OPTIMAL RECOVERY FROM A FINITE SET IN BANACH SPACES OF ENTIRE FUNCTIONS

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Abstract. We develop an approach to the problem of optimal recovery of continuous linear functionals in Banach spaces through information on a finite number of given functionals. The results obtained are applied to the problem of the best analytic continuation from a finite set in the complex space $\mathbb{C}^n$, $n \geq 1$, for classes of entire functions of exponential type which belong to the space L^p, $1 < p < \infty$, on the real subspace of $\mathbb{C}^n$. These latter are known as Wiener classes.

Contents

Introduction

The theory of optimal recovery is a part of approximation theory devoted to the reconstruction of functions, functionals and operators through incomplete information (precise or bearing errors) by means of the so-called optimal algorithms. The best general references here are the monographs [TW80, MR76, MR85] and survey [Tik87].

This domain of investigations is intensively developed, cf. for instance [MO02, MO03]. We will restrict our discussion to a popular problem of the theory of optimal recovery which is closely related to the problem of extrapolation of entire functions of exponential type from a finite set. This is the problem of reconstruction of delta type functionals.

Let V be a normed space of functions with domain D, U a subset of V and $L_x : U \to \mathbb{C}$ a delta-functional defined by $L_x(f) = f(x)$ for $f \in U$, where $x \in D$ is a fixed point. The problem consists of recovering the functional L_x through an available information.

For example, let a finite number of functionals $L_{x_k}(f) = f(x_k)$, $f \in U$, be given, for $k = 1, \ldots, N$, where $x_k \in D$. In this case the information space is $W = \mathbb{C}^N$ and

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the information operator $I : U \to W$ has the form

$$I(f) = (f(x_1), \ldots, f(x_N)) \tag{0.1}$$

for $f \in U$.

Any mapping $A : W \to \mathbb{C}$ is called an algorithm. By the optimal error of reconstruction of the functional L_x is meant the number

$$\Omega(x, U) = \inf \sup \{|f(x) - A \circ I(f)| : \ f \in U\}, \tag{0.2}$$

the infimum is over all mappings $A : W \to \mathbb{C}$.

A mapping $A_0 : W \to \mathbb{C}$ is called the optimal algorithm if the infimum in (0.2) is achieved for $A = A_0$. An element $f_0 \in U$ is said to be extremal, if

$$\Omega(x, U) = |f_0(x) - A_0 \circ I(f_0)|$$

holds.

In the case of real space V S.A. Smolyak proved that one can always choose the mapping A in (0.2) linear, if U is a convex circled set and $I : U \to W$ an information operator of the form (0.1). Recall that U is called circled if $\lambda f \in U$ for all $f \in U$ and all λ with $|\lambda| \le 1$.

In [Osi76, MO91] this result is generalised to all complex normed spaces. If e.g. $U = \{f \in V : \ \|f\| \le R\}$ is a ball about 0, then $\Omega(x, U) = \inf \{E(a; x, U) : \ a \in \mathbb{C}^N\}$, where

$$E(a; x, U) = \sup \Big\{|f(x) - \sum_{k=1}^{N} a_k \, f(x_k)| : \ f \in U\Big\}. \tag{0.3}$$

If U is a normed space of analytic functions in a domain $D \subset \mathbb{C}^n$, then $\Omega(x, U)$ is actually the unremovable error of the best analytic continuation (also optimal extrapolation) of functions $f \in U$ from the finite set $S = \{x_1, \ldots, x_N\}$ to the point $x \in D \setminus S$. For the space V of all bounded functions in a simply connected domain D of the complex plane, Osipenko [Osi76] evaluated the error $\Omega(x, U)$ in the case, where $U = \{f \in V : \ \|f\|_{L^\infty(D)} \le R\}$ is a ball around the origin, and found the optimal algorithm of analytic continuation from the set S to a fixed point x of its complement $D \setminus S$.

In [Mae97, Mae00] conditions for optimal extrapolation of Wiener class W^2_σ of entire functions are studied. For simplicity we formulate the main result in the case $n = 1$.

Let $\sigma > 0$. By W^2_σ is meant the space of all entire functions $f \in L^2(\mathbb{R}^n)$ which are of exponential type $\le \sigma$, i.e., satisfy

$$|f(z)| \le C_f \, \exp(\sigma |\Im z|) \tag{0.4}$$

for all $z \in \mathbb{C}$.

Let $S = \{z_1, \ldots, z_N\}$ be a set of pairwise different points in $\mathbb{C}$ and z_0 an arbitrary point of $\mathbb{C} \setminus S$. Let moreover $U = \{f \in W^2_\sigma : \ \|f\|_2 \le R\}$ be a ball of radius $R > 0$, and Z the space of all functions $f \in W^2_\sigma$ which vanish on S. The following theorem is proved in [Mae97].

Theorem 0.1. *Let G and G_0 be the Gram matrices of the systems of functions $\{\exp(\imath z_k x)\}_{k=1}^N$ and $\{\exp(\imath z_k x)\}_{k=0}^N$ on the interval $[-\sigma, \sigma]$, respectively. Then holds:*

1) *The elements of G and G_0 are precisely $(\exp(\imath z_j x), \exp(\imath z_k x)) = h(z_j, \overline{z}_k)$ for $j, k = 0, \ldots, N$, where $(\cdot, \cdot)$ is the scalar product in $L^2[-\sigma, \sigma]$ and*

$$h(z, \overline{w}) = \begin{cases} 2 \dfrac{\sin \sigma (z - \overline{w})}{z - \overline{w}}, & \text{if} \quad z \ne \overline{w}; \\ 2\sigma, & \text{if} \quad z = \overline{w}. \end{cases}$$

2) *The unremovable error* $\Omega(z_0,U)$ *of analytic continuation of functions in* U *from the finite set* $S=\{z_1,\dots,z_N\}$ *to the point* z_0 *is given by*

$$\Omega(z_0,U)=\sup_{f\in U\cap Z}|f(z_0)|=R\sqrt{\frac{\det G_0}{\det G}}.$$

Moreover, the equality $\Omega(z_0,U)=|f(z_0)|$ *holds only for those* $f\in U\cap Z$ *which are of the form*

$$f(z)=Re^{\imath\theta}\frac{1}{\sqrt{\det G\det G_0}}\det\begin{pmatrix} h(z,\overline{z}_0) & h(z,\overline{z}_1) & \dots & h(z,\overline{z}_N)\\ h(z_1,\overline{z}_0) & & & \\ \dots & & G & \\ h(z_N,\overline{z}_0) & & & \end{pmatrix}$$

on all of $\mathbb{C}$.

3) *The value* $f(z_0)$ *of any function* $f\in U$ *satisfies* $|f(z_0)-A_0\circ I(f)|\le\Omega(z_0,U)$, *where*

$$A_0\circ I(f)=-\frac{\det\begin{pmatrix} 0 & f(z_1) & \dots & f(z_N)\\ h(z_1,\overline{z}_0) & & & \\ \dots & & G & \\ h(z_N,\overline{z}_0) & & & \end{pmatrix}}{\det G}$$

is the optimal linear algorithm.

In the paper [FM01] Theorem 0.1 is extended to arbitrary Hilbert spaces of holomorphic functions with reproducing kernel. Moreover, the data need not be precise.

In the one-dimensional case conditions for the best analytic continuation from a finite set are also derived for a generalised Wiener class W^p_σ of entire functions of exponential type $\le\sigma$ whose restrictions to the real axis belong to the space $L^p(\mathbb{R})$, with $1<p<2$. See [Mae06].

In the present paper we bring together two areas in which the problem of the best analytic continuation from a finite subset of the complex space is of standing interest. The first of the two is approximation theory, more precisely, best approximation in normed spaces by elements of vector subspaces, cf. [Sin70, Tik87]. The second area is complex analysis, namely the theory of entire functions. The results obtained apply to present Theorem 0.1 in the context of Banach spaces W^p_σ, where $1<p<\infty$, also in many dimensions.

1. Optimal recovery of linear functionals

Let $T:V\to B$ be an algebraic isomorphism of a vector space V onto a normed space B, both V and B being over the same field $\mathbb{K}$ where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. We give a norm to V by setting $\|f\|_V:=\|Tf\|_B$ for $f\in V$, thus making V a normed space, too.

It is easy to verify that $\|L\|_{V'}=\|L\circ T^{-1}\|_{B'}$ for each continuous linear functional L on V Here, B' stands for the dual space for B, which is Banach under functional norm. This enables one to reduce extremal problems in the dual space of V to those in B'.

Suppose U is a closed ball of radius $R>0$ about the origin in V. Given a finite number of independent functionals $L_1,\dots,L_N$ on U, we consider the problem of recovering any fixed functional L on U through $L_1,\dots,L_N$. For instance, in Theorem 0.1 the original space is $V=W^2_\sigma$, $B=L^2[-\sigma,\sigma]$ and $T=\mathcal{F}$ is the Fourier transform. As $L_1,\dots,L_N$ and L one takes the evaluation functionals at the points $z_1,\dots,z_N$ and z_0, respectively.

The extreme case is when the data $I(f)=(L_1(f),\dots,L_N(f))$ determine $f\in U$ uniquely, i.e., $f=M(I(f))$, with M being a map of $\mathbb{C}^N$ to U. Then the formula

$L(f) = L \circ M(I(f))$ would uniquely restore L through the functionals $L_1, \ldots, L_N$ on U. For instance, any polynomial $\wp$ of one variable of degree $N-1$ over $\mathbb{K}$ is uniquely determined by its values at N fixed points. In other words, the value of $\wp$ at any other point is uniquely determined through the information on the values of $\wp$ at N given points.

Consider the main characteristics of optimal recovery which generalise those mentioned in the introduction. In general there might be diverse algorithms A recovering L on U through $L_1, \ldots, L_N$. We restrict our attention to those algorithms A which are linear, i.e.,

$$A \; = \; \ell(a; L_1, \ldots, L_N) \; := \; \sum_{k=1}^{N} a_k L_k$$

for $a = (a_1, \ldots, a_N) \in \mathbb{K}^N$.

By the error $E(a; L, U)$ of an algorithm A is naturally meant the supremum of $|L(f) - A \circ I(f)|$ over all $f \in U$.

The optimal (unremovable) error $\Omega(L, U)$ of reconstruction of the functional L is defined to be the infimum of $E(a; L, U)$ over all $a \in \mathbb{K}^N$. If the infimum is achieved for some $a = \alpha$ of $\mathbb{K}^N$, i.e., $\Omega(L, U) = E(\alpha; L, U)$, then the algorithm $A_0 = \ell(\alpha; L_1, \ldots, L_N)$ is called the optimal linear algorithm of recovery for the functional L.

It is clear that the optimal algorithm need not be unique. We will need a fundamental result of approximation theory in normed spaces, see for instance [Lyu92, p. 146].

Lemma 1.1. *Let B' be a normed space over $\mathbb{K}$ and $\{e_1, \ldots, e_N\}$ a linearly independent system in B'. Then, for each element $v \in B'$ there is a vector $\alpha = (\alpha_1, \ldots, \alpha_N)$ of $\mathbb{C}^N$, such that*

$$\Big\| v - \sum_{k=1}^{N} \alpha_k e_k \Big\| = \inf_{a \in \mathbb{K}^N} \Big\| v - \sum_{k=1}^{N} a_k e_k \Big\|.$$

If moreover B' is strictly convex then there is only one vector $\alpha \in \mathbb{K}^N$ with the mentioned property.

The element

$$\ell(\alpha; e_1, \ldots, e_N) = \sum_{k=1}^{N} \alpha_k e_k$$

is called the least deviating element in the linear span of $\{e_1, \ldots, e_N\}$ for v, cf. also Chebyshev polynomial. We will write it simply $\ell(\alpha)$ when no confusion can arise. For Hilbert spaces B', a constructive formula for $\ell(\alpha)$ is known, cf. Theorem 0.1.

Note that the spaces $L^q(\mathcal{X}, \mu)$ with $1 < q < \infty$ are known to be strictly convex, see [Lyu92, p. 130].

Given an optimal algorithm $A_0 = \ell(\alpha; L_1, \ldots, L_N)$, an element $f_0 \in U$ is said to be extremal if $E(\alpha; L, U) = |L(f_0) - A_0 \circ I(f_0)|$.

Let $\mathcal{L}_0 \in B'$ be a non-zero functional. Consider a set

$$\partial \|\mathcal{L}_0\|_{B'} = \{F \in B : \|F\|_B = 1, \, \mathcal{L}_0(F) = \|\mathcal{L}_0\|_{B'}\}$$

which has a simple geometric meaning. More precisely, this is the face of the closed unit ball in B lying in the support hyperplane $\{F \in B : \mathcal{L}_0(F) = \|\mathcal{L}_0\|_{B'}\}$ of the ball.

The set $\partial \|\mathcal{L}_0\|_{B'}$ may be empty. This is impossible if, e.g., the closed unit ball in B is compact in the weak topology. In particular, this is the case if B is a reflexive Banach space, which is due to the Banach-Alaoglu theorem, see e.g. [KA77, p. 241]. In this case the set $\partial \|\mathcal{L}_0\|_{B'}$ is called the subdifferential of the norm $\mathcal{L} \mapsto \|\mathcal{L}\|_{B'}$ at $\mathcal{L}_0$, see [Phe88].

Write G for the group of elements of modulus 1 in $\mathbb{K}$. In the general case the set of all extremal elements is defined by means of the set $G\,\partial\|\mathcal{L}_0\|_{B'}$ for some $\mathcal{L}_0$ depending on A_0. The juxtaposition of two sets means their 'element by element' product.

Theorem 1.2. *Let*

$$\ell(\alpha) = \sum_{k=1}^{N} \alpha_k\, L_k \circ T^{-1} \tag{1.1}$$

be a least deviating element in the linear span of $\{L_1 \circ T^{-1}, \dots, L_N \circ T^{-1}\}$ *for* $L \circ T^{-1}$. *Then:*

1) *The optimal error of recovering the functional* L *through finitely many functionals* $L_1, \dots, L_N$ *on* U *is* $\Omega(L,U) = R\,\|L \circ T^{-1} - \ell(\alpha)\|_{B'}$.

2) *The optimal linear algorithm of reconstruction of the functional* L *on* U *is given by* $A_0 = \ell(\alpha; L_1, \dots, L_N)$, *cf.* (1.1). *Moreover, it is unique provided that the space* B' *is strictly convex.*

3) *If* $\mathcal{L}_0 = L \circ T^{-1} - \ell(\alpha)$ *is non-zero and* $\partial\,\|\mathcal{L}_0\|_{B'} \neq \emptyset$, *then any extremal element* $f_0 \in U$ *has the form* $RT^{-1}F_0$, *where* $F_0 \in G\,\partial\|\mathcal{L}_0\|_{B'}$.

It readily follows from 1) that $\Omega(L,U) = 0$ if and only if L is a linear combination of $L_1, \dots, L_N$.

Proof. Fix $a \in \mathbb{K}^N$. Consider the difference

$$\Delta(a)(f) = L(f) - \sum_{k=1}^{N} a_k L_k(f)$$

for $f \in V$.

Using the formula $f = RT^{-1}F$, with $F = T(f/R) \in B$, and the definition of the norm of a continuous linear functional, we get

$$\begin{aligned} E(a; L, U) &= \sup_{f \in U} |\Delta(a)(f)| \\ &= R \sup_{\|F\|_B \le 1} \left|\Delta(a) \circ T^{-1}(F)\right| \\ &= R\, \left\|\Delta(a) \circ T^{-1}\right\|_{B'}, \end{aligned}$$

for $f \in U$ if and only if $\|T(f/R)\|_B \le 1$.

Hence it follows that

$$\begin{aligned} \Omega(L,U) &= R \inf_{a \in \mathbb{K}^N} \|L \circ T^{-1} - \ell(a; L_1 \circ T^{-1}, \dots, L_N \circ T^{-1})\|_{B'} \\ &= R\,\|L \circ T^{-1} - \ell(\alpha; L_1 \circ T^{-1}, \dots, L_N \circ T^{-1})\|_{B'}, \end{aligned} \tag{1.2}$$

the infimum is attained at some $\alpha \in \mathbb{K}^N$ by Lemma 1.1. Moreover, Lemma 1.1 implies that the extremal element $\alpha \in \mathbb{K}^N$ is unique, if the space B' is strictly convex.

By (1.2), the optimal error $\Omega(L,U)$ is equal to zero if and only if the functional $\mathcal{L}_0 = L \circ T^{-1} - \ell(\alpha)$ is zero. Then $L = \alpha_1 L_1 + \ldots + \alpha_N L_N$ on all of V. In this case the coefficients $\alpha_1, \dots, \alpha_N$ are uniquely determined by L since $L_1, \dots, L_N$ are linearly independent elements of V'.

So the assertions 1) and 2) of the theorem are true. It remains to consider the case where $\mathcal{L}_0$ is different from zero.

If $\partial\|\mathcal{L}_0\|_{B'} \neq \emptyset$, then by the very definition of the subdifferential of a norm we see that

$$\begin{aligned} \Omega(L,U) &= R\,\|\mathcal{L}_0\|_{B'} \\ &= |\mathcal{L}_0(RF)| \end{aligned}$$

for all $F \in G\,\partial \|\mathcal{L}_0\|_{B'}$. Denote by E the set of all elements $f \in U$, such that $f = RT^{-1}F$ for some $F \in G\,\partial \|\mathcal{L}_0\|_{B'}$. Then

$$\Omega(L,U) = |L(f) - A_0 \circ I(f)|$$

for all $f \in E$, i.e., E is the set of all extremal elements in U. We have thus proved the assertion 3). □

As but one application of Theorem 1.2 we mention the main result of [Mae06] on extrapolation in Wiener classes W^p_σ with $1 < p < 2$.

Example 1.3. Let $B = L^q[-\sigma,\sigma]$, where $\sigma > 0$ and $1 < q < \infty$. We identify B with the subspace of $L^q(\mathbb{R})$ consisting of all functions with support in $[-\sigma,\sigma]$. Denote by V the vector space of all tempered distributions f on $\mathbb{R}$, such that $\mathcal{F}^{-1}f \in L^q[-\sigma,\sigma]$, $\mathcal{F}^{-1}$ being the inverse Fourier transform. Then $T = \mathcal{F}^{-1}$ defines an algebraic isomorphism of V onto B, the inverse map being the Fourier transform itself. We topologise V under the norm $f \mapsto \|\mathcal{F}^{-1}f\|_B$, thus obtaining a Banach space. By the Hausdorff-Young theorem, the space V contains W^p_σ, $1/p + 1/q = 1$, provided that $1 < p < 2$. The norm on W^p_σ induced from V is weaker than that induced by the embedding to $L^p(\mathbb{R})$. Theorem 1.2 applies in this setting, thus recovering Theorem 5 of [Mae06].

2. The Paley-Wiener theorem

Let $\sigma > 0$. The classical Paley-Wiener theorem states that the space W^2_σ may be equivalently characterised as $\mathcal{F}^{-1}L^2[-\sigma,\sigma]$, where $L^2[-\sigma,\sigma]$ is thought of as the subspace of $L^2(\mathbb{R})$ consisting of all functions with support in $[-\sigma,\sigma]$, and $\mathcal{F}^{-1}$ the inverse Fourier transform,

$$\mathcal{F}^{-1}F\,(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\imath x\xi} F(\xi)d\xi$$

for $F \in L^2[-\sigma,\sigma]$.

Since $\mathcal{F}$ extends to a topological isomorphism of $\mathcal{S}'(\mathbb{R})$, the element $F \in L^2[-\sigma,\sigma]$ is uniquely determined by $\mathcal{F}^{-1}F$. Moreover, the norm of an element $f = \mathcal{F}^{-1}F$ in W^2_σ, i.e., in $L^2(\mathbb{R})$, just amounts to the $(1/\sqrt{2\pi}$-multiple of the) norm of F in $L^2[-\sigma,\sigma]$, which is due to the Plancherel theorem.

By the Hausdorff-Young theorem, the Fourier transform of a function $f \in L^p(\mathbb{R})$ belongs to $L^q(\mathbb{R})$, provided that $1 \le p \le 2$. Here, $1/p + 1/q = 1$. Moreover, $\|\hat{f}\|_{L^q(\mathbb{R})} \le (2\pi)^{1/q}\|f\|_{L^p(\mathbb{R})}$. For $p > 2$ the Fourier transform of a function in $L^p(\mathbb{R})$ may be a distribution of positive order. More precisely, if $p > 2$ and o satisfies $o > 1/2 - 1/p$, then $\hat{f}$ is a distribution of order $\le o$ in $\mathbb{R}$, and the bound cannot be relaxed, cf. [Hoe83, 7.6.6].

We thus conclude that any function $f \in W^p_\sigma$ with $1 < p < \infty$ possesses an integral representation

$$f = \mathcal{F}^{-1}\hat{f}, \tag{2.1}$$

where $\hat{f} \in \mathcal{E}'_{[-\sigma,\sigma]}$ is the Fourier transform of f. However, the Fourier transforms of functions in W^p_σ can hardly be characterised in $\mathcal{E}'_{[-\sigma,\sigma]}$ in terms of their regularity. As some suggestive evidence to this fact we mention that in [Mae06] a function $F \in L^q[-\sigma,\sigma]$ with $q > 2$ is constructed, such that $\mathcal{F}^{-1}F$ does not belong to W^p_σ with any $1 \le p \le 2$.

To characterise the Fourier transforms of functions in W^p_σ we invoke Banach spaces $l^p(\mathbb{Z})$.

3. Characterisation of Wiener classes

There is no less of generality in assuming that $\sigma = \pi$, for $f(x) \in W^p_\sigma$ if and only if $f((\pi/\sigma)x) \in W^p_\pi$.

Let $p \geq 1$. Denote by $l^p(\mathbb{Z})$ the set of all two-sided sequences $c = (c_n)_{n\in Z}$ of complex numbers, such that

$$\|c\|_{l^p(\mathbb{Z})} := \Big(\sum_{n\in\mathbb{Z}} |c_n|^p\Big)^{1/p}$$

is finite. If given the norm $c \mapsto \|c\|_{l^p(\mathbb{Z})}$, this vector space is well-known to be Banach.

To motivate the next definition, we observe that if f is an entire function of exponential type $\leq \pi$, such that $\mathcal{F}f \in L^1[-\pi, \pi]$, then for any smooth function φ on $[-\pi, \pi]$ we get

$$\langle \mathcal{F}f, \varphi\rangle = \sum_{n\in\mathbb{Z}} f(n)\hat{\varphi}(n),$$

where $\hat{\varphi}(n) = \displaystyle\int_{-\pi}^{\pi} e^{-\imath nx}\varphi(x)dx$.

Each sequence $c \in l^p(\mathbb{Z})$ defines a linear functional T_c on the space $C^\infty(\mathbb{R})$ by the formula

$$\langle T_c, \varphi\rangle := \sum_{n\in\mathbb{Z}} c_n\hat{\varphi}(n) \tag{3.1}$$

for $\varphi \in C^\infty(\mathbb{R})$.

Lemma 3.1. *Suppose* $1 < p < \infty$. *As defined above,* T_c *is a distribution on the real axis with a support in* $[-\pi, \pi]$.

Proof. If $\varphi \in C^\infty(\mathbb{R})$ then partial integration yields

$$\hat{\varphi}(n) = \frac{1}{-\imath n}\Big((-1)^n\left(\varphi(\pi) - \varphi(-\pi)\right) - \int_{-\pi}^{\pi} e^{-\imath nx}\varphi'(x)dx\Big)$$

whence

$$|\hat{\varphi}(n)| \leq \frac{1}{|n|}\Big(4\pi \sup_{x\in[-\pi,\pi]} |\varphi'(x)|\Big)$$

for all $n \neq 0$. Using the Hölder inequality, we therefore obtain

$$\begin{aligned} |\langle T_c, \varphi\rangle| &\leq |c_0|\Big(2\pi \sup_{x\in[-\pi,\pi]} |\varphi(x)|\Big) + \sum_{n\neq 0} \frac{|c_n|}{|n|}\Big(4\pi \sup_{x\in[-\pi,\pi]} |\varphi'(x)|\Big) \\ &\leq \|c\|_{l^p(\mathbb{Z})}\Big(2\pi \sup_{x\in[-\pi,\pi]} |\varphi(x)| + \Big(\sum_{n\neq 0} \frac{1}{|n|^q}\Big)^{1/q} 4\pi \sup_{x\in[-\pi,\pi]} |\varphi'(x)|\Big). \end{aligned}$$

This shows that T_c is a distribution of order ≤ 1 on $\mathbb{R}$ with a support in $[-\pi, \pi]$, as desired. □

Given a function $f \in L^1[-\pi, \pi]$, we set

$$c(\check{f}) = (\check{f}(n))_{n\in\mathbb{Z}} \tag{3.2}$$

where $\check{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{\imath nx}f(x)dx$. Obviously, $c(\check{f}) \in l^\infty(\mathbb{Z})$ and

$$\|c(\check{f})\|_{l^\infty(\mathbb{Z})} \leq \frac{1}{2\pi}\,\|f\|_{L^1[-\pi,\pi]}.$$

Lemma 3.2. *Assume that* $f \in L^1(\mathbb{R})$ *is supported in* $[-\pi, \pi]$. *Then* $T_{c(\check{f})} = f$ *in the sense of distributions on* $\mathbb{R}$.

Proof. If $\varphi \in C^\infty(\mathbb{R})$, then

$$\begin{aligned}\langle T_{c(\check{f})}, \varphi \rangle &= \sum_{n\in\mathbb{Z}} \check{f}(n)\hat{\varphi}(n) \\ &= \lim_{N\to\infty} \int_{-\pi}^{\pi} f(x)\Big(\sum_{n=-N}^{N} \frac{1}{2\pi}\hat{\varphi}(n)e^{\imath nx}\Big) dx.\end{aligned}$$

The sums in the parentheses are actually partial sums of the Fourier series of the test function φ restricted to $[-\pi,\pi]$. By the Dini theorem, this series converges to $\varphi(x)$ at each point x of the interval $[-\pi,\pi]$ with the possible exception of $x = \pm\pi$. Moreover, the partial sums of this series are bounded on the interval $[-\pi,\pi]$ uniformly in N. Indeed, the restriction of φ to $[-\pi,\pi]$ can be written in the form $\varphi(x) = \ell(x)+r(x)$, where $\ell(x) = ax$ is a linear function and $r(x)$ is a smooth function in $[-\pi,\pi]$ satisfying $r(-\pi) = r(\pi)$. It suffices to take $a = (\varphi(\pi) - \varphi(-\pi))/2\pi$. The partial sums of the Fourier series of $\ell(x)$ are bounded in $[-\pi,\pi]$ uniformly in N, as is immediately checked. On the other hand, the Fourier series of $r(x)$ converges to $r(x)$ uniformly on $[-\pi,\pi]$, hence its partial sums are bounded in this interval uniformly in N. Summarising, we conclude by the Lebesgue theorem on majorised convergence that

$$\begin{aligned}\langle T_{c(\check{f})}, \varphi \rangle &= \int_{-\pi}^{\pi} f(x)\Big(\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{2\pi}\hat{\varphi}(n)e^{\imath nx}\Big) dx \\ &= \int_{-\pi}^{\pi} f(x)\varphi(x) dx,\end{aligned}$$

as desired. □

Denote by $\mathcal{E}'^{p}_{[-\pi,\pi]}$ the space of all distributions of the form T_c on $\mathbb{R}$, where $c \in l^p(\mathbb{Z})$.

To identify $\mathcal{E}'^{p}_{[-\pi,\pi]}$ with the Fourier image of the Wiener space W^p_π, we invoke the classical Plancherel-Pólya theorem.

Theorem 3.3. *Suppose* $1 < p < \infty$. *Then:*

1) *For any sequence* $c \in l^p(\mathbb{Z})$, *the series*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (-1)^n \frac{\sin \pi z}{\pi(z-n)} \tag{3.3}$$

converges in the $L^p(\mathbb{R})$ *-norm (and uniformly on each compact set in* $\mathbb{C}$*) to a function* $f \in W^p_\pi$ *which is actually the unique solution of the interpolation problem* $f(n) = c_n$, $n \in \mathbb{Z}$.

2) *Conversely, for every function* $f \in W^p_\pi$, *the sequence* $c(f) := (f(n))_{n\in\mathbb{Z}}$ *belongs to* $l^p(\mathbb{Z})$.

Proof. See [PP38] and elsewhere. □

The equality (3.3) gains in interest if we realise that $\mathcal{F}f = T_c$, and so it just amounts to saying that

$$f = \mathcal{F}^{-1} T_{c(f)},$$

cf. (2.1). Indeed, Lemma 3.1 states that $T_c \in \mathcal{E}'_{[-\pi,\pi]}$, hence the inverse Fourier transform of T_c is well defined in $\mathcal{S}'(\mathbb{R})$. In fact, it is an entire function of exponential type $\leq \pi$ given by

$$\begin{aligned}\mathcal{F}^{-1}T_c &= \frac{1}{2\pi}\langle T_c, e^{\imath z\xi}\rangle \\ &= \sum_{n\in\mathbb{Z}} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\imath(z-n)\xi}d\xi \\ &= \sum_{n\in\mathbb{Z}} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(z-n)\xi\, d\xi \\ &= \sum_{n\in\mathbb{Z}} c_n(-1)^n \frac{\sin \pi z}{\pi(z-n)},\end{aligned}$$

as desired.

Theorem 3.4. *For $p > 1$, the Fourier transform defines an algebraic isomorphism of the space W^p_π onto $\mathcal{E}'^p_{[-\pi,\pi]}$.*

Proof. By the Plancherel-Pólya theorem, if $f \in W^p_\pi$ then the sequence $(f(n))_{n\in\mathbb{Z}}$ belongs to $l^p(\mathbb{Z})$ and f is represented by formula (3.3). It follows that $\mathcal{F}f = T_{c(f)}$ lies in $\mathcal{E}'^p_{[-\pi,\pi]}$, i.e., the Fourier transform maps W^p_π into $\mathcal{E}'^p_{[-\pi,\pi]}$. Since $W^p_\pi \hookrightarrow \mathcal{S}'(\mathbb{R})$, this map is injective. It remains to prove that it is surjective. To this end, fix a distribution $T_c \in \mathcal{E}'^p_{[-\pi,\pi]}$, i.e., let $c \in l^p(\mathbb{Z})$. Define f by formula (3.3), then $f \in W^p_\pi$ and $c(f) = c$, which is due to the Plancherel-Pólya theorem. By the above, $\mathcal{F}f = T_c$, as desired. □

If $1 < p \leq 2$, then the distributions of $\mathcal{E}'^p_{[-\pi,\pi]}$ are functions of class $L^q[-\pi,\pi]$ while they do not exhaust all of $L^q[-\pi,\pi]$. For $p > 2$, the distributions of $\mathcal{E}'^p_{[-\pi,\pi]}$ are no longer functions and even measures on $[-\pi,\pi]$, as it will be shown in Section 4.

4. A counterexample

We make use of the following well-known theorem, see for instance Theorem 6.4 on p. 326 in [Zyg59, Vol. 1].

Theorem 4.1. *Let*

$$\sum_{k=1}^{\infty} a_k \cos(n_k x) + b_k \sin(n_k x), \quad x \in [-\pi,\pi],$$

be a gap trigonometric series, i.e., $\frac{n_{k+1}}{n_k} \geq q > 1$. If it is summable by some linear summation method (e.g., by the Abel-Poisson method) on a set of positive measure, then $\sum_{k=1}^{\infty} a_k^2 + b_k^2 < \infty$.

Consider the entire function

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \frac{\sin \pi z}{\pi(z-2^k)},$$

$z \in \mathbb{C}$. By Theorem 3.3, f is in W^p_π for each $p > 2$. The trigonometric series of the Fourier transform of f on $[-\pi,\pi]$ is

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left(\cos(2^k\xi) - \imath \sin(2^k\xi)\right), \tag{4.1}$$

as is easy to see. Theorem 4.1 shows that this series fails to be summable by the Abel-Poisson summation method almost everywhere in $[-\pi,\pi]$.

Since the Fourier series of a measure in $[-\pi, \pi]$ is for all that summable by the Abel-Poisson summation method almost everywhere, we deduce that (4.1) cannot be the Fourier series of any measure in the interval $[-\pi, \pi]$.

Thus, $c(f)$ belongs to all spaces $l^p(\mathbb{Z})$ with $p > 2$. However, the distribution $T_{c(f)}$ fails to be a measure [1].

5. Estimates in Wiener spaces

Theorem 3.4 actually gives rise to two natural norms in the Wiener space W^p_π, where $p > 1$. The first of the two is the genuine norm of W^p_π, i.e., that induced by embedding to $L^p(\mathbb{R})$. The second norm is pulled back from $l^p(\mathbb{Z})$ under the algebraic isomorphism $f \mapsto \mathcal{F}f = T_{c(f)}$. Under both the norms W^p_π is complete, i.e., Banach.

Theorem 5.1. *Let $1 < p < \infty$. The norms $f \mapsto \|f\|_{L^p(\mathbb{R})}$ and $f \mapsto \|c(f)\|_{l^p(\mathbb{Z})}$ on W^p_π are equivalent.*

Proof. Let the space W^p_π be given the norm $f \mapsto \|f\|_{L^p(\mathbb{R})}$, and the space $\mathcal{E}'^p_{[-\pi,\pi]}$ be given the norm $T_c \mapsto \|c\|_{l^p(\mathbb{Z})}$. Then $\mathcal{E}'^p_{[-\pi,\pi]}$ is a Banach space, for so is the space $l^p(\mathbb{Z})$.

Theorem 5.1 just amounts to the fact that the Fourier transform maps W^p_π continuously into $\mathcal{E}'^p_{[-\pi,\pi]}$. Indeed, since the Fourier transform is an algebraic isomorphism of these spaces, it follows by the open mapping theorem that the inverse Fourier transform maps $\mathcal{E}'^p_{[-\pi,\pi]}$ continuously onto W^p_π. Therefore, the norms $f \mapsto \|f\|_{L^p(\mathbb{R})}$ and $f \mapsto \|\mathcal{F}f\|_{\mathcal{E}'^p_{[-\pi,\pi]}}$ on W^p_π are equivalent. It remains to recall that $\mathcal{F}f = T_{c(f)}$ for all $f \in W^p_\pi$.

To prove that the mapping $\mathcal{F} : W^p_\pi \to \mathcal{E}'^p_{[-\pi,\pi]}$ is continuous, we invoke the closed graph theorem. Pick a sequence $\{f_\nu\}_{\nu\in\mathbb{N}}$ in W^p_π, such that f_ν converges to $f \in W^p_\pi$ in the $L^p(\mathbb{R})$-norm and $\mathcal{F}f_\nu$ converges to T_c in $\mathcal{E}'^p_{[-\pi,\pi]}$. Our objective is to show that $\mathcal{F}f = T_c$.

By formula (3.3),

$$f_\nu(z) = \sum_{n\in\mathbb{Z}} f_\nu(n)(-1)^n \frac{\sin \pi z}{\pi(z-n)} \tag{5.1}$$

for all $z \in \mathbb{C}$, whenever $\nu = 1, 2, \ldots$. Let $\nu \to \infty$. By passing to a subsequence of $\{f_\nu\}$ we may actually assume that $f_\nu(z) \to f(x)$ for almost all $x \in \mathbb{R}$. On the other hand, the sequence

$$s_n(z) = (-1)^n \frac{\sin \pi z}{\pi(z-n)} \tag{5.2}$$

belongs to $l^q(\mathbb{Z})$ for each fixed $z \in \mathbb{C}$. Hence it follows that we can pass on the left hand side of (5.1) to the limit under the sum sign, when $\nu \to \infty$. This yields

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(-1)^n \frac{\sin \pi z}{\pi(z-n)}$$

for all $z \in \mathbb{R}$, and so for each $z \in \mathbb{C}$. Arguments similar to those after Theorem 3.3 show that $\mathcal{F}f = T_c$, which is the desired conclusion.

Hence, the graph of $\mathcal{F} : W^p_\pi \to \mathcal{E}'^p_{[-\pi,\pi]}$ is closed, and so the mapping $\mathcal{F}$ continuous, as desired. □

Theorem 5.1 goes back at least as far as [PP38], where a direct proof is given. A proof using techniques of Hardy spaces can be found in [Lev96]. This book contains an explicit estimate

$$\|c(f)\|_{l^p(\mathbb{Z})} \le (4/\pi)^{1/p} e^{2\pi} \|f\|_{L^p(\mathbb{R})}$$

[1] This example is a slight modification of an example given in [PP38].

for all $f \in W^p_\pi$. The advantage of our proof lies in the fact that it, being in the framework of an operator-theoretic approach, extends to many other integral representations.

Since the norm $f \mapsto \|c(f)\|_{l^p(\mathbb{Z})}$ in W^p_π is essentially easier to handle, from now on we endow the space W^p_π with this norm.

Corollary 5.2. *Suppose $f \in W^p_\pi$, where $1 < p < \infty$. Then the value of f at each $z \in \mathbb{C}$ is estimated by*

$$|f(z)| \leq \Big(\sum_{n \in \mathbb{Z}} \Big| \frac{\sin \pi z}{\pi (z-n)} \Big|^q \Big)^{1/q} \|c(f)\|_{l^p(\mathbb{Z})}.$$

Proof. For the proof, it is sufficient to represent f by formula (3.3) and apply the Hölder inequality. □

Given an entire function f of exponential type in $\mathbb{C}^n$, we can inductively apply Theorem 3.3 in each variable to f, thus extending this theorem to functions of several variables. We omit obvious formulations, referring the reader to [PP38]. Theorem 5.1 remains valid, too.

6. Optimal estimate of extrapolation

In this section we extend Theorem 0.1 to extrapolation from a finite set in Wiener spaces W^p_π. Pick a finite set $S = \{z_1, \ldots, z_N\}$ of pairwise distinct points in $\mathbb{C}$ and a point $z_0 \in \mathbb{C} \setminus S$.

Theorem 6.1. *Suppose $p > 1$ and let*

$$\ell_n(\alpha) = \sum_{k=1}^{N} \alpha_k \, s_n(z_k) \tag{6.1}$$

be the sequence that is nearest to the sequence $s_n(z_0)$ in the $l^q(\mathbb{Z})$-metric, where $1/p + 1/q = 1$. For $U = \{f \in W^p_\pi : \|c(f)\|_{l^p(\mathbb{Z})} \leq R\}$ with $R > 0$, the following is true:

1) *The unremovable error $\Omega(z_0, U)$ of optimal extrapolation of functions $f \in U$ from the finite set $S = \{z_1, \ldots, z_N\}$ to the point z_0 is*

$$\Omega(z_0, U) = R \, \|s(z_0) - \ell(\alpha)\|_{l^q(\mathbb{Z})}.$$

2) *The optimal linear algorithm of analytic continuation from the set S to the point z_0 is defined by*

$$A_0 \circ I(f) = \sum_{k=1}^{N} \alpha_k \, f(z_k)$$

for $f \in U$, cf. (6.1).

3) *The extremal functions $f_0 \in U$ have the property that their values at the points $n \in \mathbb{Z}$ are given by*

$$f_0(n) = \lambda e^{-\imath \arg \delta_n} \Big(\frac{|\delta_n|}{\|\delta\|_{l^q(\mathbb{Z})}} \Big)^{q-1},$$

where $\delta_n = s_n(z_0) - \ell_n(\alpha)$ and $|\lambda| = R$.

Proof. Consider the linear map $T : W^p_\pi \to l^p(\mathbb{Z})$ given by $Tf = c(f)$. By Theorem 3.3, this is an algebraic isomorphism. Our choice of the norm in W^p_π implies that this isomorphism is actually an isometry. We are now in a position to apply Theorem 1.2, which readily yields assertions 1) and 2). To show assertion 3) it suffices to combine Theorem 1.2, 3) with familiar properties of spaces $l^p(\mathbb{Z})$, which we formulate in the general case.

Lemma 6.2. *Let $(\mathcal{X},\mu)$ be a space with measure, $1 < p \le \infty$ and $1/p+1/q = 1$. If $g \in L^q(\mathcal{X},\mu)$, then $l(f) = \int_{\mathcal{X}} f(x)\overline{g(x)}\,d\mu$ defines a continuous linear functional on $L^p(\mathcal{X},\mu)$, such that $|l(f)| \le \|f\|_{L^p(\mathcal{X},\mu)}\,\|g\|_{L^q(\mathcal{X},\mu)}$ for all $f \in L^p(\mathcal{X},\mu)$. Equality holds if and only if $f(x) = \lambda e^{\imath \arg g(x)}\Big(\dfrac{|g(x)|}{\|g\|_{L^q(\mathcal{X},\mu)}}\Big)^{q-1}$ for almost all $x \in \mathcal{X}$, where $|\lambda| = \|f\|_{L^p(\mathcal{X},\mu)}$.*

Proof. See for instance [KA77, pp. 185, 257]. □

Hence it follows that the norm of the functional l on $L^p(\mathcal{X},\mu)$ just amounts to $\|g\|_{L^q(\mathcal{X},\mu)}$. Moreover, the subdifferential of the norm $\|\cdot\|_{L^p(\mathcal{X},\mu)'}$ at l consists of the only element

$$f(x) = e^{\imath \arg g(x)}\Big(\frac{|g(x)|}{\|g\|_{L^q(\mathcal{X},\mu)}}\Big)^{q-1}.$$

This completes the proof of Theorem 6.1. □

7. Algebraic formulas

Theorem 0.1 still makes sense in purely algebraic context. Namely, let V be a $\mathbb{K}$-vector space and $\{L_1,\dots,L_N\}$ a linearly independent system of linear functionals on V. We consider the problem of recovering a linear functional L on a subset U of V through $L_1,\dots,L_N$.

Set $Z := \ker L_1 \cap \ldots \cap \ker L_N$. This is a vector subspace of codimension N in V, and so complemented. We fix an algebraic complement C of Z in V, so that $V = C \oplus Z$. Note that this decomposition is actually topological if V is a Fréchet space.

Lemma 7.1. *Let $\{f_1,\dots,f_N\}$ be a basis in C. The Gram matrix*

$$G = \begin{pmatrix} L_1(f_1) & \dots & L_N(f_1) \\ & \dots & \\ L_1(f_N) & \dots & L_N(f_N) \end{pmatrix}$$

is invertible, i.e., the determinant of G is different from zero.

Proof. This is an easy exercise in linear algebra. □

The following formulas make the decomposition $V = C \oplus Z$ explicit. They are known in the case of Hilbert spaces, cf. [Gan66, p. 228].

Lemma 7.2. *Let $\{f_1,\dots,f_N\}$ be a basis in C. Then each $f \in V$ can be uniquely written in the form*

$$f = \frac{-1}{\det G}\ \det\begin{pmatrix} 0 & L_1(f) & \dots & L_N(f) \\ f_1 & & & \\ \dots & & G & \\ f_N & & & \end{pmatrix} + \pi_Z(f), \qquad (7.1)$$

where $\pi_Z(f) \in Z$.

Proof. Indeed, there are unique constants $c_1,\dots,c_N \in \mathbb{K}$ and an element $\pi_Z(f) \in Z$, such that $f = c_1 f_1 + \ldots + c_N f_N + \pi_Z(f)$. Applying the functionals $L_1,\dots,L_N$ to this equality, we get a linear system for the unknown coefficients $c_1,\dots,c_N$. More precisely,

$$\begin{array}{rcccccc} L_1(f) & = & c_1 L_1(f_1) & + & \dots & + & c_N L_1(f_N), \\ & & & & \dots & & \\ L_N(f) & = & c_1 L_N(f_1) & + & \dots & + & c_N L_N(f_N). \end{array}$$

Solving this system by Cramer's rule and using the expansion theorem for determinants gives (7.1). □

From (7.1) we easily deduce that

$$\pi_Z(f) = \frac{1}{\det G} \det \begin{pmatrix} f & L_1(f) & \dots & L_N(f) \\ f_1 & & & \\ \dots & & G & \\ f_N & & & \end{pmatrix} \tag{7.2}$$

for every $f \in V$.

Denote by $\mathcal{A}$ the set of all algorithms $A : \mathbb{K}^N \to \mathbb{K}$ recovering L on U through $L_1, \dots, L_N$. The error $E(A; L, U)$ of an algorithm A is the supremum of $|L(f) - A \circ I(f)|$ over all $f \in U$. This definition does not assume any topology in the space V. One needs a topology in V, when applying methods of functional analysis.

By the dual problem for the problem of recovering a fixed linear functional L on U through linear functionals $L_1, \dots, L_N$ is meant the problem of evaluating

$$\sup_{f \in U \cap Z} |L(f)|.$$

Our basic assumption on the geometry of the set U is that U is circular. Recall that U is called circular if $\lambda f \in U$ for all $f \in U$ and all $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. For instance, the closed ball of radius $R > 0$ about the origin in a normed space V is a circular set.

As mentioned in the introduction, the assertion 1) of the following theorem goes back at least as far as [Osi76]. Since it of great importance, we give an explicit version of this result.

Theorem 7.3. *Suppose U is a circular set in V invariant under π_Z, and G the Gram matrix $(L_j(f_i))_{\substack{i=1,\dots,N \\ j=1,\dots,N}}$ introduced in Lemma 7.1. Then:*

1) *There is a linear algorithm $A_0 = \ell(\alpha; L_1, \dots, L_N)$ with the property that $\Omega(L, U) = E(A_0; L, U)$. Moreover,*

$$\Omega(L, U) = \sup_{f \in U \cap Z} |L(f)|.$$

2) *The optimal error of recovering the functional L through $L_1, \dots, L_N$ on U is given by*

$$\Omega(L, U) = \sup_{f \in U} \left| \frac{1}{\det G} \det \begin{pmatrix} L(f) & L_1(f) & \dots & L_N(f) \\ L(f_1) & & & \\ \dots & & G & \\ L(f_N) & & & \end{pmatrix} \right|.$$

3) *The value $L(f)$ at any element $f \in U$ satisfies $|L(f) - A_0 \circ I(f)| \le \Omega(L, U)$, where*

$$A_0 \circ I(f) = \frac{-1}{\det G} \det \begin{pmatrix} 0 & L_1(f) & \dots & L_N(f) \\ L(f_1) & & & \\ \dots & & G & \\ L(f_N) & & & \end{pmatrix}$$

is the optimal linear algorithm.

Proof. We first prove that

$$\sup_{f \in U \cap Z} |L(f)| = \inf_{c \in \mathbb{K}} \sup_{f \in U \cap Z} |L(f) - c|. \tag{7.3}$$

For this purpose, fix $f \in U \cap Z$ with $L(f) \neq 0$, and $c \in \mathbb{K}$ different from zero. Since $L_1, \dots, L_N$ are linear, the set $U \cap Z$ is still circular. From this it follows that the element

$$f_c = -f \, \exp \imath(\arg c - \arg L(f))$$

belongs to $U \cap Z$. After elementary transformations we get

$$\begin{array}{rcl} |L(f) - c| & \leq & |L(f)| + |c| \\ & = & |L(f_c) - c| \\ & \leq & |L(f_c)| + |c|, \end{array}$$

whence

$$\sup_{f \in U \cap Z} |L(f) - c| = \sup_{f \in U \cap Z} \left(|L(f)| + |c| \right).$$

This establishes (7.3).

If $A : \mathbb{K}^N \to \mathbb{K}$ is an algorithm recovering L in U through $L_1, \dots, L_N$, then

$$\begin{array}{rcl} |L(f) - A \circ I(f)| & = & |L(f) - A(L_1(f), \dots, L_N(f))| \\ & = & |L(f) - A(0, \dots, 0)| \end{array}$$

for all $f \in U \cap Z$. Combining this with equality (7.3) yields

$$\begin{array}{rcl} \Omega(L, U) & \geq & \inf\limits_{A \in \mathcal{A}} \sup\limits_{f \in U \cap Z} |L(f) - A(0, \dots, 0)| \\ & = & \inf\limits_{c \in \mathbb{K}} \sup\limits_{f \in U \cap Z} |L(f) - c| \\ & = & \sup\limits_{f \in U \cap Z} |L(f)|. \end{array} \tag{7.4}$$

The equality in the second line is due to the fact that among all algorithms A there are those with $A(0, \dots, 0) = c$ for any given $c \in \mathbb{K}$.

Our next objective is to prove the inverse inequality. To do this, we denote by $\mathcal{A}_0$ the set of all linear algorithms $A = \ell(a; L_1, \dots, L_N)$, where $a \in \mathbb{K}^N$. As usual, we start with evaluating the residual functional. Pick any $f \in V$. Writing f in the form

$$f = c_1 f_1 + \dots + c_N f_N + \pi_Z(f),$$

cf. (7.1), we obtain

$$\Big| L(f) - \sum_{k=1}^{N} a_k L_k(f) \Big| = \Big| \sum_{j=1}^{N} c_j \Big(L(f_j) - \sum_{k=1}^{N} a_k L_k(f_j) \Big) + L(\pi_Z(f)) \Big|,$$

for $L_k(\pi_Z(f))$ vanishes for each $k = 1, \dots, N$. Hence it follows that

$$\Big| L(f) - \sum_{k=1}^{N} a_k L_k(f) \Big| \leq \Big| \sum_{j=1}^{N} c_j \Big(L(f_j) - \sum_{k=1}^{N} a_k L_k(f_j) \Big) \Big| + |L(\pi_Z(f))| \tag{7.5}$$

for all $f \in U$.

We are interested in finding the optimal algorithm rather than the errors of particular algorithms. Hence we choose $a_1, \dots, a_N$ in such a way that the part of f lying in C would not contribute to the residual function. Namely, we determine $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{K}^N$ from

$$\begin{array}{rcl} L(f_1) - \sum\limits_{k=1}^{N} \alpha_k L_k(f_1) & = & 0, \\ & \dots & \\ L(f_N) - \sum\limits_{k=1}^{N} \alpha_k L_k(f_N) & = & 0. \end{array} \tag{7.6}$$

By Lemma 7.1, this system has a unique solution $\alpha \in \mathbb{K}$. To this element there corresponds the linear algorithm $A_0 = \ell(\alpha; L_1, \dots, L_N)$. Therefore, (7.5) immediately implies

$$\begin{aligned}\Omega(L,U) &\le \inf_{A\in\mathcal{A}_0}\sup_{f\in U}|L(f)-A\circ I(f)| \\ &\le E(A_0;L,U) \\ &\le \sup_{f\in U}|L(\pi_Z(f))| \\ &= \sup_{f\in U\cap Z}|L(f)|, \qquad (7.7)\end{aligned}$$

since $\pi_Z(f)\in U\cap Z$ for all $f\in U$. The inverse inequality is proved in (7.4) whence the assertion 1) follows.

By Cramer's rule, (7.6) yields

$$\begin{aligned}A_0\circ I(f) &= \sum_{k=1}^{N}\alpha_k L_k(f) \\ &= \frac{-1}{\det G}\det\begin{pmatrix} 0 & L_1(f) & \dots & L_N(f) \\ L(f_1) & & & \\ \dots & & G & \\ L(f_N) & & & \end{pmatrix}\end{aligned}$$

for all $f\in V$. This is precisely the formula of assertion 3).

By (7.7),

$$\Omega(L,U)=\sup_{f\in U}|L(\pi_Z(f))|$$

which establishes the assertion 2) when combined with (7.2). Furthermore, if $f\in U$ then

$$\begin{aligned}|L(f)-A_0\circ I(f)| &= |L(\pi_Z(f))| \\ &\le \Omega(L,U),\end{aligned}$$

completing the proof of 3). □

If V is a Hilbert space and the decomposition $V=C\oplus Z$ is orthogonal, then any closed ball about the origin in V is obviously invariant under π_Z. This is no longer the case if V is Banach. However, given any fixed decomposition $V=C\oplus Z$, it is easy to show an equivalent norm on V, for which any closed ball about the origin is π_Z-invariant. This is simply $c+z\mapsto\|c\|_V+\|z\|_V$, which is equivalent to the genuine norm $c+z\mapsto\|c+z\|_V$ by the open map theorem. The invariance condition is thus not particularly restrictive.

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References

[FM01] Fedotov, A. M., and Maergoiz, L. S., *Optimal error of analytic continuation by non-exact data in Hilbert spaces of holomorphic functions*, Sibirsk. Math. Zh. **42** (2001), no. 5, 1106–1116.

[Gan66] Gantmacher, F. R., *Matrix Theory*, Nauka, Moscow, 1966.

[Hoe83] Hörmander, L., *The Analysis of Linear Partial Differential Operators, I*, Springer-Verlag, Berlin et al., 1983.

[KA77] Kantorovich, L. V., and Akilov, G. P., *Functional Analysis*, Nauka, Moscow, 1977.

[Lev96] Levin, B. Ya., *Lectures on Entire Functions*, Translation of mathematical monographs, 150, AMS, Providence, R.I., 1996.

[Lyu92] Lyubich, Yu. I., *Linear Functional Analysis*, In: Encyclopaedia of Mathematical Sciences, Vol. 19, Springer-Verlag, Berlin et al., 1992, 1–283.

[Mae97] Maergoiz, L. S., *Extremal properties of entire functions of Wiener class and their applications*, Doklady RAN **356** (1997), no. 2, 161–165.

[Mae00] Maergoiz, L. S., *Optimal estimate for extrapolation from a finite set in the Wiener class*, Sibirsk. Math. Zh. **41** (2000), no. 6, 1363–1373.

[Mae06] Maergoiz, L. S., *An Analogue of the Paley-Wiener Theorem for Entire Functions of Class W^p_σ, $1 < p < 2$, and Some Applications*, Computational Methods and Function Theory **6** (2006), no. 2, 459–469.

[MO91] Magaril-Il'yaev, G. G., and Osipenko, K. Yu., *On optimal recovery of functionals by non-exact data*, Mat. Zametki **50** (1991), no. 1, 85–93.

[MO02] Magaril-Il'yaev, G. G., and Osipenko, K. Yu. *Optimal recovery of functions and their derivatives from Fourier coefficients prescribed with an error*, Sb. Math. **193** (2002), no. 3, 387–407.

[MO03] Magaril-Il'yaev, G. G., and Osipenko, K. Yu. *Optimal reconstruction of functions and their derivatives from approximate information about the spectrum, and inequalities for the derivatives*, Funct. Anal. Appl. **37** (2003), no. 3, 203–214.

[MO75] Marchuk, A. G., and Osipenko, K. Yu., *The best approximation of functions given on a finite set up to certain inaccuracy*, Mat. Zametki **17** (1975), no. 3, 359–368.

[MR85] Micchelli, C. A., and Rivlin, T. J., *Lectures on optimal recovery*, Lect. Notes Math. **1129** (1985), p. 21–93.

[MR76] Micchelli, C. A., and Rivlin, T. J. (eds), *Optimal Estimations in Approximation Theory*, New York, Plenum Press, 1976.

[Osi76] Osipenko, K. Yu., *The best approximation of analytic functions by information on their values in a finite number of points*, Mat. Zametki **19** (1976), no. 1, 29–40.

[PP37] Plancherel, M., and Pólya, G., *Fonctions entiéres et intégrales de Fourier multiples, I*, Commentarii mathematici Helvetici **9** (1937), 224–248.

[PP38] Plancherel, M., and Pólya, G., *Fonctions entiéres et intégrales de Fourier multiples, II*, Commentarii mathematici Helvetici **10** (1938), 110–163.

[Phe88] Phelps, R. R., *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics, V. 1364, Springer-Verlag, New York-Berlin, 1988.

[Sin70] Singer, Ivan, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag, New York-Berlin, 1970, 415 pp.

[Tar95] Tarkhanov, N., *The Cauchy Problem for Solutions of Elliptic Equations*, Akademie Verlag, Berlin, 1995.

[Tik87] Tikhomirov, V. M., *Approximation theory*, In: Current Problems of Mathematics. Fundamental Directions. Vol. 14, VINITI, Moscow, 1987, 103–260.

[TW80] Traub, J. F., and Wozniakowski, H., *A General Theory of Optimal Algorithms*, Academic Press, New York, 1980.

[Zyg59] Zygmund, A., *Trigonometric Series*, Vols. I & II, Cambridge University Press, New York, 1959.

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