The Quantisation of Edge Symbols

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Abstract

We investigate operators on manifolds with edges from the point of view of the symbolic calculus induced by the singularities. We discuss new aspects of the quantisation of edge-degenerate symbols which lead to continuous operators in weighted edge spaces.

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Introduction

This paper is aimed at studying a number of new properties of edge amplitude functions belonging to the calculus of operators on a manifold with edges. (The notation 'manifold' is used here for convenience; in general our spaces are manifolds only outside their subspaces of singularities). The investigation is motivated by the program of establishing operator algebras on configurations with higher singularities, e.g., when the cones of local wedges have cross section with singularities. The expectation is that the ideas of the cone and edge pseudo-differential calculus with smooth model cones, cf. [8], [9], are more or less iterative. However, the symbolic structures in the higher floors of the calculus are by no means straightforward. Recall that already for manifolds with smooth edges there are different ways of constructing quantisations of corresponding edge-degenerate symbols; they reflect different aspects of the calculus . Compared with [11] an alternative quantisation is given in [4]; remainders belong to the class of flat Green edge symbols. The result of [4] is applied in [6] and [5] to the construction of holomorphic representatives of corner Mellin symbols.

Here we study edge symbols in another way, namely, with an additional localisation near the diagonal with respect to the cone axis variable, depending on the edge covariable. In higher corner algebras localisations of a similar kind seem to be necessary, cf. [1], [2].

The (operator-valued) symbols of the edge algebra from [7] or [9] are specific families of pseudo-differential operators on infinite cones $X^\wedge := \mathbb{R}_+ \times X \ni (r, x)$ with $r \to 0$ representing the tip and $r \to \infty$ a conical exit to infinity, where the cross section X (in the simplest case) is a closed compact C^∞ manifold. These symbols may be interpreted as a specific parameter-dependent quantisation of edge-degenerate scalar symbols (or, alternatively, of an edge-degenerate family, of classical pseudo-differential operators on X). Quantisations are usually not canonical, even in simpler situations. The shape of the edge quantisation that was established in the above mentioned expositions (and that we briefly recall below) gives rise to an algebra of continuous operators in weighted edge spaces, with all the desirable features, such as ellipticity with respect to a principal symbolic hierarchy and existence of a parametrix within the calculus. However, some aspects of the theory remained complicated, for instance, the rigorous proof of the composition behaviour and other necessary elements. For that reason the authors of [4] proposed another edge quantisation which entails the composition result in a very simple way. The hope was that a similar strategy might work also for manifolds with singularities of higher order. Unfortunately, the attempt to generalise [4], for instance, for singularities of order 2 (i.e., when the base X of the model cone itself has singularities of conical or edge type) leads to an enormous blow-up of technicalities, and the construction of corresponding alternative quantisations along the lines of [4] compared with the 'usual' ones seems to be more complicated than a direct approach. Moreover, the analogue of the usual edge quantisation for the edge calculus of second generation as studied in [1] gives rise to difficulties; so there was to be invented a modified approach. Since this is new in an analogous form also in the edge calculus of first generation, it is desirable to have a look at such an edge quantisation and to characterise the remainders compared with the former one. This is one of the points of the present paper, and we prove that the remainders are flat Green symbols. Moreover, we give a new relatively elementary proof for the composition theorem of edge symbols of the first generation.

By a manifold with corners we understand a topological space M that contains a subspace M' of singularities such that $M \setminus M'$ is a C^∞ manifold, and near every $m \in M'$ the space M is modelled on a cone $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ or a wedge $X^\Delta \times \Omega$, for an open set $\Omega \subseteq \mathbb{R}^q$, where X is again a manifold with corners (of 'lower order' than M). In addition we require specific properties of the transition maps belonging to different such local representations.

Examples are manifolds with conical singularities or edges; in this case we assume the base spaces X of the model cones to be closed, compact C^∞ manifolds. Manifolds with higher singularities can be obtained by iteratively forming cones or wedges and pasting them together to corresponding global spaces.

A well known example is the case of differential operators on a manifold B with conical singularities. Assume, for convenience, that B has one conical point v, and let $\mathbb{B}$ denote the associated stretched manifold which is a C^∞ manifold with compact boundary $\partial\mathbb{B} \cong X$ and int $\mathbb{B} \cong B \setminus \{v\}$. Then B is equal to the quotient space $\mathbb{B}/\partial\mathbb{B}$ (with $\partial\mathbb{B}$ collapsed to the point v), and $\mathbb{B}$ is locally near $\partial\mathbb{B}$ identified with $\overline{\mathbb{R}}_+ \times X$.

A differential operator A with smooth coefficients on int$\mathbb{B}$ is said to be of Fuchs type, if locally near $\partial\mathbb{B}$ in the splitting of variables $(r,x) \in \mathbb{R}_+ \times X$ the operator has the form

$$A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r)\Big(-r\frac{\partial}{\partial r}\Big)^j \tag{1}$$

with coefficients $a_j \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$ (here $\mathrm{Diff}^\nu(\cdot)$ denotes the space of all differential operators of order ν with smooth coefficients on the C^∞ manifold in the brackets). In this case the principal symbolic structure consists of a pair

$$\sigma(A) = (\sigma_\psi(A), \sigma_{\mathrm{c}}(A)),$$

where $\sigma_\psi(A)$ is the standard homogeneous principal symbol of order μ and

$$\sigma_{\mathrm{c}}(A)(z) := \sum_{j=0}^{\mu} a_j(0) z^j \tag{2}$$

the principal conormal symbol. The function (2) is operator-valued and acts between the standard Sobolev spaces on X as a family of continuous operators

$$\sigma_{\mathrm{c}}(A)(z) : H^s(X) \to H^{s-\mu}(X)$$

depending on the variable $z \in \mathbb{C}$.

Another example is the case of a C^∞ manifold with boundary, locally near the boundary modelled on $\overline{\mathbb{R}}_+ \times \Omega \ni (r,y)$, $\Omega \subseteq \mathbb{R}^q$ open. Then, if A is a differential operator of order μ with smooth coefficients up to the boundary, the principal symbolic hierarchy of A is a pair

$$\sigma(A) = (\sigma_\psi(A), \sigma_\partial(A)).$$

The first component is again the standard homogeneous principal symbol of order μ. Moreover, writing A near the boundary as

$$A = \sum_{|\alpha| \le \mu} a_\alpha(r,y) D^\alpha_{(r,y)}, \tag{3}$$

we set

$$\sigma_\partial(A)(y,\eta) := \sum_{\substack{|\alpha|=\mu \\ \alpha=(k,\gamma)}} a_\alpha(0,y) D_r^k \eta^\gamma, \tag{4}$$

$(y,\eta) \in T^*\Omega \setminus 0$, called the homogeneous principal boundary symbol of A. The symbol (4) is operator-valued and acts between the standard Sobolev spaces on $\mathbb{R}_+$ as a family of continuous operators

$$\sigma_\partial(A)(y,\eta) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+), \tag{5}$$

depending on $(y,\eta) \in T^*\Omega \setminus 0$; here $H^s(\mathbb{R}_+) := H^s(\mathbb{R})\big|_{\mathbb{R}_+}$.

A third category of examples are operators on manifolds with edges (cf. also Section 1 below). Manifolds with (smooth) edges can be regarded as a generalisation of manifolds with (smooth) boundary. In this case the local model near the edge is equal to $X^\Delta \times \Omega$ for a closed compact C^∞ manifold X. A manifold with boundary just corresponds to the case $\dim X = 0$.

We often employ the stretched manifolds rather than the manifolds with singularities themselves. In the case of a wedge $X^\Delta \times \Omega$ the associated stretched space is equal to $\overline{\mathbb{R}}_+ \times X \times \Omega$. In the corresponding splitting of variables (r, x, y) the typical differential operators A are assumed to be of the form

$$A = r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(r,y)\Big(-r\frac{\partial}{\partial r}\Big)^j (rD_y)^\alpha, \tag{6}$$

with coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \operatorname{Diff}^{\mu-(j+|\alpha|)}(X))$. Such operators will also be called edge-degenerate. Their symbolic hierarchy is then a pair

$$\sigma(A) = (\sigma_\psi(A), \sigma_\wedge(A)) \tag{7}$$

with $\sigma_\psi(A)$ being the homogeneous principal symbol. Moreover,

$$\sigma_\wedge(A)(y,\eta) := r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(0,y)\Big(-r\frac{\partial}{\partial r}\Big)^j (r\eta)^\alpha \tag{8}$$

is the so-called homogeneous principal edge symbol which represents a family of continuous operators

$$\sigma_\wedge(A)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge), \tag{9}$$

$(y,\eta) \in T^*\Omega \setminus 0$, acting between weighted Sobolev spaces on the (open infinite stretched) cone $X^\wedge := \mathbb{R}_+ \times X$, cf. Section 1.3 below. Comparing (8) with (1) we see that there is a family of subordinate conormal symbols

$$\sigma_{\mathrm{c}}\sigma_\wedge(A)(y,z) = \sum_{j=0}^{\mu} a_{j0}(0,y)z^j \tag{10}$$

from the interpretation of (9) as an operator of Fuchs type for every fixed $(y,\eta) \in T^*\Omega \setminus 0$. In the case of ellipticity the adequate weights $\gamma \in \mathbb{R}$ for (9) are to be fixed in connection with the non-bijectivity points of (10) in the complex plane. The difference between the notation 'boundary' and 'edge' symbol is motivated by the fact that the choice of spaces is different. If we write the operator (3) in edge-degenerate form (6) (with coefficients $a_{j\alpha}(r,y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$) we also obtain an edge symbol (9).

In the present paper we concentrate on edge-degenerate operators (6) and their pseudo-differential versions. The analogy with boundary value problems will not play a major role, but it is worth to recall that the edge calculus for $\dim X = 0$ corresponds to the pseudo-differential calculus of boundary value problems without the transmission property at the boundary, cf. [10].

1 Operators on manifolds with edges

1.1 Manifolds with conical singularities and edges

Let X be a topological space. As in the introduction we set

$$X^\wedge := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X) \quad \text{and} \quad X^\wedge := \mathbb{R}_+ \times X.$$

In the following discussion, for convenience, topological spaces are assumed to be countable unions of compact sets; C^∞ manifolds are assumed to be oriented and equipped with a Riemannian metric.

The definition of a manifold W with edge Y is based on a certain kind of (locally trivial) X^Δ-bundles L over Y; here Y and X are C^∞ manifold.

In order to describe the specific structure we first consider a (locally trivial) $\overline{\mathbb{R}}_+ \times X$-bundle $\mathbb{L}$ over Y, the stretched space associated with L.

The transition maps

$$l : \overline{\mathbb{R}}_+ \times X \times \Omega \to \overline{\mathbb{R}}_+ \times X \times \widetilde{\Omega}$$

between trivialisations of $\mathbb{L}$, $\Omega, \widetilde{\Omega} \subseteq \mathbb{R}^q$ open, $q = \dim Y$, are asssumed to be restrictions of transition maps $\mathbb{R} \times X \times \Omega \to \mathbb{R} \times X \times \widetilde{\Omega}$ belonging to an $\mathbb{R} \times X$-bundle $2\mathbb{L}$ over Y to $\overline{\mathbb{R}}_+ \times X \times \Omega$. Then $\mathbb{L}$ contains a subspace $\mathbb{L}_{\mathrm{sing}}$ which is an X-bundle over Y, represented by the trivialisations $\{0\} \times X \times \Omega$. Let us set $\mathbb{L}_{\mathrm{reg}} := \mathbb{L} \setminus \mathbb{L}_{\mathrm{sing}}$; this is an $X^\wedge$-bundle over Y with the trivialisations $X^\wedge \times \Omega$.

From the construction of $\mathbb{L}$ and $\mathbb{L}_{\mathrm{sing}}$ it follows that we obtain an X^Δ-bundle L by passing to the quotient space $\overline{\mathbb{R}}_+ \times X \to X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ in every fibre of $\mathbb{L}$. The bundle L contains Y as a subspace, interpreted as an edge, and $L \setminus Y$ is a C^∞ manifold which can be identified with $\mathbb{L}_{\mathrm{reg}}$ in a natural way. More precisely, we have a projection

$$\pi : \mathbb{L} \to L, \tag{11}$$

fibrewise defined by $\overline{\mathbb{R}}_+ \times X \to X^\Delta$, and π restricts to the bundle projection

$$\pi_{\mathrm{sing}} : \mathbb{L}_{\mathrm{sing}} \to Y \tag{12}$$

and to an isomorphism of $X^\wedge$-bundles

$$\pi_{\mathrm{reg}} : \mathbb{L}_{\mathrm{reg}} \to L \setminus Y. \tag{13}$$

Definition 1.1 *Let W be a topological space and $Y \subset W$ a subspace. Then W is called a manifold with edge Y, if*

(i) *$W \setminus Y$ and Y are C^∞ manifolds;*

(ii) *there exists a neighbourhood V of Y in W and a homeomorphism*

$$\chi : V \to L$$

to an X^Δ-bundle L over Y for a C^∞ manifold X, such that χ restricts to diffeomorphisms

$$\chi_0 : V \cap Y \to Y, \quad \chi_{\mathrm{reg}} : V \setminus Y \to L \setminus Y.$$

Incidentally we will write

$$W' = Y \tag{14}$$

From W we can pass to a so called stretched manifold $\mathbb{W}$ when we first replace V by the stretched set $\mathbb{V}$ that is defined to be a C^∞ manifold with boundary, diffeomorphic to $\mathbb{L}$, such that $\mathbb{V} \setminus \partial\mathbb{V}$ is identified with $V \setminus Y$ and $\partial\mathbb{V}$ isomorphic to $\mathbb{L}_{\mathrm{sing}}$ as an X-bundle over Y.

In other words, $\mathbb{W}$ is a C^∞ manifold with boundary $\partial\mathbb{W} =: \mathbb{W}_{\mathrm{sing}}$ which is an X-bundle over Y. Let us set $\mathbb{W} \setminus \partial\mathbb{W} =: \mathbb{W}_{\mathrm{reg}}$. We then have a canonical continuous map

$$\pi : \mathbb{W} \to W$$

which restricts to the bundle projection

$$\pi_{\mathrm{sing}} : \mathbb{W}_{\mathrm{sing}} \to Y$$

and to a diffeomorphism

$$\pi_{\mathrm{reg}} : \mathbb{W}_{\mathrm{reg}} \to W \setminus Y.$$

With $\mathbb{W}$ we can associate the double $2\mathbb{W}$ which is a C^∞ manifold (without boundary) by gluing together two copies $\mathbb{W}_\pm$ of $\mathbb{W}$ along the common boundary $\partial\mathbb{W}$.

Example 1.2 *For $W = X^\Delta \times \Omega$ we have $\mathbb{W} = \overline{\mathbb{R}}_+ \times X \times \Omega$ and $2\mathbb{W} = \mathbb{R} \times X \times \Omega$.*

Remark 1.3 *In the case $\dim Y = 0$ we speak about manifolds with conical singularities.*

Remark 1.4 *Let W be a manifold with edge Y. Then for every C^∞ manifold M the Cartesian product $W \times M$ is a manifold with edge $Y \times M$.*

Definition 1.5 *Let W_i be manifolds with edges Y_i, $i = 1, 2$, and let X_i be the base of the model cone for W_i, $i = 1, 2$. A continuous map*

$$T : W_1 \to W_2$$

is called an $\mathfrak{M}_1$-morphism if there is a differentiable map

$$\mathbb{T} : \mathbb{W}_1 \to \mathbb{W}_2$$

between the respective stretched manifolds as manifolds with C^∞ boundary such that

$$\mathbb{T}\big|_{\partial\mathbb{W}_1} : \partial\mathbb{W}_1 \to \partial\mathbb{W}_2$$

is a homomorphism between the corresponding X_i-bundles (in particular, $T\big|_{Y_1} : Y_1 \to Y_2$ is then a differentiable map). $T : W_1 \to W_2$ is called an $\mathfrak{M}_1$-isomorphism if there is an $\mathfrak{M}_1$-morphism $T^{-1} : W_2 \to W_1$ which is a two sided inverse to T.

In this case we also write $W_1 \cong_{\mathfrak{M}_1} W_2$.

In this way, the manifolds with edges form a category $\mathfrak{M}_1$ with the subcategory of manifolds with conical singularities.

Let W be a manifold with edge Y of dimension $q > 0$. Then the above mentioned bijection $\mathbb{V} \underset{\cong}{\to} \mathbb{L}$ allows us to define stretched wedge neighbourhoods $\mathbb{U} \subset \mathbb{V}$ that correspond to a trivialisations of $\mathbb{L}$, i.e.,

$$\mathbb{U} \cong_{\mathfrak{M}_1} \overline{\mathbb{R}}_+ \times X \times \Omega \tag{15}$$

for open sets $\Omega \subseteq \mathbb{R}^q$. Set $\mathbb{U}_{\mathrm{reg}} := \mathbb{U} \setminus \partial\mathbb{W}$.

1.2 The typical differential operators

Let W be a manifold with edge Y and $\mathbb{W}$ its stretched manifold. By

$$\mathrm{Diff}^{\mu}_{\mathrm{deg}}(W)$$

we denote the space of all differential operators $A \in \mathrm{Diff}^{\mu}(W \setminus Y)$ such that for every (stretched) wedge neighbourhood $\mathbb{U}_{\mathrm{reg}}$ in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$ the operator A has the form

$$A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y)\Big(- r\frac{\partial}{\partial r}\Big)^j (rD_y)^{\alpha} \tag{16}$$

with coefficients $a_{j\alpha} \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$.

In local cooradinates $(r, x, y) \in \mathbb{R}_+ \times \Sigma \times \Omega$, $\Sigma \subseteq \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^q$ open ($n = \dim X$, $q = \dim Y$) the homogeneous principal symbol $\sigma_{\psi}(A)$ of A of order μ has the form

$$\sigma_{\psi}(A)(r, x, y, \varrho, \xi, \eta) = r^{-\mu}\tilde{\sigma}_{\psi}(A)(r, x, y, r\varrho, \xi, r\eta)$$

for a function $\tilde{\sigma}_{\psi}(A)(r, x, y, \tilde{\varrho}, \xi, \tilde{\eta})$ that is smooth up to $r = 0$ (and homogeneous of order μ in $(\tilde{\varrho}, \xi, \tilde{\eta})$). In addition, as noted in the introduction, we have the homogeneous principal edge symbol $\sigma_{\wedge}(A)(y, \eta)$, $(y, \eta) \in T^*Y \setminus 0$, the second component of the principal symbolic hierarchy (7). Note that for

$$a(y, \eta) := r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y)\big(-r\frac{\partial}{\partial r}\big)^j (r\eta)^{\alpha},$$

and $(\kappa_{\lambda}u)(r, x) := \lambda^{\frac{n+1}{2}} u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$, we have

$$\sigma_{\wedge}(A)(y, \eta) = \lim_{\lambda \to \infty} \lambda^{-\mu}\kappa_{\lambda} a(y, \lambda\eta)\kappa_{\lambda}^{-1},$$

and

$$\sigma_{\wedge}(A)(y, \lambda\eta) = \lambda^{\mu}\kappa_{\lambda}\sigma_{\wedge}(A)(y, \eta)\kappa_{\lambda}^{-1},$$

for all $\lambda \in \mathbb{R}_+$.

1.3 Weighted Sobolev spaces

We now formulate Sobolev spaces on stretched cones $X^{\wedge} = \mathbb{R}_+ \times X \ni (r, x)$ and wedges $X^{\wedge} \times \mathbb{R}^q \ni (r, x, y)$, first for the case of a smooth compact manifold X. To this end we first consider the cylindrical Sobolev space

$$H^s(\mathbb{R} \times X)$$

defined as the set of all $u(t, .) \in H^s_{\mathrm{loc}}(\mathbb{R}_t \times X)$ such that

$$(\varphi u) \circ (1 \times \chi)^{-1} \in H^s(\mathbb{R}_t \times \mathbb{R}^n)$$

for every chart $\chi : U \to \mathbb{R}^n$ on X and every $\varphi \in C_0^{\infty}(U)$; here $(1 \times \chi)(r, .) := (r, \chi(.))$. Let

$$(S_{\beta}u)(t) := e^{-(\frac{1}{2}-\beta)t}u(e^{-t}),\ t \in \mathbb{R}$$

for any $\beta \in \mathbb{R}$. Then we set

$$\mathcal{H}^{s,\gamma}(X^\wedge) := \left(S_{\gamma-\frac{n}{2}}\right)^{-1} H^s(\mathbb{R} \times X)$$

for $n = \dim X$.

Let us interpret these spaces in connection with the Mellin transform on $\mathbb{R}_+ \ni r$,

$$Mu(z) = \int_0^\infty r^{z-1} u(r) dr.$$

For $u \in C_0^\infty(\mathbb{R}_+)$ the function, $Mu(z)$ is an entire function, and we have $Mu\big|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for

$$\Gamma_\beta := \{z \in \mathbb{C} : \mathrm{Re} z = \beta\} \tag{17}$$

for every $\beta \in \mathbb{R}$, uniformly in compact intervals. Here and in the sequel the 'weight line' Γ_β is treated as a real axis if the spaces, e.g., the Schwartz space, Sobolev spaces, etc., or amplitude functions are given with respect to the corresponding variable.

The Mellin transform will also be applied to vector- or operator-valued functions depending on $r \in \mathbb{R}$; then the covariable z will often vary on a certain weight line. We have $-r\frac{\partial}{\partial r} = M^{-1} z M$, and $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s \in \mathbb{N}$ is equal to the subspace of all $u(r,x) \in r^{-\frac{n}{2}+\gamma} L^2(X^\wedge)$ such that $(-r\partial_r)^k D_x^\alpha u(r,x) \in r^{-\frac{n}{2}+\gamma} L^2(X^\wedge)$ for all $k + |\alpha| \leq s$. Here D_x^α runs over all elements of $\mathrm{Diff}^{|\alpha|}(X)$, and $L^2(X^\wedge)$ refers to the measure $drdx$. From this definition we can recover $\mathcal{H}^{s,\gamma}(X^\wedge)$ for arbitrary $s \in \mathbb{R}$ by duality and interpolation. It will be adequate to modify the spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ at infinity by setting

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(X^\wedge)\ v \in H^s_{\mathrm{cone}}(X^\wedge)\}$$

for a space $H^s_{\mathrm{cone}}(X^\wedge)$ that is defined as follows.

Set $B := \{x \in \mathbb{R}^n : |x| < 1\}$ and $B^\vee := \{(r, rx) \in \mathbb{R}^{1+n} : (r,x) \in \mathbb{R}_+ \times B\ \}$ which is a conical set in $\mathbb{R}^{1+n}$. Let $\chi : U \to B$ be a chart on X, and consider $1 \times \chi : \mathbb{R}_+ \times U \to \mathbb{R}_+ \times B$. Together with

$$\beta : (r,x) \to (r, rx), \quad \mathbb{R}_+ \times B \to B^\vee$$

we have the composition

$$\beta \circ (1 \times \chi) : \mathbb{R}_+ \times U \to B^\vee.$$

Then $H^s_{\mathrm{cone}}(X^\wedge)$ is defined to be the subspace of all $u \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}$ such that $(1-\omega)\varphi u \circ (1 \times \chi)^{-1} \circ \beta^{-1} \in H^s(\mathbb{R}^{1+n})$ for every chart $\chi : U \to B$ and arbitrary $\varphi \in C_0^\infty(U)$. Another equivalent characterisation of $H^s_{\mathrm{cone}}(X^\wedge)$ is given in Remark 2.1 below.

Let us introduce some convenient terminology in connection with the variety of spaces that will occur in our calculus.

If E_0, E_1 are Fréchet spaces, embedded in a Hausdorff topological vector space H, we set

$$E_0 + E_1 = \{e_0 + e_1 : e_0 \in E_0, e_1 \in E_1\}. \tag{18}$$

There is then an algebraic isomorphism $E_0 + E_1 \cong E_0 \oplus E_1 / \Delta$ for $\Delta := \{(e, -e) : e \in E_0 \cap E_1\}$, and we endow (18) with the Fréchet topology of the quotient space. We then call (18) the non-direct sum of E_0 and E_1.

Moreover, let E be a Fréchet space that is a left module over an algebra A. Then $[a]E$ for $a \in A$ will denote the closure of the space $\{ae : e \in E\}$ in E. In a similar sense we employ the notation $E[b]$ or $[a]E[b]$ for $a, b \in E$ if E is a right or two-sided A-module.

Example 1.6 *The spaces $\mathcal{H}^s_{\text{cone}}(X^\wedge)$ are two sided modules over the algebra of all $\varphi(r,x) \in C^\infty(\overline{\mathbb{R}}_+ \times X)$ that do not depend on r for $r > R$ for some $R > 0$. In particular, for every cut-off function $\omega(r)$ we can form the spaces $[\omega]\mathcal{H}^{s,\gamma}(X^\wedge)$ and $[1-\omega]H^s_{\text{cone}}(X^\wedge)$, and we have*

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]H^s_{\text{cone}}(X^\wedge). \tag{19}$$

Clearly the space (19) *is independent of the specific choice of ω.*

Remark 1.7 *If E_0 and E_1 are Hilbert spaces also $E_0 + E_1$ becomes a Hilbert space by the identification with the orthogonal complement of Δ in $E_0 \oplus E_1$.*

Thus, if we fix the cut-off function ω, we get a Hilbert space structure in $\mathcal{K}^{s,\gamma}(X^\wedge)$ via (19). For $s = \gamma = 0$ we take the scalar product from the identification

$$\mathcal{K}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times X),$$

where $L^2(\mathbb{R}_+ \times X)$ refers to $drdx$ with dx being connected with a fixed Riemannian metric on X.

For purposes below we tacitly identify a coordinate neighbourhood U on X with $B \subset \mathbb{R}^n$, with the coordinates x via a chart $\chi : U \to B$. Then, the above characterisation of the space $H^s_{\text{cone}}(X^\wedge)$ for large r cone it is enough to look at $(1-\omega)\varphi H^s_{\text{cone}}(B^\wedge)$ vor any $\varphi \in C_0^\infty(B)$ and a cut-off function ω.

We then have

$$u \in (1-\omega)\varphi H^s_{\text{cone}}(X^\wedge) \Longleftrightarrow (\beta^* u)(r, \frac{\tilde{x}}{r}) \in H^s(\mathbb{R}^{1+n}_{r,\tilde{x}}) \tag{20}$$

for the map $\beta : B^\wedge \to B^\vee \subset \mathbb{R}^{1+n}$.

Remark 1.8 *Let $A \in \operatorname{Diff}^\mu_{\deg}(\overline{\mathbb{R}}_+ \times X \times \Omega)$ be given in the form* (16)*, and set*

$$a(y,\eta) := r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r,y)\Big(-r\frac{\partial}{\partial r}\Big)^j (r\eta)^\alpha. \tag{21}$$

Assume that there is an $R > 0$ such that the coefficients $a_{j\alpha}$ are independent of r for $r > R$. Then

$$D_y^\alpha D_\eta^\beta a(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu+|\beta|,\gamma-\mu+|\beta|}(X^\wedge),$$

$(y,\eta) \in \Omega \times \mathbb{R}^q$, is a family of continuous operators for every $s, \gamma \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{N}^q$.

In fact, writing $a(y,\eta) = \tilde{\omega}a(y,\eta) + (1-\tilde{\omega})a(y,\eta)$, for some cut-off function $\tilde{\omega}(r)$ we first have

$$\tilde{\omega}a(y,\eta) : \mathcal{H}^{s,\gamma}(X^\wedge) \to \tilde{\omega}\mathcal{H}^{s-\mu,\gamma-\mu}(X^\wedge)$$

which is fairly obvious. On the other hand, to obtain

$$(1-\tilde{\omega})a(y,\eta) : H^s_{\text{cone}}(X^\wedge) \to (1-\tilde{\omega})H^s_{\text{cone}}(X^\wedge) \tag{22}$$

we express $a(y,\eta)$ in local coordinates on X via a chart $\chi : U \to B$. If we write

$$a_{j\alpha}(r,y) = \sum_{|\gamma|\le\mu-(j+|\alpha|)} a_{j\alpha;\gamma}(r,x,y)D_x^\gamma$$

with coefficients $a_{j\alpha;\gamma} \in C^\infty(\overline{\mathbb{R}}_+ \times U \times \Omega)$ for a coordinate neighbourhood $U \subset X$, it suffices to consider the summands

$$r^{-\mu}a_{j\alpha;\gamma}(r,x,y)D_x^\gamma\Big(-r\frac{\partial}{\partial r}\Big)^j(r\eta)^\alpha \tag{23}$$

separately. For simplicity, let us interpret x as coordinates in the unit ball $B \subset \mathbb{R}^n$. Then (23) is a family of operators in $\mathbb{R}_+ \times B \ni (r,x)$. Applying the substitution $\tilde{x} = rx$ and the characterisation (20), for (22) it is enough to observe that

$$(1-\tilde{\omega})\varphi\Big(\frac{\tilde{x}}{r}\Big)r^{-\mu}a_{j\alpha}\Big(r,\frac{\tilde{x}}{r},y\Big)r^{|\gamma|}D_{\tilde{x}}^\gamma\Big(-r\frac{\partial}{\partial r}\Big)^j(r\eta)^\alpha$$

for $\varphi(x) \in C_0^\infty(U)$ respects the standard Sobolev spaces up to $r = \infty$.

1.4 Abstract edge spaces and symbols with twisted homogeneity

A Hilbert space E is said to be equipped with a group action if there is given a strongly continuous group of isomorphisms $\kappa_\lambda : E \to E$, $\lambda \in \mathbb{R}_+$, such that $\kappa_\lambda\kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for all $\lambda,\lambda' \in \mathbb{R}_+$. An example is the space $E = \mathcal{K}^{s,\gamma}(X^\wedge)$ with

$$(\kappa_\lambda u)(r,x) = \lambda^{\frac{n+1}{2}}u(\lambda r,x),\ \lambda \in \mathbb{R}_+$$

for $n = \dim X$. More generally, if E is a Fréchet space, written as a projective limit $\varprojlim_{j\in\mathbb{N}} E^j$ of Hilbert spaces E^j with continuous embeddings $E^{j+1} \hookrightarrow E^j$ for all j, and if $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a group action on E^0 which restricts to group actions on E^j for every j we say that E is equipped with a group action.

An example is the space

$$\mathcal{S}_\varepsilon^\gamma(X^\wedge) := \varprojlim_{j\in\mathbb{N}} E^j \tag{24}$$

for $E^j := \langle r\rangle^{-j}\mathcal{K}^{j,\gamma+\varepsilon-(1+j)^{-1}}(X^\wedge)$, $\varepsilon > 0$. Let us define

$$\mathcal{S}_{\mathcal{O}}(X^\wedge) := \varprojlim_{\varepsilon>0} \mathcal{S}_\varepsilon^\gamma(X^\wedge) \tag{25}$$

(the notation indicates that the left hand side is independent of γ).

Definition 1.9 *Let E be a Hilbert space with group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$. Then $\mathcal{W}^s(\mathbb{R}^q,E)$ (the 'abstract' edge Sobolev space on $\mathbb{R}^q$ of smoothness $s \in \mathbb{R}$) is defined to be the completion of $\mathcal{S}(\mathbb{R}^q,E)$ with respect to the norm*

$$\Big\{\int\langle\eta\rangle^{2s}\big\|\kappa_{\langle\eta\rangle}^{-1}\hat{u}(\eta)\big\|_E^2 d\eta\Big\}^{\frac{1}{2}};$$

here $\hat{u}(\eta) = F_{y\to\eta}u(\eta)$ is the Fourier transform in $\mathbb{R}^q$, and $\kappa_{\langle\eta\rangle}^{-1}$ acts on the values of $\hat{u}(\eta)$ for every η.

Definition 1.10 (i) *Let E and $\widetilde{E}$ be Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. Then the space of (operator-valued) symbols $S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ for an open set $U \subseteq \mathbb{R}^p$, $\mu \in \mathbb{R}$, is defined to be the set all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ such that*

$$\sup\Big\{\langle\eta\rangle^{-\mu+|\beta|}\big\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\big\|_{\mathcal{L}(E,\tilde{E})} : (y,\eta) \in K \times \mathbb{R}^q\Big\}$$

is finite for every $K \Subset U$ and all multi-indices $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$.

(ii) *$S^{(\mu)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \widetilde{E})$ denotes the set of all functions $f_{(\mu)} \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E, \widetilde{E}))$ such that*

$$f_{(\mu)}(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda f_{(\mu)}(y,\eta)\kappa^{-1}_\lambda$$

for all $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$, $\lambda \in \mathbb{R}_+$.

(iii) *The space $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \widetilde{E})$ of classical symbols is defined as the set of all $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ such that there are elements $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \widetilde{E})$, $j \in \mathbb{N}$, such that*

$$a(y,\eta) - \sum_{j=0}^{N} \chi(\eta)a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; E, \widetilde{E})$$

for every $N \in \mathbb{N}$. Here $\chi(\eta)$ is any excision function, i.e., $\chi \in C^\infty(\mathbb{R}^q)$, $\chi(\eta) = 0$ for $|\eta| < c_0$, $\chi(\eta) = 1$ for $|\eta| > c_1$ for certain $0 < c_0 < c_1$.

In the case $E = \mathbb{C}$ we always set $\kappa_\lambda = \mathrm{id}_{\mathbb{C}}$ for every $\lambda \in \mathbb{R}_+$. Then for $E = \widetilde{E} = \mathbb{C}$ the Definition 1.10 reproduces the standard spaces of scalar symbols. If a notation or a relation is valid both in the classical and the general case we also write '(cl)' as subscript. Let $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \widetilde{E})$ denote the corresponding spaces of symbols $a(\eta)$ that are independent of y, i.e., with constant coefficients.

Example 1.11 *Let us set*

$$E := \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-},\ \widetilde{E}_\varepsilon := \mathcal{S}^{\gamma-\mu}_\varepsilon(X^\wedge) \oplus \mathbb{C}^{j_+}$$

and

$$F := \mathcal{K}^{s,-\gamma+\mu}(X^\wedge) \oplus \mathbb{C}^{j_+},\ \widetilde{F}_\varepsilon := \mathcal{S}^{-\gamma}_\varepsilon(X^\wedge) \oplus \mathbb{C}^{j_-}$$

for $\gamma, \mu \in \mathbb{R}$ and $\varepsilon > 0$. A family of operators

$$g(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-} \to \mathcal{K}^{\infty,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_+}$$

(continuous for every $s \in \mathbb{R}$) is called a Green symbol if there is an $\varepsilon = \varepsilon(g) > 0$ such that

$$g(y,\eta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \widetilde{E}_\varepsilon), g^*(y,\eta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; F, \widetilde{F}_\varepsilon) \tag{26}$$

for all $s \in \mathbb{R}$. A Green symbol is flat (of infinite order) if the conditions (26) hold for all $\varepsilon > 0$. Especially, if $g(y,\eta)$ is an upper corner, that means $g(y,\eta), g^*(y,\eta) \in S^\mu_{\mathrm{cl}}(\mathcal{K}^{s,\beta}(X^\wedge), \mathcal{S}_{\mathcal{O}}(X^\wedge))$ for all $s, \beta \in \mathbb{R}$.

Remark 1.12 *The operator of multiplication by a function $\varphi \in C_0^\infty(\overline{\mathbb{R}}_+)$ belongs to $S^0(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge))$ for every $s, \gamma \in \mathbb{R}$. If $g(y,\eta)$ is a Green symbol in the sense of Example* 1.11 *also*

$$\operatorname{diag}(\varphi, 1) g(y,\eta) \operatorname{diag}(\tilde{\varphi}, 1)$$

is a Green symbol for every $\varphi, \tilde{\varphi} \in C_0^\infty(\overline{\mathbb{R}}_+)$

Parallel to the spaces of symbols of Definition 1.10 we have vector-valued analogues of Sobolev spaces, based on a Hilbert space E with group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$. Recall the corresponding Definition from [7]. By $\mathcal{W}^s(\mathbb{R}^q, E)$ we denote the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm

$$\left\{ \int \langle \eta \rangle^{2s} \| \kappa_{\langle \eta \rangle}^{-1} \widehat{u}(\eta) \|_E^2 d\eta \right\}^{1/2},$$

$\widehat{u}(\eta) := (F_{y \to \eta} u)(\eta)$. For an open set $\Omega \subseteq \mathbb{R}^q$ we also have the spaces

$$\mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \quad \text{and} \quad \mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$$

of vector-valued analogues of the spaces $H^s_{\mathrm{comp}}(\Omega)$ and $H^s_{\mathrm{loc}}(\Omega)$, respectively.

1.5 Conical exits to infinity

In this section we want to deepen the information of Remark 1.8 in the sense of a more systematic discussion between edge symbols, standard operators in polar coordinates, and symbols within the exit pseudo-differential calculus.

Let us first consider $\mathbb{R}^{n+1}_{\tilde{x}}$ regarded as a space with 'conical exit' to infinity $|\tilde{x}| \to \infty$.

Definition 1.13 *The space*

$$S^{\mu;\nu}(\mathbb{R}^{n+1}_{\tilde{x}} \times \mathbb{R}^{n+1}_{\tilde{\xi}}), \tag{27}$$

$\mu, \nu \in \mathbb{R}$, is defined to be the set of all $a(\tilde{x}, \tilde{\xi}) \in C^\infty(\mathbb{R}^{n+1}_{\tilde{x}} \times \mathbb{R}^{n+1}_{\tilde{\xi}})$ such that

$$\sup \big\{ \langle \tilde{x} \rangle^{-\nu + |\alpha|} \langle \tilde{\xi} \rangle^{-\mu + |\beta|} |D^\alpha_{\tilde{x}} D^\beta_{\tilde{\xi}} a(\tilde{x}, \tilde{\xi})| : (\tilde{x}, \tilde{\xi}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \big\} < \infty$$

for every $\alpha, \beta \in \mathbb{N}^{n+1}$. We also say that a symbol $a(\tilde{x}, \tilde{\xi}) \in S^\mu(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ has the exit property, if it belongs to the space (27)*; then μ is called the pseudo-differential order, ν the exit order of a.*

More generally,

$$S^{\mu;\nu,\nu'}(\mathbb{R}^{n+1}_{\tilde{x}} \times \mathbb{R}^{n+1}_{\tilde{x}'} \times \mathbb{R}^{n+1}_{\tilde{\xi}})$$

denotes the set of all $a(\tilde{x}, \tilde{x}', \tilde{\xi}) \in C^\infty(\mathbb{R}^{3(n+1)}_{\tilde{x}, \tilde{x}', \xi})$ such that

$$\sup \{ \langle \tilde{x} \rangle^{-\nu + |\alpha|} \langle \tilde{x}' \rangle^{-\nu' + |\alpha'|} \langle \tilde{\xi} \rangle^{-\mu + |\beta|} |D^\alpha_{\tilde{x}} D^{\alpha'}_{\tilde{x}'} a(\tilde{x}, \tilde{x}', \tilde{x})| : (\tilde{x}. \tilde{x}', \tilde{\xi}) \in \mathbb{R}^{3(n+1)} \} < \infty$$

for every $\alpha, \alpha', \beta \in \mathbb{N}^{n+1}$.

Moreover, we set

$$S^{\mu;\nu}_{\mathrm{cl}_{\tilde{\xi}, \tilde{x}}}(\mathbb{R}^{n+1}_{\tilde{x}} \times \mathbb{R}^{n+1}_{\tilde{\xi}}) := S^\nu_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{x}}) \hat{\otimes}_\pi S^\mu_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{\xi}})$$

which is the space of classical symbols in $\tilde{\xi}$ and $\tilde{x}$.

Example 1.14 *Let $\omega(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function such that $\omega(t) = 1$ for $t < \frac{1}{2}$, $\omega(t) = 0$ for $t > \frac{2}{3}$, and set*

$$\psi(r, r') := \omega\left(\frac{(r-r')^2}{1+(r-r')^2}\right).$$

Then ψ represents an element of $S^{0;0,0}(\mathbb{R}_r \times \mathbb{R}_{r'} \times \mathbb{R}_\varrho)$ (which is independent of the covariable ϱ).

Let us set

$$H^{s;g}(\mathbb{R}^{n+1}) := \langle \tilde{x} \rangle^{-g} H^s(\mathbb{R}^{n+1}).$$

Theorem 1.15 *For every $a(\tilde{x}, \tilde{\xi}) \in S^{\mu;\nu}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ the associated pseudo-differential operator $\mathrm{Op}(a)$ induces continuous operators*

$$\mathrm{Op}(a) : H^{s;g}(\mathbb{R}^{n+1}) \to H^{s-\mu;g-\nu}(\mathbb{R}^{n+1})$$

for all $s, g \in \mathbb{R}$.

Remark 1.16 *By virtue of*

$$\mathcal{S}(\mathbb{R}^{n+1}) = \varprojlim_{N \in \mathbb{N}} H^{N;N}(\mathbb{R}^{n+1})$$

it follows that $\mathrm{Op}(a)$ also induces a continuous operator

$$\mathrm{Op}(a) : \mathcal{S}(\mathbb{R}^{n+1}) \to \mathcal{S}(\mathbb{R}^{n+1}).$$

Let us set

$$L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1}) := \left\{ \mathrm{Op}(a) : a(\tilde{x}, \tilde{\xi}) \in S^{\mu;\nu}_{(\mathrm{cl}_{\tilde{\xi};\tilde{x}})}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \right\}.$$

As is known, $\mathrm{Op}(\cdot)$ induces an isomorphism

$$\mathrm{Op} : S^{\mu;\nu}_{(\mathrm{cl}_{\tilde{\xi};\tilde{x}})}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \to L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1}) \tag{28}$$

for every $\mu, \nu \in \mathbb{R}$, including $\mu = -\infty$ or $\nu = -\infty$. Note that $L^{-\infty;-\infty}(\mathbb{R}^{n+1} = \bigcap_{\mu,\nu} L^{\mu;\nu}(\mathbb{R}^{n+1})$ is equal to the space of all integral operators with kernel in $\mathcal{S}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$.

Definition 1.17 *An element $a(\tilde{x}, \tilde{\xi}) \in S^{\mu;\nu}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ is called elliptic, if there exists a $p(\tilde{x}, \tilde{\xi}) \in S^{-\mu;-\nu}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ such that*

$$a(\tilde{x}, \tilde{\xi}) p(\tilde{x}, \tilde{\xi}) = 1 \bmod S^{\mu-1;\nu-1}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}).$$

An operator $A \in L^{\mu;\nu}(\mathbb{R}^{n+1})$ is called elliptic if the associated symbol $a(\tilde{x}, \tilde{\xi}) \in S^{\mu;\nu}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ is elliptic (cf. the bijection (28)).

For classical symbols $a(\tilde{x}, \tilde{\xi}) \in S^{\mu;\nu}_{\mathrm{cl}_{\tilde{\xi};\tilde{x}}}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ we have a triple of homogeneous components,

$$\sigma(a) := (\sigma_\psi(a), \sigma_\mathrm{e}(a), \sigma_{\psi\mathrm{e}}(a))$$

given on $\mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\setminus\{0\})$, $(\mathbb{R}^{n+1}\setminus\{0\})\times\mathbb{R}^{n+1}$, and $(\mathbb{R}^{n+1}\setminus\{0\})\times(\mathbb{R}^{n+1}\setminus\{0\})$, respectively. The homogeneous principal symbol $\sigma_\psi(a)(\tilde{x},\tilde{\xi})$ in $\tilde{\xi}$ order μ is as usual,

$$\sigma_\psi(a)(\tilde{x},\lambda\tilde{\xi}) = \lambda^\mu\sigma_\psi(a)(\tilde{x},\tilde{\xi}) \text{ for all } (\tilde{x},\tilde{\xi}) \in \mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\setminus\{0\}), \lambda\in\mathbb{R}_+.$$

Analogously, the principal exit symbol $\sigma_{\mathrm{e}}(a)(\tilde{x},\tilde{\xi})$ in $\tilde{x}$ of order ν has the property

$$\sigma_{\mathrm{e}}(a)(\delta\tilde{x},\tilde{\xi}) = \delta^\nu\sigma_{\mathrm{e}}(a)(\tilde{x},\tilde{\xi}) \text{ for all } (\tilde{x},\tilde{\xi}) \in (\mathbb{R}^{n+1}\setminus\{0\})\times\mathbb{R}^{n+1}, \delta\in\mathbb{R}_+.$$

The third component is the homogeneous principal part of $\sigma_{\mathrm{e}}(a)(\tilde{x},\tilde{\xi})$ in $\tilde{\xi}$ of order μ and has the homogeneity

$$\sigma_{\psi\mathrm{e}}(a)(\delta\tilde{x},\lambda\tilde{\xi}) = \delta^\nu\lambda^\mu\sigma_{\psi\mathrm{e}}(a)(\tilde{x},\tilde{\xi}) \text{ for all } (\tilde{x},\tilde{\xi}) \in (\mathbb{R}^{n+1}\setminus\{0\})\times(\mathbb{R}^{n+1}\setminus\{0\}),\ \lambda,\delta\in\mathbb{R}_+.$$

Remark 1.18 *A symbol $a(\tilde{x},\tilde{\xi}) \in S^{\mu;\nu}_{\mathrm{cl}_{\tilde{\xi};\tilde{x}}}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$ is elliptic in the sense of Definition* 1.17 *if and only if*

$$\begin{aligned}\sigma_\psi(a)(\tilde{x},\tilde{\xi}) &\neq 0 \text{ for all } (\tilde{x},\tilde{\xi}) \in \mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\setminus\{0\}),\\ \sigma_{\mathrm{e}}(a)(\tilde{x},\tilde{\xi}) &\neq 0 \text{ for all } (\tilde{x},\tilde{\xi}) \in (\mathbb{R}^{n+1}\setminus\{0\})\times\mathbb{R}^{n+1},\\ \sigma_{\psi\mathrm{e}}(a)(\tilde{x},\tilde{\xi}) &\neq 0 \text{ for all } (\tilde{x},\tilde{\xi}) \in (\mathbb{R}^{n+1}\setminus\{0\})\times(\mathbb{R}^{n+1}\setminus\{0\}).\end{aligned}$$

These three conditions are independent.

Theorem 1.19 *Let $A \in L^{\mu;\nu}(\mathbb{R}^{n+1})$, $\mu,\nu\in\mathbb{R}$. Then the following conditions are equivalent:*

(i) *the operator A is elliptic;*

(ii) *A induces a Fredholm operator*

$$A : H^{s;g}(\mathbb{R}^{n+1}) \to H^{s-\mu;g-\nu}(\mathbb{R}^{n+1}) \tag{29}$$

for some fixed $s = s_0$, $g = g_0 \in \mathbb{R}$.

Theorem 1.20 (i) *Let $A \in L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1})$ be elliptic. Then A has a parametrix $P \in L^{-\mu;-\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1})$ in the sense*

$$PA-1,\ AP-1 \in L^{-\infty;-\infty}(\mathbb{R}^{n+1}).$$

(ii) *The Fredholm property of the operator* (29) *for some $s = s_0$, $g = g_0 \in \mathbb{R}$ entails the Fredholm property of* (29) *for all $s,g\in\mathbb{R}$.*

(iii) *Let $A \in L^{\mu;\nu}(\mathbb{R}^{n+1})$ be elliptic. Then* $\ker A$ *and* $\operatorname{coker} A$ *of the Fredholm operator* (29) *is independent of s,g, and there are subspaces of finite dimension $V,W\subset\mathcal{S}(\mathbb{R}^{n+1})$ such that* $\ker A = V$ *and*

$$\operatorname{im} A + W = H^{s-\mu;g-\nu}(\mathbb{R}^{n+1}),\ W\cap\operatorname{im} A = \{0\}.$$

We now interpret the stretched cone $X^\wedge \ni (r,x)$ as a space with conical exit $r\to\infty$. In order to avoid clumpsy precautions for $r\to 0$ we first pass to the cylinder $\mathbb{R}\times X$ and later on localise the operators on the plus side.

Definition 1.21 *We say that the cylinder $\mathbb{R} \times X$ is equipped with the structure of a manifold with conical exits $r \to \pm\infty$ if there is given a locally finite atlas of charts*

$$\chi_\iota : \mathbb{R} \times U_\iota \to \Gamma_\iota, \ \iota \in I,$$

for coordinate neighbourhoods U_ι on X and open sets $\Gamma_\iota \subset \mathbb{R}^{n+1}$ such that for a constant $R > 0$ independent of ι for every $\iota, \tilde{\iota} \in I$ we have:

(i) $\Gamma_\iota \cap \{\tilde{x} \in \mathbb{R}^{n+1} : |\tilde{x}| \geq R\} = \{\lambda\tilde{x} : \tilde{x} \in \Gamma_{\iota,R}, \lambda \geq 1\}$ *for* $\Gamma_{\iota,R} := \Gamma_\iota \cap \{|\tilde{x}| = R\}$*;*

(ii) $\chi_\iota(\lambda r, x) = \lambda\chi_\iota(r, x)$ *for every* $\lambda \geq 1$, $|r| \geq R$*;*

(iii) *the transition maps*

$$\tau_{\tilde{\iota},\iota} := \chi_{\tilde{\iota}}\chi_\iota^{-1} : \chi_\iota(\mathbb{R} \times (U_\iota \cap U_{\tilde{\iota}})) \to \chi_{\tilde{\iota}}(\mathbb{R} \times (U_\iota \cap U_{\tilde{\iota}}))$$

have the property

$$\tau_{\tilde{\iota},\iota}(\lambda\tilde{x}) = \lambda\tau_{\tilde{\iota},\tilde{\iota}}(\tilde{x})$$

for every $|\tilde{x}| \geq R$, $\lambda \geq 1$.

If $\mathbb{R} \times X$ is equipped with a structure in that sense we also write $X_\asymp$ instead of $\mathbb{R} \times X$.

Let us fix a locally finite partition of unity $\{\varphi'_\iota\}_{\iota \in I}$ on X subordinate to $\{U_\iota\}_{\iota \in I}$, moreover, let $\{\psi'_\iota\}_{\iota \in I}$ be a system of functions $\psi'_\iota \in C_0^\infty(U_\iota)$ such that $\psi'_\iota \equiv 1$ on $\operatorname{supp} \varphi'_\iota$. Moreover, set

$$\varphi_\iota(r, x) = \varphi'_\iota(x), \ \psi_\iota(r, x) = \psi_\iota(x)$$

for all $(r, x) \in \mathbb{R} \times X$.

Let us endow $X_\asymp$ with a Riemannian metric that is equal to a cone metric $g_{X_\asymp} := dr^2 + r^2 g_X$ for $|r| > R$ for a Riemannian metric g_X on X. If dx denotes a corresponding measure on X the associated measure on $X_\asymp$ for $|r| > R$ is of the form

$$|r|^n dr dx \tag{30}$$

for $n = \dim X$.

Now let $L^{\mu;\nu}_{(\mathrm{cl})}(\Gamma_\iota)$ denote the restriction of $L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1})$ to Γ_ι (in the sense of operators $A : C_0^\infty(\Gamma_\iota) \to C^\infty(\Gamma_\iota)$), and let

$$L^{\mu;\nu}_{(\mathrm{cl})}(X_\asymp)$$

denote the space of all operators $\sum_{\iota \in I} \varphi_\iota \{(\chi_\iota^{-1})_* A_\iota\} \psi_\iota + C$ for arbitrary $A_\iota \in L^{\mu;\nu}_{(\mathrm{cl})}(\Gamma_\iota)$ and C having a kernel in $\mathcal{S}((\mathbb{R} \times X) \times (\mathbb{R} \times X)) := \mathcal{S}(\mathbb{R} \times \mathbb{R}, C^\infty(X \times X))$.

The concept of operators on manifolds with conical exits to infinity will be necessary also in the set-up of operator-valued symbols, cf. Definition 1.1. Let us recall some basic technicalities (the formulations will be slightly more general than in the scalar case; so we also give additional information for $E = \widetilde{E} = \mathbb{C}$ and trivial $\kappa_\lambda = \tilde{\kappa}_\lambda$).

Let E and $\widetilde{E}$ be Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda \in \mathbb{R}_+}$, respectively. Then

$$S^{\mu;\nu,\nu'}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \tag{31}$$

for $\mu, \nu, \nu' \in \mathbb{R}$ denotes the set of all $a(y, y', \eta) \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ such that

$$\left\| \tilde{\kappa}^{-1}_{\langle \eta \rangle} \{ D^\alpha_y D^{\alpha'}_{y'} D^\beta_\eta a(y, y', \eta) \} \kappa_{\langle \eta \rangle} \right\|_{\mathcal{L}(E, \widetilde{E})} \le c \langle \eta \rangle^{\mu - |\beta|} \langle y \rangle^{\nu - |\alpha|} \langle y' \rangle^{\nu' - |\alpha'|} \tag{32}$$

for all $(y, y', \eta) \in \mathbb{R}^{3q}$ and $\alpha, \alpha', \beta \in \mathbb{N}^q$, with constants $c = c(\alpha, \alpha', \beta) > 0$. Similarly we have spaces of the kind

$$S^{\mu;\nu}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \tag{33}$$

where the elements $a(y, \eta)$ satisfy analogues of the estimates (32) (obtained by simply omitting y').

Elements of (31) are regarded as double symbols of corresponding pseudo-differential operators $\mathrm{Op}(a)$. There are then unique left and right symbols

$$a_L(y, \eta) \quad \text{and} \quad a_R(y', \eta)$$

belonging to $S^{\mu;\nu+\nu'}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ such that $\mathrm{Op}(a) = \mathrm{Op}(a_L) = \mathrm{Op}(a_R)$. For instance, $a_L(y, \eta)$ can be calculated as an oscillatory integral $a_L(y, \eta) = \iint e^{-iz\zeta} a(y, y + z, \eta + \zeta) dz đ\zeta$; a similar expression holds for $a_R(y', \eta)$. Setting

$$L^{\mu;\nu}(\mathbb{R}^q; E, \widetilde{E}) := \{ \mathrm{Op}(a) : S^{\mu;\nu}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \}$$

we have an isomorphism

$$\mathrm{Op} : S^{\mu;\nu}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \to L^{\mu;\nu}(\mathbb{R}^q; E, \widetilde{E}).$$

2 Edge quantisation

2.1 Pseudo-differential edge symbols

We now pass to the algebra of pseudo-differential edge symbols. These will be parameter-dependent families of operators on the infinite (stretched) cone $X^\wedge$ with a specific dependence on edge variables and covariables $(y, \eta) \in \Omega \times \mathbb{R}^q$. In this section X is assumed to be a closed compact C^∞ manifold.

In order to model families of edge-degenerate pseudo-differential operators we start from the space

$$L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\varrho,\eta}) \tag{34}$$

of classical parameter-dependent pseudo-differential operators on X which is a Fréchet space in a natural way. With $p(r, y, \varrho, \eta) \in C^\infty(\mathbb{R}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\varrho,\eta})$ we can associate a family of pseudo-differential operators

$$\mathrm{op}_r(p)(y, \eta) \in C^\infty(\Omega, L^\mu_{\mathrm{cl}}(X^\wedge; \mathbb{R}^q)).$$

Remark 2.1 *Let $\tilde{p}(\tilde{\varrho}, \tilde{\eta}) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}})$ be parameter-dependent elliptic of order μ and set*

$$b(r, \varrho, \eta) := r^{-\mu} \tilde{p}(r\varrho, r\eta).$$

Then for every $\eta \neq 0$ there is an $R(\eta) > 0$ such that

$$b(r, \varrho, \eta) : H^s(X) \to H^{s-\mu}(X)$$

is a family of isomorphisms for all $r \in \mathbb{R}$, $r \geq R$ and $\varrho \in \mathbb{R}$. In addition for a suitable choice of cut-off functions $\omega(r)$ and $\tilde{\omega}(r)$ the expression

$$\|u\|_{H^s_{\mathrm{cone}}(X^\wedge)} := \|\omega_{|\eta|}u\|_{H^s(\mathbb{R}_+\times X)} + \|(1-\omega_{|\eta|})\mathrm{op}_r(b)(\eta)(1-\tilde{\omega}_{|\eta|})u\|_{L^2(\mathbb{R}_+\times X)}$$

is an equivalent norm in the space $H^s_{\mathrm{cone}}(X^\wedge)$.

Incidentally the parameter ϱ plays the role of $\mathrm{Im}z$ for a complex variable $z = \beta + i\varrho$ with fixed $\beta \in \mathbb{R}$. In this case instead of (34) we also write

$$L^\mu_{\mathrm{cl}}(X; \Gamma_\beta \times \mathbb{R}^q), \tag{35}$$

cf. also the notation (17). Elements $f(r, y, z, \eta) \in C^\infty(\mathbb{R}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^q))$ will be regarded as amplitude functions of Mellin pseudo-differential operators

$$\mathrm{op}^\gamma_M(f)(y,\eta)u(r) := \int \left(\frac{r}{r'}\right)^{-(\frac{1}{2}-\gamma+i\varrho)} f(r, y, \frac{1}{2} - \gamma + i\varrho, \eta)u(r')\frac{dr'}{r'}đ\varrho, \tag{36}$$

$đ\varrho = (2\pi)^{-1}d\varrho$. Also in this case we have

$$\mathrm{op}^\gamma_M(f)(y,\eta) \in C^\infty(\Omega, L^\mu_{\mathrm{cl}}(X^\wedge; \mathbb{R}^q)).$$

Definition 2.2 *By $L^\mu_{\mathrm{cl}}(X; \mathbb{C}_z \times \mathbb{R}^q_\eta)$ we denote the space of all operator families $f(z,\eta) \in \mathcal{A}(\mathbb{C}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q_\eta))$ such that*

$$f(\beta + i\varrho, \eta) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\varrho,\eta})$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals. Moreover, let $\mathcal{M}^{-\infty}(X; \Gamma_\delta)_\varepsilon$ for any $\beta \in \mathbb{R}$, $\varepsilon > 0$, denote the set of all

$$f(z) \in \mathcal{A}(\{\delta - \varepsilon < \mathrm{Re}z < \delta + \varepsilon\}, \quad L^{-\infty}(X))$$

such that

$$f(\beta + i\varrho) \in L^{-\infty}(X; \mathbb{R}_\varrho)$$

for every $\beta \in (\delta - \varepsilon, \delta + \varepsilon)$, uniformly in compact β-intervals. Finally, we set

$$\mathcal{M}^{-\infty}(X; \Gamma_\delta) := \cup_{\varepsilon>0}\mathcal{M}^{-\infty}(X; \Gamma_\delta)_\varepsilon.$$

The spaces $L^\mu_{\mathrm{cl}}(X; \mathbb{C}\times\mathbb{R}^q)$ and $\mathcal{M}^{-\infty}(X; \Gamma_\delta)_\varepsilon$ are Fréchet with natural semi-norm systems that immediately follow from the definition.

Remark 2.3 (i) *If $\omega(r), \tilde{\omega}(r)$ are arbitrary cut-off functions and $f(y,z) \in C^\infty(\Omega)$, $\mathcal{M}^{-\infty}(X; \Gamma_{\frac{n+1}{2}-\gamma}))$, $\Omega \subseteq \mathbb{R}^q$ open, we have*

$$m(y,\eta) := r^{-\nu}\omega_{[\eta]}\mathrm{op}^{\gamma-\frac{n}{2}}_M(f)(y)\tilde{\omega}_{[\eta]} \in S^\nu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\nu}(X^\wedge)) \tag{37}$$

for every $s \in \mathbb{R}$. For the principal part of order ν we have

$$\sigma_\wedge(m)(y,\eta) = r^{-\nu}\omega_{|\eta|}\mathrm{op}^{\gamma-\frac{n}{2}}_M(f)(y)\tilde{\omega}_{|\eta|}. \tag{38}$$

(ii) *Let* $h(r,y,z,\eta) := \widetilde{h}(r,y,z,r\eta)$ *for* $\widetilde{h}(r,y,z,\widetilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\mathbb{C}_z \times \mathbb{R}^q_{\widetilde{\eta}}))$*, and let* $\varphi_0(r), \varphi_1(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$*, and* $\varphi_1 \equiv 0$ *on supp* φ_0 *(e.g.,* φ_1 *may be a cut-off function,* $\varphi_0 \in C_0^\infty(\mathbb{R}_+)$*). Then*

$$g(y,\eta) := \varphi_1(r[\eta])r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\varphi_0(r[\eta])$$

is a flat Green symbol of order μ*.*

The following result may be interpreted as a Mellin quantisation.

Theorem 2.4 *For every* $p(r,y,\varrho,\eta)$ *of the form*

$$p(r,y,\varrho,\eta) = \tilde{p}(r,y,\tilde{\varrho},\tilde{\eta})\big|_{\tilde{\varrho}=r\varrho,\tilde{\eta}=r\eta}$$

for a $\tilde{p}(r,y,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$ *there exists an* $h(r,y,z,\eta)$ *of the form*

$$h(r,y,z,\eta) = \tilde{h}(r,y,z,\tilde{\eta})|_{\tilde{\eta}=r\eta}$$

for an $\tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\mathbb{C}\times\mathbb{R}^q))$ *such that*

$$\mathrm{op}_r(p)(y,\eta) = \mathrm{op}_M^\gamma(h)(y,\eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q)),$$

for every $\gamma \in \mathbb{R}$*, and* $\tilde{h}(r,y,z\tilde{\eta})$ *is unique* $\bmod C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{-\infty}(X;\mathbb{C}\times\mathbb{R}^q))$*. Moreover, setting*

$$p_0(r,y,\varrho,\eta) := \tilde{p}(0,y,r\varrho,r\eta),\ \ h_0(r,y,z,\eta) := \tilde{h}(0,y,z,r\eta) \tag{39}$$

we also have

$$\mathrm{op}_r(p_0)(y,\eta) = \mathrm{op}_M^\gamma(h_0)(y,\eta) \bmod C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q))$$

for every $\gamma \in \mathbb{R}$*.*

A proof may be found in [8], see also [9], or [4].

Let us fix cut-off functions

$$\omega(r),\ \tilde{\omega}(r)$$

such that $\tilde{\omega} \equiv 1$ on $\mathrm{supp}\,\omega$; in that case we write $\omega \prec \widetilde{\omega}$. Set

$$\chi(\eta) = 1-\omega(r),\ \ \tilde{\chi}(\eta) = 1-\tilde{\tilde{\omega}}(r) \tag{40}$$

for another cut-off function $\tilde{\tilde{\omega}}(r)$ such that $\tilde{\tilde{\omega}} \prec \omega$. Moreover, choose cut-off functions $\sigma(r)$, $\tilde{\sigma}(r)$.

Let

$$p(r,y,\varrho,\eta),\ \ h(r,y,z,\eta) \tag{41}$$

be operator families related to each other as in Theorem 2.4, and set

$$a(y,\eta) := r^{-\mu}\sigma\big\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\tilde{\omega}_{[\eta]} + \chi_{[\eta]}\mathrm{op}_r(p)(y,\eta)\tilde{\chi}_{[\eta]}\big\}\tilde{\sigma} \tag{42}$$

$n = \dim X$, where $\omega_c(r) := \omega(rc)$, $\chi_c(r) := \chi(rc)$, etc., for any $c>0$.

Remark 2.5 *Let $a(y,\eta)$ be an operator function of the form (42). Then we have*

$$a(y,\eta) \in C^\infty(\Omega, L^\mu_{\mathrm{cl}}(X^\wedge;\mathbb{R}^q)).$$

The parameter-dependent homogeneous principal symbol of $a(y,\eta)$

$$\sigma_\psi(a)(r,x,y,\varrho,\xi,\eta) \in C^\infty(T^*(\mathbb{R}_+ \times X \times \Omega \times \mathbb{R}^{1+n+q} \setminus 0))$$

has the form

$$\sigma_\psi(a)(r,x,y,\varrho,\xi,\eta) = \sigma(r)\tilde{\sigma}(r)r^{-\mu}\tilde{p}_{(\mu)}(r,x,y,r\varrho,\xi,r\eta),$$

where $\tilde{p}_{(\mu)}(r,x,y,\tilde{\varrho},\xi,\tilde{\eta})$ denotes the parameter-dependent homogeneous principal symbol of $\tilde{p}(r,y,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$ of order μ.

Theorem 2.6 *We have*

$$a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

for every $s \in \mathbb{R}$. Moreover, for every $\varepsilon > 0$ we have

$$a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{S}^\gamma_\varepsilon(X^\wedge), \mathcal{S}^{\gamma-\mu}_\varepsilon(X^\wedge)).$$

A proof of Theorem 2.6 may be found in [3, Section 2.1.3].

Theorem 2.6 can be regarded as a quantisation for edge-degenerate families of pseudo-differential operators as in Theorem 2.4. In fact, Theorem 2.6 gives rise to continuous operators

$$\mathrm{Op}_y(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge)) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)) \tag{43}$$

for all $s, \gamma \in \mathbb{R}$. In other words, the correspondence $p \to a \to \mathrm{Op}(a)$ represents an operator convention that associates with p continuous operators (43).

Let us set

$$\sigma_\wedge(a)(y,\eta) := r^{-\mu}\{\omega_{|\eta|}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(y,\eta)\tilde{\omega}_{|\eta|} + \chi_{|\eta|}\mathrm{op}_r(p_0)(y,\eta)\tilde{\chi}_{|\eta|}\}$$

for the families h_0 and p_0 as in Theorem 2.4 and $(y,\eta) \in T^*\Omega \setminus 0$. Then we have

$$\sigma_\wedge(a)(y,\lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\wedge(a)(y,\eta)\kappa_\lambda^{-1} \tag{44}$$

for all $\lambda \in \mathbb{R}_+$.

2.2 A new edge quantisation

The new quatisation of edge degenerate symbols consists of taking an operator family of the form

$$a(y,\eta) := r^{-\mu}\sigma\left\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)(y,\eta)\widetilde{\omega}_{[\eta]} + \chi_{[\eta]}\psi_{[\eta]}\mathrm{op}_r(p)(y,\eta)\right\}\widetilde{\sigma} \tag{45}$$

instead of (42), where $\omega, \widetilde{\omega}, \widetilde{\omega}, \sigma, \widetilde{\sigma}$ and p, h are as before, while

$$\psi_{[\eta]}(r,r') := \psi(r[\eta], r'[\eta]),$$

with the function $\psi(r,r')$ of Example 1.14.

In order to compare (42) and (45) we analyse operator families associated with

$$p(r,\varrho,\eta) := \tilde{p}(r,r\varrho,r\eta)$$

for an element $\tilde{p}(r,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$ (it is enough to assume the y-independent case). We assume that

$$\tilde{p}(r,\tilde{\varrho},\tilde{\eta}) \text{ is independent of } r \text{ for } r \geq R$$

some constant $R > 0$.

Theorem 2.7 *Let $\sigma, \tilde{\sigma}, \chi, \tilde{\chi}$ be as in* (42) *and form*

$$g(\eta) := r^{-\mu}\sigma(r)\chi_{[\eta]}(r)(1-\psi_{[\eta]}(r,r'))\mathrm{op}_r(p)(\eta)\tilde{\chi}_{[\eta]}(r')\tilde{\sigma}(r').$$

Then we have

$$g(\eta), g^*(\eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{S}_\mathcal{O}(X^\wedge)) \tag{46}$$

for every $s,\gamma \in \mathbb{R}$, i.e., $g(\eta)$ is a flat Green symbol of order μ, (cf. Example 1.11 *and the notation* (25)*).*

Proof. We first show the property

$$g(\eta), g^*(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{S}_\mathcal{O}(X^\wedge)).$$

Let us consider $g(\eta)$; the corresponding relation for $g^*(\eta)$ can be obtained in an analogous manner. It is enough to show

$$g(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)_U,\mathcal{S}_\mathcal{O}(X^\wedge)_U)) \tag{47}$$

for every coordinate neighbourhood U on X where

$$\mathcal{K}^{s,\gamma}(X^\wedge)_U := \{u(r,x) \in \mathcal{K}^{s,\gamma}(X^\wedge) : \mathrm{supp} u \subseteq \overline{\mathbb{R}}_+ \times U\}, \tag{48}$$

$$\mathcal{S}_\mathcal{O}(X^\wedge)_U := \{u(r,x) \in \mathcal{S}_\mathcal{O}(X^\wedge) : \mathrm{supp} u \subseteq \overline{\mathbb{R}}_+ \times U\}. \tag{49}$$

Here we use the fact that there is an a tlas $\{U_1,\ldots,U_N\}$ on X in such a way that for every two indices $1 \leq j,\, k \leq N$ also $U_j \cup U_k$ is a coordinate neighbourhood on X. Without loss of generality we may assume that the coordinate neighbourhood U in (48), (49) are contained in other coordinate neighbourhoods $\widetilde{U}$ such that $\overline{U}$ are compact subsets. Now, if we pass to local coodinates on X we can write

$$\begin{aligned} g(\eta)u(r,x) = r^{-\mu}\sigma(r)\chi_{[\eta]}(r) \iint & e^{i(r-r')\varrho + i(x-x')\xi} \\ & \tilde{p}(r,x,x',r\varrho,\xi,r\eta)(1-\psi_{[\eta]}(r,r'))\tilde{\chi}_{[\eta]}(r')\tilde{\sigma}(r') \\ & u(r',x')dr'dx'đ\varrho đ\xi, \end{aligned} \tag{50}$$

where $\mathrm{supp} u(r,x)$ is contained in $\overline{\mathbb{R}}_+ \in K$ for a compact set $K \subset \mathbb{R}^n$. Here $\tilde{p}(r,x,x',\tilde{\varrho},\xi,\tilde{\eta})$ is a classical symbol in $(\tilde{\varrho},\xi,\tilde{\eta}) \in \mathbb{R}^{1+n+q}$ of order μ. After multiplying $\tilde{p}(r,x,x',\tilde{\varrho},\xi,\tilde{\eta})$ from the left and the right by localising functions $\varphi_0(x)$ and $\psi_0(x')$, respectively, belonging $C_0^\infty(K)$, we can assume that $\tilde{p}(r,x,x',\tilde{\varrho},\xi,\tilde{\eta})$ has compact support with respect to (x,x').

We want to compute the distributional kernel of (50). To this end we choose an N and write $e^{i(r-r')\varrho} = |r-r'|^{-2N} D_{\varrho}^{2N} e^{i(r-r')\varrho}$. Inserting this in (50) and integrating by parts under the integral givces us

$$\begin{aligned} g(\eta)u(r,x) = & r^{-\mu}\sigma(x)\chi_{[\eta]}(r) \iint |r[\eta]-r'[\eta]|^{-2N}(1-\psi_{[\eta]}(r,r')) \\ & e^{i(r-r')\varrho+i(x-x')\xi}(r[\eta])^{2N}(D_{\tilde{\varrho}}^{2N}\tilde{p})(r,x,x',r\varrho,\xi,r\eta) \\ & \tilde{\chi}_{[\eta]}(r')\tilde{\sigma}(r')u(r',x')dr'dx'đ\varrho đ\xi. \end{aligned}$$

The kernel of this operator can be written as

$$\begin{aligned} K(g)(r,r',x,x';\eta) = r^{-\mu}\sigma(r)\chi_{[\eta]}(r) & \iint |r[\eta]-r'[\eta]|^{2N}(1-\psi_{[\eta]}(r,r')) \\ & e^{i(r-r')\varrho+i(x-x')\xi}(r[\eta])^{2N}(D_{\tilde{\varrho}}^{2N}\tilde{p}) \\ & (r,x,x',r\varrho,\xi,r\eta)\tilde{\chi}_{[\eta]}(r')\tilde{\sigma}(r')đ\varrho đ\xi. \end{aligned} \tag{51}$$

By virtue of the symbolic estimates

$$|D_{\tilde{\varrho}}^{2N}\tilde{p}(r,x,x',r\varrho,\xi,r\eta)| \le c\langle r\varrho,\xi,r\eta\rangle^{\mu-2N} \tag{52}$$

for all $x,x' \in K$ and $r \in \mathrm{supp}\sigma\chi_{[\eta]}$ we see that the integral (51) converges, together with all derivatives in r,r',x,x' up to some order M when $N=N(M)$ is chosen sufficiently large. This shows that for every fixed $\eta \in \mathbb{R}^q$ the kernel of (51) belongs to $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n)$, and this remains true if η varies in a compact set $\subset \mathbb{R}^q$. It is also clear that the kernel smoothly depends on $\eta \in \mathbb{R}^q$. Thus we have verified the relation (47). For the formal adjoint we can argue in a similar manner.

For the proof of (46) we first generate the homogeneous principal symbol of order μ

$$\sigma_\wedge(g)(\eta) = \lim_{\lambda\to\infty} \lambda^{-\mu}\kappa_\lambda^{-1} g(\lambda\eta)\kappa_\lambda \text{ for } \eta \neq 0.$$

In local coordinates on X it takes the form

$$\begin{aligned} \sigma_\wedge(g)(\eta)u(r,x) = r^{-\mu}\chi_{|\eta|}(r) & \iint e^{i(r-r')\varrho+i(x-x')\xi}(1-\psi_{|\eta|}(r,r')) \\ & \tilde{p}(0,x,x';r\varrho,\xi,r\eta)\tilde{\chi}_{|\eta|}(r')u(r',x')dr'dx'đ\varrho đ\xi. \end{aligned}$$

Integration by parts gives us again

$$\begin{aligned} \sigma_\wedge(g)(\eta)u(r,x) = r^{-\mu}\chi_{|\eta|}(r) & \iint |r|\eta|-r'|\eta||^{-2N}(1-\psi_{|\eta|}(r,r')) \\ & e^{i(r-r')\varrho+i(x-x')\xi}(r|\eta|)^{2N}(D_{\tilde{\varrho}}^{2N}\tilde{p})(0,x,x',r\varrho,\xi,r\eta)\tilde{\chi}_{|\eta|}(r') \\ & u(r',x')dr'dx'đ\varrho đ\xi \end{aligned}$$

for every $N \in \mathbb{N}$. Using the estimate

$$|r-r'|^{-2N}(1-\psi(r,r')) \le \langle r\rangle^{-N}\langle r'\rangle^{-N} \tag{53}$$

and the symbolic estimate (52) we can easily verify that $\sigma_\wedge(g)(\eta)$ has a kernel in $\mathcal{S}_{\mathcal{O}}(X^\wedge)_U \hat{\otimes}_\pi \mathcal{S}_{\mathcal{O}}(X^\wedge)_U$. Assuming for the moment that $\tilde{p} = \tilde{p}(\tilde{\varrho},\tilde{\eta})$ is independent of r we have

$$g(\eta) - \chi(\eta)\sigma_\wedge(g)(\eta) \in S^{-\infty}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}_{\mathcal{O}}(X^\wedge))$$

for any excision function $\chi(\eta) \in C^\infty(\mathbb{R}^q)$. Otherwise, we can successively compute the lower order terms by a Taylor expansion

$$\tilde{p}(r, \tilde{\varrho}, \tilde{\eta}) = \sum_{l=0}^{L} r^l \tilde{p}_l(\tilde{\varrho}, \tilde{\eta}) + r^{L+1} \tilde{p}_{(L+1)}(r, \tilde{\varrho}, \tilde{\eta}).$$

The above process applied to $r^l \tilde{p}_l(\tilde{\varrho}, \tilde{\eta})$ then gives us the homogeneous components of $g(\eta)$ of order $\mu - l$. Another elementary calculation with $r^{L+1}\tilde{p}_{(L+1)}(r, \tilde{\varrho}, \tilde{\eta})$ yields a remainder in $S^{\mu-(L+1)}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}_{\mathcal{O}}(X^\wedge))$ which shows that our symbol is classical. For the formal adjoints we can do the same. $\square$

Proposition 2.8 *Let $\tilde{p}(r, \tilde{\varrho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, L^{-\infty}(X; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$ be independent of r for $r \geq R$ for some $R > 0$. Then*

$$g(\eta) := r^{-\mu} \sigma(r) \chi_{[\eta]}(r) \mathrm{op}_r(p)(\eta) \tilde{\chi}_{[\eta]}(r') \tilde{\sigma}(r') \tag{54}$$

is a flat Green symbol of order μ; in particular, it satisfies the relations (46).

Proof. It is evident that $g(\eta)$ has the property (46). It remains to show that $g(\eta)$ is a classical symbol. Let us first consider the case that $\tilde{p} = \tilde{p}(\tilde{\varrho}, \tilde{\eta})$ is independent of r. Then for $\eta \neq 0$ we can set

$$g_{(\mu)}(\eta) := r^{-\mu} \chi_{|\eta|}(r) \mathrm{op}_r(p) \tilde{\chi}_{|\eta|}(r').$$

Write

$$g_{(\mu)}(\eta) = r^{-\mu} \chi_{|\eta|} \psi_{|\eta|}(r, r') \mathrm{op}_r(p)(\eta) \tilde{\chi}_{|\eta|} + r^{-\mu} \chi_{|\eta|} (1 - \psi_{|\eta|}(r, r')) \mathrm{op}_r(p)(\eta) \tilde{\chi}_{|\eta|} \tag{55}$$

with the function $\psi(r, r')$ of Example 1.14, and $\psi_{|\eta|}(r, r') := \psi(r|\eta|, r'|\eta|)$. Observe that we have

$$\psi(r, r') \varphi(r) \in \mathcal{S}(\mathbb{R} \times \mathbb{R}) \tag{56}$$

for every $\varphi \in \mathcal{S}(\mathbb{R})$. The first summand on the right of (55) is a Schwartz function in r, r' with values in $L^{-\infty}(X)$. In fact, $\chi_{|\eta|}(r) \tilde{p}(\tilde{\varrho}, r\eta)$ belongs to $\mathcal{S}(\mathbb{R}_r, L^{-\infty}\ (X; \mathbb{R}_{\tilde{\varrho}}))$ for every fixed $\eta \neq 0$; then applying a relation of the kind (56) gives us this property. For the second summand on the right of (55) we can proceed in a similar manner as in the proof of Theorem 2.7.

We obtain altogether that the kernel of $g_{(\eta)}(\eta)$ belongs to $\mathcal{S}_{\mathcal{O}}(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}_{\mathcal{O}}(X^\wedge)$ for every $\eta \neq 0$. Thus

$$f(\eta) := r^{-\mu} \chi_{[\eta]} \mathrm{op}_r(p)(\eta) \tilde{\chi}_{[\eta]} \tag{57}$$

is a Green symbol, since $f(\eta), f^*(\eta)$ have the property (46) (with f instead of g) and $f(\eta) = g_{(\mu)}(\eta)$ for large $|\eta|$. Applying Remark 1.12 to $\varphi := |\eta|$, $\tilde{\varphi} := \tilde{\sigma}$ we obtain the (54) itself has the desired property.

In the case of an r-dependence of $\tilde{p}(r, \tilde{\varrho}, \tilde{\eta})$ we assume that $\tilde{p}(r, \tilde{\varrho}, \tilde{\eta})$ is independent of r for large r; otherwise we subtract $\tilde{p}(\infty, \tilde{\varrho}, \tilde{\eta})$ which is constant in r and can be treated as before and then consider the difference $\tilde{p}(r, \tilde{\varrho}, \tilde{\eta}) - \tilde{p}(\infty, \tilde{\varrho}, \tilde{\eta})$. Then we can write

$$\tilde{p}(r, \tilde{\varrho}, \tilde{\eta}) = \sum_{j=0}^{\infty} \lambda_j \varphi_j(r) \tilde{p}_j(\tilde{\varrho}, \tilde{\eta})$$

with $\lambda_j \in \mathbb{C}$, $\sum |\lambda_j| < \infty$, $\varphi_j \in C_0^\infty(\overline{\mathbb{R}}_+)$, $\tilde{p}_j(\tilde{\varrho}, \tilde{\eta}) \in L^{-\infty}(X; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}})$ tending to 0 in the respective spaces as $j \to \infty$. This easily reduces the general case to r-indeplendent $\tilde{p}$ when we apply other standard arguments, in particular, that $r^{-\mu}\chi_{[\eta]}\mathrm{op}_r(p_j)(\eta)\tilde{\chi}_{[\eta]}$ tends to zero for $j \to \infty$ in the space of Green operators of the kind discussed before. $\square$

2.3 Edge symbols as parameter-dependent cone operator families

By an edge amplitude function of order μ, referring to the weight data $\boldsymbol{g} = (\gamma, \gamma - \mu)$, we understand an operator family of the form

$$a(y,\eta) = r^{-\mu}\sigma\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\tilde{\omega}_{[\eta]} + \chi_{[\eta]}\mathrm{op}_r(p)(y,\eta)\tilde{\chi}_{[\eta]}\}\tilde{\sigma} + m(y,\eta) + g(y,\eta),$$

where the first expression on the right hand side is as in (42), moreover, $m(y,\eta)$ is a smoothing Mellin edge symbol and $g(y,\eta)$ a Green symbol as in Example 1.11 (for $j_- = j_+ = 0$).

Let $\mathcal{R}^\mu(\Omega \times \mathbb{R}^q; \boldsymbol{g})$ denote the space of all those $a(y,\eta)$. Let us set

$$\sigma(a) := (\sigma_\psi(a), \sigma_\wedge(a)), \tag{58}$$

where $\sigma_\psi(a)$ is defined as in Remark 2.5; we take into account that $\sigma_\psi(m+g) = 0$. Moreover, we set

$$\begin{aligned}\sigma_\wedge(a)(y,\eta) := r^{-\mu}\{&\omega_{|\eta|}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(y,\eta)\tilde{\omega}_{|\eta|} \\ &+ \chi_{|\eta|}\mathrm{op}_r(p_0)(y,\eta)\tilde{\chi}_{|\eta|}\} + \sigma_\wedge(m+g)(y,\eta)\end{aligned}$$

with the notation (39) and $\sigma_\wedge(m+g)(y,\eta)$ as the homogeneous principal symbol of $(m+g)(y,\eta)$ as a classical symbol, cf. also (38) for $\mu = \nu$.

Theorem 2.9 *Let $a(y,\eta) \in \mathcal{R}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ and assume that $\sigma(a) = 0$. Then we have*

$$a(y,\eta) \in S^{\mu-1}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)),$$

for every $s \in \mathbb{R}$, and $a(y,\eta)$ takes values in the space of compact operators $\mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$, $s \in \mathbb{R}$.

Proof. The relation $\sigma_\psi(a) = 0$ implies that

$$\sigma(r)\tilde{\sigma}(r)p(r,y,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{\mu-1}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$$

and

$$\sigma(r)\tilde{\sigma}(r)\tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{\mu-1}_{\mathrm{cl}}(X; \mathbb{C} \times \mathbb{R}^q_{\tilde{\eta}})).$$

This yields

$$\tilde{p}(0,y,\tilde{\varrho},\tilde{\eta}) \in C^\infty(\Omega, L^{\mu-1}_{\mathrm{cl}}(X; \mathbb{R}^{1+q})),\ \tilde{h}(0,y,z,\tilde{\eta}) \in C^\infty(\Omega, L^{\mu-1}_{\mathrm{cl}}(X; \mathbb{C} \times \mathbb{R}^q)).$$

Now $\sigma_\wedge(a)(y,\eta) = 0$ implies that

$$r^{-\mu}\{\omega_{|\eta|}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(y,\eta)\tilde{\omega}_{|\eta|} + \chi_{|\eta|}\mathrm{op}_r(p_0)(y,\eta)\tilde{\chi}_{|\eta|}\} = -\sigma_\wedge(m+g)(y,\eta). \tag{59}$$

Both sides of the latter relation are operators in the cone algebra on $X^\wedge$. That means their subordinate symbols (interior, conormal, exit) symbols coincide. In particular, we have

$$\sigma_{\mathrm{c}}(r^{-\mu}\omega_{|\eta|}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(y,\eta)\tilde{\omega}_{|\eta|})(z) = \tilde{h}_0(0,y,z,0) = -\sigma_{\mathrm{c}}\sigma_\wedge(m(y,\eta))(z).$$

In other words, setting $h_{00}(y,z) := \tilde{f}(0,y,z,0)$, we have

$$h_{00}(y,z) = -f(y,z) \tag{60}$$

(when $m(y,z)$ is given as $r^{-\mu}\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(y)\tilde{\omega}_{[\eta]}$ for an $f \in C^\infty(\Omega, \mathcal{M}^{-\infty}(X; \Gamma_{\frac{n+1}{2}-\gamma})_\varepsilon)$ which we assume without loss of generality). In addition we may assume that $\sigma \equiv 1$, $\tilde{\sigma} \equiv 1$ on $\operatorname{supp}\omega_{[\eta]}$ and $\operatorname{supp}\tilde{\omega}_{[\eta]}$. It follows that

$$\delta(\eta)\{r^{-\mu}\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0 - h_{00})y,\eta)\tilde{\omega}_{[\eta]} + \chi_{[\eta]}\mathrm{op}_r(p_0)(y,\eta)\tilde{\chi}_{[\eta]}\} + g(y,\eta)\} \tag{61}$$

is a Green symbol of order $\mu - 1$ for every excision function $\delta(\eta)$ in $\mathbb{R}^q$. In fact, the relation (59) shows that

$$r^{-\mu}\{\omega_{|\eta|}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0 - h_{00})\tilde{\omega}_{|\eta|} + \chi_{|\eta|}\mathrm{op}_r(p_0)(y,\eta)\tilde{\chi}_{|\eta|}\} = -\sigma_\wedge(g)(y,\eta).$$

This entails the identity (61) for all η with $|\eta| \geq$ const for a constant > 0. On the other hand, using the technique of proving Theorem 2.9 we see that the left hand side of (61) is an operator-valued symbol, even classical in this situation, with $-\sigma_\wedge(g)(y,\eta)$ as the homogeneous principal symbol. Then the same is true of

$$\begin{aligned} g_0(y,\eta) \quad &=: \quad r^{-\mu}\sigma\delta(\eta)\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0 - h_{00})(y,\eta)\tilde{\omega}_{[\eta]} + \chi_{[\eta]}\mathrm{op}_r(p_0)(y,\eta)\tilde{\chi}_{[\eta]}\}\tilde{\sigma} \\ &+ \quad \delta(\eta)g(y,\eta), \end{aligned}$$

cf. Remark 1.12.

Now the symbol $a(y,\eta)$ can be written in the form

$$\begin{aligned} &r^{-\mu}\sigma\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h - h_0)(y,\eta)\tilde{\omega}_{[\eta]} + \chi_{[\eta]}\mathrm{op}_r(p - p_0)(y,\eta)\tilde{\chi}_{[\eta]}\}\tilde{\sigma} \\ &+ \delta(\eta)\{r^{-\mu}\sigma\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0 - h_{00})(y,\eta)\tilde{\omega}_{[\eta]} + \chi_{[\eta]}\mathrm{op}_r(p_0)(y,\eta)\tilde{\chi}_{[\eta]}\}\tilde{\sigma}\} \\ &+ g(y,\eta) + (1 - \delta(\eta))\{r^{-\mu}\sigma\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0 - h_{00})(y,\eta)\tilde{\omega}_{[\eta]} \\ &+ \chi_{[\eta]}\mathrm{op}_r(p_0)(y,\eta)\tilde{\chi}_{[\eta]}\}\tilde{\sigma} + g(y,\eta)\} \\ = {} &r^{-\mu}\sigma\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h - h_0)(y,\eta)\tilde{\omega}_{[\eta]} + \chi_{[\eta]}\mathrm{op}_r(p - p_0)(y,\eta)\tilde{\chi}_{[\eta]}\}\tilde{\sigma} \\ &+ g_0(y,\eta) + (1 - \delta(\eta))\{r^{-\mu}\sigma\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-n/2}(h_0 - h_{00})(y,\eta) \\ &+ \chi_{[\eta]}\mathrm{op}_r(p_0)(y,\eta)\tilde{\chi}_{[\eta]}\}\tilde{\sigma}\} \end{aligned} \tag{62}$$

The summand $g_0(y,\eta)$ is a Green symbol of order $\mu - 1$ and takes values in compact operators. Also

$$r^{-\mu}\sigma\{\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h - h_0)(y,\eta)\tilde{\omega}_{[\eta]}\}\tilde{\sigma} \tag{63}$$

is compact for every (y,η), since we can write $h - h_0 = r\tilde{h}_{(1)}(r,y,z,r\eta)$ for an $\tilde{h}_{(1)}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{\mu-1}_{\mathrm{cl}}(X; \mathbb{C} \times \mathbb{R}^\eta_\xi))$ which yields an improvement of the weight at $r=0$, together with the improvement of the order. For a similar reason also

$$r^{-\mu}\sigma\{\chi_{[\eta]}\mathrm{op}_r(p-p_0)(y,\eta)\tilde{\chi}_{[\eta]}\}\tilde{\sigma} \tag{64}$$

is compact for every (y,η). Finally, (63) and (64) are operator-valued symbols of order $\mu-1$, by similar arguments as for the proof of Theorem 2.6, and the last summand on the right of (62) takes values in compact operators and is of order $-\infty$ in η. $\square$

3 Composition properties

3.1 Composition of edge symbols

In this section we analyse the composition properties of edge amplitude functions of the form (42). It will be more convenient here to employ the cut-off functions $\omega(r), \widetilde{\omega}(r), \widetilde{\widetilde{\omega}}(r)$ rather than the excision functions (40).

Moreover, since the presence of the variables $y\in\Omega$ only causes minor modifications of the arguments, we content ourselves with the y-independent case. Let

$$\begin{aligned} a(\eta) &:= r^{-\mu}\sigma\{a_M(\eta)+a_\psi(\eta)\}\widetilde{\sigma}, \\ b(\eta) &:= r^{-\nu}\sigma\{b_M(\eta)+b_\psi(\eta)\}\widetilde{\sigma}, \end{aligned}$$

where $\sigma(r)$ and $\widetilde{\sigma}(r)$ are arbitrary cut-off functions, and

$$\begin{aligned} a_M(\eta) &:= \omega_{[\eta]}\mathrm{op}_M^{\gamma-\nu-\frac{n}{2}}(h_1)(\eta)\widetilde{\omega}_{[\eta]}, & a_\psi(\eta) &:= (1-\omega_{[\eta]})\mathrm{op}_r(p_1)(\eta)(1-\widetilde{\widetilde{\omega}}_{[\eta]}), \\ b_M(\eta) &:= \omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)(\eta)\widetilde{\omega}_{[\eta]}, & b_\psi(\eta) &:= (1-\omega_{[\eta]})\mathrm{op}_r(p_2)(\eta)(1-\widetilde{\widetilde{\omega}}_{[\eta]}). \end{aligned}$$

Here $p_j(r,\rho,\eta)=\widetilde{p}_i(r,r\rho,r\eta)$ for families $\widetilde{p}_i(r,\widetilde{\rho},\widetilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+, L_{\mathrm{cl}}^{\mu_i}(X;\mathbb{R}^{1+q}_{\widetilde{\rho},\widetilde{\eta}}))$ with $\mu_1=\mu$, $\mu_2=\nu$ and $h_i(r,z,\eta)=\widetilde{h}_i(r,z,r\eta)$ for elements $\widetilde{h}_i(r,z,\widetilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+, L_{\mathrm{cl}}^{\mu_i}(X;\mathbb{C}\times\mathbb{R}^q))$, $i=1,2$, where we assume that h_i is the Mellin quantisation of p_i in the sense of Theorem 2.4.

Theorem 3.1 *We have* (*with respect to the pointwise composition of operator functions*)

$$(ab)(\eta)=\sigma r^{-(\mu+\nu)}\{c_M(\eta)+c_\psi(\eta)\}\widetilde{\sigma}+g(\eta)$$

where

$$c_M(\eta)=\omega_{[\eta]}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\widetilde{\omega}_{[\eta]}+(1-\omega_{[\eta]})\mathrm{op}_r(p)(\eta)(1-\widetilde{\widetilde{\omega}}_{[\eta]})$$

for operator functions $p(r,\rho,\eta)$ and $h(r,z,\eta)$ which are of the same nature as those in Theorem 2.4, now of order $\mu+\nu$, and $g(\eta)$ is a flat Green symbol of order $\mu+\nu$. We have

$$\sigma_\psi(ab)=\sigma_\psi(a)\sigma_\psi(b), \quad \sigma_\wedge(a)(\eta)\sigma_\wedge(b)(\eta)=\sigma_\wedge(ab)(\eta). \tag{65}$$

Proof. For the proof we employ the abbreviations $a=a(\eta)$ for

$$a=\omega a_0\widetilde{\omega}+(1-\omega)a_1(1-\widetilde{\widetilde{\omega}}),$$

where the cut-off functions $\omega, \widetilde{\omega}, \widetilde{\widetilde{\omega}}$ are have the meanning $\omega = \omega_{[\eta]}$, etc., and

$$a_0 := \sigma r^{-\mu} \mathrm{op}_M^{\gamma-\nu-\frac{n}{2}}(h_1)\widetilde{\sigma}, a_1 := \sigma r^{-\mu} \mathrm{op}_r(p_1)\widetilde{\sigma},$$

and, similarly,

$$b = \omega b_0 \widetilde{\omega} + (1-\omega) b_1 (1-\widetilde{\widetilde{\omega}})$$

for

$$b_0 := \sigma r^{-\nu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\widetilde{\sigma}, b_1 := \sigma r^{-\mu} \mathrm{op}_r(p_2)\widetilde{\sigma}.$$

In the following computations we systematically employ the properties $\widetilde{\widetilde{\omega}} \prec \omega \prec \omega$ which implies $\omega\widetilde{\omega} = \omega$, $\widetilde{\widetilde{\omega}}\omega = \widetilde{\widetilde{\omega}}$. We then obtain $ab = P + \sum_{k=1}^{6} G_k$ after elementary rearrangements of summands in the composition ab for

$$P := \omega a_0 \widetilde{\omega} b_0 \widetilde{\omega} + (1-\omega) a_1 (1-\widetilde{\widetilde{\omega}}) b_1 (1-\widetilde{\widetilde{\omega}})$$

and

$$\begin{aligned}
G_1 &:= \widetilde{\widetilde{\omega}} a_0 (\widetilde{\omega}-\omega) b_1 (1-\widetilde{\widetilde{\omega}}) + (\omega-\widetilde{\widetilde{\omega}}) a_0 (\widetilde{\omega}-\omega) b_1 (1-\widetilde{\omega}),\\
G_2 &:= (1-\widetilde{\omega}) a_1 (\omega-\widetilde{\widetilde{\omega}}) b_0 \widetilde{\omega} + (\widetilde{\omega}-\omega) a_1 (\omega-\widetilde{\widetilde{\omega}}) b_0 \widetilde{\widetilde{\omega}},\\
G_3 &:= (\omega-\widetilde{\widetilde{\omega}}) a_0 (\omega-\widetilde{\omega}) b_0 \widetilde{\widetilde{\omega}} + \widetilde{\widetilde{\omega}} a_0 (\omega-\widetilde{\omega}) b_0 \widetilde{\omega},\\
G_4 &:= (1-\widetilde{\omega}) a_1 (\widetilde{\widetilde{\omega}}-\omega) b_1 (1-\widetilde{\widetilde{\omega}}) + (\widetilde{\omega}-\omega) a_1 (\widetilde{\widetilde{\omega}}-\omega) b_1 (1-\widetilde{\omega}),\\
G_5 &:= (\widetilde{\omega}-\omega) a_0 (\widetilde{\widetilde{\omega}}-\omega)(b_1-b_0)(\widetilde{\omega}-\widetilde{\widetilde{\omega}}),\\
G_6 &:= (\omega-\widetilde{\widetilde{\omega}})(a_1-a_0)(\widetilde{\omega}-\omega) b_0 (\widetilde{\omega}-\widetilde{\widetilde{\omega}}).
\end{aligned}$$

We now write

$$P = \omega a_0 b_0 \widetilde{\omega} + (1-\omega) c_1 (1-\widetilde{\widetilde{\omega}}) + G_7 + G_8$$

for

$$G_7 := \omega a_0 (\widetilde{\omega}-1) b_0 \widetilde{\omega}, \quad G_8 := (1-\omega)\{a_1 (1-\widetilde{\widetilde{\omega}}) b_1 - c_1\}(1-\widetilde{\widetilde{\omega}}),$$

and

$$c_1(\eta) = \sigma r^{-(\mu+\nu)} \mathrm{op}_r(p)(\eta)\widetilde{\sigma} \tag{66}$$

for an operator function $p(r,\rho,\eta) = \widetilde{p}(r, r\rho, r\eta)$ which is defined by an element

$$\widetilde{p}(r, \widetilde{\rho}, \widetilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, L_{\mathrm{cl}}^{\mu+\nu}(X; \mathbb{R}^{1+q}_{\widetilde{\rho},\widetilde{\eta}})), \tag{67}$$

obtained by computing the Leibniz product $\#_r$ with respect to r,

$$r^{-(\mu+\nu)}\widetilde{p}(r, r\rho, r\eta) = r^{-\mu}\widetilde{p}_1(r, r\rho, r\eta)\sharp_r \widetilde{\sigma}(r)\sigma(r) r^{-\nu}\widetilde{p}_2(r, r\rho, r\eta).$$

In particular, $\widetilde{p}(r, \widetilde{\rho}, \widetilde{\eta})$ can be chosen to be smooth up to $r = 0$. Moreover, we have

$$\begin{aligned}
a_0 b_0 &= \sigma r^{-\mu} \mathrm{op}_M^{\gamma-\nu-\frac{n}{2}}(h_1)(\eta)\widetilde{\sigma}\sigma r^{-\nu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)(\eta)\widetilde{\sigma}\\
&= \sigma r^{-(\mu+\nu)} \mathrm{op}_M^{\gamma-\frac{n}{2}}(T^\nu h_1)(h) \mathrm{op}_M^{\gamma-\frac{n}{2}}(\sigma\widetilde{\sigma} h_2)(\eta)\widetilde{\sigma}.
\end{aligned}$$

We therefore obtain a Mellin symbol $h(r,z,\eta) = \widetilde{h}(r,z,r\eta)$ for an $\widetilde{h}(r,w,\widetilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, L_{\mathrm{cl}}^{\mu+\nu}(X; \mathbb{C}\times\mathbb{R}^q))$ such that

$$a_0 b_0 = \sigma r^{-(\mu+\nu)} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\widetilde{\sigma}. \tag{68}$$

Summing up it follows that

$$ab = \omega c_0 \widetilde{\omega} + (1-\omega)c_1(1-\widetilde{\widetilde{\omega}}) + g,$$

where $c_0(\eta)$ and $c_1(\eta)$ are given by (65) and (66), respectively, and

$$g = \sum_{j=1}^{8} G_j. \tag{69}$$

It turns out that (69) is a flat Green symbol. This will be verified in the following section.

The relation (65) follows from the fact that we can apply the above computations for the compositions to corresponding operator functions, where $[\eta]$ is replaced by $|\eta|$, $\eta \neq 0$, and all the first r-variables are frozen at zero.

□

3.2 Characterisation of remainders

In order to characterise (69) as a flat Green symbol we consider the summands separately. Let us write $G_1 := G_1' + G_1''$ for

$$G_1' := \widetilde{\widetilde{\omega}} a_0(\widetilde{\omega} - \omega) b_1 (1-\widetilde{\widetilde{\omega}}), G_1'' := (\omega - \widetilde{\widetilde{\omega}}) a_0 (\widetilde{\omega} - \omega) b_1 (1 - \widetilde{\omega}).$$

We have

$$G_1' = CD \quad \text{for} \quad C := \widetilde{\widetilde{\omega}} a_0 (\widetilde{\omega} - \omega), D := \varphi b_1 (1-\widetilde{\omega})$$

for any $\varphi \in C_0^\infty(\mathbb{R}_+)$ that is equal to 1 on $\operatorname{supp}(\widetilde{\omega} - \omega)$. The function φ is interpreted (similarly as the cut-off functions) as an η-dependent factor, i.e., $\varphi = \varphi_\eta$ for $\varphi_\eta(r) := \varphi(r[\eta])$. The family of operators C is a flat Green symbol, see Example 1.11 and Remark 2.3. It is then easy to verify that the composition with D gives again a Green symbol.

For G_1'' we choose another cut-off function ω_0 that is equal to 1 on $\operatorname{supp}\omega$ and such that $\widetilde{\omega}$ is equal to 1 on $\operatorname{supp}\omega_0$ (this is always possible). Then we can write $G_1'' = H + L$ for

$$\begin{aligned} H &:= (\omega - \widetilde{\widetilde{\omega}}) a_0 (1-\omega_0)(\widetilde{\omega} - \omega) b_1 (1-\widetilde{\omega}), \\ L &:= (\omega - \widetilde{\widetilde{\omega}}) a_0 \omega_0 (\widetilde{\omega} - \omega) b_1 (1 - \widetilde{\omega}). \end{aligned}$$

For any $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi \equiv 1$ on $\operatorname{supp}(\widetilde{\omega} - \omega)$ we have $H = CD$ for

$$C := (\omega - \widetilde{\widetilde{\omega}}) a_0 (1-\omega_0)\varphi, D := (\widetilde{\omega} - \omega) b_1 (1-\widetilde{\omega}),$$

(in this proof, C and D occur in different meaning which will be clear by the context). Since $(1-\omega)$ vanishes on $\operatorname{supp}(\omega - \widetilde{\omega})$, the factor C is smoothing and can be treated in a similar manner as the corresponding factor occurring in G_1'. As above it is agin easy to verify that then also H is a flat Green symbol. Moreover, taking some $\varphi \in C_0^\infty(\mathbb{R}_+)$, $\varphi \equiv 1$ on $\operatorname{supp}(\widetilde{\omega} - \omega)$, we can write $L = DC$ for

$$D := (\omega - \widetilde{\widetilde{\omega}}) a_0 (\widetilde{\omega} - \omega),\ C := \omega_0 \varphi b_1 (1-\widetilde{\omega}).$$

Since $1 - \widetilde{\omega}$ vanishes on $\operatorname{supp}\omega_0$, the family C is smoothing and a flat Green symbol. This implies the same for L.

The argument for G_k, $1 < k \leq 8$, are similar and left to the reader.

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