Behavior of the Solution to a Chemotaxis Model with Reproduction term *

Chen Hua, Li Jun-Feng and Liu Wei-An[†]
School of Mathematics and Statistics, Wuhan University
Wuhan 430072, P.R.China

1 Introduction

In this paper, we are concerned with the following chemotaxis equation with a ratio-dependent logistic reaction term

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\nabla(u\nabla \ln \dfrac{u}{w}) + u(a - b\dfrac{u}{w}) & (x,t) \in Q_T \\ \dfrac{\partial w}{\partial t} = \beta u - \delta w & (x,t) \in Q_T \\ u\nabla \ln(\dfrac{u}{w}) \cdot \vec{n} = 0 & (x,t) \in \Gamma_T \\ u(x,0) = u_0(x) > 0 & x \in \bar{\Omega} \\ w(x,0) = w_0(x) > 0 & x \in \bar{\Omega} \end{cases} \tag{1}$$

where u represents the density or population of a biological species, which could be a cell, or a germ, or an insect, or a vegetarian animal; while w represents an attractive resource of the species, e.g., hormone for cell, attractant or a plant as food for the insect, food resource for the animal. D, a, b and β, δ are parameters, which are supposed to be constants here. Ω represents a bounded domain in R^k(k might be 1,2, or 3) with sufficiently smooth boundary $\partial\Omega$ and $\vec{n}$ denotes the outward normal vector of $\partial\Omega$, while $Q_T = \Omega \times (0,T)$, $\Gamma_T = \partial\Omega \times (0,T)$.

The so-called chemotaxis phenomenon is quite common in bio-systems. All living systems can the environment where they live and respond to it. The response usually involves movement towards or away from an external stimulus. The mechanism for such response is called taxis. The purposes of taxis range from movement toward food to avoidance of noxious substances to large-scale aggregation for survival. There are many different kinds of taxis, such as aerotaxis, chemotaxis, geotaxis and haptotaxis. Any taxis involves at least two components:(1) an external signal and (2) the response of the

*Supported by the National Natural Science Foundation of China(10471108)
†Email: 1. chenhua@whu.edu.cn, 2. jfli007@163.com, 3. liuweian@public.wh.hb.cn

organism to the signal. The response also involves two steps:(1)detection of the signal and (2) the transduction of the external signal into an internal signal that controls pattern of movements. When the response involves the detection of a chemical, it is called chemotaxis. The term chemotaxis is used broadly in the mathematical literature to describe general chemosensitive movement responses. Chemotaxis can be either positive or negative. Models for chemotaxis have been applied to bacteria, slime molds, skin pigmentation patterns, leukocytes, etc. Many papers appear to demonstrate the role of chemotaxis using mathematical analysis or experimental stimulation.

The classical chemotaxis equation was introduced by Keller and Segel[4] as a model to describe the aggregation of slime mold amoebae Dicrocoelium Discoidal due to an attractive chemical substance.

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_u \triangle u - \nabla(u\chi(w)\nabla w) \\ \dfrac{\partial w}{\partial t} = D_w \triangle w + hu - kw \end{cases} \tag{2}$$

where D_u and D_w are the diffusion coefficients of the cells u and attractant w, respectively, and $hu - kw$ is the kinetics term of w.

Other and Steven [9] considered the following *Other-Steven model*

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial}{\partial x}(\dfrac{\partial u}{\partial x} - u\dfrac{\beta}{\alpha + \beta w}\dfrac{\partial w}{\partial x}) & x \in (0,1), t > 0 \\ \dfrac{\partial w}{\partial t} = F(u, w) & x \in (0,1), t > 0 \\ u\dfrac{\partial}{\partial x}\ln(\dfrac{u}{\alpha + \beta w}) = 0 & x = 0, 1 \\ u(x, 0) = u_0(x) > 0 & x \in (0,1) \\ w(x, 0) = w_0(x) > 0 & x \in (0,1) \end{cases} \tag{3}$$

They concluded that when w had linear growth $\dfrac{\partial w}{\partial t} = u - \mu w$ and $\mu > 0$, $\alpha > 0$, the system had the constant solution $(u, w) = (u_0, u_0/\mu)$ which was asymptotically stable, if $\mu = 0$, then there was no time-independent solution, but there was a unstable space-independent solution $(u, w) = (u_0, w_0 + u_0 t)$ for any $\alpha \geq 0$ provided that $u(x, 0) \equiv u_0, w(x, 0) \equiv w_0$. However, when w had exponential growth $\dfrac{\partial w}{\partial t} = uw$ with boundary conditions $u\dfrac{\partial}{\partial x}\ln(\dfrac{u}{w}) = 0$ and initial data $u(x, 0) \equiv u_0 > 0, w(x, 0) \equiv w_0 > 0$, the space-independent solution $(u_0, w_0 e^{u_0 t})$ was unstable.

Sleeman and Levine [13] considered a particular system modelling angiogenesis:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial}{\partial x}(u\dfrac{\partial}{\partial x}(\ln u/\Phi(w))) & (x,t) \in (0,l) \times (0,T) \\ \dfrac{\partial w}{\partial t} = \lambda p w - \mu w & (x,t) \in (0,l) \times (0,T) \\ u\frac{\partial}{\partial x}(\ln u/\Phi(w)) = 0 & x = 0, l \\ u(x,0) = u_0(x) > 0 & x \in (0,l) \\ w(x,0) = w_0(x) \geq 0 & x \in (0,l) \end{cases} \tag{4}$$

They found a explicit solution which could be global or blow up in finite time. In [18], Yang Yin etc. considered system (1) provided $a = b = 0$, and concluded the solution was either global or blew up in finite time. There are other papers [1],[7],[10],[15] concerned with the similar problem in which there is no diffusion term in the control equation.

In all works mentioned above, only chemotaxis were discussed without any reproduction or wither away. In this paper, we consider a model involving chemotaxis and an external source.

By making use of a smart function transform introduced by Yang Yin (cf.[17]) and comparative method, the asymptotical behavior of the solution to the problem (1) is discussed. The following conclusion is arrived at

Definition. *Suppose that $(u(x,t), w(x,t))$ is a solution to (1).*

- *$(u(x,t), w(x,t))$ is said to be blow-up at the finite time if there is a finite T such that the solution exists in the interval $[0,T)$ and at least one of $u(x,t), w(x,t)$ goes to $+\infty$ as $t \to T^-$.*
- *$(u(x,t), w(x,t))$ is said to be quenching at the finite time if there is a finite T such that the solution exists in the interval $[0,T)$ and keep to be bounded while at least one of its derivative $\frac{\partial u}{\partial t}, \frac{\partial w}{\partial t}$ goes to $+\infty$ as $t \to T^-$.*
- *$(u(x,t), w(x,t))$ is said to be global if it exists for $t > 0$.*

Main Result. *Suppose that $(u(x,t), w(x,t))$ is a solution to (1), then*

- *as $b + \beta \geq 0$, $(u(x,t), w(x,t))$ exists globally;*
- *as $b + \beta < 0$, $a + \delta \geq 0$ $(u(x,t), w(x,t))$ blows up if $\beta \geq 0$, or quenches if $\beta < 0$ at a finite time;*
- *as $b+\beta < 0$ and $a+\delta < 0$, whether globally $(u(x,t), w(x,t))$ exists or not depends on the ratio of initial data $u_0(x)/w_0(x)$.*

The paper is organized as follows. By a function transform, in the next section, an auxiliary equation is arrived at, to which comparative method works. In the last section, various asymptotical behaviors of the solution, going to $+\infty$ or 0 when it globally exists and blowing-up or quenching when it non-globally exists, depended on the value of the parameters are discussed in detail.

2 Global existence and blow-up or quenching of the solution

To discuss the behavior of the solution to (1), define a function transformation

$$p(x,t) = \frac{u(x,t)}{w(x,t)}$$

Then, we have the following problem:

$$\begin{cases} \dfrac{\partial p}{\partial t} = D\Delta p + D\nabla \ln w \cdot \nabla p + (a+\delta)p - (b+\beta)p^2 & (x,t) \in Q_T \\ w(x,t) = w_0(x)e^{\int_0^t (\beta p(x,\tau)-\delta)d\tau} & (x,t) \in Q_T \\ \dfrac{\partial p}{\partial n} = 0 & (x,t) \in \Gamma_T \\ p(x,0) = p_0(x) > 0 & x \in \bar{\Omega} \\ w(x,0) = w_0(x) > 0 & x \in \bar{\Omega} \end{cases} \tag{5}$$

Clearly, $(u(x,t), w(x,t))$ is a solution of system (1) if and only if $(p(x,t), w(x,t))$ is a solution of system (5).

For any given bounded positive smooth function $w(x,t)$ on the domain Q_T, the problem

$$\begin{cases} \dfrac{\partial p}{\partial t} = D\Delta p + D\nabla \ln w \cdot \nabla p + (a+\delta)p - (b+\beta)p^2 & (x,t) \in Q_T \\ \dfrac{\partial p}{\partial n} = 0 & (x,t) \in \Gamma_T \\ p(x,0) = p_0(x) > 0 & x \in \bar{\Omega} \end{cases} \tag{6}$$

is a well posed problem of semi-linear reaction-diffusion equation, to which the comparative method works. It can been seen easily that, to (6), there are a sub-solution $\underline{p}(x,t)$ and a super-solution $\bar{p}(x,t)$ free of $w(x,t)$, who read

$$\bar{p}(x,t) = \begin{cases} \dfrac{\bar{p}_0}{1+\bar{p}_0(b+\beta)t} & \text{as } a+\delta = 0 \\ \\ \dfrac{\bar{p}_0(a+\delta)e^{(a+\delta)t}}{a+\delta+\bar{p}_0(b+\beta)(e^{(a+\delta)t}-1)} & \text{as } a+\delta \neq 0 \end{cases} \tag{7}$$

$$\underline{p}(x,t) = \begin{cases} \dfrac{\underline{p}_0}{1+\underline{p}_0(b+\beta)t} & \text{as } a+\delta = 0 \\ \\ \dfrac{\underline{p}_0(a+\delta)e^{(a+\delta)t}}{a+\delta+\underline{p}_0(b+\beta)(e^{(a+\delta)t}-1)} & \text{as } a+\delta \neq 0 \end{cases} \tag{8}$$

By Comparison, we can conclude that there is a unique solution $p(x,t)$ to (6) for every given $w(x,t)$ satisfying

$$\underline{p}(x,t) \le p(x,t) \le \bar{p}(x,t)$$

And then, it can be seen that

$$\begin{cases} w_0(x)e^{\int_0^t(\beta\underline{p}(x,\tau)-\delta)d\tau} \le w(x,t) \le w_0(x)e^{\int_0^t(\beta\bar{p}(x,\tau)-\delta)d\tau}, \\ \underline{p}(x,t)w_0(x)e^{\int_0^t(\beta\underline{p}(x,\tau)-\delta)d\tau} \le u(x,t) \le \bar{p}(x,t)w_0(x)e^{\int_0^t(\beta\bar{p}(x,\tau)-\delta)d\tau}, \end{cases} \quad \text{as } \beta > 0 \tag{9}$$

$$\begin{cases} w_0(x)e^{\int_0^t(\beta\bar{p}(x,\tau)-\delta)d\tau} \le w(x,t) \le w_0(x)e^{\int_0^t(\beta\underline{p}(x,\tau)-\delta)d\tau}, \\ \underline{p}(x,t)w_0(x)e^{\int_0^t(\beta\bar{p}(x,\tau)-\delta)d\tau} \le u(x,t) \le \bar{p}(x,t)w_0(x)e^{\int_0^t(\beta\underline{p}(x,\tau)-\delta)d\tau}, \end{cases} \quad \text{as } \beta < 0 \tag{10}$$

From the above inequalities, we see that, if $\underline{p}(x,t)$ and $\bar{p}(x,t)$ are bounded on the interval $[0,T]$, the solution $(u(x,t),w(x,t))$ to (1) exists, and are positive and bounded; and then, they could be extended to a larger interval $[0,T_1)$ $(0 < T < T_1 \le +\infty)$ till $\underline{p}(x,t)$ and $\bar{p}(x,t)$ go to $+\infty$. Moreover, we can arrive at the following lemma.

Lemma. *If there is a finite T such that $\lim_{t\to T}\underline{p}(x,t) = +\infty$, then the solution to (1) blows up as $\beta > 0$ and quenches as $\beta < 0$. If $\bar{p}(x,t)$ is bounded for $t \in (0,+\infty)$, there is a global solution to (1).*

Proof. Suppose that there is a finite T such that $\lim_{t\to T}\underline{p}(x,t) = +\infty$. From (9), we see that, if $\beta > 0$, $w(x,t)$ and $u(x,t)$ both go to $+\infty$ as $t \to T^-$. Thus, the solution to (1) blows up at finite time T.

From (10), we see that, if $\beta < 0$, $w(x,t)$ and $u(x,t)$ both go to 0 and $\frac{u}{w} \to +\infty$ which means $\dfrac{\partial u}{\partial t} \to +\infty$ as $t \to T^-$. That is, the solution to (1) quenches at finite time T.

If $\bar{p}(x,t)$ is bounded for $t \in (0,+\infty)$, from (9) and (10), $w(x,t)$ and $u(x,t)$ are bounded on any bounded interval $[0,T]$. So, they can be extended. Hence, they exist globally.

3 Detailed asymptotical behavior of the solution

In this section, we discuss in detail the asymptotical behavior of the solution to (1).

3.1 The case $a+\delta > 0, b+\beta > 0$

In the case of $a+\delta > 0, b+\beta > 0$, we can see by comparison that

$$\lim_{t\to+\infty} p(x,t) = \frac{a+\delta}{b+\beta}$$

thanks to that so do both $\bar{p}(x,t)$ and $\underline{p}(x,t)$.

If $a\beta > b\delta$, there is $T > 0$ such that

$$\beta p(x,t) - \delta > \frac{a\beta - b\delta}{2(b+\beta)} > 0, \text{ as } t > T$$

Then, it can be seen that

$$\lim_{t\to+\infty} w(x,t) = \lim_{t\to+\infty} w_0(x) e^{\int_0^t (\beta p(x,\tau)-\delta)d\tau} = +\infty$$

and

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} w(x,t)p(x,t) = +\infty$$

If $a\beta < b\delta$, there is $T > 0$ such that

$$\beta p(x,t) - \delta < \frac{a\beta - b\delta}{2(b+\beta)} < 0, \text{ as } t > T$$

Then,

$$\lim_{t\to+\infty} w(x,t) = \lim_{t\to+\infty} w_0(x) e^{\displaystyle\int_0^t (\beta p(x,\tau)-\delta)d\tau} = 0$$

and

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} w(x,t)p(x,t) = 0$$

If $a\beta = b\delta$,

$$\begin{aligned}
& \int_0^t (\beta \bar{p}(x,\tau) - \delta) d\tau \\
= & \int_0^t \left(\frac{\beta \bar{p}_0 (a+\delta) e^{(a+\delta)\tau}}{a+\delta+\bar{p}_0(b+\beta)(e^{(a+\delta)\tau}-1)} - \delta \right) d\tau \\
= & \frac{\beta}{b+\beta} \ln \frac{a+\delta+\bar{p}_0(b+\beta)(e^{(a+\delta)t}-1)}{a+\delta} - \delta t \\
= & \frac{\beta}{b+\beta} \ln \frac{a+\delta+\bar{p}_0(b+\beta)(e^{(a+\delta)t}-1)}{a+\delta} - \ln e^{\delta t} \\
= & \frac{\beta}{b+\beta} \ln \frac{(a+\delta-\bar{p}_0(b+\beta))e^{-\frac{(b+\beta)\delta}{\beta}t} + \bar{p}_0(b+\beta)}{a+\delta}
\end{aligned} \tag{11}$$

then

$$\begin{aligned}
& \int_0^{+\infty} (\beta \bar{p}(\tau) - \delta) d\tau \\
= & \lim_{t\to+\infty} \frac{\beta}{b+\beta} \ln \frac{(a+\delta-\bar{p}_0(b+\beta))e^{-\frac{(b+\beta)\delta}{\beta}t} + \bar{p}_0(b+\beta)}{a+\delta} \\
= & \frac{\beta}{b+\beta} \ln \frac{(b+\beta)\bar{p}_0}{a+\delta}
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\int_0^{+\infty} (\beta \underline{p}(\tau) - \delta) d\tau \\
= &\lim_{t\to+\infty} \frac{\beta}{b+\beta} \ln \frac{(a+\delta-\underline{p}_0(b+\beta))e^{-\frac{(b+\beta)\delta}{\beta}t} + \underline{p}_0(b+\beta)}{a+\delta} \\
= &\frac{\beta}{b+\beta} \ln \frac{(b+\beta)\underline{p}_0}{a+\delta}
\end{aligned}
$$

Then, it can be seen that there is a function $W(x)$ as smooth as $w(x,t)$ such that

$$\lim_{t\to+\infty} w(x,t) = W(x)$$

and

$$
\begin{aligned}
w_0(x)\left(\frac{(b+\beta)\underline{p}_0}{a+\delta}\right)^{\frac{\beta}{b+\beta}} &\le W(x) \le w_0(x)\left(\frac{(b+\beta)\bar{p}_0}{a+\delta}\right)^{\frac{\beta}{b+\beta}}, \text{ if } \beta > 0 \\
w_0(x)\left(\frac{(b+\beta)\bar{p}_0}{a+\delta}\right)^{\frac{\beta}{b+\beta}} &\le W(x) \le w_0(x)\left(\frac{(b+\beta)\underline{p}_0}{a+\delta}\right)^{\frac{\beta}{b+\beta}}, \text{ if } \beta < 0
\end{aligned}
$$

If $\beta = 0$, then $b = 0$ or $\delta = 0$ due to $a\beta - b\delta = 0$. Thanks to $b + \beta > 0$, the case of $\beta = b = 0$ is impossible. If $\beta = \delta = 0$, then $w(x,t) = w_0(x)$ and $p(x,t) \to a/b$ as $t \to +\infty$, here $u(x,t) \to \frac{a}{b} w_0(x)$.

Theorem 1. *Suppose that $a + \delta > 0$ and $b + \beta > 0$, then there is a unique global solution $(u(x,t), w(x,t))$ to the problem (1) which satisfying the following property:*

- *if $a\beta - b\delta < 0$, there are constants $0 < M_1 \le M_2 < +\infty$,$0 < \tilde{M}_1 \le \tilde{M}_2 < +\infty$ and $0 < \lambda_1 < |a\delta - b\beta| < \lambda_2 < +\infty$ such that*

  $$M_1 e^{-\lambda_1 t} \le w(x,t) \le M_2 e^{-\lambda_2 t}, \qquad \tilde{M}_1 e^{-\lambda_1 t} \le u(x,t) \le \tilde{M}_2 e^{-\lambda_2 t}$$

  thus

  $$\lim_{t\to+\infty} w(x,t) = 0, \qquad \lim_{t\to+\infty} u(x,t) = 0$$

- *if $a\beta - b\delta > 0$, there are constants $0 < M_1 \le M_2 < +\infty$,$0 < \tilde{M}_1 \le \tilde{M}_2 < +\infty$ and $0 < \lambda_1 < |a\delta - b\beta| < \lambda_2 < +\infty$ such that*

  $$M_1 e^{\lambda_1 t} \le w(x,t) \le M_2 e^{\lambda_2 t}, \qquad \tilde{M}_1 e^{\lambda_1 t} \le u(x,t) \le \tilde{M}_2 e^{\lambda_2 t}$$

  thus

  $$\lim_{t\to+\infty} w(x,t) = +\infty, \qquad \lim_{t\to+\infty} u(x,t) = +\infty$$

- *if $a\beta - b\delta = 0$, there is a bounded and positive permanent solution $(U(x), W(x))$ to (1), i.e.,*

$$\lim_{t\to+\infty} u(x,t) = U(x), \qquad \lim_{t\to+\infty} w(x,t) = W(x)$$

with, as $\beta > 0$

$$\begin{cases} U(x) = \dfrac{a+\delta}{b+\beta} W(x), \\ w_0(x)\left(\frac{(b+\beta)\underline{p}_0}{a+\delta}\right)^{\frac{\beta}{b+\beta}} \le W(x) \le w_0(x)\left(\frac{(b+\beta)\bar{p}_0}{a+\delta}\right)^{\frac{\beta}{b+\beta}}, \end{cases}$$

as $\beta < 0$

$$\begin{cases} U(x) = \dfrac{a+\delta}{b+\beta} W(x), \\ w_0(x)\left(\dfrac{(b+\beta)\bar{p}_0}{a+\delta}\right)^{\frac{\beta}{b+\beta}} \le W(x) \le w_0(x)\left(\dfrac{(b+\beta)\underline{p}_0}{a+\delta}\right)^{\frac{\beta}{b+\beta}} \end{cases}$$

as $\beta = 0$ (here $\delta = 0$)

$$U(x) = \frac{a}{b} w_0(x), \quad W(x) = w_0(x)$$

3.2 The case $a + \delta > 0, b + \beta = 0$

In the case of $a + \delta > 0, b + \beta = 0$, there is a unique solution $p(x,t)$ to (6) satisfying

$$\underline{p}_0 e^{(a+\delta)t} \le p(x,t) \le \bar{p}_0 e^{(a+\delta)t}$$

which implies

$$\lim_{t\to+\infty} p(x,t) = +\infty$$

Suppose that $\beta > 0$.

$$\lim_{t\to+\infty} w(x,t) = +\infty, \qquad \lim_{t\to+\infty} u(x,t) = +\infty$$

Suppose that $\beta < 0$, it can been seen that

$$\begin{aligned} u(x,t) &\le \bar{p}(x,t) w_0(x) e^{\int_0^t (\beta \underline{p}(x,\tau) - \delta) d\tau} \\ &= \bar{p}_0 e^{at} w_0(x) e^{\frac{\underline{p}_0 \beta}{a+\delta}(e^{(a+\delta)t} - 1)} \end{aligned}$$

thus

$$\lim_{t\to+\infty} w(x,t) = 0, \qquad \lim_{t\to+\infty} u(x,t) = 0$$

Suppose that $\beta = 0$, then $b = 0$. If $\delta < 0$,

$$\lim_{t\to+\infty} w(x,t) = \lim_{t\to+\infty} w_0(x)e^{-\delta t} = +\infty$$

and

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} w(x,t)p(x,t) = +\infty$$

If $\delta = 0$, then

$$w(x,t) \equiv w_0(x)$$

and

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} w(x,t)p(x,t) = +\infty$$

If $\delta > 0$,

$$\lim_{t\to+\infty} w(x,t) = \lim_{t\to+\infty} w_0(x)e^{-\delta t} = 0$$

$$\lim_{t\to+\infty} u(x,t) = \begin{cases} +\infty, & \text{as } a > 0 \\ 0, & \text{as } a < 0 \end{cases}$$

while $u(x,t) < +\infty$ as $a = 0$, since

$$\underline{p}_0 w_0(x)e^{at} \leq u(x,t) \leq \bar{p}_0 w_0(x)e^{at}$$

Theorem 2. *Suppose that $a + \delta > 0$ and $b + \beta = 0$, then there is a unique global solution $(u(x,t), w(x,t))$ to the problem (1) which satisfying the following property:*

- *providing $\beta < 0$,*

$$\lim_{t\to+\infty} w(x,t) = 0, \qquad \lim_{t\to+\infty} u(x,t) = 0$$

- *providing $\beta > 0$,*

$$\lim_{t\to+\infty} w(x,t) = +\infty, \qquad \lim_{t\to+\infty} u(x,t) = +\infty$$

- *providing $\beta = 0$, then as $\delta < 0$,*

$$\lim_{t\to+\infty} w(x,t) = +\infty, \qquad \lim_{t\to+\infty} u(x,t) = +\infty;$$

as $\delta > 0$,

$$\lim_{t\to+\infty} w(x,t) = 0$$

while

$$\lim_{t\to+\infty} u(x,t) = \begin{cases} +\infty, & \text{if } a > 0 \\ 0, & \text{if } a < 0 \end{cases}$$

and $u(x,t) < +\infty$ if $a = 0$;
as $\delta = 0$,

$$w(x,t) \equiv w_0(x)$$

and

$$\lim_{t\to+\infty} u(x,t) = +\infty$$

3.3 The case $a+\delta>0, b+\beta<0$

In the case of $a+\delta>0, b+\beta<0$, there are

$$\tilde{T}_1=\frac{1}{a+\delta}\ln\left(1-\frac{a+\delta}{\bar{p}_0(b+\beta)}\right)\leq\hat{T}_1=\frac{1}{a+\delta}\ln\left(1-\frac{a+\delta}{\underline{p}_0(b+\beta)}\right)$$

such that

$$\lim_{t\to\tilde{T}_1}\bar{p}(x,t)=+\infty,\qquad \lim_{t\to\hat{T}_1}\underline{p}(x,t)=+\infty$$

that implies that there is a T_1 with $\tilde{T}_1\leq T_1\leq\hat{T}_1$ such that

$$\lim_{t\to T_1}p(x,t)=+\infty.$$

If $\beta>0$, then we have that

$$\begin{aligned} w(x,t) &\geq w_0(x)e^{\int_0^t(\beta\underline{p}(x,\tau)-\delta)d\tau} \\ &= w_0(x)\left(\frac{a+\delta+\underline{p}_0(b+\beta)(e^{(a+\delta)t}-1)}{a+\delta}\right)^{\frac{\beta}{b+\beta}} \\ &\to +\infty,\qquad \text{as } t\to\hat{T}_1 \end{aligned}\tag{12}$$

and

$$u(x,t)=P(x,t)w(x,t)\to+\infty,\qquad \text{as } t\to T_1$$

If $\beta<0$,

$$\begin{aligned} w(x,t) &\leq w_0(x)e^{\int_0^t(\beta\bar{p}(x,\tau)-\delta)d\tau} \\ &= w_0(x)\left(\frac{a+\delta+\bar{p}_0(b+\beta)(e^{(a+\delta)t}-1)}{a+\delta}\right)^{\frac{\beta}{b+\beta}} \\ &\to 0\qquad \text{as } t\to\hat{T}_1 \end{aligned}$$

while

$$u(x,t)=p(x,t)w_0(x)e^{\int_0^t(\beta p(x,\tau)-\delta)d\tau}\to 0,\qquad \text{as } t\to T_1$$

since $p(x,t)$ goes to $+\infty$ rationally while $w(x,t)$ goes to 0 exponentially.

If $\beta=0$, it is easy to see that

$$0<\lim_{t\to T_1^-}w(x,t)=w_0(x)e^{-\delta T_1}<+\infty$$

$$\lim_{t\to T_1^-}u(x,t)=+\infty$$

Theorem 3. *Suppose that $a+\delta > 0$ and $b+\beta < 0$, then the solution $(u(x,t), w(x,t))$ to (1) blows up if $\beta \geq 0$ or quenches if $\beta < 0$ at a finite time T_1 which satisfies*

$$\frac{1}{a+\delta}\ln\left(1-\frac{a+\delta}{\bar{p}_0(b+\beta)}\right) \leq T_1 \leq \frac{1}{a+\delta}\ln\left(1-\frac{a+\delta}{\underline{p}_0(b+\beta)}\right)$$

More detail,

- *if* $\beta > 0$,
$$\lim_{t\to T_1^-} w(x,t) = +\infty, \quad \lim_{t\to T_1^-} u(x,t) = +\infty,$$
- *if* $\beta = 0$,
$$\lim_{t\to T_1^-} w(x,t) = w_0(x)e^{-\delta T_1}, \quad \lim_{t\to T_1^-} u(x,t) = +\infty,$$
- *if* $\beta < 0$,
$$\lim_{t\to T_1^-} w(x,t) = 0, \quad \lim_{t\to T_1^-} u(x,t) = 0, \quad \lim_{t\to T_1^-} \frac{\partial u}{\partial t} = \infty.$$

3.4 The case $a+\delta < 0, b+\beta > 0$

In the case of $a+\delta < 0, b+\beta > 0$, there is a unique global solution $p(x,t)$ to (6) satisfying

$$\lim_{t\to+\infty} p(x,t) = 0$$

Suppose that $\delta > 0$, then

$$\lim_{t\to+\infty} w(x,t) = 0, \qquad \lim_{t\to+\infty} u(x,t) = 0$$

Suppose that $\delta = 0$, then $a < 0$. As $\beta \geq 0$,

$$\begin{aligned}
& w_0(x)e^{\int_0^t \beta\underline{p}(x,\tau)d\tau} \\
= \; & w_0(x)\left(\frac{a+\underline{p}_0(b+\beta)(e^{at}-1)}{a}\right)^{\frac{\beta}{b+\beta}} \\
\leq \; & w(x,t) \leq w_0(x)e^{\int_0^t \beta\bar{p}(x,\tau)d\tau} \\
= \; & w_0(x)\left(\frac{a+\bar{p}_0(b+\beta)(e^{at}-1)}{a}\right)^{\frac{\beta}{b+\beta}}
\end{aligned}$$

as $\beta < 0$,

$$w_0(x)e^{\int_0^t \beta\bar{p}(x,\tau)d\tau}$$

$$
\begin{aligned}
&= w_0(x)\left(\frac{a+\bar{p}_0(b+\beta)(e^{at}-1)}{a}\right)^{\frac{\beta}{b+\beta}} \\
&\le w(x,t) \le w_0(x)e^{\int_0^t \beta \underline{p}(x,\tau)d\tau} \\
&= w_0(x)\left(\frac{a+\underline{p}_0(b+\beta)(e^{at}-1)}{a}\right)^{\frac{\beta}{b+\beta}}
\end{aligned}
$$

thus, we see that

$$|w(x,t)| < +\infty, \qquad \lim_{t\to+\infty} u(x,t) = 0$$

Suppose that $\delta < 0$, then

$$\lim_{t\to+\infty} w(x,t) = +\infty$$

As $\beta \ge 0$,

$$
\begin{aligned}
& \underline{p}(x,t)w_0(x)e^{\int_0^t (\beta \underline{p}(x,\tau)-\delta)d\tau} \\
&= \underline{p}_0 w_0(x)|a+\delta|^{\frac{b}{b+\beta}} \cdot \left|a+\delta+\underline{p}_0(b+\beta)(e^{(a+\delta)t}-1)\right|^{\frac{-b}{b+\beta}} e^{at} \\
&\le u(x,t) \le \bar{p}(x,t)w_0(x)e^{\int_0^t \beta \bar{p}(x,\tau)d\tau} \\
&= \bar{p}_0 w_0(x)|a+\delta|^{\frac{b}{b+\beta}} \cdot \left|a+\delta+\bar{p}_0(b+\beta)(e^{(a+\delta)t}-1)\right|^{\frac{-b}{b+\beta}} e^{at}
\end{aligned}
$$

as $\beta < 0$,

$$
\begin{aligned}
& \underline{p}(x,t)w_0(x)e^{\int_0^t (\beta \bar{p}(x,\tau)-\delta)d\tau} \\
&= \underline{p}_0 w_0(x)|a+\delta|^{\frac{b}{b+\beta}} \cdot \left|a+\delta+\bar{p}_0(b+\beta))(e^{(a+\delta)t}-1)\right|^{\frac{-b}{b+\beta}} e^{at} \\
&\le u(x,t) \le \bar{p}(x,t)w_0(x)e^{\int_0^t \beta \bar{p}(x,\tau)d\tau} \\
&= \bar{p}_0 w_0(x)|a+\delta|^{\frac{b}{b+\beta}} \cdot \left|a+\delta+\underline{p}_0(b+\beta))(e^{(a+\delta)t}-1)\right|^{\frac{-b}{b+\beta}} e^{at}
\end{aligned}
$$

thus

$$\lim_{t\to+\infty} u(x,t) = \begin{cases} +\infty, & a>0, \\ 0, & a<0 \end{cases}$$

while $\lim_{t\to+\infty} u(x,t)$ is bounded if $a = 0$.

Theorem 4. *Suppose that $a+\delta < 0$ and $b+\beta > 0$, then there is a unique global solution $(u(x,t), w(x,t))$ to the problem (1) which satisfying the following property:*

- *provided that $\delta > 0$,*

$$\lim_{t\to+\infty} w(x,t) = 0, \qquad \lim_{t\to+\infty} u(x,t) = 0$$

- *provided that* $\delta = 0$,

$$0 < w(x,t) < +\infty, \qquad \lim_{t\to+\infty} u(x,t) = 0$$

- *provided that* $\delta < 0$,

$$\lim_{t\to+\infty} w(x,t) = +\infty, \qquad \lim_{t\to+\infty} u(x,t) = \begin{cases} +\infty, & a > 0, \\ 0, & a < 0 \end{cases}$$

while $\lim_{t\to+\infty} u(x,t)$ *is bounded if* $a = 0$.

3.5 The case $a + \delta < 0, b + \beta = 0$

In the case of $a + \delta < 0, b + \beta = 0$, there is a unique global solution $p(x,t)$ to (6) satisfying

$$\underline{p}_0 e^{(a+\delta)t} \le p(x,t) \le \bar{p}_0 e^{(a+\delta)t}$$

thus

$$\lim_{t\to+\infty} p(x,t) = 0$$

and

$$0 < \left| \exp\{\int_0^t \beta p(x,\tau) d\tau\} \right| < +\infty$$

which implies that there are two bounded positive functions $0 < W_1(x,t) \le W_2(x,t) < +\infty$ such that

$$W_1(x,t) e^{-\delta t} \le w(x,t) \le W_2(x,t) e^{-\delta t}$$

Suppose that $\delta > 0$, we see that

$$\lim_{t\to+\infty} w(x,t) = 0, \qquad \lim_{t\to+\infty} u(x,t) = 0$$

Suppose that $\delta = 0$, then $a < 0$ and $\underline{p}(x,t) = \underline{p}_0 e^{at}, \bar{p}(x,t) = \bar{p}_0 e^{at}$. So, we see that, if $\beta \ge 0$,

$$w_0(x) e^{\frac{\underline{p}_0 \beta}{|a|}} \le \lim_{t\to+\infty} w(x,t) \le w_0(x) e^{\frac{\bar{p}_0 \beta}{|a|}}$$

if $\beta < 0$,

$$w_0(x) e^{\frac{\bar{p}_0 \beta}{|a|}} \le \lim_{t\to+\infty} w(x,t) \le w_0(x) e^{\frac{\underline{p}_0 \beta}{|a|}}$$

while

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} p(x,t) w(x,t) = 0$$

Suppose that $\delta < 0$, then

$$\lim_{t\to+\infty} w(x,t) = +\infty$$

while

$$\underline{p}_0 w_0(x)e^{at}e^{\int_0^t \beta p(x,\tau)d\tau} \leq u(x,t) \leq \bar{p}_0 w_0(x)e^{at}e^{\int_0^t \beta p(x,\tau)d\tau}$$

thus

$$\lim_{t\to+\infty} u(x,t) = \begin{cases} +\infty, & a>0, \\ 0, & a<0 \end{cases}$$

while $\lim_{t\to+\infty} u(x,t)$ is bounded if $a=0$.

Theorem 5. *Suppose that $a+\delta<0$ and $b+\beta=0$, then there is a unique global solution $(u(x,t),w(x,t))$ to the problem (1) satisfying the following property:*

- *provided that $\delta>0$,*

$$\lim_{t\to+\infty} w(x,t)=0, \qquad \lim_{t\to+\infty} u(x,t)=0$$

- *provided that $\delta=0$,*

$$0<w(x,t)<+\infty, \qquad \lim_{t\to+\infty} u(x,t)=0$$

- *provided that $\delta<0$,*

$$\lim_{t\to+\infty} w(x,t)=+\infty, \qquad \lim_{t\to+\infty} u(x,t) = \begin{cases} +\infty, & a>0, \\ 0, & a<0 \end{cases}$$

  while $\lim_{t\to+\infty} u(x,t)$ *is bounded if* $a=0$.

3.6 The case $a+\delta<0, b+\beta<0$

Suppose that $a+\delta<0, b+\beta<0$, if $\bar{p}_0 \leq \dfrac{a+\delta}{b+\beta}$, we see from (8) and (7), that

$$\lim_{t\to+\infty} \bar{p}(x,t)=0, \qquad \lim_{t\to+\infty} \underline{p}(x,t)=0$$

which implies that there is a unique global solution $p(x,t)$ to (6) satisfying

$$\lim_{t\to+\infty} p(x,t)=0.$$

Similar to the analysis mentioned above, we see that there are 4 bounded positive functions $P_i(x,t), W_i(x,t), i=1,2$ such that, as $t\to+\infty$,

$$P_1(x,t)e^{(a+\delta)t} \leq p(x,t) \leq P_2(x,t)e^{(a+\delta)t}$$

$$W_1(x,t)e^{-\delta t} \leq w(x,t) \leq W_2(x,t)e^{-\delta t}$$

whereupon

$$P_1(x,t)W_1(x,t)e^{at} \leq u(x,t) \leq P_2(x,t)W_2(x,t)e^{at}$$

$$\lim_{t\to+\infty} w(x,t) = \begin{cases} 0 & \text{as } \delta > 0 \\ \textit{bounded} & \text{as } \delta = 0 \\ +\infty & \text{as } \delta < 0 \end{cases}$$

Moreover, if $\delta \geq 0$, the case of $a+\delta < 0$ means $a < 0$. And then, we see that

$$\lim_{t\to+\infty} u(x,t) = \begin{cases} 0 & \text{as } \delta \geq 0 \text{ or } a\delta > 0 \\ +\infty & \text{as } \delta < 0 \text{ and } a > 0 \end{cases}$$

Theorem 6. *Suppose that $a+\delta < 0$ and $b+\beta < 0$ and $\bar{p}_0 < \frac{a+\delta}{b+\beta}$, then there is a global solution $(u(x,t), w(x,t))$ to the problem (1) satisfying*

$$\lim_{t\to+\infty} w(x,t) = \begin{cases} 0 & \textit{as } \delta > 0 \\ \textit{bounded} & \textit{as } \delta = 0 \\ +\infty & \textit{as } \delta < 0 \end{cases}$$

$$\lim_{t\to+\infty} u(x,t) = \begin{cases} 0 & \textit{as } \delta \geq 0 \textit{ or } a\delta > 0 \\ +\infty & \textit{as } \delta < 0 \textit{ and } a > 0 \end{cases}$$

If $\underline{p}_0 > \frac{a+\delta}{b+\beta}$, there are

$$\tilde{T}_2 = \frac{1}{a+\delta} \ln\left(1 - \frac{a+\delta}{\bar{p}_0(b+\beta)}\right), \qquad \hat{T}_2 = \frac{1}{a+\delta} \ln\left(1 - \frac{a+\delta}{\underline{p}_0(b+\beta)}\right)$$

such that

$$\lim_{t\to\tilde{T}_2^-} \bar{p}(x,t) = +\infty, \qquad \lim_{t\to\hat{T}_2^-} \underline{p}(x,t) = +\infty$$

which implies that there is $T_2 \in [\tilde{T}_2, \hat{T}_2]$ such that

$$\lim_{t\to T_2^-} p(x,t) = +\infty$$

that is to say that thee is not a global solution to (6).

If $\beta > 0$,

$$w_0(x) e^{\int_0^t (\beta \underline{p}(x,\tau) - \delta) d\tau} \leq w(x,t) \leq w_0(x) e^{\int_0^t (\beta \bar{p}(x,\tau) - \delta) d\tau}$$

which implies that

$$\lim_{t\to T_2^-} w(x,t) = +\infty$$

and

$$\lim_{t\to T_2^-} u(x,t) = \lim_{t\to T_2^-} p(x,t) w(x,t) = +\infty$$

If $\beta < 0$,

$$w_0(x)e^{\int_0^t (\beta\bar{p}(x,\tau)-\delta)d\tau} \leq w(x,t) \leq w_0(x)e^{\int_0^t (\beta\underline{p}(x,\tau)-\delta)d\tau}$$

which implies that

$$\lim_{t\to T_2^-} w(x,t) = 0$$

And then, we see that

$$\lim_{t\to T_2^-} u(x,t) = 0$$

since $w(x,t)$ goes to $+\infty$ exponentially with respect to $p(x,t)$.

If $\beta = 0$, $w(x,t) = w_0(x)e^{-\delta t}$ is bounded for $t \in [0, T_2]$ while

$$\lim_{t\to T_2^-} u(x,t) = +\infty$$

Sum up above, we have the following theorem.

Theorem 7. *Suppose that $a+\delta < 0$ and $b+\beta < 0$ and $\underline{p}_0 > \frac{a+\delta}{b+\beta}$, then the solution $(u(x,t), w(x,t))$ to the problem (1) blow-up if $\beta \geq 0$ or quench if $\beta < 0$ at a finite time. In detail, there is a finite T_2 satisfying*

$$\frac{1}{a+\delta}\ln\left(1-\frac{a+\delta}{\bar{p}_0(b+\beta)}\right) \leq T_2 \leq \frac{1}{a+\delta}\ln\left(1-\frac{a+\delta}{\underline{p}_0(b+\beta)}\right)$$

such that

- *provided that $\beta > 0$,*

$$\lim_{t\to T_2^-} w(x,t) = +\infty, \qquad \lim_{t\to T_2^-\infty} u(x,t) = +\infty$$

- *provided that $\beta = 0$,*

$$\lim_{t\to T_2^-} w(x,t) = w_0(x)e^{-\delta T_2}, \qquad \lim_{t\to T_2^-} u(x,t) = +\infty$$

- *provided that $\beta < 0$,*

$$\lim_{t\to T_2^-} w(x,t) = 0, \qquad \lim_{t\to T_2^-} u(x,t) = 0, \qquad \lim_{t\to T_2^-} \frac{\partial u}{\partial t} = \infty$$

3.7 The case $a + \delta = 0, b + \beta > 0$

In the case that $a + \delta = 0, b + \beta > 0$,

$$\bar{p}(x,t) = \frac{\bar{p}_0}{1 + \bar{p}_0(b+\beta)t}, \qquad \underline{p}(x,t) = \frac{\underline{p}_0}{1 + \underline{p}_0(b+\beta)t}$$

which implies

$$\lim_{t\to+\infty} p(x,t) = 0$$

Then, it is seen easily that

$$\lim_{t\to+\infty} w(x,t) = \lim_{t\to+\infty} w_0(x) e^{\int_0^t (\beta p(x,\tau) - \delta) d\tau} = \begin{cases} 0 & \text{as } \delta > 0 \\ +\infty & \text{as } \delta < 0 \end{cases}$$

As $\delta = 0$ and $\beta > 0$,

$$\begin{aligned} w(x,t) &\geq w_0(x) e^{\int_0^t \beta \underline{p}(x,\tau) d\tau} \\ &= w_0(x) \left(1 + \underline{p}_0(b+\beta)t\right)^{\frac{\beta}{b+\beta}} \\ &\to +\infty, \qquad \text{as } t \to +\infty \end{aligned} \tag{13}$$

while as $\delta = 0$ and $\beta < 0$,

$$\begin{aligned} w(x,t) &\leq w_0(x) e^{\int_0^t \beta \underline{p}(x,\tau) d\tau} \\ &= w_0(x) \left(1 + \underline{p}_0(b+\beta)t\right)^{\frac{\beta}{b+\beta}} \\ &\to 0, \qquad \text{as } t \to +\infty \end{aligned} \tag{14}$$

and as $\delta = 0$ and $\beta = 0$, $w(x,t) = w_0(x)$.

Then, consider the behavior of $u(x,t)$. It is clear that, as $\delta > 0$,

$$\lim_{t\to+\infty} u(x,t) = 0$$

If $\delta < 0$, since $p(x,t)$ goes to 0 rationally while $w(x,t)$ goes to $+\infty$ exponentially,

$$\lim_{t\to+\infty} u(x,t) = +\infty$$

If $\delta = 0$ and $\beta \leq 0$, since $p(x,t)$ goes to 0 and $w(x,t)$ goes to 0 or is bounded,

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} p(x,t) w(x,t) = 0$$

If $\delta = 0$ and $\beta > 0$, as $b > 0$

$$\begin{aligned} u(x,t) &\geq \underline{p}(x,t) w_0(x) e^{\int_0^t \beta \underline{p}(x,\tau) d\tau} \\ &= \underline{p}_0 w_0(x) \left(1 + \underline{p}_0(b+\beta)t\right)^{\frac{b}{b+\beta}} \\ &\to +\infty, \qquad \text{as } t \to +\infty \end{aligned} \tag{15}$$

or as $b < 0$

$$\begin{aligned} u(x,t) &\leq \bar{p}(x,t)w_0(x)e^{\int_0^t \beta\bar{p}(x,\tau)d\tau} \\ &= \bar{p}_0 w_0(x)\left(1+\bar{p}_0(b+\beta)t\right)^{\frac{b}{b+\beta}} \\ &\to 0, \qquad \text{as } t\to+\infty \end{aligned} \tag{16}$$

or as $b=0$,

$$\underline{p}_0 w_0(x) \leq u(x,t) \leq \bar{p}_0 w_0(x)$$

Thus, we can sum up above as follows.

Theorem 8. *Suppose that $a+\delta=0$ and $b+\beta>0$, then there is a unique global solution $(u(x,t),w(x,t))$ to the problem (1) satisfying the following property:*

- *provided that $\delta>0$,*

$$\lim_{t\to+\infty} w(x,t)=0, \qquad \lim_{t\to+\infty} u(x,t)=0$$

- *provided that $\delta<0$,*

$$\lim_{t\to+\infty} w(x,t)=+\infty, \qquad \lim_{t\to+\infty} u(x,t)=+\infty$$

- *provided that $\delta=0$,*

$$\lim_{t\to+\infty} w(x,t)=\begin{cases} +\infty, & \textit{as } b>0 \\ w_0(x), & \textit{as } b=0 \\ 0, & \textit{as } b<0 \end{cases} \qquad \lim_{t\to+\infty} u(x,t)=\begin{cases} +\infty, & \textit{as } b>0, \\ \textit{bounded}, & \textit{as } b=0, \\ 0, & \textit{as } b<0 \end{cases}$$

3.8 The case $a+\delta=0, b+\beta<0$

In the case of $a+\delta=0, b+\beta<0$,

$$\bar{p}(x,t)=\frac{\bar{p}_0}{1-\bar{p}_0|b+\beta|t}, \qquad \underline{p}(x,t)=\frac{\underline{p}_0}{1-\underline{p}_0|b+\beta|t}$$

which implies that there are

$$\tilde{T}_3=\frac{1}{\bar{p}_0|b+\beta|}, \qquad \hat{T}_3=\frac{1}{\underline{p}_0|b+\beta|}$$

such that

$$\lim_{t\to\tilde{T}_3^-} \bar{p}(x,t)=+\infty, \qquad \lim_{t\to\hat{T}_3^-} \underline{p}(x,t)=+\infty$$

Then, we conclude by comparison that there is $T_3\in[\tilde{T}_3,\hat{T}_3]$ such that

$$\lim_{t\to T_3^-} p(x,t)=+\infty$$

Hence, we can see that

$$\lim_{t\to T_3^-} w(x,t) = \lim_{t\to T_3^-} w_0(x) e^{\int_0^t (\beta p(x,\tau)-\delta)d\tau} = \begin{cases} +\infty, & \text{as } \beta > 0 \\ w_0(x)e^{-\delta T_3}, & \text{as } \beta = 0 \\ 0, & \text{as } \beta < 0 \end{cases}$$

As for $u(x,t)$, it is obvious that, if $\beta \geq 0$,

$$\lim_{t\to T_3^-} u(x,t) = +\infty$$

If $\beta < 0$, since $p(x,t)$ goes to ∞ rationally and $w(x,t)$ goes to 0 exponentially, so $u(x,t)$ goes to 0. Hence,

$$\lim_{t\to T_3^-} u(x,t) = \lim_{t\to T_3^-} p(x,t) w_0(x) e^{\int_0^t (\beta p(x,\tau)-\delta)d\tau} = \begin{cases} +\infty, & \text{as } \beta \geq 0 \\ 0, & \text{as } \beta < 0 \end{cases}$$

To sum up, we have the following Theorem 8.

Theorem 9. *Suppose that $a+\delta = 0$ and $b+\beta < 0$, then the solution $(u(x,t), w(x,t))$ to the problem (1) blow-up as $\beta \geq 0$ or quench as $\beta < 0$ at a finite time. In detail, there is a finite T_3 satisfying*

$$\frac{1}{\bar{p}_0|b+\beta|} \leq T_3 \leq \frac{1}{\underline{p}_0|b+\beta|}$$

such that

- *provided that $\beta > 0$,*

$$\lim_{t\to T_3^-} w(x,t) = +\infty, \qquad \lim_{t\to T_3^-} u(x,t) = +\infty$$

- *provided that $\beta = 0$,*

$$\lim_{t\to T_3^-} w(x,t) = w_0(x)e^{-\delta T_3}, \qquad \lim_{t\to T_3^-} u(x,t) = +\infty$$

- *provided that $\beta < 0$,*

$$\lim_{t\to T_3^-} w(x,t) = 0, \qquad \lim_{t\to T_3^-} u(x,t) = 0, \qquad \lim_{t\to T_3^-} \frac{\partial u}{\partial t} = \infty$$

3.9 The case $a + \delta = 0, b + \beta = 0$

In the case of $a + \delta = 0, b + \beta = 0$,

$$\underline{p}_0 \leq p(x,t) \leq \bar{p}_0.$$

Obviously, $p(x,t) \equiv \frac{\delta}{\beta} = \frac{a}{b}$ is a stationary solution of (6). Let

$$p(x,t) = \frac{\delta}{\beta} + \eta(x,t)$$

then, the function $\eta(x,t)$ is subjected to

$$\begin{cases} \dfrac{\partial \eta}{\partial t} = D\Delta\eta + D\nabla \ln w \cdot \nabla \eta & (x,t) \in Q_T \\ w(x,t) = w_0(x) e^{\int_0^t \beta\eta(x,\tau)d\tau} & (x,t) \in Q_T \\ \dfrac{\partial \eta}{\partial n} = 0 & (x,t) \in \Gamma_T \\ \eta(x,0) = \eta_0(x) = p_0(x) - \dfrac{\delta}{\beta} & x \in \bar{\Omega} \\ w(x,0) = w_0(x) > 0 & x \in \bar{\Omega} \end{cases} \tag{17}$$

For any given function $w(x,t)$, it can be seen by the classical method that

$$\eta(x,t) = \sum_{n=0}^{+\infty} b_n \phi_n(x) e^{-\lambda_n t}, \quad x \in \Omega, \ t > 0$$

solves the following initial-boundary value problem

$$\begin{cases} \dfrac{\partial \eta}{\partial t} = D\Delta\eta + D\nabla \ln w \cdot \nabla \eta & (x,t) \in Q_T \\ \dfrac{\partial \eta}{\partial n} = 0 & (x,t) \in \Gamma_T \\ \eta(x,0) = \eta_0(x) = p_0(x) - \dfrac{\delta}{\beta} & x \in \bar{\Omega} \end{cases} \tag{18}$$

where

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lambda_n \to +\infty, \text{ as } n \to +\infty$$

and

$$\phi_n(x), \quad \int_\Omega |\phi_n(x)|^2 dx = 1, \quad n = 0, 1, 2, \cdots,$$

are eigenvalues and corresponding eigenfunctions, respectively, of the following eigenvalue problem

$$\begin{cases} D\Delta\phi + D\nabla \ln w \cdot \nabla\phi = -\lambda\phi & x \in \Omega \\ \dfrac{\partial \phi}{\partial n} = 0 & x \in \partial\Omega \end{cases} \tag{19}$$

while

$$b_n = \int_\Omega \eta_0(x)\phi_n(x)dx, \quad n = 0, 1, 2, \cdots$$

Then,

$$\int_0^t \eta(x,\tau)d\tau = b_0\phi_0 t + \sum_{n=1}^{\infty}\frac{b_n\phi_n(x)}{\lambda_n}(1-e^{-\lambda_n t})$$

which suggests that there is bounded positive function

$$M(x,t) = w_0(x)\prod_{n=1}^{\infty}\exp(\frac{\beta b_n\phi_n(x)}{\lambda_n}(1-e^{-\lambda_n t})$$

such that

$$w(x,t) = M(x,t)e^{\beta b_0\phi_0 t}$$

Thus, we can see that, as $\beta \neq 0$,

$$w(x,t) \to \begin{cases} +\infty & \text{if } \beta b_0\phi_0 > 0, \\ \textit{bounded} & \text{if } \beta b_0\phi_0 = 0, \\ 0 & \text{if } \beta b_0\phi_0 < 0, \end{cases}$$

Since that $\phi_0 = |\Omega|^{-1}$ and

$$b_0 = \int_\Omega \eta_0(x)\phi_0 dx = \frac{1}{|\Omega|}\int_\Omega p_0(x)dx - \frac{\delta}{\beta}$$

it can be seen that, as $\beta > 0$,

$$w(x,t) \to \begin{cases} +\infty & \text{if } \bar{p}_0 > 0, \\ \textit{bounded} & \text{if } \bar{p}_0 = 0, \\ 0 & \text{if } \bar{p}_0 < 0, \end{cases}$$

as $\beta < 0$,

$$w(x,t) \to \begin{cases} 0 & \text{if } \bar{p}_0 > 0, \\ \textit{bounded} & \text{if } \bar{p}_0 = 0, \\ +\infty & \text{if } \bar{p}_0 < 0, \end{cases}$$

where

$$\bar{p}_0 = \frac{1}{|\Omega|}\int_\Omega p_0(x)dx$$

And then, we see that $u(x,t)$ behave asymptotically as same as $w(x,t)$ does due to that $p(x,t)$ is bounded.

As $\beta = 0$, since $w(x,t) = w_0(x)e^{-\delta t}$, we see that

$$w(x,t) \to \begin{cases} 0 & \text{if } \delta > 0, \\ w_0(x) & \text{if } \delta = 0, \\ +\infty & \text{if } \delta < 0, \end{cases} \qquad u(x,t) \to \begin{cases} 0 & \text{if } \delta > 0, \\ \textit{bounded} & \text{if } \delta = 0, \\ +\infty & \text{if } \delta < 0, \end{cases}$$

Theorem 10. *Suppose that $a + \delta = 0$ and $b + \beta = 0$, then there is a unique global solution $(u(x,t), w(x,t))$ to the problem (1) satisfying the following property:*

- *provided that* $\beta > 0$,

$$\lim_{t\to+\infty} w(x,t) = \begin{cases} +\infty & \textit{if } \bar{p}_0 > 0, \\ \textit{bounded} & \textit{if } \bar{p}_0 = 0, \\ 0 & \textit{if } \bar{p}_0 < 0, \end{cases} \qquad \lim_{t\to+\infty} u(x,t) = \begin{cases} +\infty & \textit{if } \bar{p}_0 > 0, \\ \textit{bounded} & \textit{if } \bar{p}_0 = 0, \\ 0 & \textit{if } \bar{p}_0 < 0, \end{cases}$$

- *provided that* $\beta < 0$,

$$\lim_{t\to+\infty} w(x,t) = \begin{cases} 0 & \textit{if } \bar{p}_0 > 0, \\ \textit{bounded} & \textit{if } \bar{p}_0 = 0, \\ +\infty & \textit{if } \bar{p}_0 < 0, \end{cases} \qquad \lim_{t\to+\infty} u(x,t) = \begin{cases} 0 & \textit{if } \bar{p}_0 > 0, \\ \textit{bounded} & \textit{if } \bar{p}_0 = 0, \\ +\infty & \textit{if } \bar{p}_0 < 0, \end{cases}$$

- *provided that* $\beta = 0$,

$$\lim_{t\to+\infty} w(x,t) = \begin{cases} 0, & \textit{as } \delta > 0 \\ w_0(x), & \textit{as } \delta = 0 \\ +\infty, & \textit{as } \delta < 0 \end{cases} \qquad \lim_{t\to+\infty} u(x,t) = \begin{cases} 0, & \textit{as } \delta > 0, \\ \textit{bounded}, & \textit{as } \delta = 0, \\ +\infty, & \textit{as } \delta < 0 \end{cases}$$

summarizing the above, the main result is concluded.

Acknowledgement. *The work was completed when the third author was visiting the Institute of Mathematics, University of Potsdam, Germany, partially supported by the Deutsche Forschungsgemeinschaft, who appreciates the invitation by Prof. Bert-Wolfgang Schulze and the support of the DFG.*

References

[1] R. Erban and H. G. Othmer. From indidual to collective behavior in bacterial chemotaxis. SIAM J. Appl. Math. 2004;65(NO 2):361-391

[2] T. Hillen and Kevin J. Painter, A Parabolic Model with Bounded chemotaxis-Prevention of overcrowding. Adv. Appl. Math. 2001;26:280-301

[3] T. Hillen and A. Stevens, Hyperbolic Models for Chemotaxis in 1-D. Nonl. Anal.: Real world Appl. 2000;1(No 3):409-433

[4] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. Journal of Theoretical Biology 1970;26:339-415

[5] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc. Trans., Vol. 23, Amer. Math. Soc. Providence. R. I., 1968

[6] J. D. Murray, Mathematical Biology. Springer, Berlin, Heidelberg, New york, 1993

[7] T. Nagai, Global Existence of Solutions to a Parabolic Systems for Chemotaxis in Two Space Dimentions. Nonl. Anal. Theory, Methods & Applications, 1997;30(No 8):5381-5388

[8] T. Nagai and T. Senba, Global Existence and Blowup of Radial Solutions to a Parabolic-Elliptic System of Chemotaxis. Adv. Math. Sci. Appl. 1998;8:145-156

[9] H. G. Othmer and A. Sevens. Aggregation, blowup and collapse: the ABC's of taxis in reinforced random walks. SIAM Journal on Applied Mathematics 1997;51:1044-1081

[10] K. J. Painter and Jonathan A. Sherratt, Modelling the movement of interacting cell populations. J. Theor. Biol.225(2003):327-339

[11] K. J. Painter, P.K.Maini and H.G.Othmer, A chemotactic Model for the Advance and Retreat of Primitive Streak in Avain development. Bull. Math. Biol. 2000;62:501-525

[12] K. J. Painter and Thomas Hillen, Models for Chemosensitive Movement. Canadian Appl. Math. Quart.2003;10(No 4):501-543

[13] B. D. Sleeman and H. A. Levine. Partial differential equations of chemotaxis and angiogenesis. Mathematical Methods in the Applied Science 2001;24:405-426

[14] A. Stevens, Trail Following and Aggregation of myxobacteria. J. Biol. Systems, 1995;3:1059-1068

[15] A. Stevens, A Stochastic Cellular Automaton, Modelling Gliding and Aggregation of myxobacteria. SIAM J. Appl. Math. 2000;61:172-182

[16] A. Stevens, The Derivative of Chemotaxis Equations as Limit Dynamics of Moderately Interacting Stochastic Many-Particles Systems. SIAM J. Appl. Math. 2000;61(No 1):183-212

[17] Yin Yang, Some Results on Reaction-Diffusion Systems Modelling Chemotaxis. Ph.D. Thesis. Wuhan University. 2000

[18] Yang Yin, Chen Hua, and Liu Weian, On Existence of Global solutions and Blowup to a System of Reaction-Diffusion Equations Modelling Chemotaxis. SIAM J. on Math. Anal. 2001;33(No 4): 763-785

[19] Yang Yin, Chen Hua, Liu Weian and B.D. Sleeman, the Solvability of some Chemotaxis System. J.D.E. 2005;212:432-451