The existence and regularity of multiple solutions for a class of infinitely degenerate elliptic equations *

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Abstract

Let $X = (X_1, \dots, X_m)$ be an infinitely degenerate system of vector fields, we study the existence and regularity of multiple solutions of Dirichlet problem for a class of semi-linear infinitely degenerate elliptic operators associated with the sum of square operator $\Delta_X = \sum_{j=1}^m X_j^* X_j$.

Keywords: degenerate elliptic equations, Logarithmic Sobolev inequality.

1 Introduction

In this paper, we study the existence and regularity of solutions for a class of semi-linear infinitely degenerate elliptic operators. Consider a system of vector fields $X = (X_1, \dots, X_m)$ defined on an open domain $\tilde{\Omega} \subset \mathbb{R}^d$. We suppose that this system satisfies the following Logarithmic regularity estimates,

$$\|(\log \Lambda)^s u\|_{L^2(\Omega)}^2 \leq C \left\{ \sum_{j=1}^m \|X_j u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right\}, \forall u \in C_0^\infty(\tilde{\Omega}), \quad (1.1)$$

where $\Lambda = (e^2 + |D|^2)^{1/2} = \langle D \rangle$. The results of [4, 6, 7, 8, 9] gave some sufficient conditions for the estimates (1.1). We remark that if $s > 1$, the estimate (1.1) implies the hypoellipticity of the infinitely degenerate elliptic operator $\Delta_X = \sum_{j=1}^m X_j^* X_j$, where X_j^* is the formal adjoint of X_j .

Definition 1.1. If Γ is a smooth surfaces of $\tilde{\Omega}$, we say that Γ is non characteristic for the system of vector fields X , if for any point $x_0 \in \Gamma$ there exists at least one vector field in $X = (X_1, \dots, X_m)$ which is transversal to Γ at x_0 .

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Definition 1.2. Let now $\Gamma = \bigcup_{j \in J} \Gamma_j$ be the union of a family of smooth surface in $\tilde{\Omega}$. We say that Γ is non characteristic for X , if for any point $x_0 \in \Gamma$, there exists at least one vector field of $X_1, \dots, X_m$ which is transversal to Γ_j at x_0 for all j in which $x_0 \in \Gamma_j$.

We say that the vector fields $X = (X_1, \dots, X_m)$ satisfies the finite type of Hörmander's condition on an open domain $\omega \subset \tilde{\Omega}$ in $\mathbb{R}^d$ if the rank of the Lie algebra generated by the vector fields $X = (X_1, \dots, X_m)$ and its finite times commutators is equal to the space dimension d at every point in ω .

A typical example is the vector fields in $\mathbb{R}^3$, i.e. $X_1 = \partial_{x_1}$, $X_2 = \partial_{x_2}$, $X_3 = \exp(-|x_1|^{-1/s})\partial_{x_3}$ with $s > 0$. The operator Δ_X in this example is degenerate infinitely on $\Gamma_0 = \{x_1 = 0\}$, and the vector fields $X = (X_1, X_2, X_3)$ satisfies the finite type of Hörmander's condition in $\mathbb{R}^3 \setminus \Gamma_0$.

The example with infinitely degeneracy on a union of surfaces $\Gamma = \bigcup_{j \in J} \Gamma_j$ is the system in $\mathbb{R}^2$ such that $X_1 = \partial_{x_1}$, $X_2 = \exp(-(x_1^2 \sin^2(\frac{\pi}{x_1}))^{\frac{1}{2s}})\partial_{x_2}$, we have $\Gamma_j = \{x_1 = \frac{1}{j}\}$ for $j \in \mathbb{Z} \setminus \{0\}$, $\Gamma_0 = \{x_1 = 0\}$, then X_1 is transverse to all $\Gamma_j, j \in \mathbb{Z}$, and X_2 vanishes infinitely on $\Gamma = \bigcup_{j \in \mathbb{Z}} \Gamma_j$. The vector fields $X = (X_1, X_2)$ satisfies the finite type of Hörmander's condition in $\mathbb{R}^2 \setminus \Gamma$.

Related to the systems of vector fields $X = (X_1, \dots, X_m)$, Morimoto and Xu introduce the following function space (cf.[10]),

$$H_X^1(\tilde{\Omega}) = \left\{ u \in L^2(\tilde{\Omega}), X_j u \in L^2(\tilde{\Omega}), j = 1, \dots, m \right\},$$

which is a Hilbert space with norm $\|u\|_{H_X^1}^2 = \|u\|_{L^2}^2 + \|Xu\|_{L^2}^2$, and $\|Xu\|_{L^2}^2 = \sum_{j=1}^m \|X_j u\|_{L^2}^2$. Take $\Omega \subset\subset \tilde{\Omega}$ as a bounded open subset and suppose that $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X , Morimoto and Xu define the space $H_{X,0}^1(\Omega)$ as a closure of $C_0^\infty(\Omega)$ in $H_X^1(\Omega)$, which is also a Hilbert space.

If the system of vector fields X satisfies the estimates (1.1), we have the following Logarithmic Sobolev inequality;

Proposition 1.1. (cf.[10]) *Suppose that the system of vector fields $X = (X_1, \dots, X_m)$ verifies the estimates (1.1) for some $s > 1/2$. Then there exists $C_0 > 0$ such that*

$$\int_{\Omega} |v|^2 \left| \log\left(\frac{|v|}{\|v\|_{L^2(\Omega)}}\right) \right|^{2s-1} \leq C_0 \left\{ \sum_{j=1}^m \|X_j v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 \right\}, \quad (1.2)$$

for all $v \in H_{X,0}^1(\Omega)$.

Using the Logarithmic Sobolev inequality above, Morimoto and Xu [10] have studied the following semi-linear Dirichlet problems,

$$\Delta_X u = au \log |u| + bu, u|_{\partial\Omega} = 0, \quad (1.3)$$

where constant coefficients $a, b \in \mathbb{R}$. They have obtained,

Proposition 1.2. (cf.[10]) *We suppose that the system of vector fields $X = (X_1, \dots, X_m)$ satisfies the following conditions:*

- $\tilde{H}$ -1) $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X ;
- $\tilde{H}$ -2) the system of vector fields X satisfies the finite type of Hörmander's condition on $\tilde{\Omega}$ except an union of smooth surfaces Γ which are non characteristic for X ;
- $\tilde{H}$ -3) the system of vector fields X satisfies the estimate (1.1) for $s > 3/2$.

Suppose $a \neq 0$ in (1.3), then the semi-linear Dirichlet problem (1.3) posses at least one non trivial weak solution $u \in H_{X,0}^1(\Omega) \cap L^\infty(\Omega)$. Moreover, if $a > 0$, we have $u \in C^\infty(\Omega \setminus \Gamma) \cap C^0(\bar{\Omega} \setminus \Gamma)$ and $u > 0$ for all $x \in \Omega \setminus \Gamma$.

Next, it will be useful for us to introduce following Poincaré's inequality,

Proposition 1.3. (cf.[10]) *Under the hypotheses $\tilde{H}$ -1), $\tilde{H}$ -2) and $\tilde{H}$ -3), the first eigenvalue λ_1 of the operator Δ_X is strictly positive, which is equivalent to following Poincaré's inequality*

$$\|\varphi\|_{L^2}^2 \leq \frac{1}{\lambda_1} \|X\varphi\|_{L^2}^2, \quad \forall \varphi \in H_{X,0}^1(\Omega). \quad (1.4)$$

In this paper, we shall study the following semi-linear Dirichlet problem

$$-\Delta_X u = a(x)u \log |u| + b(x)u + g(x), \quad \text{in } \Omega, \quad (1.5)$$

$$u|_{\partial\Omega} = 0. \quad (1.6)$$

Our main result is as follows.

Theorem 1.4. *Suppose that the system of vector fields $X = (X_1, \dots, X_m)$ satisfies the following conditions:*

- H -1) $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X ;
- H -2) the system of vector fields X satisfies the finite type of Hörmander's condition on $\tilde{\Omega}$ except an union of smooth surfaces Γ which are non characteristic for X ;
- H -3) the system of vector fields X satisfies the estimate (1.1) for $s \geq 5/2$;
- H -4) $a(x), b(x) \in L^\infty(\Omega)$, and there exist $a_0, b_0 \in \mathbb{R}_+$, such that $a(x) \geq a_0$, and $b(x) \geq b_0$, a.e. in Ω . Then

- 1) there exists $C > 0$ such that the problem (1.5) and (1.6) has at least two solutions in $H_{X,0}^1(\Omega)$, for any $g \not\equiv 0$ satisfying $\|g\|_{L^2(\Omega)} < C$;
- 2) the problem (1.5) and (1.6) has at least one non-negative solution $u \in H_{X,0}^1(\Omega)$; furthermore, if $g(x) \in L^\infty(\Omega)$, then the non-negative solution $u(x) \in L^\infty(\Omega)$.
- 3) If $a(x), b(x), g(x) \in C^\infty(\Omega)$, and there exists $g_0 > 0$ such that $g(x) \geq g_0$, then we have $u \in C^\infty(\Omega \setminus \Gamma) \cap C^0(\bar{\Omega} \setminus \Gamma)$ and $u(x) > 0$ for all $x \in \Omega \setminus \Gamma$.

The proof of Theorem 1.4 relies essentially on the Ekeland Variational Principle (cf.[5]) and on the Mountain Pass Theorem without the Palais-Smale condition, established by Brezis-Nirenberg [3], namely

Proposition 1.5. (cf.[5]) *Let V be a complete metric space, and $F : V \rightarrow \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function, $\not\equiv +\infty$, bounded from below. For any $\epsilon > 0$, there is some point $v \in V$ with*

$$F(v) \leq \inf_V F + \epsilon. \quad (1.7)$$

$$\forall w \in V, F(w) \geq F(v) - \epsilon d(v, w). \quad (1.8)$$

Proposition 1.6. (cf.[3]) *Let Φ be a C^1 function on a Banach space E . Suppose there exists a neighborhood U of 0 in E and a constant ρ such that $\Phi(u) \geq \rho$ for every u in the boundary of U ,*

$$\Phi(0) < \rho, \quad \text{and} \quad \Phi(v) < \rho \quad \text{for some } v \notin U.$$

Set

$$c = \inf_{\mathbb{P} \in M} \max_{W \in \mathbb{P}} \Phi(w) \geq \rho,$$

where M denotes the class of paths joining 0 to v .

Conclusion: there is a sequence $\{u_i\}$ in E such that

$$\Phi(u_i) \rightarrow c \quad \text{and} \quad \Phi'(u_i) \rightarrow 0 \quad \text{in } E^*.$$

2 Auxiliary results

Definition 2.1. We say that $u \in H_{X,0}^1(\Omega)$ is a weak solution of (1.5) and (1.6) if

$$\int_{\Omega} \sum_{j=1}^m X_j u X_j v dx - \int_{\Omega} a(x) u v \log |u| dx - \int_{\Omega} b(x) u v dx - \int_{\Omega} g(x) v dx = 0,$$

for all $v \in C_0^\infty(\Omega)$.

We define the function $J_\eta, H_{X,0}^1(\Omega) \rightarrow \mathbb{R}$, $0 \leq \eta < 1$ by

$$\begin{aligned} J_\eta(u) &= \int_\Omega \sum_{j=1}^m (X_j u)^2 dx - \int_\Omega a(x) u^2 \log(|u| + \eta) dx + \int_\Omega \frac{a(x) u^2 |u|}{2(|u| + \eta)} \\ &\quad - \int_\Omega b(x) u^2 dx - 2 \int_\Omega g(x) u dx. \end{aligned}$$

A simple calculation shows that as $0 < \eta < 1$, $J_\eta \in C^1(H_{X,0}^1(\Omega), \mathbb{R})$ and its derivative is given by,

$$\begin{aligned} \langle J'_\eta(u), v \rangle &= 2 \int_\Omega \sum_{j=1}^m (X_j u)(X_j v) - 2 \int_\Omega a(x) uv \log(|u| + \eta) dx \\ &\quad + \int_\Omega \frac{a(x) u |u| v \eta}{2(|u| + \eta)^2} dx - 2 \int_\Omega b(x) uv dx - 2 \int_\Omega g(x) v dx, \end{aligned}$$

for all $u, v \in H_{X,0}^1(\Omega)$.

We have denoted by $\langle \cdot, \cdot \rangle$ the duality pairing between $H_{X,0}^1(\Omega)$ and $H_{X,0}^{-1}(\Omega)$, and $H_{X,0}^{-1}(\Omega)$ is the dual space of $H_{X,0}^1(\Omega)$, i.e. $H_{X,0}^{-1}(\Omega) = (H_{X,0}^1(\Omega))^*$. We use the notation $\rightharpoonup$ as the weak convergence and the notation $\rightarrow$ as the strong convergence in Banach space.

Definition 2.2. If F is a C^1 functional on some Banach space E and c is a real number, we say that a sequence $\{u_n\}$ in E is a $(PS)_c$ sequence of F if $F(u_n) \rightarrow c$ and $F'(u_n) \rightarrow 0$ in E^* .

Remark: If $\{u_n\}$ is a bounded sequence in $H_{X,0}^1(\Omega)$, then there exists a subsequence (denote still by $\{u_n\}$) such that $u_n \rightharpoonup u_0$ in $H_{X,0}^1(\Omega)$, $u_n \rightarrow u_0$ in $L^2(\Omega)$.

Lemma 2.1. Let $M > 0$ and let $\{v_j, j \in \mathbb{N}\}$ be a sequence in $H_{X,0}^1(\Omega)$, satisfying

$$\|v_j\|_{H_{X,0}^1(\Omega)}^2 \leq M.$$

Then $\{|v_j| \log |v_j|\}$ is uniformly integrable.

Proof.

$$\begin{aligned} \int_\Omega |v_j| |\log |v_j||^2 &\leq \frac{1}{2} |\Omega| + \frac{1}{2} \int_\Omega v_j^2 |\log |v_j||^4 dx \\ &= \frac{1}{2} |\Omega| + \frac{1}{2} \int_\Omega v_j^2 \log \frac{|v_j|}{\|v_j\|_{L^2}} + \log \|v_j\|_{L^2}^4 dx \\ &\leq \frac{1}{2} |\Omega| + 4 \int_\Omega v_j^2 \log^4 \frac{|v_j|}{\|v_j\|_{L^2}} + 4 |\log \|v_j\|_{L^2}|^4 \|v_j\|_{L^2}^2 \\ &\leq \frac{1}{2} |\Omega| + 4C_0 (\|Xv_j\|_{L^2}^2 + \|v_j\|_{L^2}^2) + 4 |\log \|v_j\|_{L^2}|^4 \|v_j\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}|\Omega| + 4C_0(\|Xv_j\|_{L^2}^2 + \|v_j\|_{L^2}^2) + \frac{4}{2^4}|\log \|v_j\|_{L^2}^2|^4 \|v_j\|_{L^2}^2 \\
&\leq \frac{1}{2}|\Omega| + 4C_0M + \frac{4}{2^4}[(4e^{-1})^4 + (\log M)^4 M] \\
&= \tilde{M},
\end{aligned}$$

where $C_0 > 0$ is a positive constant given by Proposition 1.1. We use the fact $t(\log t)^4 \leq l \log^4 l + (4e^{-1})^4$ for any $0 \leq t \leq l$.

Now, we prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subset \Omega$, the measure of A , $\mu(A) < \delta$, then

$$\int_A |v_j| |\log |v_j|| < \epsilon, \quad \forall j.$$

But for any $\epsilon > 0$, there exists $t_0 > e^2$ such that

$$\frac{1}{\log t} < \epsilon, \quad \forall t \geq t_0.$$

Take now $\delta = \epsilon(t_0 \log t_0)^{-1}$, $\mu(A) < \delta$ and

$$A_j = A \cap \{|v_j| \leq t_0\}, \quad B_j = A \cap \{|v_j| > t_0\},$$

then we have,

$$\begin{aligned}
&\int_{A_j} |v_j| |\log |v_j|| \leq t_0 \log t_0 \mu(A_j) < \epsilon, \\
&\int_{B_j} |v_j| |\log |v_j|| \leq \epsilon \int_{B_j} |v_j| |\log |v_j||^2 < \epsilon \tilde{M}.
\end{aligned}$$

The proof of Lemma 2.1 is complete.

Lemma 2.2. *If $a(x) \in L^\infty(\Omega)$, $\zeta \in C_0^\infty(\Omega)$, $\|u_n\|_{H_{X,0}^1(\Omega)} < M$, M is a positive constant independent of n , then there exists a convergent subsequence (denote still by $\{u_n\}$) such that $u_n \rightharpoonup u_0$ in $H_{X,0}^1(\Omega)$ and*

$$\lim_{n \rightarrow \infty} \int_{\Omega} a(x) u_n \zeta \log(|u_n| + 1/2^n) dx = \int_{\Omega} a(x) u_0 \zeta \log(|u_0|) dx.$$

Proof. We have

$$\begin{aligned}
&\int_{\Omega} |a(x) u_n \zeta| |\log(|u_n| + 2^{-n})|^2 dx \leq C \int_{\Omega} |u_n| |\log(|u_n| + 2^{-n})|^2 dx \\
&\leq C \int_{\{x: |u_n| + 2^{-n} \leq 1\}} |u_n| |\log(|u_n| + 2^{-n})|^2 dx \\
&+ C \int_{\{x: |u_n| + 2^{-n} \geq 1\}} |u_n| |\log(|u_n| + 2^{-n})|^2 dx
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_{\{x: |u_n|+2^{-n} \leq 1\}} |u_n| |\log(|u_n|)|^2 dx \\
&+ C \int_{\{x: |u_n|+2^{-n} \geq 1\}} |u_n| |\log(2|u_n|)|^2 dx \\
&\leq C \int_{\{x: |u_n|+2^{-n} \leq 1\}} |u_n| |\log(|u_n|)|^2 dx \\
&+ C \int_{\{x: |u_n|+2^{-n} \geq 1\}} |u_n| (\log^2 2 + |\log(|u_n|)|^2) dx \\
&\leq C \int_{\Omega} |u_n| |\log(|u_n|)|^2 dx + C \left(\int_{\Omega} |u_n|^2 dx + |\Omega| \right),
\end{aligned}$$

since $a(x) \in L^\infty(\Omega)$, $\zeta \in C_0^\infty(\Omega)$. By the proof of Lemma 2.1, we know there exists $\tilde{M}$, such that

$$\int_{\Omega} |a(x)u_n \zeta| |\log(|u_n| + 2^{-n})|^2 dx \leq \tilde{M}.$$

Next, we prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subset \Omega$, $\mu(A) < \delta$, then

$$\int_A |a(x)u_n \zeta| |\log(|u_n| + 2^{-n})| dx < \epsilon, \quad \forall n.$$

But for any $\epsilon > 0$, there exists $t_0 > e^2$ such that

$$\frac{1}{\log t} < \epsilon, \quad \forall t \geq t_0.$$

Take now $\delta = \epsilon \{a_\infty \max_{x \in \Omega} |\zeta(x)| [(t_0 + 2^{-1})^2 + e^{-1}]\}^{-1}$, $\mu(A) < \delta$, $a_\infty = \|a(x)\|_{L^\infty(\Omega)}$ and

$$A_n = A \cap \{|u_n| \leq t_0\}, \quad B_n = A \cap \{|u_n| > t_0\},$$

then we have,

$$\begin{aligned}
&\int_{A_n} |a(x)u_n \zeta| |\log(|u_n| + 2^{-n})| dx \\
&\leq a_\infty \max_{x \in \Omega} |\zeta(x)| \int_{A_n} |u_n| |\log(|u_n| + 2^{-n})| dx \\
&\leq a_\infty \max_{x \in \Omega} |\zeta(x)| \int_{A_n} [(|u_n| + 2^{-n})^2 + e^{-1}] \\
&\leq a_\infty \max_{x \in \Omega} |\zeta(x)| [(|t_0| + 2^{-1})^2 + e^{-1}] \mu(A_n) \\
&< \epsilon,
\end{aligned}$$

$$\int_{B_n} |a(x)u_n \zeta| |\log(|u_n| + 2^{-n})| dx < \epsilon \int_{B_n} |a(x)u_n \zeta| |\log(|u_n| + 2^{-n})|^2 dx < \epsilon \tilde{M}.$$

Similarly, we can prove that

Lemma 2.3. For any fixed $0 < \eta \ll 1$, $a(x) \in L^\infty(\Omega)$, $\zeta \in C_0^\infty(\Omega)$, $\|u_n\|_{H_{X,0}^1(\Omega)} < M$, M is a positive constant independent of n , there exists a convergent subsequence (denote still by $\{u_n\}$) such that $u_n \rightharpoonup u_0$ in $H_{X,0}^1(\Omega)$, and

$$\lim_{n \rightarrow \infty} \int_{\Omega} a(x) u_n \zeta \log(|u_n| + \eta) dx = \int_{\Omega} a(x) u_0 \zeta \log(|u_0| + \eta) dx.$$

Lemma 2.4. For any fixed $0 < \eta \ll 1$, $a(x) \in L^\infty(\Omega)$, $u(x) \in H_{X,0}^1(\Omega)$, $u_n \in C_0^\infty(\Omega)$ and $\|u_n - u\|_{H_{X,0}^1(\Omega)} \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} a(x) u u_n \log(|u_n| + \eta) dx = \int_{\Omega} a(x) u^2 \log(|u| + \eta) dx.$$

Lemma 2.5. If $a(x) \in L^\infty(\Omega)$, $u(x) \in H_{X,0}^1(\Omega)$, $u_n \in C_0^\infty(\Omega)$ and $\|u_n - u\|_{H_{X,0}^1(\Omega)} \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} a(x) u u_n \log(|u_n|) dx = \int_{\Omega} a(x) u^2 \log(|u|) dx.$$

Similar to Lemma 2.1, we have

Lemma 2.6. Let $M > 0$ and let $\{v_j, j \in \mathbb{N}\}$ be a sequence in $H_{X,0}^1(\Omega)$ satisfying

$$\|v_j\|_{H_{X,0}^1(\Omega)}^2 \leq M.$$

Then there exists a convergent sub-sequence $\{v_{j_k}\}$ such that $v_{j_k} \rightharpoonup v_0 \in H_{X,0}^1(\Omega)$ and

$$\lim_{j_k \rightarrow \infty} \int_{\Omega} |v_{j_k}|^2 |\log|v_{j_k}|| = \int_{\Omega} |v_0|^2 |\log|v_0||,$$

and

$$\int_{\Omega} |v_0|^2 |\log|v_0|| \leq CM,$$

where C is a positive constant independent of j .

Proof. Using the fact $|t \log t| \leq t^2 + e^{-1}$, for $\forall t > 0$, we have

$$\begin{aligned} \int_{\Omega} |v_j|^2 |\log|v_j||^2 &= \int_{\Omega} |v_j|^2 \left| \log \frac{|v_j|}{\|v_j\|_{L^2}} + \log \|v_j\|_{L^2} \right|^2 \\ &\leq 2 \int_{\Omega} |v_j|^2 \left| \log \frac{|v_j|}{\|v_j\|_{L^2}} \right|^2 + 2 \|v_j\|_{L^2(\Omega)}^2 |\log \|v_j\|_{L^2}|^2 \\ &\leq 2C_0 (\|X v_j\|_{L^2}^2 + \|v_j\|_{L^2}^2) + 2(M + e^{-1})^2 \\ &\leq 2C_0 M + 4(M^2 + e^{-2}) \\ &= \tilde{M}, \end{aligned}$$

C_0 is a positive constant given by Proposition 1.1. The rest of the proof is similar to the proof of Lemma 2.1.

Next, we can prove that for any $\epsilon > 0$, there exists $\delta > 0$, such that if $A \subset \Omega$, $\mu(A) < \delta$, then

$$\int_A |v_j|^2 |\log |v_j|| dx < \epsilon, \quad \forall j.$$

Actually for any $\epsilon > 0$, there exists $t_0 > e^2$, such that

$$\frac{1}{\log t} < \epsilon, \quad \forall t \geq t_0.$$

Take now $\delta = \epsilon(t_0^2 \log t_0 + \frac{1}{2}e^{-1})^{-1}$, $\mu(A) < \delta$ and

$$A_j = A \cap \{|v_j| \leq t_0\}, \quad B_j = A \cap \{|v_j| > t_0\},$$

then we have,

$$\int_{A_j} |v_j|^2 |\log |v_j|| dx \leq \int_{A_j} (t_0^2 \log t_0 + \frac{1}{2}e^{-1}) < (t_0^2 \log t_0 + \frac{1}{2}e^{-1})\mu(A_j) < \epsilon,$$

$$\int_{B_j} |v_j|^2 |\log |v_j|| dx \leq \epsilon \int_{B_j} |v_j|^2 |\log |v_j||^2 dx < \epsilon \tilde{M}.$$

Thus we have

Lemma 2.7. *For any fixed $0 < \eta < 1$, $a(x) \in L^\infty(\Omega)$, $u_n \in H_{X,0}^1(\Omega)$ and $\|u_n\|_{H_{X,0}^1(\Omega)} < M$, (M is a positive constant independent of n) there exists a convergent subsequence (denote still by $\{u_n\}$) such that $u_n \rightharpoonup u_0$ in $H_{X,0}^1(\Omega)$ and*

$$\lim_{n \rightarrow \infty} \int_{\Omega} a(x) u_n u_0 \log(|u_n| + \eta) dx = \int_{\Omega} a(x) u_0^2 \log(|u_0| + \eta) dx.$$

Lemma 2.8. *If $a(x) \in L^\infty(\Omega)$, $\|u_n\|_{H_{X,0}^1(\Omega)} < M$, M is a positive constant independent of n , then there exists a convergent subsequence (denote still by $\{u_n\}$) such that $u_n \rightharpoonup u_0$ in $H_{X,0}^1(\Omega)$ and*

$$\lim_{n \rightarrow \infty} \int_{\Omega} a(x) u_n^2 \log(|u_n| + 1/2^n) dx = \int_{\Omega} a(x) u_0^2 \log(|u_0|) dx.$$

3 The existence of solutions

For any fixed $0 < \epsilon < 1$, $0 < \eta \ll 1$ and $u \in H_{X,0}^1(\Omega)$, by using Young's inequality, Proposition 1.1 and Proposition 1.3, we have,

$$\begin{aligned}
J_\eta(u) &= \|Xu\|_{L^2(\Omega)}^2 - \int_\Omega a(x)u^2 \log(|u| + \eta)dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u| + \eta)} - \int_\Omega b(x)u^2 dx \\
&\quad - 2 \int_\Omega g(x)u dx \\
&= \|Xu\|_{L^2(\Omega)}^2 - \int_{|u|>\eta} a(x)u^2 \log(|u| + \eta)dx - \int_{|u|\leq\eta} a(x)u^2 \log(|u| + \eta)dx \\
&\quad + \int_\Omega \frac{a(x)u^2|u|}{2(|u| + \eta)} - \int_\Omega b(x)u^2 dx - 2 \int_\Omega g(x)u dx \\
&\geq \|Xu\|_{L^2(\Omega)}^2 - \int_{|u|>\eta} a(x)u^2 \log 2|u|dx - \log 2\eta \int_{|u|\leq\eta} a(x)u^2 dx \\
&\quad + \int_\Omega \frac{a(x)u^2|u|}{2(|u| + \eta)} - \int_\Omega b(x)u^2 dx - 2 \int_\Omega g(x)u dx \\
&\geq \|Xu\|_{L^2(\Omega)}^2 - \log 2 \int_{|u|>\eta} a(x)u^2 dx - \int_{|u|>\eta} a(x)u^2 (\log \frac{|u|}{\|u\|_{L^2}} + \log \|u\|_{L^2})dx \\
&\quad - a_0 \log 2\eta \int_{|u|\leq\eta} u^2 dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u| + \eta)} - \int_\Omega b(x)u^2 dx - 2 \int_\Omega g(x)u dx \\
&> \|Xu\|_{L^2(\Omega)}^2 - a_\infty \log 2 \int_\Omega u^2 dx - \frac{\epsilon}{C_0} \int_\Omega u^2 \log^2 \frac{|u|}{\|u\|_{L^2}} - \frac{C_0}{4\epsilon} \int_\Omega a^2(x)u^2 \\
&\quad - \log \|u\|_{L^2} \int_{|u|>\eta} a(x)u^2 - a_0 \log 2\eta \int_{|u|\leq\eta} u^2 dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u| + \eta)} \\
&\quad - b_\infty \int_\Omega u^2 dx - \int_\Omega g^2(x)dx - \int_\Omega u^2(x)dx \\
&> \|Xu\|_{L^2(\Omega)}^2 - a_\infty \log 2 \int_\Omega u^2 dx - \epsilon(\|Xu\|_{L^2}^2 + \|u\|_{L^2}^2) - \frac{C_0 a_\infty^2}{4\epsilon} \int_\Omega u^2 \\
&\quad - \log \|u\|_{L^2} \int_{|u|>\eta} a(x)u^2 - a_0 \log 2\eta \int_{|u|\leq\eta} u^2 dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u| + \eta)} \\
&\quad - b_\infty \int_\Omega u^2 dx - \int_\Omega g^2(x)dx - \int_\Omega u^2(x)dx \\
&> (1 - \epsilon) \frac{\lambda_1}{1 + \lambda_1} \|u\|_{H_{X,0}^1(\Omega)}^2 - C_1 \|u\|_{L^2(\Omega)}^2 - \log \|u\|_{L^2} \int_{|u|>\eta} a(x)u^2 dx \\
&\quad - a_0 \log 2\eta \int_{|u|\leq\eta} u^2 dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u| + \eta)} - \|g\|_{L^2(\Omega)}^2
\end{aligned}$$

$$\begin{aligned}
&> (1 - \epsilon) \frac{\lambda_1}{1 + \lambda_1} \|u\|_{H_{X,0}^1(\Omega)}^2 - C_1 \|u\|_{L^2(\Omega)}^2 - \log \|u\|_{H_{X,0}^1(\Omega)} \int_{|u|>\eta} a(x) u^2 dx \\
&- a_0 \log 2\eta \int_{|u|\leq\eta} u^2 dx + \int_{\Omega} \frac{a(x) u^2 |u|}{2(|u| + \eta)} - \|g\|_{L^2(\Omega)}^2,
\end{aligned}$$

where $C_1 = a_\infty \log 2 + \epsilon + \frac{C_0}{4\epsilon} a_\infty^2 + b_\infty + 1$, $C_0 > 0$ is a positive constant given by Proposition 1.1, $a_\infty = \|a\|_{L^\infty}$, $b_\infty = \|b\|_{L^\infty}$.

If we set $B_R = \{u \in H_{X,0}^1(\Omega), \|u\|_{H_{X,0}^1(\Omega)} < R\}$, the estimate above shows that, as η is small enough, there exist $R = R(\epsilon) > 0$, and $\delta = \delta(R) > 0$ such that $J_\eta(u)|_{\partial B_R} \geq \delta > 0$ for all g with $\|g\|_{L^2(\Omega)} \leq C$. For example, we can take,

$$R(\epsilon) = \exp\left\{\frac{C_1}{-a_0}\right\}, \quad C = C(\epsilon) = \frac{R}{2} \sqrt{\frac{\lambda_1(1 - \epsilon)}{1 + \lambda_1}},$$

$$\delta(R) = \frac{\lambda_1(1 - \epsilon)}{8(1 + \lambda_1)} R^2(\epsilon), \quad \eta < \frac{1}{2} \exp\left\{\frac{C_1}{-a_0}\right\}.$$

Define $c_\eta = c_\eta(R) = \inf_{u \in \bar{B}_R} J_\eta(u)$, then $c_\eta \leq J_\eta(0) = 0$. The set $\bar{B}_R$ becomes a complete metric space with respect to the distance,

$$\text{dist}(u, v) = \|u - v\|_{H_{X,0}^1(\Omega)} \text{ for any } u, v \in \bar{B}_R.$$

On the other hand, J_η is lower semi-continuous and bounded from below on $\bar{B}_R$. So, by Proposition 1.5 (cf. [5] Theorem 1.1), for any positive integer n there exists $\{u_{\eta,n}\}$, satisfying

$$c_\eta \leq J_\eta(u_{\eta,n}) \leq c_\eta + \frac{1}{n} \quad (3.1)$$

$$J_\eta(w) \geq J_\eta(u_{\eta,n}) - \frac{1}{n} \|u_{\eta,n} - w\|_{H_{X,0}^1(\Omega)} \text{ for all } w \in \bar{B}_R. \quad (3.2)$$

We claim that $0 < \|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} < R$ for any n large enough. Indeed, if $\|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} = R$ for infinitely many n , we may assume, without loss of generality, that $\|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} = R$ for all $n \geq 1$. It follows that $J_\eta(u_{\eta,n}) \geq \delta > 0$. Combining this with (3.1) and letting $n \rightarrow \infty$, we have $0 \geq c_\eta \geq \delta > 0$ which is a contradiction.

We now prove that $J'_\eta(u_{\eta,n}) \rightarrow 0$ as $n \rightarrow \infty$ in $H_{X,0}^{-1}(\Omega)$. Indeed, for any $u \in H_{X,0}^{-1}(\Omega)$ with $\|u\|_{H_{X,0}^1(\Omega)} = 1$, let $w_n = u_{\eta,n} + tu$. For a fixed n , we have $\|w_n\|_{H_{X,0}^1(\Omega)} \leq \|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} + t < R$, where $t > 0$ is small enough. From (3.2) we obtain

$$J_\eta(u_{\eta,n} + tu) \geq J_\eta(u_{\eta,n}) - \frac{t}{n} \|u\|_{H_{X,0}^1(\Omega)},$$

that is

$$\frac{J_\eta(u_{\eta,n} + tu) - J_\eta(u_{\eta,n})}{t} \geq -\frac{1}{n} \|u\|_{H_{X,0}^1(\Omega)} = -\frac{1}{n}.$$

Letting $t \searrow 0$, we deduce that $\langle J'_\eta(u_{\eta,n}), u \rangle \geq -1/n$ and a similar argument for $t \nearrow 0$ produces $|\langle J'_\eta(u_{\eta,n}), u \rangle| \leq 1/n$ for any $u \in H_{X,0}^1(\Omega)$ with $\|u\|_{H_{X,0}^1(\Omega)} = 1$. So

$$\|J'_\eta(u_{\eta,n})\|_{-1} = \sup_{\substack{u \in H_{X,0}^1(\Omega) \\ \|u\|_{H_{X,0}^1(\Omega)}=1}} |\langle J'_\eta(u_{\eta,n}), u \rangle| \leq \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.3)$$

Thus, $\{u_{\eta,n}\}$ is a $(PS)_{c_\eta}$ sequence in $H_{X,0}^1(\Omega)$, i.e.

$$J_\eta(u_{\eta,n}) \rightarrow c_\eta, \text{ and } J'_\eta(u_{\eta,n}) \rightharpoonup 0 \text{ in } H_{X,0}^{-1}(\Omega). \quad (3.4)$$

Since $\|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} \leq R$, $\{u_{\eta,n}\}$ is a bounded sequence in $H_{X,0}^1(\Omega)$, and passing to a subsequence (denote still by $\{u_{\eta,n}\}$), we may assume that $u_{\eta,n} \rightharpoonup u_{\eta,0}$ in $H_{X,0}^1(\Omega)$ for some $u_{\eta,0} \in H_{X,0}^1(\Omega)$. So, by Lemma 2.3, we know that $J'_\eta(u_{\eta,0}) = 0$, i.e.

$$\begin{aligned} 2 \int_\Omega \sum_{j=1}^m (X_j u_{\eta,0})(X_j v) &= 2 \int_\Omega a(x) u_{\eta,0} v \log(|u_{\eta,0}| + \eta) dx + \int_\Omega \frac{a(x) u_{\eta,0}^2 v \eta}{2(|u_{\eta,0}| + \eta)^2} dx \\ &- 2 \int_\Omega b(x) u_{\eta,0} v dx - 2 \int_\Omega g(x) v dx = 0, \end{aligned}$$

for all $v \in C_0^\infty(\Omega)$.

We know $\{u_{\eta,0}\}$ is also bounded in $H_{X,0}^1(\Omega)$. For $\eta = \eta_i = \frac{1}{2^i}$, $\frac{1}{2^i} < \frac{1}{2} \exp\{\frac{C_1}{-a_0}\}$, passing to a subsequence (denote still by $\{u_{\eta,n}\}$), we may assume that $u_{\eta_i,0} \rightharpoonup u_0$ in $H_{X,0}^1(\Omega)$ as $i \rightarrow \infty$. Now by Lemma 2.2, we have,

$$\int_\Omega \sum_{j=1}^m (X_j u_0)(X_j v) - \int_\Omega a(x) u_0 v \log |u_0| dx - \int_\Omega b(x) u_0 v dx - \int_\Omega g(x) v dx = 0, \quad (3.5)$$

u_0 is a weak solution of (1.5) and (1.6).

We can prove that $J_0(u_0) = c_0$. Actually, we have

$$\begin{aligned} J_\eta(u_{\eta,n}) + \frac{1}{2} \|J'_\eta(u_{\eta,n})\|_{-1} \|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} &\geq J_\eta(u_{\eta,n}) - \frac{1}{2} \langle J'_\eta(u_{\eta,n}), u_{\eta,n} \rangle \\ &= \int_\Omega \frac{a(x) u_{\eta,n}^2 |u_{\eta,n}|}{2(|u_{\eta,n}| + \eta)} - \int_\Omega \frac{a(x) u_{\eta,n}^2 |u_{\eta,n}| \eta}{4(|u_{\eta,n}| + \eta)^2} - \int_\Omega g u_{\eta,n}. \end{aligned}$$

Letting $n \rightarrow \infty$, we know

$$c_\eta \geq \int_\Omega \frac{a(x) u_{\eta,0}^2 |u_{\eta,0}|}{2(|u_{\eta,0}| + \eta)} - \int_\Omega \frac{a(x) u_{\eta,0}^2 |u_{\eta,0}| \eta}{4(|u_{\eta,0}| + \eta)^2} - \int_\Omega g u_{\eta,0}. \quad (3.6)$$

By Lemma 2.7, we have

$$\begin{aligned} 0 &= \langle J'_{\eta_i}(u_{\eta_i,0}), u_{\eta_i,0} \rangle = 2 \|X u_{\eta_i,0}\|_{L^2}^2 - 2 \int_\Omega a(x) u_{\eta_i,0}^2 \log(|u_{\eta_i,0}| + \eta_i) dx \\ &+ \int_\Omega \frac{a(x) u_{\eta_i,0}^2 |u_{\eta_i,0}| \eta_i}{2(|u_{\eta_i,0}| + \eta_i)^2} dx - 2 \int_\Omega b(x) u_{\eta_i,0}^2 dx - 2 \int_\Omega g(x) u_{\eta_i,0} dx. \end{aligned}$$

Therefore

$$\begin{aligned} J_{\eta_i}(u_{\eta_i,0}) &= \int_{\Omega} \frac{a(x)u_{\eta_i,0}^2|u_{\eta_i,0}|}{2(|u_{\eta_i,0}| + \eta_i)} dx - \int_{\Omega} \frac{a(x)u_{\eta_i,0}^2|u_{\eta_i,0}|\eta_i}{4(|u_{\eta_i,0}| + \eta_i)^2} dx \\ &\quad - \int_{\Omega} g(x)u_{\eta_i,0} dx. \end{aligned} \quad (3.7)$$

By (3.5), (3.6) and (3.7), we have:

$$\begin{aligned} 0 \geq c_0 &= \inf_{u \in \bar{B}_R} J_0(u) \geq \lim_{i \rightarrow \infty} \inf_{u \in \bar{B}_R} J_{\eta_i}(u) = \lim_{i \rightarrow \infty} c_{\eta_i} \\ &\geq \frac{1}{2} \int_{\Omega} a(x)u_0^2 dx - \int_{\Omega} g(x)u_0 dx = J_0(u_0). \end{aligned}$$

Since $u_0 \in \bar{B}_R$, it follows that $J_0(u_0) = c_0$.

On the other hand, letting $\tilde{u} \in H_{X,0}^1(\Omega)$, $\|\tilde{u}\|_{H_{X,0}^1(\Omega)} = R$, and $t > 0$, we have

$$\begin{aligned} J_{\eta}(t\tilde{u}) &< J_0(t\tilde{u}) = t^2 \left[\|X\tilde{u}\|_{L^2(\Omega)}^2 - \log t \int_{\Omega} a(x)\tilde{u}^2 - \int_{\Omega} a(x)\tilde{u}^2 \log |\tilde{u}| \right. \\ &\quad \left. + \frac{1}{2} \int_{\Omega} a(x)\tilde{u}^2 - \int_{\Omega} b(x)\tilde{u}^2 - 2 \int_{\Omega} g(x)\tilde{u}/t \right] \\ &< t^2 \left[\|X\tilde{u}\|_{L^2(\Omega)}^2 - \log t \int_{\Omega} a(x)\tilde{u}^2 - \int_{\Omega} a(x)\tilde{u}^2 \log |\tilde{u}| \right. \\ &\quad \left. + \frac{1}{2} \int_{\Omega} a(x)\tilde{u}^2 - \int_{\Omega} b(x)\tilde{u}^2 + \frac{1}{t} \left(\int_{\Omega} g^2(x) + \int_{\Omega} \tilde{u}^2 \right) \right]. \end{aligned}$$

We can find $\bar{t} >> 1$, such that $J_{\eta}(t\tilde{u}) < J_0(t\tilde{u}) < 0$ for all $t \geq \bar{t}$. Letting $\bar{u} = \bar{t}\tilde{u}$, then we have $\|\bar{u}\|_{H_{X,0}^1(\Omega)} > R$ and $J_{\eta}(\bar{u}) < 0$.

We put

$$\varrho = \{\gamma \in C([0, 1], H_{X,0}^1(\Omega)) : \gamma(0) = 0, \gamma(1) = \bar{t}\tilde{u}, \}, \quad (3.8)$$

$$\bar{c}_{\eta} = \inf_{\gamma \in \varrho} \sup_{u \in \gamma} J_{\eta}(u). \quad (3.9)$$

For $\gamma_0 = \{t\bar{t}\tilde{u} : 0 \leq t \leq 1\}$, we have

$$\begin{aligned} \sup_{u \in \gamma_0} J_{\eta}(u) &\leq \sup_{u \in \gamma_0} J_0(u) = \sup_{0 \leq t \leq 1} \left[(t\bar{t})^2 \|X\tilde{u}\|_{L^2(\Omega)}^2 - (t\bar{t})^2 \log(t\bar{t}) \int_{\Omega} a(x)\tilde{u}^2 \right. \\ &\quad \left. - (t\bar{t})^2 \int_{\Omega} a(x)\tilde{u}^2 \log |\tilde{u}| + \frac{(t\bar{t})^2}{2} \int_{\Omega} a(x)\tilde{u}^2 - (t\bar{t})^2 \int_{\Omega} b(x)\tilde{u}^2 - 2(t\bar{t}) \int_{\Omega} g(x)\tilde{u} \right] \\ &\leq \bar{t}^2 \|X\tilde{u}\|_{L^2(\Omega)}^2 + \frac{1}{2e} \int_{\Omega} a(x)\tilde{u}^2 + \bar{t}^2 \int_{\Omega} a(x)\tilde{u}^2 |\log |\tilde{u}|| + \frac{\bar{t}^2}{2} \int_{\Omega} a(x)\tilde{u}^2 \\ &\quad + \bar{t} \int_{\Omega} g^2 + \bar{t} \int_{\Omega} \tilde{u}^2. \end{aligned}$$

So there exists a positive constant B (which is independent of η), satisfying

$$\bar{c}_\eta \leq B. \quad (3.10)$$

It follows from the Proposition 1.6 (cf. [3] Theorem 2.2) that there is a $(PS)_{c_\eta}$ sequence $\{u_{\eta,n}\}$ of $J_\eta(u)$ such that

$$J_\eta(u_{\eta,n}) = \bar{c}_\eta + o(1) \text{ and } J'_\eta(u_{\eta,n}) \rightarrow 0 \quad \text{in} \quad H_{X,0}^{-1}(\Omega).$$

We have

$$\begin{aligned} J_\eta(u) - \frac{1}{2} \langle J'_\eta(u), u \rangle &= \int_\Omega \frac{a(x)u^2|u|}{2(|u| + \eta)} dx - \int_\Omega \frac{a(x)u^2|u|\eta}{4(|u| + \eta)^2} dx - \int_\Omega g(x)u dx \\ &> \int_\Omega \frac{a(x)u^2|u|}{4(|u| + \eta)} dx - \frac{a_0}{16} \int_\Omega u^2 dx - \frac{4}{a_0} \int_\Omega g^2 dx \\ &= \int_{|u|>\eta} \frac{a(x)u^2|u|}{4(|u| + \eta)} dx + \int_{|u|\leq\eta} \frac{a(x)u^2|u|}{4(|u| + \eta)} dx - \frac{a_0}{16} \int_{|u|>\eta} u^2 dx - \frac{a_0}{16} \int_{|u|\leq\eta} u^2 dx \\ &\quad - \frac{4}{a_0} \int_\Omega g^2 dx \\ &> \frac{1}{4} \int_{|u|>\eta} \frac{a(x)u^2|u|}{2|u|} dx - \frac{a_0}{16} \int_{|u|>\eta} u^2 dx - \frac{a_0}{16} \int_{|u|\leq\eta} u^2 dx - \frac{4}{a_0} \int_\Omega g^2 dx \\ &> \frac{a_0}{16} \int_{|u|>\eta} u^2 dx - \frac{a_0\eta^2|\Omega|}{16} - \frac{4}{a_0} \|g\|_{L^2}^2. \end{aligned}$$

So, we have

$$\begin{aligned} &\bar{c}_\eta + o(1) + \frac{1}{2} \|J'_\eta(u_{\eta,n})\|_{-1} \|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} + \frac{a_0|\Omega|}{16} + \frac{4}{a_0} \|g\|_{L^2}^2 \\ &\geq J_\eta(u_{\eta,n}) - \frac{1}{2} \langle J'_\eta(u_{\eta,n}), u_{\eta,n} \rangle + \frac{a_0|\Omega|}{16} + \frac{4}{a_0} \|g\|_{L^2}^2 \\ &> \frac{a_0}{16} \int_{|u|>\eta} u_{\eta,n}^2 dx. \end{aligned}$$

By (3.10), we have

$$\begin{aligned} &\int_\Omega |u_{\eta,n}|^2 dx = \int_{|u|>\eta} |u_{\eta,n}|^2 dx + \int_{u\leq\eta} |u_{\eta,n}|^2 dx \\ &< \frac{16}{a_0} \left[\bar{c}_\eta + o(1) + \frac{1}{2} \|J'_\eta(u_{\eta,n})\|_{-1} \|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} + \frac{a_0|\Omega|}{16} + \frac{4}{a_0} \|g\|_{L^2}^2 \right] + \eta^2 |\Omega| \\ &< \frac{16}{a_0} \left[B + o(1) + \frac{1}{2} \|J'_\eta(u_{\eta,n})\|_{-1} \|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} + \frac{a_0|\Omega|}{16} + \frac{4}{a_0} \|g\|_{L^2}^2 \right] + |\Omega| \\ &< C + C \|J'_\eta(u_{\eta,n})\|_{-1} \|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} + o(1), \end{aligned} \quad (3.11)$$

where C is a positive constant which is independent of η and n , and dependent of $|\Omega|$, $\|g\|_{L^2}^2$, a_0 , and B . Similar to the estimate of $J_\eta(u)$ at the beginning of this section, we have (if taking $\epsilon = \frac{1}{2}$)

$$\begin{aligned} B + o(1) &> \bar{c}_\eta + o(1) = J_\eta(u_{\eta,n}) \geq \frac{\lambda_1}{2(1 + \lambda_1)} \|u_{\eta,n}\|_{H_{X,0}^1}^2 - C_1 \|u_{\eta,n}\|_{L^2}^2 \\ &- a_\infty \|u_{\eta,n}\|_{L^2}^2 |\log \|u_{\eta,n}\|_{L^2}| - \|g\|_{L^2(\Omega)}^2, \end{aligned}$$

where $C_1 = a_\infty \log 2 + \frac{C_0}{2} a_\infty^2 + b_\infty + \frac{3}{2}$, to be independent of η and n , and C_0 and λ_1 are given by Proposition 1.1 and Proposition 1.3 respectively.

Furthermore, using the fact $|t \log t| \leq t^2 + e^{-1}$ for $t \geq 0$, we have

$$\begin{aligned} \frac{\lambda_1}{2(1 + \lambda_1)} \|u_{\eta,n}\|_{H_{X,0}^1}^2 &\leq B + o(1) + C_1 \|u_{\eta,n}\|_{L^2}^2 + a_\infty \|u_{\eta,n}\|_{L^2}^2 |\log \|u_{\eta,n}\|_{L^2}| + \|g\|_{L^2(\Omega)}^2 \\ &\leq B + o(1) + C_1 \|u_{\eta,n}\|_{L^2}^2 + \frac{1}{2} a_\infty (\|u_{\eta,n}\|_{L^2}^4 + e^{-1}) + \|g\|_{L^2(\Omega)}^2 \\ &< C + o(1) + C \|u_{\eta,n}\|_{L^2}^2 + C \|u_{\eta,n}\|_{L^2}^4, \end{aligned}$$

where C is independent of η and n .

By (3.11), we have

$$\|u_{\eta,n}\|_{H_{X,0}^1}^2 \leq C + o(1) + C \|J'_\eta(u_{\eta,n})\|_{-1} \|u_{\eta,n}\|_{H_{X,0}^1(\Omega)} + C \|J'_\eta(u_{\eta,n})\|_{-1}^2 \|u_{\eta,n}\|_{H_{X,0}^1(\Omega)}^2.$$

Since $J'_\eta(u_{\eta,n}) \rightarrow 0$ in $H_{X,0}^{-1}(\Omega)$, thus there exists $N_0 > 0$ such that $\|u_{\eta,n}\|_{H_{X,0}^1}^2 \leq M$, if $n > N_0$, where M is a constant, independent of η and n . That means $\{u_{\eta, N_0+j}\}_{j \in \mathbb{N}}$ is a bounded sequence in $H_{X,0}^1(\Omega)$. Hence there exists a subsequence (we still denote by $\{u_{\eta,n}\}$), such that $u_{\eta,n} \rightharpoonup u_{\eta,0}$ in $H_{X,0}^1(\Omega)$ for some $u_{\eta,0} \in H_{X,0}^1(\Omega)$. By Lemma 2.3, we have $J'_\eta(u_{\eta,0}) = 0$, that is

$$\begin{aligned} &2 \int_\Omega \sum_{j=1}^m (X_j u_{\eta,0})(X_j v) - 2 \int_\Omega a(x) u_{\eta,0} v \log(|u_{\eta,0}| + \eta) dx \\ &+ \int_\Omega \frac{a(x) u_{\eta,0} |u_{\eta,0}| v \eta}{2(u_{\eta,0} + \eta)^2} dx - 2 \int_\Omega b(x) u_{\eta,0} v dx - 2 \int_\Omega g(x) v dx = 0, \end{aligned} \quad (3.12)$$

for any $v \in C_0^\infty(\Omega)$.

For $\eta = \eta_i = \frac{1}{2^i}$, $\frac{1}{2^i} < \frac{1}{2} \exp\{\frac{C_1}{-a_0}\}$, we know $\{u_{\eta_i,0}\}$ is also bounded in $H_{X,0}^1(\Omega)$. Passing to a subsequence, we may assume that $u_{\eta_i,0} \rightharpoonup u_1$ in $H_{X,0}^1(\Omega)$ as $i \rightarrow \infty$. Now by Lemma 2.2, we have,

$$\begin{aligned} &\int_\Omega \sum_{j=1}^m (X_j u_1)(X_j v) - \int_\Omega a(x) u_1 v \log |u_1| dx - \int_\Omega b(x) u_1 v dx \\ &- \int_\Omega g(x) v dx = 0 \end{aligned} \quad (3.13)$$

for all $v \in C_0^\infty(\Omega)$. That means u_1 is a weak solution of problem (1.5) and (1.6).

Next, we prove $u_{\eta_i,0} \rightarrow u_1$ in $H_{X,0}^1(\Omega)$. In fact, $C_0^\infty(\Omega)$ is dense in $H_{X,0}^1(\Omega)$, thus from Lemma 2.4 and Lemma 2.5, we know that (3.12) and (3.13) are also true for any $v \in H_{X,0}^1(\Omega)$.

Especially, we have

$$\begin{aligned} & 2 \int_{\Omega} \sum_{j=1}^m (X_j u_{\eta_i,0})^2 - 2 \int_{\Omega} a(x) u_{\eta_i,0}^2 \log(|u_{\eta_i,0}| + \eta_i) dx \\ & + \int_{\Omega} \frac{a(x) u_{\eta_i,0}^2 |u_{\eta_i,0}| \eta_i}{2(|u_{\eta_i,0}| + \eta_i)^2} dx - 2 \int_{\Omega} b(x) u_{\eta_i,0}^2 dx - 2 \int_{\Omega} g(x) u_{\eta_i,0} dx = 0, \end{aligned} \quad (3.14)$$

$$\int_{\Omega} \sum_{j=1}^m (X_j u_1)^2 - \int_{\Omega} a(x) u_1^2 \log |u_1| dx - \int_{\Omega} b(x) u_1^2 dx - \int_{\Omega} g(x) u_1 dx = 0. \quad (3.15)$$

Letting $i \rightarrow \infty$ in (3.14), and from Lemma 2.8 and (3.15), we have

$$\|X_j u_{\eta_i,0}\|_{L^2(\Omega)} \rightarrow \|X_j u_1\|_{L^2(\Omega)}, \quad i \rightarrow \infty,$$

which means $u_{\eta_i,0} \rightarrow u_1$ in $H_{X,0}^1(\Omega)$.

Now by Proposition 1.6 ([3]), we have

$$J_0(u_1) = \lim_{i \rightarrow \infty} J_{\eta_i}(u_{\eta_i,0}) = \bar{c}_0 > 0 \geq J_0(u_0),$$

that means the problem (1.5) and (1.6) has at least two solutions in $H_{X,0}^1(\Omega)$.

If we replace, at the beginning, B_R by $B_R^+ = \{u \in H_{X,0}^1(\Omega), \|u\|_{H_{X,0}^1(\Omega)} < R, u \geq 0\}$, thus it is similar to the proof of existence of the solution u_0 , we can deduce that the problem (1.5) and (1.6) has a non-negative solution in $H_{X,0}^1(\Omega)$.

4 Boundedness and regularity of weak solutions

Similar to the proof of [10], we can deduce the boundedness and regularity of weak solutions.

By using the interpolation inequality, the condition H-3) and the Logarithmic Sobolev inequality (1.2) give that, for any $N \geq 1$, there exists C_N such that,

$$\int_{\Omega} v^2 \log^2 \left(\frac{|v|}{\|v\|_{L^2}} \right) \leq \frac{1}{N} \|Xv\|_{L^2}^2 + C_N \|v\|_{L^2}^2, \quad (4.1)$$

for all $v \in H_{X,0}^1(\Omega)$.

In order to prove that the solution $u \in L^\infty(\Omega)$, it suffices to show that, under the assumptions of Theorem 1.4, there exists $\bar{A} > 0$ such that the estimate

$$\|u\|_{L^p} \leq \bar{A} \quad (4.2)$$

holds for any $p \geq 2$. In fact, for $\epsilon > 0$, $\Omega_\epsilon = \{x \in \Omega; |u(x)| \geq \bar{A} + \epsilon\}$, it follows from (4.2) that $|\Omega_\epsilon| \leq (\frac{\bar{A}}{\bar{A} + \epsilon})^p \rightarrow 0$ (as $p \rightarrow \infty$) and hence we have $\|u\|_{L^\infty} \leq \bar{A}$.

We prove the estimate (4.2) by the following three steps. First, for any $p \geq 1$, $m \in \mathbb{N}$, we shall use u^{2p-1} or $u^{2p-1} \log^{2m}(u^p)$ as test function for the equation (1.5). Since we do not know if $u^{2p-1} \log^{2m}(u^p) \in H_{X,0}^1(\Omega)$, so we replace the function u by $u_{(k)}$, where $k > 1$ and $u_{(k)}(x) = u(x)$ if $x \in \{x \in \Omega; |u(x)| < k\}$ and $u_{(k)}(x) = k$ if $x \in \{x \in \Omega; |u(x)| \geq k\}$. Then it is easy to check (see [6] and [7, Theorem 7 and Theorem 8]) that $u_{(k)}^{2p-1} \log^{2m}(u_{(k)}^p) \in H_{X,0}^1(\Omega)$ for all $p > 1$, $m \in \mathbb{N}$. In the case of $p = 1$, we use $u(\log^m u)_{(k)}^2 \in H_{X,0}^1(\Omega)$ as the test function. To simplify the notation, we shall drop the subscript and use $u^{2p-1} \log^{2m}(u^p)$ as the test function. We have

Proposition 4.1. *Under the hypotheses H-1), H-2), H-3), H-4) of Theorem 1.4, and $g(x) \in L^\infty(\Omega)$, $u \in H_{X,0}^1(\Omega)$, $u \geq 0$, $\|u\|_{L^2(\Omega)} \neq 0$ be a weak solution of the equation (1.5). Suppose that for some $p_0 \geq 1$, there exists A_0, A_1 such that*

$$0 < A_1 \leq \|u\|_{L^{2p_0}} \leq A_0.$$

Then

$$\begin{aligned} & \int_{\Omega} |X(\bar{u})^{p_0}|^2 + \int_{\Omega} (\bar{u})^{2p_0} \log^2(\bar{u}^{p_0}) \\ & \leq 2C_2 + a_\infty^2 + 2p_0[b_\infty + a_\infty |\log A_0| + (1 + |\Omega|)g_\infty/A_1], \end{aligned} \quad (4.3)$$

where $a_\infty = \|a\|_{L^\infty}$, $b_\infty = \|b\|_{L^\infty}$, $g_\infty = \|g\|_{L^\infty}$ and the constant C_2 is given by (4.1) and $\bar{u} = u/\|u\|_{L^{2p_0}}$.

Proof. We have $\bar{u} \in H_{X,0}^1(\Omega)$, $\|\bar{u}\|_{L^{2p_0}} = 1$, and $\bar{u}$ is a weak solution of equation

$$-\Delta_X \bar{u} = a(x)\bar{u} \log \bar{u} + (a(x) \log \|u\|_{L^{2p_0}} + b(x))\bar{u} + \frac{g(x)}{\|u\|_{L^{2p_0}}}. \quad (4.4)$$

Take $\bar{u}^{2p_0-1}$ as the test function, we have

$$\begin{aligned} & \frac{2p_0-1}{p_0^2} \int_{\Omega} |X\bar{u}^{p_0}|^2 = \frac{1}{p_0} \int_{\Omega} a(x)\bar{u}^{2p_0} \log \bar{u}^{p_0} \\ & + \int_{\Omega} (a(x) \log \|u\|_{L^{2p_0}} + b(x))\bar{u}^{2p_0} + \frac{1}{\|u\|_{L^{2p_0}}} \int_{\Omega} g(x)\bar{u}^{2p_0-1}, \end{aligned}$$

where

$$\begin{aligned} & \frac{1}{\|u\|_{L^{2p_0}}} \int_{\Omega} g(x) \bar{u}^{2p_0-1} \leq \frac{g_{\infty}}{A_1} \left[\int_{\bar{u}>1} |\bar{u}^{2p_0-1}| + \int_{\bar{u}\leq 1} |\bar{u}^{2p_0-1}| \right] \\ & \leq \frac{g_{\infty}}{A_1} \left(\int_{\bar{u}>1} \bar{u}^{2p_0} + |\Omega| \right) \leq \frac{g_{\infty}}{A_1} \left(\int_{\Omega} \bar{u}^{2p_0} + |\Omega| \right) = \frac{(1+|\Omega|)g_{\infty}}{A_1}. \end{aligned}$$

Furthermore

$$\int_{\Omega} |X \bar{u}^{p_0}|^2 \leq \frac{1}{2} \int_{\Omega} \bar{u}^{2p_0} \log^2(\bar{u}^{p_0}) + \frac{1}{2} a_{\infty}^2 + p_0 a_{\infty} |\log A_0| + p_0 b_{\infty} + \frac{(1+|\Omega|)p_0 g_{\infty}}{A_1}. \quad (4.5)$$

On the other hand, the Logarithmic Sobolev inequality (4.1) gives

$$\int_{\Omega} (u^{p_0})^2 \log^2\left(\frac{|u^{p_0}|}{\|u^{p_0}\|_{L^2}}\right) \leq \frac{1}{2} \|X(u^{p_0})\|_{L^2}^2 + C_2 \|u^{p_0}\|_{L^2}^2.$$

Note that $\|u^{p_0}\|_{L^2} = \|u\|_{L^{2p_0}}^{p_0}$ and $\bar{u} = u/\|u\|_{L^{2p_0}}$, we have

$$\int_{\Omega} \bar{u}^{2p_0} \log^2(\bar{u}^{p_0}) \leq \frac{1}{2} \|X(\bar{u}^{p_0})\|_{L^2}^2 + C_2. \quad (4.6)$$

Adding (4.5) and (4.6), we have the desired estimate (4.3).

Proposition 4.2. *We have for any $m \in \mathbb{N}$,*

$$\int_{\Omega} |X(\bar{u}^{p_0})|^2 \log^{2m-2}(\bar{u}^{p_0}) + \int_{\Omega} \bar{u}^{2p_0} \log^{2m}(\bar{u}^{p_0}) \leq M_1^{2m} P(m, p_0) (m!)^2 \quad (4.7)$$

where $P(m, p_0) = p_0^m$ if $m \leq \sqrt{p_0}$, $P(m, p_0) = p_0^{\sqrt{p_0}}$ if $m > \sqrt{p_0}$, and

$$M_1 \geq (2|\Omega| + 4C_2 + 2C_4 + 10 + 6a_{\infty}^2 + 8b_{\infty} + 8a_{\infty} |\log A_0| + 4g_{\infty}(1+|\Omega|)/A_1)^{\frac{1}{2}}.$$

Proof. From the estimate $0 < A_1 \leq \|u\|_{L^{2p_0}} \leq A_0$, we have the estimate (4.7) for $m = 1$. By induction, we suppose that (4.7) is also hold for $m \in \mathbb{N}$, then we need to prove that (4.7) is hold for $m + 1$. Here we simplify the notation again, i.e. $\bar{u}$ and p_0 would be replaced by u and p in the equation (4.4). We take $u^{2p-1} \log^{2m}(u^p)$ as the test function in (4.4), then

$$\begin{aligned} & \frac{2p-1}{p^2} \int_{\Omega} |Xu^p|^2 \log^{2m}(u^p) + \frac{2m}{p} \int_{\Omega} |Xu^p|^2 \log^{2m-1}(u^p) \\ &= \frac{1}{p} \int_{\Omega} a(x) u^{2p} \log^{2m+1}(u^p) + \int_{\Omega} (a(x) \log \|u\|_{L^{2p}} + b(x)) u^{2p} \log^{2m}(u^p) \\ &+ \int_{\Omega} \frac{g(x)}{\|u\|_{L^{2p}}} u^{2p-1} \log^{2m}(u^p). \end{aligned}$$

That is

$$\begin{aligned} \int_{\Omega} |Xu^p|^2 \log^{2m}(u^p) &\leq \frac{1}{2} \int_{\Omega} |Xu^p|^2 \log^{2m}(u^p) + 2m^2 \int_{\Omega} |Xu^p|^2 \log^{2m-2}(u^p) \\ &+ \frac{1}{4} \int_{\Omega} u^{2p} \log^{2m+2}(u^p) + (a_{\infty}^2 + pa_{\infty} \log A_0 + pb_{\infty}) \int_{\Omega} u^{2p} \log^{2m}(u^p) \\ &+ \frac{pg_{\infty}}{A_1} \int_{\Omega} u^{2p-1} \log^{2m}(u^p). \end{aligned}$$

Using the fact $l^l \leq e^l l!$, we have

$$\begin{aligned} \int_{\Omega} u^{2p-1} \log^{2m}(u^p) &= \int_{|u|<1} u^{2p-1} \log^{2m}(u^p) + \int_{|u|\geq 1} u^{2p-1} \log^{2m}(u^p) \\ &\leq 2^{2m} (m!)^2 |\Omega| + \int_{\Omega} u^{2p} \log^{2m}(u^p) < (1 + |\Omega|) M_1^{2m} P(m, p) (m!)^2, \end{aligned}$$

so that

$$\begin{aligned} \int_{\Omega} |Xu^p|^2 \log^{2m}(u^p) &\leq \frac{1}{2} \int_{\Omega} (u^p)^2 \log^{2m+2}(u^p) + [4m^2 + 2a_{\infty}^2 + \\ &2(pa_{\infty} |\log A_0| + pb_{\infty} + pg_{\infty} (1 + |\Omega|)/A_1)] M_1^{2m} P(m, p) (m!)^2. \end{aligned} \quad (4.8)$$

We study now the term $\int_{\Omega} u^{2p} \log^{2m+2}(u^p)$. Set $\Omega = \Omega_1 \cup \Omega_2^+ \cup \Omega_2^-$ with $\Omega_1 = \{x \in \Omega; u(x) \leq 1\}$ and

$$\begin{aligned} \Omega_2^+ &= \{x \in \Omega; u(x) > 1, |\log^m(u^p)| \leq \|u^p \log^m(u^p)\|_{L^2}\}, \\ \Omega_2^- &= \{x \in \Omega; u(x) > 1, |\log^m(u^p)| > \|u^p \log^m(u^p)\|_{L^2}\}. \end{aligned}$$

Then

$$\int_{\Omega_1} u^{2p} \log^{2m+2}(u^p) \leq |\Omega| ((m+1)!)^2.$$

For the second part, (4.3) gives

$$\begin{aligned} \int_{\Omega_2^+} u^{2p} \log^{2m+2}(u^p) &\leq \|u^p \log^m(u^p)\|_{L^2}^2 \int_{\Omega} u^{2p} \log^2(u^p) \\ &\leq (2C_2 + a_{\infty}^2 + 2pb_{\infty} + 2pa_{\infty} |\log A_0| + (1 + |\Omega|)g_{\infty}/A_1) M_1^{2m} P(m, p) (m!)^2. \end{aligned}$$

Next, for the third part, we use the Logarithmic Sobolev inequality (4.1) for $N = 4$,

$$\begin{aligned} \int_{\Omega_2^-} u^{2p} \log^{2m+2}(u^p) &\leq \int_{\Omega_2^-} (u^p \log^m u^p)^2 \log^2 \left(\frac{u^p \log^m(u^p)}{\|u^p \log^m(u^p)\|_{L^2}} \right) \\ &\leq \frac{1}{4} \|X(u^p \log^m u^p)\|_{L^2}^2 + C_4 \|u^p \log^m u^p\|_{L^2}^2 \\ &\leq \frac{1}{2} \int_{\Omega} |X(u^p)|^2 \log^{2m}(u^p) + m^2 \int_{\Omega} |X(u^p)|^2 \log^{2m-2}(u^p) \\ &+ C_4 \int_{\Omega} u^{2p} \log^{2m}(u^p) \\ &\leq \frac{1}{2} \int_{\Omega} |X(u^p)|^2 \log^{2m}(u^p) + (C_4 + m^2) M_1^{2m} P(m, p) (m!)^2. \end{aligned}$$

Sum up the three parts above, we get

$$\begin{aligned} \int_{\Omega} u^{2p} \log^{2m+2}(u^p) &\leq \frac{1}{2} \int_{\Omega} |X(u^p)|^2 \log^{2m}(u^p) + |\Omega|((m+1)!)^2 \\ &+ [2C_2 + C_4 + m^2 + a_{\infty}^2 + 2pb_{\infty} + 2pa_{\infty} |\log A_0| \\ &+ (1 + |\Omega|)g_{\infty}/A_1] M_1^{2m} P(m, p) (m!)^2. \end{aligned}$$

which implies by (4.8),

$$\begin{aligned} \int_{\Omega} u^{2p} \log^{2m+2}(u^p) + \int_{\Omega} |X(u^p)|^2 \log^{2m}(u^p) &\leq [2\Omega + 4C_2 + 2C_4 + 10 \quad (4.9) \\ &+ 6a_{\infty}^2 + 8b_{\infty} + 8a_{\infty} |\log A_0| + 2g_{\infty}(1 + |\Omega|)/A_1] M_1^{2m} P(m+1, p) ((m+1)!)^2. \end{aligned}$$

Proposition 4.2 is proved.

Proposition 4.3. *Under the hypotheses of Proposition 4.1, if for some $p_0 \geq 1$ and $A_0 \geq e^{12}$ we have*

$$\|u\|_{L^{2p_0}} \leq A_0,$$

then for

$$M_1 \geq [2|\Omega| + 4C_2 + 2C_4 + 10 + 6a_{\infty}^2 + 8b_{\infty} + 8a_{\infty} \log A_0 + 2g_{\infty}(1 + |\Omega|)/A_1]^{\frac{1}{2}},$$

and $\delta = 1/2M_1$, we have

$$\int_{\Omega} u^{2p_0(1+\delta)} \leq A_0^{2p_0(1+\delta)(1+(\frac{1}{p_0(1+\delta)})^{\frac{1}{3}})} \quad (4.10)$$

Proof. For any $\delta > 0$, the estimate (4.7) gives that

$$\begin{aligned} \left(\int_{\Omega} |\bar{u}^{p_0(1+\delta)}|^2 dx \right)^{\frac{1}{2}} &= \left(\int_{\Omega} |\bar{u}^{p_0} \bar{u}^{\delta p_0}|^2 dx \right)^{\frac{1}{2}} = \left(\int_{\Omega} |\bar{u}^{p_0} e^{\delta \log(\bar{u}^{p_0})}|^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega} |\bar{u}^{p_0} \sum_{m=0}^{\infty} \frac{(\delta \log(\bar{u}^{p_0}))^m}{m!}|^2 dx \right)^{\frac{1}{2}} \leq \sum_{m=0}^{\infty} \left(\int_{\Omega} \bar{u}^{2p_0} \frac{(\delta \log(\bar{u}^{p_0}))^{2m}}{(m!)^2} dx \right)^{\frac{1}{2}} \\ &\leq \sum_{m=0}^{\infty} \frac{\delta^m}{m!} \left(\int_{\Omega} \bar{u}^{2p_0} \log^{2m}(\bar{u}^{p_0}) dx \right)^{\frac{1}{2}} \leq \sum_{m=0}^{\infty} \delta^m M_1^m P(m, p_0) \leq p_0^{\sqrt{p_0}} \sum_{m=0}^{\infty} (\delta M_1)^m. \end{aligned}$$

For $\delta = 1/2M_1$, we have finally

$$\int_{\Omega} u^{2p_0(1+\delta)} dx \leq 4p_0^{2\sqrt{p_0}} A_0^{2p_0(1+\delta)}.$$

Since for any $p_0 > 1$,

$$4p_0^{2\sqrt{p_0}} = 4e^{2\sqrt{p_0} \log p_0} \leq (e^{12})^{2p_0^{\frac{2}{3}}},$$

which implies the estimate (4.10) if $A_0 \geq e^{12}$.

We set now for $k \in \mathbb{N}$,

$$p_k = p_0(1 + \delta)^k, \quad A_k = A_0^{1+p_0^{-1/3} \sum_{j=1}^k (\frac{1}{1+\delta})^{j/3}},$$

then Proposition 4.3 implies that

$$\begin{aligned} \int_{\Omega} u^{2p_0(1+\delta)^{k+1}} &= \int_{\Omega} u^{2p_k(1+\delta)} \leq A_k^{2p_k(1+\delta)(1+(\frac{1}{p_k(1+\delta)})^{1/3})} \\ &\leq A_0^{2p_0(1+\delta)^{k+1}(1+p_0^{-1/3} \sum_{j=1}^{k+1} (\frac{1}{1+\delta})^{j/3})}, \end{aligned}$$

where $\delta = \frac{1}{2}M_1$ and

$$M_1 \geq [2|\Omega| + 4C_2 + 2C_4 + 10 + 6a_{\infty}^2 + 8b_{\infty} + 8a_{\infty}|\log A_k| + 2g_{\infty}(1 + |\Omega|)/A_1]^{1/2}. \quad (4.11)$$

We have now for $\delta = \frac{1}{2}M_1 \leq 1/4$,

$$\begin{aligned} \frac{\log A_k}{\log A_0} &= 1 + p_0^{-1/3} \sum_{j=1}^k (\frac{1}{1+\delta})^{j/3} \leq 1 + p_0^{-1/3} \sum_{j=1}^{\infty} (\frac{1}{1+\delta})^{j/3} \\ &= 1 + p_0^{-1/3} \frac{(\frac{1}{1+\delta})^{1/3}}{1 - (\frac{1}{1+\delta})^{1/3}} \leq 1 + 4p_0^{-1/3}M_1 \leq 5M_1, \end{aligned}$$

where M_1 is independent of k , thus we have proved for any $k \in \mathbb{N}$,

$$\int_{\Omega} u^{2p_0(1+\delta)^k} \leq (A_0^{5M_1})^{2p_0(1+\delta)^k}.$$

If we choose $A_0 = e^{12}$, then the estimate (4.2) holds for $\bar{A} = e^{60M_1}$.

The regularity of the solution for the problem (1.5) and (1.6) can be deduced by following result:

Proposition 4.4. *Suppose $a(x)$, $b(x)$, $g(x) \in C^{\infty}(\Omega)$, and there exist a_0 , b_0 , $g_0 > 0$, such that $a(x) \geq a_0$, $b(x) \geq b_0$, $g(x) \geq g_0$ in Ω . Let $u \in H_{X,0}^1(\Omega)$, $u \geq 0$, $\|u\|_{L^2} \neq 0$ be a weak solution of the problem (1.5) and (1.6), and $\partial\Omega$ is non characteristic. Then $u \in C^{\infty}(\Omega \setminus \Gamma) \cap C^0(\bar{\Omega} \setminus \Gamma)$, and $u(x) > 0$ for all $x \in \Omega \setminus \Gamma$.*

Proof. Suppose $x_0 \in \Omega \setminus \Gamma$, then there exists a neighborhood $V_0 \subset \Omega \setminus \Gamma$ of x_0 , for $\varphi \in C_0^{\infty}(V_0)$ we shall prove that $v = \varphi u \in C^{\infty}(V_0)$. It follows from equation (1.5) that,

$$-\Delta_X v = a(x)\varphi u \log u + b(x)\varphi u + g(x)\varphi + \sum_{j=1}^m \varphi_j X_j u + \varphi_0 u = f_0 + \sum_{j=1}^m X_j f_j,$$

with $\varphi_j \in C^\infty(V_0)$, $f_j \in L^\infty(V_0)$, $j = 0, \dots, m$. Since the system of vector fields X satisfies the finitely type Hörmander's condition on V_0 , the regularity result of [8] (see also [7, 9]) implies that $u \in C^\epsilon(V_0)$ for some $\epsilon > 0$. If we have $u(x) \geq \alpha > 0$ for $x \in V_0$, then by $t \log t \in C^\infty(t \geq \alpha)$, we can deduce $u \log u \in C^\epsilon(V_0)$, thus we can prove by recurrence that $u \in C^\infty(V_0)$. For $x_0 \in \partial\Omega \setminus \Gamma$, we have also $u \in C^\epsilon(V_0 \cap \bar{\Omega})$, but we know only $u \log u \in C^0(V_0 \cap \bar{\Omega})$, so we can not obtain the C^∞ regularity of u near to the boundary $\partial\Omega$. Therefore the Proposition 4.4 will be deduced by the following Lemma directly.

Lemma 4.5. *Suppose $a(x), b(x), g(x)$ satisfy the conditions of Proposition 4.4, and $u \in C^0(\Omega_1)$, $u \geq 0$ is a non trivial weak solution of the equation (1.5) on an open set $\Omega_1 \subset \Omega$, then $u(x) > 0$ for all $x \in \Omega_1$.*

Proof. Suppose that $u(x_0) = 0$ for some $x_0 \in \Omega_1$, then for any $\epsilon > 0$, there exists a small neighborhood $U_0 \subset \Omega_1$ of x_0 such that $0 \leq u(x) \leq \epsilon$ on $\bar{U}_0$. Since $g(x)$ is continuous on $\bar{U}_0$, there exists $\alpha > 0$ such that $g(x) \geq \alpha$ on $\bar{U}_0$.

Choosing ϵ small enough such that in U_0 , we have

$$a(x)u \log u + b(x)u < 0,$$

and

$$a(x)u \log u + b(x)u + g(x) \geq 0.$$

That is $\Delta_X u \leq 0$ in U_0 . But x_0 is a minimum point of u , the maximum principle of Bony [10] implies that $u \equiv 0$ in U_0 . That means u is a trivial solution by continuous of u in Ω_1 .

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