On a paper of Krupchyk, Tarkhanov, and Tuomela

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These remarks are due to the similarity of the article from Krupchyk, Tarkhanov, and Tuomela [5] with parts of Sections 3.2.3.1 and 3.2.3.2 of the author's joint monograph with Rempel [6]. The general background are (classical) pseudo-differential boundary value problems on a smooth manifold M with boundary in the sense of Boutet de Monvel [2]. Those form an operator algebra with a principal symbolic structure $\sigma = (\sigma_{\text{int}}, \sigma_\partial)$, with the interior and boundary symbols σ_{int} and σ_∂, respectively. Writing an operator A in that calculus as a 2×2 block matrix, the upper left corner is a pseudo-differential operator on $\text{int} M$ with the transmission property at the boundary, plus a singular Green operator G which is smoothing over $\text{int} M$. The other entries off the diagonal signify trace and potential operators, while the lower right corner is a pseudo-differential operator on ∂M.

The interior symbol $\sigma_{\text{int}}(A)$ is the standard homogeneous principal symbol of the upper left corner, and $\sigma_\partial(A)$ the so-called boundary symbol; the latter is a family of 2×2 block matrix operators, acting between Sobolev spaces in normal direction to the boundary, plus the fibres of involved bundles over the boundary. The composition of two operators A and B in that framework belongs to the calculus again (provided that the bundles in the middle fit together, and one of the factors is properly supported) and we have $\sigma(AB) = \sigma(A)\sigma(B)$ (the latter composition is component-wise). Moreover, if A is of order 0, and if the order of differentiation in the (integral representation of) the Green and the trace operator, transversal to the boundary, is zero, then the L^2-adjoint A^* belongs to the calculus, where $\sigma(A^*) = \sigma(A)^*$. Ellipticity of A means the bijectivity of $\sigma_{\text{int}}(A)$ over $T^*M \setminus 0$ and of $\sigma_\partial(A)$ over $T^*(\partial M) \setminus 0$ (0 represents the corresponding zero-section). Operators A in Boutet de Monvel's calculus induce continuous mappings between standard Sobolev spaces of distributional sections in the involved vector bundles over M and ∂M, respectively. Ellipticity of A entails the existence of a parametrix in the calculus and also the Fredholm property when M is compact (in Sobolev spaces of sufficiently large smoothness). All this is done in [2].

In [6] we studied the abstract Fredholm complexes in Hilbert spaces (Section 3.2.3.1) and complexes of operators in Boutet de Monvel's calculus (Section 3.2.3.2). Recall that complexes are sequences of operators $A_0, \ldots, A_{N-1}$ such that $A_{k+1}A_k = 0$ for all k (for notational purposes we set $A_j = 0$ for $j < 0$ and $j \geq N$). In the case of operators with a symbolic structure, compatible with compositions, the relation $A_{k+1}A_k = 0$ entails $\sigma(A_{k+1})\sigma(A_k) = 0$. Ellipticity

of a complex is defined as the exactness of the symbolic complex $\sigma(A_k)$. As is shown in [6], such a complex has a parametrix of operators in the calculus, i.e., a sequence $P_0, \ldots, P_{N-1}$ with P_k mapping the spaces in the opposite direction than A_k and $A_{k-1}P_{k-1} + P_k A_k = 1 + C_k$ for $\sigma(C_k) = 0$ for all k. In particular, it follows that $\sigma(A_{k-1})\sigma(P_{k-1}) + \sigma(P_k)\sigma(A_k) = 1$ for all k.

An elliptic 'quasi'-complex in the terminology of [5] is a sequence A_k such that the complexes $\sigma(A_k) = (\sigma_{\mathrm{int}}(A_k), \sigma_{\partial}(A_k))$ are exact but the relations $A_{k+1}A_k = 0$ are not required. However, it is an immediate consequence of the Hodge theory from [6] together with the shape of the operators that furnish the parametrix of [6] (which is also available for sequences that are only 'quasi' for a certain $k < N - 1$) that an elliptic quasi complex can always be turned to an elliptic complex when we change A_k by a lower order operator. This purely functional analytic observation (always true in analogous form for operators in an algebra with a symbolic structure that determines operators modulo lower order remainders) is mentioned in [5] as Theorem 2.2 with reference to another paper from Tarkhanov, apparently not treated as something trivial.
The authors of [5] show the same thing once again as Theorem 8.1 with a very long proof claiming Hodge theory and parametrices for elliptic complexes as achievements of their paper. The new ideas (results and methods) of [5] are contained in the complement of the following equivalences.

[6], Definition 1, page 282 = [5], Definition 3.1

[6], proof of Theorem 2, page 283 contains [5], Theorem 3.2

[6], Theorem 2, page 283 contains [5], Lemma 4.1 and Theorem 4.3

[6], constructions of Section 3.2.3.2 entail [5], Corollary 4.4 and Theorem 4.5 as trivial consequences; in particular, [6], Theorem 2, page 283 contains [5], Theorem 4.5

[6], Theorem 4, page 272 is the same as [5], Lemma 5.1

[6], proof of Theorem 3, page 272 coincides with the constructions of [5], Theorem 5.2 and Lemma 5.3

[6], Proposition 5, page 274 contains [5], Corollary 6.1.

There are other noticeable aspects of the paper [5], for instance, how well known technical tools on operators in Boutet de Monvel's calculus are quoted. For instance, in [5], Lemma 1.2 ('important for us') the authors recall the continuity in Sobolev spaces. [5], Lemma 3.1 states such a continuity again, quoting a witness from 1996. Taking into account Lemma 1.2 and the fact that there are pseudo-differential order reducing operators of any order on the boundary, Lemma 3.1 explains that the composition of continuous operators is again continuous. In any case, continuity in Sobolev spaces is contained in [2] (see also

Eskin's book [4], Lemma 23.8). Another point is the comments in [5] with reference to Corollary 6.1 on Dynin's paper [3]; here it remains unclear why the authors copied the corresponding observations from [6], pages 274 and 284, without giving any new, additional information. Let us finally note that there are other references on elliptic complexes or Fredholm complexes, e.g., the paper of Atiyah and Bott [1] on pseudo-differential complexes on closed manifolds, which suggests the algebraic structure of useful parametrices also in more general cases. In [6] this is combined with isomorphisms of complexes, induced by reductions of orders in Boutet de Monvel's calculus (see the second operator in [2], formula (5.10), that can be turned to an isomorphism by adding a finite rank smoothing operator which belongs to the calculus, only using arguments known by [2]). This step is crucial for obtaining Laplacians for corresponding reduced complexes of order 0. Neither [1] nor the latter aspect are cited in [5].

References

[1] Atiyah, M.F. and Bott, R., A Lefschetz fixed point formula for elliptic complexes I, Ann. of Math. (2), 86, 2, 374-407, 1967

[2] Boutet de Monvel, L., Boundary problems for pseudo-differential operators, Acta Math., 126, 11-51, 1971

[3] Dynin, A.S., Elliptic boundary problems for pseudo-differential complexes, Funkts. Analiz, 6, 1, 75-76, 1972

[4] Eskin, G.I., Boundary value problems for elliptic pseudodifferential equations, Transl. of Nauka, Moskva, 1973, Math. Monographs, Amer. Math. Soc., 52, Providence, Rhode Island, 980 begin

[5] Krupchyk, K. and Tarkhanov, N. and Tuomela, J., Elliptic quasicomplexes in Boutet de Monvel algebra, Journal of Functional Analysis, 247, 202-230, 2007

[6] Rempel, S. and Schulze, B.-W., Index theory of elliptic boundary problems, Berlin, Akademie-Verlag, 1982