On rays of minimal growth for elliptic cone operators

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Abstract. We present an overview of some of our recent results on the existence of rays of minimal growth for elliptic cone operators and two new results concerning the necessity of certain conditions for the existence of such rays.

1. Introduction

The aim of this article is twofold. On the one hand, we present an overview of some of the results contained in [5, 6] on the subject in the title, and of the geometric perspective we developed in the course of the investigations leading to the aforementioned papers. We illustrate the main ideas of our approach by means of examples concerning Laplacians on a compact 2-manifold. Already this simple situation exhibits the structural richness and complexity of the general theory.

On the other hand, we offer some improvements, cf. Theorems 4.3 and 5.5, regarding necessary and sufficient conditions for a closed sector $\Lambda \subset \mathbb{C}$ to be a sector of minimal growth for a certain class of elliptic cone operators A and for the associated model operator $A_\wedge$.

Recall that a closed sector of the form

$$\Lambda = \{\lambda \in \mathbb{C} : \lambda = re^{i\theta} \text{ for } r \geq 0,\ \theta \in \mathbb{R},\ |\theta - \theta_0| \leq a\} \tag{1.1}$$

is called a *sector of minimal growth* (or of maximal decay) for a closed operator

$$A : \mathcal{D} \subset H \to H,$$

where H is a Hilbert space and $\mathcal{D}$ is dense in H, if there is a constant $R > 0$ such that $A - \lambda$ is invertible for every $\lambda \in \Lambda_R = \{\lambda \in \Lambda : |\lambda| \geq R\}$, and the resolvent $(A-\lambda)^{-1}$ satisfies either of the equivalent estimates

$$\big\|(A-\lambda)^{-1}\big\|_{\mathscr{L}(H)} \leq C/|\lambda|, \quad \big\|(A-\lambda)^{-1}\big\|_{\mathscr{L}(H,\mathcal{D})} \leq C \tag{1.2}$$

for some $C > 0$ and all $\lambda \in \Lambda_R$.

We are interested in cone operators on smooth manifolds with boundary. Specifically, let M be a smooth n-manifold with boundary $Y = \partial M$ and let $E \to$

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M be a Hermitian vector bundle over M. Fix a defining function x for Y. A differential *cone operator* of order m acting on sections of $E \to M$ is an operator of the form $A = x^{-m} P$ with P in the class $\mathrm{Diff}_b^m(M; E)$ of totally characteristic differential operators of order m, cf. Melrose [13]. We write $A \in x^{-m} \mathrm{Diff}_b^m(M; E)$. More explicitly, in the interior $\mathring{M}$ of M, A is a differential operator with smooth coefficients, and near the boundary, in local coordinates $(x, y) \in (0, \varepsilon) \times Y$, it is of the form

$$A = x^{-m} \sum_{k+|\alpha| \le m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha \tag{1.3}$$

with coefficients $a_{k\alpha}$ smooth up to $x = 0$; here $D_x = -i\partial/\partial x$ and likewise D_{y_j}. We will say that A (or P) has *coefficients independent of x near Y*, if the coefficients $a_{k\alpha}$ in (1.3) do not depend on x (this notion depends on the choice of tubular neighborhood map, defining function x, and connection on E. For a precise definition see [5]).

This paper consists of 5 sections. In Section 2 we review some basic properties of cone operators while in Section 3 we discuss the associated model operators. The new results on rays of minimal growth can be found in Sections 4 and 5. Apart from the works explicitly cited in the text, our list of references contains additional items referring to related works on resolvents and rays of minimal growth for elliptic operators.

2. Preliminaries on cone operators

Let A be a differential cone operator. As introduced in [5], the principal symbol of A, ${}^c\boldsymbol{\sigma}(A)$, is defined on the c-cotangent ${}^cT^*M$ of M rather than on the cotangent itself. Over $\mathring{M}$ it is essentially the usual principal symbol, and equal to

$$\sum_{k+|\alpha|=m} a_{k\alpha}(x, y)\xi^k \eta^\alpha$$

near the boundary Y, see (1.3).

Example. Let M be a compact 2-manifold with boundary $Y = S^1$. Let $g_Y(x)$ be a smooth family of Riemannian metrics on S^1 such that $g_Y(0)$ is the standard metric, dy^2. We equip M with a "cone metric" g that near Y takes the form $g = dx^2 + x^2 g_Y(x)$ (g is a regular Riemannian metric in the interior of M). Then, near Y, the Laplace-Beltrami operator Δ has the form

$$x^{-2}\big((xD_x)^2 + a(x, y)(xD_x) + \Delta_Y(x)\big), \tag{2.1}$$

where $a(x, y)$ is a smooth function with $a(0, y) = 0$ and $\Delta_Y(x)$ is the nonnegative Laplacian on S^1 associated with $g_Y(x)$. In this case, near the boundary, we have

$${}^c\boldsymbol{\sigma}(\Delta) = \xi^2 + \boldsymbol{\sigma}(\Delta_Y(x)).$$

Ellipticity and boundary spectrum

An operator $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ is *c-elliptic* if ${}^c\boldsymbol{\sigma}(A)$ is invertible on ${}^cT^*M\backslash 0$. Moreover, the family $A-\lambda$ is said to be *c-elliptic with parameter* $\lambda \in \Lambda \subset \mathbb{C}$ if ${}^c\boldsymbol{\sigma}(A)-\lambda$ is invertible on $({}^cT^*M \times \Lambda)\backslash 0$.

Associated with $A = x^{-m}P$ there is an operator-valued polynomial

$$\mathbb{C} \ni \sigma \mapsto \hat{P}(\sigma) \in \operatorname{Diff}^m(Y;E|_Y)$$

called the *conormal symbol* of P (and of A). If we write A as in (1.3), then

$$\hat{P}(\sigma) = \sum_{k+|\alpha|\le m} a_{k\alpha}(0,y)\sigma^k D_y^\alpha.$$

If A is c-elliptic, then $\hat{P}(\sigma)$ is invertible for all $\sigma \in \mathbb{C}$ except a discrete set $\operatorname{spec}_b(A)$, the *boundary spectrum of* A, cf. [13]; $\hat{P}(\sigma)$ is a holomorphic family of elliptic operators on Y and $\sigma \to \hat{P}(\sigma)^{-1}$ is a meromorphic operator-valued function on $\mathbb{C}$.

Example. The Laplacian (2.1) is clearly c-elliptic. If y is the angular variable on S^1, then

$$\hat{P}(\sigma) = \sigma^2 + \Delta_Y(0) = \sigma^2 + D_y^2,$$

and the boundary spectrum of Δ is given by

$$\operatorname{spec}_b(\Delta) = \{\pm ik : k \in \mathbb{N}_0\}.$$

Closed extensions

Let $\mathfrak{m}$ be a positive b-density on M, that is, $x\mathfrak{m}$ is a smooth everywhere positive density on M. Let $L_b^2(M;E)$ be the L^2 space of sections of E with respect to the Hermitian form on E and the density $\mathfrak{m}$. Consider A initially defined on $C_0^\infty(\mathring{M};E)$ and look at it as an unbounded operator on the Hilbert space

$$x^{-m/2}L_b^2(M;E) = L^2(M;E;x^{2m}\mathfrak{m}).$$

The particular weight $x^{-m/2}$ is just a convenient normalization and represents no loss. If we are interested in A on $x^\mu L_b^2(M;E)$ for $\mu \in \mathbb{R}$, we can base all our analysis on the space $x^{-m/2}L_b^2(M;E)$ by considering the operator $x^{-\mu-m/2}A\,x^{\mu+m/2}$.

Typically, A has a large class of closed extensions

$$A_{\mathcal{D}} : \mathcal{D} \subset x^{-m/2}L_b^2(M;E) \to x^{-m/2}L_b^2(M;E). \tag{2.2}$$

There are two canonical closed extensions, namely the ones with domains

$$\mathcal{D}_{\min}(A) = \text{closure of } C_0^\infty(\mathring{M};E) \text{ with respect to } \|\cdot\|_A,$$
$$\mathcal{D}_{\max}(A) = \{u \in x^{-m/2}L_b^2(M;E) : Au \in x^{-m/2}L_b^2(M;E)\},$$

where $\|u\|_A = \|u\| + \|Au\|$ is the graph norm in $\mathcal{D}_{\max}(A)$. Both domains are dense in $x^{-m/2}L_b^2(M;E)$, and for any closed extension (2.2),

$$\mathcal{D}_{\min}(A) \subseteq \mathcal{D} \subseteq \mathcal{D}_{\max}(A).$$

Let

$$\mathfrak{D}(A) = \{\mathcal{D} \subset \mathcal{D}_{\max}(A) : \mathcal{D} \text{ is a vector space and } \mathcal{D}_{\min}(A) \subset \mathcal{D}\}.$$

The elements of $\mathfrak{D}(A)$ are in one-to-one correspondence with the subspaces of $\mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$. If the operator A is fixed and there is no possible ambiguity, we will omit A from the notation and will write simply $\mathcal{D}_{\min}$, $\mathcal{D}_{\max}$, and $\mathfrak{D}$.

Theorem 2.1 (Lesch [11])**.** *If $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ is c-elliptic, then*

$$\dim \mathcal{D}_{\max}/\mathcal{D}_{\min} < \infty$$

and all closed extensions of A are Fredholm. Moreover,

$$\operatorname{ind} A_{\mathcal{D}} = \operatorname{ind} A_{\mathcal{D}_{\min}} + \dim \mathcal{D}/\mathcal{D}_{\min}. \tag{2.3}$$

Modulo $\mathcal{D}_{\min}$, the elements of $\mathcal{D}_{\max}$ are determined by their asymptotic behavior near the boundary of M. The structure of these asymptotics depends on the conormal symbols of A and on the part of $\operatorname{spec}_b(A)$ in the strip $\{|\Im\sigma| < m/2\}$. More details will be discussed in the next section.

Corollary 2.2. *If A is c-elliptic and symmetric (formally selfadjoint), then*

$$\operatorname{ind} A_{\mathcal{D}_{\max}} = -\operatorname{ind} A_{\mathcal{D}_{\min}} \;\; \textit{and} \;\; \operatorname{ind} A_{\mathcal{D}_{\min}} = -\frac{1}{2}\dim \mathcal{D}_{\max}/\mathcal{D}_{\min}.$$

Example. Consider the cone Laplacian Δ, cf. (2.1). Then (3.2) and (3.6) imply

$$\dim \mathcal{D}_{\max}(\Delta)/\mathcal{D}_{\min}(\Delta) = 2$$

and thus, by the previous corollary,

$$\operatorname{ind} \Delta_{\min} = -1 \quad \text{and} \quad \operatorname{ind} \Delta_{\max} = 1. \tag{2.4}$$

If $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ is c-elliptic, the embedding $\mathcal{D}_{\max} \hookrightarrow x^{-m/2}L_b^2(M;E)$ is compact. Therefore, for every $\mathcal{D} \in \mathfrak{D}$ and $\lambda \in \mathbb{C}$, the operator $A_{\mathcal{D}} - \lambda$ is also Fredholm with $\operatorname{ind}(A_{\mathcal{D}} - \lambda) = \operatorname{ind} A_{\mathcal{D}}$. Consequently, if $\operatorname{spec}(A_{\mathcal{D}}) \neq \mathbb{C}$, then we necessarily have $\operatorname{ind} A_{\mathcal{D}} = 0$. For this reason, we will primarily be interested in the set of domains

$$\mathfrak{G} = \{\mathcal{D} \in \mathfrak{D} : \operatorname{ind} A_{\mathcal{D}} = 0\} \tag{2.5}$$

which is empty unless $\operatorname{ind} A_{\mathcal{D}_{\min}} \leq 0$ and $\operatorname{ind} A_{\mathcal{D}_{\max}} \geq 0$. Let $d'' = -\operatorname{ind} A_{\mathcal{D}_{\min}}$.

Using that the map $\mathfrak{D} \ni \mathcal{D} \mapsto \mathcal{D}/\mathcal{D}_{\min}$ is a bijection, we identify $\mathfrak{G}$ with the complex Grassmannian of d''-dimensional subspaces of $\mathcal{D}_{\max}/\mathcal{D}_{\min}$.

Example. For Δ we have

$$\mathfrak{G}(\Delta) \cong \mathbb{CP}^1 = S^2.$$

Note that by (2.3) and (2.4), $\operatorname{ind} \Delta_{\mathcal{D}} = 0$ if and only if $\dim \mathcal{D}/\mathcal{D}_{\min} = 1$.

We finish this section with the following proposition that gives a first glimpse of the complexity of the spectrum of elliptic cone operators.

Proposition 2.3. *If A is c-elliptic and* $\dim \mathfrak{G} > 0$*, then for any $\lambda \in \mathbb{C}$ there is a domain $\mathcal{D} \in \mathfrak{G}$ such that $\lambda \in \operatorname{spec}(A_{\mathcal{D}})$. If, in addition, A is symmetric on $\mathcal{D}_{\min}$, then for any $\lambda \in \mathbb{R}$ there is a $\mathcal{D} \in \mathfrak{G}$ such that $A_{\mathcal{D}}$ is selfadjoint and $\lambda \in \operatorname{spec}(A_{\mathcal{D}})$.*

A proof is given in [5, Propositions 5.7 and 6.7]. A surprising consequence of the second statement is that for any arbitrary negative number λ there is always a selfadjoint extension of A having λ as eigenvalue, even if A is positive on $\mathcal{D}_{\min}$.

3. The model operator

Let $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ be c-elliptic. The *model operator* $A_\wedge$ associated with A is an operator on N_+Y, the closed inward normal bundle of Y, that in local coordinates takes the form

$$A_\wedge = x^{-m} \sum_{k+|\alpha|\le m} a_{k\alpha}(0,y)(xD_x)^k D_y^\alpha,$$

if A is written as in (1.3). A Taylor expansion in x (at $x=0$) of the coefficients of the operator A induces a decomposition

$$x^m A = \sum_{k=0}^{N-1} P_k x^k + x^N \tilde{P}_N \quad \text{for every } N \in \mathbb{N}, \tag{3.1}$$

where each P_k has coefficients independent of x near Y. Thus the model operator can be written, near Y, as $A_\wedge = x^{-m}P_0$. In other words, $A_\wedge$ can be thought of as the "most singular" part of A.

We trivialize N_+Y as $Y^\wedge = [0,\infty)\times Y$. The operator $A_\wedge \in x^{-m}\operatorname{Diff}_b^m(Y^\wedge;E)$ acts on $C_0^\infty(\mathring{Y}^\wedge;E)$ and can be extended as a densely defined closed operator in $x^{-m/2}L_b^2(Y^\wedge;E)$. The space $L_b^2(Y^\wedge;E)$ is the L^2 space with respect to a density of the form $\frac{dx}{x}\otimes \pi^*\mathfrak{m}_Y$ and the canonically induced Hermitian form on $\pi^*(E|_Y)$, where $\pi : Y^\wedge \to Y$ is the projection on the factor Y. The density $\mathfrak{m}_Y$ is related to $\mathfrak{m}$ and, by abuse of notation, we denote $\pi^*(E|_Y)$ by E, cf. [5]. Again, there are two canonical domains $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$ and we denote by $\mathfrak{D}_\wedge$ the set of subspaces of $\mathcal{D}_{\wedge,\max}$ that contain $\mathcal{D}_{\wedge,\min}$. There is a natural (and useful) linear isomorphism

$$\theta : \mathcal{D}_{\max}/\mathcal{D}_{\min} \to \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min},$$

cf. Section 5. As a consequence we have

$$\dim \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} = \dim \mathcal{D}_{\max}/\mathcal{D}_{\min} \tag{3.2}$$

which by Theorem 2.1 is finite. It is known (cf. Lesch [11]) that $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ is isomorphic to a finite dimensional space $\mathcal{E}_{\wedge,\max}$ consisting of functions of the form

$$\varphi = \sum_{\substack{\sigma\in \operatorname{spec}_b(A)\\ |\Im\sigma|<m/2}} \left(\sum_{k=0}^{m_\sigma} c_{\sigma,k}(y)\log^k x\right) x^{i\sigma} \tag{3.3}$$

where $c_{\sigma,k} \in C^\infty(Y;E)$. More precisely, for every $u \in \mathcal{D}_{\wedge,\max}$ there is a function $\varphi \in \mathcal{E}_{\wedge,\max}$ such that $u(x,y) - \omega(x)\varphi(x,y) \in \mathcal{D}_{\wedge,\min}$ for some (hence any) cut-off function $\omega \in C_0^\infty([0,1))$, $\omega = 1$ near 0. The function φ is uniquely determined by the equivalence class $u + \mathcal{D}_{\wedge,\min}$.

We identify

$$\mathcal{E}_{\wedge,\max} = \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$$

and let

$$\pi_{\wedge,\max} : \mathcal{D}_{\wedge,\max} \to \mathcal{E}_{\wedge,\max}$$

be the canonical projection.

Contrary to the situation in Theorem 2.1, the closed extensions of $A_\wedge$ do not need to be Fredholm. However, if $A-\lambda$ is c-elliptic with parameter, then the canonical extensions $A_{\wedge,\min}-\lambda$ and $A_{\wedge,\max}-\lambda$ are both Fredholm for $\lambda\neq 0$, cf. [6, Remark 5.26]. Moreover, we have

$$\operatorname{ind}(A_{\wedge,\min}-\lambda)=\operatorname{ind}A_{\mathcal{D}_{\min}}, \tag{3.4}$$

cf. Corollary 5.35 in [6].

Example. On $Y^\wedge=[0,\infty)\times S^1$ with the cone metric $dx^2+x^2dy^2$, the Laplace-Beltrami operator is given by

$$\Delta_\wedge = x^{-2}\big((xD_x)^2+\Delta_Y\big), \tag{3.5}$$

where Δ_Y is the nonnegative Laplacian on S^1. $\Delta_\wedge$ is precisely the model operator associated with the cone Laplacian Δ discussed in the previous section, cf. (2.1). It is easy to check that for any cut-off function $\omega\in C_0^\infty([0,1))$, the functions

$$\omega(x)\cdot 1,\ \Delta_\wedge(\omega(x)\cdot 1),\ \omega(x)\log x,\ \text{ and } \Delta_\wedge(\omega(x)\log x)$$

are all in the space $x^{-1}L_b^2(Y^\wedge)$. Thus $\omega(x)\cdot 1$ and $\omega(x)\log x$ are elements of $\mathcal{D}_{\wedge,\max}$. In fact,

$$\mathcal{E}_{\wedge,\max}=\operatorname{span}\{1,\log x\}. \tag{3.6}$$

Observe that $\Delta-\lambda$ is c-elliptic with parameter $\lambda\in\mathbb{C}\backslash\mathbb{R}_+$ and therefore the closed extensions of $\Delta_\wedge-\lambda$ are Fredholm for every $\lambda\in\mathbb{C}\backslash\overline{\mathbb{R}}_+$.

The model operator has a dilation/scaling property that can be exploited to analyze its closed extensions and their resolvents from a geometric point of view. In order to describe this property we first introduce the one-parameter group of isometries

$$\mathbb{R}_+\ni\varrho\mapsto\kappa_\varrho : x^{-m/2}L_b^2(Y^\wedge;E)\to x^{-m/2}L_b^2(Y^\wedge;E)$$

which on functions is defined by

$$(\kappa_\varrho f)(x,y)=\varrho^{m/2}f(\varrho x,y). \tag{3.7}$$

It is easily verified that the operator $A_\wedge$ satisfies the relation

$$\kappa_\varrho A_\wedge=\varrho^{-m}A_\wedge\kappa_\varrho.$$

This implies

$$A_\wedge-\lambda=\varrho^m\kappa_\varrho(A_\wedge-\lambda/\varrho^m)\kappa_\varrho^{-1} \tag{3.8}$$

for every $\varrho>0$ and $\lambda\in\mathbb{C}$. This homogeneity property, called *κ-homogeneity*, will be used systematically to describe the closed extensions of $A_\wedge$ with nonempty resolvent sets.

It is convenient to introduce the set

$$\text{bg-res}\,A_\wedge=\{\lambda\in\mathbb{C}: A_{\wedge,\min}-\lambda \text{ is injective and } A_{\wedge,\max}-\lambda \text{ is surjective}\},$$

the *background resolvent set* of $A_\wedge$, cf. [5].

Lemma 3.1 (Lemma 7.3 in [5]). *If* $\lambda \in \text{bg-res}\, A_\wedge$ *and* $\mathcal{D} \in \mathfrak{D}_\wedge$, *then* $A_{\wedge,\mathcal{D}} - \lambda$ *is Fredholm. The set* $\text{bg-res}\, A_\wedge$ *is a disjoint union of open sectors,*

$$\text{bg-res}\, A_\wedge = \bigcup_{\alpha \in \mathfrak{I} \subset \mathbb{N}} \mathring{\Lambda}_\alpha.$$

This lemma follows immediately from (3.8).

For $\lambda \in \text{bg-res}\, A_\wedge$ and $\mathcal{D} \in \mathfrak{D}_\wedge$ we have

$$\operatorname{ind}(A_{\wedge,\mathcal{D}} - \lambda) = \operatorname{ind}(A_{\wedge,\min} - \lambda) + \dim \mathcal{D}/\mathcal{D}_{\wedge,\min}. \tag{3.9}$$

Moreover, the map

$$\mathring{\Lambda}_\alpha \ni \lambda \mapsto \operatorname{ind}(A_{\wedge,\mathcal{D}} - \lambda)$$

is constant since the embedding $\mathcal{D} \hookrightarrow x^{-m/2} L^2_b(Y^\wedge; E)$ is continuous. Now, in analogy with (2.5) we define

$$\mathfrak{G}_{\wedge,\alpha} = \{\mathcal{D} \in \mathfrak{D}_\wedge : \operatorname{ind}(A_{\wedge,\mathcal{D}} - \lambda) = 0 \text{ for } \lambda \in \mathring{\Lambda}_\alpha\}$$

and let $d''_\alpha = -\operatorname{ind}(A_{\wedge,\min} - \lambda)$ for $\lambda \in \mathring{\Lambda}_\alpha$. We identify $\mathfrak{G}_{\wedge,\alpha}$ with the complex Grassmannian of d''_α-dimensional subspaces of $\mathcal{E}_{\wedge,\max}$.

The canonical domains $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$ are both κ-invariant. Thus the group action κ_ϱ induces an action on $\mathcal{E}_{\wedge,\max}$. In general, κ_ϱ does not preserve the elements of $\mathfrak{D}_\wedge$. In fact, the set of κ-invariant domains in $\mathfrak{D}_\wedge$ is an analytic variety because it consists of the stationary points of a holomorphic flow, cf. Section 7 in [5]. To better analyze the resolvents of the closed extensions of $A_\wedge$ over the open sector $\mathring{\Lambda}_\alpha$, we will consider the manifold $\mathfrak{G}_{\wedge,\alpha}$ together with the flow generated by the induced action of κ_ϱ given by $\kappa_\varrho(\mathcal{D}/\mathcal{D}_{\wedge,\min}) = \kappa_\varrho(\mathcal{D})/\mathcal{D}_{\wedge,\min}$.

Example. The background resolvent set of $\Delta_\wedge$ is the open sector $\mathbb{C}\backslash\overline{\mathbb{R}}_+$; this is easily seen after noting that $\Delta_\wedge$ is the standard Laplacian in $\mathbb{R}^2$ written in polar coordinates. Moreover, since $\operatorname{ind}(\Delta_{\wedge,\min} - \lambda) = -1$ for every $\lambda \in \mathbb{C}\backslash\overline{\mathbb{R}}_+$, we have that $\mathcal{D} \in \mathfrak{D}_\wedge$ belongs to $\mathfrak{G}_\wedge$ if and only if $\dim \mathcal{D}/\mathcal{D}_{\wedge,\min} = 1$. Thus

$$\mathfrak{G}_\wedge \cong \mathbb{CP}^1 = S^2.$$

We identify $\mathcal{E}_{\wedge,\max}$ with $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ and use (3.6) to write

$$\mathcal{E}_{\wedge,\max} = \operatorname{span}\{1, \log x\}.$$

For $\mathcal{D} \in \mathfrak{G}_\wedge$ we then have

$$\pi_{\wedge,\max}\mathcal{D} = \operatorname{span}\{\zeta_0 \cdot 1 + \zeta_1 \log x\} \quad \text{for some } \zeta_0, \zeta_1 \in \mathbb{C},\ (\zeta_0, \zeta_1) \neq 0. \tag{3.10}$$

Hence, with κ as defined in (3.7), we get

$$\pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D} = \operatorname{span}\{(\zeta_0 - \zeta_1 \log \varrho) \cdot 1 + \zeta_1 \log x\}. \tag{3.11}$$

Clearly, the only κ-invariant domain in $\mathfrak{G}_\wedge$ is the domain $\mathcal{D}_F$ such that

$$\pi_{\wedge,\max}\mathcal{D}_F = \operatorname{span}\{1\};$$

$\mathcal{D}_F$ is precisely the domain of the Friedrichs extension of $\Delta_\wedge$, cf. [7]. Every domain $\mathcal{D} \in \mathfrak{G}_\wedge$ with $\zeta_1 \neq 0$ in (3.10) generates a nontrivial orbit as given by (3.11). In order to describe the flow of κ on these nonstationary points, rewrite (3.10) as

$$\pi_{\wedge,\max}\mathcal{D} = \operatorname{span}\left\{\tfrac{\zeta_0}{\zeta_1}\cdot 1 + \log x\right\}.$$

Then the projection to $\mathcal{E}_{\wedge,\max}$ of the dilation $\kappa_\varrho^{-1}\mathcal{D}$ is given by

$$\pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D} = \operatorname{span}\left\{\left(\tfrac{\zeta_0}{\zeta_1} - \log\varrho\right)\cdot 1 + \log x\right\}. \tag{3.12}$$

If $[\zeta_0 : \zeta_1] \in \mathbb{CP}^1$ is the point corresponding to $\mathcal{D}$, then $\kappa_\varrho^{-1}\mathcal{D}$ is represented by $[\zeta_0 - \zeta_1\log\varrho : \zeta_1]$. In other words, in the situation at hand, the flow generated by κ on $\mathfrak{G}_\wedge \cong \mathbb{CP}^1$ consists of curves that in projective coordinates are lines parallel to the real axis, see Figure 1.

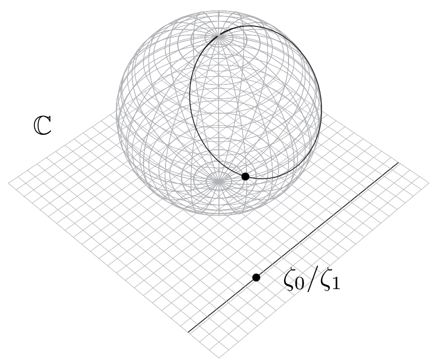

FIGURE 1. Orbit in $\mathfrak{G}_\wedge(\Delta_\wedge)$ generated by $\mathcal{D} \leftrightarrow \zeta_0/\zeta_1 \in \mathbb{C}$.

Observe that the Friedrichs extension corresponds to the point $[1 : 0] \in \mathbb{CP}^1$. Using

$$\pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D} = \operatorname{span}\left\{1 + \tfrac{\zeta_1}{\zeta_0 - \zeta_1\log\varrho}\log x\right\} \quad \text{if } \varrho \neq e^{\zeta_0/\zeta_1}$$

we see that

$$\pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D} \to \operatorname{span}\{1\} = \pi_{\wedge,\max}\mathcal{D}_F \quad \text{as } \varrho \to \infty \text{ or } \varrho \to 0. \tag{3.13}$$

Let $\mathcal{D}_0 \in \mathfrak{G}_\wedge$ be such that $\pi_{\wedge,\max}\mathcal{D}_0 = \operatorname{span}\{\log x\}$. This domain gives a selfadjoint extension of $\Delta_\wedge$ which on the sphere corresponds to the point $[0 : 1]$. The circle consisting of the orbit of $\mathcal{D}_0$ together with $\mathcal{D}_F$ is the set of domains of selfadjoint extensions of $\Delta_\wedge$.

4. Ray conditions on the model cone

In this section we will discuss the existence of sectors of minimal growth for the model operator $A_\wedge \in x^{-m}\operatorname{Diff}_b^m(Y^\wedge; E)$ associated with a c-elliptic cone operator. We fix a component $\mathring{\Lambda}_\alpha$ of bg-res $A_\wedge$ and let $\mathfrak{G}_\wedge = \mathfrak{G}_{\wedge,\alpha}$.

Let Λ be a closed sector such that $\Lambda\backslash 0 \subset \mathring{\Lambda}_\alpha$, cf. (1.1), let $\operatorname{res} A_{\wedge,\mathcal{D}}$ be the resolvent set of $A_{\wedge,\mathcal{D}}$. The κ-invariant domains are the simplest domains to analyze.

Proposition 4.1 (Proposition 8.4 in [5])**.** *Suppose $\mathcal{D} \in \mathfrak{G}_\wedge$ is κ-invariant. If there exists $\lambda_0 \in \mathring{\Lambda}_\alpha$ such that $A_{\wedge,\mathcal{D}} - \lambda_0$ is invertible, then $\mathring{\Lambda}_\alpha \subset \operatorname{res} A_{\wedge,\mathcal{D}}$ and Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}}$.*

If $\mathcal{D} \in \mathfrak{G}_\wedge$ is not κ-invariant, the situation is more complicated. Nonetheless, in [5] we found a condition necessary and sufficient for a sector Λ to be a sector of minimal growth for $A_{\wedge,\mathcal{D}}$. This condition is expressed in terms of finite dimensional spaces and projections that we proceed to discuss briefly.

For $\lambda \in \text{bg-res}\, A_\wedge$ we let

$$\mathcal{K}_{\wedge,\lambda} = \ker(A_{\wedge,\max} - \lambda).$$

Then

$$\operatorname{res} A_{\wedge,\mathcal{D}} = \text{bg-res}\, A_\wedge \cap \{\lambda : \mathcal{K}_{\wedge,\lambda} \cap \mathcal{D} = 0\},$$

and for $\lambda \in \operatorname{res} A_{\wedge,\mathcal{D}}$ we have

$$\mathcal{D}_{\wedge,\max} = \mathcal{K}_{\wedge,\lambda} \oplus \mathcal{D}. \tag{4.1}$$

Projecting on $\mathcal{E}_{\wedge,\max}$, this direct sum induces the decomposition

$$\mathcal{E}_{\wedge,\max} = \pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda} \oplus \pi_{\wedge,\max}\mathcal{D}, \tag{4.2}$$

and the projection on $\pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda}$ according to (4.2) is given by the map

$$\begin{aligned} \hat{\pi}_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}} &: \mathcal{E}_{\wedge,\max} \to \mathcal{E}_{\wedge,\max} \\ (u + \mathcal{D}_{\wedge,\min}) &\mapsto \pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}} u + \mathcal{D}_{\wedge,\min}, \end{aligned} \tag{4.3}$$

where $\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}}$ is the projection on $\mathcal{K}_{\wedge,\lambda}$ according to (4.1).

The following theorem gives a condition on the operator norm of (4.3) for a sector Λ to be a sector of minimal growth for $A_{\wedge,\mathcal{D}}$. Define

$$\|u\|_\lambda^2 = \|u\|^2 + |\lambda|^{-2}\|A_\wedge u\|^2$$

for $\lambda \neq 0$ and $u \in \mathcal{D}_{\wedge,\max}$.

Theorem 4.2. *Let $\mathcal{D} \in \mathfrak{G}_\wedge$, let Λ be a closed sector with $\Lambda\backslash 0 \subset \mathring{\Lambda}_\alpha$. Then Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}}$ if and only if there are C, $R > 0$ such that $\Lambda_R \subset \operatorname{res} A_{\wedge,\mathcal{D}}$, and*

$$\big\|\hat{\pi}_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}}\big\|_{\mathscr{L}(\mathcal{E}_{\wedge,\max},\|\cdot\|_\lambda)} \leq C \quad \text{for } \lambda \in \Lambda_R, \tag{4.4}$$

where $\hat{\pi}_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}}$ is the projection (4.3)*.*

This theorem is a rephrasing of [5, Theorem 8.7]. There the condition (4.4) appears in the equivalent form

$$\big\|\hat{\pi}_{\mathcal{K}_{\wedge,\hat{\lambda}},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\big\|_{\mathscr{L}(\mathcal{E}_{\wedge,\max})} \leq C \quad \text{for } \lambda \in \Lambda_R, \tag{4.4$'$}$$

where $\hat\lambda = \lambda/|\lambda|$, and $\hat\pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}$ is the projection on $\mathcal{K}_{\wedge,\hat\lambda}$ induced (following the steps (4.1)–(4.3)) by the direct sum

$$\mathcal{D}_{\wedge,\max} = \mathcal{K}_{\wedge,\varrho^{-m}\lambda} \oplus \kappa_\varrho^{-1}\mathcal{D} \tag{4.5}$$

for $\lambda \in \operatorname{res} A_{\wedge,\mathcal{D}}$ and $\varrho > 0$. This decomposition is a consequence of (4.1) and the κ-invariance of $\mathcal{D}_{\wedge,\max}$, as follows. First, the κ-homogeneity of $A_\wedge - \lambda$, cf. (3.8), implies

$$\kappa_\varrho^{-1}(\mathcal{K}_{\wedge,\lambda}) = \mathcal{K}_{\wedge,\varrho^{-m}\lambda} \text{ for } \varrho > 0.$$

Furthermore, if $\mathcal{D} \in \mathfrak{G}_\wedge$ and $\lambda \in \text{bg-res}\, A_\wedge$, then

$$\varrho^{-m}\lambda \in \operatorname{res} A_{\wedge,\kappa_\varrho^{-1}\mathcal{D}} \Longleftrightarrow \lambda \in \operatorname{res} A_{\wedge,\mathcal{D}}.$$

In particular,

$$\mathcal{K}_{\wedge,\varrho^{-m}\lambda} \cap \kappa_\varrho^{-1}\mathcal{D} = \{0\} \Longleftrightarrow \mathcal{K}_{\wedge,\lambda} \cap \mathcal{D} = \{0\},$$

as claimed.

The equivalence of (4.4) and (4.4′) follows immediately from the identity

$$\kappa^{-1}_{|\lambda|^{1/m}} \pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}}\, \kappa_{|\lambda|^{1/m}} = \pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}$$

using the relation (3.8) and the fact that κ is an isometry on $x^{-m/2}L^2_b$. The virtue of (4.4′) is that the norm is fixed, while the advantage of (4.4) lies in that it gives a more explicit dependence on λ and deals with a projection on a subspace of $\mathcal{E}_{\wedge,\max}$ with *fixed* complement $\pi_{\wedge,\max}\mathcal{D}$.

In [5, Corollary 8.22] it is proved that Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}}$ if and only if there are constants C, $R > 0$ such that $\Lambda_R \subset \operatorname{res} A_{\wedge,\mathcal{D}}$ and

$$\big\|\kappa^{-1}_{|\lambda|^{1/m}}(A_{\wedge,\mathcal{D}} - \lambda)^{-1}\big\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\wedge,\max})} \le C/|\lambda|, \quad \lambda \in \Lambda_R.$$

It can be shown that this estimate is equivalent to (4.4) and (4.4′).

Example. We consider again the model Laplacian $\Delta_\wedge$ from the previous section. Recall that $\text{bg-res}\,\Delta_\wedge = \mathbb{C}\backslash\overline{\mathbb{R}}_+$. For $\lambda \in \mathbb{C}\backslash\overline{\mathbb{R}}_+$, we have

$$\pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda} = \operatorname{span}\{-k_0\log(-\lambda) + k_1 \log x\} \quad \text{for some } k_0, k_1 > 0,$$

where log means the principal branch of the logarithm. Moreover, by (3.12),

$$\pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D} = \operatorname{span}\Big\{\big(\tfrac{\zeta_0}{\zeta_1} - \log\varrho\big)\cdot 1 + \log x\Big\}.$$

The projection in (4.4′) can be computed explicitly. Namely, if $u = \alpha_0 + \alpha_1 \log x \in \mathcal{E}_{\wedge,\max} = \operatorname{span}\{1, \log x\}$ and $\lambda = \varrho^m\lambda_0$, then

$$\hat\pi_{\mathcal{K}_{\wedge,\lambda_0},\kappa_\varrho^{-1}\mathcal{D}} u = \frac{-\alpha_0 + \alpha_1(\frac{\zeta_0}{\zeta_1} - \log\varrho)}{k_0\log(-\lambda_0) + k_1(\frac{\zeta_0}{\zeta_1} - \log\varrho)}\,(-k_0\log(-\lambda_0) + k_1\log x)\,. \tag{4.6}$$

Let Λ be a closed sector in $\mathbb{C}\backslash\mathbb{R}_+$ containing the half-plane $\{\Re\lambda < 0\}$. Since the family of projections (4.6) is bounded as $\varrho \to \infty$, uniformly for $|\lambda_0| = 1$ in Λ, regardless of the specific choice of α_0, α_1, Theorem 4.2 implies that every closed extension $\Delta_{\wedge,\mathcal{D}}$, $\mathcal{D} \in \mathfrak{G}_\wedge$, of the model Laplacian admits Λ as a sector of minimal growth.

Equivalent geometric condition

We identify $\mathfrak{G}_\wedge$ with the Grassmannian $\mathrm{Gr}_{d''}(\mathcal{E}_{\wedge,\max})$ where $d'' = -\operatorname{ind}(A_{\wedge,\min}-\lambda)$ for $\lambda \in \mathring{\Lambda}_\alpha \subset \text{bg-res}\, A_\wedge$. Let $d' = \dim \mathcal{K}_{\wedge,\lambda}$. The condition that in the Grassmannian $\mathrm{Gr}_{d''}(\mathcal{E}_{\wedge,\max})$, the curve

$$[R,\infty) \ni \varrho \mapsto \pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D}$$

does not approach the set

$$\mathscr{V}_{\mathcal{K}_{\wedge,\lambda}} = \{D \in \mathrm{Gr}_{d''}(\mathcal{E}_{\wedge,\max}) : D \cap \pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda} \neq 0\}$$

as $\varrho \to \infty$, is sufficient for the validity of (4.4′). This is [5, Theorem 8.28]. The following theorem states that the condition is also necessary.

For $D \in \mathrm{Gr}_{d''}(\mathcal{E}_{\wedge,\max})$ let

$$\Omega^-(D) = \big\{D' \in \mathrm{Gr}_{d''}(\mathcal{E}_{\wedge,\max}) : \exists\, \{\varrho_k\}_{k=1}^\infty \subset \mathbb{R}_+ \text{ such that } \varrho_k \to \infty \text{ and } \kappa_{\varrho_k}^{-1} D \to D' \text{ as } k \to \infty\big\}.$$

Theorem 4.3. *Let $\lambda_0 \in \mathring{\Lambda}_\alpha$. The ray through λ_0 is a ray of minimal growth for $A_{\wedge,\mathcal{D}}$ if and only if $\Omega^-(\pi_{\wedge,\max}\mathcal{D}) \cap \mathscr{V}_{\mathcal{K}_{\wedge,\lambda_0}} = \varnothing$.*

Proof. Let $\lambda_0 \in \mathring{\Lambda}_\alpha$ and $\mathcal{D} \in \mathfrak{G}_\wedge$. For simplicity, we use the notation

$$D = \pi_{\wedge,\max}\mathcal{D}, \quad \mathscr{V} = \mathscr{V}_{\mathcal{K}_{\wedge,\lambda_0}}, \quad \mathcal{K} = \pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda_0} \text{ and } \pi_{\mathcal{K},D} = \hat{\pi}_{\mathcal{K}_{\wedge,\lambda_0},\mathcal{D}}.$$

Suppose $\Omega^-(D) \cap \mathscr{V} = \varnothing$. Since $\Omega^-(D)$ and $\mathscr{V}$ are closed sets, there are a neighborhood $\mathcal{U}$ of $\mathscr{V}$ and a constant $R > 0$ such that if $\varrho > R$ then $\kappa_\varrho^{-1} D \notin \mathscr{V}$. Then Lemma 5.24 in [5] gives that $\|\pi_{\mathcal{K},\kappa_\varrho^{-1}D}\|$ is uniformly bounded as $\varrho \to \infty$, and therefore, by Theorem 4.2 the ray through λ_0 is a ray of minimal growth for $A_{\wedge,\mathcal{D}}$.

Assume now that there are C, $R > 0$ such that $\Lambda_R \subset \operatorname{res} A_{\wedge,\mathcal{D}}$ and the condition (4.4′) is satisfied. Suppose $\Omega^-(D) \cap \mathscr{V} \neq \varnothing$ and let $D_0 \in \Omega^-(D) \cap \mathscr{V}$. Since $D_0 \in \mathscr{V}$, we have $D_0 \cap \mathcal{K} \neq \{0\}$. On the other hand, $D_0 \in \Omega^-(D)$ implies that there is a sequence $\{\varrho_k\}_{k=1}^\infty \subset \mathbb{R}_+$ such that $\varrho_k \to \infty$ and $D_k = \kappa_{\varrho_k}^{-1} D \to D_0$ as $k \to \infty$. Note that for ϱ_k large we have $\varrho_k^m \lambda_0 \in \operatorname{res} A_{\wedge,\mathcal{D}}$, so $\lambda_0 \in \operatorname{res} A_{\wedge,\kappa_{\varrho_k}^{-1}\mathcal{D}}$ and therefore, $D_k \notin \mathscr{V}$.

Pick $v \in D_0 \cap \mathcal{K}$ with $\|v\| = 1$. Let π_{D_k} denote the orthogonal projection on D_k. Since $D_k \to D_0$ as $k \to \infty$, we have $\pi_{D_k} \to \pi_{D_0}$, so $v_k = \pi_{D_k} v \to \pi_{D_0} v = v$ as $k \to \infty$. Since $D_k \notin \mathscr{V}$, $v_k - v \neq 0$ and $\pi_{\mathcal{K},D_k} v_k = 0$. Hence

$$\pi_{\mathcal{K},D_k}\left(\frac{v - v_k}{\|v - v_k\|}\right) = \frac{v}{\|v - v_k\|} \to \infty \text{ as } k \to \infty,$$

since $\|v\| = 1$ and $v_k \to v$ as $k \to \infty$. But this implies that $\|\pi_{\mathcal{K},D_k}\| \to \infty$ contradicting the boundedness of the norm in (4.4′). Thus $\Omega^-(D) \cap \mathscr{V} = \varnothing$. ◻

Example. Let $\Delta_\wedge$ be the model Laplacian and let $\mathcal{D} \in \mathfrak{G}_\wedge$. In this case, the limiting set $\Omega^-(\pi_{\wedge,\max}\mathcal{D})$ consists of the one element of $\mathbb{CP}^1$ corresponding to the Friedrichs extension of $\Delta_\wedge$, cf. (3.13). From this new perspective, it is evident that every closed extension of $\Delta_\wedge$ must admit a sector of minimal growth.

5. Rays of minimal growth

We continue to assume that $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ is c-elliptic.

Unlike the case of a differential operator with smooth coefficients on a closed manifold, that a ray Γ is a ray of minimal growth for the principal symbol ${}^c\boldsymbol{\sigma}(A)$ of A is not expected to imply that Γ is a ray of minimal growth for A. In this context, it is useful to think of $A_\wedge$ as a symbol (the wedge symbol) associated with A, cf. Schulze [15], so that it is natural to impose ray conditions on $A_\wedge$. For this to work, however, we need to transfer the information about the given domain $\mathcal{D}$ of A on M to equivalent information for $A_\wedge$ on $Y^\wedge$, and vice versa.

Theorem 5.1 (Theorem 4.12 in [5])**.** *There is a natural isomorphism*

$$\theta^{-1} : \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} \to \mathcal{D}_{\max}/\mathcal{D}_{\min}$$

given by a finite iterative procedure that involves the boundary spectrum of A and the decomposition (3.1)*. In particular, if A has coefficients independent of x near Y, then θ is the identity map.*

Example. Let M be a compact 2-manifold with boundary $Y = S^1$. Let A be a differential operator in $x^{-2}\operatorname{Diff}_b^2(M)$ that over the interior of M coincides with some Laplacian, and near Y, is of the form

$$A = x^{-2}\big((xD_x)^2 + q(x)\Delta_Y\big),$$

where Δ_Y is the standard nonnegative Laplacian on S^1 and q is a smooth function. We assume q to have the form

$$q(x) = \alpha^2 + \beta x + x^2\gamma(x),$$

where α, β are constants such that $\frac{1}{2} < \alpha < 1$, $\beta \neq 0$, and $\gamma(0) = 1$. The associated model operator is then given by

$$A_\wedge = x^{-2}\big((xD_x)^2 + \alpha^2\Delta_Y\big),$$

and $\operatorname{spec}_b(A) = \{\pm i\alpha k : k \in \mathbb{N}_0\}$. Since $\frac{1}{2} < \alpha < 1$, only the set $\{-i\alpha, 0, i\alpha\}$ is relevant for the spaces $\mathcal{E}_{\max}$ and $\mathcal{E}_{\wedge,\max}$, cf. (3.3). Here, similar to $\mathcal{E}_{\wedge,\max}$, the space $\mathcal{E}_{\max}$ consists of singular functions and is isomorphic to the quotient $\mathcal{D}_{\max}/\mathcal{D}_{\min}$. If y denotes the angular variable on S^1,

$$\mathcal{E}_{\wedge,\max} = \operatorname{span}\{1, \log x, e^{iy}x^\alpha, e^{-iy}x^\alpha, e^{iy}x^{-\alpha}, e^{-iy}x^{-\alpha}\},$$
$$\mathcal{E}_{\max} = \operatorname{span}\Big\{1, \log x, e^{\pm iy}x^\alpha, e^{\pm iy}x^{-\alpha}\big(1 - \tfrac{\beta}{2\alpha-1}x\big)\Big\}.$$

In this case, $\theta : \mathcal{E}_{\max} \to \mathcal{E}_{\wedge,\max}$ acts as the identity on $\operatorname{span}\{1, \log x, e^{\pm iy}x^{\alpha}\}$, but

$$\theta\Big(e^{\pm iy}x^{-\alpha}\big(1 - \tfrac{\beta}{2\alpha-1}x\big)\Big) = e^{\pm iy}x^{-\alpha}.$$

The map θ induces an isomorphism

$$\Theta : \mathfrak{D} \to \mathfrak{D}_\wedge$$

that we use to define $\mathcal{D}_\wedge = \Theta\mathcal{D}$ for any given $\mathcal{D} \in \mathfrak{D}$. The operator $A_{\wedge,\mathcal{D}_\wedge}$ is the closed extension of $A_\wedge$ in $x^{-m/2}L^2_b(Y^\wedge; E)$ uniquely associated with $A_{\mathcal{D}}$.

As in [6, Section 6], and motivated by the importance of κ_ϱ in studying the model operator $A_\wedge$, we introduce on $\mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$ the one-parameter group

$$\tilde\kappa_\varrho = \theta^{-1}\kappa_\varrho\theta \ \text{ for } \varrho > 0.$$

Similar to the situation on the model cone, the spectrum and resolvent of the closed extensions of A can be geometrically analyzed by considering the manifold $\mathfrak{G}$, cf. (2.5), together with the flow generated by $\tilde\kappa_\varrho$.

An interesting consequence of Theorem 5.1 is the following.

Proposition 5.2. *If $A - \lambda$ is c-elliptic with parameter $\lambda \neq 0$, then*

$$\operatorname{ind}(A_{\wedge,\mathcal{D}_\wedge} - \lambda) = \operatorname{ind} A_{\mathcal{D}}.$$

Proof. The existence of θ implies $\dim \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} = \dim \mathcal{D}/\mathcal{D}_{\min}$. Now, the proposition follows by combining this identity with the relative index formulas (2.3) and (3.9), together with the equation (3.4). ☐

The following theorem describes the pseudodifferential structure of the resolvent of a cone operator A and gives tangible conditions over a given sector Λ on the symbols ${}^c\boldsymbol{\sigma}(A)$ and $A_\wedge$ for A to have Λ as a sector of minimal growth.

Theorem 5.3 (Theorem 6.9 in [6])**.** *Let $A \in x^{-m}\operatorname{Diff}^m_b(M; E)$ be such that $A - \lambda$ is c-elliptic with parameter $\lambda \in \Lambda$. If Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$, then it is a sector of minimal growth for $A_{\mathcal{D}}$. Moreover,*

$$(A_{\mathcal{D}} - \lambda)^{-1} = B(\lambda) + G_{\mathcal{D}}(\lambda),$$

where $B(\lambda)$ is a parametrix of $A_{\mathcal{D}_{\min}} - \lambda$ with $B(\lambda)(A_{\mathcal{D}_{\min}} - \lambda) = 1$ for λ sufficiently large, and $G_{\mathcal{D}}(\lambda)$ is a pseudodifferential regularizing operator of finite rank.

The following lemma gives further information about the behavior at large of the resolvent along a sector of minimal growth.

Given two cut-off functions ω_0 and ω_1, the notation $\omega_1 \prec \omega_0$ will indicate that $\omega_0 = 1$ in a neighborhood of the support of ω_1.

Lemma 5.4. *Let $A \in x^{-m}\operatorname{Diff}^m_b(M; E)$ be c-elliptic and let Λ be a sector of minimal growth for $A_{\mathcal{D}}$. For every pair of cut-off functions $\omega_1 \prec \omega_0$, supported near the boundary, we have*

$$(1 - \omega_0)(A_{\mathcal{D}} - \lambda)^{-1}\omega_1 \in \mathscr{S}\big(\Lambda, \mathscr{L}(x^{-m/2}L^2_b, \mathcal{D}_{\max})\big),$$

where $\mathscr{S}$ stands for Schwartz (rapidly decreasing as $|\lambda| \to \infty$).

Proof. Since Λ is a sector of minimal growth for $A_{\mathcal{D}}$, the family $A-\lambda$ must be c-elliptic with parameter $\lambda \in \Lambda$, and $A_{\wedge,\min}-\lambda$ must be injective for every $\lambda \in \Lambda$, $\lambda \neq 0$. A proof of this can be found in [6, Theorem 4.1].

As a consequence (cf. [6, Section 5]), there is a parametrix $B(\lambda)$ such that $B(\lambda)(A_{\mathcal{D}_{\min}}-\lambda)=1$ for large $\lambda \in \Lambda$, and

$$(1-\omega_0)B(\lambda)\omega_1 \in \mathscr{S}\big(\Lambda, \mathscr{L}(x^{-m/2}L^2_b, \mathcal{D}_{\max})\big) \tag{5.1}$$

for all cut-off functions $\omega_1 \prec \omega_0$ supported near the boundary. We now make use of the identity

$$(A_{\mathcal{D}}-\lambda)^{-1} = B(\lambda) + (1-B(\lambda)(A-\lambda))(A_{\mathcal{D}}-\lambda)^{-1}.$$

Multiplying by $(1-\omega_0)$ from the left and by ω_1 from the right, (5.1) proves the assertion for the first term involving $B(\lambda)$. On the other hand, since $1-B(\lambda)(A-\lambda)$ vanishes on $\mathcal{D}_{\min}$ for large λ, we have for such λ,

$$\begin{aligned}(1-\omega_0)(1-B(\lambda)(A-\lambda)) &= (1-\omega_0)(1-B(\lambda)(A-\lambda))\omega_2 \\ &= -(1-\omega_0)B(\lambda)(A-\lambda)\omega_2 \\ &= -(1-\omega_0)B(\lambda)\omega_1(A-\lambda)\omega_2\end{aligned}$$

whenever $\omega_2 \prec \omega_1$. Thus, by (5.1),

$$(1-\omega_0)(1-B(\lambda)(A-\lambda)) : \mathcal{D}_{\max} \to \mathcal{D}_{\max}$$

is rapidly decreasing as $|\lambda| \to \infty$. Finally, the assertion of the lemma can be completed using the fact that $(A_{\mathcal{D}}-\lambda)^{-1}\omega_1 : x^{-m/2}L^2_b \to \mathcal{D}_{\max}$ is uniformly bounded. ☐

Necessity of the conditions

The converse of Theorem 5.3 involves proving that the minimal growth of the resolvent $(A_{\mathcal{D}}-\lambda)^{-1}$ over a sector Λ implies a corresponding behavior for the inverse of ${}^c\boldsymbol{\sigma}(A)-\lambda$ and for the resolvent $(A_{\wedge,\mathcal{D}_\wedge}-\lambda)^{-1}$.

While in [6, Theorem 4.1] we established the necessity of the condition on ${}^c\boldsymbol{\sigma}(A)$, we did not address the question whether Λ must necessarily be a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$. In the next theorem we prove that this is indeed the case when A has coefficients independent of x near $Y=\partial M$.

Theorem 5.5. *Let $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$ be c-elliptic with coefficients independent of x near Y. If Λ is a a sector of minimal growth for $A_{\mathcal{D}}$, then $A-\lambda$ is c-elliptic with parameter $\lambda \in \Lambda$, and Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$.*

Proof. As stated in the proof of Lemma 5.4, the assumption on the resolvent of $A_{\mathcal{D}}$ implies that $A-\lambda$ is c-elliptic with parameter $\lambda \in \Lambda$ and that $A_{\wedge,\min}-\lambda$ is injective for every $\lambda \neq 0$. Thus we only need to prove the statement about $A_{\wedge,\mathcal{D}_\wedge}$.

By Proposition 5.2, and since $\operatorname{ind} A_{\mathcal{D}}=0$, we have $\operatorname{ind}(A_{\wedge,\mathcal{D}_\wedge}-\lambda)=0$ for $\lambda \neq 0$. For this reason, in order to show that Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$, it suffices to find (for large $\lambda \in \Lambda$) a right-inverse of $A_{\wedge,\mathcal{D}_\wedge}-\lambda$ that is uniformly bounded in $\mathscr{L}\big(x^{-m/2}L^2_b, \mathcal{D}_\wedge\big)$ as $|\lambda| \to \infty$.

Since A is assumed to have coefficients independent of x near the boundary, there is a cut-off function ω_0 such that

$$A\omega_0 = A_\wedge\omega_0 \quad \text{and} \quad \omega_0\mathcal{D} = \omega_0\mathcal{D}_\wedge.$$

Let ω_1, ω_2 be cut-off functions with $\omega_2 \prec \omega_1 \prec \omega_0$. Then the operator

$$B(\lambda) = \omega_1(A_\mathcal{D} - \lambda)^{-1}\omega_2$$

can be regarded as an operator on M with values in $\mathcal{D}$ or as an operator on $Y^\wedge$ with values in $\mathcal{D}_\wedge$. Depending on the context we will write $B(\lambda)$ as

$$B_\mathcal{D}(\lambda) : x^{-m/2}L^2_b(M;E) \to \mathcal{D} \quad \text{or} \quad B_{\mathcal{D}_\wedge}(\lambda) : x^{-m/2}L^2_b(Y^\wedge;E) \to \mathcal{D}_\wedge.$$

On M we consider

$$\begin{aligned}(A_\mathcal{D} - \lambda)B_\mathcal{D}(\lambda) &= \omega_0(A_\mathcal{D} - \lambda)\omega_1(A_\mathcal{D} - \lambda)^{-1}\omega_2\\ &= \omega_2 - \omega_0(A_\mathcal{D} - \lambda)(1-\omega_1)(A_\mathcal{D} - \lambda)^{-1}\omega_2\\ &= \omega_2 + R(\lambda)\end{aligned}$$

with $R(\lambda) = -\omega_0(A_\mathcal{D} - \lambda)(1-\omega_1)(A_\mathcal{D} - \lambda)^{-1}\omega_2$. By Lemma 5.4, $R(\lambda)$ is rapidly decreasing in the norm as $|\lambda| \to \infty$.

Because of the presence and nature of the cut-off functions ω_0 and ω_2, $R(\lambda)$ can also be regarded as an operator on $Y^\wedge$, say $R_\wedge(\lambda) \in \mathscr{S}\big(\Lambda, \mathscr{L}(x^{-m/2}L^2_b)\big)$. Now, using that $(A_\mathcal{D} - \lambda)\omega_1 = (A_{\wedge,\mathcal{D}_\wedge} - \lambda)\omega_1$, we get on $Y^\wedge$ the identity

$$(A_{\wedge,\mathcal{D}_\wedge} - \lambda)B_{\mathcal{D}_\wedge}(\lambda) = \omega_2 + R_\wedge(\lambda). \tag{5.2}$$

Furthermore, we have

$$\|B_{\mathcal{D}_\wedge}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\wedge,\max})} = O(1) \ \text{ as } |\lambda| \to \infty,$$

since, by assumption, $\|B_\mathcal{D}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\max})}$ has the same asymptotic behavior.

On the other hand, as $A - \lambda$ is c-elliptic with parameter, by [6, Theorem 5.24] there is a family of pseudodifferential operators $B_{2,\wedge}(\lambda) : x^{-m/2}L^2_b \to \mathcal{D}_{\wedge,\min}$ (uniformly bounded in λ) such that $(A_\wedge - \lambda)B_{2,\wedge}(\lambda) - 1$ is regularizing, and for $\omega_3 \prec \omega_2$, the families $\omega_3 B_{2,\wedge}(\lambda)(1-\omega_2)$ and $\big[(A_\wedge - \lambda)B_{2,\wedge}(\lambda) - 1\big](1-\omega_2)$ are rapidly decreasing in the norm as $|\lambda| \to \infty$. Thus, as $A_{\wedge,\mathcal{D}_\wedge}(1-\omega_3) = A_\wedge(1-\omega_3)$,

$$(A_{\wedge,\mathcal{D}_\wedge} - \lambda)(1-\omega_3)B_{2,\wedge}(\lambda)(1-\omega_2) = (1-\omega_2) + S_\wedge(\lambda) \tag{5.3}$$

with $S_\wedge(\lambda) \in \mathscr{S}\big(\Lambda, \mathscr{L}(x^{-m/2}L^2_b)\big)$. Finally, the operator family

$$Q_\wedge(\lambda) = B_{\mathcal{D}_\wedge}(\lambda) + (1-\omega_3)B_{2,\wedge}(\lambda)(1-\omega_2) : x^{-m/2}L^2_b \to \mathcal{D}_{\wedge,\max}$$

is bounded in the norm as $|\lambda| \to \infty$ and by (5.2) and (5.3) we have

$$(A_{\wedge,\mathcal{D}_\wedge} - \lambda)Q_\wedge(\lambda) - 1 \in \mathscr{S}\big(\Lambda, \mathscr{L}(x^{-m/2}L^2_b)\big).$$

By a Neumann series argument, it follows that $A_{\wedge,\mathcal{D}_\wedge} - \lambda : \mathcal{D}_\wedge \to x^{-m/2}L^2_b$ has a uniformly bounded right-inverse for large $\lambda \in \Lambda$. ∎

References

[1] M. Agranovich and M. Vishik, *Elliptic problems with a parameter and parabolic problems of general type*, Russ. Math. Surveys **19** (1963), 53–159.

[2] S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.

[3] J. Brüning and R. Seeley, *The expansion of the resolvent near a singular stratum of conical type*, J. Funct. Anal. **95** (1991), no. 2, 255–290.

[4] J. Gil, *Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators*, Math. Nachr. **250** (2003), 25–57.

[5] J. Gil, T. Krainer, and G. Mendoza, *Geometry and spectra of closed extensions of elliptic cone operators*, preprint math.AP/0410178 at arXiv.org, to appear in Canadian Journal of Mathematics.

[6] J. Gil, T. Krainer, and G. Mendoza, *Resolvents of elliptic cone operators*, preprint math.AP/0410176 at arXiv.org.

[7] J. Gil and G. Mendoza, *Adjoints of elliptic cone operators*, Amer. J. Math. **125** (2003) 2, 357–408.

[8] G. Grubb, *Functional calculus of pseudodifferential boundary problems*, 2nd ed., Progress in Mathematics, vol. 65. Birkhäuser, Basel, 1996.

[9] V. Kondrat'ev, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Mosc. Math. Soc. **16** (1967), 227–313.

[10] T. Krainer, *Resolvents of elliptic boundary problems on conic manifolds*, preprint math.AP/0503021 at arXiv.org.

[11] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Math. vol 136, B.G. Teubner, Stuttgart, Leipzig, 1997.

[12] P. Loya, *On the resolvent of differential operators on conic manifolds*, Comm. Anal. Geom. **10** (2002), no. 5, 877–934.

[13] R. Melrose, *Transformation of boundary value problems*, Acta Math. **147** (1981), no. 3-4, 149–236.

[14] E. Schrohe and J. Seiler, *The resolvent of closed extensions of cone differential operators*, Canad. J. Math. **57** (2005), no. 4, 771–811.

[15] B.-W. Schulze, *Pseudo-differential operators on manifolds with edges*, Proc. Symp. 'Partial Differential Equations', Holzhau 1988 (Leipzig), Teubner-Texte zur Math., vol. 112, Teubner, 1989, pp. 259–288.

[16] R. Seeley, *Complex powers of an elliptic operator*, Singular Integrals, AMS Proc. Symp. Pure Math. X, 1966, Amer. Math. Soc., Providence, 1967, pp. 288–307.

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