

# The Mixed Problem for a Degenerate Operator Equation<sup>\*†</sup>

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## Abstract

We consider a mixed problem for a degenerate differential operator equation of higher order. We establish some embedding theorems in weighted Sobolev spaces and show existence and uniqueness of the generalized solution of this problem. We also give a description of the spectrum for the corresponding operator.

## 1 Introduction

In this article we consider the mixed problem for the operator equation

$$Pu \equiv (-1)^m D_t^m (t^\alpha D_t^m) u + t^\alpha Au = f, \quad (1)$$

where  $t \in (0, b)$ ,  $\alpha \geq 0$ ,  $D_t \equiv d/dt$ ,  $f \in L_{2,-\alpha}((0, b), \mathcal{H})$ , and  $A : \mathcal{H} \rightarrow \mathcal{H}$  is a linear operator in a separable Hilbert space  $\mathcal{H}$ .

Our approach, similar to that used in [3], [15] for the case  $m = 1$  and in [12] for  $m = 2$ , is based on the consideration of the one-dimensional equation (1), i.e., when  $A$  is the operator of multiplication by a number  $a$  (see [8]). Note that in [3], [12] and [13] have been considered Dirichlet problem for the equation (1). It has been proved in [14] that the spectrum of the operator  $\mathbb{L}u \equiv (-1)^m t^{2m-\alpha} D_t^m (t^\alpha D_t^m) u$ ,  $\mathbb{L} : L_{\alpha-2m} \rightarrow L_{\alpha-2m}$ ,  $\alpha \neq 1, 3, \dots, 2m-1$  is purely continuous and coincides with the ray  $[4^{-m}(1-\alpha)^2(3-\alpha)^2 \dots (2m-1-\alpha)^2; +\infty)$ .

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In Section 2 we define the weighted Sobolev spaces  $\dot{W}_\alpha^m$ ,  $W_\alpha^m(b)$ ,  $W_\alpha^m(0)$ ,  $W_\alpha^m$ , describe the behaviour of the functions from these spaces close to  $t = 0$  and prove some embedding and compactness theorems. Furthermore we define the generalized solution for the one-dimensional equation (1) and give a sufficient condition on the number  $a$  which guarantees existence and uniqueness of the generalized solution for every  $f \in L_{2,-\alpha}$ .

In Section 3 under some conditions on the spectrum of the operator  $A$  we prove unique solvability of the operator equation (1) for every  $f \in L_{2,-\alpha}((0, b), \mathcal{H})$  and give the description of the spectrum for the corresponding operator  $\mathbb{P} = t^{-\alpha}P$ .

Note that the operator  $A$  in the equation (1) in general is unbounded in  $\mathcal{H}$ .

## 2 The One-dimensional Case

### 2.1 Spaces $\dot{W}_\alpha^m$ , $W_\alpha^m(b)$ , $W_\alpha^m(0)$ and $W_\alpha^m$

Let  $\dot{C}^m[0, b]$  be the set of  $m$ -times continuously differentiable functions  $u(t)$  defined on  $[0, b]$  and satisfying the conditions

$$u^{(k)}(t)\Big|_{t=0} = u^{(k)}(t)\Big|_{t=b} = 0, \quad k = 0, 1, \dots, m-1. \quad (2)$$

Let  $\dot{W}_\alpha^m$ ,  $\alpha \geq 0$ , be the completion of  $\dot{C}^m[0, b]$  in the norm

$$|u, W_\alpha^m|^2 = \int_0^b t^\alpha |u^{(m)}(t)|^2 dt. \quad (3)$$

Denote  $L_{2,\beta} = \{f, |f, L_{2,\beta}|^2 = \int_0^b t^\beta |f(t)|^2 dt < \infty\}$ .

**Proposition 2.1** *For every  $u \in \dot{W}_\alpha^m$  close to  $t = 0$  we have*

$$|u^{(j)}(t)|^2 \leq C_j t^{2m-2j-1-\alpha} |u, \dot{W}_\alpha^m|^2 \quad (4)$$

where  $\alpha \neq 1, 3, \dots, 2m-1$ ,  $j = 0, 1, \dots, m-1$  and in the case  $\alpha = 2n+1$ ,  $n = 0, 1, \dots, m-1$ ,  $t^{2m-2j-2n-2} |\ln t|$  instead of  $t^{2m-2j-1-\alpha}$  in the expression (4),  $j = 0, 1, \dots, m-n-1$  (see [13]).

**Proposition 2.2** *For every  $\alpha \neq 1, 3, \dots, 2m-1$  we have a continuous embedding*

$$\dot{W}_\alpha^m \subset L_{2,\alpha-2m} \quad (5)$$

which is not compact. Moreover for every  $\beta > \alpha - 2m$  the embedding  $\dot{W}_\alpha^m \subset L_{2,\beta}$  is compact.

For the proof of these propositions see [5], [9] and [13]. Note also that here for  $\alpha \neq 1, 3, \dots, 2m - 1$  we have used Hardy's inequality (see [7])

$$\int_0^b t^{\beta-2} |u(t)|^2 dt \leq \frac{4}{(\beta-1)^2} \int_0^b t^\beta |u'(t)|^2 dt, \quad \beta \neq 1 \quad (6)$$

where the number  $4(\beta-1)^{-2}$  is the best possible constant. Note that for  $\alpha = 1, 3, \dots, 2m - 1$  the embedding (5) fails.

**Remark 2.3** *In the space  $\dot{W}_\alpha^m$  for  $\alpha \neq 1, 3, \dots, 2m - 1$  we can define an equivalent norm*

$$\|u\|_\alpha^2 \equiv \int_0^b \left| (t^{\alpha/2} u)^{(m)} \right|^2 dt. \quad (7)$$

Indeed, to estimate the norm (7) by the norm (3) we first apply the Leibniz rule for differentiation of the expression  $(t^{\alpha/2} u)^{(m)}$ , then use Hardy's inequality (6) for the derivatives  $u^{(k)}(t)$ ,  $k = 0, 1, \dots, m - 1$ , and estimate all terms by  $\int_0^b t^\alpha |u^{(m)}(t)|^2 dt$ . To estimate (3) by the norm (7) we first set  $u(t) = t^{-\alpha/2} v(t)$  and after differentiation again use Hardy's inequality for the derivatives  $v^{(k)}(t)$ ,  $k = 0, 1, \dots, m - 1$ , estimate all terms by  $\int_0^b |v^{(m)}(t)|^2 dt$  and then return to the function  $u(t)$ .

Note that Remark 2.3 for the case  $\alpha < 1$  was proved in [6].

Remark 2.3 allows us to write

$$\dot{W}_\alpha^m = t^{-\alpha/2} \dot{W}^m(0, b).$$

Denote by  $W_\alpha^m$  the completion of  $C^m[0, b]$  in the norm

$$|u, W_\alpha^m|^2 = \int_0^b \left( t^\alpha |u^{(m)}(t)|^2 + t^\alpha |u(t)|^2 \right) dt. \quad (8)$$

**Proposition 2.4** *For every  $u \in W_\alpha^m$  close to  $t = 0$  we have*

$$|u^{(j)}(t)|^2 \leq (B_j + C_j t^{2m-2j-1-\alpha}) |u, W_\alpha^m|^2, \quad (9)$$

where  $\alpha \neq 1, 3, \dots, 2m - 1$ ,  $j = 0, 1, \dots, m - 1$  and in the case  $\alpha = 2n + 1$ ,  $n = 0, 1, \dots, m - 1$ ,  $t^{2m-2j-2n-2} |\ln t|$  instead of  $t^{2m-2j-1-\alpha}$  in the expression (9),  $j = 0, 1, \dots, m - n - 1$ .

*Proof.* Let  $0 < h < b/2m$ . Then there are constants  $a_{jk}$  and  $b_{jk}$ ,  $j = 1, 2, \dots, m - 1$ ,  $k = 0, 1, \dots, m - 1$ , such that for every  $t \in (0, b/2)$  we have the following equalities (see [9])

$$u^{(j)}(t) = \sum_{k=0}^{m-1} \left[ a_{jk} u(t + kh) + b_{jk} \int_t^{t+kh} (t + kh - x)^{m-1} u^{(m)}(x) dx \right]. \quad (10)$$

For  $x \in [t, t + kh]$  we have  $0 \leq t + kh - x \leq (m - 1)h$  and, therefore,

$$\left| \int_t^{t+kh} (t + kh - x)^{m-1} u^{(m)}(x) dx \right| \leq c \int_t^b |u^{(m)}(x)| dx. \quad (11)$$

From (10) together with the inequality (11) we have

$$|u^{(j)}(t)|^2 \leq c \left( \sum_{k=0}^{m-1} |u(t + kh)|^2 + \left( \int_t^b |u^{(m)}(x)| dx \right)^2 \right). \quad (12)$$

Multiplying both sides of (12) by  $t^\alpha$  and integrating over  $(0, b/2)$  for  $j = 1, 2, \dots, m - 1$  we get

$$|u^{(j)}, L_{2,\alpha}(0, b/2)|^2 \leq c \left( |u, L_{2,\alpha}|^2 + \int_0^{b/2} t^\alpha \left( \int_t^b |u^{(m)}(x)| dx \right)^2 dt \right). \quad (13)$$

Now for  $\alpha \neq 1$  we can write

$$\begin{aligned} & \int_0^{b/2} t^\alpha \left( \int_t^b |u^{(m)}(x)| dx \right)^2 dt \leq \\ & \leq c \int_0^{b/2} t^\alpha (b^{1-\alpha} - t^{1-\alpha}) \left( \int_t^b x^\alpha |u^{(m)}(x)|^2 dx \right) dt. \end{aligned}$$

In the case  $\alpha = 1$  we proceed in the same way. Therefore, from the inequality (13) we get the following estimate for the intermediate derivatives  $u^{(j)}(t)$ ,  $j = 1, 2, \dots, m - 1$  for every  $\alpha \geq 0$

$$|u^{(j)}, L_{2,\alpha}(0, b/2)| \leq c |u, W_\alpha^m|. \quad (14)$$

Inequality (14) is true also for the case  $t \in (b/2, b)$ , because then we indeed do not have a weight (see [2]).

Let  $\alpha \neq 1, 3, \dots, 2m - 1$  and  $t_0 \in (0, b]$ . Then we can write

$$u^{(m-1)}(t) - u^{(m-1)}(t_0) = \int_{t_0}^t u^{(m)}(x) dx. \quad (15)$$

Therefore, we get

$$\begin{aligned} |u^{(m-1)}(t) - u^{(m-1)}(t_0)|^2 & \leq \left| \int_{t_0}^t x^{-\alpha} dx \int_{t_0}^t x^\alpha |u^{(m)}(x)|^2 dx \right| \leq \\ & \leq c |t^{1-\alpha} - t_0^{1-\alpha}| |u, W_\alpha^m|^2. \end{aligned}$$

From the equality (15) it follows that

$$\left|u^{(m-1)}(t)\right|^2 \leq 2 \left( \left|u^{(m-1)}(t_0)\right|^2 + \left|\int_{t_0}^t u^{(m)}(\tau) d\tau\right|^2 \right). \quad (16)$$

The second term of the right-hand side of (16) is already estimated. Now we estimate  $|u^{(m-1)}(t_0)|^2$ . Multiplying both sides of (16) by  $t^\alpha$  (after interchanging  $t$  and  $t_0$ ) and integrating over  $(0, b)$  we get

$$\left|u^{(m-1)}(t_0)\right|^2 \int_0^b t^\alpha dt \leq 2 \left|u^{(m-1)}, L_{2,\alpha}\right|^2 + 2 \int_0^b t^\alpha \left|\int_{t_0}^t u^{(m)}(\tau) d\tau\right|^2 dt,$$

which together with (14) proves that  $|u^{(m-1)}(t_0)|^2 \leq c|u, W_\alpha^m|^2$ . From (16) now we conclude that  $|u^{(m-1)}(t)|^2 \leq (c_1 + c_2 t^{1-\alpha})|u, W_\alpha^m|^2$ , i.e., the inequality (9) for  $j = m - 1$  is proved. To show (9) for  $j = m - 2$  we employ the inequality (16) (for  $m - 1$  instead of  $m$ ), then using (14) we estimate  $|u^{(m-2)}(t_0)|^2$  through  $|u, W_\alpha^m|$ . Then, using (9) for  $j = m - 1$  we can write

$$\left|u^{(m-2)}(t)\right|^2 \leq 2 \left( c|u, W_\alpha^m|^2 + |t - t_0| |u, W_\alpha^m|^2 \left| \int_{t_0}^t (c_1 + c_2 t^{1-\alpha}) dt \right| \right),$$

i.e., we get (9) for  $j = m - 2$ . In similar way we prove the inequality for  $1 \leq j \leq m - 3$ . If  $\alpha = 2n + 1$  for some  $n = 0, 1, \dots, m - 1$  then by successively estimating of  $|u^{(j)}(t)|, j = 0, 1, \dots, m - 1$ , starting with  $|u^{(m-n-1)}(t)|$  up to  $|u(t)|$  instead of  $t^{1-\alpha}$  we get  $|\ln t|$  and then continue the proof in the same way. Note also that the numbers  $B_j$  and  $C_j$  for  $j = 0, 1, \dots, m - 1$  do not depend on  $u \in W_\alpha^m$ .

*The proof is complete.*

Denote by  $s_\alpha$  (the number of the ‘‘maintained conditions’’)  $s_\alpha = m - [\frac{\alpha+1}{2}]$  for  $0 \leq \alpha < 2m - 1$  and  $s_\alpha = 0$  for  $\alpha \geq 2m - 1$ , where  $[a]$  is the integer part of the number  $a$ . From Proposition 2.2 it follows that in the case  $\alpha < 1$  (weak degeneracy)  $u^{(j)}(0)$  exist for all  $j = 0, 1, \dots, m - 1$ , while for  $\alpha \geq 1$  (strong degeneracy) not all  $u^{(j)}(0)$  exist. More precisely, for  $0 \leq \alpha < 2m - 1$  the derivatives at zero  $u^{(j)}(0)$  exist only for  $j = 0, 1, \dots, s_\alpha - 1$ , while for  $\alpha \geq 2m - 1$  all  $u^{(j)}(0), j = 0, 1, \dots, m - 1$  in general may be infinite.

**Proposition 2.5** *The embedding*

$$W_\alpha^m \subset L_{2,\alpha} \quad (17)$$

*is compact for every  $\alpha \geq 0$ .*

*Proof.* To show the compactness of the embedding (17) we first observe that this embedding is compact for every interval  $(1/n, b)$ , where  $n \in \mathbb{N}$ ,  $n \geq n_0$ ,  $1/n_0 < b$  (see [2]). Let now  $\{u_k\}_{k=1}^\infty$  be some bounded sequence in  $W_\alpha^m$  and  $|u_k, W_\alpha^m| \leq M$ . We have to prove that there is some convergent in  $L_{2,\alpha}$  subsequence. For every  $n \geq n_0$  from  $\{u_k\}_{k=1}^\infty$  we can choose a subsequence  $\{u_{nk}\}_{k=1}^\infty$  which is convergent in  $L_{2,\alpha}(1/n, b)$ . Let us prove that the diagonal-sequence  $\{u_{nn}\}_{n=n_0}^\infty$  is convergent in  $L_{2,\alpha}$ . We only need to verify that  $|u_{nn} - u_{mm}, L_{2,\alpha}(0, \varepsilon)| \rightarrow 0$  as  $\varepsilon \rightarrow +0$  for every  $m, n \geq n_0$ . Indeed, using the inequality (9) for  $j = 0$  we have (for  $\alpha \neq 1, 3, \dots, 2m - 1$ )

$$|u_{nn} - u_{mm}, L_{2,\alpha}(0, \varepsilon)|^2 \leq 2M^2 \int_0^\varepsilon t^\alpha (B_0 + C_0 t^{2m-1-\alpha}) dt \leq C\varepsilon.$$

For the case  $\alpha = 2n + 1, n = 0, 1, \dots, m - 1$  we use the inequality  $|u(t)|^2 \leq (B_0 + C_0 t^{2m-2n-2} |\ln t|) |u, W_\alpha^m|^2$ .

*The proof is complete.*

**Remark 2.6** *The embedding*

$$W_\alpha^m \subset L_{2,\beta} \tag{18}$$

*is compact for every  $\alpha > 2m - 1$  and  $\beta > \alpha - 2m$ .*

Indeed, the embedding (18) follows from the inequality (9) for  $j = 0$ . To prove the compactness of the embedding (18) it is enough to show (see the proof of Proposition 2.5) that

$$|u_{nn} - u_{mm}, L_{2,\beta}(0, \varepsilon)|^2 \leq 2M^2 \int_0^\varepsilon t^\beta (B_0 + C_0 t^{2m-1-\alpha}) dt \leq C\varepsilon^{\beta+2m-\alpha},$$

because  $\beta + 1 > \alpha - 2m + 1 > 2m - 1 - 2m + 1 = 0$ ,  $\beta + 2m - 1 - \alpha + 1 > \alpha - 2m + 2m - \alpha = 0$  and  $\beta > \beta + 2m - 1 - \alpha$ .

Observe that in the case  $\beta = \alpha - 2m$  and  $\alpha \leq 2m - 1$  in contrast to the embedding (5) the embedding (18) fails (see [9]). In this case we only have  $W_\alpha^m \subset L_{2,\beta}$ ,  $\beta > -1$ . However, for  $\alpha > 2m - 1$  we have the embedding  $W_\alpha^m \subset L_{2,\alpha-2m}$  (not compact) which can be proved by using of Hardy's inequality (6) (see [5] and [9]).

Note also that with the help of the diagonal sequence we obtain the compactness of the embedding

$$H^{s,\delta}(\mathbb{R}^n) \subset H^{s_1,\delta_1}(\mathbb{R}^n), \quad s > s_1, \delta > \delta_1, s, \delta \in \mathbb{R},$$

where  $H^{s,\delta}(\mathbb{R}^n) = (1 + |x|^2)^{-\delta/2} H^s(\mathbb{R}^n)$  and  $H^s(\mathbb{R}^n)$  is the Sobolev space, because this embedding is compact on  $B_r = \{x \in \mathbb{R}^n, |x| \leq r\}$  for every  $r > 0$ . Another proof is given in [10].

Denote by  $W_\alpha^m(0)$  the completion of  $\{u \in C^m[0, b], u^{(k)}(t)|_{t=0} = 0, k = 0, 1, \dots, m-1\}$  in the norm (8). The definition of the space  $W_\alpha^m(0)$  implies that the functions  $u \in W_\alpha^m(0)$  near to  $t = 0$  have the same estimate as the functions  $u \in \dot{W}_\alpha^m$ , while in contrast to the space  $\dot{W}_\alpha^m$  for the functions  $u \in W_\alpha^m(0)$  the conditions  $u^{(k)}(t)|_{t=b} = 0, k = 0, 1, \dots, m-1$  in general fail. Note also that the codimension of the space  $W_\alpha^m(0)$  in  $W_\alpha^m$  is equal to  $\text{codim} W_\alpha^m(0) = \dim(W_\alpha^m/W_\alpha^m(0)) = s_\alpha$ , where  $W_\alpha^m/W_\alpha^m(0)$  is the quotient space. Therefore, using the definition of the numbers  $s_\alpha$  it follows that for  $\alpha \geq 2m - 1$  the spaces  $W_\alpha^m(0)$  and  $W_\alpha^m$  coincide.

Denote by  $W_\alpha^m(b)$  the completion of  $\{u \in C^m[0, b], u^{(k)}(t)|_{t=b} = 0, k = 0, 1, \dots, m-1\}$  in the norm (3). Now for the functions  $u \in W_\alpha^m(b)$  near to  $t = 0$  we have the inequalities (9) and  $u^{(k)}(t)|_{t=b} = 0, k = 0, 1, \dots, m-1$ . The codimension of the space  $\dot{W}_\alpha^m$  in  $W_\alpha^m(b)$  is equal to  $\text{codim} \dot{W}_\alpha^m = \dim(W_\alpha^m(b)/\dot{W}_\alpha^m) = s_\alpha$ , therefore, for  $\alpha \geq 2m - 1$  we have  $\dot{W}_\alpha^m = W_\alpha^m(b)$ .

From the definition of the space  $W_\alpha^m(0)$  immediately follows that we have a continuous embedding  $W_\alpha^m(0) \subset L_{2,\alpha-2m}, \alpha \neq 1, 3, \dots, 2m-1$ , which is not compact. As a consequence of Proposition 2.5 we have the compact embeddings  $W_\alpha^m(0), W_\alpha^m(b) \subset L_{2,\alpha}$  for every  $\alpha \geq 0$ .

## 2.2 Mixed Problem of First Type

Now we consider the mixed problem of first type for the following special case of the one-dimensional equation (1)

$$Bu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + t^\alpha u = f, \quad f \in L_{2,-\alpha}. \quad (19)$$

**Definition 2.7** *A function  $u \in W_\alpha^m(0)$  is called a generalized solution of the mixed problem of first type for the equation (19) if for every  $v \in W_\alpha^m(0)$  we have*

$$(t^\alpha u^{(m)}, v^{(m)}) + (t^\alpha u, v) = (f, v) \quad (20)$$

where  $(\cdot, \cdot)$  is the scalar product in  $L_2(0, b)$  (see [11]).

**Proposition 2.8** *The generalized solution of the mixed problem of first type for the equation (19) exists and is unique for every  $f \in L_{2,-\alpha}$ .*

The uniqueness of the generalized solution immediately follows from Definition 2.7. To prove the existence first we note that the linear functional

$l_f(v) \equiv (f, v)$  is continuous in  $W_\alpha^m(0)$  because

$$|l_f(v)| \leq |f, L_{2,-\alpha}| |v, L_{2,\alpha}| \leq |f, L_{2,-\alpha}| |v, W_\alpha^m(0)|.$$

Using Riesz's lemma on the representation of linear continuous functionals we can write it in the form

$$l_f(v) \equiv (f, v) = \{u, v\}_\alpha, \quad v \in W_\alpha^m(0),$$

where  $\{u, v\}_\alpha$  is the scalar product in  $W_\alpha^m(0)$ . The element  $u \in W_\alpha^m(0)$  realizing the linear functional  $l_f(v)$  gives us the generalized solution.

If the generalized solution  $u(t)$  is classical then from (20) we conclude that for  $\alpha = 0$  the function  $u(t)$  fulfills the following conditions (see [11])

$$u^{(k)}(t) \Big|_{t=0} = u^{(2m-k-1)}(t) \Big|_{t=b} = 0, \quad k = 0, 1, \dots, m-1, \quad (21)$$

i.e., we have Dirichlet conditions at the left endpoint of the segment  $[0, b]$  and Neumann conditions at the right endpoint. Note that the conditions (21) are of Sturm type and, therefore, regular (see [4]).

**Definition 2.9** *We say that  $u \in W_\alpha^m(0)$  belongs to  $D(B)$ , if the equality (20) is satisfied for some  $f \in L_{2,-\alpha}$ . In this case we will write  $Bu = f$ .*

According to Definition 2.9 we have an operator

$$B : D(B) \subset W_\alpha^m(0) \subset L_{2,\alpha} \rightarrow L_{2,-\alpha}.$$

To get an operator in the same space we set  $g(t) = t^{-\alpha} f(t)$ . It is evident that  $g(t)$  belongs to  $L_{2,\alpha}$  and  $|f, L_{2,-\alpha}| = |g, L_{2,\alpha}|$ . Therefore, we get an operator  $\mathbb{B} \equiv t^{-\alpha} B : D(\mathbb{B}) = D(B) \subset W_\alpha^m \subset L_{2,\alpha} \rightarrow L_{2,\alpha}$  with  $\mathbb{B}u = g$  in  $L_{2,\alpha}$ .

**Proposition 2.10** *The operator  $\mathbb{B} : L_{2,\alpha} \rightarrow L_{2,\alpha}$  is selfadjoint and positive. Moreover, the inverse operator  $\mathbb{B}^{-1} : L_{2,\alpha} \rightarrow L_{2,\alpha}$  is compact.*

*Proof.* Symmetry and positivity of the operator  $\mathbb{B}$  follow from Definition 2.9. Let  $v \in \mathcal{D}(\mathbb{B}^*)$  and  $\mathbb{B}^*v = 0$ . Then for every  $u \in \mathcal{D}(\mathbb{B})$  we have  $(\mathbb{B}u, v) = (u, \mathbb{B}^*v)$ . From Proposition 2.8 it follows that  $v = 0$ , i.e., the operator  $\mathbb{B}^*$  is invertible. Now using (8) and the equality (20) with  $v = u$  we get

$$|u, W_\alpha^m|^2 = (f, u) \leq |f, L_{2,-\alpha}| |u, L_{2,\alpha}| \leq |g, L_{2,\alpha}| |u, W_\alpha^m|.$$



Therefore, we have

$$|u, L_{2,\alpha}| \leq |\mathbb{B}u, L_{2,\alpha}|, \quad (22)$$

i.e., the operator  $\mathbb{B}^{-1}$  is bounded. Thus the operator  $(\mathbb{B}^*)^{-1}$  is defined on  $L_{2,\alpha}$  (see [4]). Now the self-adjointness of the operator  $\mathbb{B}$  is a consequence of its symmetry. The compactness of the operator  $\mathbb{B}^{-1} : L_{2,\alpha} \rightarrow L_{2,\alpha}$  follows from the inequality (22) and from the compactness of the embedding  $W_\alpha^m(0) \subset W_\alpha^m$  (see Proposition 2.5).

*The proof is complete.*

**Corollary 2.11** *The operator  $\mathbb{B}$  has a discrete spectrum, and the system of the corresponding eigenfunctions is dense in  $L_{2,\alpha}$ .*

This follows from the connection of the spectra of the operators  $\mathbb{B}$  and  $\mathbb{B}^{-1}$  and from the properties of compact selfadjoint operators (see [4]).

Note that if  $\lambda$  is an eigenvalue and  $u(t)$  a corresponding eigenfunction of the operator  $\mathbb{B}$  then we have

$$(-1)^m (t^\alpha u^{(m)})^{(m)} + t^\alpha u = \lambda t^\alpha u. \quad (23)$$

From the inequality (22) and Definition 2.7 it follows that  $\lambda \geq 1$ . For  $0 \leq \alpha < 1$  the number  $\lambda = 1 \notin \sigma\mathbb{B}$  ( $\sigma\mathbb{B}$  is the spectrum of the operator  $\mathbb{B}$ ) and for  $\alpha \geq 1$  it is an eigenvalue for the operator  $\mathbb{B}$  with the multiplicity  $m - s_\alpha$  (for the definition of the number  $s_\alpha$  see Subsection 2.1), since  $t^{s_\alpha} P_{m-s_\alpha-1}(t)$  for every polynomial  $P_{m-s_\alpha-1}(t)$  of order  $m - s_\alpha - 1$  is an eigenfunction (see Proposition 2.1). Therefore, for the solvability of the equation

$$(-1)^m (t^\alpha u^{(m)})^{(m)} = f, \quad f \in L_{2,-\alpha}, \quad (24)$$

we get the following result:

**Proposition 2.12** *The generalized solution of the mixed problem of first type for the equation (24) for  $\alpha \geq 1$  exists if and only if  $(f, P_{m-s_\alpha-1}(t)) = 0$  for any polynomial  $P_{m-s_\alpha-1}(t)$  of order  $m - s_\alpha - 1$ .*

Here we have used both  $(g, P_{m-s_\alpha-1}(t))_\alpha = (f, P_{m-s_\alpha-1}(t))$  since  $t^\alpha g(t) = f(t)$  ( $(\cdot, \cdot)_\alpha$  is the scalar product in  $L_{2,\alpha}$ ) and the definition of the operator  $\mathbb{B}$ . Note that the generalized solution of the mixed problem of first type for the equation (24) for  $\alpha \geq 1$  is unique up to an arbitrary additive polynomial of order  $m - s_\alpha - 1$ .

Now we can consider the general case of the one-dimensional equation (1)

$$Pu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + at^\alpha u = f, \quad f \in L_{2,-\alpha}, \quad (25)$$

because the number  $1 - a$  can be regarded as a spectral parameter for the operator  $\mathbb{B}$ . Therefore, if  $1 - a \notin \sigma\mathbb{B}$  then the equation (25) is uniquely solvable for every  $f \in L_{2,-\alpha}$ .

### 2.3 Mixed Problem of Second Type

Now we consider the mixed problem of second type for the another special case of the one-dimensional equation (1)

$$Su \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} = f, \quad f \in L_{2,-\alpha}. \quad (26)$$

**Definition 2.13** *A function  $u \in W_\alpha^m(b)$  is called a generalized solution of the mixed problem of second type for the equation (26) if for every  $v \in W_\alpha^m(b)$  we have*

$$(t^\alpha u^{(m)}, v^{(m)}) + (t^\alpha u, v) = (f, v). \quad (27)$$

In the same way as in Subsection 2.2 we prove the existence and uniqueness of the generalized solution for every  $f \in L_{2,-\alpha}$  and define a corresponding operator

$$S : D(S) \subset W_\alpha^m(b) \subset L_{2,\alpha} \rightarrow L_{2,-\alpha}.$$

For the classical solution  $u(t)$  in the case  $\alpha = 0$  from (27) we get the conditions (see [11])

$$u^{(2m-k-1)}(t) \Big|_{t=0} = u^{(k)}(t) \Big|_{t=b} = 0, \quad k = 0, 1, \dots, m-1, \quad (28)$$

i.e., we have Neumann conditions at the left endpoint of the segment  $[0, b]$  and Dirichlet conditions at the right endpoint.

Define  $\mathbb{S} \equiv t^{-\alpha} S$ ,  $\mathbb{S} : L_{2,\alpha} \rightarrow L_{2,\alpha}$ . In similar way we prove that the operator  $\mathbb{S} : L_{2,\alpha} \rightarrow L_{2,\alpha}$  is selfadjoint, positive and the inverse operator  $\mathbb{S}^{-1}$  is compact. Therefore, the operator  $\mathbb{S}$  has a discrete spectrum.

Now we can consider the general equation (25) because the number  $-a$  can be regarded as a spectral parameter for the operator  $\mathbb{S}$ . Hence, if  $-a \notin \sigma \mathbb{S}$  then the mixed problem of second type for the equation (25) is uniquely solvable for every  $f \in L_{2,-\alpha}$ .

## 3 The Operator Equation

In this section we consider the operator version of the equation (1)

$$Pu \equiv (-1)^m D_t^m (t^\alpha D_t^m) u + t^\alpha Au = f, \quad f \in L_{2,-\alpha}((0, b), \mathcal{H}), \quad \alpha \geq 0. \quad (29)$$

Suppose that the operator  $A : \mathcal{H} \rightarrow \mathcal{H}$  commutes with  $D_t$  and has a complete system of eigenfunctions  $\{\varphi_k\}_{k=1}^\infty$ , forming a Riesz basis (see [4]) in  $\mathcal{H}$ . Hence

$A\varphi_k = a_k\varphi_k$ ,  $k \in \mathbb{N}$ , for every  $x \in \mathcal{H}$  we have  $x = \sum_{k=1}^{\infty} x_k\varphi_k$ , and there are some positive constants  $c_1$  and  $c_2$  such that

$$c_1 \sum_{k=1}^{\infty} |x_k|^2 \leq \|x\|^2 \leq c_2 \sum_{k=1}^{\infty} |x_k|^2. \quad (30)$$

Hence for every  $u \in L_{2,\alpha}((0, b), \mathcal{H})$ ,  $f \in L_{2,-\alpha}((0, b), \mathcal{H})$  we have

$$u = \sum_{k=1}^{\infty} u_k(t)\varphi_k, \quad f = \sum_{k=1}^{\infty} f_k(t)\varphi_k, \quad k \in \mathbb{N}. \quad (31)$$

Therefore, the operator equation (29) can be decomposed into an infinite chain of ordinary differential equations

$$P_k u_k \equiv (-1)^m (t^\alpha u_k^{(m)})^{(m)} + a_k t^\alpha u_k = f_k, \quad f_k \in L_{2,-\alpha}, \quad k \in \mathbb{N}. \quad (32)$$

For the equations (32) we can define the generalized solutions  $u_k(t)$ ,  $k \in \mathbb{N}$ , of the mixed problem of first or second type (see Section 2).

**Definition 3.1** *A function  $u \in L_{2,\alpha}((0, b), \mathcal{H})$  is called a generalized solution of the mixed problem of first or second type for the equation (29) if the functions  $u_k(t)$ ,  $k \in \mathbb{N}$ , in the representation (31) are generalized solutions of the mixed problem of first or second type for the equations (32).*

**Proposition 3.2** *The operator equation (29) is uniquely solvable for every  $f \in L_{2,-\alpha}((0, b), \mathcal{H})$  if and only if the equations (32) are uniquely solvable for every  $f_k \in L_{2,-\alpha}$ ,  $k \in \mathbb{N}$ , and the inequalities*

$$|u_k, L_{2,\alpha}| \leq c |f_k, L_{2,-\alpha}| \quad (33)$$

*are satisfied uniformly with respect to  $k \in \mathbb{N}$ .*

For the proof of Proposition 3.2 see [4].

Let the numbers  $\{\lambda_k\}_{k=1}^{\infty}$  and  $\{\mu_k\}_{k=1}^{\infty}$  are the eigenvalues of the operators  $\mathbb{B}$  and  $\mathbb{S}$  (see Section 2). Suppose that

$$\rho(1 - a_k, \lambda_m) > \varepsilon, \quad k, m \in \mathbb{N}, \quad (34)$$

$$\rho(-a_k, \mu_m) > \varepsilon, \quad k, m \in \mathbb{N}, \quad (35)$$

where  $\varepsilon > 0$  and  $\rho$  is the distance in the complex plane.

**Theorem 3.3** *Under the condition (34) ((35)) the generalized solution of the mixed problem of first type (second type) for the operator equation (29) exists and is unique for every  $f \in L_{2,-\alpha}((0, b), \mathcal{H})$ .*

First note that under the condition (34) ((35)) the equations (32) are uniquely solvable for every  $f_k \in L_{2,-\alpha}$ ,  $k \in \mathbb{N}$  and the inequalities (33) are satisfied. Now the proof of Theorem 3.3 follows from Proposition 3.2.

Let  $g = t^{-\alpha}f$ ,  $f \in L_{2,-\alpha}((0, b), \mathcal{H})$ . Then  $g \in L_{2,\alpha}((0, b), \mathcal{H})$  and we define an operator

$$\mathbb{P} \equiv t^{-\alpha}P : D(\mathbb{P}) = D(P) \subset L_{2,\alpha}((0, b), \mathcal{H}) \rightarrow L_{2,\alpha}((0, b), \mathcal{H}),$$

with  $\mathbb{P}u = g$  in  $L_{2,\alpha}((0, b), \mathcal{H})$ . It follows from the condition (34) ((35)) that for the generalized solution  $u \in W_{\alpha}^m(0)$  ( $u \in W_{\alpha}^m(b)$ ) of the mixed problem of first type (second type) we have

$$|u, L_{2,\alpha}((0, b), \mathcal{H})| \leq c|g, L_{2,\alpha}((0, b), \mathcal{H})|. \quad (36)$$

The operator  $\mathbb{P}^{-1} : L_{2,\alpha}((0, b), \mathcal{H}) \rightarrow L_{2,\alpha}((0, b), \mathcal{H})$  in general is not compact in contrast to Proposition 2.10 (it will be compact only in the case when the space  $\mathcal{H}$  is finite-dimensional). If the operator  $A : \mathcal{H} \rightarrow \mathcal{H}$  is selfadjoint we can describe the spectrum of the operator  $\mathbb{P}$ .

**Proposition 3.4** *The spectrum of the operator  $\mathbb{P}$  for the mixed problem of first type (of second type) is the subset of the direct sum of the spectra  $\sigma\mathbb{B}$  and  $\sigma(A - I)$  ( $\sigma\mathbb{S}$  and  $\sigma A$ ), i.e.,*

$$\sigma\mathbb{P} \subset \sigma\mathbb{B} + \sigma(A - I) \equiv \{\lambda_1 + \lambda_2 - 1 : \lambda_1 \in \sigma\mathbb{B}, \lambda_2 \in \sigma A\}, (\sigma\mathbb{P} \subset \sigma\mathbb{S} + \sigma A).$$

The proof of Proposition 3.4 immediately follows from the equality  $\mathbb{P} = \mathbb{B} \otimes I_{\mathcal{H}} + I_{L_{2,\alpha}} \otimes (A - I)$  ( $\mathbb{P} = \mathbb{S} \otimes I_{\mathcal{H}} + I_{L_{2,\alpha}} \otimes A$ ) (see [1]).

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