Operators with Corner-degenerate Symbols

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Abstract

We establish elements of a new approch to ellipticity and parametrices within operator algebras on a manifold with higher singularities, only based on some general axiomatic requirements on parameter-dependent operators in suitable scales of spaces. The idea is to model an iterative process with new generations of parameter-dependent operator theories, together with new scales of spaces that satisfy analogous requirements as the original ones, now on a corresponding higher level.
The "full" calculus is voluminous; so we content ourselves here with some typical aspects such as symbols in terms of order reducing families, classes of relevant examples, and operators near the conical exit to infinity.

Contents

Introduction

This paper is aimed at studying operators with certain degenerate operator-valued amplitude functions, motivated by the iterative calculus of pseudo-differential operators on manifolds with higher singularities. Here, in contrast to [36], [37], we develop the aspect of symbols, based on "abstract" reductions of orders which makes the approch transparent from a new point of view. To illustrate the idea, let us first consider, for example, the Laplacian on a manifold with conical singularities (say, without boundary). In this case the ellipticity does not only refer to the "standard" principal homogeneous symbol but also to the so-called conormal symbol. The latter one, contributed by the conical point, is operator-valued and singles out the weights in Sobolev spaces, where the operator has the Fredholm property. Another example of ellipticity with different principal symbolic components is the case of boundary value problems. The boundary (say, smooth), interpreted as an edge, contributes the operator-valued boundary (or edge) symbol which is responsible for the nature of boundary conditions (for instance, of Dirichlet or Neumann type in the case of the Laplacian). In general, if the configuration has polyhedral singularities of order k, we have to expect a principal symbolic hierarchy of length $k+1$, with components contributed by the various strata. In order to characterise the solvability of elliptic equations, especially, the regularity of solutions in suitable scales of spaces, it is adequate to embed the problem in a pseudo-differential calculus, and to construct a parametrix. For higher singularities this is a program of tremendous complexity. It is therefore advisable to organise general elements of the calculus by means of an axiomatic framework which contains the typical features, such as the cone- or edge-degenerate behaviour of symbols but ignores the (in general) huge tail of $k-1$ iterative steps to reach the singularity level k. The "concrete" (pseudo-differential) calculus of operators on manifolds with conical or edge singularities may be found in several papers and monographs, see, for instance, [27], [31], [30], [5]. Operators on manifolds of singularity order 2 are studied in [32], [36], [15], [6]. Theories of that kind are also possible for boundary value problems with the transmission property at the (smooth part of the) boundary, see, for instance, [26], [12], [8]. This is useful in numerous applications, for instance, to models of elasticity or crack theory, see [12], [9], [7]. Elements of operator structures on manifolds with higher singularities are developed, for instance, in [35], [1]. The nature of such theories depends very much on specific assumptions on the degeneracy of the involved symbols. There are worldwide different schools studying operators on singular manifolds, partly motivated by problems of geometry, index theory, and topology, see, for instance, Melrose [16], Melrose and Piazza [17], Nistor [22], Nazaikinskij, Savin, Sternin [18], [19], [20], and many others. We do not study here operators of "multi-Fuchs" type, often associated with "corner manifolds". Our operators are of a rather different behaviour with respect to the degeneracy of symbols. Nevertheless the various theories have intersections and common sources, see the paper of Kondratiev [13] or papers and monographs of other representatives of a corresponding Russian school, see, for instance, [24], [25]. Let us briefly recall a few basic facts on operators on manifolds with conical singularities or edges.

Let M be a manifold with conical singularity $v \in M$, i.e., $M \setminus \{v\}$ is smooth, and M is close to v modelled on a cone $X^{\Delta} := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ with base X, where X is a closed compact C^∞ manifold. We then have differential operators of order $\mu \in \mathbb{N}$ on $M \setminus \{v\}$, locally near v in the splitting of variables $(r, x) \in \mathbb{R}_+ \times X$ of the form

$$A := r^{-\mu} \sum_{j=0}^{\mu} a_j(r) \left(-r\frac{\partial}{\partial r}\right)^j \tag{0.1}$$

with coefficients $a_j \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$ (here $\mathrm{Diff}^\nu(\cdot)$ denotes the space of all differential operators of order ν on the manifold in parentheses, with smooth coefficients). Observe that when we consider a Riemannian metric on $\mathbb{R}_+ \times X := X^\wedge$ of the form $dr^2 + r^2 g_X$, where g_X is a Riemannian metric on X, then the associated Laplace-Beltrami operator is just of the form (0.1) for $\mu = 2$. For such operators we have the homogeneous principal symbol $\sigma_\psi(A) \in C^\infty(T^*(M \setminus \{v\}) \setminus 0)$, and locally near v in the variables (r, x) with covariables (ρ, ξ) the function

$$\tilde{\sigma}_\psi(A)(r, x, \rho, \xi) := r^\mu \sigma_\psi(A)(r, x, r^{-1}\rho, \xi)$$

which is smooth up to $r = 0$. If a symbol (or an operator function) contains r and ρ in the combination $r\rho$ we speak of degeneracy of Fuchs type.

It is interesting to ask the nature of an operator algebra that contains Fuchs type differential operators of the from (0.1) on X^{Δ}, together with the parametrices of elliptic elements. An analogous problem is meaningful on M. Answers may be found in [31], including the tools of the resulting so-called cone algebra. As noted above the ellipticity close to the tip $r = 0$ is connected with a second symbolic structure, namely, the conormal symbol

$$\sigma_{\mathrm{c}}(A)(w) := \sum_{j=0}^{\mu} a_j(0) w^j : H^s(X) \to H^{s-\mu}(X) \tag{0.2}$$

which is a family of operators, depending on $w \in \Gamma_{\frac{n+1}{2}-\gamma}$, $\Gamma_\beta := \{w \in \mathbb{C} : \mathrm{Re} w = \beta\}$, $n = \dim X$. Here $H^s(X)$ are the standard Sobolev spaces of smoothness $s \in \mathbb{R}$ on X. Ellipticity of A with respect to a weight $\gamma \in \mathbb{R}$ means that (0.2) is a family of isomorphisms for all $w \in \Gamma_{\frac{n+1}{2}-\gamma}$.

The ellipticity on the infinite cone X^{Δ} refers to a further principal symbolic structure, to be observed when $r \to \infty$. The behaviour in that respect is not symmetric under the substitution $r \to r^{-1}$. Also the present axiomatic approch will refer to "abstract" conical exits to infinity based on specific insight on a relationship between edge-degeneracy and such conical exits, known from the edge calculus of [28], [30] (see also [2] in a higher singular case). A differential operator on an open stretched wedge $\mathbb{R}_+ \times X \times \Omega \ni (r,x,y)$, $\Omega \subseteq \mathbb{R}^q$ open, is called edge-degenerate, if it has the form

$$A = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y)\left(-r\frac{\partial}{\partial r}\right)^j (rD_y)^\alpha, \tag{0.3}$$

$a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$. Observe that (0.3) can be written in the form $A = r^{-\mu}\mathrm{Op}_{r,y}(p)$ for an operator-valued symbol p of the form $p(r,y,\rho,\eta) = \tilde{p}(r,y,r\rho,r\eta)$ and $\tilde{p}(r,y,\tilde{\rho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$,

$$\mathrm{Op}_{r,y}(p)u(r,y) = \iint e^{i(r-r')\rho+i(y-y')\eta}p(r,y,\rho,\eta)u(r',y')dr'dy'đ\rho đ\eta.$$

Here $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^l_\lambda)$ means the space of classical parameter-dependent pseudo-differential operators on X of order μ, with parameter $\lambda \in \mathbb{R}^l$, that is, locally on X the operators are given in terms of amplitude functions $a(x,\xi,\lambda)$, where (ξ,λ) is treated as an $(n+l)$-dimensional covariable, and we have $L^{-\infty}(X;\mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(X))$ with $L^{-\infty}(X)$ being the (Fréchet) space of smoothing operators on X.
The notion of parameter-dependent operators of the form (0.1), with a parameter $\eta \in \mathbb{R}^q$ is motivated by edge-degenerate operators. Omitting now the variable y such operator families have the form

$$A(\eta) = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r)\left(-r\frac{\partial}{\partial r}\right)^j (r\eta)^\alpha. \tag{0.4}$$

This can also be written $A(\eta) = r^{-\mu}\mathrm{Op}_r(p)(\eta), p(r,\rho,\eta) := \tilde{p}(r,r\rho,r\eta)$, for a suitable $\tilde{p}(r,\tilde{\rho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$. In this form we also reach parameter-dependent pseudo-differential operators of Fuchs type. As we know from the calculus on the infinite cone a definition of adequate distribution spaces at $r = \infty$, denoted by $H^s_{\mathrm{cone}}(\mathbb{R}_+ \times X)(:= H^s_{\mathrm{cone}}(\mathbb{R} \times X)|_{\mathbb{R}_+\times X})$, can be formulated in terms of parameter-dependent isomorphisms on the base X of the cone as follows. $H^s_{\mathrm{cone}}(\mathbb{R}\times X)$, $s \in \mathbb{R}$, is defined to be the completion of $\mathcal{S}(\mathbb{R}, C^\infty(X))$ with respect to the norm

$$\left\{\int \|[r]^{-s}\left(\mathrm{Op}_r(p)u\right)(r,x)\|^2_{L^2(X)}dr\right\}^{\frac{1}{2}}$$

for a family $p(r,\rho,\eta) := \tilde{p}(r\rho,r\eta)$, $\eta \in \mathbb{R}^q \setminus \{0\}$ fixed, where $\tilde{p}(\tilde{\rho},\tilde{\eta}) \in L^s_{\mathrm{cl}}(X,\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ is parameter-dependent elliptic of order s, with parameters $(\tilde{\rho},\tilde{\eta})$, chosen in such a way that $\tilde{p}(\tilde{\rho},\tilde{\eta}) : H^t(X) \to H^{t-s}(X)$ is an isomorphism for every $(\tilde{\rho},\tilde{\eta}) \in \mathbb{R}^{1+q}$, $t \in \mathbb{R}$. Changing $\eta \neq 0$, or the family $\tilde{p}$ itself, gives rise to equivalent norms in the space $H^s_{\mathrm{cone}}(\mathbb{R} \times X)$. This will be the background of a definition of analogues of such spaces in the abstract set up (see Definition 2.20 below).

This paper is organised as follows. In Chapter 2 we introduce spaces of symbols based on families of reductions of orders in given scales of (analogues of Sobolev)

spaces.
Chapter 3 is devoted to the specific effects of an axiomatic cone calculus at the conical exit to infinity. The cone at infinity is represented by a real axis $\mathbb{R} \ni r$, and the operators take values in vector-valued analogues of Sobolev spaces in r.

As indicated above, our results are designed as a step of a larger concept of abstract edge and corner theories, organised in an iterative manner. The full calculus employs the one for $r \to \infty$ in combination with Mellin operators on $\mathbb{R}_+$ near $r = 0$. However, the continuation of the calculus in that sense needs more space than available in the present note. We believe that the structures for $r \to \infty$ in the present form are completely new, despite of the efforts with analogous intentions in the papers [1], [2]. The main difficulty was to invent convenient classes of symbols with a specific intertwining of variables and covariables together with extra parameters $\eta \neq 0$ which play the role of future edge covariables in homogeneous edge symbols of higher generation (see (0.4) as an example).

1 Elements of the cone calculus

1.1 Scales and order reducing families

Let $\mathfrak{E}$ denote the set of all families $\mathcal{E} = (E^s)_{s\in\mathbb{R}}$ of Hilbert spaces with continuous embeddings $E^{s'} \hookrightarrow E^s$, $s' \geq s$, so that $E^\infty := \bigcap_{s\in\mathbb{R}} E^s$ is dense in every E^s, $s \in \mathbb{R}$ and that there is a dual scale $\mathcal{E}^* = (E^{*s})_{s\in\mathbb{R}}$ with a non-degenerate sesquilinear pairing $(.,.)_0 : E^0 \times E^{*0} \to \mathbb{C}$, such that $(.,.)_0 : E^\infty \times E^{*\infty} \to \mathbb{C}$, extends to a non-degenerate sesquilinear pairing

$$E^s \times E^{*-s} \to \mathbb{C}$$

for every $s \in \mathbb{R}$, where $\sup_{f\in E^{*-s}\setminus\{0\}} \frac{|(u,f)_0|}{\|f\|_{E^{*-s}}}$ and $\sup_{g\in E^s\setminus\{0\}} \frac{|(g,v)_0|}{\|g\|_{E^s}}$ are equivalent norms in the spaces E^s and E^{*-s}, respectively; moreover, if $\mathcal{E} = (E^s)_{s\in\mathbb{R}}$, $\widetilde{\mathcal{E}} = (\widetilde{E}^s)_{s\in\mathbb{R}}$ are two scales in consideration and $a \in \mathcal{L}^\mu(\mathcal{E}, \widetilde{\mathcal{E}}) := \bigcap_{s\in\mathbb{R}} \mathcal{L}(E^s, \widetilde{E}^{s-\mu})$, for some $\mu \in \mathbb{R}$, then

$$\sup_{s\in[s',s'']} \|a\|_{s,s-\mu} < \infty$$

for every $s' \leq s''$; here $\|.\|_{s,\tilde{s}} := \|.\|_{\mathcal{L}(E^s,\widetilde{E}^{\tilde{s}})}$. Later on, in the case $s = \tilde{s} = 0$ we often write $\|.\| := \|.\|_{0,0}$.
Let us say that a scale $\mathcal{E} \in \mathfrak{E}$ is said to have the compact embedding property, if the embeddings $E^{s'} \hookrightarrow E^s$ are compact when $s' > s$.

Remark 1.1. Every $a \in \mathcal{L}^\mu(\mathcal{E}, \widetilde{\mathcal{E}})$ has a formal adjoint $a^* \in \mathcal{L}^\mu(\widetilde{\mathcal{E}}^*, \mathcal{E}^*)$, obtained by $(au, v)_0 = (u, a^*v)_0$ for all $u \in E^\infty, v \in \widetilde{E}^{*\infty}$.

Remark 1.2. The space $\mathcal{L}^\mu(\mathcal{E}, \widetilde{\mathcal{E}})$ is Fréchet in a natural way for every $\mu \in \mathbb{R}$.

Definition 1.3. A system $(b^\mu(\eta))_{\mu\in\mathbb{R}}$ of operator functions $b^\mu(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}^\mu(\mathcal{E},\mathcal{E}))$ is called an order reducing family of the scale $\mathcal{E}$, if $b^\mu(\eta) : E^s \to E^{s-\mu}$ is an isomorphism for every $s,\mu \in \mathbb{R}$, $\eta \in \mathbb{R}^q$, $b^0(\eta) = \mathrm{id}$ for every $\eta \in \mathbb{R}^q$, and

(i) $D^\beta_\eta b^\mu(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}^{\mu-|\beta|}(\mathcal{E},\mathcal{E}))$ for every $\beta \in \mathbb{N}^q$;

(ii) for every $s \in \mathbb{R}, \beta \in \mathbb{N}^q$ we have

$$\max_{|\beta|\le k} \sup_{\substack{\eta\in\mathbb{R}^q \\ s\in[s',s'']}} \|b^{s-\mu+|\beta|}(\eta)\{D^\beta_\eta b^\mu(\eta)\}b^{-s}(\eta)\|_{0,0} < \infty$$

for all $k \in \mathbb{N}$, and for all real $s' \le s''$.

(iii) for every $\mu,\nu \in \mathbb{R}$, $\nu \ge \mu$, we have

$$\sup_{s\in[s',s'']} \|b^\mu(\eta)\|_{s,s-\nu} \le c\langle\eta\rangle^B$$

for all $\eta \in \mathbb{R}^q$ and $s' \le s''$ with constants $c(\mu,\nu,s), B(\mu,\nu,s) > 0$, uniformly bounded in compact s-intervals and compact μ,ν-intervals for $\nu \ge \mu$; moreover, for every $\mu \le 0$ we have

$$\|b^\mu(\eta)\|_{0,0} \le c\langle\eta\rangle^\mu$$

for all $\eta \in \mathbb{R}^q$ with constants $c > 0$, uniformly bounded in compact μ-intervals, $\mu \le 0$.

Clearly the operators b^μ in (iii) for $\nu > \mu$ or $\mu < 0$, are composed with a corresponding embedding operator.
In addition we require that the operator families $(b^\mu(\eta))^{-1}$ are equivalent to $b^{-\mu}(\eta)$, according to the following notation. Another order reducing family $(b_1^\mu(\eta))_{\mu\in\mathbb{R}}, \eta \in \mathbb{R}^q$, in the scale $\mathcal{E}$ is said to be equivalent to $(b^\mu(\eta))_{\mu\in\mathbb{R}}$, if for every $s \in \mathbb{R}, \beta \in \mathbb{N}^q$, there are constants $c = c(\beta, s)$ such that

$$\|b_1^{s-\mu+|\beta|}(\eta)\{D^\beta_\eta b^\mu(\eta)\}b_1^{-s}(\eta)\|_{0,0} \le c,$$

$$\|b^{s-\mu+|\beta|}(\eta)\{D^\beta_\eta b_1^\mu(\eta)\}b^{-s}(\eta)\|_{0,0} \le c,$$

for all $\eta \in \mathbb{R}^q$, uniformly in $s \in [s', s'']$ for every $s' \le s''$.

Remark 1.4. Parameter-dependent theories of operators are common in many concrete contexts. For instance, if Ω is an (open) C^∞ manifold, there is the space $L^\mu_{\mathrm{cl}}(\Omega, \mathbb{R}^q)$ of parameter-dependent pseudo-differential operators on Ω of order $\mu \in \mathbb{R}$, with parameter $\eta \in \mathbb{R}^q$, where the local amplitude functions $a(x, \xi, \eta)$ are classical symbols in $(\xi, \eta) \in \mathbb{R}^{n+q}$, treated as covariables, $n = \dim\Omega$, while $L^{-\infty}(\Omega, \mathbb{R}^q)$ is the space of Schwartz functions in $\eta \in \mathbb{R}^q$ with values in $L^{-\infty}(\Omega)$, the space of smoothing operators on Ω. Later on we will also consider specific examples with more control on the dependence on η, namely, when $\Omega = M \setminus \{v\}$ for a manifold M with conical singularity v.

Example. Let X be a closed compact C^∞ manifold, $E^s := H^s(X), s \in \mathbb{R}$, the scale of classical Sobolev spaces on X and $b^\mu(\eta) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q_\eta)$ a parameter-dependent elliptic family that induces isomorphisms $b^\mu(\eta) : H^s(X) \to H^{s-\mu}(X)$ for all $s \in \mathbb{R}$. Then for $\nu \geq \mu$ we have

$$\|b^\mu(\eta)\|_{\mathcal{L}(H^s(X),H^{s-\nu}(X))} \leq c\langle\eta\rangle^{\pi(\mu,\nu)}$$

for all $\eta \in \mathbb{R}^q$, uniformly in $s \in [s', s'']$ for arbitrary s', s'', as well as in compact μ- and ν-intervals for $\nu \geq \mu$, where

$$\pi(\mu, \nu) := \max(\mu, \mu - \nu) \tag{1.1}$$

with a constant $c = c(\mu, \nu, s', s'') > 0$. Observe that $\sup_{\xi\in\mathbb{R}^p} \frac{\langle\xi,\eta\rangle^\mu}{\langle\xi\rangle^\nu} \leq \langle\eta\rangle^{\pi(\mu,\nu)}$ for all $\eta \in \mathbb{R}^q$.

Remark 1.5. Let $b^s(\tilde{\tau}, \tilde{\eta}) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\tau},\tilde{\eta}})$ be an order reducing family as in the above example, now with the parameter $(\tilde{\tau}, \tilde{\eta}) \in \mathbb{R}^{1+q}$ rather than η, and of order $s \in \mathbb{R}$. Then, setting $b^s(t, \tau, \eta) := b^s(t\tau, t\eta)$ the expression

$$\Big\{ \int \|[t]^{-s}\mathrm{Op}_t(b^s)(\eta^1)u\|^2_{L^2(X)}dt \Big\}^{\frac{1}{2}}$$

for $\eta^1 \in \mathbb{R}^q \setminus \{0\}$, $|\eta^1|$ sufficiently large, is a norm on the space $\mathcal{S}(\mathbb{R}, C^\infty(X))$. Let $H^s_{\mathrm{cone}}(\mathbb{R} \times X)$ denote the completion of $\mathcal{S}(\mathbb{R}, C^\infty(X))$ in this norm. Observe that this space is independent of the choice of η^1, $|\eta^1|$ sufficiently large. For reference below we also form weighted variants $H^{s;g}_{\mathrm{cone}}(\mathbb{R} \times X) := \langle t\rangle^{-g}H^s_{\mathrm{cone}}(\mathbb{R} \times X)$, $g \in \mathbb{R}$, and set

$$H^{s;g}_{\mathrm{cone}}(\mathbb{R}_+ \times X) := H^{s;g}_{\mathrm{cone}}(\mathbb{R} \times X)|_{\mathbb{R}_+\times X}. \tag{1.2}$$

As is known, cf. [12], the spaces $H^{s;g}_{\mathrm{cone}}(\mathbb{R} \times X)$ are weighted Sobolev spaces in the calculus of pseudo-differential operators on $\mathbb{R}_+ \times X$ with $|t| \to \infty$ being interpreted as a conical exit to infinity.

Another feature of order reducing families, known, for instance, in the case of the above example, is that when $U \subseteq \mathbb{R}^p$ is an open set and $m(y) \in C^\infty(U)$ a strictly positive function, $m(y) \geq c$ for $c > 0$ and for all $y \in U$, the family $b^s_1(y, \eta) := b^s(m(y)\eta)$, $s \in \mathbb{R}$, is order reducing in the sense of Definition 1.3 and equivalent to $b(\eta)$ for every $y \in U$, uniformly in $y \in K$ for any compact subset $K \subset U$. A natural requirement is that when $m > 0$ is a parameter, there is a constant $M = M(s', s'') > 0$ such that

$$\|b^s(\eta)b^{-s}(m\eta)\|_{0,0} \leq c \max(m, m^{-1})^M \tag{1.3}$$

for every $s \in [s', s'']$, $m \in \mathbb{R}_+$, and $\eta \in \mathbb{R}^q$.

We now turn to another example of an order reducing family, motivated by the calculus of pseudo-differential operators on a manifold with edge (here in "abstract" form), where all the above requirements are satisfied, including the latter one.

Definition 1.6. (i) If H is a Hilbert space and $\kappa := \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ a group of isomorphisms $\kappa_\lambda : H \to H$, such that $\lambda \to \kappa_\lambda h$ defines a continuous function $\mathbb{R}_+ \to H$ for every $h \in H$, and $\kappa_\lambda\kappa_\rho = \kappa_{\lambda\rho}$ for $\lambda, \rho \in \mathbb{R}$, we call κ a group action on H.

(ii) Let $\mathcal{H} = (H^s)_{s\in\mathbb{R}} \in \mathfrak{E}$ and assume that H^0 is endowed with a group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ that restricts (for $s > 0$) or extends (for $s < 0$) to a group action on H^s for every $s \in \mathbb{R}$. In addition, we assume that κ is a unitary group action on H^0. We then say that $\mathcal{H}$ is endowed with a group action.

If H and κ are as in Definition 1.6 (i), it is known that there are constants $c, M > 0$, such that

$$\|\kappa_\lambda\|_{\mathcal{L}(H)} \leq c \max(\lambda, \lambda^{-1})^M \tag{1.4}$$

for all $\lambda \in \mathbb{R}_+$.
Let $\mathcal{W}^s(\mathbb{R}^q, H)$ denote the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} := \Big\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H d\eta \Big\}^{\frac{1}{2}};$$

$\hat{u}(\eta) = \mathcal{F}_{y\to\eta}u(\eta)$ is the Fourier transform in $\mathbb{R}^q$. The space $\mathcal{W}^s(\mathbb{R}^q, H)$ will be referred to as edge space on $\mathbb{R}^q$ of smoothness $s \in \mathbb{R}$ (modelled on H). Given a scale $\mathcal{H} = (H^s)_{s\in\mathbb{R}} \in \mathfrak{E}$ with group action we have the edge spaces

$$W^s := \mathcal{W}^s(\mathbb{R}^q, H^s),\ s \in \mathbb{R}.$$

If necessary we also write $\mathcal{W}^s(\mathbb{R}^q, H^s)_\kappa$. The spaces form again a scale $\mathcal{W} := (W^s)_{s\in\mathbb{R}} \in \mathfrak{E}$.
For purposes below we now formulate a class of operator-valued symbols

$$S^\mu(U \times \mathbb{R}^q; H, \widetilde{H})_{\kappa,\tilde{\kappa}} \tag{1.5}$$

for open $U \subseteq \mathbb{R}^p$ and Hilbert spaces H and $\widetilde{H}$, endowed with group actions $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}, \tilde{\kappa} = \{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, as follows. The space (1.5) is defined to be the set of all $a(y, \eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ such that

$$\sup_{(y,\eta)\in K\times\mathbb{R}^q} \langle\eta\rangle^{-\mu+|\beta|} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y, \eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\tilde{H})} < \infty \tag{1.6}$$

for every $K \Subset U, \alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$.

Remark 1.7. Analogous symbols can also be defined in the case when $\widetilde{H}$ is a Fréchet space with group action, i.e., $\widetilde{H}$ is written as a projective limit of Hilbert spaces $\widetilde{H}_j, j \in \mathbb{N}$, with continuous embeddings $\widetilde{H}_j \hookrightarrow \widetilde{H}_0$, where the group action on $\widetilde{H}_0$ restricts to group actions on $\widetilde{H}_j$ for every j. Then $S^\mu(U \times \mathbb{R}^q; H, \widetilde{H}) := \varprojlim_{j\in\mathbb{N}} S^\mu(U \times \mathbb{R}^q; H, \widetilde{H}_j)$.

Consider an operator function $p(\xi,\eta) \in C^\infty(\mathbb{R}^{p+q}_{\xi,\eta}, \mathcal{L}^\mu(\mathcal{H},\mathcal{H}))$ that represents a symbol

$$p(\xi,\eta) \in S^\mu(\mathbb{R}^{p+q}_{\xi,\eta}; H^s, H^{s-\mu})_{\kappa,\kappa}$$

for every $s \in \mathbb{R}$, such that $p(\xi,\eta) : H^s \to H^{s-\mu}$ is a family of isomorphisms for all $s \in \mathbb{R}$, and the inverses $p^{-1}(\xi,\eta)$ represent a symbol

$$p^{-1}(\xi,\eta) \in S^{-\mu}(\mathbb{R}^{p+q}_{\xi,\eta}; H^s, H^{s+\mu})_{\kappa,\kappa}$$

for every $s \in \mathbb{R}$. Then $b^\mu(\eta) := \mathrm{Op}_x(p)(\eta)$ is a family of isomorphisms

$$b^\mu(\eta) : W^s \to W^{s-\mu},\ \eta \in \mathbb{R}^q,$$

with the inverses $b^{-\mu}(\eta) := \mathrm{Op}_x(p^{-1})(\eta)$.

Proposition 1.8. (i) *We have*

$$\|b^\mu(\eta)\|_{\mathcal{L}(W^0,W^0)} \leq c\langle\eta\rangle^\mu \tag{1.7}$$

for every $\mu \leq 0$, with a constant $c(\mu) > 0$.

(ii) *For every $s,\mu,\nu \in \mathbb{R}$, $\nu \geq \mu$, we have*

$$\|b^\mu(\eta)\|_{\mathcal{L}(W^s,W^{s-\nu})} \leq c\langle\eta\rangle^{\pi(\mu,\nu)+M(s)+M(s-\mu)} \tag{1.8}$$

for all $\eta \in \mathbb{R}^q$, with a constant $c(\mu,s) > 0$, and $M(s) \geq 0$ defined by

$$\|\kappa_\lambda\|_{\mathcal{L}(H^s,H^s)} \leq c\lambda^{M(s)} \text{ for all } \lambda \geq 1.$$

Proof. (i) Let us check the estimate (1.7). For the computations we denote by $j : H^{-\mu} \hookrightarrow H^0$ the embedding operator. We have for $u \in W^0$

$$\begin{aligned}
\|b^\mu(\eta)u\|^2_{W^0} &= \int \|jp(\xi,\eta)(\mathcal{F}u)(\xi)\|^2_{H^0} d\xi \\
&= \int \|\kappa^{-1}_{\langle\xi,\eta\rangle} j \kappa_{\langle\xi,\eta\rangle} \kappa^{-1}_{\langle\xi,\eta\rangle} p(\xi,\eta) \kappa_{\langle\xi,\eta\rangle} \kappa^{-1}_{\langle\xi,\eta\rangle} (\mathcal{F}u)(\xi)\|^2_{H^0} d\xi \\
&\leq \int \|\kappa^{-1}_{\langle\xi,\eta\rangle} j \kappa_{\langle\xi,\eta\rangle}\|^2_{\mathcal{L}(H^{-\mu},H^0)} \|\kappa^{-1}_{\langle\xi,\eta\rangle} p(\xi,\eta) \kappa_{\langle\xi,\eta\rangle} \kappa^{-1}_{\langle\xi,\eta\rangle} (\mathcal{F}u)(\xi)\|^2_{H^{-\mu}} d\xi \\
&\leq c \int \|\kappa^{-1}_{\langle\xi,\eta\rangle} p(\xi,\eta) \kappa_{\langle\xi,\eta\rangle}\|^2_{\mathcal{L}(H^0,H^{-\mu})} \|\kappa^{-1}_{\langle\xi,\eta\rangle} (\mathcal{F}u)(\xi)\|^2_{H^0} d\xi \\
&\leq c \sup_{\xi \in \mathbb{R}^p} \langle\xi,\eta\rangle^{2\mu} \|u\|^2_{W^0}.
\end{aligned}$$

Thus $\|b^\mu(\eta)\|_{\mathcal{L}(W^0,W^0)} \leq c \sup_{\xi\in\mathbb{R}^p} \langle\xi,\eta\rangle^\mu \leq c\langle\eta\rangle^\mu$, since $\mu \leq 0$.

(ii) Let $j : H^{s-\mu} \hookrightarrow H^{s-\nu}$ denote the canonical embedding. For every fixed $s \in \mathbb{R}$ we have

$$\begin{aligned}\|b^\mu(\eta)u\|^2_{W^{s-\nu}} &= \int \langle\xi\rangle^{2(s-\nu)} \|\kappa^{-1}_{\langle\xi\rangle} jp(\xi,\eta)(\mathcal{F}_{x\to\xi}u)(\xi)\|^2_{H^{s-\nu}} d\xi \\ &= \int \langle\xi\rangle^{2(s-\nu)} \|\kappa^{-1}_{\langle\xi\rangle} jp(\xi,\eta)\kappa_{\langle\xi\rangle}\langle\xi\rangle^{-s}\langle\xi\rangle^{s}\kappa^{-1}_{\langle\xi\rangle}(\mathcal{F}_{x\to\xi}u)(\xi)\|^2_{H^{s-\nu}} d\xi \\ &= \sup_{\xi\in\mathbb{R}^p} \langle\xi\rangle^{-2\nu}\|\kappa^{-1}_{\langle\xi\rangle} jp(\xi,\eta)\kappa_{\langle\xi\rangle}\|^2_{\mathcal{L}(H^s,H^{s-\nu})} \int \langle\xi\rangle^{2s}\|\kappa^{-1}_{\langle\xi\rangle}\mathcal{F}_{x\to\xi}u(\xi)\|^2_{H^s} d\xi\end{aligned}$$

We have

$$\begin{aligned}&\|\kappa^{-1}_{\langle\xi\rangle}\big(jp(\xi,\eta)\big)\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^{s-\nu})} \\ &\qquad \le \|\kappa^{-1}_{\langle\xi\rangle} j\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^{s-\mu},H^{s-\nu})}\|\kappa^{-1}_{\langle\xi\rangle}p(\xi,\eta)\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^{s-\mu})} \\ &\qquad\qquad \le c\|\kappa^{-1}_{\langle\xi\rangle}p(\xi,\eta)\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^{s-\mu})}\end{aligned}$$

with a constant $c > 0$.

We employed here that $\|\kappa^{-1}_{\langle\xi\rangle} j\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^{s-\mu},H^{s-\nu})} \le c$ for all $\xi \in \mathbb{R}^p$. Moreover, we have

$$\begin{aligned}&\|\kappa^{-1}_{\langle\xi\rangle}p(\xi,\eta)\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^{s-\mu})} \\ &\le \|\kappa^{-1}_{\langle\xi\rangle}\kappa_{\langle\xi,\eta\rangle}\|_{\mathcal{L}(H^{s-\mu},H^{s-\mu})}\|\kappa^{-1}_{\langle\xi,\eta\rangle}p(\xi,\eta)\kappa_{\langle\xi,\eta\rangle}\|_{\mathcal{L}(H^s,H^{s-\mu})}\|\kappa^{-1}_{\langle\xi,\eta\rangle}\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^s)} \\ &\qquad \le c\langle\xi,\eta\rangle^{\mu}\|\kappa_{\langle\xi,\eta\rangle\langle\xi\rangle^{-1}}\|_{\mathcal{L}(H^{s-\mu},H^{s-\mu})}\|\kappa_{\langle\xi,\eta\rangle^{-1}\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^s)} \\ &\qquad\qquad \le c\langle\xi,\eta\rangle^{\mu}\Big(\frac{\langle\xi,\eta\rangle}{\langle\xi\rangle}\Big)^{M(s-\mu)+M(s)}.\end{aligned}$$

As usual, $c > 0$ denotes different constants (they may also depend on s); the numbers $M(s)$, $s \in \mathbb{R}$, are determined by the estimates

$$\|\kappa_\lambda\|_{\mathcal{L}(H^s,H^s)} \le c\lambda^{M(s)} \text{ for all } \lambda \ge 1.$$

We obtain altogether that

$$\|b^\mu(\eta)\|_{\mathcal{L}(W^s,W^{s-\nu})} \le c \sup_{\xi\in\mathbb{R}^n} \frac{\langle\xi,\eta\rangle^{\mu}}{\langle\xi\rangle^{\nu}}\Big(\frac{\langle\xi,\eta\rangle}{\langle\xi\rangle}\Big)^{M(s-\mu)+M(s)} \le c\langle\eta\rangle^{\pi(\mu,\nu)+M(s-\mu)+M(s)}.$$

□

It can be proved that the operators in Proposition 1.8 also have the uniformity properties with respect to s, μ, ν in compact sets, imposed in Definition 1.3.

1.2 Symbols based on order reductions

We now turn to operator valued symbols, referring to scales

$$\mathcal{E} = (E^s)_{s\in\mathbb{R}}, \widetilde{\mathcal{E}} = (\widetilde{E}^s)_{s\in\mathbb{R}} \in \mathfrak{E}.$$

For purposes below we slightly generalise the concept of order reducing families by replacing the parameter space $\mathbb{R}^q \ni \eta$ by $\mathbb{H} \ni \eta$, where

$$\mathbb{H} := \{\eta = (\eta', \eta'') \in \mathbb{R}^{q'+q''} : q = q' + q'', \eta'' \neq 0\}. \tag{1.9}$$

In other words for every $\mu \in \mathbb{R}$ we fix order-reducing families $b^\mu(\eta)$ and $\tilde{b}^\mu(\eta)$ in the scales $\mathcal{E}$ and $\widetilde{\mathcal{E}}$, respectively, where η varies over $\mathbb{H}$, and the properties of Definition 1.3 are required for all $\eta \in \mathbb{H}$. In many cases we may admit the case $\mathbb{H} = \mathbb{R}^q$ as well.

Definition 1.9. By $S^\mu(U \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})$ for open $U \subseteq \mathbb{R}^p, \mu \in \mathbb{R}$, we denote the set of all $a(y,\eta) \in C^\infty(U \times \mathbb{H}, \mathcal{L}^\mu(\mathcal{E}, \widetilde{\mathcal{E}}))$ such that

$$D_y^\alpha D_\eta^\beta a(y,\eta) \in C^\infty(U \times \mathbb{H}, \mathcal{L}^{\mu-|\beta|}(\mathcal{E}, \widetilde{\mathcal{E}})), \tag{1.10}$$

and for every $s \in \mathbb{R}$ we have

$$\max_{|\alpha|+|\beta|\le k} \sup_{\substack{y\in K, \eta\in\mathbb{H}, \eta\ge h \\ s\in[s',s'']}} \|\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_y^\alpha D_\eta^\beta a(y,\eta)\} b^{-s}(\eta)\|_{0,0} \tag{1.11}$$

is finite for all $K \Subset U$, $k \in \mathbb{N}, h > 0$.

Let $S^\mu(\mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})$ denote the subspace of all elements of $S^\mu(U \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})$ that are independent of y.
Observe that when $(b^\mu(\eta))_{\mu\in\mathbb{R}}$ is an order reducing family parametrised by $\eta \in \mathbb{H}$ then we have

$$b^\mu(\eta) \in S^\mu(\mathbb{H}; \mathcal{E}, \mathcal{E}) \tag{1.12}$$

for every $\mu \in \mathbb{R}$.

Remark 1.10. The space $S^\mu(U \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})$ is Fréchet with the semi-norms

$$a \to \max_{|\alpha|+|\beta|\le k} \sup_{\substack{(y,\eta)\in K\times\mathbb{H}, |\eta|\ge h \\ s\in[s',s'']}} \|\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_y^\alpha D_\eta^\beta a(y,\eta)\} b^{-s}(\eta)\|_{0,0} \tag{1.13}$$

parametrised by $K \Subset U$, $s \in \mathbb{Z}$, $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, $h > 0$, which are the best constants in the estimates (1.11). We then have

$$S^\mu(U \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}}) = C^\infty(U, S^\mu(\mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})) = C^\infty(U)\hat{\otimes}_\pi S^\mu(\mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}}).$$

We will also employ other variants of such symbols, for instance, when $\Omega \subseteq \mathbb{R}^m$ is an open set,

$$S^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}}) := C^\infty(\overline{\mathbb{R}}_+ \times \Omega, S^\mu(\mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})).$$

In order to emphasise the similarity of our considerations for $\mathbb{H}$ with the case $\mathbb{H} = \mathbb{R}^q$ we often write again $\mathbb{R}^q$ and later on tacitly use the corresponding results for $\mathbb{H}$ in general.

Remark 1.11. Let $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q)$ be a polynomial in η of order μ and $\mathcal{E} = (E^s)_{s\in\mathbb{R}}$ a scale and identify $D_y^\alpha D_\eta^\beta a(y,\eta)$ with $\big(D_y^\alpha D_\eta^\beta a(y,\eta)\big)\iota$ with the embedding $\iota : E^s \to E^{s-\mu+|\beta|}$. Then we have

$$\begin{aligned}&\|b^{s-\mu+|\beta|}(\eta)\big(D_y^\alpha D_\eta^\beta a(y,\eta)\big)b^{-s}(\eta)\|_{0,0} \\ &\qquad \le |D_y^\alpha D_\eta^\beta a(y,\eta)|\|b^{-\mu+|\beta|}(\eta)\|_{0,0} \le c\langle\eta\rangle^{\mu-|\beta|}\langle\eta\rangle^{-\mu+|\beta|} = c\end{aligned}$$

for all $\beta \in \mathbb{N}^q$, $|\beta| \le \mu$, $y \in K \Subset U$ (see Definition 1.3 (iii)). Thus $a(y,\eta)$ is canonically identified with an element of $S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \mathcal{E})$.

Proposition 1.12. *We have*

$$S^{-\infty}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}}) := \bigcap_{\mu\in\mathbb{R}} S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}}) = C^\infty(U, \mathcal{S}(\mathbb{R}^q, \mathcal{L}^{-\infty}(\mathcal{E}, \widetilde{\mathcal{E}}))).$$

Proof. Let us show the assertion for y-independent symbols; the y-dependent case is then straightforward. For notational convenience we set $\widetilde{\mathcal{E}} = \mathcal{E}$; the general case is analogous. First let $a(\eta) \in S^{-\infty}(\mathbb{R}^q; \mathcal{E}, \mathcal{E})$, which means that $a(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}^{-\infty}(\mathcal{E}, \mathcal{E}))$ and

$$\|b^{s+N}(\eta)\{D_\eta^\beta a(\eta)\}b^{-s}(\eta)\|_{0,0} < c \tag{1.14}$$

for all $s \in \mathbb{R}$, $N \in \mathbb{N}$, $\beta \in \mathbb{N}^q$ and show that

$$\sup_{\eta\in\mathbb{R}^q} \|\langle\eta\rangle^M D_\eta^\beta a(\eta)\|_{s,t} < \infty \tag{1.15}$$

for every $s, t \in \mathbb{R}$, $M \in \mathbb{N}$, $\beta \in \mathbb{N}^q$. To estimate (1.15) it is enough to assume $t > 0$. We have

$$\|\langle\eta\rangle^M D_\eta^\beta a(\eta)\|_{s,t} = \|b^{-kt}(\eta)b^{kt}(\eta)\langle\eta\rangle^M D_\eta^\beta a(\eta)b^{-s}(\eta)b^s(\eta)\|_{s,t} \tag{1.16}$$

for every $k \in \mathbb{N}, k \ge 1$, it is sufficient to show that the right hand side is uniformly bounded in $\eta \in \mathbb{R}^q$ for sufficiently large choice of k. The right hand side of (1.16) can be estimated by

$$\|b^{-t}(\eta)\|_{0,t}\|b^{(1-k)t}(\eta)\|_{0,0}\|b^{kt}(\eta)D_\eta^\beta a(\eta)b^{-s}(\eta)\|_{0,0}\|b^s(\eta)\|_{s,0}.$$

Using $\|b^{kt}(\eta)D_\eta^\beta a(\eta)b^{-s}(\eta)\|_{0,0} \leq c$, which is true by assumption and the estimates

$$\|b^s(\eta)\|_{s,0} \leq c\langle\eta\rangle^B, \quad \|b^{-t}(\eta)\|_{0,t} \leq c\langle\eta\rangle^{B'},$$

with different $B, B' \in \mathbb{R}$ and $\|b^{(1-k)t}(\eta)\|_{0,0} \leq c\langle\eta\rangle^{(1-k)t}$ (see Definition 1.3 (iii)) we obtain altogether

$$\|\langle\eta\rangle^M D_\eta^\beta a(\eta)\|_{s,t} \leq c\langle\eta\rangle^{M+B+B'+(1-k)t}$$

for some $c > 0$. Choosing k large enough it follows that the exponent on the right hand side is < 0, i.e., we obtain uniform boundedness in $\eta \in \mathbb{R}^q$.
To show the reverse direction suppose that $a(\eta)$ satisfies (1.15), and let $\beta \in \mathbb{N}^q$, $M, s, t \in \mathbb{R}$ be arbitrary. We have

$$\|b^t(\eta)D_\eta^\beta a(\eta)b^{-s}(\eta)\|_{0,0} \leq \\ \|b^t(\eta)\langle\eta\rangle^{-M}\|_{t,0}\|\langle\eta\rangle^{2M}D_\eta^\beta a(\eta)\|_{s,t}\|\langle\eta\rangle^{-M}b^{-s}(\eta)\|_{0,s}. \quad (1.17)$$

Now using (1.15) and the estimates

$$\|b^t(\eta)\langle\eta\rangle^{-M}\|_{t,0} \leq c\langle\eta\rangle^{A-M}, \quad \|\langle\eta\rangle^{-M}b^{-s}(\eta)\|_{0,s} \leq c\langle\eta\rangle^{A'-M},$$

with constants $A, A' \in \mathbb{R}$, we obtain

$$\|b^t(\eta)D_\eta^\beta a(\eta)b^{-s}(\eta)\|_{0,0} \leq c\langle\eta\rangle^{A+A'-2M}.$$

Choosing M large enough we get uniform boundedness of (1.17) in $\eta \in \mathbb{R}^q$ which completes the proof. □

Proposition 1.13. *Let $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ and $\mu \leq 0$. Then we have*

$$\|a(y,\eta)\|_{0,0} \leq c\langle\eta\rangle^\mu$$

for all $y \in K \Subset U, \eta \in \mathbb{R}^q$, with a constant $c = c(s, K) > 0$.

Proof. For simplicity we consider the y-independent case. It is enough to show that $\|a(\eta)u\|_{\widetilde{E}^0} \leq c\langle\eta\rangle^\mu\|u\|_{E^0}$ for all $u \in E^\infty$. Let $j : E^{-\mu} \to E^0$ denote the embedding operator. We then have

$$\begin{aligned}\|a(\eta)u\|_{\widetilde{E}^0} =& \|a(\eta)b^{-\mu}(\eta)jb^\mu(\eta)u\|_{\widetilde{E}^0} \\ \leq& \|a(\eta)b^{-\mu}(\eta)\|_{\mathcal{L}(E^0,\widetilde{E}^0)}\|jb^\mu(\eta)u\|_{E^0} \leq c\langle\eta\rangle^\mu\|u\|_{E^0}.\end{aligned}$$

□

Proposition 1.14. *A symbol $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$, $\mu \in \mathbb{R}$, satisfies the estimates*

$$\|a(y,\eta)\|_{s,s-\nu} \leq c\langle\eta\rangle^A \quad (1.18)$$

for every $\nu \geq \mu$, for every $y \in K \Subset U, \eta \in \mathbb{R}^q$, $s \in \mathbb{R}$, with constants $c = c(s,\mu,\nu) > 0$, $A = A(s,\mu,\nu,K) > 0$ that are uniformly bounded when s, μ, ν vary over compact sets, $\nu \geq \mu$.

Proof. For simplicity we consider again the y-independent case. Let $j : \widetilde{E}^{s-\mu} \hookrightarrow \widetilde{E}^{s-\nu}$ be the embedding operator. Then we have

$$\|a(\eta)\|_{s,s-\nu} = \|j\tilde{b}^{-s+\mu}(\eta)\tilde{b}^{s-\mu}(\eta)a(\eta)b^{-s}(\eta)b^{s}(\eta)\|_{s,s-\nu}$$
$$\leq \|j\tilde{b}^{-s+\mu}(\eta)\|_{0,s-\nu}\|\tilde{b}^{s-\mu}(\eta)a(\eta)b^{-s}(\eta)\|_{0,0}\|b^{s}(\eta)\|_{s,0}.$$

Applying (1.11) and Definition 1.3 (iii) we obtain (1.18) with $A = B(-s+\mu, -s+\nu, 0) + B(s, s, 0)$, together with the uniform boundedness of the involved constants. □

Also here it can be proved that the involved constants in Propositions 1.13, 1.14 are uniform in compact sets with respect to s, μ, ν.

Proposition 1.15. *The symbol spaces have the following properties:*

(i) $S^{\mu}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}}) \subseteq S^{\mu'}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ *for every* $\mu' \geq \mu$;

(ii) $D_y^{\alpha} D_{\eta}^{\beta} S^{\mu}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}}) \subseteq S^{\mu-|\beta|}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ *for every* $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$;

(iii) $S^{\mu}(U \times \mathbb{R}^q; \mathcal{E}_0, \widetilde{\mathcal{E}}) S^{\nu}(U \times \mathbb{R}^q; \mathcal{E}, \mathcal{E}_0) \subseteq S^{\mu+\nu}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ *for every* $\mu, \nu \in \mathbb{R}$ (*the notation on the left hand side of the latter relation means the space of all* (y, η)*-wise compositions of elements in the respective factors*).

Proof. For simplicity we consider symbols with constant coefficients. Let us write $\|\cdot\| := \|\cdot\|_{0,0}$, etc.

(i) $a(\eta) \in S^{\mu}(\mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ means (1.10) and (1.11); this implies

$$\|\tilde{b}^{s-\mu'+|\beta|}(\eta)\{D_{\eta}^{\beta}a(\eta)\}b^{-s}(\eta)\| = \|\tilde{b}^{\mu-\mu'}(\eta)\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_{\eta}^{\beta}a(\eta)\}b^{-s}(\eta)\|$$
$$\leq c\langle\eta\rangle^{\mu-\mu'}\|\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_{\eta}^{\beta}a(\eta)\}b^{-s}(\eta)\| \leq c\|\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_{\eta}^{\beta}a(\eta)\}b^{-s}(\eta)\|.$$

We employed $\mu - \mu' \leq 0$ and the property (iv) in Definition 1.3.

(ii) The estimates (1.10) can be written as

$$\|\tilde{b}^{s-(\mu-|\beta|)}(\eta)\{D_{\eta}^{\beta}a(\eta)\}b^{-s}(\eta)\| \leq c$$

which just means that $D_{\eta}^{\beta}a(\eta) \in S^{\mu-|\beta|}(\mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$.

(iii) Given $a(\eta) \in S^{\mu}(\mathbb{R}^q; \mathcal{E}_0, \widetilde{\mathcal{E}}), \tilde{a}(\eta) \in S^{\nu}(\mathbb{R}^q; \mathcal{E}, \mathcal{E}_0)$ we have (with obvious meaning of notation)

$$\|\tilde{b}_0^{s-\nu+|\gamma|}(\eta)\{D_{\eta}^{\gamma}\tilde{a}(\eta)\}b^{-s}(\eta)\| \leq c, \ \|\tilde{b}^{s-\mu+|\delta|}(\eta)\{D_{\eta}^{\delta}a(\eta)\}b_0^{-s}(\eta)\| \leq c$$

for all $\gamma, \delta \in \mathbb{N}^q$. If $\alpha \in \mathbb{N}^q$ is any multi-index, $D_\eta^\alpha(a\tilde{a})(\eta)$ is a linear combination of compositions $D_\eta^\delta a(\eta) D_\eta^\gamma \tilde{a}(\eta)$ with $|\gamma| + |\delta| = |\alpha|$. It follows that

$$\begin{aligned}
&\|\tilde{b}^{s-(\mu+\nu)+|\alpha|}(\eta) D_\eta^\delta a(\eta)\{D_\eta^\gamma \tilde{a}(\eta)\} b^{-s}(\eta)\| \\
&\quad = \|\tilde{b}^{s-(\mu+\nu)+|\alpha|}(\eta) D_\eta^\delta a(\eta) b_0^{-s+\nu-|\gamma|}(\eta) b_0^{s-\nu+|\gamma|}(\eta) D_\eta^\gamma \tilde{a}(\eta) b^{-s}(\eta)\| \\
&\quad \leq \|\tilde{b}^{t-\mu+|\alpha|-|\gamma|}(\eta) D_\eta^\delta a(\eta) b_0^{-t}(\eta)\| \, \|b_0^{s-\nu+|\gamma|}(\eta) D_\eta^\gamma \tilde{a}(\eta) b^{-s}(\eta)\| \qquad (1.19)
\end{aligned}$$

for $t = s - \nu + |\gamma|$; the right hand side is bounded in η, since $|\alpha| - |\gamma| = |\delta|$. □

Remark 1.16. Observe from (1.19) that the semi-norms of compositions of symbols can be estimated by products of semi-norms of the factors.

1.3 An example from the parameter-dependent cone calculus

We now construct a specific family of reductions of orders between weighted spaces on a compact manifold M with conical singularity v, locally near v modelled on a cone

$$X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$$

with a smooth compact manifold X as base. The parameter η will play the role of covariables of the calculus of operators on a manifold with edge; that is why we talk about an example from the edge calculus. The associated "abstract" cone calculus according to what we did so far in the Sections 1.1 and 1.2 and then below in Chapter 3 will be a contribution to the calculus of corner operators of second generation. It will be convenient to pass to the stretched manifold $\mathbb{M}$ associated with M which is a compact C^∞ manifold with boundary $\partial\mathbb{M} \cong X$ such that when we squeeze down $\partial\mathbb{M}$ to a single point v we just recover M. Close to $\partial\mathbb{M}$ the manifold $\mathbb{M}$ is equal to a cylinder $[0,1) \times X \ni (t,x)$, a collar neighbourhood of $\partial\mathbb{M}$ in M. A part of the considerations will be performed on the open stretched cone $X^\wedge := \mathbb{R}_+ \times X \ni (t,x)$ where we identify $(0,1) \times X$ with the interior of the collar neighbourhood (for convenience, without indicating any pull backs of functions or operators with respect to that identification). Let $\widetilde{M} := 2\mathbb{M}$ be the double of $\mathbb{M}$ (obtained by gluing together two copies $\mathbb{M}_\pm$ of $\mathbb{M}$ along the common boundary $\partial\mathbb{M}$, where we identify $\mathbb{M}$ with $\mathbb{M}_+$); then $\widetilde{M}$ is a closed compact C^∞ manifold. On the space M we have a family of weighted Sobolev spaces $H^{s,\gamma}(M), s, \gamma \in \mathbb{R}$, that may be defined as

$$H^{s,\gamma}(M) := \{\sigma u + (1-\sigma)v : u \in \mathcal{H}^{s,\gamma}(X^\wedge), v \in H^s_{\text{loc}}(M \setminus \{v\})\},$$

where $\sigma(t)$ is a cut-off function (i.e., $\sigma \in C_0^\infty(\overline{\mathbb{R}}_+), \sigma \equiv 1$ near $t = 0$), $\sigma(t) = 0$ for $t > 2/3$. Here $\mathcal{H}^{s,\gamma}(X^\wedge)$ is defined to be the completion of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \| b^{\mu}_{\mathrm{base}}(\operatorname{Im} w)(\mathcal{M}u)(w)\|^2_{L^2(X)} dw \right\}^{\frac{1}{2}}, \tag{1.20}$$

$n = \dim X$, where $b^{\mu}_{\mathrm{base}}(\tau) \in L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}_{\tau})$ is a family of reductions of order on X, similarly as in the example in Section 1.1 (in particular, $b^{s}_{\mathrm{base}}(\tau) : H^s(X) \to H^0(X) = L^2(X)$ is a family of isomorphisms). Moreover, $\mathcal{M}$ is the Mellin transform, $(\mathcal{M}u)(w) = \int_0^\infty t^{w-1}u(t)dt$, $w \in \mathbb{C}$ the complex Mellin covariable, and

$$\Gamma_\beta := \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}$$

for any real β. From $t^\delta \mathcal{H}^{s,\gamma}(X^\wedge) = \mathcal{H}^{s,\gamma+\delta}(X^\wedge)$ for all $s, \gamma, \delta \in \mathbb{R}$ it follows the existance of a strictly positive function $\mathrm{h}^\delta \in C^\infty(M \setminus \{v\})$, such that the operator of multiplication by h^δ induces an isomorphism

$$\mathrm{h}^\delta : H^{s,\gamma}(M) \to H^{s,\gamma+\delta}(M) \tag{1.21}$$

for every $s, \gamma, \delta \in \mathbb{R}$.
Moreover, again according to the same example, now for any smooth compact manifold $\widetilde{M}$ we have an order reducing family $\tilde{b}(\eta)$ in the scale of Sobolev spaces $H^s(\widetilde{M}), s \in \mathbb{R}$. More generally, we employ parameter-dependent families $\tilde{a}(\eta) \in L^{\mu}_{\mathrm{cl}}(\widetilde{M}; \mathbb{R}^q)$. The symbols $a(\eta)$ that we want to establish in the scale $H^{s,\gamma}(M)$ on our compact manifold M with conical singularity v will be essentially (i.e., modulo Schwartz functions in η with values in globally smoothing operators on M) constructed in the form

$$a(\eta) := \sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} + (1-\sigma)a_{\mathrm{int}}(\eta)(1-\tilde{\tilde{\sigma}}), \tag{1.22}$$

$a_{\mathrm{int}}(\eta) := \tilde{a}(\eta)|_{\mathrm{int}\mathbb{M}}$, with cut-off functions $\sigma(t), \tilde{\sigma}(t), \tilde{\tilde{\sigma}}(t)$ on the half axis, supported in $[0, 2/3)$, with the property

$$\tilde{\tilde{\sigma}} \prec \sigma \prec \tilde{\sigma}$$

(here $\sigma \prec \tilde{\sigma}$ means the $\tilde{\sigma}$ is equal to 1 in a neighbourhood of supp σ).
The "edge" part of (1.22) will be defined in the variables $(t, x) \in X^\wedge$. Let us choose a parameter-dependent elliptic family of operators of order μ on X

$$\tilde{p}(t, \tilde{\tau}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\tau},\tilde{\eta}})).$$

Setting

$$p(t, \tau, \eta) := \tilde{p}(t, t\tau, t\eta) \tag{1.23}$$

we have what is known as an edge-degenerate family of operators on X. We now employ the following Mellin quantisation theorem.

Definition 1.17. Let $M_{\mathcal{O}}^{\mu}(X;\mathbb{R}^q)$ defined as the set of all $h(z,\eta) \in \mathcal{A}(\mathbb{C}, L_{\mathrm{cl}}^{\mu}(X;\mathbb{R}^q))$ such that $h(\beta+i\tau,\eta) \in L_{\mathrm{cl}}^{\mu}(X;\mathbb{R}_{\tau,\eta}^{1+q})$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals (here $\mathcal{A}(\mathbb{C}, E)$ with any Fréchet space E denotes the space of all E-valued holomorphic functions in $\mathbb{C}$, in the Fréchet topology of uniform convergence on compact sets).

Observe that also $M_{\mathcal{O}}^{\mu}(X;\mathbb{R}^q)$ is a Fréchet space in a natural way. Given an $f(t,t',z,\eta) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L_{\mathrm{cl}}^{\mu}(X;\Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^q))$ we set

$$\mathrm{op}_M^{\gamma}(f)(\eta)u(r) := \int_{\mathbb{R}}\int_0^\infty (\frac{t}{t'})^{-(\frac{1}{2}-\gamma+i\tau)} f(t,t',\frac{1}{2}-\gamma+i\tau,\eta)u(t')\frac{dt'}{t'} đ\tau$$

which is regarded as a (parameter-dependent) weighted pseudo-differential operator with symbol f, referring to the weight $\gamma \in \mathbb{R}$. There exists an element

$$\tilde{h}(t,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\tilde{\eta}}^q)) \tag{1.24}$$

such that, when we set

$$h(t,z,\eta) := \tilde{h}(t,z,t\eta) \tag{1.25}$$

we have

$$\mathrm{op}_M^{\gamma}(h)(\eta) = \mathrm{Op}_t(p)(\eta) \tag{1.26}$$

mod $L^{-\infty}(X^\wedge;\mathbb{R}_\eta^q)$, for every weight $\gamma \in \mathbb{R}$. Observe that when we set

$$p_0(t,\tau,\eta) := \tilde{p}(0,t\tau,t\eta), \;\; h_0(t,z,\eta) := \tilde{h}(0,z,t\eta)$$

we also have $\mathrm{op}_M^{\gamma}(h_0)(\eta) = \mathrm{Op}_t(p_0)(\eta) \mod L^{-\infty}(X^\wedge;\overline{\mathbb{R}}_+)$, for all $\gamma \in \mathbb{R}$.
Let us now choose cut-off functions $\omega(t), \tilde{\omega}(t), \tilde{\tilde{\omega}}(t)$ such that $\tilde{\tilde{\omega}} \prec \omega \prec \tilde{\omega}$.
Fix the notation $\omega_\eta(t) := \omega(t[\eta])$, and form the operator function

$$\begin{aligned} a_{\mathrm{edge}}(\eta) := \omega_\eta(t)t^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}_\eta(t) \\ + t^{-\mu}\big(1-\omega_\eta(t)\big)\mathrm{Op}_t(p)(\eta)\big(1-\tilde{\tilde{\omega}}_\eta(t)\big) + m(\eta) + g(\eta). \end{aligned} \tag{1.27}$$

Here $m(\eta)$ and $g(\eta)$ are smoothing Mellin and Green symbols of the edge calculus. The definition of $m(\eta)$ is based on smoothing Mellin symbols $f(z) \in M^{-\infty}(X;\Gamma_\beta)$. Here $M^{-\infty}(X;\Gamma_\beta)$ is the subspace of all $f(z) \in L^{-\infty}(X;\Gamma_\beta)$ such that for some $\varepsilon > 0$ (depending on f) the function f extends to an

$$l(z) \in \mathcal{A}(U_{\beta,\varepsilon}, L^{-\infty}(X))$$

where $U_{\beta,\varepsilon} := \{z \in \mathbb{C} : |\mathrm{Re} z - \beta| < \varepsilon\}$ and

$$l(\delta + i\tau) \in L^{-\infty}(X;\mathbb{R}_\tau)$$

for every $\delta \in (\beta-\varepsilon, \beta+\varepsilon)$, uniformly in compact subintervals. By definition we then have $f(\beta+i\tau) = l(\beta+i\tau)$; for brevity we often denote the holomorphic extension l of f again by f. For $f \in M^{-\infty}(X; \Gamma_{\frac{n+1}{2}-\gamma})$ we set

$$m(\eta) := t^{-\mu}\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega}_\eta$$

for any cut-off functions $\omega, \tilde{\omega}$.
In order to explain the structure of $g(\eta)$ in (1.27) we first introduce weighted spaces on the infinite stretched cone $X^\wedge = \mathbb{R}_+ \times X$, namely,

$$\mathcal{K}^{s,\gamma;g}(X^\wedge) := \omega\mathcal{H}^{s,\gamma}(X^\wedge) + (1-\omega)H^{s;g}_{\mathrm{cone}}(X^\wedge) \tag{1.28}$$

for any $s, \gamma, g \in \mathbb{R}$, and a cut-off function ω, see (1.20) which defines $\mathcal{H}^{s,\gamma}(X^\wedge)$ and the formula (1.2). Moreover, we set $\mathcal{K}^{s,\gamma}(X^\wedge) := \mathcal{K}^{s,\gamma;0}(X^\wedge)$. The operator families $g(\eta)$ are so-called Green symbols in the covariable $\eta \in \mathbb{R}^q$, defined by

$$g(\eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{S}^{\gamma-\mu+\varepsilon}(X^\wedge)), \tag{1.29}$$

$$g^*(\eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta; \mathcal{K}^{s,-\gamma+\mu;g}(X^\wedge), \mathcal{S}^{-\gamma+\varepsilon}(X^\wedge)), \tag{1.30}$$

for all $s, \gamma, g \in \mathbb{R}$, where g^* denotes the $\eta-$wise formal adjoint with respect to the scalar product of $\mathcal{K}^{0,0;0}(X^\wedge) = r^{-\frac{n}{2}}L^2(\mathbb{R}_+ \times X)$ and $\varepsilon = \varepsilon(g) > 0$. Here

$$\mathcal{S}^\beta(X^\wedge) := \omega\mathcal{K}^{\infty,\beta}(X^\wedge) + (1-\omega)\mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))$$

for any cut-off function ω. The notion of operator-valued symbols in (1.29), (1.30) refers to (1.5) in its generalisation to Fréchet spaces $\widetilde{H}$ (rather than Hilbert spaces) with group actions (see Remark 1.7) that is in the present case given by

$$\kappa_\lambda : u(t,x) \to \lambda^{\frac{n+1}{2}+g}u(\lambda t, x),\ \lambda \in \mathbb{R}_+ \tag{1.31}$$

$n = \dim X$, both in the spaces $\mathcal{K}^{s;\gamma,g}(X^\wedge)$ and $\mathcal{S}^{\gamma-\mu+\varepsilon}(X^\wedge)$.
The following Theorem 1.18 is crucial for proving that our new order reduction family is well defined. Therefore we will sketch the main steps of the proof, which is based on the edge calculus. Various aspects of the proof can be found in the literature, for example in Kapanaze and Schulze [11, Proposition 3.3.79], Schrohe and Schulze [29], Harutyunyan and Schulze [8]. Among the tools we have the pseudo-differential operators on $X^\wedge$ interpreted as a manifold with conical exit to infinity $r \to \infty$; the general background may be found in Schulze [34]. The calculus of such exit operators goes back to Parenti [23], Cordes [3], Shubin [40], and others.

Theorem 1.18. *We have*

$$\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} \in S^\mu(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu;g}(X^\wedge)) \tag{1.32}$$

for every $s, g \in \mathbb{R}$, *more precisely,*

$$D_\eta^\beta \{\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma}\} \in S^{\mu-|\beta|}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{s-\mu+|\beta|,\gamma-\mu;g}(X^\wedge)) \tag{1.33}$$

for all $s, g \in \mathbb{R}$ *and all* $\beta \in \mathbb{N}^q$. (*The spaces of symbols in* (1.32), (1.33) *refer to the group action* (1.31)).

Proof. To prove the assertions it is enough to consider the case without $m(\eta)+g(\eta)$, since the latter sum maps to $\mathcal{K}^{\infty,\gamma;g}(X^\wedge)$ anyway. The first part of the Theorem is known, see, for instance, [8] or [4]. Concerning the relation (1.33) we write

$$\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} = \sigma\{a_{\mathrm{c}}(\eta) + a_\psi(\eta)\}\tilde{\sigma} \tag{1.34}$$

with

$$a_{\mathrm{c}}(\eta) := t^{-\mu}\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}_\eta,$$
$$a_\psi(\eta) := t^{-\mu}(1-\omega_\eta)\mathrm{Op}_t(p)(\eta)(1-\tilde{\tilde{\omega}}_\eta)$$

and it suffices to take the summands separately. In order to show (1.33) we consider, for instance, the derivative $\partial/\partial\eta_j =: \partial_j$ for some $1 \le j \le q$. By iterating the process we then obtain the assertion. We have

$$\partial_j \sigma\{a_{\mathrm{c}}(\eta) + a_\psi(\eta)\}\tilde{\sigma} = \sigma\{\partial_j a_{\mathrm{c}}(\eta) + \partial_j a_\psi(\eta)\}\tilde{\sigma} = b_1(\eta) + b_2(\eta) + b_3(\eta)$$

with

$$b_1(\eta) := \sigma t^{-\mu}\Big\{\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\partial_j\tilde{\omega}_\eta + (1-\omega_\eta)\mathrm{Op}_t(p)(\eta)\partial_j(1-\tilde{\tilde{\omega}}_\eta)\Big\}\tilde{\sigma},$$
$$b_2(\eta) := \sigma t^{-\mu}\Big\{\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(\partial_j h)(\eta)\tilde{\omega}_\eta + (1-\omega_\eta)\mathrm{Op}_t(\partial_j p)(\eta)(1-\tilde{\tilde{\omega}}_\eta)\Big\}\tilde{\tilde{\sigma}},$$
$$b_3(\eta) := \sigma t^{-\mu}\Big\{(\partial_j\omega_\eta)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}_\eta + (\partial_j(1-\omega_\eta))\mathrm{Op}_t(p)(\eta)(1-\tilde{\tilde{\omega}}_\eta)\Big\}\tilde{\sigma}.$$

In $b_1(\eta)$ we can apply a pseudo-locality argument which is possible since $\partial_j\tilde{\omega}_\eta \equiv 0$ on $\mathrm{supp}\,\omega_\eta$ and $\partial_j(1-\tilde{\tilde{\omega}}_\eta) \equiv 0$ on $\mathrm{supp}\,(1-\omega_\eta)$; this yields (together with similar considerations as for the proof of (1.32))

$$b_1(\eta) \in S^{\mu-1}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu;g}(X^\wedge)).$$

Moreover we obtain

$$b_2(\eta) \in S^{\mu-1}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{s-\mu+1,\gamma-\mu;g}(X^\wedge))$$

since $\partial_j h$ and $\partial_j p$ are of order $\mu - 1$ (again combined with arguments for (1.32)). Concerning $b_3(\eta)$ we use the fact that there is a $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi \equiv 1$ on $\mathrm{supp}\,\partial_j\omega$, $\tilde{\omega} - \psi \equiv 0$ on $\mathrm{supp}\,\partial_j\omega$ and $(1-\tilde{\tilde{\omega}}) - \psi \equiv 0$ on $\mathrm{supp}\,\partial_j\omega$. Thus, when we set $\psi_\eta(t) := \psi(t[\eta])$, we obtain $b_3(\eta) := c_3(\eta) + c_4(\eta)$ with

$$c_3(\eta) := \sigma t^{-\mu}\Big\{(\partial_j\omega_\eta)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\psi_\eta - (\partial_j\omega_\eta)\mathrm{Op}_t(p)(\eta)\psi_\eta\Big\}\tilde{\sigma},$$
$$c_4(\eta) := \sigma t^{-\mu}\Big\{(\partial_j\omega_\eta)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)[\tilde{\omega}_\eta - \psi_\eta] - (\partial_j\omega_\eta)\mathrm{Op}_t(p)(\eta)[(1-\tilde{\tilde{\omega}}_\eta) - \psi_\eta]\Big\}\tilde{\sigma}.$$

Here, using $\partial_j \omega_\eta = (\omega')_\eta \partial_j(t[\eta])$ which yields an extra power of t on the left of the operator, together with pseudo-locality, we obtain

$$c_4(\eta) \in S^{\mu-1}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu;g}(X^\wedge)).$$

To treat $c_3(\eta)$ we employ that both $\partial_j \omega_\eta$ and ψ_η are compactly supported on $\mathbb{R}_+$. Using the property (1.26), we have

$$c_3(\eta) = \sigma t^{-\mu}(\partial_j \omega_\eta)\big\{\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) - \mathrm{Op}_t(p)(\eta)\big\}\psi_\eta \tilde{\sigma} \\ \in S^{\mu-1}(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu;g}(X^\wedge)).$$

□

Definition 1.19. A family of operators $c(\eta) \in \mathcal{S}(\mathbb{R}^q, \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^{s,\gamma}(M), H^{\infty,\delta}(M)))$ is called a smoothing element in the parameter-dependent cone calculus on M associated with the weight data $(\gamma, \delta) \in \mathbb{R}^2$, written $c \in C_G(M, (\gamma, \delta); \mathbb{R}^q)$, if there is an $\varepsilon = \varepsilon(c) > 0$ such that

$$c(\eta) \in \mathcal{S}(\mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(M), H^{\infty,\delta+\varepsilon}(M))), \\ c^*(\eta) \in \mathcal{S}(\mathbb{R}^q, \mathcal{L}(H^{s,-\delta}(M), H^{\infty,-\gamma+\varepsilon}(M)));$$

for all $s \in \mathbb{R}$; here c^* is the η-wise formal adjoint of c with respect to the $H^{0,0}(M)$-scalar product.

The η-wise kernels of the operators $c(\eta)$ are in $C^\infty((M \setminus \{v\}) \times (M \setminus \{v\}))$. However, they are of flatness ε in the respective distance variables to v, relative to the weights δ and γ, respectively. Let us look at a simple example to illustrate the structure. We choose elements $k \in \mathcal{S}(\mathbb{R}^q, H^{\infty,\delta+\varepsilon}(M))$, $k' \in \mathcal{S}(\mathbb{R}^q, H^{\infty,-\gamma+\varepsilon}(M))$ and assume for convenience that k and k' vanish outside a neighbourhood of v, for all $\eta \in \mathbb{R}^q$. Then with respect to a local splitting of variables (t, x) near v we can write $k = k(\eta, t, x)$ and $k' = k'(\eta, t', x')$, respectively. Set

$$c(\eta)u(t, x) := \iint k(\eta, t, x) k'(\eta, t', x') u(t', x') t'^n dt' dx'$$

with the formal adjoint

$$c^*(\eta)v(t', x') := \iint \overline{k'(\eta, t', x') k(\eta, t, x)} v(t, x) t^n dt dx.$$

Then $c(\eta)$ is a smoothing element in the parameter-dependent cone calculus.
By $C^\mu(M, (\gamma, \gamma - \mu); \mathbb{R}^q)$ we denote the set of all operator families

$$a(\eta) = \sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} + (1 - \sigma) a_{\mathrm{int}}(\eta)(1 - \tilde{\tilde{\sigma}}) + c(\eta) \tag{1.35}$$

where a_{edge} is of the form (1.27), $a_{\mathrm{int}} \in L^\mu_{\mathrm{cl}}(M \setminus \{v\}; \mathbb{R}^q)$, while $c(\eta)$ is a parameter-dependent smoothing operator on M, associated with the weight data $(\gamma, \gamma - \mu)$.

Theorem 1.20. *Let M be a compact manifold with conical singularity. Then the η-dependent families* (1.22) *which define continuous operators*

$$a(\eta): H^{s,\gamma}(M) \to H^{s-\nu,\gamma-\nu}(M) \tag{1.36}$$

for all $s \in \mathbb{R}$, $\nu \geq \mu$, have the properties:

$$\|a(\eta)\|_{\mathcal{L}(H^{s,\gamma}(M),H^{s-\nu,\gamma-\nu}(M))} \leq c\langle\eta\rangle^{B} \tag{1.37}$$

for all $\eta \in \mathbb{R}^q$, and $s \in \mathbb{R}$, with constants $c = c(\mu,\nu,s) > 0$, $B = B(\mu,\nu,s)$, and, when $\mu \leq 0$

$$\|a(\eta)\|_{\mathcal{L}(H^{0,0}(M),H^{0,0}(M))} \leq c\langle\eta\rangle^{\mu} \tag{1.38}$$

for all $\eta \in \mathbb{R}$, $s \in \mathbb{R}$, with constants $c = c(\mu, s) > 0$.

Proof. The result is known for the summand $(1-\sigma)a_{\mathrm{int}}(\eta)(1-\tilde{\tilde{\sigma}})$ as we see from the example in Section 1.1. Therefore, we may concentrate on

$$p(\eta) := \sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} : H^{s,\gamma}(M) \to H^{s-\nu,\gamma-\nu}(M).$$

To show (1.37) we pass to

$$\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\nu,\gamma-\nu}(X^\wedge).$$

Then Theorem 1.18 shows that we have symbolic estimates, especially

$$\|\kappa_{\langle\eta\rangle}^{-1} p(\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))} \leq c\langle\eta\rangle^{\mu}.$$

We have

$$\|p(\eta)\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\nu,\gamma-\nu}(X^\wedge))} \leq \|p(\eta)\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))},$$

and

$$\begin{aligned}\|p(\eta)\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))} &= \|\kappa_{\langle\eta\rangle}\kappa_{\langle\eta\rangle}^{-1} p(\eta)\kappa_{\langle\eta\rangle}\kappa_{\langle\eta\rangle}^{-1}\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))}\\ &\leq \|\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))}\|\kappa_{\langle\eta\rangle}^{-1} p(\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s,\gamma}(X^\wedge))}\\ &\qquad \|\kappa_{\langle\eta\rangle}^{-1}\|_{\mathcal{L}(\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge),\mathcal{K}^{s,\gamma}(X^\wedge))} \leq c\langle\eta\rangle^{\mu+\widetilde{M}+M}.\end{aligned}$$

Here we used that $\kappa_{\langle\eta\rangle}, \kappa_{\langle\eta\rangle}^{-1}$ satisfy estimates like (1.4).
For (1.38) we employ that κ_λ is operating as a unitary group on $\mathcal{K}^{0,0}(X^\wedge)$. This gives us

$$\begin{aligned}\|p(\eta)\|_{\mathcal{L}(\mathcal{K}^{0,0}(X^\wedge),\mathcal{K}^{0,0}(X^\wedge))} &= \|\kappa_{\langle\eta\rangle}^{-1} p(\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{0,0}(X^\wedge),\mathcal{K}^{0,0}(X^\wedge))}\\ &\leq \|\kappa_{\langle\eta\rangle}^{-1} p(\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{0,0}(X^\wedge),\mathcal{K}^{-\mu,-\mu}(X^\wedge))} \leq c\langle\eta\rangle^{\mu}.\end{aligned}$$

□

Theorem 1.21. *For every $k \in \mathbb{Z}$ there exists an $f_k(z) \in M^{-\infty}(X; \Gamma_{\frac{n+1}{2}-\gamma})$ such that for every cut-off functions $\omega, \tilde{\omega}$ the operator*

$$A := 1 + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega} : H^{s,\gamma}(M) \to H^{s,\gamma}(M) \quad (1.39)$$

is Fredholm and of index k, for all $s \in \mathbb{R}$.

Proof. We employ the result (cf. [33]) that for every $k \in \mathbb{Z}$ there exists an $f_k(z)$ such that

$$\widetilde{A} := 1 + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega} : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge) \quad (1.40)$$

is Fredholm of index k. Recall that the proof of the latter result follows from a corresponding theorem in the case $\dim X = 0$. The Mellin symbol f_k is constructed in such a way that $1+f_k(z) \neq 0$ for all $z \in \Gamma_{\frac{1}{2}-\gamma}$ and the argument of $1+f_k(z)|_{\Gamma_{\frac{1}{2}-\gamma}}$ varies from 1 to $2\pi k$ when $z \in \Gamma_{\frac{1}{2}-\gamma}$ goes from $\mathrm{Im} z = -\infty$ to $\mathrm{Im} z = +\infty$. The choice of $\omega, \tilde{\omega}$ is unessential; so we assume that $\omega, \tilde{\omega} \equiv 0$ for $r \geq 1 - \varepsilon$ with some $\varepsilon > 0$. Let us represent the cone $\widetilde{M} := X^\Delta$ as a union of $\big([0, 1 + \frac{\varepsilon}{2}) \times X\big)/(\{0\} \times X) =: \widetilde{M}_-$ and $(1 - \frac{\varepsilon}{2}, \infty) \times X =: \widetilde{M}_+$. Then

$$\widetilde{A}|_{\widetilde{M}_-} = 1 + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}, \ \widetilde{A}|_{\widetilde{M}_+} = 1. \quad (1.41)$$

Moreover, without loss of generality, we represent M as a union $\big([0, 1 + \frac{\varepsilon}{2}) \times X\big)/(\{0\} \times X) \cup M_+$ where M_+ is an open C^∞ manifold which intersects $\big([0, 1 + \frac{\varepsilon}{2}) \times X\big)/(\{0\} \times X) =: M_-$ in a cylinder of the form $(1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2}) \times X$. Let B denote the operator on M, defined by

$$B_- := A|_{M_-} = 1 + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}, \ B_+ := A|_{M_+} = 1 \quad (1.42)$$

We are then in a special situation of cutting and pasting of Fredholm operators. We can pass to manifolds with conical singularities N and $\widetilde{N}$ by setting

$$N = \widetilde{M}_- \cup M_+, \ \widetilde{N} = M_- \cup \widetilde{M}_+$$

and transferring the former operators in (1.41), (1.42) to N and $\widetilde{N}$, respectively, by gluing together the $\pm$ pieces of $\widetilde{A}$ and A to belong to $\widetilde{M}_\pm$ and $M_\pm$ to corresponding operators $\widetilde{B}$ on $\widetilde{N}$ and B on N. We then have the relative index formula

$$\mathrm{ind} A - \mathrm{ind} B = \mathrm{ind} \widetilde{A} - \mathrm{ind} \widetilde{B} \quad (1.43)$$

(see [21]). In the present case $\widetilde{A}$ and $\widetilde{M}$ are the same as B and N where $\widetilde{B}$ and $\widetilde{N}$ are the same as A and M. It follows that

$$\mathrm{ind} \widetilde{A} - \mathrm{ind} \widetilde{B} = \mathrm{ind} B - \mathrm{ind} A. \quad (1.44)$$

From (1.43), (1.44) it follows that $\mathrm{ind} A = \mathrm{ind} B = \mathrm{ind} \widetilde{A}$. □

Theorem 1.22. *There is a choice of m and g such that the operators* (1.22) *form a family of isomorphisms*

$$a(\eta) : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M) \tag{1.45}$$

for all $s \in \mathbb{R}$ and all $\eta \in \mathbb{R}^q$.

Proof. We choose a function

$$p(t, \tau, \eta, \zeta) := \tilde{p}(t\tau, t\eta, \zeta)$$

similarly as (1.23) where $\tilde{p}(\tilde{\tau}, \tilde{\eta}, \zeta) \in L^{\mu}_{\text{cl}}(X; \mathbb{R}^{1+q+l}_{\tilde{\tau},\tilde{\eta},\zeta}), l \geq 1$, is a parameter-dependent elliptic with parameters $\tilde{\tau}, \tilde{\eta}, \zeta$. For purposes below we specify $\tilde{p}(t, \tilde{\tau}, \tilde{\eta}, \zeta)$ in such a way that the parameter-dependent homogeneous principal symbol in $(t, x, \tilde{\tau}, \xi, \tilde{\eta}, \zeta)$ for $(\tilde{\tau}, \xi, \tilde{\eta}, \zeta) \neq 0$ is equal to

$$(|\tilde{\tau}|^2 + |\xi|^2 + |\tilde{\eta}|^2 + |\zeta|^2)^{\frac{\mu}{2}}.$$

We now form an element

$$\tilde{h}(t, z, \tilde{\eta}, \zeta) \in M^{\mu}_{\mathcal{O}}(X; \mathbb{R}^{q+l}_{\tilde{\eta},\zeta})$$

analogously as (1.24) such that

$$h(t, z, \eta, \zeta) := \tilde{h}(t, z, t\eta, \zeta)$$

satisfies

$$\text{op}^{\gamma}_{M}(h)(\eta, \zeta) = \text{Op}_t(p)(\eta, \zeta)$$

mod $L^{-\infty}(X^\wedge; \mathbb{R}^{q+l}_{\eta,\zeta})$. For every fixed $\zeta \in \mathbb{R}^l$ this is exactly as before, but in this way we obtain corresponding ζ-dependent families of such objects. It follows

$$\sigma b_{\text{edge}}(\eta, \zeta)\tilde{\sigma} = t^{-\mu}\sigma \left\{\omega_\eta \text{op}^{\gamma-\frac{n}{2}}_{M}(h)(\eta, \zeta)\tilde{\omega}_\eta + \chi_\eta \text{Op}_t(p)(\eta, \zeta)\tilde{\chi}_\eta\right\}\tilde{\sigma}$$

with

$$\chi_\eta(t) := 1 - \omega_\eta(t), \;\; \tilde{\chi}_\eta(t) := 1 - \tilde{\tilde{\omega}}_\eta(t).$$

Let us form the principal edge symbol

$$\sigma_\wedge(\sigma b_{\text{edge}}\tilde{\sigma})(\eta, \zeta) = t^{-\mu}\left\{\omega_{|\eta|}\text{op}^{\gamma-\frac{n}{2}}_{M}(h)(\eta, \zeta)\tilde{\omega}_{|\eta|} + \chi_{|\eta|}\text{Op}_t(p)(\eta, \zeta)\tilde{\chi}_{|\eta|}\right\}$$

for $|\eta| \neq 0$ which gives us a family of continuous operators

$$\sigma_\wedge(\sigma b_{\text{edge}}\tilde{\sigma})(\eta, \zeta) : \mathcal{K}^{s,\gamma;g}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu;g}(X^\wedge) \tag{1.46}$$

which is elliptic as a family of classical pseudo-differential operators on $X^\wedge$. In addition it is exit elliptic on $X^\wedge$ with respect to the conical exit of $X^\wedge$ to infinity.

In order that (1.46) is Fredholm for the given weight $\gamma \in \mathbb{R}$ and all $s, g \in \mathbb{R}$ it is necessary and sufficient that the subordinate conormal symbol

$$\sigma_{\mathrm{c}}\sigma_{\wedge}(\sigma b_{\mathrm{edge}}\tilde{\sigma})(z,\zeta) : H^s(X) \to H^{s-\mu}(X)$$

is a family of isomorphisms for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$. This is standard information from the calculus on the stretched cone $X^\wedge$. By definition the conormal symbol is just

$$\tilde{h}(0,z,0,\zeta) : H^s(X) \to H^{s-\mu}(X). \tag{1.47}$$

Since by construction $\tilde{h}(\beta + i\tau, 0, \zeta)$ is parameter-dependent elliptic on X with parameters $(\tau, \zeta) \in \mathbb{R}^{1+l}$, for every $\beta \in \mathbb{R}$ (uniformly in finite β-intervals) there is a $C > 0$ such that (1.47) becomes bijective whenever $|\tau, \zeta| > C$. In particular, choosing ζ large enough it follows the bijectivity for all $\tau \in \mathbb{R}$, i.e., for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$. Let us fix ζ^1 in that way and write again

$$p(t,\tau,\eta) := p(t,\tau,\eta,\zeta^1),\ \ h(t,z,\eta) := h(z,t\eta,\zeta^1).$$

We are now in the same situation we started with, but we know in addition that (1.46) is a family of Fredholm operators of a certain index, say, $-k$ for some $k \in \mathbb{Z}$. With the smoothing Mellin symbol $f_k(z)$ as in (1.40) we now form the composition

$$\sigma b_{\mathrm{edge}}(\eta)\tilde{\sigma}(1 + \omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}_\eta) \tag{1.48}$$

which is of the form

$$\sigma b_{\mathrm{edge}}(\eta)\tilde{\sigma} + \omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega}_\eta + g(\eta) \tag{1.49}$$

for another smoothing Mellin symbol $f(z)$ and a certain Green symbol $g(\eta)$. Here, by a suitable choice of $\omega, \tilde{\omega}$, without loss of generality we assume that $\sigma \equiv 1$ and $\tilde{\sigma} \equiv 1$ on $\operatorname{supp}\omega_\eta \cup \operatorname{supp}\tilde{\omega}_\eta$, for all $\eta \in \mathbb{R}^q$. Since (1.48) is a composition of parameter-dependent cone operators the associated edge symbol is equal to

$$F(\eta) := \sigma_\wedge(\sigma b_{\mathrm{edge}}\tilde{\sigma})(\eta)(1 + \omega_{|\eta|}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}_{|\eta|}) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{1.50}$$

which is a family of Fredholm operators of index 0. By construction (1.50) depends only on $|\eta|$. For $\eta \in S^{q-1}$ we now add a Green operator g_0 on $X^\wedge$ such that

$$F(\eta) + g_0(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is an isomorphism; it is known that such g_0 (of finite rank) exists (for $N = \dim\ker F(\eta)$ it can be written in the form $g_0 u := \sum_{j=1}^N (u, v_j)w_j$, where $(\cdot,\cdot)$ is the $K^{0,0}(X^\wedge)$-scalar product and $(v_j)_{j=1,\ldots,N}$ and $(w_j)_{j=1,\ldots,N}$ are orthonormal systems of functions in $C_0^\infty(X^\wedge)$). Setting

$$g(\eta) := \sigma\vartheta(\eta)|\eta|^\mu \kappa_{|\eta|} g_0 \kappa_{|\eta|}^{-1}\tilde{\sigma}$$

with an excision function $\vartheta(\eta)$ in $\mathbb{R}^q$ we obtain a Green symbol with $\sigma_\wedge(g)(\eta) = |\eta|^\mu \kappa_{|\eta|} g_0 \kappa_{|\eta|}^{-1}$ and hence

$$\sigma_\wedge(F(\eta) + g(\eta)) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is a family of isomorphisms for all $\eta \in \mathbb{R}^q \setminus \{0\}$. Setting

$$a_{\mathrm{edge}}(\eta) := \Big[t^{-\mu}\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega}_\eta + \chi_\eta \mathrm{Op}_t(p)(\eta)\tilde{\chi}_\eta\Big]\Big(1 + \omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}_\eta\Big) + |\eta|^\mu \vartheta(\eta)\kappa_{|\eta|} g_0 \kappa_{|\eta|}^{-1} \quad (1.51)$$

we obtain an operator family

$$\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} = F(\eta) + g(\eta)$$

as announced before. Next we choose a parameter-dependent elliptic $a_{\mathrm{int}}(\eta) \in L^\mu_{\mathrm{cl}}(M\setminus\{v\}; \mathbb{R}^q_\eta)$ such that its parameter-dependent homogeneous principal symbol close to $t = 0$ (in the splitting of variables (t, x)) is equal to

$$(|\tau|^2 + |\xi|^2 + |\eta|^2)^{\frac{\mu}{2}}.$$

Then we form

$$a(\eta) := \sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} + (1-\sigma)a_{\mathrm{int}}(\eta)(1-\tilde{\tilde{\sigma}})$$

with $\sigma, \tilde{\sigma}, \tilde{\tilde{\sigma}}$ as in (1.22). This is now a parameter-dependent elliptic element of the cone calculus on M with parameter $\eta \in \mathbb{R}^q$. It is known, see the explanations after this proof, that there is a constant $C > 0$ such that the operators (1.45) are isomorphisms for all $|\eta| \geq C$. Now, in order to construct $a(\eta)$ such that (1.45) are isomorphisms for all $\eta \in \mathbb{R}^q$ we simply perform the construction with $(\eta, \lambda) \in \mathbb{R}^{q+r}, r \geq 1$ in place of η, then obtain a family $a(\eta, \lambda)$ and define $a(\eta) := a(\eta, \lambda^1)$ with a $\lambda^1 \in \mathbb{R}^r, |\lambda^1| \geq C$. □

Let us now give more information on the above mentioned space

$$C^\mu(M, \boldsymbol{g}; \mathbb{R}^q),\ \boldsymbol{g} = (\gamma, \gamma-\mu),$$

of parameter-dependent cone operators on M of order $\mu \in \mathbb{R}$, with the weight data $\boldsymbol{g}$. The elements $a(\eta) \in C^\mu(M, \boldsymbol{g}; \mathbb{R}^q)$ have a principal symbolic hierarchy

$$\sigma(a) := (\sigma_\psi(a), \sigma_\wedge(a)) \quad (1.52)$$

where $\sigma_\psi(a)$ is the parameter-dependent homogeneous principal symbol of order μ, defined through $a(\eta) \in L^\mu_{\mathrm{cl}}(M \setminus \{v\}; \mathbb{R}^q)$. This determines the reduced symbol

$$\tilde{\sigma}_\psi(a)(t, x, \tau, \xi, \eta) := t^\mu \sigma_\psi(a)(t, x, t^{-1}\tau, \xi, t^{-1}\eta)$$

given close to v in the splitting of variables (t, x) with covariables (τ, ξ). By construction $\tilde{\sigma}_\psi(a)$ is smooth up to $t = 0$. The second component $\sigma_\wedge(a)(\eta)$ is defined as

$$\sigma_\wedge(a)(\eta) := t^{-\mu}\omega_{|\eta|}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(\eta)\tilde{\omega}_{|\eta|} \\ + t^{-\mu}(1-\omega_{|\eta|})\mathrm{Op}_t(p_0)(\eta)(1-\tilde{\tilde{\omega}}_{|\eta|}) + \sigma_\wedge(m+g)(\eta)$$

where $\sigma_\wedge(m+g)(\eta)$ is just the (twisted) homogeneous principal symbol of $m+g$ as a classical operator-valued symbol.

The element $a(\eta)$ of $C_G(M,\boldsymbol{g};\mathbb{R}^q)$ represent families of continuous operators

$$a(\eta) : H^{s,\gamma}(M) \to H^{s,\gamma-\mu}(M) \tag{1.53}$$

for all $s \in \mathbb{R}$.

Definition 1.23. An element $a(\eta) \in C^\mu(M,\boldsymbol{g};\mathbb{R}^q)$ is called elliptic, if

(i) $\sigma_\psi(a)$ never vanishes as a function on $T^*((M \setminus \{v\}) \times \mathbb{R}^q) \setminus 0$ and if $\tilde{\sigma}_\psi(a)$ does not vanish for all (t,x,τ,ξ,η), $(\tau,\xi,\eta) \neq 0$, up to $t=0$;

(ii) $\sigma_\wedge(a)(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$ is a family of isomorphisms for all $\eta \neq 0$, and any $s \in \mathbb{R}$.

Theorem 1.24. *If $a(\eta) \in C^\mu(M,\boldsymbol{g};\mathbb{R}^q)$, $\boldsymbol{g} = (\gamma,\gamma-\mu)$ is elliptic, there exists an element $a^{(-1)}(\eta) \in C^{-\mu}(M,\boldsymbol{g}^{-1};\mathbb{R}^q)$ $\boldsymbol{g}^{-1} := (\gamma-\mu,\gamma)$, such that*

$$1 - a^{(-1)}(\eta)a(\eta) \in C_G(M,\boldsymbol{g}_l;\mathbb{R}^q), \quad 1 - a(\eta)a^{(-1)}(\eta) \in C_G(M,\boldsymbol{g}_r;\mathbb{R}^q),$$

where $\boldsymbol{g}_l := (\gamma,\gamma)$, $\boldsymbol{g}_r := (\gamma-\mu,\gamma-\mu)$.

The proof employs known elements of the edge symbolic calculus (cf. [34]); so we do not recall the details here. Let us only note that the inverses of $\sigma_\psi(a), \tilde{\sigma}_\psi(a)$ and $\sigma_\wedge(a)$ can be employed to construct an operator family $b(\eta) \in C^{-\mu}(M,\boldsymbol{g}^{-1};\mathbb{R}^q)$ such that

$$\sigma_\psi(a)^{(-1)} = \sigma_\psi(b), \quad \tilde{\sigma}_\psi(a)^{(-1)} = \tilde{\sigma}_\psi(b), \quad \sigma_\wedge(a)^{(-1)} = \sigma_\wedge(b).$$

This gives us $1 - b(\eta)a(\eta) =: c_0(\eta) \in C^{-1}(M,\boldsymbol{g}_l;\mathbb{R}^q)$, and a formal Neumann series argument allows us to improve $b(\eta)$ to a left parametrix $a^{(-1)}(\eta)$ by setting $a^{(-1)}(\eta) := \left(\sum_{j=0}^\infty c_0^j(\eta)\right) b(\eta)$ (using the existence of the asymptotic sum in $C^0(M,\boldsymbol{g};\mathbb{R}^q)$). In a similar manner we can construct a right parametrix, i.e., $a^{(-1)}(\eta)$ is as desired.

Corollary 1.25. *If $a(\eta)$ is as in Theorem 1.24, then (1.53) is a family of Fredholm operators of index 0, and there is a constant $C > 0$ such that the operators (1.53) are isomorphisms for all $|\eta| \geq C$, $s \in \mathbb{R}$.*

Corollary 1.26. *If we perform the construction of Theorem* 1.24 *with the parameter* $(\eta, \lambda) \in \mathbb{R}^{q+l}$, $l \geq 1$, *rather than* η, *Corollary* 1.25 *yields that* $a(\eta, \lambda)$ *is invertible for all* $\eta \in \mathbb{R}^q$, $|\lambda| \geq C$. *Then, setting* $a(\eta) := a(\eta, \lambda^1)$, $|\lambda^1| \geq C$ *fixed, we obtain* $a^{-1}(\eta) \in C^{-\mu}(M, \boldsymbol{g}^{-1}; \mathbb{R}^q)$.

Observe that the operator functions of Theorem 1.20 refer to scales of spaces with two parameters, namely, $s \in \mathbb{R}$, the smoothness, and $\gamma \in \mathbb{R}$, the weight. Compared with Definition 1.9 we have here an additional weight. There are two ways to make the different view points compatible. One is to apply weight reducing isomorphisms

$$\mathrm{h}^{-\gamma} : H^{s,\gamma}(M) \to H^{s,\gamma-\mu}(M) \tag{1.54}$$

in (1.21). Then, passing from

$$a(\eta) : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M) \tag{1.55}$$

to

$$b^\mu(\eta) := \mathrm{h}^{-\gamma+\mu} a(\eta) \mathrm{h}^\gamma : H^{s,0}(M) \to H^{s-\mu,0}(M) \tag{1.56}$$

we obtain operator functions between spaces only referring to s but with properties as required in Definition 1.9 (which remains to be verified).

Remark 1.27. The spaces $E^s := H^{s,0}(M)$, $s \in \mathbb{R}$, form a scale with the properties at the beginning of Section 1.1.

Another way is to modify the abstract framework by admitting scales $E^{s,\gamma}$ rather than E^s, where in general γ may be in $\mathbb{R}^k$ (which is motivated by the higher corner calculus). We do not study the second possibility here but we only note that the variant with $E^{s,\gamma}$-spaces is very similar to the one without γ.
Let us now look at operator functions of the form (1.56).

Theorem 1.28. *The operators* (1.56) *constitute an order reducing family in the spaces* $E^s := H^{s,0}(M)$, *where the properties* (i)-(iii) *of Definition* 1.3 *are satisfied.*

Proof. In this proof we concentrate on the properties of our operators for every fixed s, μ, ν with $\nu \geq \mu$. The uniformity of the involved constants can easily be deduced; however, the simple (but lengthy) considerations will be left out.
(i) We have to show that

$$D_\eta^\beta b^\mu(\eta) = D_\eta^\beta \{\mathrm{h}^{-\gamma+\mu} a(\eta) \mathrm{h}^\gamma\} \in C^\infty(\mathbb{R}^q, \mathcal{L}(E^s, E^{s-\mu+|\beta|}))$$

for all $s \in \mathbb{R}$, $\beta \in \mathbb{N}^q$. According to (1.22) the operator function is a sum of two contributions. The second summand

$$(1-\sigma)\mathrm{h}^{-\gamma+\mu} a_{\mathrm{int}}(\eta) \mathrm{h}^\gamma (1-\tilde{\tilde{\sigma}})$$

is a parameter-dependent family in $L^{\mu}_{\mathrm{cl}}(2\mathbb{M};\mathbb{R}^q)$ and obviously has the desired property. The first summand is of the form

$$\sigma \mathrm{h}^{-\gamma+\mu}\{a_{\mathrm{edge}}(\eta)+m(\eta)+g(\eta)\}\mathrm{h}^{\gamma}\tilde{\sigma}.$$

From the proof of Theorem 1.20 we have

$$D^{\beta}_{\eta}\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} \in S^{\mu-|\beta|}(\mathbb{R}^q;\mathcal{K}^{s,\gamma;g}(X^{\wedge}),\mathcal{K}^{s-\mu+|\beta|,\gamma-\mu;g}(X^{\wedge}))$$

for every $\beta \in \mathbb{N}^q$. In particular, these operator functions are smooth in η and the derivates improve the smoothness in the image by $|\beta|$. This gives us the desired property of $\sigma \mathrm{h}^{-\gamma+\mu}a_{\mathrm{edge}}(\eta)\mathrm{h}^{\gamma}\tilde{\sigma}$. The C^{∞} dependence of $m(\eta)+g(\eta)$ in η is clear (those are operator-valued symbols), and they map to $\mathcal{K}^{\infty,\gamma-\mu;g}(X^{\wedge})$ anyway. Therefore, the desired property of $\sigma \mathrm{h}^{-\gamma+\mu}\{m(\eta)+g(\eta)\}\mathrm{h}^{\gamma}\tilde{\sigma}$ is satisfied as well.
(ii) This property essentially corresponds to the fact that the product in consideration close to the conical point is a symbol in η of order zero and that the group action in $\mathcal{K}^{0,0}(X^{\wedge})$-spaces is unitary. Outside the conical point the boundedness is as in the example in Section 1.1.
(iii) The proof of this property close to the conical point is of a similar structure as Proposition 1.8, since our operators are based on operator-valued symbols referring to spaces with group action. The contribution outside the conical point is as in the example in Section 1.1. □

Remark 1.29. For $E^s := \mathcal{H}^{s,0}(M)$, $s \in \mathbb{R}$, $\mathcal{E} = (E^s)_{s\in\mathbb{R}}$, the operator functions $b^{\mu}(\eta)$ of the form (1.56) belong to $S^{\mu}(\mathbb{R}^q;\mathcal{E},\mathcal{E})$ (see the notation after Definition 1.9).

2 Operators referring to a conical exit to infinity

2.1 Symbols with weights at infinity

Let $\mathcal{E} = (E^s)_{s\in\mathbb{R}}$ be a scale in $\mathfrak{C}$ with the compact embedding property (see Section 1.1), and choose a family of order reducing operators

$$b^s(\rho,\eta,\lambda),\ (\rho,\eta,\lambda) \in \mathbb{R}^{1+q}\times(\mathbb{R}^l\setminus\{0\}),\ s\in\mathbb{R}, \tag{2.1}$$

$q,l \in \mathbb{N}\setminus\{0\}$ (see Definition 1.3, here with (ρ,η,λ) instead of η). Let us form the operator family

$$p^s(r,\rho,\eta,\lambda) := b^s([r]\rho,[r]\eta,[r]\lambda) \tag{2.2}$$

for $r \in \mathbb{R}$ (recall that $r \to [r]$ is a strictly positive function in $C^{\infty}(\mathbb{R})$ such that $[r] = |r|$ for $|r| > R$ for some $R > 0$).

Theorem 2.1. *The operator*

$$[r]^s \mathrm{Op}_r(p^{-s})(\eta, \lambda) : L^2(\mathbb{R}, E^0) \to L^2(\mathbb{R}, E^0) \tag{2.3}$$

is continuous for every $s \geq 0$ *and satisfies the estimate*

$$\| [r]^s \mathrm{Op}_r(p^{-s})(\eta, \lambda) \|_{\mathcal{L}(L^2(\mathbb{R}, E^0))} \leq c |\eta, \lambda|^{-s} \tag{2.4}$$

for all $(\eta, \lambda) \in \mathbb{R}^q \times (\mathbb{R}^l \setminus \{0\})$, $|\lambda| \geq 1$, *for some constant* $c > 0$.

For the proof we employ the following variant of the Calderón-Vaillancourt theorem for operators with operator-valued symbols (cf. Hwang [10] for scalar symbols, Seiler [39] in the operator-valued case,) see also [8, Section 2.2.2].

Theorem 2.2. *Let* H *and* $\widetilde{H}$ *be Hilbert spaces with group actions* κ *and* $\tilde{\kappa}$, *respectively. Assume that a function* $a(y, \eta) \in C^\infty(\mathbb{R}^{2q}, \mathcal{L}(H, \widetilde{H}))$ *satisfies the estimate*

$$\pi(a) := \sup \left\{ \| \tilde{\kappa}^{-1}_{\langle \eta \rangle} \{ D_y^\alpha D_\eta^\beta a(y, \eta) \} \kappa_{\langle \eta \rangle} \|_{\mathcal{L}(H, \widetilde{H})} : (y, \eta) \in \mathbb{R}^{2q}, \alpha \leq \boldsymbol{\alpha}, \beta \leq \boldsymbol{\beta} \right\} < \infty$$

for $\boldsymbol{\alpha} := (M+1, \dots, M+1)$, $\boldsymbol{\beta} := (1, \dots, 1)$, *with* $M \in \mathbb{N}$ *being a constant such that* (1.4) *holds for* $\tilde{\kappa}$. *Then* $\mathrm{Op}(a)$ *induces a continuous operator*

$$\mathrm{Op}(a) : \mathcal{W}^0(\mathbb{R}^q, H) \to \mathcal{W}^0(\mathbb{R}^q, \widetilde{H}),$$

and we have $\| \mathrm{Op}(a) \|_{\mathcal{L}(\mathcal{W}^0(\mathbb{R}^q, H), \mathcal{W}^0(\mathbb{R}^q, \widetilde{H}))} \leq c \pi(a)$ *for a constant* $c > 0$ *independent of* a.

Proof of Theorem 2.1. In order to show the continuity of (2.3) and the estimate (2.4) for the operator norm we apply Theorem 2.2 to the case $H = \widetilde{H} = E^0$ and $\kappa = \tilde{\kappa} = \mathrm{id}$; then $M = 1$. Setting for the moment

$$a(r, \rho, \eta, \lambda) := [r]^s b^{-s}([r]\rho, [r]\eta, [r]\lambda)$$

we have $a(r, \rho, \eta, \lambda) \in C^\infty(\mathbb{R} \times \mathbb{R}^{1+q+l}, \mathcal{L}(E^0, E^0))$ and

$$\| a(r, \rho, \eta, \lambda) \|_{0,0} \leq c [r]^s \langle [r]\rho, [r]\eta, [r]\lambda \rangle^{-s} \tag{2.5}$$

(see Definition 1.3 (iii)). From

$$\sup_{(r,\rho) \in \mathbb{R}^2} [r]^s \langle [r]\rho, [r]\eta, [r]\lambda \rangle^{-s} \leq |\eta, \lambda|^{-s}$$

for all $(\eta, \lambda) \in \mathbb{R}^{q+l}$, $|\lambda| \geq 1$, we obtain

$$\sup_{(r,\rho) \in \mathbb{R}^2} \| a(r, \rho, \eta, \lambda) \|_{0,0} \leq c |\eta, \lambda|^{-s}$$

for those η, λ. A similar estimate is needed for the derivatives $D_r^k D_\rho^m a(r, \rho, \eta, \lambda)$ for all $0 \leq k, m \leq 1$. For simplicity, we consider the case $q = l = 1$. With the notation $\tilde{\rho} = [r]\rho, \tilde{\eta} = [r]\eta, \tilde{\lambda} = [r]\lambda$ we obtain

$$\partial_r([r]^s b^{-s}([r]\rho, [r]\eta, [r]\lambda)) =$$
$$[r]^s \partial_r [r]\big((\rho \frac{\partial}{\partial \tilde{\rho}} + \eta \frac{\partial}{\partial \tilde{\eta}} + \lambda \frac{\partial}{\partial \tilde{\lambda}}) b^{-s}\big)([r]\rho, [r]\eta, [r]\lambda) + (\partial_r [r]^s) b^{-s}([r]\rho, [r]\eta, [r]\lambda). \tag{2.6}$$

The last summand on the right hand side of (2.6) can be estimated in a similar manner as before, since $\sup_{r \in \mathbb{R}} |(\partial_r [r]^s)[r]^{-s}| < \infty$. Concerning the derivatives of b^{-s} with respect to (ρ, η, λ) we can employ the fact that

$$b^{-s}(\tilde{\rho}, \tilde{\eta}, \tilde{\lambda}) \in S^{-s}(\mathbb{R}^{1+q+l}; \mathcal{E}, \mathcal{E}),$$

(see (1.12)). Then the first order derivatives in $(\tilde{\rho}, \tilde{\eta}, \tilde{\lambda})$ belong to $S^{-s-1}(\mathbb{R}^{1+q+l}; \mathcal{E}, \mathcal{E})$ and hence, according to Proposition 1.13,

$$\|D^\alpha_{\tilde{\rho}, \tilde{\eta}, \tilde{\lambda}} b^{-s}(\tilde{\rho}, \tilde{\eta}, \tilde{\lambda})\|_{0,0} \leq c \langle \tilde{\rho}, \tilde{\eta}, \tilde{\lambda} \rangle^{-s-1},$$

for any multi-index α with $|\alpha| = 1$. This gives us, for instance, for the first summands on the right hand side of (2.6)

$$\begin{aligned} \sup \|(\partial_r [r])[r]^s \{(\rho(\partial_{\tilde{\rho}} b^{-s}) + \eta(\partial_{\tilde{\eta}} b^{-s}) + \lambda(\partial_{\tilde{\lambda}} b^{-s}))([r]\rho, [r]\eta, [r]\lambda)\}\|_{0,0} \\ \leq c \sup [r]^{s+1} \{|\rho| + |\eta| + |\lambda|\} \langle [r]\rho, [r]\eta, [r]\lambda \rangle^{-s-1} \\ \leq c \sup [r]^s \langle [r]\rho, [r]\eta, [r]\lambda \rangle^{-s} \leq c|\eta, \lambda|^{-s} \end{aligned}$$

for all $(\eta, \lambda) \in \mathbb{R}^{q+l}$, $|\lambda| \geq 1$.

Let us now consider the derivative in ρ. In this case we have

$$\begin{aligned} \sup \|\frac{\partial}{\partial \rho}[r]^s b^{-s}([r]\rho, [r]\eta, [r]\lambda)\|_{0,0} = \sup \|[r]^{s+1}\big(\frac{\partial}{\partial \tilde{\rho}} b^{-s}\big)([r]\rho, [r]\eta, [r]\lambda)\|_{0,0} \\ \leq \sup [r]^{s+1} \langle [r]\rho, [r]\eta, [r]\lambda \rangle^{-s-1} \leq c|\eta, \lambda|^{-s-1} \end{aligned}$$

for all $(\eta, \lambda) \in \mathbb{R}^{q+l}$, $|\lambda| \geq 1$. The other derivatives can be treated in a similar manner. We thus obtain altogether the estimate (2.4). □

Remark 2.3. The computations in the latter proof show that for $s \geq 0$

$$\|D^\alpha_{\eta,\lambda}\big([r]^s \mathrm{Op}_r(p^{-s})(\eta, \lambda)\big)\|_{\mathcal{L}(L^2(\mathbb{R}, E^0))} \leq c|\eta, \lambda|^{-s-|\alpha|}$$

for all $(\eta, \lambda) \in \mathbb{R}^{q+l}$, $|\lambda| \geq 1$, with a constant $c > 0$.

Definition 2.4. Let us set $\mathbb{H} = \{\eta \in \mathbb{R}^q : \eta'' \neq 0\}$, where $q = q' + q''$, $\eta = (\eta', \eta'') \in \mathbb{R}^{q'+q''}$, $q'' > 0$ (see the fomula (1.9)). By

$$S^{\mu;\nu}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}}; \mathbb{H})_{\text{cone}} \tag{2.7}$$

$\mu, \nu \in \mathbb{R}$, we denote the set of all operator functions of the form

$$a(r, \rho, \eta) = [r]^{-\mu} \tilde{a}(r, [r]\rho, [r]\eta), \tag{2.8}$$

such that

$$\|\tilde{b}^{s-\mu+|\beta|}([r]\rho, [r]\eta) D_r^l D_{\rho,\eta}^{\beta} \left\{[r]^{-\mu} \tilde{a}(r, [r]\rho, [r]\eta)\right\} b^{-s}([r]\rho, [r]\eta)\| \leq c\langle r\rangle^{\nu-\mu+|\beta|-l} \tag{2.9}$$

for all $(r, \rho, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{H}$, $|\eta| \geq h$, $h > 0$, and all $l \in \mathbb{N}, \beta \in \mathbb{N}^{1+q}, s \in [s', s'']$, with constants $c = c(l, \beta, s', s'', h) > 0$. Here, as usual, we write $\|.\| = \|.\|_{0,0}$.

In an analogous manner we define the subspace

$$S_{\text{cl}}^{\mu;\nu}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}}; \mathbb{H})_{\text{cone}} \tag{2.10}$$

of elements of (2.7) such that the function $\tilde{a}(r, \tilde{\rho}, \tilde{\eta})$ in (2.8) is classical in r of order ν, which means that there is a sequence of homogeneous components $\tilde{a}_{(\nu-j)}(r, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\mathbb{R} \setminus \{0\}, C^\infty(\mathbb{R}_{\tilde{\rho}} \times \mathbb{R}_{\tilde{\eta}}))$, $j \in \mathbb{N}$, such that

$$\tilde{a}_{(\nu-j)}(\lambda r, \tilde{\rho}, \tilde{\eta}) = \lambda^{\nu-j} \tilde{a}_{(\nu-j)}(r, \tilde{\rho}, \tilde{\eta})$$

for all $\lambda \in \mathbb{R}_+$, and the functions $\tilde{a}_{(\nu-j)}(\pm 1, \tilde{\rho}, \tilde{\eta})$ satisfy the estimates

$$\|\tilde{b}^{s-\mu+|\beta|}([r]\rho, [r]\eta) D_r^l D_{\rho,\eta}^{\beta} \left\{[r]^{-\mu} \tilde{a}_{(\nu-j)}(\pm 1, [r]\rho, [r]\eta)\right\} b^{-s}([r]\rho, [r]\eta)\| \leq \langle r\rangle^{-\mu+|\beta|-l} \tag{2.11}$$

for all (r, ρ, η), l, β, and s as before, and that for any excision function $\chi(r)$ in the variable $r \in \mathbb{R}$ the difference

$$a(r, \rho, \eta) - [r]^{-\mu} \chi(r) \sum_{j=0}^{N} \tilde{a}_{(\nu-j)}(r, [r]\rho, [r]\eta) \tag{2.12}$$

belongs to $S^{\mu;\nu-(N+1)}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}}; \mathbb{H})_{\text{cone}}$ in the former sense, for every $N \in \mathbb{N}$.
If an assertion refers to classical as well as to general symbols we write $S_{(\text{cl})}^{\mu;\nu}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}}; \mathbb{H})_{\text{cone}}$.
It can easily be proved, using (1.3), that when $\delta(r) \in S_{\text{cl}}^1(\mathbb{R})$ is a strictly positive function such that $\delta^{-1}(r) \in S_{\text{cl}}^{-1}(\mathbb{R})$, the space $S^{\mu;\nu}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}}; \mathbb{H})_{\text{cone}}$ can be equivalently defined as the set of all functions of the form $\delta^{-\mu}(r)\tilde{a}(r, \delta(r)\rho, \delta(r)\eta)$, where $\tilde{a}$ depends in a similar manner on $(\delta(r)\rho, \delta(r)\eta)$ as the former one on $([r]\rho, [r]\eta)$.

Remark 2.5. The space (2.7) is Fréchet in a natural way with the semi-norm system

$$\pi(a) := \sup \|\langle r\rangle^{-\nu+\mu-|\beta|+l}\tilde{b}^{s-\mu+|\beta|}([r]\rho,[r]\eta) \\ D_r^l D_{\rho,\eta}^{\beta}\{[r]^{-\mu}\tilde{a}(r,[r]\rho,[r]\eta)\}b^{-s}([r]\rho,[r]\eta)\|, \quad (2.13)$$

where the supremum is taken over all $r \in \mathbb{R}$, $(\rho,\eta) \in \mathbb{R}\times\mathbb{H}$, $|\eta| \geq h, h > 0$, $l \in \mathbb{N}$, $\beta \in \mathbb{N}^{1+q}$, $s \in [s', s'']$. Also the subspace (2.10) of (2.7) is Fréchet with the semi-norms (2.13) together with the semi-norms from the homogeneous components in r (see (2.11)) as well as from the (non-classical) remainders (2.12).

Let

$$S^{\mu}(\mathbb{R}_{[r]\rho}\times\mathbb{H}_{[r]\eta};\mathcal{E},\widetilde{\mathcal{E}}) \quad (2.14)$$

denote the subspace of all $a(r,\rho,\eta) \in S^{\mu;\mu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}$ that are of the form $\tilde{a}([r]\rho,[r]\eta)$ with $\tilde{a}$ as in (2.8). If we mean that for a function (2.8) the semi-norms (2.13) are finite, we write $S_{(\text{cl})}^{\nu-\mu}(\mathbb{R}, S^{\mu}(\mathbb{R}_{[r]\rho}\times\mathbb{H}_{[r]\eta};\mathcal{E},\widetilde{\mathcal{E}}))$ rather than (2.7).

Example. Let $\tilde{p}(\tilde{\rho},\tilde{\eta}) \in L_{\text{cl}}^{\mu}(X;\mathbb{R}_{\tilde{\rho},\tilde{\eta}}^{1+q})$, and

$$a(r,\rho,\eta) := [r]^{\nu-\mu}\tilde{p}([r]\rho,[r]\eta).$$

Then we have $a(r,\rho,\eta) \in S_{\text{cl}}^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}$ for $\mathcal{E} = \widetilde{\mathcal{E}} = \big(H^s(X)\big)_{s\in\mathbb{R}}$ and $\mathbb{H} = \mathbb{R}^q \setminus \{0\}$ ($\eta = 0$ is ruled out, because the relevant properties that are of interest here are valied only in this case).

Other interesting examples come from the parameter-depedent cone or edge operators, see Chapter 2.

Proposition 2.6. *The spaces of Definition* 2.4 *have the following properties:*

(i) $S_{(\text{cl})}^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}} \subseteq S_{(\text{cl})}^{\mu+k;\nu+j+k}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}$ *for every* $j,k \in \overline{\mathbb{R}}_+$ *in the general and* $j,k \in \mathbb{N}$ *in the classical case;*

(ii) $D_r^k D_{\rho,\eta}^{\beta} S_{(\text{cl})}^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}} \subseteq S_{(\text{cl})}^{\mu-|\beta|;\nu-k}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}$ *for every* $\mu,\nu \in \mathbb{R}$, $k \in \mathbb{N}$, $\beta \in \mathbb{N}^{1+q}$*;*

(iii) $S_{(\text{cl})}^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E}_0,\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}} S_{(\text{cl})}^{\tilde{\mu};\tilde{\nu}}(\mathbb{R}\times\mathbb{R};\mathcal{E},\mathcal{E}_0;\mathbb{H})_{\text{cone}}$
$\subseteq S_{(\text{cl})}^{\mu+\tilde{\mu};\nu+\tilde{\nu}}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}$ *for every* $\mu,\nu,\tilde{\mu},\tilde{\nu} \in \mathbb{R}$.

Proof. Let us check the assertions for general symbols; the classical case is left to the reader. (i) The proof is similar as that of Proposition 1.15(i).
(ii) Let $a(r,\rho,\eta) \in S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}$. By definition we have (2.8), and (2.9) is finite. By induction it is enough to check the assertion for first order derivatives.

We have $(\frac{\partial}{\partial r}\tilde{a})(r,[r]\rho,[r]\eta) \in S^{\nu-1}(\mathbb{R}, S^{\mu}(\mathbb{R}_{[r]\rho} \times \mathbb{H}_{[r]\eta}; \mathcal{E}, \widetilde{\mathcal{E}}))$. Thus, when we differentiate (2.8) with respect to r we may forget about the first r-variable in $\tilde{a}$ and simply compute a derivative of the form

$$\frac{\partial}{\partial r}\{[r]^{-\mu}\tilde{a}([r]\rho,[r]\eta)\} \tag{2.15}$$

for an r-independent $\tilde{a}$ of the form (2.14). Assume for simplicity $q=1$. For (2.15) we then obtain

$$(\frac{\partial}{\partial r}[r]^{-\mu})\tilde{a}([r]\rho,[r]\eta) + [r]^{-\mu}(\frac{\partial}{\partial r}[r])(\rho\frac{\partial \tilde{a}}{\partial \tilde{\rho}} + \eta\frac{\partial \tilde{a}}{\partial \tilde{\eta}})([r]\rho,[r]\eta). \tag{2.16}$$

The factor in the first summand on the right of (2.16) can be rewritten as

$$\frac{\partial}{\partial r}[r]^{-\mu} = \big(\frac{\partial}{\partial r}[r]^{-\mu}\big)([r]^{\mu})[r]^{-\mu}$$

but $\big(\frac{\partial}{\partial r}[r]^{-\mu}\big)[r]^{\mu} \in S^{-1}_{\mathrm{cl}}(\mathbb{R})$; so this contributes -1 to the order in r. To treat the second summand in (2.16) we observe that

$$[r]\big(\rho\frac{\partial \tilde{a}}{\partial \tilde{\rho}} + \eta\frac{\partial \tilde{a}}{\partial \tilde{\eta}}\big)([r]\rho,[r]\eta)$$

is an element of (2.14) (see Remark 1.11 and Proposition 1.15 (ii)). Therefore, we gain a factor $[r]^{-1}$. Using $(\frac{\partial}{\partial r}[r])[r]^{-1} \in S^{-1}_{\mathrm{cl}}(\mathbb{R})$ then we see that the r-derivative is as desired.

The first order derivative of $a(r,\rho,\eta)$ in ρ has the form

$$[r]^{-\mu+1}\big(\frac{\partial \tilde{a}}{\partial \tilde{\rho}}\big)(r,[r]\rho,[r]\eta). \tag{2.17}$$

By virtue of $\frac{\partial \tilde{a}}{\partial \tilde{\rho}}(r,[r]\rho,[r]\eta) \in S^{\nu}(\mathbb{R}, S^{\mu-1}(\mathbb{R}_{[r]\rho} \times \mathbb{H}_{[r]\eta}; \mathcal{E}, \widetilde{\mathcal{E}}))$ (see also Proposition 1.15 (ii)) it follows that (2.17) is of analogous form as (2.8) with $\mu-1$ instead of μ. The other derivatives can be treated in a similar manner.
(iii) In order to show that $(a\tilde{a})(r,\rho,\eta)$ has the asserted property for $a(r,\rho,\eta) \in S^{\mu;\nu}(\mathbb{R}\times\mathbb{R}; \mathcal{E}_0, \widetilde{\mathcal{E}}; \mathbb{H})_{\mathrm{cone}}$, $\tilde{a}(r,\rho,\eta) \in S^{\tilde{\mu};\tilde{\nu}}(\mathbb{R}\times\mathbb{R}; \mathcal{E}, \mathcal{E}_0; \mathbb{H})_{\mathrm{cone}}$ we assume for convenience that $\mathcal{E} = \mathcal{E}_0 = \widetilde{\mathcal{E}}$; the general case is completely analogous.
Writing

$$a(r,\rho,\eta) = [r]^{-\mu}p(r,[r]\rho,[r]\eta), \quad \tilde{a}(r,\rho,\eta) = [r]^{-\tilde{\mu}}\tilde{p}(r,[r]\rho,[r]\eta)$$

with

$$p(r,\tilde{\rho},\tilde{\eta}) \in S^{\nu}(\mathbb{R}, V), \quad \tilde{p}(r,\tilde{\rho},\tilde{\eta}) \in S^{\tilde{\nu}}(\mathbb{R}, \widetilde{V}), \tag{2.18}$$

$$V := S^{\mu}(\mathbb{R}_{[r]\rho} \times \mathbb{H}_{[r]\eta}; \mathcal{E}, \mathcal{E}), \quad \widetilde{V} := S^{\tilde{\mu}}(\mathbb{R}_{[r]\rho} \times \mathbb{H}_{[r]\eta}; \mathcal{E}, \mathcal{E}),$$

it follows that

$$(a\tilde{a})(r,\rho,\eta) = [r]^{-(\mu+\tilde{\mu})}(p\tilde{p})(r,[r]\rho,[r]\eta).$$

Then a straightforward computation shows that

$$(p\tilde{p})(r,[r]\rho,[r]\eta) \in S^{\nu+\tilde{\nu}}(\mathbb{R},\tilde{\tilde{V}}), \quad \tilde{\tilde{V}} := S^{\mu+\tilde{\mu}}(\mathbb{R}_{[r]\rho}\times\mathbb{H}_{[r]\eta};\mathcal{E},\mathcal{E}).$$

□

In the calculus of operators with such symbols it is desirable also to have double symbols. We need them only in the form

$$a(r,\rho,\eta)b(r',\rho',\eta) =: c(r,r',\rho,\rho',\eta) \tag{2.19}$$

for $a(r,\rho,\eta) := \tilde{a}(r,[r]\rho,[r]\eta)$, $b(r',\rho',\eta) := \tilde{b}(r',[r']\rho',[r']\eta)$ for some $\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E}_0,\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$, $\tilde{b}(r',\tilde{\rho}',\tilde{\eta}) \in S^{\tilde{\mu};\tilde{\nu}}(\mathbb{R}\times\mathbb{R};\mathcal{E},\mathcal{E}_0;\mathbb{H})_{\mathrm{cone}}$. The composition of associated operators in terms of the symbolic structure will be studied in Section 2.2 below.

Observe that the space $S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$ is embedded in another class of operator families, defined to be the set of all $a(r,\rho,\eta) \in C^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{H},\mathcal{L}^\mu(\mathcal{E},\widetilde{\mathcal{E}}))$ such that (writing $\|.\|_{s,t} := \|.\|_{\mathcal{L}(E^s,\widetilde{E}^t)}$)

$$\sup\langle r\rangle^{-N-|\beta|}\langle\rho,\eta\rangle^{-M}\|D_r^j D_{\rho,\eta}^\beta a(r,\rho,\eta)\|_{s,s-\mu} \tag{2.20}$$

is finite for certain $N,M \in \mathbb{N}$ and every $j \in \mathbb{N}$, $\beta \in \mathbb{N}^{1+q}$, where sup is taken over all $(r,\rho,\eta) \in \mathbb{R}\times\mathbb{R}\times\mathbb{H}$, $|\eta| \geq h$ for any fixed $h > 0$, and $s \in [s',s'']$ for arbitrary $s' \leq s''$, with orders N,M depending on μ,ν as well as on the chosen smoothness interval $[s',s'']$.
Let us check (2.20), for instance, for $j = \beta = 0$. In this case for $\xi := ([r]\rho,[r]\eta)$, $a = \tilde{a}(r,\xi)$ we have

$$\begin{aligned}\|a(r,\rho,\eta)\|_{s,s-\mu} &= \|\tilde{b}^{-s+\mu}(\xi)\tilde{b}^{s-\mu}(\xi)\tilde{a}(r,\xi)b^{-s}(\xi)b^s(\xi)\|_{s,s-\mu}\\ &\leq \|\tilde{b}^{-s+\mu}(\xi)\|_{0,s-\mu}\|\tilde{b}^{s-\mu}(\xi)\tilde{a}(r,\xi)b^{-s}(\xi)b^s(\xi)\|_{0,0}\|b^s(\xi)\|_{s,0}\\ &\leq c\langle r\rangle^{\nu-\mu}\langle[r]\rho,[r]\eta\rangle^{B_1+B_2}\end{aligned} \tag{2.21}$$

using

$$\|\tilde{b}^{-s+\mu}(\xi)\|_{0,s-\mu} \leq c\langle\xi\rangle^{B_1}, \quad \|b^s(\xi)\|_{s,0} \leq \langle\xi\rangle^{B_2}$$

for some $B_1,B_2 > 0$, uniformly in $s \in [s',s'']$. The right hand side of (2.21) can be estimated by

$$c\langle r\rangle^{\nu-\mu+B_1+B_2}\langle\rho,\eta\rangle^{B_1+B_2}$$

which allows us to set $N = \nu-\mu+B_1+B_2$, $M = B_1+B_2$.
Concerning the derivates, using

$$\partial_r\tilde{a}(r,[r]\rho,[r]\eta) = \big(\partial_r\tilde{a}\big)(r,[r]\rho,[r]\eta) + \partial_r[r]\big(\partial_{\tilde{\rho}}\tilde{a} + \sum_{l=1}^q \partial_{\tilde{\eta}_l}\tilde{a}\big)(r,[r]\rho,[r]\eta)$$

or

$$\partial_\rho \tilde{a}(r,[r]\rho,[r]\eta) = [r]\big(\partial_{\tilde{\rho}}\tilde{a}\big)(r,[r]\rho,[r]\eta)$$

we see that the estimates remain true with the same N, M for all $j, k \in \mathbb{N}$.

Let $S^{\mu;\boldsymbol{M},\boldsymbol{N}}(\mathbb{R}\times\mathbb{R}\times\mathbb{H};\mathcal{E},\widetilde{\mathcal{E}})$ denote the set of all $a(r,\rho,\eta) \in C^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{H},\mathcal{L}^\mu(\mathcal{E},\widetilde{\mathcal{E}}))$ satisfying the symbolic estimates (2.20); here $\boldsymbol{M} := \{M(s',s'') : s' \le s''\}$, $\boldsymbol{N} := \{N(s',s'') : s' \le s''\}$ is the system orders M, N in (2.20) which depends on $[s',s'']$.

2.2 Operators in weighted spaces

With a symbol $a(r,\rho,\eta) \in S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$ we associate a family of pseudo-differential operators in the usual way, namely,

$$\mathrm{Op}(a)(\eta)u(r) = \iint e^{i(r-r')\rho} a(r,\rho,\eta)u(r')dr'đ\rho = \iint e^{-ir'\rho} a(r,\rho,\eta)u(r'+r)dr'đ\rho$$

first for $u \in \mathcal{S}(\mathbb{R},E^\infty)$.

Theorem 2.7. *Let $a(r,\rho,\eta) \in S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$. Then*

$$\mathrm{Op}_r(a)(\eta) : \mathcal{S}(\mathbb{R},E^s) \to \mathcal{S}(\mathbb{R},\widetilde{E}^{s-\mu})$$

is a family of continuous operators for every $s \in \mathbb{R}$.

The proof is relatively simple, based on the fact that even the respective operators for $a(r,\rho,\eta) \in S^{\mu;\boldsymbol{M},\boldsymbol{N}}(\mathbb{R}\times\mathbb{R}\times\mathbb{H};\mathcal{E},\widetilde{\mathcal{E}})$ define such continuous operators.

Theorem 2.8. *Let $a(r,\rho,\eta) \in S^{\mu;0}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$, $\mu \le 0$ and $g \in \mathbb{R}$. Then*

$$\mathrm{Op}_r(a)(\eta) : \langle r\rangle^{-g}L^2(\mathbb{R},E^0) \to \langle r\rangle^{-g}L^2(\mathbb{R},\widetilde{E}^0)$$

is a family of continuous operators, and we have

$$\|\mathrm{Op}_r(a)(\eta)\|_{\mathcal{L}(\langle r\rangle^{-g}L^2(\mathbb{R},E^0),\langle r\rangle^{-g}L^2(\mathbb{R},\widetilde{E}^0))} \le c|\eta|^\mu \tag{2.22}$$

for all $\eta \in \mathbb{H}$, $|\eta| \ge h$ for any $h > 0$, with a constant $c = c(h) > 0$.

Proof. For $g = 0$ the proof is completely analogous to that of Theorem 2.1. For $g \ne 0$ we use the fact that $\langle r\rangle^{-g}$ can be regarded as an element of $S^{0;-g}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$ for any $g \in \mathbb{R}$. Then, since $\langle r\rangle^{-g} : L^2(\mathbb{R},E^0) \to \langle r\rangle^{-g}L^2(\mathbb{R},\widetilde{E}^0)$ is an isomorphism it suffices to show that $\langle r\rangle^{-g}\mathrm{Op}(a)\langle r\rangle^{g} = \mathrm{Op}(a_g)$ for some $a_g \in S^{\mu;0}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$. However, this will be a consequence of Theorem 2.16 below which implies that $\langle r\rangle^{-g}a(r,\rho,\eta)\#\langle r\rangle^{g} \in S^{\mu;0}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$. ☐

In the following we systematically refer to oscillatory integral techniques analogously as in Kumano-go [14]. Vector-valued generalisations are more or less straightforward; however, we employ a rather subtle variant in terms of degenerate symbols; this makes it necessary to recall some basic constructions.
Let V be a Fréchet space, defined with a countable semi-norm system $(\pi_j)_{j\in\mathbb{N}}$.

Definition 2.9. Given sequences $\boldsymbol{\mu} := (\mu_j)_{j\in\mathbb{N}}$, $\boldsymbol{\nu} := (\nu_j)_{j\in\mathbb{N}}$, we define the space

$$S^{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbb{R}^{2q}; V)$$

of V-valued amplitude functions to be the set of all $a(x,\xi) \in C^\infty(\mathbb{R}^{2q}, V)$ such that

$$\pi_j\big(D_x^\alpha D_\xi^\beta a(x,\xi)\big) \leq c\langle\xi\rangle^{\mu_j}\langle x\rangle^{\nu_j} \tag{2.23}$$

for all $(x,\xi) \in \mathbb{R}^{2q}$, $\alpha,\beta \in \mathbb{N}^q$, with constants $c(\alpha,\beta,j) > 0$, for all $j \in \mathbb{N}$. Moreover, we set

$$S^{\boldsymbol{\infty};\boldsymbol{\infty}}(\mathbb{R}^{2q}; V) := \bigcup_{\boldsymbol{\mu},\boldsymbol{\nu}} S^{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbb{R}^{2q}; V)$$

where the union is taken over all $\boldsymbol{\mu},\boldsymbol{\nu}$.

Remark 2.10. The space $S^{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbb{R}^{2q}; V)$ is Fréchet for every fixed $\boldsymbol{\mu},\boldsymbol{\nu}$, with the semi-norm system $\sup_{(x,\xi)\in\mathbb{R}^{2q}}\langle x\rangle^{-\nu_j}\langle\xi\rangle^{-\mu_j}\pi_j\big(D_x^\alpha D_\xi^\beta a(x,\xi)\big)$, for all $\alpha,\beta \in \mathbb{N}^q$, $j \in \mathbb{N}$ (together with the semi-norms of $C^\infty(\mathbb{R}^{2q}, V)$).

The following observations and constructions may be found in Seiler [38], see also [8].

Proposition 2.11. (i) *$a \in S^{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbb{R}^{2q}; V)$ implies $D_x^\alpha D_\xi^\beta a \in S^{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbb{R}^{2q}; V)$ for every $\alpha,\beta \in \mathbb{N}^q$.*

(ii) *If $V, \widetilde{V}$ are Fréchet spaces and $T : V \to \widetilde{V}$ is a continuous operator, then $a \in S^{\boldsymbol{\infty};\boldsymbol{\infty}}(\mathbb{R}^{2q}; V)$ implies $Ta := ((x,\xi) \to T(a(x,\xi))) \in S^{\boldsymbol{\infty};\boldsymbol{\infty}}(\mathbb{R}^{2q}; \widetilde{V})$; more precisely, $a \to Ta$ defines a continuous operator*

$$S^{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbb{R}^{2q}; V) \to S^{\tilde{\boldsymbol{\mu}};\tilde{\boldsymbol{\nu}}}(\mathbb{R}^{2q}, \widetilde{V})$$

for every $(\boldsymbol{\mu};\boldsymbol{\nu})$, with a resulting pair of orders $(\tilde{\boldsymbol{\mu}};\tilde{\boldsymbol{\nu}})$ (recall that the semi-norm systems are fixed in the respective Fréchet spaces).

(iii) *Let V be the projective limit of Fréchet spaces V_j with respect to linear maps $T_j : V \to V_j$, $j \in I$, (with I being a countable index set). Then $a \in S^{\boldsymbol{\infty};\boldsymbol{\infty}}(\mathbb{R}^{2q}; V)$ is equivalent to $T_j a \in S^{\boldsymbol{\infty};\boldsymbol{\infty}}(\mathbb{R}^{2q}; V_j)$ for every $j \in I$.*

(iv) *If V_0, V_1, V be Fréchet spaces and $\langle\cdot,\cdot\rangle : V_0 \times V_1 \to V$ a continuous bilinear map, then $a_k \in S^{\boldsymbol{\infty};\boldsymbol{\infty}}(\mathbb{R}^{2q}, V_k)$, $k = 0, 1$, implies $\langle a_0, a_1\rangle \in S^{\boldsymbol{\infty};\boldsymbol{\infty}}(\mathbb{R}^{2q}; V)$; more precisely, $(a_0, a_1) \to \langle a_0, a_1\rangle$ induces continuous maps*

$$S^{\boldsymbol{\mu}_0;\boldsymbol{\nu}_0}(\mathbb{R}^{2q}; V_0) \times S^{\boldsymbol{\mu}_1;\boldsymbol{\nu}_1}(\mathbb{R}^{2q}; V_1) \to S^{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbb{R}^{2q}; V)$$

for every two pairs of sequences $(\boldsymbol{\mu}_0;\boldsymbol{\nu}_0), (\boldsymbol{\mu}_1;\boldsymbol{\nu}_1)$, with some resulting $(\boldsymbol{\mu};\boldsymbol{\nu})$.

(v) *Let W be a closed subspace of V; then $a \in S^{\infty;\infty}(\mathbb{R}^{2q}; V)$ implies $[a] \in S^{\infty;\infty}(\mathbb{R}^{2q}; V/W)$, where $[a]$ denotes the image under the quotient map $V \to V/W$.*

Definition 2.12. A function $\chi_\varepsilon(x) : (0,1] \times \mathbb{R}^m \to \mathbb{C}$ is called regularising, if

(i) $\chi_\varepsilon(x) \in \mathcal{S}(\mathbb{R}^m)$ for every $0 < \varepsilon \leq 1$;

(ii) $\sup_{(\varepsilon,x)\in(0,1]\times\mathbb{R}^m} |D_x^\alpha \chi_\varepsilon(x)| < \infty$ for every $\alpha \in \mathbb{N}^m$;

(iii) $\lim_{\varepsilon\to 0} D_x^\alpha \chi_\varepsilon(x) \to \begin{cases} 1 & \text{for } \alpha = 0 \\ 0 & \text{for } \alpha \neq 0, \end{cases}$ pointwise in $\mathbb{R}^m$.

An example of a regularising function in the sense of the latter definition is $\chi(\varepsilon x)$ for any $\chi(x) \in \mathcal{S}(\mathbb{R}^m)$ with $\chi(0) = 1$.

Remark 2.13. If $\chi_\varepsilon(x,\xi)$ is any regularising function on $(0,1] \times \mathbb{R}^{2q}$, and $a(x,\xi) \in S^{\infty;\infty}(\mathbb{R}^{2q}; V)$, then we can form the oscillatory integral

$$\mathrm{Os}[a] = \lim_{\varepsilon\to 0} \iint e^{-ix\xi} \chi_\varepsilon(x,\xi) a(x,\xi) dx đ\xi. \tag{2.24}$$

Remark 2.14. In the regularisation of $\iint e^{-ix\xi} a(x,\xi) dx đ\xi$ we first assume that $a(x,\xi) \in \mathcal{S}(\mathbb{R}^{2q}; V)$, use the identities

$$e^{-ix\xi} = \langle\xi\rangle^{-2M}(1-\Delta_x)^M e^{-ix\xi}, \;\; e^{-ix\xi} = \langle x\rangle^{-2N}(1-\Delta_\xi)^N e^{-ix\xi},$$

and integrate by parts. This yields

$$\iint e^{-ix\xi} a(x,\xi) dx đ\xi = \iint e^{-ix\xi} \langle x\rangle^{-2N}(1-\Delta_\xi)^N \langle\xi\rangle^{-2M}(1-\Delta_x)^M a(x,\xi) dx đ\xi$$

for every $N, M \in \mathbb{N}$. It follows that the right hand side converges with respect to the semi-norm π_j for $N = N_j, M = M_j$ sufficiently large, for any fixed $j \in \mathbb{N}$. This implies

$$\begin{aligned} &\lim_{\varepsilon\to 0} \iint e^{-ix\xi} \chi_\varepsilon(x,\xi) a(x,\xi) dx đ\xi \\ &\quad = \lim_{\varepsilon\to 0} \iint e^{-ix\xi} \langle x\rangle^{-2N_j}(1-\Delta_\xi)^{N_j} \langle\xi\rangle^{-2M_j}(1-\Delta_x)^{M_j} \chi_\varepsilon(x,\xi) a(x,\xi) dx đ\xi \end{aligned}$$

with convergence with respect to π_j. Similarly as in the scalar case, Lebesgue's theorem on dominated convergence gives us the convergence of the right hand side for arbitrary $a(x,\xi) \in S^{\infty;\infty}(\mathbb{R}^{2q}; V)$. Thus the left hand side exists as well.

A consequence is the following theorem.

Theorem 2.15. *For every $a(x,\xi) \in S^{\boldsymbol{\infty};\boldsymbol{\infty}}(\mathbb{R}^{2q};V)$ the oscillatory integral* (2.24) *exists as an element of V and is independent of the choice of χ. Moreover, $a(x,\xi) \to \mathrm{Os}[a]$ induces a continuous map*

$$\mathrm{Os}[\cdot] : S^{\boldsymbol{\mu};\boldsymbol{\nu}}(\mathbb{R}^{2q};V) \to V$$

for every $\boldsymbol{\mu},\boldsymbol{\nu}$.

One of the main issues here is to ensure that the operators $\mathrm{Op}(a)(\eta)$ with symbols $a(r,\rho,\eta) \in S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$ form a calculus which is closed under the usual operations, especially compositions. To formulate the corresponding result it will be easier to first admit symbols of the larger class $S^{\mu;\boldsymbol{M},\boldsymbol{N}}(\mathbb{R}\times\mathbb{R}\times\mathbb{H};\mathcal{E},\widetilde{\mathcal{E}})$ and then to obtain the result for symbols in $S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$ itself.
As mentioned before we apply here elements of Kumano-go's technique on oscillatory integrals, especially with double symbols in variables and covariables. We only need such symbols in form of pointwise compositions

$$a(r,\rho,\eta)b(r',\rho',\eta)$$

for

$$a(r,\rho,\eta) \in S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E}_0,\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}, \tag{2.25}$$

$$b(r',\rho',\eta) \in S^{\tilde{\mu};\tilde{\nu}}(\mathbb{R}\times\mathbb{R};\mathcal{E},\mathcal{E}_0;\mathbb{H})_{\mathrm{cone}}. \tag{2.26}$$

Using $a \in S^{\mu;\boldsymbol{M},\boldsymbol{N}}, b \in S^{\tilde{\mu};\widetilde{\boldsymbol{M}},\widetilde{\boldsymbol{N}}}$ for suitable $\boldsymbol{M},\boldsymbol{N}$ and $\widetilde{\boldsymbol{M}},\widetilde{\boldsymbol{N}}$ we first carry out the computations in that more general set-up and then obtain that the respective subclasses remain preserved.

For simplicity the operators are considered for $u \in \mathcal{S}(\mathbb{R},E^\infty)$, cf. Theorem 2.7. We have

$$\begin{aligned}
&\mathrm{Op}(a)(\eta)\mathrm{Op}(b)(\eta)u(r)\\
&\quad= \iint e^{i(r-r')\rho}a(r,\rho,\eta)\left\{\iint e^{i(r'-r'')\rho'}b(r',\rho',\eta)u(r'')dr''đ\rho'\right\}dr'đ\rho\\
&\quad= \iiiint e^{i(r-r')\rho+i(r'-r'')\rho'}a(r,\rho,\eta)b(r',\rho',\eta)u(r'')dr''đ\rho'dr'đ\rho
\end{aligned}$$

with integration in the order r'',ρ',r',ρ. This implies

$$\begin{aligned}
&\mathrm{Op}(a)(\eta)\mathrm{Op}(b)(\eta)u(r)\\
&\quad= \iiint e^{i(r-r')\rho+ir'\rho'}a(r,\rho,\eta)b(r',\rho',\eta)\hat{u}(\rho')đ\rho'dr'đ\rho.
\end{aligned} \tag{2.27}$$

An analogue of a corresponding expression in Kumano-go [14] gives us

$$\begin{aligned}
&\mathrm{Op}(a)(\eta)\mathrm{Op}(b)(\eta)u(r)\\
&\quad= \iint e^{i(t\rho+t'\rho')}a(r,\rho,\eta)b(r+t,\rho',\eta)u(r+t+t')dtdt'đ\rhođ\rho'
\end{aligned}$$

as an oscillatory integral. Setting

$$a\#b(r,\rho,\eta) := \iint e^{-it\tau} a(r,\rho+\tau,\eta) b(r+t,\rho,\eta) dt đ\tau \tag{2.28}$$

and applying a substitution in the variables it follows that

$$\begin{aligned} \mathrm{Op}(a\#b)(\eta) &= \int e^{ir\rho'} \left\{ \iint e^{-it\tau} a(r,\rho'+\tau,\eta) b(r+t,\rho',\eta) dt đ\tau \right\} \hat{u}(\rho') đ\rho' \\ &= \int e^{ir\rho} \left\{ \int e^{-ir'\rho} \left\{ \int e^{ir'\rho'} a(r,\rho,\eta) b(r',\rho',\eta) \hat{u}(\rho') đ\rho' \right\} dr' \right\} đ\rho \end{aligned} \tag{2.29}$$

(see the formula (2.27)).
Now, as usual, Taylor's formula gives us

$$a(r,\rho+\tau,\eta) = \sum_{k=0}^{N} \frac{\tau^k}{k!} \left(\partial_\rho^k a\right)(r,\rho,\eta) + \frac{\tau^{N+1}}{N!} \int_0^1 (1-\theta)^N \left(\partial_\rho^N a\right)(r,\rho+\theta\tau,\eta) d\theta$$

and hence

$$\begin{aligned} a\#b(r,\rho,\eta) &= \sum_{k=0}^{N} \left(\partial_\rho^k a\right)(r,\rho,\eta) \iint e^{-it\tau} \frac{\tau^k}{k!} b(r+t,\rho,\eta) dt đ\tau \\ &+ \iint e^{-it\tau} \frac{\tau^{N+1}}{N!} \left\{ \int_0^1 (1-\theta)^N \left(\partial_\rho^{N+1} a\right)(r,\rho+\theta\tau,\eta) d\theta \right\} b(r+t,\rho,\eta) dt đ\tau. \end{aligned} \tag{2.30}$$

Applying $D_r^k u(r) = \iint e^{-it\tau} \tau^k u(r+t) dt đ\tau$ in the sum on the right of (2.30) and integrating by parts in the second term we obtain

$$a\#b(r,\rho,\eta) = \sum_{k=0}^{N} \frac{1}{k!} \partial_\rho^k a(r,\rho,\eta) D_r^k b(r,\rho,\eta) + r_N(r,\rho,\eta)$$

with

$$\begin{aligned} r_N(r,\rho,\eta) = \frac{1}{N!} \iint e^{-it\tau} &\left\{ \int_0^1 (1-\theta)^N \left(\partial_\rho^{N+1} a\right)(r,\rho+\theta\tau,\eta) d\theta \right\} \\ &\left(D_r^{N+1} b\right)(r+t,\rho,\eta) dt đ\tau. \end{aligned}$$

Theorem 2.16. *Let* $a(r,\rho,\eta) \in S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E}_0,\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}$, $b(r,\rho,\eta) \in S^{\tilde{\mu};\tilde{\nu}}(\mathbb{R}\times\mathbb{R};\mathcal{E},\mathcal{E}_0;\mathbb{H})_{\mathrm{cone}}$. *Then for the Leibniz product* (2.28) *we have*

$$a\#b(r,\rho,\eta) \in S^{\mu+\tilde{\mu};\nu+\tilde{\nu}}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\mathrm{cone}}.$$

Proof. By virtue of Proposition 2.6 (iii) the sum on the right of (2.28) has the asserted property. Therefore, it suffices to show that for every π from the system of semi-norms on the space $S^{\mu+\tilde{\mu};\nu+\tilde{\nu}}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}$ we have $\pi(r_N)<\infty$ when $N=N(\pi)$ is large enough. However, this is the case, as a straightforward (but lengthy) computation shows, using the shape of π, see the formula (2.13), and the regularisation process, described in Remark 2.14. □

Remark 2.17. The computation that verifies $\pi(r_N)<\infty$ shows, in fact, more, namely, that for every $M\in\mathbb{N}$ and every semi-norm π_{M+1} in the space

$$S^{\mu+\tilde{\mu}-M;\nu+\tilde{\nu}-M}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}$$

we have $\pi_{M+1}(r_N)<\infty$ provided that $N=N(\pi_{M+1})\geq M$ is large enough. This gives us

$$\pi_{M+1}\left(a\#b-\sum_{k=0}^{M}\frac{1}{k!}(\partial_\rho^k a)D_r^k b\right)=\pi_{M+1}\left(r_N+\sum_{k=M+1}^{N}\frac{1}{k!}(\partial_\rho^k a)D_r^k b\right)$$
$$\leq \pi_{M+1}(r_N)+\pi_{M+1}\left(\sum_{k=M+1}^{N}\frac{1}{k!}(\partial_\rho^k a)D_r^k b\right)<\infty,$$

since by Proposition 2.6 (iii) the second summand on the right of the latter inequality is finite, and hence

$$a\#b(r,\rho,\eta)-\sum_{k=0}^{M}\frac{1}{k!}(\partial_r^k a)(r,\rho,\eta)D_r^k b(r,\rho,\eta)$$
$$\in S^{\mu+\tilde{\mu}-(M+1);\nu+\tilde{\nu}-(M+1)}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}. \quad (2.31)$$

Theorem 2.18. *The operator* (2.3) *for* $s\geq 0$ *is injective for all* $(\eta,\lambda)\in\mathbb{R}^{q+l}$, $|\lambda|\geq C$, *for a sufficiently large* $C>0$.

Proof. By virtue of Theorem 2.16 the composition

$$\mathrm{Op}_r([r]^{-s}p^s)(\eta,\lambda)\mathrm{Op}_r([r]^s p^{-s})(\eta,\lambda) \quad (2.32)$$

is an operator with amplitude function

$$[r]^{-s}p^s(r,\rho,\eta,\lambda)\#[r]^s p^{-s}(r,\rho,\eta,\lambda)=1-c(r,\rho,\eta,\lambda), \quad (2.33)$$

$c(r,\rho,\eta,\lambda)\in S^{-1;-1}(\mathbb{R}\times\mathbb{R};\mathcal{E},\mathcal{E};\mathbb{R}^q\times(\mathbb{R}^l\setminus\{0\}))$. From Theorem 2.8 we have the estimate (2.22) with (η,λ) in place of η, for $\mu=-1$. Thus the composition (2.32) becomes an isomorphism in $L^2(\mathbb{R},E^0)$ for sufficiently large $|\lambda|$ and for all $\eta\in\mathbb{R}^q$. This implies the injectivity of the operator (2.3). □

Corollary 2.19. *Let $s, g \in \mathbb{R}$, and form the composition*

$$\mathrm{Op}_r([r]^{-s+g}p^s)(\eta,\lambda)\mathrm{Op}_r([r]^{s-g}p^{-s})(\eta,\lambda) \tag{2.34}$$

as a continuous operator $\mathcal{S}(\mathbb{R}, E^\infty) \to \mathcal{S}(\mathbb{R}, E^\infty)$ (see Theorem 2.7). Then (2.34) extends to a continuous and injective operator $L^2(\mathbb{R}, E^0) \to L^2(\mathbb{R}, E^0)$ for all $(\eta, \lambda) \in \mathbb{R}^q \times (\mathbb{R}^l \setminus \{0\})$, $|\lambda| \geq C$, for a suitable constant $C > 0$.

In the following definition we employ the symbols (2.2).

Definition 2.20. Let us set $B^{s;g}(\eta,\lambda) := \mathrm{Op}_r([r]^{-s+g}p^s)(\eta,\lambda)$ for $s, g \in \mathbb{R}, (\eta,\lambda) \in \mathbb{R}^q \times (\mathbb{R}^l \setminus \{0\})$, $|\lambda| \geq C$, where $C > 0$ is a constant as in Corollary 2.19. Then $H^{s;g}_{\mathrm{cone}}(\mathbb{R}, \mathcal{E})$ is defined to be the completion of $\mathcal{S}(\mathbb{R}, E^\infty)$ with respect to the norm

$$\|B^{s;g}(\eta^1, \lambda^1)u\|_{L^2(\mathbb{R},E^0)}$$

for any fixed $\eta^1 \in \mathbb{R}^q$ and $\lambda^1 \in \mathbb{R}^l$, $|\lambda^1| \geq C$.

From the construction if follows that

$$B^{s;g}(\eta, \lambda^1) : H^{s;g}_{\mathrm{cone}}(\mathbb{R}, \mathcal{E}) \to L^2(\mathbb{R}, E^0) \tag{2.35}$$

is a family of isomorphisms for every $|\lambda^1|$ sufficiently large.
By construction we have

$$[r]^{-s+g}p^s(r, \rho, \eta, \lambda) \in S^{s;g}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \mathcal{E}; \mathbb{H})_{\mathrm{cone}}$$

for $\mathbb{H} = \mathbb{R}^q \times (\mathbb{R}^l \setminus \{0\})$. In the following we impose a requirement on the choice of the operator family $B^{s;g}(\eta,\lambda)$, namely, that for every $s, g \in \mathbb{R}$ there exists a symbol $f^{-s;-g}(r, \rho, \eta, \lambda) \in S^{-s;-g}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \mathcal{E}; \mathbb{H})_{\mathrm{cone}}$ such that

$$\left(B^{s;g}(\eta, \lambda^1)\right)^{-1} = \mathrm{Op}_r(f^{-s;-g})(\eta, \lambda^1) : L^2(\mathbb{R}, E^0) \to H^{s;g}_{\mathrm{cone}}(\mathbb{R}, \mathcal{E})$$

for all $\eta \in \mathbb{R}^q$ and those $\lambda^1 \in \mathbb{R}^l \setminus \{0\}$ where (2.35) is invertible. In applications this is a fairly mild condition which is connected with the property (also a requirement in the abstract approch) that within the calculus there is an asymptotic summation of symbols (or operators) when the involved orders μ and weights ν tend to $-\infty$. In order to simplify notation we assume $B^{s;0}(\eta,\lambda)$ to be costructed (according to Definition 2.20) first for $s \geq 0$, where for $s = 0$ we simply take the identity; then we set $B^{s;0}(\eta,\lambda) = \mathrm{Op}(f^{s;0})(\eta,\lambda)$ for $s < 0$, and finally $B^{s;g}(\eta,\lambda) := \langle r\rangle^g B^{s;0}(\eta,\lambda)$ for arbitrary $s, g \in \mathbb{R}$.

Remark 2.21. The space $H^{s;g}_{\mathrm{cone}}(\mathbb{R}, \mathcal{E})$ is independent of the specific η^1, λ^1 and also of the choice of the order reducing family (2.1) that is involved in $B^{s;g}$ (more precisely, (2.1) may be replaced by an equivalent family).

Theorem 2.22. *For every* $a(r,\rho,\eta) \in S^{\mu;\nu}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}};\mathbb{H})_{\text{cone}}$ *the operator*

$$\operatorname{Op}_r(a)(\eta) : \mathcal{S}(\mathbb{R},E^\infty) \to \mathcal{S}(\mathbb{R},\widetilde{E}^\infty)$$

extends to a continuous mapping

$$\operatorname{Op}(a)(\eta) : H^{s;g}_{\text{cone}}(\mathbb{R},\mathcal{E}) \to H^{s-\mu;g-\nu}_{\text{cone}}(\mathbb{R},\widetilde{\mathcal{E}})$$

for every $s,g\in\mathbb{R}$ *and every fixed* $\eta\in\mathbb{H}$.

Proof. First observe that we have

$$H^{s;g}_{\text{cone}}(\mathbb{R},\mathcal{E}) = \langle r\rangle^{-g} H^{s;0}_{\text{cone}}(\mathbb{R},\mathcal{E}).$$

Similarly as in the proof of Theorem 2.8 it suffices to consider the case $g=0,\nu=0$. It is clear that for $|\lambda^1|$ sufficiently large we get norms

$$H^{s;0}_{\text{cone}}(\mathbb{R},\mathcal{E}) \ni u \to \|B^{s;0}(\eta,\lambda^1)u\|_{L^2(\mathbb{R},E^0)}$$

on the space $H^{s;0}_{\text{cone}}(\mathbb{R},\mathcal{E})$ which are equivalent for every two fixed $\eta=\eta^1$ or η^2 in $\mathbb{H}$. Then we can write

$$\begin{aligned}\|\operatorname{Op}(a)(\eta)u\|_{H^{s-\mu;0}_{\text{cone}}(\mathbb{R},\widetilde{\mathcal{E}})} &\sim \|B^{s-\mu;0}(\eta,\lambda^1)\operatorname{Op}(a)(\eta)u\|_{L^2(\mathbb{R},E^0)}\\ &= \|B^{s-\mu;0}(\eta,\lambda^1)\operatorname{Op}(a)(\eta)B^{-s;0}(\eta,\lambda^1)B^{s;0}(\eta,\lambda^1)u\|_{L^2(\mathbb{R},E^0)}\\ &\le c\|B^{s;0}(\eta,\lambda^1)u\|_{L^2(\mathbb{R},E^0)} \sim c\|u\|_{H^{s;0}_{\text{cone}}(\mathbb{R},\mathcal{E})},\end{aligned}$$

where $c := \|B^{s-\mu;0}(\eta,\lambda^1)\operatorname{Op}(a)(\eta)B^{-s;0}(\eta,\lambda^1)\|_{\mathcal{L}(L^2(\mathbb{R},E^0),L^2(\mathbb{R},\widetilde{E}^0))}$ is finite. In fact, the operator under the latter norm is equal to

$$\operatorname{Op}\left([r]^{-s+\mu}p^{s-\mu}(r,\rho,\eta,\lambda^1)\#a(r,\rho,\eta)\#[r]^s p^{-s}(r,\rho,\eta,\lambda^1)\right);$$

by Corollary 2.19 the corresponding symbol belongs to $S^{0;0}(\mathbb{R}\times\mathbb{R};\mathcal{E},\widetilde{\mathcal{E}},\mathbb{H})_{\text{cone}}$, and we can apply Theorem 2.8. ☐

Theorem 2.23. *There are continuous embeddings*

$$H^{s';g'}_{\text{cone}}(\mathbb{R},\mathcal{E}) \hookrightarrow H^{s;g}_{\text{cone}}(\mathbb{R},\mathcal{E}) \tag{2.36}$$

for all $s'\ge s$, $g'\ge g$ *that are compact when* $s'>s$, $g'>g$, *and if the scale* $\mathcal{E}$ *has the compact embedding property.*

Proof. For $u\in\mathcal{S}(\mathbb{R},E^\infty)$ we can write

$$\begin{aligned}\|B^{s;g}(\eta,\lambda^1)u\|_{L^2(\mathbb{R},E^0)} &= \|B^{s;g}(\eta,\lambda^1)B^{-s';-g'}(\eta,\lambda^1)B^{s';g'}(\eta,\lambda^1)\|_{L^2(\mathbb{R},E^0)}\\ &\le c\|B^{s';g'}(\eta,\lambda^1)\|_{L^2(\mathbb{R},E^0)}\end{aligned}$$

for $c = \|B^{s;g}(\eta, \lambda^1)B^{-s';-g'}(\eta, \lambda^1)\|_{\mathcal{L}(L^2(\mathbb{R},E^0),L^2(\mathbb{R},E^0))}$. By virtue of Theorem 2.16 we have

$$B^{s;g}(\eta, \lambda^1)B^{-s';-g'}(\eta, \lambda^1) = \mathrm{Op}(h)(\eta, \lambda^1)$$

for some $h(r, \rho, \eta, \lambda) \in S^{s-s';g-g'}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \mathcal{E}; \mathbb{H})_{\mathrm{cone}}$. Since the latter space is contained in $S^{s-s';0}(\mathbb{R}\times\mathbb{R}; \mathcal{E}, \mathcal{E}; \mathbb{H})_{\mathrm{cone}}$ (see Proposition 2.6 (i)) the operator $\mathrm{Op}(h)$ in continuous in $L^2(\mathbb{R}, E^0)$ by Theorem 2.8. This implies $c < \infty$, and hence we have a continuous embedding (2.36) for $s' \geq s, g' \geq g$. The compactness for $s' > s, g' > g$ follows from the fact that the embedding can also be interpreted as the composition of operators

$$B^{-s;-g}(B^{s;g}B^{-s';-g'})B^{s';g'}$$

(always depending on (η, λ^1)), where the operator

$$B^{s;g}(\eta, \lambda^1)B^{-s';-g'}(\eta, \lambda^1) = \mathrm{Op}(h)(\eta, \lambda^1) : L^2(\mathbb{R}, E^0) \to L^2(\mathbb{R}, E^0)$$

is compact, since the weight and the order of the symbol h are strictly negative, and h takes values in compact operators $E^0 \to E^{s'-s} \hookrightarrow E^0$ (to be proved by similar arguments as in [34, Theorem 1.3.61]). □

2.3 Ellipticity in the exit calculus

In this section we assume that the scales $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ have the compact embedding property.

Definition 2.24. An element

$$a(r, \rho, \eta) \in S^{\mu;\nu}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}}; \mathbb{H})_{\mathrm{cone}}$$

is said to be elliptic with parameter $\eta \in \mathbb{R}^q \setminus \{0\}$, if there is an element

$$p(r, \rho, \eta) \in S^{-\mu;-\nu}(\mathbb{R} \times \mathbb{R}; \widetilde{\mathcal{E}}, \mathcal{E}; \mathbb{H})_{\mathrm{cone}}$$

such that

$$1 - p(r, \rho, \eta)a(r, \rho, \eta) =: c(r, \rho, \eta) \in S^{-1;-1}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \mathcal{E}; \mathbb{H})_{\mathrm{cone}},$$
$$1 - a(r, \rho, \eta)p(r, \rho, \eta) =: \tilde{c}(r, \rho, \eta) \in S^{-1;-1}(\mathbb{R} \times \mathbb{R}; \widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}; \mathbb{H})_{\mathrm{cone}}.$$

Remark 2.25. The conditions in Definition 2.24 imply that

$$a(r, \rho, \eta) : E^s \to E^{s-\mu}$$

is a family of Fredholm operators for all $s \in \mathbb{R}$, $(r, \rho, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{H}$ because the remainders $c, \tilde{c}$ are pointwise compact, since they consist of families of continuous operators $E^s \to E^{s+1}$ and $\widetilde{E}^s \to \widetilde{E}^{s+1}$, respectively, for all s.

Theorem 2.26. *Let $A(\eta) = \mathrm{Op}_r(a)(\eta)$, and let*

$$a(r, \rho, \eta) \in S^{\mu;\nu}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}}; \mathbb{H})_{\mathrm{cone}}$$

be elliptic. Then $A(\eta)$ induces a family of Fredholm operators

$$A(\eta) : H^{s;g}_{\mathrm{cone}}(\mathbb{R}, \mathcal{E}) \to H^{s-\mu;g-\nu}_{\mathrm{cone}}(\mathbb{R}, \widetilde{\mathcal{E}})$$

for every $s \in \mathbb{R}$ and $\eta \in \mathbb{H}$.

Proof. Let us set $P(\eta) = \mathrm{Op}_r(p)(\eta)$. Then according to Theorem 2.16 and Remark 2.17 we have

$$1 - P(\eta)A(\eta) = \mathrm{Op}(c_0)(\eta)$$

for a symbol $c_0(r, \rho, \eta)$ that is equal to $c(r, \rho, \eta) \mod S^{-1;-1}(\mathbb{R} \times \mathbb{R}; \mathcal{E}, \mathcal{E}; \mathbb{H})_{\mathrm{cone}}$. Similarly as in the proof of Theorem 2.23 it follows that $\mathrm{Op}(c_0)(\eta)$ is a family of compact operators in the space $H^{s;g}_{\mathrm{cone}}(\mathbb{R}, \mathcal{E})$, $s \in \mathbb{R}$. Analogously we obtain that $1 - A(\eta)P(\eta) = \mathrm{Op}(\tilde{c}_0)(\eta)$ for a symbol $\tilde{c}_0(r, \rho, \eta) \in S^{-1;-1}(\mathbb{R} \times \mathbb{R}; \widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}; \mathbb{H})_{\mathrm{cone}}$ is compact in the space $H^{s-\mu;g-\mu}_{\mathrm{cone}}(\mathbb{R}, \widetilde{\mathcal{E}})$, $s \in \mathbb{R}$. This gives us the Fredholm property of $A(\eta)$. □

Remark 2.27. There are other properties of elliptic operators, analogously as in the standard context on a closed C^∞ manifold, such as independence of kernel and cokernel (as the kernel of the formal adjoint) on s and g; those are finite-dimensional subspaces of $\mathcal{S}(\mathbb{R}, E^\infty)$ and $\mathcal{S}(\mathbb{R}, E^{*\infty})$, respectively.

Let us finally note that in the higher corner calculus (to be elaborated elsewhere) the present operators are localised near $r = \infty$ and glued together with Mellin operators in a neighbourhood of $r = 0$. Together with weighted spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E})$, defined in an analogous manner as $\mathcal{H}^{s,\gamma}(X^\wedge)$ (see the formula (1.20)), the analogues of the spaces (1.28) then are defined by

$$\mathcal{K}^{s,\gamma;g}(\mathbb{R}_+, \mathcal{E}) = \omega\mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E}) + (1 - \omega)H^{s;g}_{\mathrm{cone}}(\mathbb{R}, \mathcal{E})|_{\mathbb{R}_+}$$

for some cut-off function ω on the half-axis.

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