -\frac{1}{2}$ and $(\hat{y}, \hat{v}^a) \in \Omega$, where Ω is given by (4.12). Then the data of the four velocity vector u^α and the Makino variable w are: $y = \phi^{-4(\gamma-1)}\hat{y}$, $v^b = \phi^{-2}\hat{v}^b$,*
$$(w, \bar{u}^a) := \Phi^{-1}(y, v^a) \qquad \textit{and} \qquad \bar{u}^0 = 1 + h_{ab}\bar{u}^a \bar{u}^b$$
and the data for the energy and momentum densities are: $z = y^{\frac{2}{\gamma-1}}$, $j^a = zv^a$. These data satisfy the compatibility conditions (4.6) and (4.7). In addition, by Moser type estimate Theorem B.7 and Proposition B.5 $(w, \bar{u}^a) \in H_{s-1,\delta+2}$ and $\bar{u}^0 - 1 \in H_{s-1,\delta+2}$ and therefore if $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, then these data have the needed regularity so they can serve as initial data for the hyperbolic system (2.16) for the perfect fluid.

4.3 Solutions to the elliptic systems

This section is devoted to the solutions the linear elliptic systems (4.25) and (4.29). The assumption on the given metric $\bar{h}_{ab}$ is that $(\bar{h}_{ab} - I) \in H_{s,\delta}$. So according to Theorem 6.7 of Section 6, the operator $\Delta_{\bar{h}} : H_{s,\delta} \to H_{s-2,\delta+2}$ is semi Fredholm. In fact, it is an isomorphism, this can be shown in a similar manner to Step 1 of Section 6.3. We now consider the operator
$$L := -\Delta_{\bar{h}} + \frac{1}{8}R(\bar{h}) : H_{s,\delta} \to H_{s-2,\delta+2}, \tag{4.35}$$

which is also semi Fredholm. If $R(\bar{h}) \geq 0$, then L is injective. A weaker condition is that L does not have non-positive eigenvalues, this known as the *Brill-Cantor condition* [8]. The variational formulation of this property is:

Definition 4.5 ***(Brill-Cantor condition)*** *A metric $\bar{h}_{ab}$ satisfies the Brill-Cantor condition if*

$$\inf_{u\neq 0} \frac{\int \left(|Du|^2_{\bar{h}} + \frac{1}{8}R(\bar{h})u^2\right) d\mu_{\bar{h}}}{\|u\|^2_{\bar{h}}} > 0, \tag{4.36}$$

where the infimum is taken over all $u \in C^1_0(\mathbb{R}^3)$, $|Du|^2_{\bar{h}} = \bar{h}^{ab}\partial_a u \partial_b u$, $\|u\|^2_{\bar{h}} = \int u^2 d\mu_{\bar{h}}$ and $\mu_{\bar{h}}$ is the volume element with respect to the metric $\bar{h}_{ab}$.

This condition is invariant under conformal transformations, a fact which has been proved for example in [12]

Theorem 4.6 ***(Construction of a metric having zero scalar curvature)*** *Assume the given metric $\bar{h}_{ab}$ satisfies $(\bar{h}_{ab} - \delta_{ab}) \in H_{s,\delta}$, $s \geq 2$, $\delta > -\frac{3}{2}$ and $\bar{h}_{ab}$ satisfies the Brill-Cantor condition (4.36). Then there exists a scalar function α such that $\alpha - 1 \in H_{s,\delta}$, $\alpha(x) > 0$ and the metric $\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}$ has a scalar curvature zero.*

Proof The desired α is a solution to the elliptic equation (4.25). By setting $u = \alpha + 1$ this equation goes to

$$Lu = -\Delta_{\bar{h}} u + \frac{1}{8}R(\bar{h})u = -\frac{1}{8}R(\bar{h}). \tag{4.37}$$

We define for $\tau \in [0,1]$, $L_\tau u = -\Delta_{\bar{h}} u + \frac{\tau}{8}R(\bar{h})u$. If $L_\tau u = 0$, then by Lemma 6.9, $u \in H_{s,-1}$ so

$$0 = (u, L_\tau u) = \int \left(|Du|^2_{\bar{h}} + \frac{\tau}{8}R(\bar{h})u^2\right) d\mu_{\bar{h}}. \tag{4.38}$$

Now, if $\int R(\bar{h})u^2 d\mu_{\bar{h}} \geq 0$, then obviously (4.38) implies that $u \equiv 0$. Otherwise $\int R(\bar{h})u^2 d\mu_{\bar{h}} < 0$, then there is sequence $\{u_n\} \subset C^\infty_0$ such that $u_n \to u$ in $H_{s,-1}$ - norm and

$$\int \left(|Du|^2_{\bar{h}} + \frac{1}{8}R(\bar{h})u^2\right) d\mu_{\bar{h}} = \lim_n \int \left(|Du_n|^2_{\bar{h}} + \frac{1}{8}R(\bar{h})u_n^2\right) d\mu_{\bar{h}} > 0 \tag{4.39}$$

by the Brill-Cantor condition (4.36). Substituta (4.39) in (4.38) yields

$$0 = \int \left(|Du|^2_{\bar{h}} + \frac{1}{8}R(\bar{h})u^2\right) d\mu_{\hat{h}} + \frac{(\tau - 1)}{8} \int R(\bar{h})u^2 d\mu_{\bar{h}}, \tag{4.40}$$

which is certainly a contradiction. Thus L_τ is injective for each $\tau \in [0,1]$, $L_0 = -\Delta_{\bar{h}}$ is isomorphism, hence $L_1 = -\Delta_{\bar{h}} + \frac{1}{8}R(\bar{h})$ is isomorphism by Theorem 6.8.

Having proved the existence, we now show that $\alpha = u + 1$ is nonnegative. The set $\{x : \alpha(x) < 0\}$ has compact support since $\lim_{x\to\infty} u(x) = 0$ by the embedding Theorem B.13.

So letting $w = -\min(\alpha, 0)$, we have $w \in H_0^1(\mathbb{R}^3)$ and if the set $\{x : \alpha(x) < 0\}$ is not empty, then $w \not\equiv 0$ and then the Brill-Cantor condition gives

$$\int_{\{\alpha<0\}} \left(|Dw|^2_{\bar{h}} + \frac{1}{8} R(\bar{h}) w^2 \right) d\mu_{\bar{h}} > 0. \tag{4.41}$$

On the other hand, according to Definition 6.10 of weak solutions,

$$0 = \int \left((D\alpha, Dw)_{\bar{h}} + \frac{1}{8} R(\bar{h}) \alpha w \right) d\mu_{\bar{h}} = -\int_{\{\alpha<0\}} \left(|Dw|^2_{\bar{h}} + \frac{1}{8} R(\bar{h}) w^2 \right) d\mu_{\bar{h}}. \tag{4.42}$$

So we conclude that $\alpha \geq 0$. Since $\alpha \geq 0$, we have by Harnack's inequality

$$\sup_{B_r} \alpha \leq C \inf_{B_r} \alpha$$

provided that B_r is sufficiently small ball. Hence, the set $\{\alpha(x) = 0\}$ is both open and closed, which is impossible. Thus $\alpha(x) > 0$. ■

Remark 4.7 *The conditions for applying Harnack's inequality to a second order elliptic operator*

$$Lu = \partial_a \left(A_{ab}(x) \partial_b u \right) + C(x) u$$

are boundedness of the coefficients (see e. g. [18]; Section 8) However, following carefully the proofs we found it can be applied also when the zero order coefficient belongs to $L^q_{\mathrm{loc}}(\mathbb{R}^3)$ with $q > \frac{3}{2}$. In local coordinates equation (4.25) takes the form

$$L\alpha = \partial_a \left(\sqrt{|\bar{h}|} \bar{h}^{ab} \partial_b \alpha \right) + \sqrt{|\bar{h}|} R(\bar{h}) \alpha = 0.$$

For $s \geq 2$, $\sqrt{|\bar{h}|}\bar{h}^{ab}$ are bounded and non-degenerate, while $\sqrt{|\bar{h}|} R(\bar{h}) \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$.

We turn now the Lichnerowicz Laplacian system (4.29) for which we present:

Theorem 4.8 ***(Solution of Lichnerowicz Laplacian)*** *Let $\hat{h}_{ab}$ be a Riemannian metric in $\mathbb{R}^3$ so that $(\hat{h} - I) \in H_{s,\delta}$. Let vector $\hat{j}^b \in H_{s-2,\delta+2}$, $s \geq 2$ and $\delta > -\frac{3}{2}$. Then equation (4.29) has a unique solution $W \in H_{s,\delta}$.*

Proof (of theorem 4.8) In order to verify condition (H1) of Section 6.2 we compute the principle symbol of $L_{\Delta_{\hat{h}}}$ in (4.29). For each $\xi_a \in T^*_x M$, the principle symbol is a linear map from E_x to F_x, where E_x and F_x are a fibers in $T_x M$. In local coordinates $\Delta_{\hat{h}} = \hat{h}^{ab} \partial_a \partial_b + \text{lower terms}$ and $D_a = \partial_a + \Gamma(\hat{h}^{ab}, \partial \hat{h}_{ab})$, hence

$$\left(\Delta_{L_{\hat{h}}}(\xi) \right)^b_a = |\xi|^2_{\hat{h}} \delta^b_a + \frac{1}{3} \xi^b \xi_a. \tag{4.43}$$

So

$$\left(\left(\Delta_{L_{\hat{h}}}(\xi) \right) \eta, \eta \right)_{\hat{h}} = \hat{h}^{bc} \left(L_{\Delta_{\hat{h}}}(\xi) \right)^b_a \eta^a \eta^c = |\xi|^2_{\hat{h}} |\eta|^2_{\hat{h}} + \frac{1}{3} (\xi_a \eta^a)^2 \geq |\xi|^2_{\hat{h}} |\eta|^2_{\hat{h}}. \tag{4.44}$$

Thus $\left(\Delta_{L_{\hat{h}}}(\xi)\right)_a^b$ has positive eigenvalue and therefore $L_{\Delta_{\hat{h}}}$ is strongly elliptic. Furthermore, by Proposition B.5 and (B.23) we have that if $(\hat{h}_{ab} - I) \in H_{s,\delta}$, $s \geq 2$ and $\delta > -\frac{3}{2}$, then

$$\Delta_{L_{\hat{h}}} : H_{s,\delta} \to H_{s-2,\delta+2}.$$

Hence, we may apply Theorem 6.8 in order to obtain existence of the elliptic system (4.29). For the given metric $\hat{h}_{ab}$ we define one parameter family of metrics $h_t = (1-t)I + t\hat{h}$, $0 \leq t \leq 1$, and the following associated operators with respect to these metrics: $(D_a)_t$ the covariant derivative, $\mathcal{L}_t$ the Killing operator and $L_t = \Delta_{L_{h_t}} = (D)_t \cdot \mathcal{L}_t$ the Lichnerowicz Laplacian. We want to show that L_t is injective. We recall that $-2\mathcal{L}_t$ is the formal adjoint of D_t (see e. g. [3]), in addition, if $L_t(W) = 0$, then by Lemma 6.9 implies $W \in H_{s,-1}$. Thus we may use integration by parts and get

$$\begin{aligned} 0 = (W, L_t W)_{h_t} &= \int (h_t)_{ab} W^a L_t(W)^b d\mu_{h_t} = \int (h_t)_{ab} W^a (D_c)_t \cdot (\mathcal{L}_t W)^{cb} d\mu_{h_t} \\ &= -2 \int (h_t)_{ab} (h_t)_{dc} (\mathcal{L}_t W)^{ad} (\mathcal{L}_t W)^{cb} \mu_{h_t} = -2 \int |\mathcal{L}_t W)|^2_{h_t} \mu_{h_t} \end{aligned} \tag{4.45}$$

Now, if let $\widetilde{h} = |h_t|^{-\frac{1}{3}} h_t$, then

$$\pounds_W \widetilde{h} = |h_t|^{-\frac{1}{3}} \left(\pounds_W h_t - h_t \frac{2}{3} (D_a)_t W^a \right) = |h_t|^{-\frac{1}{3}} \mathcal{L}_t(W). \tag{4.46}$$

Thus $L_t(W) = \Delta_{L_{h_t}}(W) = 0$ implies $W \equiv 0$ if and only if there are no non-trivial Killing vector fields W in $H_{s,-1}$. This fact has been proved by G. Choquet and Y.Choquet-Bruhat [9] for $s > \frac{7}{2}$, D. Christodoulou and N. O'Murchadha for $s > 3 + \frac{3}{2}$ [14], and Bartnik for $s \geq 2$ [1] (See also Maxwell [29], where he obtained the minimum regularity $s > \frac{3}{2}$). Now $L_0 = \Delta_{L_I}$ is an operator with constant coefficients, so by Lemma 6.5 is an isomorphism. ■

5 Local Existence for Hyperbolic Equations

In this section we prove an existence theorem (locally in time) for quasi linear symmetric hyperbolic system in the $H_{s,\delta}$ spaces. The known existence results in the H^s space of Fisher and Marsden [16] and Kato [23] (see also [38], [37]), cannot be applied to the $H_{s,\delta}$ spaces. The main difficulty here is the establishment of energy estimates for linear hyperbolic systems. In order to achieve it we have defined a specific inner-product in $H_{s,\delta}$ (see Definition 5.3) and in addition the Kato-Ponce commutator estimate [24], [38], has an essential role in our approach. Once the energy estimates have been established in the $H_{s,\delta}$ space, we follow Majda's [27] iteration procedure and show existence, uniqueness and continuity in that norm.

We consider the the Cauchy problem for a quasi linear (uniform) symmetric hyperbolic system of the form

$$\begin{cases} A^0(u;t,x)\partial_t u + \sum_{a=1}^{3} A^a(u;t,x)\partial_a u + B(u;t,x)u + F(u;t,x) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{5.1}$$

under the following assumptions:

(H1) A^α are symmetric matrices for $\alpha = 0, 1, 2, 3$;

(H2) $A^\alpha(u;t,x), B(u;t,x), F(u;t,x) \in C^\infty$ in each variable;

(H3) $(A^0(0;t,\cdot) - I), A^a(0;t,\cdot), B(0;t,\cdot), F(0;t,\cdot) \in H_{s,\delta}$;

(H4) $\partial_t A^0 \in L^\infty$.

The main result of this section is the well posedness of the system (5.1) in $H_{s,\delta}$ spaces:

Theorem 5.1 ***(Well posedness of first order hyperbolic symmetric systems in*** $H_{s,\delta}$***)*** *Let $s > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$ and assume hypotheses (H1)-(H4) hold. If the initial condition u_0 belongs to $H_{s,\delta}$ and satisfies*

$$\frac{1}{\mu}\delta_{\alpha\beta}u_0^\alpha u_0^\beta \leq A^0_{\alpha\beta}u_0^\alpha u_0^\beta \leq \mu\delta_{\alpha\beta}u_0^\alpha u_0^\beta, \qquad \mu \in \mathbb{R}^+ \tag{5.2}$$

then there exits a positive T which depends on the $H_{s,\delta}$-norm of the initial data and there exists a unique $u(t,x)$ a solution to (5.1) which in addition satisfies

$$u \in C([0,T], H_{s,\delta}) \cap C^1([0,T], H_{s-1,\delta+1}). \tag{5.3}$$

Remark 5.2 *Condition (H3) is sometime too restrictive for applications. We may replace it by*

(H3') $(A^0(U^0;t,\cdot) - I), A^a(U^0;t,\cdot), B(U^0;t,\cdot), F(U^0;t,\cdot) \in H_{s,\delta}$,

where U^0 is a constant vector. Setting $u = U^0 + v$, then v satisfies

$$\begin{cases} \tilde{A}^0(v;t,x)\partial_t v = \sum_{a=1}^{3} \tilde{A}^a(u;t,x)\partial_a v + \tilde{B}(v;t,x)v + \tilde{F}(v;t,x) \\ v(x,0) = u_0(x) - U^0 \end{cases}, \tag{5.4}$$

where $\tilde{A}^\alpha(v;t,x) = A^\alpha(U^0 + v;t,x)$, $\tilde{B}(v;t,x) = B(U^0 + v;t,x)$ and $\tilde{F}(v;t,x) = F(U^0 + v;t,x) + \tilde{B}(U^0 + v;t,x)$. The Moser type estimates are valid under assumptions (H3') (see Remark B.10).

5.1 Strategy

We will proceed with the following strategy:

1. The establishment of energy estimates for linear systems in the fractional weighted spaces $H_{s,\delta}$.
2. We approximate the initial data by a C_0^∞ sequence and then construct an iteration process which consists of solutions to a linear system having a C_0^∞ initial data.
3. We show that the sequence which is constructed by the iteration process is bounded in $H_{s,\delta}$-norm and weakly converges to a solution.
4. At the final stage we prove uniqueness and continuity in $H_{s,\delta}$-norm.

5.2 Energy estimates in the fractional weighted spaces

The energy estimates are indispensable means for the proof of well posedness of hyperbolic systems. In order to achieve it we introduce an inner product which depends on a matrix A. We assume $A = A(t,x)$ is $m \times m$ symmetric matrix which satisfies

$$\frac{1}{\mu}U^T U \leq U^T A U \leq \mu U^T U \tag{5.5}$$

for some positive μ. Here B^T denotes the transpose matrix. We recall that $f_\epsilon(x) = f(\epsilon x)$, the sequence $\{\psi_j\}$ is a dyadic resolution of the unity in $\mathbb{R}^3$ which is defined the Appendix A and that $\Lambda^s u = \mathcal{F}^{-1}\left((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u\right)$, where $\mathcal{F}$ denotes the Fourier transform. In this section the expression (5.6) below will serve as a norm of the space $H_{s,\delta}$:

$$\|u\|^2_{H_{s,\delta}} := \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^2 u)_{(2^j)}\|^2_{H^s}. \tag{5.6}$$

Corollary A.5 implies that (5.6) is equivalent to the norm of Definition 3.1.

Definition 5.3 ***(Inner Product)*** *For a symmetric matrix $A = A(t,x)$ which satisfies (5.5) we let*

$$\begin{aligned}
\langle u, v\rangle_{s,\delta,A} &:= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\langle \Lambda^s\left((\psi_j^2 u)_{(2^j)}\right), (A)_{2^j}\Lambda^s\left((\psi_j^2 v)_{(2^j)}\right)\right\rangle_{L^2} \\
&= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \int \left[\Lambda^s\left((\psi_j^2 u)_{(2^j)}\right)\right]^T (A)_{2^j}\left[\Lambda^s\left((\psi_j^2 v)_{(2^j)}\right)\right] dx
\end{aligned} \tag{5.7}$$

and its associated norm $\|u\|^2_{H_{s,\delta,A}} = \langle u, u\rangle_{s,\delta,A}$.

Obviously $\langle u, v\rangle_{s,\delta,A} = \langle v, u\rangle_{s,\delta,A}$ and from (5.5) we obtain the equivalence,

$$\frac{1}{\mu}\|u\|^2_{H_{s,\delta}} \leq \|u\|^2_{H_{s,\delta,A}} \leq \mu\|u\|^2_{H_{s,\delta}}. \tag{5.8}$$

We come now to the crucial estimate of this section.

Lemma 5.4 ***(An energy estimate)*** *Let $s > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$, $A^\alpha = A^\alpha(t,x)$ be $m \times m$ symmetric matrices such that $(A^0(t,\cdot) - I), A^a(t,\cdot) \in H_{s,\delta}$ and A^0 satisfies (5.5). If $u(t) = u(t,\cdot)$ is a C_0^∞ solution of the linear hyperbolic system*

$$A^0(t,x)\partial_t u = \sum_{a=1}^{3} A^a(t,x)\partial_a u, \tag{5.9}$$

then

$$\frac{d}{dt}\|u(t)\|^2_{H_{s,\delta,A^0}} \leq C\left(\mu\|u(t)\|^2_{H_{s,\delta,A^0}} + 1\right), \tag{5.10}$$

where $C = C(\|A^0 - I\|_{H_{s,\delta}}, \|A^a\|_{H_{s,\delta}}, \|\partial_t u\|_{H_{s-1,\delta}}, \|\partial_t A^0\|_{L^\infty})$.

An essential tool for deriving these estimates is the Kato & Ponce Commutator Estimate [24], [38].

Theorem 5.5 ***(Kato and Ponce)*** *For $s > 0$, $f \in H^s \cap C^1$, $g \in H^{s-1} \cap L^\infty$ we have*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2} \leq C\left\{\|\nabla f\|_{L^\infty}\|g\|_{H^{s-1}} + \|f\|_{H^s}\|g\|_{L^\infty}\right\}. \tag{5.11}$$

This estimate will be used term wise in the inner product (5.7).

Proof (of lemma 5.4) Since u is C_0^∞ we may interchange the derivation with respect to t with the inner-product (5.7) and get

$$\begin{aligned}
\frac{d}{dt}\langle u, u\rangle_{s,\delta,A^0} &= 2\langle u, \partial_t u\rangle_{s,\delta,A^0} \\
&+ \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\int \left[\Lambda^s\left((\psi_j^2 u)_{(2^j)}\right)\right]^T \left(\partial_t A^0\right)_{2^j}\left[\Lambda^s\left((\psi_j^2 u)_{(2^j)}\right)\right]dx \\
&\leq 2\langle u, \partial_t u\rangle_{s,\delta,A^0} + \|\partial_t A^0\|_{L^\infty}\left(\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^2 u)_{(2^j)}\|^2_{H^s}\right) \\
&= 2\langle u, \partial_t u\rangle_{s,\delta,A^0} + \|\partial_t A^0\|_{L^\infty}\|u\|^2_{H_{s,\delta}}
\end{aligned} \tag{5.12}$$

We turn now to the hard task of the proof, namely, the estimation of $\langle u, \partial_t u\rangle_{s,\delta,A^0}$. Put

$$E(j) = \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left((A^0)_{2^j}\right)\Lambda^s\left(\left(\psi_j^2\partial_t u\right)_{2^j}\right)\right\rangle_{L^2} \tag{5.13}$$

and let $\{\Psi_k\}$ be the sequence of functions which is defined in the proof of Theorem B.8 ($\Psi_k(x) = \frac{1}{\sum \psi_j(x)}\psi_k(x)$). It follows from the definition of the sequence $\{\psi_j\}$ (see Appendix A) that

$$\Psi_k \psi_j^2 \neq 0 \qquad \text{only when } k = j-3, ..., j+4. \tag{5.14}$$

Hence,

$$\begin{aligned}
E(j) &= \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left((A^0)_{2^j}\right)\Lambda^s\left(\left(\sum_{k=0}^{\infty}\Psi_k\right)_{2^j}\left(\psi_j^2\partial_t u\right)_{2^j}\right)\right\rangle_{L^2} \\
&= \sum_{k=j-3}^{j+4}\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left((A^0)_{2^j}\right)\Lambda^s\left(\left(\Psi_k\right)_{2^j}\left(\psi_j^2\partial_t u\right)_{2^j}\right)\right\rangle_{L^2} \\
&= \sum_{k=j-3}^{j+4}\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), (A^0)_{2^j}\left[\Lambda^s\left(\left(\Psi_k\right)_{2^j}\left(\psi_j^2\partial_t u\right)_{2^j}\right) - \left(\Psi_k\right)_{2^j}\Lambda^s\left(\psi_j^2\partial_t u\right)_{2^j}\right]\right\rangle_{L^2} \\
&+ \sum_{k=j-3}^{j+4}\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), (\Psi_k A^0)_{2^j}\Lambda^s\left(\psi_j^2\partial_t u\right)_{2^j}\right\rangle_{L^2} \\
&= E_1(j,k) + E_2(j,k).
\end{aligned}$$

This splitting will enable us to estimate $E_2(j,k)$ in terms of the $H_{s,\delta}$ norm of $A^0 - I$ while by Theorem 5.5,

$$\begin{aligned}
& |E_1(j,k)| \\
\leq\; & \left\|\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right\|_{L^2}\left\|A^0\right\|_{L^\infty}\left\|\Lambda^s\left(\left(\Psi_k\right)_{2^j}\left(\psi_j^2\partial_t u\right)_{2^j}\right) - \left(\Psi_k\right)_{2^j}\Lambda^s\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{L^2} \\
\leq\; & \left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|A^0\right\|_{L^\infty}\left\{\left\|\nabla\left(\Psi_k\right)_{2^j}\right\|_{L^\infty}\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{H^{s-1}} + \left\|\left(\Psi_k\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{L^\infty}\right\} \\
\leq\; & C\left\|A^0\right\|_{L^\infty}\left(\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{H^{s-1}}\right).
\end{aligned} \tag{5.15}$$

In the last step above we have used the below useful estimates. First, by (A.4) and (5.14),

$$\left\|\nabla\left(\Psi_k\right)_{2^j}\right\|_{L^\infty} = 2^j\left\|\nabla\Psi_k\right\|_{L^\infty} \leq C2^j 2^{-k} \leq 8C. \tag{5.16}$$

Secondly, from (A.12) we see that

$$\|f_\epsilon\|_{H^s}^2 \lesssim \begin{cases} \epsilon^{-3}\|f\|_{H^s}^2, & \epsilon \leq 1 \\ \epsilon^{2s-3}\|f\|_{H^s}^2, & \epsilon \geq 1 \end{cases}. \tag{5.17}$$

Recalling that $\psi_k(x) = \psi_1(2^{-k}x)$ and $(\psi_k(x))_{2^j} = (\psi_1(x))_{2^{j-k}}$, applying the above and combing this with (5.14) and Proposition B.1, we have

$$\begin{aligned}\left\|(\Psi_k)_{2^j}\right\|_{H^s} &= \left\|\left(\sum_j \psi_j\right)_{2^j}^{-1} (\psi_k)_{2^j}\right\|_{H^s} \le C\left\|(\psi_k)_{2^j}\right\|_{H^s} \\ &= C\left\|(\psi_1)_{2^{(j-k)}}\right\|_{H^s} \le C2^{(s-\frac{3}{2})3}\left\|\psi_1\right\|_{H^s}. \end{aligned} \tag{5.18}$$

Finally, by the Sobolev embedding

$$\|v\|_{L^\infty} \le C\|v\|_{H^s}, \tag{5.19}$$

we obtain $\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{L^\infty} \le C\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{H^{s-1}}$.

In order to use equation (5.9) we split $E_2(j,k)$ as follows:

$$\begin{aligned} E_2(j,k) &= \left\langle \Lambda^s\left((\psi_j^2 u)_{2^j}\right), \left((\Psi_k A^0)_{2^j}\right)\Lambda^s\left((\psi_j^2\partial_t u)_{2^j}\right)\right\rangle_{L^2} \\ &= \left\langle \Lambda^s\left((\psi_j^2 u)_{2^j}\right), \left[(\Psi_k A^0)_{2^j}\Lambda^s\left((\psi_j^2\partial_t u)_{2^j}\right) - \Lambda^s\left((\Psi_k A^0)_{2^j}(\psi_j^2\partial_t u)_{2^j}\right)\right]\right\rangle_{L^2} \\ &+ \left\langle \Lambda^s\left((\psi_j^2 u)_{2^j}\right), \Lambda^s\left((\Psi_k A^0)_{2^j}(\psi_j^2\partial_t u)_{2^j}\right)\right\rangle_{L^2} \\ &= E_3(j,k) + E_4(j,k). \end{aligned}$$

In the estimation of the first term $E_3(j,k)$, the Kato-Ponce commutator estimate (5.11) is being used again:

$$\begin{aligned} &|E_3(j,k)| \\ \le\ & C\left\|(\psi_j^2 u)_{2^j}\right\|_{H^s}\left\{\left\|\nabla(\Psi_k A^0)_{2^j}\right\|_{L^\infty}\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{H^{s-1}} + \left\|(\Psi_k A^0)_{2^j}\right\|_{H^s}\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{L^\infty}\right\}. \end{aligned}$$

From (5.16) and the embedding (5.19), we have

$$\begin{aligned} \left\|\nabla(\Psi_k A^0)_{2^j}\right\|_{L^\infty} &= 2^j\left\|\left(\nabla\left(\Psi_k A^0 - I\right)\right)_{2^j}\right\|_{L^\infty} + 2^j\left\|\nabla(\Psi_k)_{2^j}\right\|_{L^\infty} \\ &\le C\left\{2^j\left\|\left(\nabla\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^{s-1}} + 1\right\} \end{aligned}$$

and from (5.18)

$$\left\|(\Psi_k A^0)_{2^j}\right\|_{H^s} \le \left\|\left(\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^s} + \left\|\nabla(\Psi_k)_{2^j}\right\|_{H^s} \le \left\|\left(\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^s} + C.$$

Thus

$$\begin{aligned}
&|E_3(j,k)| \\
\leq\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \quad \left\|\left(\psi_j^2 \partial_t u\right)_{2^j}\right\|_{H^{s-1}} \quad \left\{2^j \left\|\left(\nabla \Psi_k \left(A^0 - I\right)\right)_{2^j}\right\|_{H^{s-1}} + 1\right\} \\
+\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \left\|\left(\psi_j^2 \partial_t u\right)_{2^j}\right\|_{L^\infty} \left\{\left\|\left(\Psi_k \left(A^0 - I\right)\right)_{2^j}\right\|_{H^s} + 1\right\} \\
\leq\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \quad \left\|\left(\psi_j^2 \partial_t u\right)_{2^j}\right\|_{H^{s-1}} \left\{2^j \left\|\left(\nabla \Psi_k \left(A^0 - I\right)\right)_{2^j}\right\|_{H^{s-1}} + 1\right\} \qquad (5.20) \\
+\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \left\|\left(\Psi_k \left(A^0 - I\right)\right)_{2^j}\right\|_{H^s} \left\|\partial_t u\right\|_{L^\infty} \\
+\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \left\|\left(\psi_j^2 \partial_t u\right)_{2^j}\right\|_{H^{s-1}}.
\end{aligned}$$

Now equation (5.9) is being utilized and

$$\begin{aligned}
E_4(j,k) &= \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k \psi_j^2\right)_{2^j}\left(A^0 \partial_t u\right)\right)_{2^j}\right\rangle_{L^2} \\
&= \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k \psi_j^2\right)_{2^j}\left(\sum_{a=1}^{3} A^a \partial_a u\right)_{2^j}\right)\right\rangle_{L^2} \\
&= \sum_{a=1}^{3}\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k A^a\right)_{2^j}\left(\psi_j^2 \partial_a u\right)_{2^j}\right)\right\rangle_{L^2} \qquad (5.21) \\
&= \sum_{a=1}^{3}\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left[\Lambda^s\left(\left(\Psi_k A^a\right)_{2^j}\left(\psi_j^2 \partial_a u\right)_{2^j}\right) - \left(\Psi_k A^a\right)_{2^j}\Lambda^s\left(\left(\psi_j^2 \partial_a u\right)_{2^j}\right)\right]\right\rangle_{L^2} \\
&+ \sum_{a=1}^{3}\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left[\left(\Psi_k A^a\right)_{2^j}\Lambda^s\left(\left(\psi_j^2 \partial_a u\right)_{2^j}\right)\right]\right\rangle_{L^2} \\
&= E_5(j,k,a) + E_6(j,k,a).
\end{aligned}$$

Again, by Kato-Ponce commutator estimate (5.11),

$$\begin{aligned}
&|E_5(j,k,a)| \\
\leq\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\{\left\|\nabla\left(\Psi_k A^a\right)_{2^j}\right\|_{L^\infty}\left\|\left(\psi_j^2 \partial_a u\right)_{2^j}\right\|_{H^{s-1}} + \left\|\left(\Psi_k A^a\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2 \partial_a u\right)_{2^j}\right\|_{L^\infty}\right\} \\
\leq\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\{\left\|\nabla A^a\right\|_{L^\infty} + \left\|A^a\right\|_{L^\infty}\right\} 2^j \left\|\left(\psi_j^2 \partial_a u\right)_{2^j}\right\|_{H^{s-1}} \qquad (5.22) \\
+\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\Psi_k A^a\right)_{2^j}\right\|_{H^s} \quad \left\|\partial_a u\right\|_{L^\infty}.
\end{aligned}$$

Using the commutation $\partial_a \Lambda^s = \Lambda^s \partial_a$, the symmetry of A^a and the fact that $\Lambda^s\left(\psi_j^2 u\right)$ is rapidly decreasing, we calculate $E_6(j,k,a)$ as follows:

$$\begin{aligned}
0 &= \int \partial_a \left\{ \left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]^T \left(\Psi_k A^a\right)_{2^j} \left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]\right\} dx \\
&= 2^j \int \left\{ \left[\Lambda^s\left(\left(\psi_j^2 \partial_a u\right)_{2^j}\right)\right]^T \left(\Psi_k A^a\right)_{2^j} \left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]\right\} dx \\
&+ 2^j \int \left\{ \left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]^T \left(\Psi_k A^a\right)_{2^j} \left[\Lambda^s\left(\left(\psi_j^2 \partial_a u\right)_{2^j}\right)\right]\right\} dx \\
&+ 2^j 2 \int \left\{ \left[\Lambda^s\left(\left(\left(\partial_a \psi_j\right) \psi_j u\right)_{2^j}\right)\right]^T \left(\Psi_k A^a\right)_{2^j} \left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]\right\} dx \\
&+ 2^j 2 \int \left\{ \left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]^T \left(\Psi_k A^a\right)_{2^j} \left[\Lambda^s\left(\left(\left(\partial_a \psi_j\right) \psi_j u\right)_{2^j}\right)\right]\right\} dx \\
&+ 2^j \int \left\{ \left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]^T \left(\partial_a\left(\Psi_k A^a\right)\right)_{2^j} \left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]\right\} dx.
\end{aligned}$$

Since A^a is a symmetric matrix, the first and the second terms are equal to $E_6(j,k,l)$, and the third is equal to the forth one. Hence by Proposition B.1 and Cauchy Schwarz inequality,

$$\begin{aligned}
|2E_6(j,k,a)| &\leq 2\left\|\left(\Psi_k A^a\right)_{2^j}\right\|_{L^\infty} \left\|\left(\partial_a \psi_j \psi_j u\right)_{2^j}\right\|_{H^s} \quad \left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \\
&+ \left\|\left(\partial_a\left(\Psi_k A^a\right)\right)_{2^j}\right\|_{L^\infty} \left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}^2 \\
&\leq C\left\|A^a\right\|_{L^\infty} \left\|\left(\psi_j u\right)_{2^j}\right\|_{H^s} \quad \left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \\
&+ \left\{\left\|\partial_a A^a\right\|_{L^\infty} + C\left\|A^a\right\|_{L^\infty}\right\} \left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}^2 .
\end{aligned}$$

Taking the sum $\sum 2^{(\frac{3}{2}+\delta)2j} E(j)$ we are coming across three types of summations:

1. Given $v \in H_{s_1,\delta}$, $w \in H_{s_2,\delta}$ and γ_i equals 1 or 2, then

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\|\left(\psi_j^{\gamma_1} v\right)_{2^j}\right\|_{H^{s_1}} \left\|\left(\psi_j^{\gamma_2} w\right)_{2^j}\right\|_{H^{s_2}} \\
&\leq \frac{1}{2}\left(\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\|\left(\psi_j^{\gamma_1} v\right)_{2^j}\right\|_{H^{s_1}}^2 + 2^{(\frac{3}{2}+\delta)2j} \left\|\left(\psi_j^{\gamma_2} w\right)_{2^j}\right\|_{H^{s_2}}^2\right) \\
&\leq C\left(\|v\|_{H_{s_1,\delta}}^2 + \|w\|_{H_{s_2,\delta}}^2\right),
\end{aligned}$$

where in the last inequality the equivalence of the norms (A.18) was involved.

2. Given $v \in H_{s,\delta}$ and $w \in H_{s,\delta}$, then from the scaling property (5.17) and Proposition B.1 we have

$$
\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \left\|(\psi_j^2 v)_{2^j}\right\|_{H^s} \|(\Psi_k w)_{2^j}\|_{H^s} \\
\leq\ & \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \left\|(\psi_j^2 v)_{2^j}\right\|_{H^s}^2 + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \|(\Psi_k w)_{2^j}\|_{H^s}^2 \\
\leq\ & \frac{7}{2} \|v\|_{H_{s,\delta}}^2 + C \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \|(\Psi_k w)_{2^k}\|_{H^s}^2 \\
\leq\ & \frac{7}{2} \|v\|_{H_{s,\delta}}^2 + C \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2k} \|(\psi_k w)_{2^k}\|_{H^s}^2 \\
\leq\ & C \left(7\|v\|_{H_{s,\delta}}^2 + 7\|w\|_{H_{s,\delta}}^2\right).
\end{aligned}
$$

3. Given $v \in H_{s_1,\delta}$, $w \in H_{s_2,\delta}$, $z \in H_{s_3,\delta}$ and γ_i equals 1 or 2, then by Hölder inequality and the same arguments as in type 2, we get

$$
\begin{aligned}
& \sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j}\left\|(\psi_j^{\gamma_1}v)_{2^j}\right\|_{H^{s_1}}\left\|(\psi_j^{\gamma_2}w)_{2^j}\right\|_{H^{s_2}} 2^j\left\|(\nabla(\Psi_k z))_{2^j}\right\|_{H^{s_3-1}} \\
\leq\ & \sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)j}\left\|(\psi_j^{\gamma_1}v)_{2^j}\right\|_{H^{s_1}} 2^{(\frac{3}{2}+\delta)j}\left\|(\psi_j^{\gamma_2}w)_{2^j}\right\|_{H^{s_2}} 2^{(\frac{3}{2}+\delta+1)j}\left\|(\nabla(\Psi_k z))_{2^j}\right\|_{H^{s_3-1}} \\
\leq\ & \left(\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4}\left(2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^{\gamma_1}v\right)_{2^j}\right\|_{H^{s_1}}^2\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
\times\ & \left(\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4}\left(2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^{\gamma_2}w\right)_{2^j}\right\|_{H^{s_2}}^2\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
\times\ & C\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{\left(\frac{3}{2}+\delta+1\right)2j}\left\|(\nabla(\psi_k z))_{2^k}\right\|_{H^{s_3-1}}^2\right)^{\frac{1}{2}} \\
\leq\ & \left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^{\gamma_1}v\right)_{2^j}\right\|_{H^{s_1}}^2\right)^{\frac{1}{2}} \\
\times\ & \left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^{\gamma_2}w\right)_{2^j}\right\|_{H^{s_2}}^2\right)^{\frac{1}{2}} \\
\times\ & C\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{\left(\frac{3}{2}+\delta+1\right)2k}\left\|(\nabla(\psi_k z))_{2^k}\right\|_{H^{s_3-1}}^2\right)^{\frac{1}{2}} \\
\leq\ & C\|v\|_{H_{s_1,\delta}}\quad\|w\|_{H_{s_2,\delta}}\quad\|\nabla z\|_{H_{s_3-1,\delta+1}} \\
\leq\ & C\|v\|_{H_{s_1,\delta}}\quad\|w\|_{H_{s_2,\delta}}\quad\|z\|_{H_{s_3,\delta}} \\
\leq\ & C\left(\|v\|_{H_{s_1,\delta}}^2+\left(\|w\|_{H_{s_2,\delta}}\|z\|_{H_{s_3,\delta}}\right)^2\right).
\end{aligned}
$$

Applying these three types of inequalities we have,

$$
\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j}|2E_6(j,k,a)| \leq C\left(\|A^a\|_{L^\infty}+\|\partial_a A^a\|_{L^\infty}\right)\|u\|_{H_{s,\delta}}^2\,, \tag{5.23}
$$

$$
\begin{aligned}
& \sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j}|E_5(j,k,a)| \leq C\left(\|\nabla A^a\|_{L^\infty}+\|A^a\|_{L^\infty}\right)\left\{\|u\|^2_{H_{s,\delta}}+\|\partial_a u\|^2_{H_{s-1,\delta+1}}\right\} \\
+ & \quad C\left\{\|u\|^2_{H_{s,\delta}}+\|A^a\|^2_{H_{s,\delta}}\|\partial_a u\|^2_{L^\infty}\right\} \leq C\left(\|\nabla A^a\|_{L^\infty}+\|A^a\|_{L^\infty}\right)\left\{\|u\|^2_{H_{s,\delta}}+\|u\|^2_{H_{s,\delta}}\right\} \\
+ & \quad C\left\{\|u\|^2_{H_{s,\delta}}+\|A^a\|^2_{H_{s,\delta}}\|\partial_a u\|^2_{H_{s-1,\delta+1}}\right\} \qquad (5.24) \\
\leq & \quad C\left\{2\|\nabla A^a\|_{L^\infty}+2\|A^a\|_{L^\infty}+\|A^a\|^2_{H_{s,\delta}}+1\right\}\|u\|^2_{H_{s,\delta}},
\end{aligned}
$$

here we have applied the embedding (B.30) to $\|\partial_a u\|_{L^\infty}$. Applying the same to $\|\partial_t u\|_{L^\infty}$ we have

$$
\begin{aligned}
& \sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j}|E_3(j,k)| \\
\leq & \quad C\left\{\|u\|^2_{H_{s,\delta}}+\|\partial_t u\|^2_{H_{s-1,\delta}}\left\|\nabla\left(A^0-I\right)\right\|^2_{H_{s-1,\delta+1}}\right\}+2C\left\{\|u\|^2_{H_{s,\delta}}+\|\partial_t u\|^2_{H_{s-1,\delta}}\right\} \\
+ & \quad C\left\{\|u\|^2_{H_{s,\delta}}+\left\|\left(A^0-I\right)\right\|^2_{H_{s,\delta}}\|\partial_t u\|^2_{L^\infty}\right\} \qquad (5.25) \\
\leq & \quad 2C\left\{\|u\|^2_{H_{s,\delta}}+\|\partial_t u\|^2_{H_{s-1,\delta}}\left(1+\left\|A^0-I\right\|^2_{H_{s,\delta}}\right)\right\}
\end{aligned}
$$

and finally

$$
\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}|E_1(j)| \leq C\left\|A^0\right\|_{L^\infty}\left\{\|u\|^2_{H_{s,\delta}}+\|\partial_t u\|^2_{H_{s-1,\delta}}\right\}. \tag{5.26}
$$

Recalling that

$$
\langle u,\partial_t u\rangle_{s,\delta,A^0}=\sum_{j=0}^{\infty}2^{(\frac{3}{2}+\delta)2j}E(j)=\sum_{j=0}^{\infty}2^{(\frac{3}{2}+\delta)2j}\left\langle\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right),\left((A^0)_{2^j}\right)\Lambda^s\left(\left(\psi_j^2\partial_t u\right)_{2^j}\right)\right\rangle_{L^2},
$$

then inequalities (5.23), (5.24), (5.26) and (5.26) imply that

$$
\langle u,\partial_t u\rangle_{s,\delta,A^0} \leq C\left(\|A^\alpha\|_{L^\infty},\|\nabla A^a\|_{L^\infty},\|A^a\|_{H_{s,\delta}},\|A^0-I\|_{H_{s,\delta}},\|\partial_t u\|_{H_{s-1,\delta}}\right)\left\{\|u\|^2_{H_{s,\delta}}+1\right\}.
$$

Since $s>\frac{5}{2}$ and $\delta\geq-\frac{3}{2}$ we can use Theorem B.13 (of the Appendix B) and bound the norms $\|A^\alpha\|_{L^\infty}$ and $\|\nabla A^\alpha\|_{L^\infty}$ by by the norms $\|A^0-I\|_{H_{s,\delta}}$ and $\|A^a\|_{H_{s,\delta}}$. Thus, combining these bounds with above inequality and inequality (5.12), we have obtained

$$
\frac{d}{dt}\langle u(t),u(t)\rangle_{s,\delta,A^0}\leq C\left(\|u(t)\|^2_{H_{s,\delta}}+1\right), \tag{5.27}
$$

where $C=C(\|A^a\|_{H_{s,\delta}},\|A^0-I\|_{H_{s,\delta}},\|\partial_t u\|_{H_{s-1,\delta}},\|\partial_t A^0\|_{L^\infty})$. Inserting the equivalence of norms $\|u\|^2_{H_{s,\delta}}\leq\mu\|u\|^2_{H_{s,\delta,A^0}}$ in (5.27), we obtain (5.10) which completes the proof of Lemma 5.4. ∎

We may extend the energy estimate (5.10) to a non-homogeneous symmetric hyperbolic systems.

Lemma 5.6 ***(An energy estimate)*** *Let $s > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$, $A^\alpha = A^\alpha(t,x)$ be $m \times m$ symmetric matrices such that $(A^0(t,\cdot) - I), A^a(t,\cdot) \in H_{s,\delta}$ and A^0 satisfies (5.5). Let $B(t,\cdot)$, $F(t,\cdot) \in H_{s,\delta}$. If $u(t,\cdot)$ is a C_0^∞ solution of the linear hyperbolic system*

$$A^0(t,x)\partial_t u = \sum_{a=1}^{3} A^a(t,x)\partial_a u + B(t,x)u + F(t,x), \tag{5.28}$$

then

$$\frac{d}{dt}\|u(t)\|^2_{H_{s,\delta,A^0}} \leq C\left(\mu\|u(t)\|^2_{H_{s,\delta,A^0}} + 1\right), \tag{5.29}$$

where the constant C depends on $\|A^a\|_{H_{s,\delta}}$, $\|A^0 - I\|_{H_{s,\delta}}$, $\|\partial_t u\|_{H_{s-1,\delta}}$, $\|\partial_t A^0\|_{L^\infty}$, $\|B\|_{H_{s,\delta}}$ and $\|F\|_{H_{s,\delta}}$.

Proof (of Lemma 5.6) This proof is precisely as the previous one expect the two terms

$$\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k B\right)_{2^j}\left(\psi_j^2 u\right)_{2^j}\right)\right\rangle_{L^2} \tag{5.30}$$

and

$$\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k \psi_j^2 F\right)_{2^j}\right)\right\rangle_{L^2} \tag{5.31}$$

which are added to (5.21). Using the álgebra properties of H^s spaces, we see that (5.30) is less than

$$C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|^2_{H^s}\|(\Psi_k B)_{2^j}\|_{H^s} \leq C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|^2_{H^s}\|B\|_{H_{s,\delta}};$$

and by Cauchy Schwarz inequality (5.31) is less than

$$C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\Psi_k\psi_j^2 F\right)_{2^j}\right\|_{H^s} \leq C\frac{1}{2}\left\{\left\|\left(\psi_j^2 u\right)_{2^j}\right\|^2_{H^s} + \left\|\left(\psi_j^2 F\right)_{2^j}\right\|^2_{H^s}\right\}.$$

Multiplying (5.30) and (5.31) by $2^{(\frac{3}{2}+\delta)2j}$ and taking the sum, it results with two quantities less than $\|u\|^2_{H_{s,\delta}}\|B\|_{H_{s,\delta}}$ and $\left(\|u\|^2_{H_{s,\delta}} + \|F\|^2_{H_{s,\delta}}\right)$ respectively. ■

5.3 Construction of the iteration

We assume $u_0(x)$, the initial value of (5.1), is contained in G_1, where the origin belongs to G_1 and G_1 is a compact subset of an open set G of $\mathbb{R}^m$. In addition we assume,

$$\frac{1}{\mu}U^T U \leq U^T A^0 U \leq \mu U^T U \qquad \text{for all} \quad U \in G_2, \tag{5.32}$$

where G_2 is a compact set of G such that $G_1 \Subset G_2$ and $\mu > 0$.

Remark 5.7 *Since the matrix A^0 is continuous, the initial condition (5.2) guarantees the existence of a domain G_2.*

The initial data u_0 will be approximated by a sequence $\{u_0^k\}$ of smooth functions with compact support, which converges to u_0 in $H_{s,\delta}(\mathbb{R}^3)$. It follows from the embedding $\|v\|_{L^\infty} \leq C\|v\|_{H_{s,\delta}}$ and the density Theorem B.14 that there is a positive R, $u_0^0 \in C_0^\infty(\mathbb{R}^3)$ and $\{u_0^k\}_{k=1}^\infty \subset C_0^\infty(\mathbb{R}^3)$ such that

$$\|u_0^0\|_{H_{s+1,\delta}} \leq C\|u_0\|_{H_{s,\delta}}, \tag{5.33}$$

$$\|u_0^0 - u_0\|_{H_{s,\delta}} \leq \frac{R}{\mu 8}, \tag{5.34}$$

$$\|u - u_0^0\|_{H_{s,\delta}} \leq R \Rightarrow u \in G_2 \tag{5.35}$$

and

$$\|u_0^k - u_0\|_{H_{s,\delta}} \leq 2^{-k}\frac{R}{\mu 8}. \tag{5.36}$$

The iteration procedure is defined as follows: $u^0(t,x) = u_0^0(x)$ and $u^{k+1}(t,x)$ is a solution to the linear initial value problem

$$\begin{cases} A^0(u^k;t,x)\partial_t u^{k+1} = \displaystyle\sum_{a=1}^{3} A^a(u^k;t,x)\partial_a u^{k+1} + B(u^k;t,x)u^{k+1} + F(u^k;t,x), \\ u^{k+1}(x,0) = u_0^{k+1}(x). \end{cases} \tag{5.37}$$

The existence of $\{u^k(t,x)\} \subset C_0^\infty(\mathbb{R}^3)$ follows from:

Theorem 5.8 ***(Existence of classical solutions of a linear symmetric hyperbolic system)*** *Let A^α, B and F be C^∞ functions and $v_0 \in C_0^\infty(\mathbb{R}^3)$ be an initial datum. Then the linear system*

$$\begin{cases} A^0(t,x)\partial_t v = \displaystyle\sum_{a=1}^{3} A^a(t,x)\partial_a v + B(t,x)v + F(t,x) \\ v(x,0) = v_0(x) \end{cases} \tag{5.38}$$

has a unique solution $v(t,x)$ such that $v(t,x) \in C^\infty$ and it has compact support in $\mathbb{R}^3$ for each fixed t.

For the proof we refer to John [21]. It is evident from these facts, inequalities (5.32) and (5.35) that for each k, $u^k(t,x)$ is well defined, $u^k(t,x) \in C^\infty$, $u^k(t,x)$ has compact support in $\mathbb{R}^3$ and $u^k(t,x) \in G_2$ for some positive T. We put

$$T_k = \sup\{T : \sup_{0<t<T} \|u^k(t) - u_0^0\|_{H_{s,\delta}} \leq R\}. \tag{5.39}$$

Our next issue is to show the existence of $T^* > 0$ such that $T_k \geq T^*$ for $k = 1,2,3,...$

5.4 Boundness in the $H_{s,\delta}$ norm

We introduce the following notations: $u(t) := u(t,x)$ and

$$|||u|||_{s,\delta,T} := \sup\{\|u(t)\|_{H_{s,\delta}} : 0 \leq t \leq T\}.$$

The main result of this subsection is:

Lemma 5.9 ***(Boundness in the $H_{s,\delta}$ norm)*** *There are positive constants T^* and L such that*

(A) $|||u^k - u_0^0|||_{s,\delta,T^*} \leq R$

(B) $|||\partial_t u^k|||_{s-1,\delta+1,T^*} \leq L.$

Proof (of lemma 5.9) We first prove (B). Let

$$G^{k+1} = \sum_{a=1}^{3} A^a(u^k;t,x)\partial_a u^{k+1} + B(u^k;t,x)u^{k+1} + F(u^k;t,x),$$

then by the algebra property (B.12) and Moser type estimate (B.23),

$$\begin{aligned}
& \|G^{k+1}\|_{H_{s-1,\delta+1}} \\
\leq\ & \sum_{a=1}^{3} \|A^a(u^k)\|_{H_{s,\delta}} \|\partial_a u^k\|_{H_{s-1,\delta+1}} + \|B(u^k)\|_{H_{s,\delta}} \|u^k\|_{H_{s,\delta}} + \|F(u^k)\|_{H_{s,\delta}} \\
\leq\ & \sum_{a=1}^{3} \left(C\|u^k\|_{H_{s,\delta}} + \|A^a(0)\|_{H_{s,\delta}}\right) \|u^k\|_{H_{s,\delta}} + \left(C\|u^k\|_{H_{s,\delta}} + \|B(0)\|_{H_{s,\delta}}\right) \|u^k\|_{H_{s,\delta}} \\
+\ & C\|u^k\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}}.
\end{aligned} \tag{5.40}$$

The constant C here depends on $\|A^a\|_{C^{N+1}(G_2)}$, $\|B\|_{C^{N+1}(G_2)}$, $\|F\|_{C^{N+1}(G_2)}$ and $\|u^k\|_{L^\infty}$ (see (B.18)). Since

$$\|u^k(t)\|_{H_{s,\delta}} \leq \|u^k(t) - u_0^0\|_{H_{s,\delta}} + \|u_0^0\|_{H_{s,\delta}}, \tag{5.41}$$

the induction assumption (A) and the inequality (5.33) imply that $\|u^k\|_{H_{s,\delta}} \leq R + C\|u_0\|_{H_{s,\delta}}$. Using the embedding $\|u^k\|_{L^\infty} \leq C\|u^k\|_{H_{s,\delta}}$, we see that $\|G^{k+1}\|_{H_{s-1,\delta+1}} \leq C_1(R)$, where the constant $C_1(R)$ depends upon R, condition (H3) and the initial data, but it is independent of k. From (5.37) we have

$$\partial_t u^{k+1} = \left(A^0(u^k;t,x)\right)^{-1} G^{k+1} = \left(\left(A^0(u^k;t,x)\right)^{-1} - I\right) G^{k+1} + G^{k+1}.$$

Repeating same arguments as above, we conclude that

$$\|\left(\left(A^0(u^k;t,x)\right)^{-1} - I\right) G^{k+1}\|_{H_{s-1,\delta+1}} \leq C_2(R)$$

and the constant $C_2(R)$ does not depends on k. We take $L = C_1(R) + C_2(R)$. Here we have used Moser estimate with $F(u) = A^{-1}(u) - I$, and the formula $\frac{\partial A^{-1}(u)}{\partial u} = A^{-1}(u)\frac{\partial A(u)}{\partial u}A^{-1}(u)$. Thus the constant $C(R)$ depends on $\|A^0\|_{C^{N+2}(G_2)}$ and μ.

We turn now to show (A). Let $V^{k+1} = u^{k+1} - u_0^0$, then inserting it in the equation (5.37) we have obtained

$$
\begin{aligned}
A^0(u^k;t,x)\partial_t V^{k+1} &= A^0(u^k;t,x)u_t^{k+1} = \sum_{a=1}^{3} A^a(u^k;t,x)\partial_a u^{k+1} + B(u^k;t,x)u^k + F(u^k;t,x) \\
&= \sum_{a=1}^{3} A^a(u^k;t,x)\partial_a V^{k+1} + B(u^k;t,x)V^{k+1} + F(u^k;t,x) \qquad (5.42)\\
&+ \sum_{a=1}^{3} A^a(u^k;t,x)\partial_a u_0^0 + B(u^k;t,x)u_0^0
\end{aligned}
$$

and $V^{k+1}(x,0) = u_0^{k+1}(x,0) - u_0^0(x)$. At this stage we would like employ the energy estimate Lemma 5.6. Due the the fact that the coefficients of (5.42) depend on u^k, it is obligatory to control the constant of (5.29) in terms of $\|u^k\|_{H_{s,\delta}}$. Therefore we need to bound $\|\left(A^0(u^k;t,x) - I\right)\|_{H_{s,\delta}}$, $\|A^a(u^k;t,x)\|_{H_{s,\delta}}$, $\|B(u^k;t,x)\|_{H_{s,\delta}}$, $\|F(u^k;t,x)\|_{H_{s,\delta}}$ and $\|\frac{\partial}{\partial t}A^0(u^k;t,x)\|_{L^\infty}$ by $\|u^k\|_{H_{s,\delta}}$. The first four are similar, so take for example $A^a(u^k;t,x)$: We use assumption (H2), Moser type estimate (B.19) and Remark B.10, then

$$
\|A^a(u^k;t,x)\|_{H_{s,\delta}} \leq C\left\{\|A^a\|_{C^{N+1}(G_2)}\left(1 + \|u^k\|_{L^\infty}^N\right)\right\}\|u^k\|_{H_{s,\delta}} + \|A^a(0;t,\cdot)\|_{H_{s,\delta}}. \qquad (5.43)
$$

For the last one we have

$$
\begin{aligned}
&\|\frac{\partial}{\partial t}A^0(u^k;t,x)\|_{L^\infty} = \|\frac{\partial}{\partial u}A^0(u^k;t,x)\partial_t u^k(t,x) + \partial_t A^0(u^k;t,x)\|_{L^\infty} \\
&\leq \|\frac{\partial}{\partial u}A^0(u^k;t,x)\|_{L^\infty}\|\partial_t u^k(t,x)\|_{L^\infty} + \|\partial_t A^0(u^k;t,x)\|_{L^\infty} \qquad (5.44)\\
&\leq C\|\frac{\partial}{\partial u}A^0(u^k;t,x)\|_{L^\infty}\|\partial_t u^k(t,x)\|_{H_{s-1,\delta+1}} + \|\partial_t A^0(u^k;t,x)\|_{L^\infty}.
\end{aligned}
$$

We conclude from inequalities (5.43) and (5.44), the inductions hypothesis (A) and (B), (5.35) and (H4) that the constant of (5.29) depends on R, L, $\|u_0\|_{H_{s,\delta}}$ and the $H_{s,\delta}$-norm of the coefficients, but it is independent of k. Hence, the energy estimate Lemma 5.6 implies that

$$
\frac{d}{dt}\|V^{k+1}(t)\|_{H_{s,\delta,A^0}}^2 \leq C(R,L)\left(\mu\|V^{k+1}(t)\|_{H_{s,\delta,A^0}}^2 + 1\right), \qquad (5.45)
$$

Applying Gronwall's inequality, (5.34), (5.36) and the equivalence (5.8) results in

$$\begin{aligned}
|||V^{k+1}|||^2_{s,\delta,T} &\leq \mu e^{C(R,L)\mu T}\left(\mu\|V^{k+1}(0)\|^2_{H_{s,\delta}} + T\right) \\
&= \mu e^{C(R,L)\mu T}\left(\mu\|u_0^{k+1} - u_0^0\|^2_{H_{s,\delta}} + T\right) \\
&\leq \mu e^{C(R,L)\mu T}\left(\mu\left(\|u_0^{k+1} - u_0\|^2_{H_{s,\delta}} + \|u_0^0 - u_0\|^2_{H_{s,\delta}}\right) + T\right) \\
&\leq e^{C(R,L)\mu T}\left(2\mu^2\left(\frac{R}{\mu 8}\right)^2 + \mu T\right). \qquad (5.46)
\end{aligned}$$

Therefore $|||V^{k+1}|||^2_{s,\delta,T} \leq R^2$, if

$$T \leq \frac{1}{\mu C(R,L)}\log\left(\frac{R^2}{\left(\frac{R^2}{32} + \mu T\right)}\right) \leq \frac{\log(32)}{\mu C(R,T)}.$$

Thus taking $T^* = \frac{\log(32)}{\mu C(R,T)}$ proves (A) and completes the proof of Lemma 5.9. ■

5.5 Contraction in the lower norm

We show here that $\left\{u^k\right\}$ has a contraction property in $\|\cdot\|_{0,\delta,T^{**}}$ for a positive T^{**}. In order to achieved it we need an energy estimate in $H_{0,\delta} \backsim L^2_\delta$. For that purpose we introduce the below inner-product in L^2_δ: for two vectors u and v in L^2_δ, we set

$$\langle u, v\rangle_{L^2_\delta, A^0} = \int (1+|x|)^{2\delta}\left(u^T A^0 v\right) dx, \qquad (5.47)$$

and its associated norm $\|u\|^2_{L^2_\delta, A^0} = \langle u, u\rangle_{L^2_\delta, A^0}$. The ordinary norm is denoted by $\|u\|^2_{L^2_\delta} = \langle u, u\rangle_{L^2_\delta, I}$. Since A^0 satisfies (5.32),

$$\frac{1}{\mu}\|u\|^2_{L_\delta} \leq \langle u, u\rangle_{L^2_\delta, A^0} \leq \mu\|u\|^2_{L_\delta}, \qquad (5.48)$$

and hence by Theorem A.2, $\|u\|^2_{L^2_\delta, A^0} \simeq \|u\|_{H_{0,\delta}}$.

Proposition 5.10 ***(Energy estimate in*** L^2_δ***)*** *Suppose u satisfies the linear hyperbolic system (5.38), then*

$$\frac{d}{dt}\langle u(t), u(t)\rangle_{L^2_\delta, A^0} \leq \mu C\langle u(t), u(t)\rangle_{L^2_\delta, A^0} + \|F\|^2_{L^2_\delta}, \qquad (5.49)$$

where $C = C(\|\partial_t A^0\|_{L^\infty}, \|A^a\|_{L^\infty_{-1}}, \|B\|_{L^\infty}, \|\partial_a A^a\|_{L^\infty})$.

Proof (of Proposition 5.10) Taking the derivative of (5.47) with respect to t, we get

$$\begin{aligned}
\frac{d}{dt}\langle u,u\rangle_{L^2_\delta,A^0} &= 2\langle u,\partial_t u\rangle_{L^2_\delta,A^0}+\int(1+|x|)^{2\delta}\left(u^T\partial_t A^0 u\right)dx\\
&= 2\sum_{a=1}^{3}\int(1+|x|)^{2\delta}\left(u^TA^a\partial_a u\right)dx+2\int(1+|x|)^{2\delta}\left(u^TBu\right)dx\\
&+ 2\int(1+|x|)^{2\delta}\left(u^TF\right)dx+\int(1+|x|)^{2\delta}\left(u^T\partial_t A^0u\right)dx\\
&= 2\sum_{a=1}^{3}L_{1,a}+2L_2+2L_3+L_4.
\end{aligned}$$

Clearly,

$$|L_2|\leq\|B\|_{L^\infty}\int(1+|x|)^{2\delta}|u|^2dx\leq\|B\|_{L^\infty}\|u\|^2_{L^2_\delta}$$

and in a similar way we obtain the estimates of L_4 while by Cauchy-Schwarz inequality,

$$|L_3|\leq\|u\|_{L^2_\delta}\|F\|_{L^2_\delta}\leq\frac{1}{2}\left(\|u\|^2_{L^2_\delta}+\|F\|^2_{L^2_\delta}\right).$$

Now,

$$\begin{aligned}
0 &= \int\partial_a\left((1+|x|)^{2\delta}\left(u^TA^au\right)\right)dx\\
&= 2\delta\int(1+|x|)^{2\delta-1}\frac{x_a}{|x|}\left(u^TA^au\right)dx+\int(1+|x|)^{2\delta}\left((\partial_a u)^TA^au\right)dx\\
&+ \int(1+|x|)^{2\delta}\left(u^T\partial_aA^au\right)dx+\int(1+|x|)^{2\delta}\left(u^TA^a\partial_au\right)dx,
\end{aligned}$$

and since A^0 is symmetric, the second and the fourth terms are equal to $L_{1,a}$. Hence,

$$\begin{aligned}
2|L_{1,a}| &\leq 2\delta\int(1+|x|)^{2\delta}\frac{|A^0|}{1+|x|}\left(|u|^2\right)dx+\int(1+|x|)^{2\delta}|\partial_aA^a||u|^2dx\\
&\leq \left(\|A^a\|_{L^\infty_{-1}}+\|\partial_aA\|_{L^\infty}\right)\|u\|^2_{L^2_\delta}.
\end{aligned}$$

■

In order to proof the contraction we shall also need the following proposition.

Proposition 5.11 ***(Difference estimate in*** L^2_δ***)*** *Let* $G:\mathbb{R}^m\to\mathbb{R}^m$ *be a* C^1 *mapping. Then*

$$\|G(u)-G(v)\|^2_{L^2_\delta}\leq\|\nabla G\|^2_{L^\infty}\|u-v\|^2_{L^2_\delta}. \tag{5.50}$$

Proof (of Proposition 5.11)

$$\|G(u)-G(v)\|^2_{L^2_\delta} = \int (1+|x|)^{2\delta} (G(u)-G(v))^2\, dx$$
$$= \int (1+|x|)^{2\delta} \left(\int_0^1 \nabla G\,(su+(1-s)v)\,(u-v)ds \right)^2 dx \leq \|\nabla G\|^2_{L^\infty} \|u-v\|^2_{L^2_\delta}.$$

∎

Lemma 5.12 ***(Contraction in a lower norm)*** *There is a positive T^{**}, $0<\Lambda<1$ and a positive sequence $\{\beta_k\}$ with $\sum \beta_k < \infty$ such that*

$$|||u^{k+1}-u^k|||_{0,\delta,T^{**}} \leq \Lambda |||u^k-u^{k-1}|||_{0,\delta,T^{**}} + \beta_k. \tag{5.51}$$

Here $|||u|||_{0,\delta,T^{**}} = \sup\{\|u(t)\|_{H_{0,\delta}} : 0 \leq t \leq T^{**}\}$.

Proof (of Lemma 5.12) Since u^k satisfies equation (5.37), the difference $\left[u^{k+1}-u^k\right]$ will satisfy

$$A^0(u^k)\partial_t \left[u^{k+1}-u^k\right] = \sum_{a=1}^{3} A^a(u^k)\partial_a \left[u^{k+1}-u^k\right] + B(u^k)\left[u^{k+1}-u^k\right] + F^k, \tag{5.52}$$

where

$$F^k = -\left[A^0(u^k)-A^0(u^{k-1})\right]\partial_t u^k + \sum_{a=1}^{3} \left[A^a(u^k)-A^a(u^{k-1})\right]\partial_a u^k$$
$$+ \left[B(u^k)-B(u^{k-1})\right]u^k + \left[F(u^k)-F(u^{k-1})\right].$$

Applying Proposition 5.10 to equation (5.52) above we have

$$\frac{d}{dt}\langle \left[u^{k+1}-u^k\right], \left[u^{k+1}-u^k\right]\rangle_{L^2_\delta,A^0} \leq \mu C \langle \left[u^{k+1}-u^k\right], \left[u^{k+1}-u^k\right]\rangle_{L^2_\delta,A^0} + \|F^k\|^2_{L^2_\delta}. \tag{5.53}$$

Thus Gronwall's inequality yields,

$$\|\left[u^{k+1}(t)-u^k(t)\right]\|^2_{L^2_\delta,A^0} \leq e^{\mu C t}\left[\|\left[u^{k+1}(0)-u^k(0)\right]\|^2_{L^2_\delta,A^0} + \int_0^t \|F^k(s)\|_{L^2_\delta} ds \right]. \tag{5.54}$$

The constant C in inequalities (5.53) and (5.54) depends on $\|A^a(u^k)\|_{L^\infty_{-1}}$, $\|B(u^k)\|_{L^\infty}$, $\|\partial_t(A^0(u^k))\|_{L^\infty}$ and $\|\partial_a\left(A^a(u^k)\right)\|_{L^\infty}$. The first two of them are bounded by a constant independent of k, since it follows from (A) of Lemma 5.9 that $u^k \in G_2$. The estimation of $\|\partial_t(A^0(u^k))\|_{L^\infty}$ is done in (5.44) and for the last one, since $s-1>\frac{3}{2}$, we have

$$\|\partial_a\left(A^a(u^k;t,x)\right)\|_{L^\infty} \leq \|\frac{\partial}{\partial u}A^a(u^k;t,x)\partial_a u^k\|_{L^\infty} + \|\partial_a A^a(u^k;t,x))\|_{L^\infty}$$
$$\leq C\|\frac{\partial}{\partial u}A^a(u^k;t,x)\|_{L^\infty}\|\partial_a u^k\|_{H_{s-1,\delta+1}} + \|\partial_a A^a(u^k;t,x))\|_{L^\infty}$$
$$\leq C\|\frac{\partial}{\partial u}A^a(u^k;t,x)\|_{L^\infty}\|u^k\|_{H_{s,\delta}} + \|\partial_a A^a(u^k;t,x))\|_{L^\infty}$$

Lemma 5.9 (A) implies that $\|u^k\|_{H_{s,\delta}}$ is bounded and $u^k \in G_2$, therefore the above inequality shows that $\|\partial_a \left(A^a(u^k;t,x)\right)\|_{L^\infty}$ is bounded by a constant independent of k. From Proposition 5.11 we obtain

$$\begin{aligned}
\left\|F^k\right\|_{L^2_\delta}^2 &\leq 2\left\|\left[A^0(u^k)-A^0(u^{k-1})\right]\partial_t u^k\right\|_{L^2_\delta}^2 + 2\sum_{a=1}^3\left\|\left[A^a(u^k)-A^a(u^{k-1})\right]\partial_a u^k\right\|_{L^2_\delta}^2 \\
&+ 2\left\|\left[B(u^k)-B(u^{k-1})\right]u^k\right\|_{L^2_\delta}^2 + 2\left\|\left[F(u^k)-F(u^{k-1})\right]\right\|_{L^2_\delta}^2 \\
&\leq 2\Bigg\{\|\nabla A^0\|_{L^\infty(G_2)}^2\left\|\partial_t u^k\right\|_{L^\infty}^2 + \sum_{a=1}^3\|\nabla A^a\|_{L^\infty(G_2)}^2\left\|\partial_a u^k\right\|_{L^\infty}^2 \\
&+ \|\nabla B\|_{L^\infty(G_2)}^2\left\|u^k\right\|_{L^\infty}^2 + \|\nabla F\|_{L^\infty(G_2)}^2\Bigg\}\left\|[u^k-u^{k-1}]\right\|_{L^2_\delta}^2, \qquad (5.55)
\end{aligned}$$

here ∇ is the gradient with respect to u. Since $\left\|\partial_t u^k\right\|_{L^\infty} \leq C\left\|\partial_t u^k\right\|_{H_{s-1,\delta+1}}$, $\left\|\partial_a u^k\right\|_{L^\infty} \leq C\left\|u^k\right\|_{H_{s-1,\delta+1}} \leq C\left\|u^k\right\|_{H_{s,\delta}}$ and $\left\|u^k\right\|_{L^\infty} \leq C\left\|u^k\right\|_{H_{s,\delta}}$, it follows from (5.55) and Lemma 5.9 that

$$\left\|F^k(s)\right\|_{L^2_\delta}^2 \leq C_1\left\|[u^k(s)-u^{k-1}(s)]\right\|_{L^2_\delta}^2, \qquad (5.56)$$

where the constant C_1 depends upon R and L of Lemma 5.9, but it is independent of k. By the equivalence $\|u\|_{L^2_\delta,A^0}^2 \simeq \|u\|_{H_{0,\delta}}$, (5.56) and (5.54) above, we conclude that

$$\begin{aligned}
&\left\|[u^{k+1}(t)-u^k(t)\right\|_{H_{0,\delta}}^2 \\
\leq\ & C_2 e^{\mu C t}\left[\left\|[u^{k+1}(0)-u^k(0)]\right\|_{H_{0,\delta}}^2 + C_1\int_0^t\left\|[u^k(s)-u^{k-1}(s)\right\|_{H_{0,\delta}}^2\right] \\
\leq\ & C_2 e^{\mu C t}\left[\left\|[u^{k+1}(0)-u^k(0)]\right\|_{H_{0,\delta}}^2 + C_1 t\sup_{0\leq s\leq t}\left\|[u^k(s)-u^{k-1}(s)\right\|_{H_{0,\delta}}^2\right],
\end{aligned}$$

where C_1, C_2 and C do not depend on k. Hence

$$\begin{aligned}
&|||[u^{k+1}(t)-u^k(t)|||_{0,\delta,T^{**}} \\
\leq\ & \sqrt{2C_2 e^{\mu C T^{**}}}\left[\left\|[u^{k+1}(0)-u^k(0)]\right\|_{H_{0,\delta}} + \sqrt{2C_1T^{**}}|||[u^k-u^{k-1}|||_{0,\delta,T^{**}}\right].
\end{aligned}$$

Thus, taking T^{**} sufficiently small so that $\Lambda := 2\sqrt{C_2 e^{\mu C T^{**}}}\sqrt{C_1T^{**}} < 1$ and putting $\beta_k = \sqrt{2C_2 e^{\mu C T^{**}}}\left\|[u^{k+1}(0)-u^k(0)]\right\|_{H_{0,\delta}}$ completes the proof of the Lemma. ■

Lemma 5.12 implies that $\{u^k\}$ is a Cauchy sequence in $C([0,T^{**}],H_{0,\delta})$. Combing this with the intermediate estimates $\|u\|_{H_{s',\delta}} \leq \|u\|_{H_{s,\delta}}^{\frac{s'}{s}}\|u\|_{H_{0,\delta}}^{1-\frac{s'}{s}}$ (see Proposition B.4 (*ii*)) and Lemma 5.9 (A), we conclude that $\{u^k\}$ is a Cauchy sequence in $C([0,T^{**}],H_{s',\delta})$ for any $s' < s$. Therefore there is a unique $u \in C([0,T^{**}],H_{s',\delta})$ such that

$$|||u^k-u|||_{s',\delta,T^{**}} \to 0 \qquad \text{for any} \qquad s' < s. \qquad (5.57)$$

Taking $\frac{5}{2} < s' < s$ and utilizing the embedding Theorem B.13, we have

$$u^k \to u \qquad \text{in} \qquad C\left([0,T^{**}], C^1_\beta(\mathbb{R}^3)\right) \qquad \text{for any} \quad \beta \le \delta + \frac{3}{2},$$

where $C^1_\beta(\mathbb{R}^3)$ is the class for which the norm

$$\sup_{\mathbb{R}^3}\left((1+|x|)^\beta |u(x)| + \sum_{a=1}^{3}(1+|x|)^{\beta+1}|\partial_a u(x)|\right)$$

is finite. From (5.37)

$$\partial_t u^{k+1} = \left(A^0(u^k;t,x)\right)^{-1}\left[\sum_{a=1}^{3} A^a(u^k;t,x)\partial_a u^{k+1} + B(u^k;t,x)u^{k+1} + F(u^k;t,x)\right],$$

therefore by Corollary B.11 $\partial_t u^k \to \partial_t u$ in $H_{s-1,\delta+1}$. Hence

$$\partial_t u^k \to \partial_t u \qquad \text{in} \qquad C\left([0,T^{**}], C_{\beta+1}(\mathbb{R}^3)\right) \qquad \text{for any} \quad \beta \le \delta + \frac{3}{2}.$$

Thus $u \in C^1\left(\mathbb{R}^3 \times [0,T^{**}]\right)$ is a classical solution of the nonlinear system (5.1). Moreover, it follows from Lemma 5.9 (B) that $u \in \mathrm{Lip}\left([0,T^{**}], H_{s-1,\delta+1}\right)$. Our next task is to show that u^k converges weakly to u in $H_{s,\delta}$.

5.6 Weak Convergence

Lemma 5.13 ***(Weak Convergence)*** *For any $\phi \in H_{s,\delta}$, we have*

$$\lim_k \left\langle u^k(t), \phi\right\rangle_{s,\delta} = \langle u(t), \phi\rangle_{s,\delta} \tag{5.58}$$

uniformly for $0 \le t \le T^{**}$. Consequently

$$\|u(t)\|_{H_{s,\delta}} \le \liminf_k \|u^k(t)\|_{H_{s,\delta}} \tag{5.59}$$

and hence the solution u of the initial value problem (5.1) belongs to $L^\infty\left([0,T^{**}], H_{s,\delta}\right)$.

We recall that

$$\langle u, v\rangle_{s,\delta} = \sum_j 2^{(\frac{3}{2}+\delta)2j}\left\langle(\psi_j^2 u)_{(2^j)}, (\psi_j^2 v)_{(2^j)}\right\rangle_s$$

is an inner-product on $H_{s,\delta}$. In order to show Lemma 5.13 we need the below property.

Proposition 5.14 *Let $s < \frac{s'+s''}{2}$, $u \in H_{s',\delta}$, $v \in H_{s'',\delta}$. Then we have*

$$\left|\langle u, v\rangle_{s,\delta}\right| \le \|u\|_{H_{s',\delta}}\|u\|_{H_{s'',\delta}}. \tag{5.60}$$

Proof (of Proposition 5.14) Elementary arguments show that

$$|\langle u, v\rangle_s| \leq \|u\|_{H^{s'}}\|u\|_{H^{s''}},$$

here

$$\langle u, v\rangle_s = \int (\Lambda^s u)^T \Lambda^s(v)dx = \int (1+|\xi|^2)^s \hat{u}^T \overline{\hat{v}} d\xi.$$

Applying it term-wise and using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\left|\langle u, v\rangle_{s,\delta}\right| &\leq \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left|\left\langle \left(\psi_j^2 u\right)_{2^j}, \left(\psi_j^2 v\right)_{2^j} \right\rangle_s\right| \\
&\leq \sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta\right)j} \left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^{s'}}\right) \left(2^{\left(\frac{3}{2}+\delta\right)j} \left\|\left(\psi_j^2 v\right)_{2^j}\right\|_{H^{s''}}\right) \\
&\leq \left(\sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^{s'}}^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|\left(\psi_j^2 v\right)_{2^j}\right\|_{H^{s''}}^2\right)^{\frac{1}{2}} \\
&= \|u\|_{H_{s',\delta}}\|v\|_{H_{s'',\delta}}
\end{aligned}$$

■

Proof (of Lemma 5.13) Take s' and s'' such that $s' < s < s''$ and $s < \frac{s'+s''}{2}$. For a given $\phi \in H_{s,\delta}$ and positive ϵ, we may find by Theorem B.14 (b), $\tilde{\phi} \in H_{s'',\delta}$ such that

$$\|\phi - \tilde{\phi}\|_{H_{s,\delta}} \leq \frac{\epsilon}{2R} \quad \text{and} \quad \|\tilde{\phi}\|_{H_{s'',\delta}} \leq C(\epsilon)\|\phi\|_{H_{s,\delta}}, \tag{5.61}$$

where R is the positive number appearing in (5.35). Now,

$$\begin{aligned}
\left\langle u^k(t) - u(t), \phi\right\rangle_{s,\delta} &= \left\langle u^k(t) - u(t), \tilde{\phi}\right\rangle_{s,\delta} + \left\langle u^k(t) - u(t), \left(\phi - \tilde{\phi}\right)\right\rangle_{s,\delta} \\
&= I_k + II_k.
\end{aligned}$$

Therefore Proposition 5.14, (5.61) and (5.57) imply that

$$|I_k| \leq \|u^k(t) - u(t)\|_{H_{s',\delta}}\|\tilde{\phi}\|_{H_{s'',\delta}} \leq \|u^k(t) - u(t)\|_{H_{s',\delta}} C(\epsilon)\|\phi\|_{H_{s,\delta}} \to 0.$$

While in the second estimate we use Lemma 5.9 (A) and get

$$\begin{aligned}
|II_k| &\leq \|u^k(t) - u(t)\|_{H_{s,\delta}}\|\phi - \tilde{\phi}\|_{H_{s,\delta}} \\
&\leq \left(\|u^k(t) - u_0^0\|_{H_{s,\delta}} + \|u(t) - u_0^0\|_{H_{s,\delta}}\right)\|\phi - \tilde{\phi}\|_{H_{s,\delta}} \leq \frac{2R\epsilon}{2R} = \epsilon.
\end{aligned}$$

Thus,

$$\limsup_k \left|\left\langle u^k(t) - u(t), \phi\right\rangle_{s,\delta}\right| \leq \epsilon$$

which completes the proof of the limit (5.58). ■

For each k, $\left\langle u^k(t), \phi \right\rangle_{s,\delta}$ is continuous for $t \in [0, T^{**}]$ and by Lemma 5.13 it convergences uniformly to $\langle u(t), \phi\rangle_{s,\delta}$, hence $\langle u(t), \phi\rangle_{s,\delta}$ is a continuous function of t for any $\phi \in H_{s,\delta}$ and we have obtained the following:

Theorem 5.15 ***(Existence)*** *Under conditions (H1)-(H4) and (5.32) there is* $u \in C^1\left(\mathbb{R}^3 \times [0, T^{**}]\right)$ *a classical solution to the hyperbolic system (5.1) such that* $u(t, x) \in \overline{G_2}$ *and*

$$u \in L^\infty\left([0, T^{**}], H_{s,\delta}\right) \cap C_w\left([0, T^{**}], H_{s,\delta}\right) \cap \operatorname{Lip}\left([0, T^{**}], H_{s-1,\delta+1}\right), \tag{5.62}$$

where C_w *means continuous in the weak topology of* $H_{s,\delta}$.

5.7 Well-posedness

In this section we well prove continuity in $H_{s,\delta}$-norm and uniqueness.

Theorem 5.16 ***(Uniqueness)*** *Assume conditions (H1)-(H4) and (5.32) hold. If* $u_1(t,x)$ *and* $u_2(t,x)$ *are classical solutions to the hyperbolic system (5.1) such that* $u_1, u_2 \in \overline{G_2}$, *then* $u_1 \equiv u_2$.

Proof (of Theorem 5.16) Let u_1 and u_2 be a solutions to the hyperbolic system hyperbolic system (5.1) with the same initial data and let $V(t, x) = u_1(t, x) - u_2(t, x)$. Then V satisfies the equation

$$\begin{aligned} A^0(u_1)\partial_t V &= \sum_{a=1}^{3} A^a(u_1)\partial_a V + B(u_1)V \\ &- \left[A^0(u_1) - A^0(u_2)\right]\partial_t u_1 + \sum_{a=1}^{3}[A^a(u_1) - A^a(u_2))]\partial_a u_1 \\ &+ \left[B(u_1) - B(u_2)\right]u_1 + [F(u_1) - F(u_2))] \end{aligned} \tag{5.63}$$

and $V(x, 0) = 0$. Setting

$$G = [A^a(u_1) - A^a(u_2))]\partial_a u_1 + \left[B(u_1) - B(u_2)\right]u_1 + [F(u_1) - F(u_2))]$$

and applying Proposition 5.10 to (5.63), we have

$$\frac{d}{dt}\langle V, V\rangle_{L^2_\delta, A^0(u_1)} \leq \mu C \langle V, V\rangle_{L^2_\delta, A^0(u_1)} + \|G\|^2_{L^2_\delta}.$$

Let $T \leq T^*$, then Gronwall's inequality and the equivalence (5.48) imply

$$|||V|||^2_{0,\delta,T} \leq C_1 e^{C\mu T}\int_0^T \|G(t)\|^2_{L^2_\delta} dt.$$

Similar estimation as done in (5.55) yield that $\|G(t)\|^2_{L^2_\delta} \leq C_2\|V(t)\|^2_{L^2_\delta}$. Hence,

$$|||V|||^2_{0,\delta,T} \leq C_3 e^{C\mu T} T \quad |||V|||^2_{0,\delta,T}. \tag{5.64}$$

Thus, if T is sufficiently small, then 5.64 leads to a contradiction unless $V \equiv 0$. ∎

Theorem 5.17 ***(Continuation in norm)*** *Under conditions (H1)-(H4) and (5.32), any solutions u to the hyperbolic system (5.1) which satisfies $u(t,x) \in \overline{G_2}$ and the regularity condition (5.62), satisfies in addition*

$$u \in C\left([0,T^{**}], H_{s,\delta}\right) \cap C^1\left([0,T^{**}], H_{s-1,\delta+1}\right). \tag{5.65}$$

Proof (of Theorem 5.17) We first treat the continuity $C\left([0,T^{**}], H_{s,\delta}\right)$. Since u is a solution of initial value problem (5.1) which is reversible in time, is sufficient to show that

$$\lim_{t\downarrow 0} \|u(t) - u(0)\|_{H_{s,\delta}} = \lim_{t\downarrow 0} \|u(t) - u_0\|_{H_{s,\delta}} = 0. \tag{5.66}$$

We shall use the following known argument: suppose $\{w_n\}$ is a sequence in Hilbert space which converge weakly to w_0 and $\limsup_n \|w_n\| \leq \|w_0\|$, then $\lim_n \|w_n - w_0\| = 0$. We are going to use the equivalence norm $\| \quad \cdot \quad \|_{H_{s,\delta,A^0(u(0))}}$, so we need to show

$$\limsup_{t\downarrow 0} \|u(t)\|_{H_{s,\delta,A^0(u(0))}} \leq \|u_0\|_{Hs,\delta,A^0(u(0))}. \tag{5.67}$$

Let $\{u^k(t)\}$ be the sequence which is defined by the iteration process (5.37). It follows from the uniqueness Theorem 5.16 and (5.59) that

$$\|u(t)\|_{H_{s,\delta,A^0(u(t))}} \leq \liminf_k \|u^k(t)\|_{Hs,\delta,A^0(u(t))}, \tag{5.68}$$

where the limit above is uniformly in t. Applying the the energy estimate (5.29), we have

$$\frac{d}{dt}\|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u^k(t))}} \leq C\left(\mu\|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u^k(t))}} + 1\right).$$

So Gronwall's inequality yields

$$\|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u^k(t))}} \leq e^{C\mu t}\left[\|u^{k+1}(0)\|^2_{H_{s,\delta,A^0(u^k(0))}} + t\right]. \tag{5.69}$$

Take now arbitrary $\epsilon > 0$, since $u^k(t) \to u(t)$ uniformly in $[0,T^{**}]$, we see from the inner-product (5.7) that there is k_0 such that

$$\|v(t)\|_{H_{s,\delta,A^0(u(t))}} \leq (1+\epsilon)\|v(t)\|_{H_{s,\delta,A^0(u^k(t))}} \qquad \text{for} \qquad k \geq k_0. \tag{5.70}$$

Combing (5.68), (5.69), (5.70) and (5.36) with the fact that $u^k(t) \to u(t)$ uniformly in $[0, T^{**}]$, we obtain

$$\begin{aligned}\limsup_{t\downarrow 0} \|u(t)\|^2_{H_{s,\delta,A^0(0)}} =& \limsup_{t\downarrow 0} \|u(t)\|^2_{H_{s,\delta,A^0(u(t))}} \\ \leq& \limsup_{t\downarrow 0} \left(\liminf_k \|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u(t))}}\right) \\ \leq& \limsup_{t\downarrow 0} \left(\liminf_k (1+\epsilon)^2 \|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u^k(t))}}\right) \\ \leq& \limsup_{t\downarrow 0} \left(\liminf_k e^{C\mu t}\left[(1+\epsilon)^2 \|u^{k+1}(0)\|^2_{H_{s,\delta,A^0(u^k(0))}} + t\right]\right) \\ =& \limsup_{t\downarrow 0} \left(e^{C\mu t}\left[(1+\epsilon)^2 \|u_0\|^2_{H_{s,\delta,A^0(u(0))}} + t\right]\right) \\ =& (1+\epsilon)^2 \|u_0\|^2_{H_{s,\delta,A^0(u(0))}}\end{aligned}$$

which proves (5.67).

It remains to show that $\lim_{t\to t_0}\left(\|\partial_t u(t) - \partial_t u(t_0)\|_{H_{s-1,\delta+1}}\right) = 0$. Now,

$$\partial_t u = \left(A^0(u;t,x)\right)^{-1}\left\{\sum_{a=1}^{3} A^a(u;t,x)\partial_a u + B(u;t,x)u + F(u;t,x)\right\}. \tag{5.71}$$

By the first step of the proof, $\|\partial_a u(t) - \partial_a u(t_0)\|_{H_{s-1,\delta+1}} \to 0$ and $\|u(t) - u(t_0)\|_{H_{s,\delta}} \to 0$. At this stage we apply Corollary B.11 to the right hand of (5.71) and this completes the proof of Theorem 5.17.

■

5.8 Local existence for the evolution equations of Einstein-Euler system

In the previous subsections we have established the well-posdness of first order symmetric hyperbolic systems in $H_{s,\delta}$ spaces. We would like to apply it to the evolution equations of Einstein-Euler systems (2.33) and (2.16).

The unkowns of the evolution equations are the gravitaional field $g_{\alpha\beta}$ and its first order partial derivatives $\partial_\alpha g_{\gamma\delta}$, the Makino variable w and the velocity vector u^α. We represent them by the vector

$$U = \left(g_{\alpha\beta} - \eta_{\alpha\beta}, \partial_a g_{\gamma\delta}, \partial_0 g_{\gamma\delta}, w, u^a, u^0 - 1\right), \tag{5.72}$$

here $\eta_{\alpha\beta}$ denotes the Minkowski metric.

We first probe it's initial data. Recall that the initial data for Einstein-Euler systems (2.1) and (2.4) are $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^a)$, where

$$(\bar{h}_{ab} - I, \bar{A}_{ab}, (\hat{y}, \hat{v}^a)) \in H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2} \tag{5.73}$$

and the initial data for the semi-Riemannian metric $g_{\alpha\beta}$ are given by (4.1). Therefore when $t = 0$, we have by Theorem 4.3 and Corollary **??**, that $g_{ab} - \eta_{ab} = h_{ab} - I \in H_{s,\delta}$, $\partial_0 g_{ab} \in H_{s-1,\delta+1}$ and $(w, u^a, u^0 - 1) \in H_{s-1,\delta+2}$, while $g_{0a} = 0$, $g_{00} = -1$ and $\partial_a g_{0\alpha} = 0$. So we conclude that

$$U(0,\cdot) \in H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}. \tag{5.74}$$

In this situation we cannot apply directly Theorem 5.1. We intoduce some more convenience notations: $\mathbf{g} = g_{\alpha\beta} - \eta_{\alpha,\beta}$, $\partial\mathbf{g} = \partial_\alpha g_{\gamma\delta}$ (that is, $\partial\mathbf{g}$ is the set of all first order partial derivatives), $\mathbf{v} = (w, u^a, u^0 - 1)$ and $U = (\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$.

The idea to overcome this obstacle is the following. Since $H_{s-1,\delta} \subset H_{s,\delta}$, it follows from (5.74) that

$$U(0,\cdot) \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}. \tag{5.75}$$

If we prove the existence of $U(t,x)$ which is a solution to the coupled systems (2.33) and (2.16) with initial data in the form of (5.75) and such that $U(t,\cdot) \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ and it is continuous with respect to this norm, then from inequality

$$\|\mathbf{g}\|_{H_{s,\delta}} \lesssim \left(\|\mathbf{g}\|_{H_{s-1,\delta}} + \|\partial\mathbf{g}\|_{H_{s-1,\delta+1}} \right), \tag{5.76}$$

we will get that $U(t,\cdot) \in H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ and it will be continuous with respect to the norm of $H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$. Note that (5.76) is a simple consequence of the integral representation (A.3) of the $H_{s,\delta}$ norm.

In order to achieve this we carefully examine the structure of the coupled systems (2.33) and (2.16). According to Conclusion 2.4, we can write Einstein-Euler system in the form:

$$A^0 \partial_t U = \sum_{a=1}^{3} A^a \partial_a U + BU, \tag{5.77}$$

where A^α and B are 55×55 matrices such that

$$A^0 = \left(\begin{array}{c|c|c} I_{10} & \mathbf{0}_{10\times40} & \mathbf{0}_{10\times5} \\ \hline \mathbf{0}_{40\times10} & \widetilde{A^0}(\mathbf{g}) & \mathbf{0}_{40\times5} \\ \hline \mathbf{0}_{5\times10} & \mathbf{0}_{5\times40} & \widehat{A}^0(\mathbf{g},\mathbf{v}) \end{array} \right), \qquad A^a = \left(\begin{array}{c|c|c} \mathbf{0}_{10} & \mathbf{0}_{10\times40} & \mathbf{0}_{10\times5} \\ \hline \mathbf{0}_{40\times10} & \widetilde{A^a}(\mathbf{g}) & \mathbf{0}_{40\times5} \\ \hline \mathbf{0}_{5\times10} & \mathbf{0}_{5\times40} & \widehat{A}^a(\mathbf{g},\mathbf{v}) \end{array} \right) \tag{5.78}$$

and

$$B = \left(\begin{array}{c|c|c} \mathbf{0}_{10} & \mathbf{b}_{10\times40} & \mathbf{0}_{10\times5} \\ \hline & \widetilde{B}(\mathbf{g},\partial\mathbf{g},\mathbf{v}) & \\ \hline \mathbf{0}_{5\times10} & \mathbf{0}_{5\times40} & \widehat{B}(\mathbf{g},\partial\mathbf{g}) \end{array} \right). \tag{5.79}$$

Here $\widetilde{A^\alpha}(\mathbf{g})$ is 40×40 symmetric, $\widehat{A^\alpha}(\mathbf{g},\mathbf{v})$ is 5×5 symmetric and both $\widetilde{A^0}(\mathbf{g})$ and $\widehat{A^0}(\mathbf{g},\mathbf{v})$ are positive definite matrices; $\widetilde{B}(\mathbf{g},\partial\mathbf{g},\mathbf{v})$ is 40×55 matrix, $\widehat{B}(\mathbf{g},\partial\mathbf{g})$ is 5×5 matrix and $\mathbf{b}=(\mathbf{I}_{10\times 10}\mid \mathbf{0}_{10\times 30})$ matrix.

A natural norm on the product space $H_{s-1,\delta}\times H_{s-1,\delta+1}\times H_{s-1,\delta+2}$ is

$$\|U\|^2_{s-1,\delta}=\|\mathbf{g}\|^2_{H_{s-1,\delta}}+\|\partial\mathbf{g}\|^2_{H_{s-1,\delta+1}}+\|\mathbf{v}\|^2_{H_{s-1,\delta+2}}. \tag{5.80}$$

Note that from the algebra property (B.12) and Moser type estimates (B.19) we have that $A^\alpha U, BU\in H_{s-1,\delta}\times H_{s-1,\delta+1}\times H_{s-1,\delta+2}$, whenever $U\in H_{s-1,\delta}\times H_{s-1,\delta+1}\times H_{s-1,\delta+2}$.

We formulate an inner-product in accordance with the norm (5.80) and the structure of A^0. Let $U_1=(\mathbf{g}_1,\partial\mathbf{g}_1,\mathbf{v}_1)$ and $U_2=(\mathbf{g}_2,\partial\mathbf{g}_2,\mathbf{v}_2)$, similarly to (5.7) we set

$$\begin{aligned}
&\langle U_1,U_2\rangle_{s-1,\delta,A^0}\\
:=\ &\sum_{j=0}^\infty 2^{(\frac{3}{2}+\delta)2j}\int\left[\Lambda^{s-1}\left((\psi_j^2\mathbf{g}_1)_{(2^j)}\right)\right]^T\left[\Lambda^{s-1}\left((\psi_j^2\mathbf{g}_2)_{(2^j)}\right)\right]dx\\
+\ &\sum_{j=0}^\infty 2^{(\frac{3}{2}+\delta+1)2j}\left[\Lambda^{s-1}\left((\psi_j^2\partial\mathbf{g}_1)_{(2^j)}\right)\right]^T(\widetilde{A}^0)_{(2^j)}\left[\Lambda^{s-1}\left((\psi_j^2\partial\mathbf{g}_2)_{(2^j)}\right)\right]dx\\
+\ &\sum_{j=0}^\infty 2^{(\frac{3}{2}+\delta+2)2j}\left[\Lambda^{s-1}\left((\psi_j^2\mathbf{v}_1)_{(2^j)}\right)\right](\widehat{A}^0)_{(2^j)}\left[\Lambda^{s-1}\left((\psi_j^2\mathbf{v}_2)_{(2^j)}\right)\right]dx
\end{aligned} \tag{5.81}$$

and $\|U\|^2_{s-1,\delta,A^0}=\langle U,U\rangle_{s-1,\delta,A^0}$. Since A^0 is positive definite, $\|U\|_{s-1,\delta,A^0}\sim\|U\|_{s-1,\delta}$

We can now repeat all the arguments and estimations of subsections 5.2-5.7, which are applied term-wise to the norm (5.80) and inner-product (5.81), and in this way we extend Theorem 5.1 to the product space:

Theorem 5.18 ***(Well posedness of hyperbolic systems in product spaces)*** *Let $s-1>\frac{5}{2}$, $\delta\geq-\frac{3}{2}$ and assume the coefficient of (5.77) are of the form (5.78) and (5.79). If $U_0\in H_{s-1,\delta}\times H_{s-1,\delta+1}\times H_{s-1,\delta+2}$ and satisfies*

$$\frac{1}{\mu}U_0^TU_0\leq U_0^TA^0U_0\leq\mu U_0^TU_0,\qquad \mu\in\mathbb{R}^+ \tag{5.82}$$

then there exits a positive T which depends on $\|U_0\|_{s-1,\delta}$ and a unique $U(t,x)$ a solution to (5.77) such that $U(0,x)=U_0(x)$ and in addition it satisfies

$$U\in C([0,T],H_{s-1,\delta}\times H_{s-1,\delta+1}\times H_{s-1,\delta+2})\cap C^1([0,T],H_{s-2,\delta+1}\times H_{s-2,\delta+2}\times H_{s-2,\delta+3}). \tag{5.83}$$

Corollary 5.19 ***(Solution to the gravitational field and the fluid)*** *Let $\frac{7}{2}<s<\frac{2}{\gamma-1}+\frac{3}{2}$ and $\delta>-\frac{3}{2}$. Then there exists a positive T, a unique gravitational field $g_{\alpha\beta}$ solution to (2.33) and a unique (w,u^α) solution to Euler equation (2.16) such that*

$$g_{\alpha\beta}-\eta_{\alpha\beta}\in C([0,T],H_{s,\delta})\cap C^1([0,T],H_{s-1,\delta+1}) \tag{5.84}$$

and

$$(w, u^a, u^0 - 1) \in C([0,T], H_{s-1,\delta+2}) \cap C^1([0,T], H_{s-2,\delta+3}). \tag{5.85}$$

Proof (of Corollary 5.19) Theorem 4.3 implies that the initial data for $g_{\alpha\beta}$ belong to $H_{s,\delta}$ and Corollary **??** yields that initial data for (w, u^α) are in $H_{s-1,\delta+2}$. Thus $U(0,\cdot) \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$, where U is given by (5.72). In addition, the continuity of A^0 and Theorem 4.1 imply that the vector $U(0,\cdot)$ satisfies (5.82). Therefore Theorem 5.18 with inequality (5.76) give the desired result. ∎

6 Quasi Linear Elliptic Equations in $H_{s,\delta}$

In this section we will establish the elliptic theory in $H_{s,\delta}$ which is essential for the solution of the constraint equations. We will extend earlier results in weighted Sobolev spaces of integer order which were obtained by Cantor [7], Choquet-Bruhat and Christodoulou [11] and Christodoulou and O'Murchadha [14] to the fractional ordered spaces. The essential tool is the a priori estimate (6.18) and proving it requires first to establish an analogous a priori estimate in Bessel potential spaces. Our approach is based on the techniques of Pseudodifferential Operators which have symbols with limited regularity and we are adopting ideas being presented in Taylor's books [38]and [39]. A different method was derived recently by Maxwell [29].

6.1 A priori estimates for linear elliptic systems in H^s

In this section we consider a second order homogeneous elliptic system

$$(Lu)^i = \sum_{\alpha,\beta,j} a_{ij}^{\alpha\beta}(x)\partial_\alpha\partial_\beta u^j, \tag{6.1}$$

where the indexes $i, j = 1,,, N$ and $\alpha, \beta = 1, 2, 3$ (since only $\mathbb{R}^3$ is being discussed in this paper). We will use the convention

$$Lu = A(x)D^2u, \tag{6.2}$$

where $A(x)$ is $N \times N$ block matrix with blocks A_{ij}, each one of them is 3×3 matrix, D^2u is $N \times 1$ block matrix with each block 3×3 matrix and the meaning of $A_{ij}D^2u^j$ is $\sum_{\alpha,\beta=1,2,3} a_{ij}^{\alpha\beta}\partial_\alpha\partial_\beta u^j$. The symbol of (6.1) is $N \times N$ matrix $A(x,\xi)$, defined for all $\xi \in \mathbb{C}^3$ as follows:

$$A(x,\xi)_{ij} := -\langle A_{ij} i\xi, i\xi\rangle = \sum_{\alpha,\beta} a_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta. \tag{6.3}$$

The following definitions are due to Morrey [31].

Definition 6.1

1. *The system (6.1) is* **elliptic** *provided that*

$$\det\left(A(x,\xi)\right) = \det\left(\sum_{\alpha,\beta} a_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\right) \neq 0, \qquad \textit{for all} \qquad 0 \neq \xi \in \mathbb{R}^3; \tag{6.4}$$

2. *The system (6.1) is* **strongly elliptic** *provided that for some positive* λ

$$\langle A(x,\xi)\eta,\eta\rangle = \sum_{\alpha,\beta,i,j} a_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \geq \lambda|\xi|^2|\eta|^2. \tag{6.5}$$

Our main task is to obtain a priori estimate in the Bessel potential spaces H^s for the operator (6.1) whose coefficients $a_{\alpha\beta}^{ij}$ belong to H^{s_2}. In case s and s_2 are integers, then one may prove (6.8) below by means of induction and the classical results of Douglis and Nirenberg [15], and Morrey [31]. We will employ techniques of Pseudodifferential calculus. If the coefficients of the matrix A belongs to H^{s_2}, then $A(x,\xi) \in H^{s_2}S^2_{1,0}$, that is, $\|\partial_\xi^\alpha A(\cdot,\xi)\|_{H^{s_2}} \leq C_\alpha(1+|\xi|^2)^{(2-|\alpha|)/2)}$. We follow Taylor and decompose

$$A(x,\xi) = A^{\#}(x,\xi) + A^b(x,\xi) \tag{6.6}$$

in such way that a good parametrix can be constructed for $A^{\#}(x,\xi)$, while $A^b(x,\xi)$ will have order less than 2. According to Proposition 8.2 in [39], for $s_2 > \frac{3}{2}$ there is $0<\delta<1$ such that

$$A^{\#}(x,\xi) \in S^2_{1,\delta}, \qquad A^b(x,\xi) \in H^{s_2}S^{2-\sigma\delta}_{1,\delta}, \qquad \sigma = s_2 - \frac{3}{2}$$

where $A^{\#}(x,\xi) = \sum_{k=0}^\infty J_{\epsilon_k}A(x,\xi)\phi_k(\xi)$, $\epsilon_k = c2^{-k\delta}$. Here $\{\phi_k\}$ is the Littlewood-Paley partition of unity, that is, $\phi_0 \in C_0^\infty(\mathbb{R}^3)$, $\phi_0(0)=1$, $\phi_k(\xi) = \phi_0(2^{-k}\xi) - \phi_0(2^{-k+1}\xi)$ and $\sum_{k=0}^\infty \phi_k(\xi) = 1$. The smoothing operator J_ϵ is defined as follows:

$$J_\epsilon f(x) = \phi_0(\epsilon D)f(x) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \epsilon^{-3}\widehat{\phi}_0(\frac{y}{\epsilon})f(x-y)dy,$$

where $\widehat{\phi}_0$ is the inverse Fourier transform. In order that $A^{\#}$ will have a good parametrix we need to verify that it is a strongly elliptic. Since the original operator is strongly elliptic,

$$\begin{aligned}
&\sum_{\alpha,\beta,i,j} J_{\epsilon_k}a_{ij}^{\alpha\beta}(x)\phi_k(\xi)\xi_\alpha\xi_\beta\eta^i\eta^j \\
&= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \left(\sum_{\alpha,\beta,i,j} \epsilon_k^{-3}\widehat{\phi}_0(\frac{y}{\epsilon_k})(y)a_{ij}^{\alpha\beta}(y-x)\phi_k(\xi)\xi_\alpha\xi_\beta\eta^i\eta^j\right)dy \\
&\geq \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \quad \lambda\phi_k(\xi)|\xi|^2|\eta|^2 \int \epsilon_k^{-3}\widehat{\phi}_0(\frac{y}{\epsilon_k})dy = \lambda\phi_k(\xi)|\xi|^2|\eta|^2\phi_0(0) \\
&= \quad \lambda\phi_k(\xi)|\xi|^2|\eta|^2
\end{aligned}$$

for each fixed k. Summing over the k we have,

$$\langle A^{\#}(x,\xi)\eta,\eta\rangle = \sum_{k=0}^{\infty}\sum_{\alpha,\beta,i,j}\left(J_{\epsilon_k}a_{ij}^{\alpha\beta}\right)(x)\phi_k(\xi)\xi_\alpha\xi_\beta\eta^i\eta^j \geq \sum_{k=0}^{\infty}\lambda\phi_k(\xi)|\xi|^2|\eta|^2 = \lambda|\xi|^2|\eta|^2,$$

thus (6.5) holds for $A^{\#}$. The last step assures that $\|A^{\#}(x,\xi)^{-1}\| \leq \frac{1}{\lambda|\xi|^2}$ and then it follows from the identity $\partial(A^{-1}) = A^{-1}(\partial(A))A^{-1}$ that

$$\|\partial_x^\beta\partial_\xi^\alpha(A^{\#}(x,\xi))^{-1}\| \leq C_{\alpha\beta}(1+|\xi|^2)^{(-2-|\alpha|+\delta|\beta|)/2},$$

that is, $(A^{\#}(x,\xi))^{-1} \in S^{-2}_{1,\delta}$. Hence, the operator $A^{\#}(x,D)$ has a parametrix $E^{\#}(x,D) \in OPS^{-2}_{1,\delta}$ satisfying

$$E^{\#}(x,D)A^{\#}(x,D) = I + S, \tag{6.7}$$

where $S \in OPS^{-\infty}$ (See e. g. [38] Section 0.4).

Lemma 6.2 ***(An a priori estimates in H^s)*** *Let $Lu = A(x)D^2u$ be a strongly elliptic system and assume $A \in H^{s_2}$, $s_2 > \frac{3}{2}$ and $0 \leq s-2 \leq s_2$. Then there is a constant C such that*

$$\|u\|_{H^s} \leq C\left\{\|Lu\|_{H^{s-2}} + \|u\|_{H^{s-2}}\right\}. \tag{6.8}$$

Proof (of Lemma 6.2) We decompose $A(x,D)$ as in (6.2) and let $E^{\#}(x,D)$ be the above parametrix, then by (6.7)

$$E^{\#}(x,D)A(x,D)u = u + Su + E^{\#}(x,D)A^b(x,D)u. \tag{6.9}$$

Since $E^{\#}(x,D), S : H^{s-2} \to H^s$ are bounded,

$$\|E^{\#}(x,D)A(x,D)u\|_{H^s} = \|E^{\#}(x,D)Lu\|_{H^s} \leq C\|Lu\|_{H^{s-2}} \tag{6.10}$$

and

$$\|Su\|_{H^s} \leq C\|u\|_{H^{s-2}}. \tag{6.11}$$

According to [39] Proposition 8.1, (see also [38] Proposition 2.1.J)

$$A^b(x,D) : H^{s-\sigma\delta} \to H^{s-2}.$$

Hence,

$$\|E^{\#}(x,D)A^b(x,D)u\|_{H^s} \leq C\|A^b(x,D)u\|_{H^{s-2}} \leq C\|u\|_{H^{s-\sigma\delta}}. \tag{6.12}$$

Using the intermediate estimate $\|u\|_{H^{s-\sigma\delta}} \leq \epsilon\|u\|_{H^s} + C(\epsilon)\|u\|_{H^{s-2}}$, and combining it with (6.9)-(6.11), we obtain the estimate (6.8). ∎

6.2 A priori estimates in $H_{s,\delta}$

Our main task here is to extend the a priori estimate (6.8) to $H_{s,\delta}$-spaces and for a second order elliptic systems of the form:

$$\begin{aligned}(Lu)^i &= \sum_{\alpha,\beta,j} a_{ij}^{\alpha\beta}(x)\partial_\alpha\partial_\beta u^j + \sum_{\alpha,j} b_{ij}^{\alpha}(x)\partial_\alpha u^j + \sum_j c_{ij}(x)u^j \\ &= A(x)D^2u + B(x)(Du) + C(x)u.\end{aligned} \tag{6.13}$$

Here $A(x)$ is as in the previous subsection, $B(x)$ is $N\times N$ block matrix with each block 1×3 matrix, $C(x)$ is $N\times N$ matrix and $Du = (\partial_1 u^1, ..., \partial_3 u^N)^T$. We introduce the following hypotheses:

Hypotheses (H)

(H1) $\sum a_{i,j}^{\alpha,\beta}(x)\eta^i\eta^j\xi_\alpha\xi\beta \geq \lambda|\eta|^2|\xi|^2$ (i.e. L is strongly elliptic);

(H2) $(A(\cdot) - A_\infty) \in H_{s_2,\delta_2}$, $B \in H_{s_1,\delta_1}$, $C \in H_{s_0,\delta_0}$
$s_i \geq s-2, i = 0,1,2,$ $s_2 > \frac{3}{2}, s_1 > \frac{1}{2}, s_0 \geq 0$ and $\delta_i > \frac{1}{2} - i$, $i = 0,1,2,$ the matrix A_∞ has constant coefficients and $A_\infty D^2 u$ is an elliptic system, that is, $\det\left(\sum (a_\infty)_{ij}^{\alpha,\beta}\xi_\alpha\xi_\beta\right) \neq 0$.

We shall first derive an a priori estimate for a second order homogeneous operator

$$L_2 u = A(x)D^2 u.$$

Lemma 6.3 ***(An a priori estimate for homogeneous operator in*** $H_{s,\delta}$***)*** *Assume the operator L_2 satisfies hypotheses (H) and $s \geq 2$. Then*

$$\|u\|_{H_{s,\delta}} \leq C\left\{\|L_2 u\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}}\right\}, \tag{6.14}$$

where the constant C depends on s, δ and $\|A - A_\infty\|_{H_{s_2,\delta_2}}$.

Proof (of Lemma 6.3) According to Corollary A.5,

$$\sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^4 u\right)_{2^j}\right\|_{H^s}^2$$

is an equivalent norm in $H_{s,\delta}$. The main idea of the proof is to apply Lemma 6.2 to each term of the equivalent norm above. We use the convention (6.2) and compute

$$L_2(\psi^4 u) = \psi^4 L_2\left(D^2 u\right) + \psi A(x)R(u,\psi),$$

where

$$R(u,\psi)=8\psi\left(D\psi\right)^T\left(\psi Du\right)+12\left(D\psi\right)^T\left(D\psi\right)\left(\psi u\right)+4\psi\left(D^2\psi\right)\left(\psi u\right).$$

Applying the a priori estimate (6.8), we have

$$\begin{aligned}
\|u\|^2_{H_{s,\delta}} &\lesssim \sum_{j=0}^{\infty}2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^4u\right)_{2^j}\right\|^2_{H^s}\\
&\lesssim \sum_{j=0}^{\infty}2^{\left(\frac{3}{2}+\delta\right)2j}\left\{\left\|L_2\left((\psi_j^4u)\right)_{2^j}\right\|^2_{H^{s-2}}+\left\|\left(\psi_j^4u\right)_{2^j}\right\|^2_{H^{s-2}}\right\}\\
&\lesssim \sum_{j=0}^{\infty}2^{\left(\frac{3}{2}+\delta\right)2j}\left\{2^{4j}\left\|\left(\psi_j^4L_2(u)\right)_{2^j}\right\|^2_{H^{s-2}}+\left\|\left(\psi_j^4u\right)_{2^j}\right\|^2_{H^{s-2}}\right\}\\
&+\sum_{j=0}^{\infty}2^{\left(\frac{3}{2}+\delta+2\right)2j}\left\|\left(\psi_jAR(u,\psi_j)\right)_{2^j}\right\|^2_{H^{s-2}}\\
&\lesssim \|L_2(u)\|^2_{H_{s-2,\delta+2}}+\|u\|^2_{H_{s-2,\delta}}+\|AR\|^2_{H_{s-2,\delta+2}}.
\end{aligned}\tag{6.15}$$

The assumption on s_2 and δ_2 enable us to use the algebra property (B.12) and get

$$\begin{aligned}
\|AR\|_{H_{s-2,\delta+2}}&\le C\left(\|(A-A_\infty)R\|_{H_{s-2,\delta+2}}+\|A_\infty R\|_{H_{s-2,\delta+2}}\right)\\
&\le C\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}}+\|A_\infty\|\right)\|R\|_{H_{s-2,\delta+2}}.
\end{aligned}\tag{6.16}$$

Property (A.4) of ψ_j and Proposition B.1 imply

$$\|(\psi_jR)_{2^j}\|_{H^{s-2}}\le C\left(2^{-j}\|(\psi_jDu)_{2^j}\|_{H^{s-2}}+2^{-2j}\|(\psi_ju)_{2^j}\|_{H^{s-2}}\right)$$

and hence $\|R\|_{H_{s-2,\delta+2}}\le C\left(\|u\|_{H_{s-1,\delta}}+\|u\|_{H_{s-2,\delta}}\right)$. Thus, inequalities (6.15) and (6.16) yields

$$\|u\|_{H_{s,\delta}}\le C\left\{\|L_2u\|_{H_{s-2,\delta+2}}+\left(\|A-A_\infty\|_{H_{s_2,\delta_2}}+1\right)\left(\|u\|_{Hs-1,\delta}+\|u\|_{H_{s-2,\delta}}\right)\right\}.\tag{6.17}$$

Invoking the intermediate estimate $\|u\|_{Hs-1,\delta}\le\sqrt{2\epsilon}\|u\|_{Hs,\delta}+C(\epsilon)\|u\|_{Hs-2,\delta}$ (see(B.10)) and taking ϵ so that $C\left(\|A-A_\infty\|_{H_{s_2,\delta_2}}+1\right)\sqrt{2\epsilon}\le\frac{1}{2}$, we obtain from (6.17) the desired estimate (6.14). ■

Lemma 6.4 ***(An a priori estimate in*** $H_{s,\delta}$***)*** *Assume the operator L of the form (6.13) satisfies hypotheses (H) and $s\ge 2$. Then*

$$\|u\|_{H_{s,\delta}}\le C\left\{\|Lu\|_{H_{s-2,\delta+2}}+\|u\|_{H_{s-2,\delta}}\right\},\tag{6.18}$$

where the constant C depends on s,δ and the coefficients of L.

Proof (Proof of Lemma 6.4) By Lemma 6.3,

$$\begin{aligned}\|u\|_{H_{s,\delta}} &\leq C\left\{\|L_2u\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}}\right\}\\ &\leq C\left\{\|Lu\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}} + \|(L-L_2)u\|_{H_{s-2,\delta+2}}\right\},\end{aligned}$$

where $(L-L_2)u = B(x)(Du) + C(x)u$. Hypothesis (H2) together with the algebra (B.12) give

$$\|B(x)(Du)\|_{H_{s-2,\delta+2}} \lesssim \|B\|_{H_{s_1,\delta_1}}\|Du\|_{H_{s-2,\delta+1}} \lesssim \|B\|_{H_{s_1,\delta_1}}\|u\|_{H_{s-1,\delta}}, \tag{6.19}$$

and

$$\|C(x)u\|_{H_{s-2,\delta+2}} \lesssim \|C\|\|_{H_{s_0,\delta_0}}\|u\|_{H_{s-2,\delta}}.$$

Finally, we apply the intermediate estimate (B.10) to the right hand side of (6.19) and by taking ϵ sufficiently small we obtain (6.18). ■

Lemma 6.5 ***(Isomorphism of an operator with constant coefficients)*** *Let $A_\infty u := A_\infty D^2u$ be a homogeneous elliptic system with constant coefficients. Then for any $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$, the operator $A_\infty : H_{s,\delta+2} \to H_{s-2,\delta}$ is isomorphism satisfying*

$$\|u\|_{H_{s,\delta}} \leq C\|A_\infty D^2u\|_{H_{s-2,\delta+2}}. \tag{6.20}$$

Proof (of Lemma 6.5) Both statements are true when s is an integer, see

e. g. [11], Theorem 5.1. For s between two integers m_0 and m_1, we have $s = s_\theta = \theta m_0 + (1-\theta)m_1$ and $s-2 = s_\theta - 2 = \theta(m_0-2) + (1-\theta)(m_1-2)$, where $0 < \theta < 1$. The interpolation property (A.21) implies

$$H_{s,\delta} = [H_{m_0,\delta}, H_{m_1,\delta}]_\theta \quad \text{and} \quad H_{s-2,\delta} = [H_{m_0-2,\delta}, H_{m_1-2,\delta}]_\theta.$$

Since $A_\infty^{-1} : H_{m_i-2,\delta} \to H_{m_i,\delta+2}$, $i = 0, 1$, is continuous, it follows from interpolation theory that $A_\infty^{-1} : H_{s_\theta-2,\delta} \to H_{s_\theta,\delta+2}$ is also continuous (see e. g. [41]). Hence (6.20) holds. ■

The next lemma improves the a priori estimate (6.18).

Lemma 6.6 ***(Improved a priori estimate)*** *Let L be an elliptic operator of the form (6.13) which satisfies hypotheses (H). Assume $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then for any δ' there is a constant C such that*

$$\|u\|_{H_{s,\delta}} \leq C\left\{\|Lu\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-1,\delta'}}\right\}. \tag{6.21}$$

The constant C depends on the H_{s_i,δ_i}-norm of the coefficients of L, s, δ and δ'.

Proof (of Lemma 6.6) Let $\chi_R \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function satisfying $\mathrm{supp}(\chi_R) \subset \{|x| \le 2R\}$, $\chi_R(x) = 1$ for $|x| \le R$, $0 \le \chi_R(x) \le 1$ and $\|\partial^\alpha \chi_R\|_\infty \le C_\alpha R^{-|\alpha|}$. For $u \in H_{s,\delta}$ we write

$$u = (1-\chi_R)u + \chi_R u$$

and R will be determinate later on. We start with the estimation of $\|(1-\chi_R)u\|_{H_{s,\delta}}$ and for that purpose we use the convention (6.2) and compute

$$\begin{aligned} A_\infty(D^2(1-\chi_R)u) &= (1-\chi_R)A_\infty(D^2u) - 2A_\infty(D\chi_R)^T(Du) - A_\infty(D^2\chi_R)(u) \\ &= (1-\chi_R)(Lu) + E_1 + E_2, \end{aligned} \tag{6.22}$$

where

$$E_1 = -(1-\chi_R)\left\{((A-A_\infty))(D^2u) + B(x)(Du) + C(x)u\right\}$$

and

$$E_2 = -\left\{2A_\infty(D\chi_R)^T(Du) + A_\infty(D^2\chi_R)(u)\right\}$$

Applying inequality (6.20) of Lemma 6.5,

$$\begin{aligned} \|(1-\chi_R)u\|_{H_{s,\delta}} &\le C\|A_\infty D^2((1-\chi_R)u)\|_{H_{s-2,\delta+2}} \\ &\le C\left\{\|(1-\chi_R)Lu\|_{H_{s-2,\delta+2}} + \|E_1\|_{H_{s-2,\delta+2}} + \|E_2\|_{H_{s-2,\delta+2}}\right\}. \end{aligned} \tag{6.23}$$

Since $\|(1-\chi_R)Lu\|_{H_{s-2,\delta+2}} \lesssim \|Lu\|_{H_{s-2,\delta+2}}$ (see Proposition B.2 (a)), it remains to estimate $\|E_1\|_{H_{s-2,\delta+2}}$ and $\|E_2\|_{H_{s-2,\delta+2}}$. We may choose δ_i' so that $\delta_i > \delta_i' > \frac{1}{2} - i$, $i = 0, 1, 2$ and then we put $\gamma = \min_{i=0,1,2}(\delta_i - \delta_i')$. Under these conditions we can apply the algebra property (B.12), Proposition B.2 (b) and get

$$\begin{aligned} \|E_1\|_{H_{s-2,\delta+2}} &\le C\left\|(1-\chi_R)\left\{(A-A_\infty)(D^2u) + B(Du) + Cu\right\}\right\|_{H_{s-2,\delta+2}} \\ &\le C\{\|(1-\chi_R)(A-A_\infty)\|_{H_{s_2,\delta_2'}}\|D^2u\|_{H_{s-2,\delta+2}} \\ &+ \|(1-\chi_R)B\|_{H_{s_1,\delta_1'}}\|Du\|_{H_{s-1,\delta+1}} + \|(1-\chi_R)C\|_{H_{s_0,\delta'}}\|u\|_{H_{s,\delta}}\} \\ &\le \frac{C_1}{R^\gamma}\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}} + \|B\|_{H_{s_1,\delta_1}} + \|C\|_{H_{s_0,\delta_0}}\right)\|u\|_{H_{s,\delta}} \\ &\le \frac{C_1\Lambda}{R^\gamma}\|u\|_{H_{s,\delta}}, \end{aligned} \tag{6.24}$$

where $\Lambda = \left(\|A-A_\infty\|_{H_{s_2,\delta_2}} + \|B\|_{H_{s_1,\delta_2}} + \|C\|_{H_{s_0,\delta_0}}\right)$.

Next, since $D\chi_R$ has compact support, inequality (A.23) implies that

$$\begin{aligned} \|E_2\|_{H_{s-2,\delta+2}} &\le C(R)\left\{\|2A_\infty((D\chi_R)^T(Du))\|_{H_{s-2,\delta'+1}} + \|A_\infty((D^2\chi_R)u)\|_{H_{s-2,\delta'}}\right\} \\ &\le C(R)\|A_\infty\|\left\{2\|Du\|_{H_{s-2,\delta'+1}} + \|u\|_{H_{s-2,\delta'}}\right\} \\ &\le C(R)\|A_\infty\|\|u\|_{H_{s-1,\delta'}}. \end{aligned} \tag{6.25}$$

We turn now to the estimation of $\|\chi_R u\|_{H_{s,\delta}}$. Noting that $(\chi_R u)$ has compact support, we have by (A.23), (6.18) and Proposition B.2 that

$$\|\chi_R u\|_{H_{s,\delta}} \leq C(R)\|\chi_R u\|_{H_{s,\delta'}} \leq C(R)\{\|L(\chi_R u)\|_{H_{s-2,\delta'+2}} + \|u\|_{H_{s-1,\delta'}}\}. \tag{6.26}$$

Similarly to (6.22) we compute

$$L(\chi_R u) = \chi_R L(u) + 2A((D\chi_R)^T(Du)) + A((D^2\chi_R)u) + B(D\chi_R)u. \tag{6.27}$$

We estimate each term of (6.27) separately. Once again, since $\chi_R Lu$ has compact support,

$$\|\chi_R(Lu)\|_{H_{s-2,\delta'+2}} \leq C(R)\|Lu\|_{H_{s-2,\delta+2}}. \tag{6.28}$$

Next, using the second assumption of (H), algebra (B.12) and compactness of supp(χ_R) we get

$$\begin{aligned}
&\|2A((D\chi_R)^T(Du))\|_{H_{s-2,\delta'+2}} \\
&\leq 2\|(A-A_\infty)(D\chi_R^T(Du))\|_{H_{s-2,\delta'+2}} + \|A_\infty(D\chi_R^T(Du))\|_{H_{s-2,\delta'+2}} \\
&\leq C\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\|\right)\|(D\chi_R)^T(Du)\|_{H_{s-2,\delta'+2}} \\
&\leq C(R)\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\|\right)\|Du\|_{H_{s-2,\delta'+1}} \\
&\leq C(R)\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\|\right)\|u\|_{H_{s-1,\delta'}}.
\end{aligned} \tag{6.29}$$

In a similar manner we estimate the other terms and together with inequalities (6.23)-(6.26), (6.28) and (6.29) we have

$$\begin{aligned}
\|u\|_{H_{s,\delta}} &\leq \|(1-\chi_R)u\|_{H_{s,\delta}} + \|\chi_R u\|_{H_{s,\delta}} \\
&\leq C\left\{\|Lu\|_{H_{s-2,\delta+2}} + C_2\|u\|_{H_{s-1,\delta'}} + \frac{C_1\Lambda}{R^\gamma}\|u\|_{H_{s,\delta}}\right\},
\end{aligned} \tag{6.30}$$

where C_1 and C_2 depend on the norms of the coefficients of L and in addition C_2 depends in R. We now take R such that $\frac{C_1\Lambda}{R^\gamma} \leq \frac{1}{2}$, then (6.21) follows from (6.30). ■

The next two theorems are consequence of the compact embedding, Theorem B.12, the a priori estimate (6.21) and standard arguments of Functional Analysis.

Theorem 6.7 ***(Semi Fredholm)*** *Assume the operator L satisfies hypotheses(H), $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then $L : H_{s,\delta} \to H_{s-2,\delta+2}$ is semi Fredholm, that is,*

(i) $\mathbf{dim}(\mathbf{Ker}L) < \infty$;

(ii) *If L is injective, then there is a constant C such that*

$$\|u\|_{H_{s,\delta}} \leq C\|Lu\|_{H_{s-2,\delta+2}}; \tag{6.31}$$

(iii) *L has a closed range.*

Theorem 6.8 ***(A homotopy argument)*** *Lets ≥ 2 and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Assume L be an elliptic operator of the form (6.13) that fulfilled the hypotheses (H) and L_t is a continuous family of operators which satisfy hypotheses (H) for $t \in [0,1]$, $L_1 = L$ and*

$$L_t : H_{s,\delta} \to H_{s-2,\delta+2} \text{ is injective.}$$

If

$$L_0 : H_{s,\delta} \to H_{s-2,\delta+2} \text{ is an isomorphism,}$$

then the same is true for L.

The next Lemma shows that solutions to the homogeneous system have lower growth at infinity. We follow Christodoulou and O'Murchadha's proof [14].

Lemma 6.9 ***(Lower growth of homogeneous solutions)*** *Assume L satisfies hypotheses (H), $u \in H_{s,\delta}$, $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. If $Lu = 0$, then $u \in H_{s,\delta'}$ for any $-\frac{3}{2} < \delta' < -\frac{1}{2}$.*

Proof (of Lemma 6.9) The inclusion $H_{s,\delta} \subset H_{s,\delta'}$ for $\delta' < \delta$, implies that it suffices to show the statement for $\delta' > \delta$. The conditions on δ_i imply that we may find $\delta' > \delta$ so that $\delta_i + \delta + i > \delta' + 2 - \frac{3}{2}$. Applying the algebra property (B.12) to

$$f := A_\infty u - Lu = (A_\infty - A(x)) \cdot D^2 u - B(x)(Du)^T - C(x)u,$$

we obtain that f belongs to $H_{s-2,\delta'+2}$. Now $Lu = 0$, so $A_\infty u = f$ and since $A_\infty : H_{s,\delta'} \to H_{s-2,\delta'+2}$ is isomorphism by Lemma 6.5, we conclude that hence $u \in H_{s,\delta'}$. We now replace δ by δ' repeat the above arguments. ∎

6.3 Semi Linear Elliptic Equations on Asymptotically Flat Manifolds

A Riemannian 3-manifold (M, h) is asymptotically flat (AF) if there is a compact subset K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$ and the metric h tends to the identity I at infinity. A natural definition of the last statement in our case is $h - I \in H_{s',\delta'}$. Thus the assumptions of this subsection are: $h - I \in H_{s',\delta'}$, $s' > \frac{3}{2}$ and $\delta' > -\frac{3}{2}$.

We denote by Δ_h be the Laplace-Beltrami operator on (M, h). In the coordinates (x^1, x^2, x^3) it takes the form

$$\Delta_h = \frac{1}{\sqrt{|h|}} \partial_j \left(\sqrt{|h|} h^{ij} \partial_i \right), \tag{6.32}$$

where $|h| = \det(h_{ij})$ and $h^{ij} = (h_{ij})^{-1}$. Inserting the identity $\partial_j|h| = |h|\mathrm{tr}(h^{ij}(\partial_j(h_{ij}))$ into (6.32), we have

$$\Delta_h = h^{ij}\partial_j\partial_i + \partial_j(h^{ij})\partial_i + \frac{1}{2}\mathrm{tr}(h^{ij}(\partial_j(h_{ij}))h^{ij}\partial_i. \tag{6.33}$$

Hence, by means of algebra (B.12) and Moser type estimate (B.19), the elliptic operator (6.33) satisfies hypothesis (H) of Section 6.2 provided that $s \leq s'$.

Let us introduce some more notations. We denote by $\mu_h = \sqrt{|h|}dx$ the Lebesgue measure on the manifold (M, h), $(Du \cdot Dv)_h = h^{ij}\partial_i u \partial_j v$, and $\|Du\|_h^2 = (Du \cdot Du)_h$. Integration by parts yields

$$\begin{aligned} \int (\Delta_h u)\, v d\mu_h &= \int \partial_j \left(\sqrt{|h|} h^{ij} \partial_i u \right) v dx \\ = -\int h^{ij}\partial_i u \partial_j v \sqrt{|h|} dx &= -\int (Du \cdot Dv)_h d\mu_h. \end{aligned} \tag{6.34}$$

Formula (6.34) holds whenever $v \in H_0^1(\mathbb{R}^3)$, $u \in H_{s,\delta}$ and $s \geq 1$. Therefore it enables us to define weak solutions on the manifold (M, h).

Definition 6.10 ***(Weak solutions)*** *A function u in $H_{s,\delta}$ is a weak solution of*

$$-\Delta_h u + c(x)u = f \in H_{s-2,\delta+2}$$

on (M, h), if

$$\int \left((Du \cdot Dv)_h + cuv \right) d\mu_h = \int f v d\mu_h, \tag{6.35}$$

for all $v \in H_0^1(\mathbb{R}^3)$.

Remark 6.11 *In case $u, v \in H_{s,\delta}$, $s \geq 2$ and $\delta \geq -1$, then by algebra $h^{ij}\partial_i u, \sqrt{|h|}\partial_j v \in H_{s-1,0}$. Applying the Cauchy Schwarz inequality*

$$\begin{aligned} &\int |(Du \cdot Dv)_h| d\mu_h = \int |h^{ij}\partial_i u \partial_j v| \sqrt{|h|} dx \\ &\leq \left(\int (h^{ij}\partial_i u)^2 \right)^{\frac{1}{2}} \left(\int \sqrt{|h|}\partial_j v)^2 \right)^{\frac{1}{2}} \leq \|h^{ij}\partial_i u\|_{H_{s-1,0}} \|\sqrt{|h|}\partial_j v\|_{H_{s-1,0}}, \end{aligned}$$

we see that $h^{ij}\partial_i u \partial_j v|\sqrt{|h|} \in L^1(\mathbb{R}^3)$. Similarly, the integrand of the left hand side of (6.34) belongs to $L^1(\mathbb{R}^3)$. Hence, approximating u and v by C_0^∞ functions and using Lebesgue's Dominated Convergence Theorem we have

$$\int (\Delta_h u)\, v d\mu_h = -\int (Du \cdot Dv)_h d\mu_h, \quad u, v \in H_{s,\delta}, \textit{ whenever } s \geq 2, \textit{ and } \delta \geq -1. \tag{6.36}$$

In this section we will prove existence and uniqueness for the semi-linear equation

$$-\Delta_h u = F(x,u) := \sum_{i=1}^{N} m_i(x) h_i(u), \tag{6.37}$$

where $m_i \in H_{s_0,\delta_0}$, $m_i(x) \geq 0$, $s_0 \geq 0$, $\delta_0 > \frac{1}{2}$ and for $u > -1$ the functions h_i are decreasing, nonnegative and smooth. These conditions ensure $F(\cdot,u)$ and $\frac{\partial F}{\partial p}(\cdot,u)$ are in $H_{s-2,\delta+2}$ whenever $u \in H_{s,\delta}$ and $s \geq 2$.

Theorem 6.12 ***(Existence and uniqueness)*** *Let $h - I \in H_{s',\delta'}$, $s' > \frac{3}{2}$, $\delta' > -\frac{3}{2}$, $2 \leq s \leq s'$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then equation (6.37) has a unique solution u in $H_{s,\delta}$. Furthermore, $0 \leq u \leq K$ for a nonnegative constant K.*

In order to show Theorem 6.12 we need the weak maximal principle:

Proposition 6.13 ***(Weak maximal principle)*** *Assume $c \in H_{s'-2,\delta'+2}$ is nonnegative. If $u \in H_{s,\delta}$ satisfies*

$$-\Delta_h u + cu \leq 0, \tag{6.38}$$

then $u \leq 0$.

Proof (of Proposition 6.13) For $\epsilon > 0$ we put $w = \max(u - \epsilon, 0)$. It has compact support since $\lim_{x\to\infty} u(x) = 0$. Further, $Dw = Du$ a.e. in $\{u(x) > \epsilon\}$ (see e. g. [18] or [25]. Thus, $w \in H_0^1(\mathbb{R}^3)$ and $w \geq 0$, so by (6.35)

$$0 \geq \int \left((Du, Dw)_h + cuw\right) d\mu_h = \int_{\{u\geq\epsilon\}} \left(\|Du\|_h^2 + cu^2\right) d\mu_h.$$

Therefore $u \equiv \epsilon$ in $\{u(x) \geq \epsilon\}$. Since ϵ is arbitrary, we have $u \leq 0$. ■

Proof (of Existence) The proof will be done in several steps. We define a map Φ : $\{H_{s,\delta} \times [0,1], u(x) > -1\} \to H_{s-2,\delta+2}$ by

$$\Phi(u,\tau) = -\Delta_h u - \tau F(x,u),$$

let $u(\tau)$ denotes a solution of $\Phi(u,\tau) = 0$ and put $J = \{0 \leq s \leq 1 : \Phi(u(s),s) = 0\}$. We will show that J is both open and closed set. Since $0 \in J$, $J = [0,1]$ which yields the existence result.

Step 1. *The set J is open:*

Let

$$Lw := \left(\frac{\partial \Phi}{\partial u}(u,\tau)\right)(w) = -\Delta_h w - \tau \frac{\partial F}{\partial p}(\cdot,u) w$$

and

$$L_t w = -\Delta_{\{th+(1-t)I\}} w - t\tau \frac{\partial F}{\partial p}(\cdot,u) w.$$

If $L_t w = 0$, then by Lemma (6.9) $w \in H_{s,-1}$. So we may use (6.36) and get

$$\int (L_t w)\, w d\mu_{\{th+(1-t)I\}} = \int \left(\|Dw\|^2_{\{th+(1-t)I\}} - t\tau \frac{\partial F}{\partial p}(\cdot, u) w^2 \right) d\mu_{\{th+(1-t)I\}}.$$

Since $\frac{\partial F}{\partial p} \leq 0$, the above yields that $L_t w = 0$ implies $w \equiv 0$ for each $t \in [0,1]$. In addition $L_0 = -\Delta_I = -\Delta$ is an isomorphism according to Lemma 6.5. So Theorem 6.8 implies that $L_1 = L$ is an isomorphism too. Thus J is open by the Implicit Function Theorem.

Step 2. $\|u(\tau)\|_{H_{s,\delta}} \leq C$ *for a constant* C *independent of* τ*:*

We first establish the bound in $H_{2,\delta}$-norm. The the weak maximum principle implies $u(\tau) \geq 0$ and since $F(x,p)$ is decreasing in p,

$$\|F(\cdot, u(\tau))\|_{H_{0,\delta+2}} \leq \|F(\cdot, 0)\|_{H_{0,\delta+2}} \leq \left(\sum_{i=1}^{N} h_i(0)^2 \|m_i\|^2_{H_{0,\delta+2}} \right)^{\frac{1}{2}} := K.$$

We showed in Step 1 that $\Delta_h : H_{s,\delta} \to H_{s-2,\delta+2}$ is injective, therefore from Theorem 6.7 (ii),

$$\|u(\tau)\|_{H_{2,\delta}} \leq C\| - \Delta_h u(\tau)\|_{H_{0,\delta+2}} \leq C\|F(\cdot, 0)\|_{H_{0,\delta+2}} \leq CK. \tag{6.39}$$

Now, by Moser estimate (B.19), $\|h_i(u(\tau))\|_{H_{2,\delta}} \leq C\|(u(\tau))\|_{H_{2,\delta}}$ and by algebra (B.12), $\|F(\cdot, u(\tau))\|_{H_{2,\delta}} \leq C\|u(\tau)\|_{H_{2,\delta}}$. In order to improve (6.39), we take s'' so that $s'' - 2 \leq 2$ and $s'' \leq s$. Then we may apply again (6.31) and with (6.39) we have

$$\begin{aligned} &\|u(\tau)\|_{H_{s'',\delta}} \leq C\| - \Delta_h u(\tau)\|_{H_{s''-2,\delta+2}} \leq C\| - \Delta_h u(\tau)\|_{H_{2,\delta+2}} \\ &= C\|F(\cdot, u(\tau))\|_{H_{2,\delta}} \leq C\|u(\tau)\|_{H_{2,\delta}} \leq CK. \end{aligned} \tag{6.40}$$

We have proved the boundedness in case $s'' = s$, otherwise we can repeat the same procedure as above to improve regularity until we would reach the desired regularity. It is obvious that the bound on $\|u(\tau)\|_{H_{s,\delta}}$ does not depend on τ.

Step 3. *Lipschitz continuity with respect to* τ*:*

Differentiation of the equation $\Phi(u(\tau), \tau) = 0$ with respect to τ gives

$$-\Delta_h u_\tau(\tau) - \tau \frac{\partial F}{\partial p} F(x, u(\tau)) u_\tau(\tau) = F(x, u(\tau)).$$

Now $\frac{\partial F}{\partial p} F(x,p) \leq 0$, so in the same way as we did in Step 1 we obtain that the operator $L = -\Delta_h - \tau \frac{\partial F}{\partial p} F(x, u(\tau)) : H_{s,\delta} \to H_{s-2,\delta+2}$ is injective. Hence, by 6.7 (ii),

$$\|u_\tau\|_{H_{s,\delta}} \leq C\|L(u_\tau)\|_{H_{s-2,\delta+2}} = C\|F(x, u(\tau))\|_{H_{s-2,\delta+2}}. \tag{6.41}$$

Next, Step 2 implies

$$\|F(x, u(\tau))\|_{H_{s-2,\delta+2}} \le C\|u(\tau)\|_{H_{s,\delta}} \left(\sum_{i=1}^{N} \|m_i\|_{H_{s_0,\delta_0}}\right) \le C. \tag{6.42}$$

Thus, combining (6.41) with (6.42) we get

$$\|u(\tau_1) - u(\tau_2)\|_{H_{s,\delta}} \le C|\tau_1 - \tau_2|. \tag{6.43}$$

Step 4. *The set J is closed:*

Take a sequence $\{\tau_n\} \subset J$ such that $\tau_n \to \tau_0$. By (6.43), $\{u(\tau_n)\}$ is Cauchy in $H_{s,\delta}$ and therefore it converges to $u_0 \in H_{s,\delta}$. Since the map Φ is continuous, it follows that $\Phi(u_0, \tau_0) = 0$, that is $\tau_0 \in J$. This completes the proof of the existence. ∎

Proof (of Uniqueness) Assume u_1 and u_2 are solutions to (6.37). We conduct the proof by showing that $\Omega := \{x : u_1(x) > u_2(x)\}$ is an empty set. Note that Ω is open since u_1 and u_2 are continuous. Put $w = u_1 - u_2$, then $-\Delta_h w = F(x, u_1) - F(x, u_2) \le 0$ in Ω. So $w \le 0$ in Ω by Proposition (6.13). That obviously leads to a contradiction unless Ω is empty. ∎

Appendix

A Construction of the Spaces $H_{s,\delta}$:

The weighted Sobolev spaces of integer order below were introduced by Cantor [6] and independently by Nirenberg and Walker [32]. Nirenberg and Walker initiate the study of elliptic operators in these spaces, while Cantor used them to solve the constraint equations on asymptotically flat manifolds. For an nonnegative integer m and a real δ we define a norm

$$\left(\|u\|^*_{m,\delta}\right)^2 = \sum_{|\alpha|\le m} \int \left(\langle x\rangle^{\delta+|\alpha|}|\partial^\alpha u|\right)^2 dx, \tag{A.1}$$

where $\langle x\rangle = 1 + |x|$. The space $H_{s,m}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (A.1). Note that the weight varies with the derivatives.

Here we will repeat Triebel's extension of these spaces into a fractional order, [40],[41]. Let $s = m + \lambda$, where m is a nonnegative integer and $0 < \lambda < 1$. One possibility of extending the ordinary integer order Sobolev spaces is the *Lipschitz-Sobolevskij Spaces*, having a norm

$$\|u\|^2_{m+\lambda,2} = \sum_{|\alpha|\le m} \int |\partial^\alpha u|^2 dx + \sum_{|\alpha|=m} \int\int \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{3+\lambda 2}} dxdy. \tag{A.2}$$

Hence, a reasonable definition of *weighted fractional Sobolev norm* is a combination of the norm (A.1) with (A.2):

$$\left(\|u\|^*_{s,\delta}\right)^2 = \begin{cases} \sum_{|\alpha|\le m} \int |\langle x\rangle^{\delta+|\alpha|}\partial^\alpha u|^2 dx, & s=m \\ \left.\begin{array}{l} \sum_{|\alpha|\le m} \int |\langle x\rangle^{\delta+|\alpha|}\partial^\alpha u|^2 dx \\ \quad + \sum_{|\alpha|=m} \int\int \frac{|\langle x\rangle^{m+\lambda+\delta}\partial^\alpha u(x) - \langle y\rangle^{m+\lambda+\delta}\partial^\alpha u(y)|^2}{|x-y|^{3+2\lambda}} dxdy \end{array}\right\}, & s=m+\lambda. \end{cases} \tag{A.3}$$

here m is a nonnegative integer and $0<\lambda<1$. The space $H_{s,\delta}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (A.3).

The norm (A.3) is essential for the understating of the connections between the integer and the fractional order. But it has a disadvantage, namely, the double integral makes it almost impossible to establish any property (embedding, a priori estimate, etc.) needed for PDEs. We are therefore looking for an equivalent definition of the norm (A.3).

Let $K_j=\{x : 2^{j-3}\le |x|\le 2^{j+2}\}$, $(j=1,2,...)$ and $K_0=\{x : |x|\le 4\}$. Let $\{\psi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R}^3)$ be a sequence such that $\psi_j(x)=1$ on K_j, $\operatorname{supp}(\psi_j)\subset\{x : 2^{j-4}\le|x|\le 2^{j+3}\}$, for $j\ge 1$, $\operatorname{supp}(\psi_0)\subset\{x : |x|\le 2^3\}$ and

$$|\partial^\alpha\psi_j(x)|\le C_\alpha 2^{-|\alpha|j}, \tag{A.4}$$

where the constant C_α does not depend on j.

We define now,

$$\left(\|u\|^\bigstar_{s,\delta}\right)^2 = \begin{cases} \sum_{j=0}^\infty \left(2^{\delta 2j}\|\psi_j u\|^2_{L^2} + 2^{(\delta+m)2j}\sum_{|\alpha|=m}\|\partial^\alpha(\psi_j u)\|^2_{L^2}\right), & s=m \\ \left.\begin{array}{l} \sum_{j=0}^\infty \left(2^{\delta 2j}\|\psi_j u\|^2_{L^2} + 2^{(\delta+m)2j}\sum_{|\alpha|=m}\|\partial^\alpha(\psi_j u)\|^2_{L^2}\right) \\ \quad + \sum_{j=0}^\infty 2^{(\delta+m+\lambda)2j}\left(\sum_{|\alpha|=m}\int\int \frac{|\partial^\alpha(\psi_j u)(x)-\partial^\alpha(\psi_j u)(y)|^2}{|x-y|^{3+2\lambda}}dxdy\right), \end{array}\right\} & s=m+\lambda. \end{cases} \tag{A.5}$$

Proposition A.1 ***(Equivalence of norms)*** *There are two positive constants c_0 and c_1 depending only on s, δ and the constants in (A.4) such that*

$$c_0\|u\|^\bigstar_{s,\delta}\le\|u\|^*_{s,\delta}\le c_1\|u\|^\bigstar_{s,\delta}. \tag{A.6}$$

This equivalence was proved in [40] (see also [4]).

We express these norms in terms of Fourier transform. Let

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^3} \int u(x) e^{-ix\cdot\xi} dx$$

denotes the Fourier transform, put

$$\Lambda^s u = \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}u), \tag{A.7}$$

and let H^s denotes the Bessel Potentials space having the norm

$$\|u\|_{H^s}^2 = \|\Lambda^s u\|_{L^2}^2 = \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi. \tag{A.8}$$

We also set

$$\|u\|_{h^s}^2 = \|\mathcal{F}^{-1}(|\xi|^s \mathcal{F}u)\|_{L^2}^2 = \int (|\xi|^s |\hat{u}(\xi)|)^2 d\xi.$$

It is well known that (see e. g. [19]; p. 240-241)

$$\|u\|_{h^s}^2 \simeq \begin{cases} \sum_{|\alpha|=m} \int |\partial^\alpha u|^2 dx & s = m \\ \sum_{|\alpha|=m} \int\int \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{3+2\lambda}} dx & s = m + \lambda \end{cases} \tag{A.9}$$

and since $(1 + |\xi|^2)^s \simeq (1 + |\xi|^s)$,

$$\|u\|_{H^s}^2 \simeq \left(\|u\|_{L^2}^2 + \|u\|_{h^s}^2\right). \tag{A.10}$$

Hence, by (A.5),

$$\left(\|u\|_{s,\delta}^{\bigstar}\right)^2 \simeq \sum_{j=0}^{\infty} \left(2^{\delta 2j} \|\psi_j u\|_{L^2}^2 + 2^{(\delta+s)2j} \|\psi_j u\|_{h^s}^2\right) \tag{A.11}$$

We invoke now the scaling $u_\epsilon(x) := u(\epsilon x)$ ($\epsilon > 0$), then simple calculations yields $\|u_\epsilon\|_{L^2}^2 = \epsilon^{-3}\|u\|_{L^2}^2$ and $\|u_\epsilon\|_{h^s}^2 = \epsilon^{2s-3}\|u\|_{h^s}^2$. Combining the later one with (A.10), we have

$$\|u_\epsilon\|_{H^s}^2 \simeq \epsilon^{-3} \left(\|u\|_{L^2} + \epsilon^{2s}\|u\|_{h^s}^2\right). \tag{A.12}$$

Setting $\epsilon = 2^j$, multiplying (A.12) by 2^{3j} and inserting it in (A.11), we conclude

$$\left(\|u\|_{s,\delta}^{\bigstar}\right)^2 \simeq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2^j}\|_{H^s}^2. \tag{A.13}$$

The last one is the most convenience form of norm for applications.

Definition 3.1 (Weighted Spaces, an infinite sum). *For $s \geq 0$ and $-\infty < \delta < \infty$, we define the $H_{s,\delta}$ norm by*

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j u)_{(2^j)}\|^2_{H^s}. \tag{A.14}$$

The space $H_{s,\delta}$ is the set of all temperate distributions with a finite norm given by (A.14).

Combining Proposition A.1 with (A.11) and (A.13) we get:

Theorem A.2 ***(Equivalence of norms, Triebel)*** *There are two positive constant c_0 and c_1 depending only on s, δ and the constants in (A.4) such that*

$$c_0\|u\|_{Hs,\delta} \leq \|u\|^*_{s,\delta} \leq c_1\|u\|_{Hs,\delta}. \tag{A.15}$$

Remark A.3 *Let $s' \leq s$ and $\delta' \leq \delta$, then the inclusion $H_{s,\delta} \hookrightarrow H_{s',\delta'}$ follows easily from the representations (A.8) and (A.14) of the norms.*

Remark A.4 *The functions $\{\psi_j\}$ are constructed by means of a composition of exponential functions. Hence, for any positive γ there holds*

$$c_1(\gamma,\alpha)|\partial^\alpha\psi_j^\gamma(x)| \leq |\partial^\alpha\psi_j(x)| \leq c_2(\gamma,\alpha)|\partial^\alpha\psi_j^\gamma(x)|. \tag{A.16}$$

Therefore the equivalence (A.6) remains valid with ψ_j^γ replacing ψ_j and hence

$$\sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^\gamma u)_{(2^j)}\|^2_{H^s} \simeq \left(\|u\|^\star_{s,\delta}\right)^2 \simeq \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j u)_{(2^j)}\|^2_{H^s}. \tag{A.17}$$

Corollary A.5 ***(Equivalence of norms)*** *For any positive γ, there are two positive constants c_0 and c_1 depending on s, δ and γ such that*

$$c_0\|u\|^2_{H_{s,\delta}} \leq \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^\gamma u)_{(2^j)}\|^2_{H^s} \leq c_1\|u\|^2_{H_{s,\delta}}. \tag{A.18}$$

Definition A.6 ***(Inner-Product)*** *The norm A.14 enables us to define an inner-product on $H_{s,\delta}$. We first recall that if $u, v : \mathbb{R}^3 \to \mathbb{R}^m$ are in H^s, then*

$$\langle u,v\rangle_s = \langle\Lambda^s u, \Lambda^s v\rangle_{L^2} = \int \left(\Lambda^s u\right)^T \left(\Lambda^s v\right) dx = \int (1+|\xi|^2)^s \hat{u}^T(\xi)\overline{\hat{v}(\xi)}d\xi \tag{A.19}$$

is an inner product on H^s, here U^T denotes the transpose vector. By means of this and Corollary A.5, for any positive γ, the expression

$$\langle u,v\rangle_{s,\delta} = \sum_j 2^{(\frac{3}{2}+\delta)2j}\left\langle(\psi_j^\gamma u)_{(2^j)}, (\psi_j^\gamma v)_{(2^j)}\right\rangle_s \tag{A.20}$$

is an inner-product on $H_{s,\delta}$.

Theorem A.7 ***(Complex interpolation, Triebel)*** *Let $0 < \theta < 1$, $0 \le s_0 < s_1$ and $s_\theta = \theta s_0 + (1-\theta)s_1$, then*

$$[H_{s_0,\delta}, H_{s_1,\delta}]_\theta = H_{s_\theta,\delta}, \tag{A.21}$$

where (A.21) is a complex interpolation.

As a consequence of the interpolation Theorem A.7 we get

Corollary A.8 ***(Embedding of $H_{s,\delta}$ in $H_{s-1,\delta+1}$)***

$$\|\partial_i u\|_{H_{s-1,\delta+1}} \le \|u\|_{H_{s,\delta}} \tag{A.22}$$

Proof (of Corollary A.8) Let m be a positive integer and define $T : H_{m,\delta} \to H_{m-1,\delta+1}$ by $T(u) = \partial_i u$. Using the norm (A.1) we see that $\|T(u)\|_{H_{m-1,\delta+1}} \le \|u\|_{H_{m,\delta}}$. So (A.22) follows from Theorem A.7. ■

Remark A.9 *If* $\operatorname{supp} u \subset \{|x| \le R\}$, *then for any* δ

$$c_1(R)\|u\|_{H^s} \le \|u\|_{H_{s,\delta}} \le c_2(R)\|u\|_{H^s}. \tag{A.23}$$

This follows from the integral representation of the norm (A.1) and the interpolation (A.21).

B Some Properties of $H_{s,\delta}$

We start with a well known fact in H^s spaces.

Proposition B.1 ***(Multiplication by smooth functions)*** *Let $N \ge s$ be an integer. Assume $f \in C^N(\mathbb{R}^3)$ satisfies $\sup_{|\alpha|\le N} |\partial^\alpha f| \le K$, then*

$$\|fu\|_{H^s} \le C_s K \|u\|_{H^s}. \tag{B.1}$$

Proof (of Proposition B.1) Obviously there holds $\|fu\|_{H^N} \le CK\|u\|_{H^N}$ and $\|fu\|_{L^2} \le K\|u\|_{L^2}$. Since H^s is a complex interpolation space $[L^2, H^N]_\theta = H^s$, where $\theta = \frac{N-s}{N}$ (see e.g [37]; 13.6) and in addition $u \mapsto fu$ is a linear map, it follows from Interpolation Theory that

$$\|fu\|_{H^s} \le KC^{1-\theta}\|u\|_{H^s}. \tag{B.2}$$

■

Let $\chi_R \in C^\infty(\mathbb{R}^3)$ satisfies $\chi_R(x) = 1$ for $|x| \le R$, $\chi_R(x) = 0$ for $|x| \ge 2R$ and

$$|\partial^\alpha \chi_R| \le c_\alpha R^{-|\alpha|}. \tag{B.3}$$

Proposition B.2 ***(Two useful estimates)***

(a) *Let $N \geq s$ be an integer. Assume $f \in C^N(\mathbb{R}^3)$ satisfies* $\sup |D^k f| \leq K$ *for* $k = 0, 1, ...N$, *then*

$$\|fu\|_{H_{s,\delta}} \leq C_s K \|u\|_{H_{s,\delta}}. \tag{B.4}$$

(b) *For $\delta' < \delta$*

$$\|(1-\chi_R)u\|_{H_{s,\delta'}} \leq \frac{C(\delta,\delta')}{R^{\delta-\delta'}} \|u\|_{H_{s,\delta}}. \tag{B.5}$$

Corollary B.3 ***(Multiplication by cutoff functions)***

(a)

$$\|(D^m \chi_R)u\|_{H_{s,\delta}} \leq \frac{C}{R^m} \|u\|_{H_{s,\delta}}. \tag{B.6}$$

(b)

$$\|(1-\chi_R)u\|_{H_{s,\delta}} \leq C \|u\|_{H_{s,\delta}}. \tag{B.7}$$

Proof (of Proposition B.2)

(a) By Proposition (B.1),

$$\begin{aligned} \|fu\|^2_{H_{s,\delta}} &= \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j f u)_{2^j}\right\|^2_{H^s} \leq (CK)^2 \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s} \\ &= (CK)^2 \|u\|^2_{H_{s,\delta}}. \end{aligned} \tag{B.8}$$

(b) Let J_0 be the smallest integer such that $R \leq 2^{J_0 - 3}$. Then $(1-\chi_R)\psi_j = 0$ for $j = 0, 1, ..., J_0 - 1$. Hence

$$\begin{aligned} \|(1-\chi_R)u\|^2_{H_{s,\delta'}} &= \sum_{j=J_0}^{\infty} 2^{\left(\frac{3}{2}+\delta'\right)2j} \left\|(\psi_j(1-\chi_R)u)_{2^j}\right\|^2_{H^s} \\ \leq C^2 \sum_{j=J_0}^{\infty} 2^{\left(\frac{3}{2}+\delta'\right)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s} &= C^2 \sum_{j=J_0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} 2^{(\delta'-\delta)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s} \\ \leq C^2 2^{(\delta'-\delta)2J_0} \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s} &\leq \frac{C^2}{(8R)^{(\delta-\delta')2}} \|u\|^2_{H_{s,\delta}}. \end{aligned} \tag{B.9}$$

■

B.1 Two intermediate estimates

Proposition B.4 ***(Intermediate estimates)***

(i) *Let $0 \leq s_0 < s < s_1$ and $\varepsilon > 0$, then there is a constant $C = C(\varepsilon)$ such that*

$$\|u\|_{H_{s,\delta}} \leq \sqrt{2}\varepsilon\|u\|_{H_{s_1,\delta}} + C\|u\|_{H_{s_0,\delta}}, \tag{B.10}$$

holds for all $u \in H_{s_1,\delta}$.

(ii) *Let $0 < s' < s$, then*

$$\|u\|_{H_{s',\delta}} \leq \|u\|_{H_{s,\delta}}^{\frac{s'}{s}}\|u\|_{H_{0,\delta}}^{1-\frac{s'}{s}}. \tag{B.11}$$

Proof (of Proposition B.4) Both inequalities (B.10) and (B.11) are well known in H^s spaces. We apply them to each term of the norm (A.14). Therefore

$$\begin{aligned}
\|u\|_{H_{s,\delta}}^2 &= \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|_{H^s}^2 \\
&\leq 2\epsilon^2 \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|_{H^{s_1}}^2 + 2C^2(\epsilon) \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|_{H^{s_0}}^2 \\
&= 2\epsilon^2\|u\|_{H_{s_1,\delta}}^2 + 2C^2(\epsilon)\|u\|_{H_{s_0,\delta}}^2,
\end{aligned}$$

which proves (i). In the proof of (ii) we use Hölder inequality and obtain

$$\begin{aligned}
\|u\|_{H_{s,\delta}}^2 &= \sum_j 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|_{H_s^2}^2 \\
&\leq \sum_j 2^{\left(\frac{3}{2}+\delta\right)2j\left(\frac{s'}{s}\right)} \left\|(\psi_j u)_{2^j}\right\|_{H_s}^{2\frac{s'}{s}} \quad 2^{\left(\frac{3}{2}+\delta\right)2j\left(\frac{s-s'}{s}\right)} \left\|(\psi_j u)_{2^j}\right\|_{L_2}^{2\frac{s-s'}{s}} \\
&\leq \left(\sum_j 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|_{H_s}^2\right)^{\frac{s'}{s}} \quad \left(\sum_j 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)\, u_{2^j}\right\|_{L_2}^2\right)^{\frac{s-s'}{s}} \\
&= \left(\|u\|_{H_{s,\delta}}\right)^{\frac{2s'}{s}} \quad \left(\|u\|_{L_{2,\delta}}\right)^{\frac{2(s'-1)}{s}}.
\end{aligned}$$

■

B.2 Algebra

Proposition B.5 ***(Algebra in $H_{s,\delta}$)*** *If $s_1, s_2 \geq s$, $s_1 + s_2 > s + \frac{3}{2}$ and $\delta_1 + \delta_2 \geq \delta - \frac{3}{2}$, then*

$$\|uv\|_{H_{s,\delta}} \leq C\|u\|_{H_{s_1,\delta_1}}\|v\|_{H_{s_2,\delta_2}}. \tag{B.12}$$

Proof (of Proposition B.5) By Corollary A.5,

$$\|uv\|_{H_{s,\delta}}^2 \simeq \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\| \left(\psi_j^2 uv\right)_{2^j} \right\|_{H^s}^2 . \tag{B.13}$$

We apply the classic algebra property $\|uv\|_{H^s} \le C\|u\|_{H^{s_1}}\|v\|_{H^{s_2}}$ (see e. g. [36]), to each term of the norm (B.13) and then we use Cauchy Schwarz inequality,

$$\begin{aligned}
\|uv\|_{H_{s,\delta}}^2 &\le C \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\| \left(\psi_j^2 uv\right)_{2^j} \right\|_{H^s}^2 \\
&\le C^2 \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\| (\psi_j u)_{2^j} \right\|_{H^{s_1}}^2 \left\| (\psi_j v)_{2^j} \right\|_{H^{s_2}}^2 \\
&\le C^2 \sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_1\right)2j} \left\| (\psi_j u)_{2^j} \right\|_{H^{s_1}}^2 \right) \left(2^{\left(\frac{3}{2}+\delta_2\right)2j} \left\| (\psi_j v)_{2^j} \right\|_{H^{s_2}}^2 \right) \\
&\le C^2 \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_1\right)2j} \left\| (\psi_j u)_{2^j} \right\|_{H^{s_1}}^2 \right)^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_2\right)2j} \left\| (\psi_j v)_{2^j} \right\|_{H^{s_2}}^2 \right)^2 \right)^{\frac{1}{2}} \\
&\le C^2 \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_1\right)2j} \left\| (\psi_j u)_{2^j} \right\|_{H^{s_1}}^2 \right) \right) \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_2\right)2j} \left\| (\psi_j v)_{2^j} \right\|_{H^{s_2}}^2 \right) \right) \\
&\le C^2 \|u\|_{H_{s_1,\delta_1}}^2 \|v\|_{H_{s_2,\delta_2}}^2 .
\end{aligned}$$

■

B.3 Fractional power $|u|^\gamma$

In [22] Kateb showed that if $u \in H^s \cap L^\infty$, $1 < \gamma$ and $0 < s < \gamma + \frac{1}{2}$, then

$$\||u|^\gamma\|_{H^s} \le C(\|u\|_{L^\infty})\|u\|_{H^s}. \tag{B.14}$$

Proposition B.6 ***(Fractional power in*** $H_{s,\delta}$***)*** *Let* $u \in H_{s,\delta} \cap L^\infty$, $1 < \gamma$, $0 < s < \gamma + \frac{1}{2}$ *and* $\delta \in \mathbb{R}$, *then*

$$\||u|^\gamma\|_{H_{s,\delta}} \le C(\|u\|_{L^\infty})\|u\|_{H_{s,\delta}}. \tag{B.15}$$

Proof (of Proposition B.6) Property (B.15) is a direct consequence of the equivalence (A.18) and (B.14). Because

$$\begin{aligned} \| |u|^{\gamma} \|^2_{H s,\delta} &\simeq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \| (\psi_j^{\gamma} |u|^{\gamma})_{(2^j)} \|^2_{H^s} \\ &\leq (C(\|u\|_{L^\infty}))^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \| (\psi_j u)_{(2^j)} \|^2_{H^s} \leq (C(\|u\|_{L^\infty}))^2 \|u\|^2_{H_{s,\delta}}. \end{aligned} \tag{B.16}$$

■

B.4 Moser type estimates

Y. Meyer proved the below Moser type estimate [30].See also Taylor [38].

Theorem B.7 ***(Third Moser inequality for Bessel potentials spaces)*** *Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be C^{N+1} function such that $F(0) = 0$. Let $s > 0$ and $u \in H^s \cap L^\infty$. Then*

$$\|F(u)\|_{H^s} \leq K \|u\|_{H^s}, \tag{B.17}$$

where

$$K = K_N(F, \|u\|_{L^\infty}) \leq C \|F\|_{C^{N+1}} \left(1 + \|u\|^N_{L^\infty}\right), \tag{B.18}$$

here N is a positive integer such that $N \geq [s] + 1$.

We generalize this important inequality to the $H_{s,\delta}$ spaces.

Theorem B.8 ***(Third Moser inequality in $H_{s,\delta}$)*** *Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be C^{N+1} function such that $F(0) = 0$. Let $s > 0$, $\delta \in \mathbb{R}$ and $u \in H_{s,\delta} \cap L^\infty$. Then*

$$\|F(u)\|_{H_{s,\delta}} \leq K \|u\|_{H_{s,\delta}}, \tag{B.19}$$

The constant K in (B.19) depends on one in (B.18) and in addition on δ.

Proof (of Theorem B.8) We set $\Psi_j(x) = \frac{1}{\varphi(x)} \psi_j(x)$, where $\varphi(x) = \sum_{j=0}^{\infty} \psi_j(x)$. From the properties of the sequence $\{\psi_j\}$, it follows that $1 \leq \varphi(x) \leq 7$. So the sequence $\{\Psi_j\} \subset C_0^\infty(\mathbb{R}^3)$ and $\sum_{j=0}^{\infty} \Psi_j(x) = 1$. From (A.12) we conclude that

$$\|u_\epsilon\|^2_{H^s} \leq C \max\{\epsilon^{2s-3}, \epsilon^{-3}\} \|u\|^2_{H^s} \tag{B.20}$$

and with the combination of Proposition B.1 and Meyer's TheoremB.7 we have,

$$\begin{aligned}
\|F(u)\|^2_{H_{s,\delta}} &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j(F(u))_{(2^j)}\|^2_{H^s} \\
&= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| \left(\psi_j F\left(\sum_{k=0}^{\infty} \Psi_k(x) u \right)\right)_{(2^j)} \right\|^2_{H^s} \\
&= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| \left(\psi_j F\left(\sum_{k=j-4}^{j+3} \Psi_k(x) u \right)\right)_{(2^j)} \right\|^2_{H^s} \\
&\leq CK^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \| \left(\Psi_k u\right)_{(2^j)} \|^2_{H^s} \\
&\leq CK^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \| \left((\Psi_k u)_{2^{j-k}} \right)_{(2^k)} \|^2_{H^s} \\
&\leq CK^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \max\{2^{(2s-3)(j-k)}, 2^{-3(j-k)}\} \| \left(\Psi_k u\right)_{(2^k)} \|^2_{H^s} \\
&\leq C(s)K^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \| \left(\psi_k u\right)_{(2^k)} \|^2_{H^s} \\
&\leq C(s,\delta)K^2 \sum_{j=0}^{\infty} \sum_{k=j-4}^{j+3} 2^{(\frac{3}{2}+\delta)2k} \| \left(\psi_k u\right)_{(2^k)} \|^2_{H^s} \\
&\leq 7C(s,\delta)K^2 \sum_{k=0}^{\infty} 2^{(\frac{3}{2}+\delta)2k} \| \left(\psi_k u\right)_{(2^k)} \|^2_{H^s} \leq 7C(s,\delta)K^2 \|u\|^2_{H_{s,\delta}}.
\end{aligned} \tag{B.21}$$

■

As a consequence of Theorem B.13 we can sharpen this result.

Corollary B.9 ***(Sharp version of the third Moser inequality)*** *Let $F:\mathbb{R}^m \to \mathbb{R}^l$ be C^{N+1} function, $F(0)=0$, $N \geq [s]+1$, $s > \frac{3}{2}$ and $\delta \geq -\frac{3}{2}$. Then*

$$\|F(u)\|_{H_{s,\delta}} \leq C(\|F\|_{C^{N+1}}, \|u\|_{H_{s,\delta}})\|u\|_{H_{s,\delta}}. \tag{B.22}$$

Remark B.10 *If $F(0) \neq 0$ and $F(0) \in H_{s,\delta}$, then we can apply Theorem B.8 to $\widetilde{F}(u) := F(u) - F(0)$ and get*

$$\|F(u)\|_{H_{s,\delta}} \leq \|\widetilde{F}(u)\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}} \leq K\|u\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}}. \tag{B.23}$$

We may apply Theorem B.8 to the estimate the difference $F(u) - F(v)$.

Corollary B.11 ***(A difference estimate in $H_{s,\delta}$)*** *Suppose F is a C^{N+2} function and $u, v \in H_{s,\delta} \cap L^\infty$. Then*

$$\|F(u) - F(v)\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty}) \left(\|u\|_{H_{s,\delta}} + \|v\|_{H_{s,\delta}}\right) \|u - v\|_{H_{s,\delta}}. \tag{B.24}$$

Proof (of Corollary B.11) Put $\tilde{F}(u) = F(u) - F(0) - DF'(0)u$, then it suffices to show inequality (B.24) for $\tilde{F}$. Now,

$$\tilde{F}(u) - \tilde{F}(v) = \int_0^1 \left(D\tilde{F}\left(tu + (1-t)v\right)\right)(u - v)dt = G(u, v)(u - v), \tag{B.25}$$

where $G(u, v) = \int_0^1 D\tilde{F}\left(tu + (1-t)v\right)dt$. Since $G(0, 0) = \int_0^1 D\tilde{F}(0)dt = 0$, we can apply Theorem B.8 to $G(u, v)$ and get:

$$\|G(u, v)\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty}) \left(\|u\|_{H_{s,\delta}} + \|v\|_{H_{s,\delta}}\right). \tag{B.26}$$

Applying algebra (B.12) to the right side of (B.25), we have

$$\left\|\tilde{F}(u) - \tilde{F}(v)\right\|_{H_{s,\delta}} \leq C \|G(u, v)\|_{H_{s,\delta}} \|(u - v)\|_{H_{s,\delta}} \tag{B.27}$$

and its combination with (B.26) gives (B.24). ∎

B.5 Compact embedding

Theorem B.12 ***(Compact embedding)*** *Let $0 \leq s' < s$ and $\delta' < \delta$, then the embedding*

$$H_{s,\delta} \hookrightarrow H_{s',\delta'}. \tag{B.28}$$

is compact.

Proof (of Theorem B.12) Let $\{u_n\} \subset H_{s,\delta}$ be a sequence with $\|u_n\|_{H_{s,\delta}} \leq 1$. Since $H_{s,\delta}$ is a Hilbert space there is a subsequence, denoted by $\{u_n\}$, which converges weakly to u_0. We will complete the proof by showing that $u_n \to u_0$ strongly in $H_{s',\delta'}$.

Let $\chi_R \in C_0^\infty$ such that $\chi_R(x) = 1$ for $|x| \leq R$ and $\mathrm{supp}(\chi_R) \subset B_{2R}$. For a given $\epsilon > 0$, we take R such that $2\frac{C(\delta,\delta')}{R^{\delta-\delta'}} < \epsilon$, where $C(\delta, \delta')$ is the constant of inequality (B.5). For a bounded domain Ω, it is known that the embedding $H^s(\Omega) \hookrightarrow H^{s'}(\Omega)$ is compact and from Remark A.9 it follows that $\|\chi_R u_n\|_{H^s} \leq C$, where C does not depend on n . Hence $\chi_R u_n$ converges strongly to $\hat{u}_0$ in $H^{s'}$. In addition, we have that $\chi_R u_n \to \chi_R u_0$ weakly in H^s and hence $\chi_R u_n \to \chi_R u_0$ weakly in $H^{s'}$. Thus the sequence $\{\chi_R u_n\}$ converges both strongly to $\hat{u}_0$ and weakly to $\chi_R u_0$ in $H^{s'}$, hence $\hat{u}_0 = \chi_R u_0$ (because $\lim_n \langle(\chi_R u_n - \chi_R u_0), (\hat{u}_0 - \chi_R u_0)\rangle_{s'} = \langle(\hat{u}_0 - \chi_R u_0), (\hat{u}_0 - \chi_R u_0)\rangle_{s'} = \|\hat{u}_0 - \chi_R u_0\|^2_{H^{s'}} = 0$).

By (A.23) $\lim_n \|\chi_R u_n - \chi_R u_0\|_{H_{s',\delta'}} = 0$, hence we may take n sufficiently large so that $\|\chi_R u_n - \chi_R u_0\|_{H_{s',\delta'}} < \epsilon$. Therefore

$$\begin{aligned}
&\|u_n - u_0\|_{H_{s',\delta'}} = \|(\chi_R u_n - \chi_R u_0) + (1-\chi_R)(u_n - u_0)\|_{H_{s',\delta'}} \\
&\leq \|(\chi_R u_n - \chi_R u_0)\|_{H_{s',\delta'}} + \|(1-\chi_R)(u_n - u_0)\|_{H_{s',\delta'}} \\
&< \epsilon + \frac{C}{R^{\delta-\delta'}}\|(u_n - u_0)\|_{H_{s,\delta}} \leq \epsilon + \frac{C}{R^{\delta-\delta'}}\left(\|u_n\|_{H_{s,\delta}} + \|u_0\|_{H_{s,\delta}}\right) \\
&\leq \epsilon + 2\frac{C(\delta,\delta')}{R^{\delta-\delta'}} < 2\epsilon
\end{aligned} \tag{B.29}$$

and that completes the proof. ∎

B.6 Embedding into the continuous

We introduce the following notations. For a nonnegative integer m, $0 < \sigma < 1$ and $\beta \in \mathbb{R}$, we set

$$\begin{gathered}
H_\sigma(x,u) = \sup_{\{y:|y-x|\leq \frac{1}{2}(1+|x|)\}} \tfrac{|u(x)-u(y)|}{|x-y|^\sigma} \\
\|u\|_{C_\beta} = \sup_x \left((1+|x|)^\beta |u(x)|\right) \\
\|u\|_{C^\sigma_\beta} = \|u\|_{C_\beta} + \sup_x \left((1+|x|)^{\beta+\sigma} H_\sigma(x,u)\right) \\
\|u\|_{C^m_\beta} = \sum_{|\alpha|\leq m} \sup_x \left((1+|x|)^{\beta+|\alpha|}|\partial^\alpha u(x)|\right) \\
\|u\|_{C^{m+\sigma}_\beta} = \|u\|_{C^m_\beta} + \sum_{|\alpha|=m} \sup_x \left((1+|x|)^{\beta+m+\sigma}|H_\sigma(x,\partial^\alpha u)|\right)
\end{gathered}$$

Let C^m_β, $C^{m+\sigma}_\beta$ be the functions spaces corresponding to the above norms.

Theorem B.13 ***(Embedding into the continuous)***

1. *If $s > \frac{3}{2} + m$ and $\delta + \frac{3}{2} \geq \beta$, then any $u \in H_{s,\delta}$ has a representative $\tilde{u} \in C^m_\beta$ satisfying*
$$\|\tilde{u}\|_{C^m_\beta} \leq C\|u\|_{H_{s,\delta}}. \tag{B.30}$$

2. *If $s > \frac{3}{2} + m + \sigma$ and $\delta + \frac{3}{2} \geq \beta$, then any $u \in H_{s,\delta}$ has a representative $\tilde{u} \in C^m_{\beta,\sigma}$ satisfying*
$$\|\tilde{u}\|_{C^{m+\sigma}_\beta} \leq C\|u\|_{H_{s,\delta}}. \tag{B.31}$$

Proof (of Theorem B.13) We first show (B.30) and (B.31) when $m = 0$. In order to make notations simpler we will use the convention $2^k = 0$ if $k < 0$. Recall that $\psi_j(x) = 1$

on $K_j := \{2^{j-3} \leq |x| \leq 2^{j+2}\}$. Using the known embedding $\sup_x |u(x)| \leq C\|u\|_{H^s}$ (see e. g. [26]),we have

$$\begin{aligned}
&\sup_x (1+|x|)^\beta |u(x)| \leq 2^\beta \sup_{j\geq -1} \left(2^{\beta j} \sup_{\{2^j \leq |x| \leq 2^{j+1}\}} |u(x)| \right) \\
\leq & 2^\beta \sup_{j\geq -1} \left(2^{\beta j} \sup |\psi_j(x) u(x)| \right) = 2^\beta \sup_{j\geq -1} \left(2^{\beta j} \sup |\psi_j(2^j x) u(2^j x)| \right) \\
\leq & 2^\beta C \sup_{j\geq -1} \left(2^{\beta j} \|(\psi_j u)_{2^j}\|_{H^s} \right) \leq 2^\beta C \sup_{j\geq -1} \left(2^{(\frac{3}{2}+\delta)j} \|(\psi_j u)_{2^j}\|_{H^s} \right) \leq 2^\beta C \|u\|_{H_{s,\delta}}.
\end{aligned} \tag{B.32}$$

In order to show (B.31) we use the known estimate

$$\sup_x |u(x)| + \sup_{x,y,x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\sigma} \leq C\|u\|_{H^s}.x$$

(see e. g. [26]) and repeat similar arguments to the above, then

$$\begin{aligned}
&\sup_x (1+|x|)^{(\beta+\sigma)} \sup_{\{|y-x|\leq \frac{1}{2}(1+|x|)} \frac{|u(x)-u(y)|}{|x-y|^\sigma} \\
&\leq 2^{(\beta+\sigma)} \sup_{j\geq -1} \left(2^{(\beta+\sigma)j} \sup_{\{2^j \leq |x| \leq 2^{j+1}\}} \sup_{\{|y-x|\leq \frac{1}{2}(1+|x|)\}} \frac{|u(x)-u(y)|}{|x-y|^\sigma} \right) \\
&\leq 2^{(\beta+\sigma)} \sup_{j\geq -1} \left(2^{(\beta+\sigma)j} \sup_{\{2^j \leq |x| \leq 2^{j+1}\}} \sup_{\{\frac{1}{2}\max\{2^j-1,0\} \leq |y| \leq \frac{1}{2}(1+3\cdot 2^{j+1})\}} \frac{|u(x)-u(y)|}{|x-y|^\sigma} \right) \\
&\leq 2^{(\beta+\sigma)} \sup_{j\geq -1} \left(2^{(\beta+\sigma)j} \sup_{x\in K_j} \sup_{y\in K_j} \frac{|u(x)-u(y)|}{|x-y|^\sigma} \right) \\
&\leq 2^{(\beta+\sigma)} \sup_{j\geq -1} \left(2^{(\beta+\sigma)j} \sup_x \sup_y \frac{|\psi_j(x)u(x)-\psi_j(y)u(y)|}{|x-y|^\sigma} \right) \\
&\leq 2^{(\beta+\sigma)} \sup_{j\geq -1} \left(2^{(\beta+\sigma)j} \sup_x \sup_y \frac{|\psi_j(2^j x)u(2^j x)-\psi_j(2^j y)u(2^j y)|}{|2^j x-2^j y|^\sigma} \right) \\
&= 2^{(\beta+\sigma)} \sup_{j\geq -1} \left(2^{\beta j} \sup_x \sup_y \frac{|\psi_j(2^j x)u(2^j x)-\psi_j(2^j y)u(2^j y)|}{|x-y|^\sigma} \right) \\
&\leq 2^{(\beta+\sigma)} C \sup_{j\geq -1} \left(2^{\beta j} \|(\psi_j u)_{2^j}\|_{H^s} \right) \leq 2^{(\beta+\sigma)} C \sup_{j\geq -1} \left(2^{(\frac{3}{2}+\delta)j} \|(\psi_j u)_{2^j}\|_{H^s} \right) \\
&\leq C\|u\|_{H_{s,\delta}}.
\end{aligned} \tag{B.33}$$

If $m > 1$, $s > \frac{3}{2}+m$ or $s > \frac{3}{2}+\sigma+m$ and $\delta+\frac{3}{2} \geq \beta$, then $\partial^\alpha u \in H_{s-|\alpha|,\delta+|\alpha|}$ for $1 \leq |\alpha| \leq m$. So we may apply (B.32) and (B.33) to $\partial^\alpha u$ and obtain $\|\partial^\alpha u\|_{C_{\beta+k}}$ or $\|\partial^\alpha u\|_{C^\sigma_{\beta+k}}$ are less or equal to $\|\partial^\alpha u\|_{H_{s-|\alpha|,\delta+|\alpha|}}$. ■

B.7 Density

Theorem B.14 ***(Density of C_0^∞ functions)***

(a) *The class $C_0^\infty(\mathbb{R}^3)$ is dense in $H_{s,\delta}$.*

(b) *Given $u \in H_{s,\delta}$ and $s' > s \geq 0$. Then for $\rho > 0$ there is $u_\rho \in C_0^\infty(\mathbb{R}^3)$ and a positive constant $C(\rho)$ such that*

$$\|u_\rho - u\|_{H_{s,\delta}} \leq \rho \quad \text{and} \quad \|u_\rho\|_{H_{s',\delta}} \leq C(\rho)\|u\|_{H_{s,\delta}}. \tag{B.34}$$

Property (a) was proved by Triebel [41]. We prove both of them here since (b) relies on (a).

Proof (of Theorem B.14) Let J_ϵ be the standard mollifier, that is, $\mathrm{supp}(J_\epsilon) \subset B(0,\epsilon)$, $\hat{J}_\epsilon(\xi) = \hat{J}_1(\epsilon\xi) = \hat{J}(\epsilon\xi)$ and $\hat{J}(0) = 1$. It is well known that for any $v \in H^s$, $\|J_\epsilon * v - v\|_{H^s} \to 0$ and that $J_\epsilon * v$ belongs to $C^\infty(\mathbb{R}^3)$. In addition, we claim that there is $C = C(\epsilon, s, s')$ such that

$$\|J_\epsilon * v\|_{H^{s'}} \leq C\|v\|_{H^s}. \tag{B.35}$$

Indeed, since $J \in C_0^\infty(\mathbb{R}^3)$, $|\hat{J}(\xi)| \leq C_m(1+|\xi|)^{-m}$ for any integer m. Therefore, for a given s' and ϵ, we chose m and the constant $C(\epsilon, s, s')$ so that $(1+|\xi|^2)^{s'-s}|\hat{J}(\epsilon\xi)|^2 \leq C^2(\epsilon, s, s')$. Hence

$$\begin{aligned}
\|J_\epsilon * v\|_{H^{s'}}^2 = &\int (1+|\xi|^2)^{s'}|\hat{J}(\epsilon\xi)|^2|\hat{v}(\xi)|^2 d\xi = \int (1+|\xi|^2)^{s}|\hat{v}(\xi)|^2(1+|\xi|^2)^{s'-s}\hat{J}(\epsilon\xi)|^2 d\xi \\
\leq &C^2(\epsilon, s, s')\int (1+|\xi|^2)^{s}|\hat{v}(\xi)|^2 d\xi = C^2(\epsilon, s, s')\|v\|_{H^s}^2.
\end{aligned}$$

(a) Given $u \in H_{s,\delta}$ and $\rho > 0$ we may chose N such that

$$\sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j(u)_{(2^j)}\|_{H^s}^2 \leq \rho^2.$$

Set now $u_N = \sum_{j=0}^{N}\Psi_k u$, where Ψ_k is defined as in the proof of Theorem B.8. We use Proposition B.1 and get

$$\|u - u_N\|_{H_{s,\delta}}^2 \leq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\|\left(\psi_j\left(\sum_{k=N+1}^{\infty}\Psi_k u\right)\right)_{(2^j)}\right\|_{H^s}^2$$

$$= \sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\|\left(\sum_{k=j-3}^{j+4}\psi_j\Psi_k u\right)_{(2^j)}\right\|_{H^s}^2 \leq C\sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\sum_{k=j-3}^{j+4}\left\|(\psi_j u)_{(2^j)}\right\|_{H^s}^2$$

$$\leq 7C\sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\|(\psi_j u)_{(2^j)}\right\|_{H^s}^2 = 7C\rho^2.$$

Now u_N has compact support, therefore $J_\epsilon * u_N \in C_0^\infty(\mathbb{R}^3)$ and

$$\|J_\epsilon * u_N - u_N\|^2_{H_{s,\delta}} \leq \sum_{j=0}^{N+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j (J_\epsilon * u_N - u_N))_{(2^j)} \right\|^2_{H^s} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

(b) Let $u \in H_{s,\delta}$ and $\rho > 0$, then by (a) we can chose N sufficiently large and ϵ small so that $\|J_\epsilon * u_N - u\|_{H_{s,\delta}} < \rho$ and by (B.35)

$$\|J_\epsilon * u_N\|^2_{H_{s',\delta}} \leq \sum_{j=0}^{N+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j (J_\epsilon * u_N))_{(2^j)} \right\|^2_{H^{s'}}$$

$$\leq C^2(\epsilon, s, s') \sum_{j=0}^{N+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j u_N))_{(2^j)} \right\|^2_{H^s} \leq C^2 C^2(\epsilon, s, s') \|u\|^2_{H_{s,\delta}}.$$

Thus, $u_\rho = J_\epsilon * u_N$.

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List of theorems, lemmas, corollaries