Edge Symbolic Structures of Second Generation

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Abstract

Operators on a manifold with (geometric) singularities are degenerate in a natural way. They have a principal symbolic structure with contributions from the different strata of the configuration. We study the calculus of such operators on the level of edge symbols of second generation, based on specific quantizations of the corner-degenerate interior symbols, and show that this structure is preserved under compositions.

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Contents

Introduction

This paper is aimed at studying the symbolic structure of operators on a configuration with singularities of second order. By that we understand a geometry that is locally a wedge, i.e., a Cartesian product between a cone and an edge, where the base of the cone

is a manifold W with smooth edge $Y \subset W$. In particular, if $\Xi \subseteq \mathbb{R}^p$ is an open set and $W^\Delta := (\overline{\mathbb{R}}_+ \times W)/(\{0\} \times W)$ the (infinite) cone with base W, then

$$W^\Delta \times \Xi \tag{1}$$

has singularities of second order. The space W itself is, locally near Y, modelled on a wedge $X^\Delta \times \Omega$ for a C^∞ manifold X and an open set $\Omega \subseteq \mathbb{R}^q$, $q = \dim Y$. Then (1) is locally near $Y^\Delta \times \Xi$ of the form

$$(X^\Delta \times \Omega)^\Delta \times \Xi.$$

Thus (1) can be regarded as a manifold with edge $Y^\Delta \times \Xi$, and $Y^\Delta \times \Xi$ itself is a manifold with smooth edge Ξ. It will often be convenient to talk about corresponding open stretched spaces, e.g., $X^\wedge := \mathbb{R}_+ \times X \ni (r, x)$, or, more generally, $(X^\wedge \times \Omega)^\wedge \times \Xi = \mathbb{R}_+ \times \mathbb{R}_+ \times X \times \Omega \times \Xi$ in the splitting of variables (t, r, x, y, z).

If g_X is a Riemannian metric on X, then $dr^2 + r^2 g_X + dy^2$ is an example of a wedge metric on $X^\wedge \times \Omega$. The associated Laplace-Beltrami operator can be written as

$$A := r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y)(-r\partial_r)^j (rD_y)^\alpha \tag{2}$$

(for $\mu = 2$) with coefficients

$$a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X));$$

here $\mathrm{Diff}^\nu(\cdot)$ denotes the space of differential operators of order ν on the C^∞ manifold in the brackets. (Clearly, in the present case we have simply $A = r^{-2}((r\partial_r)^2 + (\dim X - 1)r\partial_r + \Delta_X)$ with Δ_X being the Laplace-Beltrami operator on X belonging to g_X). Let $\mathrm{Diff}^\mu_{\mathrm{deg},1}(X^\wedge \times \Omega)$ denote the space of all differential operators of order μ on $X^\wedge \times \Omega$ of the form (2), equipped with the structure of a Fréchet space (for convenience, all manifolds in consideration are assumed to be countable unions of compact sets). Then, looking at $(X^\wedge \times \Omega)^\wedge \times \Xi$, $\Xi \subseteq \mathbb{R}^p$ open, with a wedge metric of the kind

$$dt^2 + t^2(dr^2 + r^2 g_X + dy^2) + dz^2, \tag{3}$$

the associated Laplace-Beltrami operator has the form

$$\boldsymbol{A} := t^{-\mu} \sum_{k+|\beta| \leq \mu} b_{k\beta}(t, z)(-t\partial_t)^k (tD_z)^\beta \tag{4}$$

(again for $\mu = 2$) with coefficients $b_{k\beta}(t, z) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \mathrm{Diff}^{\mu-(k+|\beta|)}_{\mathrm{deg},1}(X^\wedge \times \Omega))$. Let $\mathrm{Diff}^\mu_{\mathrm{deg},2}((X^\wedge \times \Omega)^\wedge \times \Xi)$ denote the space of all such differential operators (4) on $(X^\wedge \times \Omega)^\wedge \times \Xi$ of order μ.

According to (2), the operator (4) can be written as

$$\boldsymbol{A} = r^{-\mu} t^{-\mu} \sum_{j+|\alpha|+k+|\beta| \leq \mu} c_{j\alpha,k\beta}(r, y, t, z)(-r\partial_r)^j (rD_y)^\alpha (-rt\partial_t)^k (rtD_z)^\beta \tag{5}$$

with coefficients $c_{j\alpha,k\beta} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega \times \overline{\mathbb{R}}_+ \times \Xi, \mathrm{Diff}^{\mu-(j+|\alpha|+k+|\beta|)}(X))$. This shows the nature of degeneracy of geometric differential operators on the open stretched manifold $(X^\wedge \times \Omega)^\wedge \times \Xi$ close to $t = 0$ and $r = 0$. Observe that (5) gives us a natural inclusion

$$\mathrm{Diff}^\mu_{\mathrm{deg},2}((X^\wedge \times \Omega)^\wedge \times \Xi) \subset \mathrm{Diff}^\mu_{\mathrm{deg},1}(X^\wedge \times (\mathbb{R}_+ \times \Omega \times \Xi)) \qquad (6)$$

when we reinterpret $(t, y, z) \in \mathbb{R}_+ \times \Omega \times \Xi$ as edge variables of the corresponding stretched wedge $X^\wedge \times (\mathbb{R}_+ \times \Omega \times \Xi)$.

In this paper we develop some crucial elements of the pseudo-differential calculus for symbols with 'corner-degenerate' behaviour of that kind. The main focus is the structure of amplitude functions which take values in operators on the infinite stretched wedge $W^\wedge$. The definition, given in Section 2.1, represents a specific quantization of corner-degenerate symbols. In Section 2.2 we study the behaviour under compositions. The rest of the paper is devoted to necessary details on the nature of the Mellin quantization in the corner-degenerate situation, cf. Section 3.1, and on Green remainders that appear in compositions, cf. Section 3.2.

Let us recall that the adequate way of establishing operator conventions (quantizations) of corner-degenerate interior symbols is far from being evident. This is already the case for the edge calculus of first generation which corresponds to operators on configurations with smooth edges (of any codimension), cf. [14] and [8]. The edge-quantization of [8] is an alternative compared with that of [14], in order to make the composition behaviour more transparent. In higher calculi, an analogue of that idea is more difficult than the original one. Therefore, in the present set-up we do not try to apply the method of [8]. However, for the higher singular case it seems to be indispensable to change the quantization in another way, namely, to localise symbols in corner axis direction $t \in \mathbb{R}_+$ far from the origin close to the diagonal of $\mathbb{R}_+ \times \mathbb{R}_+$. This is caused by the very complex behaviour of edge-degenerate operators when the edge has a conical exit to infinity, here $t \to \infty$, which is the case when the model cone itself has singularities, cf. [4].

Similarly as the edge algebra in [14] (cf. also [16] or [5]) our results will be necessary to establish an operator algebra with extra edge conditions that is closed under the construction of parametrices, cf., analogously, Boutet Monvel's algebra [2]. In the present case the conditions are to be posed both on edges of first and second generation; this will be done in a future paper. Let us finally give a number of references that have from different point of view connections with this paper, namely, Agranovich and Vishik [1] (parameter-dependent calculus), Eskin [6], Rempel and Schulze [13], Grubb [9] (pseudo-differential calculus of boundary value problems), Witt [19] (structure of operator-valued Mellin symbols), Seiler [18] (Green operators in the cone algebra), Schulze [17] (cone calculus when the base has smooth edges), and joint works of the second author with Kapanadze [12] and Harutjunjan [10] (various models of applications, especially, crack theory, and higher corner Mellin symbols).

1 Operators on a manifold with edge

1.1 Edge symbols for differential operators

By a manifold W with edge Y of first singularity order we understand a quotient space $W = \mathbb{W}/ \sim$, where $\mathbb{W}$ is a C^∞ manifold with boundary, $\partial\mathbb{W}$ is an X-bundle over Y,

and X and Y are C^∞ manifolds. The equivalence relation '$\sim$' means $w_1 = w_2$ when $w_1, w_2 \in \mathbb{W}_{\text{reg}} := \mathbb{W} \setminus \partial\mathbb{W}$ and $\pi w_1 = \pi w_2$ when $w_1, w_2 \in \mathbb{W}_{\text{sing}} := \partial\mathbb{W}$, where $\pi : \mathbb{W}_{\text{sing}} \to Y$ denotes the canonical projection. From this definition it follows another equivalent characterisation of a manifold W with (smooth) edge Y, namely, as a topological space W such that $W \setminus Y$ and Y are C^∞ manifolds, and every $y \in Y$ has a neighbourhood V such that there is a homeomorphism $\chi : V \to X^\triangle \times \Omega$ for some open set $\Omega \subseteq \mathbb{R}^q$ which is induced by a diffeomorphism $\pi^{-1}V =: \mathbb{V} \to \overline{\mathbb{R}}_+ \times X \times \Omega$ by passing to the corresponding quotient map. There is a natural notion of morphisms and isomorphisms in the corresponding category $\mathfrak{M}_1$ of such manifolds with edges, cf. [3]. This allows us to form another category $\mathfrak{M}_2$ of manifolds K with edge Z of second singularity order. Here Z is a C^∞ manifold, moreover, $K \setminus Z \in \mathfrak{M}_1$, and K is locally near a point $z \in Z$ modelled on a wedge (1) for an open subset $\Xi \subseteq \mathbb{R}^p$ (more precisely, every $z \in \Xi$ has a neighbourhood in K that is homeomorphic to such a wedge). Concerning the transition maps

$$W^\triangle \times \Xi \to W^\triangle \times \widetilde{\Xi} \tag{7}$$

for different such local models for open $\Xi, \widetilde{\Xi} \subseteq \mathbb{R}^p$, we require that there is an isomorphism $\mathbb{R} \times W \times \Xi \to \mathbb{R} \times W \times \widetilde{\Xi}$ in $\mathfrak{M}_1$ that restricts to isomorphisms

$$\mathbb{R}_+ \times W \times \Xi \to \mathbb{R}_+ \times W \times \widetilde{\Xi} \tag{8}$$

and $\{0\} \times W \times \Xi \to \{0\} \times W \times \widetilde{\Xi}$ in $\mathfrak{M}_1$ such that (7) is the quotient map with respect to $\overline{\mathbb{R}}_+ \times W \times \Xi \to W^\triangle \times \Xi$ and $\overline{\mathbb{R}}_+ \times W \times \widetilde{\Xi} \to W^\triangle \times \widetilde{\Xi}$, respectively.

On K we choose splittings of variables

$$(t, \cdot, z) \in \mathbb{R}_+ \times W \times \Xi \tag{9}$$

for every local wedge (1). Let $W \in \mathfrak{M}_1$ be a manifold with edge Y, and let $\text{Diff}^\mu_{\text{deg},1}(W)$ denote the space of all $A \in \text{Diff}^\mu(W \setminus Y)$ that are locally near Y in the splitting of variables $(r, \cdot, y) \in X^\wedge \times \Omega$ of the form (2). Observe that when $\chi : W \to \tilde{W}$ is an isomorphism between $W, \tilde{W} \in \mathfrak{M}_1$, then the operator push forward $\chi_* : \text{Diff}^\mu_{\text{deg},1}(W) \to \text{Diff}^\mu_{\text{deg},1}(W)$ is an isomorphism.

Let $K \in \mathfrak{M}_2$ be a manifold with edge Z, and let $\text{Diff}^\mu_{\text{deg},2}(K)$ denote the subspace of all $A \in \text{Diff}^\mu_{\text{deg},1}(K \setminus Z)$ that have near Z in the local splitting of variables (9) the form (4) with coefficients $b_{k\beta}(t, z) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \text{Diff}^{\mu-(k+|\beta|)}_{\text{deg},1}(W))$. It can easily be proved that this definition makes sense, i.e., is invariant under the transition maps (8) (which correspond to the local description of isomorphisms in $\mathfrak{M}_2$). In the following we are mainly interested in the behaviour of operators in local wedges $W^\triangle \times \Xi$ (or corresponding stretched wedges $W^\wedge \times \Xi$) rather than globally on K.

The principal symbolic structure of operators $\boldsymbol{A} \in \text{Diff}^\mu_{\text{deg},2}(K)$ for $K := W^\triangle \times \Xi$ is defined as a triple

$$\sigma(\boldsymbol{A}) := (\sigma_\psi(\boldsymbol{A}), \sigma_\wedge(\boldsymbol{A}), \sigma_{\boldsymbol{\wedge}}(\boldsymbol{A})).$$

Here $\sigma_\psi(\boldsymbol{A})$ is the 'usual' homogeneous principal symbol of order μ, given on the smooth part $\mathbb{R}_+ \times (W \setminus Y) \times \Xi$ of K.

Locally near Y in the variables $(t, r, x, y, z) \in \mathbb{R}_+ \times (\mathbb{R}_+ \times \Sigma \times \Omega) \times \Xi$ (with $x \in \Sigma$ and $y \in \Omega$ being local coordinates on X and Y, respectively, $\Sigma \subseteq \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^q$ open) $\sigma_\psi(\boldsymbol{A})$ is the homogeneous principal part of a symbol of form

$$\boldsymbol{p}\,(t, r, x, y, z, \tau, \rho, \xi, \eta, \zeta) = t^{-\mu} r^{-\mu} \widetilde{\boldsymbol{p}}\,(t, r, x, y, z, tr\tau, r\rho, \xi, r\eta, tr\zeta)$$

for an element

$$\widetilde{\boldsymbol{p}}\left(t, r, x, y, z, \widetilde{\boldsymbol{\tau}}, \widetilde{\rho}, \xi, \widetilde{\eta}, \widetilde{\boldsymbol{\zeta}}\right) \in S_{\mathrm{cl}}^{\mu}\big(\overline{\mathbb{R}}_{+} \times \overline{\mathbb{R}}_{+} \times \Sigma \times \Omega \times \Xi \times \mathbb{R}_{\widetilde{\boldsymbol{\tau}}, \widetilde{\rho}, \xi, \widetilde{\eta}, \widetilde{\boldsymbol{\zeta}}}^{2+n+q+p}\big)$$

(in the present case $\widetilde{p}$ is, of course, a polynominal in $(\widetilde{\boldsymbol{\tau}}, \widetilde{\rho}, \xi, \widetilde{\eta}, \widetilde{\boldsymbol{\zeta}})$ of order μ).

Moreover, if we write $\boldsymbol{A}$ in the form (5), we have the edge symbol of first generation

$$\sigma_{\wedge}(\boldsymbol{A}) := r^{-\mu} \sum_{j+|\alpha|+k+|\beta| \leq \mu} c_{j\alpha,k\beta}(0, y, t, z)(-r\partial_r)^j(r\eta)^{\alpha}(-irt\tau)^k(rt\zeta)^{\beta}.$$

This is a family of continuous operators

$$\sigma_{\wedge}(\boldsymbol{A})(t, y, z, \tau, \eta, \zeta) : \mathcal{K}^{s,\gamma}(X^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}) \tag{10}$$

for $(t, y, z, \tau, \eta, \zeta) \in T^*(\mathbb{R}_+ \times \Omega \times \Xi) \setminus 0$, $s, \gamma \in \mathbb{R}$. (The definition of the weighted cone Sobolev spaces in (10) will be given in Section 1.2 below.) Finally, from (4) with coefficients $b_{k\beta}(t, z) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Xi, \mathrm{Diff}_{\mathrm{deg},1}^{\mu-(k+|\beta|)}(W))$ we define the edge symbol of second generation

$$\sigma_{\boldsymbol{\wedge}}(\boldsymbol{A}) := t^{-\mu} \sum_{k+|\beta| \leq \mu} b_{k\beta}(0, z)(-t\partial_t)^k(t\zeta)^{\beta},$$

which is a family of continuous operators

$$\sigma_{\boldsymbol{\wedge}}(\boldsymbol{A})(z, \zeta) : \mathcal{K}^{s,(\gamma,\theta)}(W^{\wedge}) \to \mathcal{K}^{s-\mu,(\gamma-\mu,\theta-\mu)}(W^{\wedge}) \tag{11}$$

for $(z, \zeta) \in T^*\Xi \setminus 0$, $s, \gamma, \theta \in \mathbb{R}$ (the spaces in (11) will also be defined in Section 1.2 below).

1.2 Edge spaces of second generation

In this section we formulate some necessary material on weighted cone and edge Sobolev spaces. We first recall the definition of spaces of first generation. After that we pass to the corner spaces of second generation.

Let X be a closed compact C^{∞} manifold, $n = \dim X$, and $H^s(X)$ the standard Sobolev space of smoothness $s \in \mathbb{R}$ on X. Moreover, let $L_{\mathrm{cl}}^{\mu}(X; \mathbb{R}^l)$ denote the space of all classical parameter-dependent pseudo-differential operators on X of order $\mu \in \mathbb{R}$ (that is, the local symbols $a(x, \xi, \lambda)$ contain the parameter $\lambda \in \mathbb{R}^l$, the symbolic estimates treat $(\xi, \lambda) \in \mathbb{R}^{n+l}$ as a covariable, and we set $L^{-\infty}(X; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(X))$ with $L^{-\infty}(X)$ being the space of operators with kernel in $C^{\infty}(X \times X)$). We employ the fact that for every $\mu \in \mathbb{R}$ there is an element $R^{\mu}(\lambda) \in L_{\mathrm{cl}}^{\mu}(X; \mathbb{R}^l)$ that is parameter-dependent elliptic (i.e., the homogeneous principal components $a_{(\mu)}(x, \xi, \lambda)$ of $a(x, \xi, \lambda)$ do not vanish for $(\xi, \lambda) \neq 0$) and $R^{\mu}(\lambda) : H^s(X) \to H^{s-\mu}(X)$ defines a family of isomorphisms for all $s \in \mathbb{R}$ and all $\lambda \in \mathbb{R}^l$.

For $\mu = s$ we now choose such a family $R^s(\rho)$, with parameter $\rho \in \mathbb{R}$ and denote by $\mathcal{H}^{s,\gamma}(X^{\wedge})$ for $s, \gamma \in \mathbb{R}$ the completion of $C_0^{\infty}(X^{\wedge})$ with respect to the norm $\left\{(2\pi i)^{-1} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(\mathrm{Im}\, w)(Mu)(w)\|_{L^2(X)}^2 dw\right\}^{\frac{1}{2}}$. Here $(Mu)(w) = \int_{\mathbb{R}_+} r^{w-1}u(r)dr$ is the Mellin transform, applied to $u = u(r) \in C_0^{\infty}(\mathbb{R}_+, C^{\infty}(X))$ as a $C^{\infty}(X)$-valued function,

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and $\Gamma_\beta := \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}$, $\beta \in \mathbb{R}$. Observe that the spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ are related to the (standard) cylindrical Sobolev spaces $H^s(\mathbb{R} \times X)$ by the identity

$$\mathcal{H}^{s,\gamma}(X^\wedge) = \left(S_{\gamma - \frac{n}{2}}\right)^{-1} H^s(\mathbb{R} \times X) \tag{12}$$

where

$$(S_\beta u)(p) := e^{-(\frac{1}{2} - \beta)p} u(e^{-p}), \quad p \in \mathbb{R}. \tag{13}$$

In this paper a cut-off function w on the half-axis is any $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$, $0 \le \omega(r) \le 1$, that is equal to 1 in a neighbourhood of $r = 0$. Given two cut-off functions $\omega, \widetilde{\omega}$ we write

$$\omega \prec \widetilde{\omega} \quad \text{if} \quad \widetilde{\omega} \equiv 1 \quad \text{on} \quad \operatorname{supp} \omega. \tag{14}$$

Let $H^s_{\mathrm{cone}}(X^\wedge)$ denote the subspace of all $g \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}$ such that for every coordinate neighbourhood U on X, every diffeomorphism $\chi : U \to V$ to an open set $V \subset S^n$, $\chi(x) = \alpha$, every $\varphi \in C_0^\infty(U)$ and any cut-off function $\omega(r)$ we have $\varphi(\chi^{-1}(\alpha))(1 - \omega(r))g(r, \chi^{-1}(\alpha)) \in H^s(\mathbb{R}^{n+1})$ (here (r, α) has the meaning of polar coordinates in $\mathbb{R}^{n+1} \setminus \{0\} \cong (S^n)^\wedge$).

For purposes below we set

$$X_\succeq := \mathbb{R} \times X$$

when we interpret the cylinder as a manifold with conical exits $r \to \pm\infty$. In that sense we define weighted Sobolev spaces $H^{s;\delta}(X_\succeq)$ of smoothness $s \in \mathbb{R}$ and weight $\delta \in \mathbb{R}$ at $r = \pm\infty$ by setting

$$H^{s;\delta}(X_\succeq) := \left\{ u(r,x) \in H^s_{\mathrm{loc}}(\mathbb{R} \times X) : u(r,x)|_{r>0}, u(-r,x)|_{r>0} \in \langle r \rangle^{-\delta} H^s_{\mathrm{cone}}(X^\wedge) \right\};$$

here $\langle r \rangle = (1 + r^2)^{1/2}$. Observe that $H^{0;0}(X_\succeq)$ can be identified with $\langle r \rangle^{-n/2} L^2(\mathbb{R} \times X)$ where the L^2 space is based on the measure $dr\,dx$. We now set

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \left\{ \omega f + (1 - \omega)g : f \in \mathcal{H}^{s,\gamma}(X^\wedge), \ g \in H^s_{\mathrm{cone}}(X^\wedge) \right\}$$

for every $s, \gamma \in \mathbb{R}$ and any cut-off function ω on the half-axis. In $\mathcal{K}^{s,\gamma}(X^\wedge)$ we choose a Hilbert space scalar product, such that $\mathcal{K}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(X^\wedge)$ with L^2 referring to the measure $dr\,dx$. In $\mathcal{K}^{s,\gamma}(X^\wedge)$ we have a strongly continuous group of isomorphisms $\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$, defined by $(\kappa_\delta u)(r,x) = \delta^{\frac{n+1}{2}} u(\delta r, x)$. In general, if E is a Hilbert space, equipped with a strongly continuous group of isomorphisms $\kappa_\delta : E \to E$, $\delta \in \mathbb{R}_+$, such that $\kappa_\delta \kappa_{\delta'} = \kappa_{\delta\delta'}$ for all $\delta, \delta' \in \mathbb{R}_+$, we say that E is equipped with a group action ('strongly continuous' means $\kappa_\lambda e \in C(\mathbb{R}_+, E)$ for every $e \in E$). Moreover, if E is a Fréchet space, written as a projective limit of Hilbert spaces E^j, $j \in \mathbb{N}$, with continuous embeddings $\ldots \hookrightarrow E^{j+1} \hookrightarrow E^j \hookrightarrow \ldots \hookrightarrow E^0$, we say that E is endowed with a group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$, if κ is a group action on E^0 which restricts to a group action on E^j for every $j \in \mathbb{N}$.

Definition 1.1. *Let E be a Hilbert space with group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$. Then the 'abstract edge Sobolev space' $\mathcal{W}^s(\mathbb{R}^q, E)$, $s \in \mathbb{R}$, is defined as the completion of the space $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm*

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q, E)} = \left\{ \int \langle \eta \rangle^{2s} \|\kappa_{\langle \eta \rangle}^{-1} \hat{u}(\eta)\|_E^2 \, d\eta \right\}^{\frac{1}{2}},$$

where $\langle \eta \rangle = (1 + |\eta|)^{1/2}$, $\hat{u}(\eta) = Fu(\eta) = \int e^{-iy\eta} u(y)\,dy$.

These spaces have been introduced in [14] in connection with operators on manifolds with smooth edges. Concerning more details and useful functional analytic properties, see also [11] or [15].

Applying Definition 1.1 to $E = \mathcal{K}^{s,\gamma}(X^\wedge)$ with the above mentioned group action we obtain the spaces

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)).$$

Now let W be a compact manifold with smooth edge Y and consider the stretched manifold $\mathbb{W}$. Let $p : \partial\mathbb{W} \to Y$ be the canonical projection of the X-bundle $\partial\mathbb{W}$ to the edge Y. Then for every coordinate neighbourhood U on Y we can form a neighbourhood $[0,1) \times p^{-1}(U)$ in $\mathbb{W}$ and a corresponding diffeomorphism

$$\theta : (0,1) \times p^{-1}(U) \to \mathbb{R}_+ \times X \times \mathbb{R}^q \tag{15}$$

associated with a chart $U \to \mathbb{R}^q$ (obtained by the restriction of a corresponding diffeomorphism $[0,1) \times p^{-1}(U) \to \overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q$). We now define the weighted edge space

$$\mathcal{W}^{s,\gamma}(W) \quad \text{for} \quad s, \gamma \in \mathbb{R}$$

as the subspace of all $u \in H^s_{\text{loc}}(\mathbb{W}_{\text{reg}})$ such that for arbitrary such U we have

$$(\omega\varphi u) \circ \theta^{-1} \in \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$$

for any $\varphi \in C_0^\infty(U)$ and a cut-off function $\omega(r)$ supported by $[0,1)$.

By gluing together two copies of $\mathbb{W}$ along the common boundary $\mathbb{W}_{\text{sing}}$, we obtain the double $2\mathbb{W}$ which is a closed compact C^∞ manifold. This gives us the scale of spaces $H^{s;\delta}((2\mathbb{W})_{\succcurlyeq})$, $s, \delta \in \mathbb{R}$.

Let $\omega(r) \in C^\infty(\mathbb{W})$ be a function that is equal to 1 in a neighbourhood of $\mathbb{W}_{\text{sing}}$ and supported in a collar neighbourhood of $\mathbb{W}_{\text{sing}}$. We then define the space

$$\mathcal{W}^{s,\gamma;\delta}(W_{\succcurlyeq}) \quad \text{for} \quad s, \gamma, \delta \in \mathbb{R} \tag{16}$$

to be the completion of $C_0^\infty(\mathbb{R} \times \mathbb{W}_{\text{reg}})$ with respect to the norm

$$\left\{ \|(1-\omega)u\|^2_{H^{s;\delta}((2\mathbb{W})_{\succcurlyeq})} + \sum_{j=1}^N \|\omega\varphi_j u \circ \chi_j^{-1} \circ \beta^{-1}\|^2_{\langle t,\widetilde{y}\rangle^{-\delta}\mathcal{W}^s(\mathbb{R}_t \times \mathbb{R}^q_{\widetilde{y}}, \mathcal{K}^{s,\gamma}(\mathbb{R}_{+,\widetilde{r}} \times X))} \right\}^{1/2}.$$

Here $\{\varphi_1, \ldots, \varphi_N\}$ is a partition of unity subordinate to an open covering $\{U_1, \ldots, U_N\}$ of Y by coordinate neighbourhoods, and we fix maps

$$\chi_j : (t, w) \to (t, r, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times X \times \mathbb{R}^q$$

where w varies on a subset of a collar neighbourhood of $\mathbb{W}_{\text{sing}}$ in $\mathbb{W}$, which has the form $(0,1) \times X \times U_j$ and is mapped under χ_j to $\mathbb{R}_+ \times X \times \mathbb{R}^q$ where the last component is given by charts $\theta_j : U_j \to \mathbb{R}^q$ on Y.

Moreover

$$\beta : \mathbb{R} \times \mathbb{R}_+ \times X \times \mathbb{R}^q \to \mathbb{R} \times \mathbb{R}_+ \times X \times \mathbb{R}^q$$

is defined by $(t, r, x, y) \to (t, [t]r, x, [t]y)$. Here $t \to [t]$ denotes a strictly positive C^∞ function in t such that $[t] = |t|$ for $|t| \geq c$ for a constant $c > 0$ (the same notation will be used later on also in arbitrary dimensions).

Another important ingredient of the corner Sobolev spaces of second generation in (11) are spaces of the kind

$$\mathcal{H}^{s,(\gamma,\theta)}(W^\wedge) := (S_{\theta-\frac{1}{2}\dim W})^{-1}\mathcal{W}^{s,\gamma}(\mathbb{R}\times W), \qquad (17)$$

with S_β being given by (13) and $\mathcal{W}^{s,\gamma}(\mathbb{R}\times W)$ as an analogue of the cylindrical Sobolev space occurring in (12), here with W in place of X. More precisely, $\mathcal{W}^{s,\gamma}(\mathbb{R}\times W)$ consists of all distributions u on $\mathbb{R}\times\mathbb{W}_{\mathrm{reg}}$ such that $(1-\omega)u \in H^s(\mathbb{R}\times 2\mathbb{W})|_{\mathbb{R}\times\mathbb{W}_{\mathrm{reg}}}$ and

$$(\omega u)\circ\chi^{-1}\in\mathcal{W}^s(\mathbb{R}\times\mathbb{R}^q,\mathcal{K}^{s,\gamma}(X^\wedge))$$

for any chart $\chi:\mathbb{R}\times U\to\mathbb{R}\times\mathbb{R}^q$ on $\mathbb{R}\times Y$, $\chi(p,y)=(p,\chi'(y))$, $\chi':U\to\mathbb{R}^q$.

Definition 1.2. (i) *We set*

$$\mathcal{K}^{s,(\gamma,\theta)}(W^\wedge) := \Big\{\sigma(t)f(t,w)+(1-\sigma(t))g(t,w):f\in\mathcal{H}^{s,(\gamma,\theta)}(W^\wedge),$$
$$g\in\mathcal{W}^{s,\gamma;0}(W_{\precsim})|_{t>0}\Big\}$$

for arbitrary $s,\gamma,\theta\in\mathbb{R}$; *here* $\sigma(t)$ *is any cut-off function on* $\mathbb{R}_+$.

(ii) *We set*

$$\mathcal{S}^{(\gamma,\theta)}(W^\wedge) := \sigma\mathcal{K}^{\infty,(\gamma,\theta)}(W^\wedge)+(1-\sigma)\mathcal{W}^{\infty,\gamma;\infty}(W_{\precsim})|_{t>0}$$

for any cut-off function $\sigma(t)$ *where* $\mathcal{W}^{\infty,\gamma;\infty}(W_{\precsim}) := \bigcap_{N\in\mathbb{N}}\mathcal{W}^{N,\gamma;N}(W_{\precsim})$, *cf. the formula* (16), *and, analogously,* $\mathcal{K}^{\infty,(\gamma,\theta)}(W^\wedge) := \bigcap_{N\in\mathbb{N}}\mathcal{K}^{N,(\gamma,\theta)}(W^\wedge)$.

Remark 1.3. *We have*

$$\mathcal{S}^{(\gamma,\theta)}(W^\wedge) = \bigcap_{N\in\mathbb{N}}\langle t\rangle^{-N}\mathcal{K}^{N,(\gamma,\theta)}(W^\wedge).$$

This gives $\mathcal{S}^{(\gamma,\theta)}(W^\wedge)$ *the structure of a Fréchet space.*

In the spaces $\mathcal{W}^{s,\gamma;\delta}(W_{\precsim})$, $\mathcal{H}^{s,(\gamma,\theta)}(W^\wedge)$, $\mathcal{K}^{s,(\gamma,\theta)}(W^\wedge)$, we can introduce scalar products in which these are Hilbert spaces. Let us take the scalar products of the spaces of smoothness and weights zero as reference scalar products of corresponding sesquilinear pairings.

As scalar product between u,v in the space $\mathcal{W}^{0,0;0}(W_{\precsim})$ we can take

$$((1-\omega)u,(1-\omega)v)_{H^{0;0}((2\mathbb{W})_{\precsim})}$$
$$+\sum_{j=1}^N(\omega\varphi_j u\circ\chi_j^{-1}\circ\beta^{-1},\omega\varphi_j v\circ\chi_j^{-1}\circ\beta^{-1})_{\mathcal{W}^0(\mathbb{R}_{t,\tilde{y}}^{1+q},\mathcal{K}^{0,0}(\mathbb{R}_{+,\tilde{r}}\times X))}.$$

In a similar manner we can proceed with the other spaces. For instance, in $\mathcal{K}^{0;(0,0)}(W^\wedge)$ we take the scalar product

$$(u,v)_{\mathcal{K}^{0,(0,0)}(W^\wedge)} := (\sigma u,\sigma v)_{\mathcal{H}^{0,(0,0)}(W^\wedge)}+((1-\sigma)u,(1-\sigma)v)_{\mathcal{W}^{0,0;0}(W_{\precsim})|_{t>0}}.$$

The scalar product in $\mathcal{H}^{0,(0,0)}(W^\wedge)$ comes from a scalar product in $\mathcal{W}^{0,0}(\mathbb{R}\times W)$ via (17), where $\mathcal{W}^{0,0}(\mathbb{R}\times\mathbb{W})$ is identified with $h^{-\frac{n}{2}}L^2(\mathbb{R}\times\mathbb{W})$ for a function h on $\mathbb{W}_{\mathrm{reg}}$ which is C^∞, strictly positive, and equal to r in a neighbourhood of $\mathbb{W}_{\mathrm{sing}}$.

Proposition 1.4. *There is a non-degenerate sesquilinear pairing*

$$\mathcal{K}^{s,(\gamma,\theta)}(W^\wedge) \times \mathcal{K}^{-s,(-\gamma,-\theta)}(W^\wedge) \to \mathbb{C}$$

via the $\mathcal{K}^{0,(0,0)}(W^\wedge)$ - scalar product.

The proof is straightforward and left to the reader.

1.3   Parameter-dependent edge operators

Motivated by the form of corner-degenerate differential operators (4), along the linies of [14] (see also [5] or [16]), we now form families of pseudo-differential operators on a manifold W with edge Y. This will be done first locally on Y, i.e., on an open set $\Omega \subseteq \mathbb{R}^q$ and then globally on W. Local edge-degenerate families are modelled on functions

$$\widetilde{\boldsymbol{p}}(t,r,y,z,\widetilde{\tau},\widetilde{\rho},\widetilde{\eta},\widetilde{\zeta}) \in C^\infty\big(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega \times \Xi, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{2+q+p}_{\widetilde{\tau},\widetilde{\rho},\widetilde{\eta},\widetilde{\zeta}})\big),$$

that are first given in terms of standard parameter-dependent operators on X, with the parameters $(\widetilde{\tau},\widetilde{\rho},\widetilde{\eta},\widetilde{\zeta}) \in \mathbb{R}^{2+q+p}$, smoothly depending on $(t,r,y,z) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega \times \Xi$. Later on, in the corner-degenerate set-up, the covariables are used in the meaning

$$\widetilde{\tau} = rt\tau, \quad \widetilde{\rho} = r\rho, \quad \widetilde{\eta} = r\eta, \quad \widetilde{\zeta} = rt\zeta.$$

We then set

$$\boldsymbol{p}(t,r,y,z,\widetilde{\tau},\rho,\eta,\widetilde{\zeta}) := \widetilde{\boldsymbol{p}}(t,r,y,z,r\widetilde{\tau},r\rho,r\eta,r\widetilde{\zeta}) \qquad (18)$$

which gives us an edge-degenerate family of operators on X, with $\mathbb{R}_+ \times \Omega \times \Xi \ni (t,y,z)$ being interpreted as the edge and $\mathbb{R}_+ \ni r$ as the axis of the local model cone $X^\wedge = \mathbb{R}_+ \times X$. In order to formulate edge amplitude functions of first generation, we employ the following Mellin quantization result, cf. [14], [8].

Theorem 1.5. *Let $\boldsymbol{p}$ be as in (18), and let $\varphi \in C^\infty_0(\mathbb{R}_+)$ be a function such that $\varphi \equiv 1$ near 1. Then there exists a family of operators of the form*

$$\boldsymbol{h}(t,r,y,z,\widetilde{\tau},v,\eta,\widetilde{\zeta}) = \widetilde{\boldsymbol{h}}(t,r,y,z,r\widetilde{\tau},v,r\eta,r\widetilde{\zeta})$$

for an $\widetilde{\boldsymbol{h}}(t,r,y,z,\widetilde{\tau},v,\widetilde{\eta},\widetilde{\zeta}) \in C^\infty\big(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega \times \Xi, L^\mu_{\mathrm{cl}}(X;\mathbb{R}_{\widetilde{\tau}} \times \mathbb{C}_v \times \mathbb{R}^{q+p}_{\widetilde{\eta},\widetilde{\zeta}})\big)$ such that

$$\mathrm{op}_r(\boldsymbol{p})(t,y,z,\widetilde{\tau},\eta,\widetilde{\zeta}) - \mathrm{op}^{\frac{1}{2}}_M(\boldsymbol{h})(t,y,z,\widetilde{\tau},\eta,\widetilde{\zeta}) \in C^\infty\big(\mathbb{R}_+ \times \Omega \times \Xi, L^\mu_{\mathrm{cl}}(X^\wedge;\mathbb{R}^{1+p+q}_{\widetilde{\tau},\eta,\widetilde{\zeta}})\big).$$

The remainder in the latter expression has the form

$$\mathrm{op}_r([1 - \varphi(r'/r)]\boldsymbol{p}(t,r,y,z,\widetilde{\tau},\rho,\eta,\widetilde{\zeta})).$$

Let us choose cut-off functions $\omega_0, \omega_1, \omega_2$ on the r-half-axis, $\omega_2 \prec \omega_1 \prec \omega_0$, and form the operator functions

$$\boldsymbol{a}_0(t,y,z,\widetilde{\tau},\eta,\widetilde{\zeta}) = r^{-\mu}\omega_1(r[\widetilde{\tau},\eta,\widetilde{\zeta}])\,\mathrm{op}^{\gamma-\frac{n}{2}}_M(\boldsymbol{h})(t,y,z,\widetilde{\tau},\eta,\widetilde{\zeta})\omega_0(r'[\widetilde{\tau},\eta,\widetilde{\zeta}]),$$

$$\boldsymbol{a}_1(t,y,z,\widetilde{\tau},\eta,\widetilde{\zeta}) = r^{-\mu}(1 - \omega_1(r[\widetilde{\tau},\eta,\widetilde{\zeta}]))\,\mathrm{op}_r(\boldsymbol{p})(t,y,z,\widetilde{\tau},\eta,\widetilde{\zeta})(1 - \omega_2(r'[\widetilde{\tau},\eta,\widetilde{\zeta}])),$$

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for $n = \dim X$, and set

$$\boldsymbol{a}(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta}) := \sigma_1(r)\{(\boldsymbol{a}_0 + \boldsymbol{a}_1)(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta})\}\sigma_0(r')$$

for some cut-off functions $\sigma_0(r)$, $\sigma_1(r)$. In this way we obtain an operator-valued symbol

$$\boldsymbol{a}(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta}) \in S^\mu(\mathbb{R}_+ \times \Omega \times \Xi \times \mathbb{R}^{1+p+q}_{\widetilde{\tau}, \eta, \widetilde{\zeta}}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu, \gamma-\mu}(X^\wedge)),$$

for $s \in \mathbb{R}$. The latter notation is used in the following sense. Let E and $\widetilde{E}$ be Hilbert spaces with group actions $\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ and $\{\widetilde{\kappa}_\delta\}_{\delta \in \mathbb{R}_+}$, respectively. In the present case we have $E = \mathcal{K}^{s,\gamma}(X^\wedge)$, $\widetilde{E} = \mathcal{K}^{s-\mu, \gamma-\mu}(X^\wedge)$, and the group action is given by $u(r, x) \to \delta^{\frac{n+1}{2}} u(\delta r, x)$, $\delta \in \mathbb{R}_+$, on both spaces, $n = \dim X$. Then

$$S^\mu(U \times \mathbb{R}^m; E, \widetilde{E}),$$

for an open set $U \subseteq \mathbb{R}^d$, is defined as the set of all $a(x, \xi) \in C^\infty(U \times \mathbb{R}^m; \mathcal{L}(E, \widetilde{E}))$ such that

$$\sup_{x \in K, \xi \in \mathbb{R}^n} \langle \xi \rangle^{-\mu+|\beta|} \|\widetilde{\kappa}^{-1}_{\langle\xi\rangle}\{D_x^\alpha D_\xi^\beta a(x, \xi)\}\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(E, \widetilde{E})}$$

is finite for arbitrary $K \Subset U$, $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^m$. Moreover, $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^m; E, \widetilde{E})$ denotes the subspace of so-called classical symbols, defined by asymptotic expansions into terms $\chi(\xi)a_{(\mu-j)}(x, \xi)$, $j \in \mathbb{N}$, for an excision function $\chi(\xi)$ and functions $a_{(\mu-j)}(x, \xi)$ that are homogeneous in the sense

$$a_{(\mu-j)}(x, \delta\xi) = \delta^{\mu-j}\widetilde{\kappa}_\delta a_{(\mu-j)}(x, \xi)\kappa_\delta^{-1}$$

for all $(x, \xi) \in U \times (\mathbb{R}^m \setminus \{0\})$, $\delta \in \mathbb{R}_+$. We also employ symbols for the case where, for instance, E or $\widetilde{E}$ are Fréchet spaces with group action (cf. the notation in Section 1.2); the extension of the definition is straightforward, cf. [15].

Examples are the spaces

$$\mathcal{S}^\gamma(X^\wedge) := \omega\mathcal{K}^{\infty,\gamma}(X^\wedge) + (1 - \omega)\mathcal{S}(\mathbb{R}_+, C^\infty(X))$$

for a cut-off function $\omega(r)$, written as projective limits of $\omega\mathcal{K}^{N,\gamma}(X^\wedge) + (1 - \omega)\langle r \rangle^{-N} H^N(\mathbb{R}_+, H^N(X))$ over $N \in \mathbb{N}$ (here $H^s(\mathbb{R}_+, E) := \{u|_{\mathbb{R}_+} : u \in H^s(\mathbb{R}, E)\}$ for any Hilbert space E). An equivalent characterisation is

$$\mathcal{S}^\gamma(X^\wedge) = \bigcap_{N \in \mathbb{N}} \langle r \rangle^{-N} \mathcal{K}^{N,\gamma}(X^\wedge).$$

The first part of the following definition comes from the 'standard' edge calculus, see [5], or [16].

Definition 1.6.   (i) *The space* $\boldsymbol{R}^\mu_G(U \times \mathbb{R}^m, (\gamma, \beta))$, $U \subseteq \mathbb{R}^d$ *open, is the set of all*

$$\boldsymbol{g}(x, \xi) \in \bigcap_{s \in \mathbb{R}} C^\infty(U \times \mathbb{R}^m, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\beta}(X^\wedge)))$$

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(*called Green symbols of the edge calculus of first generation*) *of order* $\mu \in \mathbb{R}$ *with weights* (γ, β) *if*

$$\boldsymbol{g}(x, \xi) \in \bigcap_{s \in \mathbb{R}} S_{\mathrm{cl}}^{\mu}(U \times \mathbb{R}^{m}, \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{S}^{\beta+\varepsilon}(X^{\wedge})),$$

$$\boldsymbol{g}^{*}(x, \xi) \in \bigcap_{s \in \mathbb{R}} S_{\mathrm{cl}}^{\mu}(U \times \mathbb{R}^{m}, \mathcal{K}^{s,-\beta}(X^{\wedge}), \mathcal{S}^{-\gamma+\varepsilon}(X^{\wedge})),$$

for an $\varepsilon = \varepsilon(\boldsymbol{g}) > 0$; *here* $\boldsymbol{g}^{*}$ *is the pointwise formal adjoint with respect to the* $\mathcal{K}^{0,0}(X^{\wedge})$-*scalar product. A similar definition makes sense also when* $U \subseteq \mathbb{R}^{d}$ *is replaced by* $\overline{\mathbb{R}}_{+} \times U'$ *for an open set* $U' \subseteq \mathbb{R}^{d-1}$.

(ii) *Let* $V \subseteq \mathbb{R}^{e}$ *be open, and fix reals* $\gamma, \theta, \beta, \delta$. *The space* $R_{G}^{\mu}(V \times \mathbb{R}^{p}, (\gamma, \theta), (\beta, \delta))$, *is the set of all*

$$g(z, \zeta) \in \bigcap_{s \in \mathbb{R}} C^{\infty}\left(U \times \mathbb{R}^{p}, \mathcal{L}(\mathcal{K}^{s,(\gamma,\theta)}(W^{\wedge}), \ \mathcal{K}^{\infty,(\beta,\delta)}(W^{\wedge}))\right)$$

(*called Green symbols of the edge calculus of second generation*) *if it has the properties*

$$g(z, \zeta) \in \bigcap_{s \in \mathbb{R}} S_{\mathrm{cl}}^{\mu}(V \times \mathbb{R}^{p}; \mathcal{K}^{s,(\gamma,\theta)}(W^{\wedge}), \ \mathcal{S}^{(\beta+\varepsilon_1, \delta+\varepsilon_2)}(W^{\wedge})), \tag{19}$$

$$g^{*}(z, \zeta) \in \bigcap_{s \in \mathbb{R}} S_{\mathrm{cl}}^{\mu}(V \times \mathbb{R}^{p}; \mathcal{K}^{s,(-\beta,-\delta)}(W^{\wedge}), \ \mathcal{S}^{(-\gamma+\varepsilon_1, -\theta+\varepsilon_2)}(W^{\wedge})), \tag{20}$$

(*cf. Definition 1.2 (ii)*), *for certain* $\varepsilon_i = \varepsilon_i(g) > 0$, $i = 1, 2$; *here* g^{*} *is the pointwise adjoint with respect to the* $\mathcal{K}^{0,(0,0)}(W^{\wedge})$- *scalar product. A similar definition makes sense also when* V *is replaced by* $\overline{\mathbb{R}}_{+} \times V'$ *for an open set* $V' \subseteq \mathbb{R}^{e-1}$, *with symbolic estimates that are required uniformly up to* 0 *on* $\overline{\mathbb{R}}_{+}$. *Let* $R_{G,\mathcal{O}}^{\mu}(V \times \mathbb{R}^{p}, (\gamma, \theta), (\beta, \delta))$ *denote the subspace of all* $g(z, \zeta) \in R_{G}^{\mu}(V \times \mathbb{R}^{p}, (\gamma, \theta), (\beta, \delta))$ *such that* (19) *and* (20) *hold for arbitrary* $\varepsilon_2 > 0$. *Green symbols of that kind will also be called flat.*

Below, if the weights $(\gamma, \theta), (\beta, \delta)$ are fixed and known from the context, we also write $R_{G,\mathcal{O}}^{\mu}$ instead of $R_{G,\mathcal{O}}^{\mu}(V \times \mathbb{R}^{p}; (\gamma, \theta), (\beta, \delta))$.

Proposition 1.7. (i) *Let* $\boldsymbol{g}(x, \xi) \in \boldsymbol{R}_{G}^{\mu}(U \times \mathbb{R}^{m}, (\gamma, \beta))$; *then* $r^{k}\boldsymbol{g}(x, \xi)r^{l} \in \boldsymbol{R}_{G}^{\mu}(U \times \mathbb{R}^{n}, (\gamma - l, \beta + k))$ *for every* $l, k \in \mathbb{R}$;

(ii) *let* $g(z, \zeta) \in R_{G}^{\mu}(V \times \mathbb{R}^{p}, (\gamma, \theta), (\beta, \delta))$; *then* $t^{k}g(z, \zeta)t^{l} \in R_{G}^{\mu}(V \times \mathbb{R}^{p}, (\gamma, \theta - l), (\beta, \delta + k))$ *for every* $l, k \in \mathbb{R}$.

Let $M^{-\infty}(X; \Gamma_{\beta})$ denote the subspace of all $f(v) \in L^{-\infty}(X; \Gamma_{\beta})$ $\left(:= \mathcal{S}(\Gamma_{\beta}, L^{-\infty}(X))\right)$ such that for an $\epsilon = \epsilon(f) > 0$ there is an extension $f(v) \in \mathcal{A}(\{v : \beta - \epsilon < \operatorname{Re} v < \beta + \epsilon\}, L^{-\infty}(X))$ with the property $f(\delta + i\rho) \in L^{-\infty}(X; \Gamma_{\delta})$ for every $\beta - \epsilon < \delta < \beta + \epsilon$, uniformly in compact subintervals. Then, for any $f(x, v) \in C^{\infty}\left(U, M^{-\infty}(X; \Gamma_{\frac{n+1}{2} - \gamma})\right)$ and arbitrary cut-off functions $\omega(r), \widetilde{\omega}(r)$, the operator family

$$\boldsymbol{g}(x, \xi) := r^{\beta - \gamma + \varepsilon}\omega(r[\xi])\operatorname{op}_{M}^{\gamma - \frac{n}{2}}(f)(x)\widetilde{\omega}(r'[\xi]),$$

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$\varepsilon > 0$, is a Green symbol in the sense of Definition 1.6 (i). Of course, for $\varepsilon = 0$ this is not the case. These operator functions are then a typical ingredient of the edge calculus on W, namely, the so-called smoothing Mellin operators. An operator $G \in L^{-\infty}(W \setminus Y)$ is called smoothing in the edge calculus on W, associated with the weight data (γ, β), if G and its formal adjoint G^* induce continuous operators

$$G : \mathcal{W}^{s,\gamma}(W) \rightarrow \mathcal{W}^{\infty,\beta+\varepsilon}(W), \quad G^* : \mathcal{W}^{s,-\beta}(W) \rightarrow \mathcal{W}^{\infty,-\gamma+\varepsilon}(W)$$

for all $s \in \mathbb{R}$ and some $\varepsilon = \varepsilon(G) > 0$. The formal adjoint is defined by the $\mathcal{W}^{0,0}$-pairing with a fixed scalar product (from $r^{-\frac{n}{2}} L^2(\mathbb{W})$).

More generally, $\mathcal{Y}^{-\infty}(W; \mathbb{R}^l)$ will denote the space of all Schwartz functions $G(\lambda)$ on $\mathbb{R}^l \ni \lambda$ with values in the space of smoothing operators just defined. The weight data are assumed to be known by the context; otherwise we write $\mathcal{Y}^{-\infty}(W, (\gamma, \beta); \mathbb{R}^l)$. We will use below that this is an inductive limit of Fréchet spaces; so it makes sense to talk about C^∞ functions with values in this space. Let us now define edge operators with parameters together with their symbolic structure. Near the edge in local stretched coordinates they are defined as operators $\mathrm{Op}_y(\boldsymbol{a})(t, z, \widetilde{\tau}, \widetilde{\zeta})$ for edge symbols of the form

$$
\begin{aligned}
\boldsymbol{a}(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta}) &= \sigma_1(r)\{(\boldsymbol{a}_0 + \boldsymbol{a}_1)(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta})\}\sigma_2(r') \\
&+ \boldsymbol{m}(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta}) + \boldsymbol{g}(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta})
\end{aligned}
\tag{21}
$$

for

$$\boldsymbol{m}(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta}) = r^{-\mu}\omega(r[\widetilde{\tau}, \eta, \widetilde{\zeta}]) \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(t, y, z)\widetilde{\omega}(r'[\widetilde{\tau}, \eta, \widetilde{\zeta}]),$$

with a Mellin amplitude function $f(t, r, y, z, v) \in C^\infty\big(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega \times \Xi, M^{-\infty}(X; \Gamma_{\frac{n+1}{2}-\gamma})\big)$, and

$$\boldsymbol{g}(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta}) = r^{-\mu}\boldsymbol{g}_0(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta}),$$

where $\boldsymbol{g}_0(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta})$ is a Green symbol in the sense of Definition 1.6 (i) with $U := \overline{\mathbb{R}}_+ \times \Omega \times \Xi \ni x = (t, y, z)$, $\xi = (\widetilde{\tau}, \eta, \widetilde{\zeta}) \in \mathbb{R}^{1+p+q}$ and weight data $\gamma = \beta$. Let $\boldsymbol{R}^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \Xi \times \mathbb{R}^{1+q+p}_{\widetilde{\tau},\eta,\widetilde{\zeta}}, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu)$ denote the space of all operator functions of the form (21). In a similar manner we can define $\boldsymbol{R}^\mu(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega \times \Xi \times \mathbb{R}^{1+q+p}_{\widetilde{\tau},\eta,\widetilde{\zeta}}, \boldsymbol{g})$ when $t \in \overline{\mathbb{R}}_+$ is replaced by $(t, t') \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$. We also use notation like $\boldsymbol{R}^\mu(\mathbb{R}_+ \times \Omega \times \Xi \times \mathbb{R}^{1+q+p}_{\widetilde{\tau},\eta,\widetilde{\zeta}}, \boldsymbol{g})$ and $\boldsymbol{R}^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \times \Xi \times \mathbb{R}^{1+q+p}_{\widetilde{\tau},\eta,\widetilde{\zeta}}, \boldsymbol{g})$ when the dependence of the involved amplitude functions in t or (t, t') is assumed to be smooth in $t \in \mathbb{R}_+$ or $(t, t') \in \mathbb{R}_+ \times \mathbb{R}_+$.

In the following definition we choose an open covering of Y by charts $\theta_j : U_j \rightarrow \mathbb{R}^q$, $j = 1, \ldots, N$, a subordinate partition of unity $\{\varphi_1, \ldots, \varphi_N\}$ and a system of functions $\{\psi_1, \ldots, \psi_N\}$, $\psi_j \in C_0^\infty(U_j)$, such that $\psi_j \equiv 1$ on $\mathrm{supp}\, \varphi_j$ for all j.

Definition 1.8. *By* $C^\infty\big(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^\mu(W, \boldsymbol{g}; \mathbb{R}_{\widetilde{\tau}} \times \mathbb{R}^p_{\widetilde{\zeta}})\big) \ni \widetilde{a}(t, z, \widetilde{\tau}, \widetilde{\zeta})$ *we denote the space of all operator families*

$$\mathcal{W}^{s,\gamma}(W) \rightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(W),$$

continuous for all $s \in \mathbb{R}$, *that are of the form*

$$\sigma \widetilde{a}_{\mathrm{edge}}(t, z, \widetilde{\tau}, \widetilde{\zeta})\widetilde{\sigma} + (1-\sigma)\widetilde{a}_{\mathrm{int}}(t, z, \widetilde{\tau}, \widetilde{\zeta})(1-\widetilde{\widetilde{\sigma}}) + \widetilde{g}(t, z, \widetilde{\tau}, \widetilde{\zeta}), \tag{22}$$

where $\sigma(r), \widetilde{\sigma}(r), \widetilde{\widetilde{\sigma}}(r)$ *are cut-off functions such that* $\widetilde{\widetilde{\sigma}} \prec \sigma \prec \widetilde{\sigma}$ *and*

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(i)

$$\widetilde{a}_{\text{edge}}(t, z, \widetilde{\tau}, \widetilde{\zeta}) = \sum_{j=1}^{N} \varphi_j(\theta_j^{-1})_* \operatorname{Op}_y(\boldsymbol{a}_j)(t, z, \widetilde{\tau}, \widetilde{\zeta}) \psi_j$$

for arbitrary edge symbols $\boldsymbol{a}_j(t, y, z, \widetilde{\tau}, \eta, \widetilde{\zeta}) \in \boldsymbol{R}^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \Xi \times \mathbb{R}^{1+p+q}_{\widetilde{\tau}, \eta, \widetilde{\zeta}}, \boldsymbol{g})$ of the form (21);

(ii) $\widetilde{a}_{\text{int}}(t, z, \widetilde{\tau}, \widetilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, L^\mu_{\text{cl}}(W \setminus Y; \mathbb{R}^{1+p}_{\widetilde{\tau}, \widetilde{\zeta}}))$;

(iii) $\widetilde{g}(t, z, \widetilde{\tau}, \widetilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^{-\infty}(W, \boldsymbol{g}; \mathbb{R}^{1+p}_{\widetilde{\tau}, \widetilde{\zeta}}))$.

2 Edge symbols of second generation

2.1 Quantization of corner-degenerate symbols

Our next objective is to formulate a Mellin quantization result for operator families

$$p(t, z, \tau, \zeta) := \widetilde{p}(t, z, t\tau, t\zeta) \quad \text{with} \quad \widetilde{p}(t, z, \widetilde{\tau}, \widetilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^\mu(W, \boldsymbol{g}; \mathbb{R}^{1+p}_{\widetilde{\tau}, \widetilde{\zeta}})), \qquad (23)$$

where $\mathcal{Y}^\mu(W, \boldsymbol{g}; \mathbb{R}^{1+p})$ refers to the weight data $\boldsymbol{g} = (\gamma, \gamma - \mu)$ with respect to the edge $Y \subset W$. Let $C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^\mu(W, \boldsymbol{g}; \mathbb{C}_w \times \mathbb{R}^p_\zeta))$ denote the space of all families of operators $h(t, z, w, \zeta)$ that are entire in w with values in $C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^\mu(W, \boldsymbol{g}; \mathbb{R}^p_\zeta))$ such that

$$h(t, z, \delta + i\tau, \zeta) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^\mu(W, \boldsymbol{g}; \mathbb{R}_\tau \times \mathbb{R}^p_\zeta))$$

for every real δ, and uniformly in compact intervals. The definition of holomorphic dependence of operator functions in $w \in \mathbb{C}$ can be given in more detail by using analogues of the ingredients of Definition 1.8 with $w \in \mathbb{C}$ in place of $\widetilde{\tau}$. In particular, this needs a corresponding version of the $\boldsymbol{R}^\mu$-classes, first in the form

$$\boldsymbol{R}^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \Xi \times \Gamma_\beta \times \mathbb{R}^{p+q}_{\eta, \widetilde{\zeta}}, \boldsymbol{g}) \qquad (24)$$

where $\widetilde{\tau}$ is replaced by $\operatorname{Im} w$ for $w \in \Gamma_\beta$ for some $\beta \in \mathbb{R}$. Then

$$\boldsymbol{R}^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \Xi \times \mathbb{C} \times \mathbb{R}^{p+q}_{\eta, \widetilde{\zeta}}, \boldsymbol{g})$$

is the space of all entire functions $\boldsymbol{f}(t, y, z, w, \eta, \widetilde{\zeta})$ in w with values in $\boldsymbol{R}^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \Xi \times \mathbb{R}^{p+q}_{\eta, \widetilde{\zeta}}, \boldsymbol{g})$ such that $\boldsymbol{f}(t, y, z, \beta + i\widetilde{\tau}, \eta, \widetilde{\zeta})$ belongs to (24) for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.

The following theorem is an analogue of [8, Theorem 3.2], here for the calculus of second generation.

Theorem 2.1. *Let $p(t, z, \tau, \zeta)$ be as in (23), and let $\varphi \in C_0^\infty(\mathbb{R}_+)$ be a function such that $\varphi \equiv 1$ near 1. Then there exists a family of operator-valued Mellin symbols of the form*

$$h(t, z, w, \zeta) = \widetilde{h}(t, z, w, t\zeta) \quad \text{with} \quad \widetilde{h}(t, z, w, \widetilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^\mu(W, \boldsymbol{g}; \mathbb{C}_w \times \mathbb{R}^p_{\widetilde{\zeta}})) \quad (25)$$

such that

$$\operatorname{op}_t(p)(z,\zeta) - \operatorname{op}_M^{\frac{1}{2}}(h)(z,\zeta) = \operatorname{op}_t(r)(z,\zeta)$$

for $r(t,t',z,\tau,\zeta) := (1 - \varphi(t'/t))p(t,z,\tau,\zeta)$, and the operator family $\operatorname{op}_t(r)(z,\zeta)$ belongs to $C^\infty\big(\Xi,\mathcal{Y}^{-\infty}(W^\wedge,\boldsymbol{g};\mathbb{R}_\zeta^p)\big)$.

The proof will be given in Section 3.1 below.

Let us choose a function $\omega(t,t') \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ as in [4, Lemma 2.10], i.e., $\omega(t,t') := \psi\left(\frac{(t-t')^2}{1+(t-t')^2}\right)$ for some $\psi \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\psi(t) = 1$ for $t < \frac{1}{2}$, $\psi(t) = 0$ for $t > \frac{3}{2}$, and set

$$a(z,\zeta) := \sigma_1(t)\{a_0(z,\zeta) + a_1(z,\zeta)\}\sigma_0(t') \tag{26}$$

for

$$a_0(z,\zeta) = t^{-\mu}\omega_1(t[\zeta]) \operatorname{op}_M^{\theta-\frac{d}{2}}(h)(z,\zeta)\omega_0(t'[\zeta]), \tag{27}$$

$$a_1(z,\zeta) = t^{-\mu}\left(1 - \omega_1(t[\zeta])\right)\omega(t[\zeta],t'[\zeta]) \operatorname{op}_t(p)(z,\zeta)\left(1 - \omega_2(t'[\zeta])\right), \tag{28}$$

$d := \dim W$, where σ_0, σ_1 and $\omega_0, \omega_1, \omega_2$ are cut-off functions on the t-half axis, such that $\omega_2 \prec \omega_1 \prec \omega_0$.

We then have the following result:

Proposition 2.2. [4, Theorem 3.8] *We have*

$$a(z,\zeta) \in S^\mu\big(\Xi \times \mathbb{R}^p; \mathcal{K}^{s,(\gamma,\theta)}(W^\wedge), \mathcal{K}^{s-\mu,(\gamma-\mu,\theta-\mu)}(W^\wedge)\big)$$

for every $s \in \mathbb{R}$.

For

$$p_0(t,z,\tau,\zeta) := \widetilde{p}(0,z,t\tau,t\zeta), \quad h_0(t,z,w,\zeta) = \widetilde{h}(0,z,w,t\zeta)$$

we define

$$\sigma_\wedge(a)(z,\zeta) := t^{-\mu}\Big\{\omega_1(t|\zeta|) \operatorname{op}_M^{\theta-\frac{d}{2}}(h_0)(t,\zeta)\omega_0(t'|\zeta|)$$
$$+ (1 - \omega_1(t|\zeta|))\omega(t|\zeta|,t'|\zeta|) \operatorname{op}_t(p_0)(z,\zeta)(1 - \omega_2(t'|\zeta|))\Big\}.$$

Remark 2.3. *The edge symbol $\sigma_\wedge(a)(z,\zeta)$ induces a family of continuous operators*

$$\sigma_\wedge(a)(z,\zeta) : \mathcal{K}^{s,(\gamma,\theta)}(W^\wedge) \to \mathcal{K}^{s-\mu,(\gamma-\mu,\theta-\mu)}(W^\wedge)$$

for every $s \in \mathbb{R}$.

Let us endow the spaces $\mathcal{K}^{s,(\gamma,\theta)}(W^\wedge)$ with the group action

$$\kappa_\lambda : \mathcal{K}^{s,(\gamma,\theta)}(W^\wedge) \to \mathcal{K}^{s,(\gamma,\theta)}(W^\wedge),$$

defined by $(\kappa_\lambda u)(t,\cdot) = \lambda^{\frac{d+1}{2}} u(\lambda t,\cdot), \lambda \in \mathbb{R}_+$. For $(z,\zeta) \in T^*\Xi \setminus 0$ we then have

$$\sigma_\wedge(a)(z,\lambda\zeta) = \lambda^\mu \kappa_\lambda \sigma_\wedge(a)(z,\zeta)\kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+, (z,\zeta) \in T^*\Xi \setminus 0$.

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2.2 Composition results

We now formulate a result on the compositions of operator-valued amplitude functions of the kind (26). To this end we reformulate the operators as

$$a(z,\zeta) = \omega_1(t[\zeta])A_0(z,\zeta)\omega_0(t'[\zeta]) + (1 - \omega_1(t[\zeta]))A_1(z,\zeta)(1 - \omega_2(t'[\zeta]))$$

for

$$A_0(z,\zeta) := \sigma_1(t)t^{-\mu} \operatorname{op}_M^{\theta-\frac{d}{2}}(h)(z,\zeta)\sigma_0(t'), \tag{29}$$

$$A_1(z,\zeta) := \sigma_1(t)t^{-\mu}\omega(t[\zeta],t'[\zeta]) \operatorname{op}_t(p)(z,\zeta)\sigma_0(t'), \tag{30}$$

under the same conditions as before. In particular, we assume $\omega_2 \prec \omega_1 \prec \omega_0$. For the composition we consider another operator function

$$b(z,\zeta) = \omega_1(t[\zeta])B_0(z,\zeta)\omega_0(t'[\zeta]) + (1 - \omega_1(t[\zeta]))B_1(z,\zeta)(1 - \omega_2(t'[\zeta])),$$

for

$$B_0(z,\zeta) := \widetilde{\sigma}_1(t)t^{-\nu} \operatorname{op}_M^{\theta+\nu-\frac{d}{2}}(f)(z,\zeta)\widetilde{\sigma}_0(t'),$$

$$B_1(z,\zeta) := \widetilde{\sigma}_1(t)t^{-\nu}\widetilde{\omega}(t[\zeta],t'[\zeta]) \operatorname{op}_t(q)(z,\zeta)\sigma_0(t'),$$

where q is of a similar structure as p, now of order ν, and f is an associated Mellin symbol as in Theorem 2.1. We fix the involved weights in such a way that

$$b(z,\zeta) \in S^\nu\big(\Xi \times \mathbb{R}^p; \mathcal{K}^{s,(\gamma+\nu,\theta+\nu)}(W^\wedge), \mathcal{K}^{s-\nu,(\gamma,\theta)}(W^\wedge)\big)$$

for all real s, cf. Proposition 2.2.

Theorem 2.4. *Given $a(z,\zeta)$ and $b(z,\zeta)$ as before, there is a $c(z,\zeta)$ of analogous structure, now of order $\mu + \nu$ and referring to the pair of weights $(\gamma + \nu, \theta + \nu)$, $(\gamma - \mu, \theta - \mu)$, such that for the pointwise composition we have*

$$a(z,\zeta)b(z,\zeta) = c(z,\zeta) + g(z,\zeta) \tag{31}$$

for a Green symbol $g(z,\zeta) \in R_{G,\mathcal{O}}^{\mu+\nu}\left(\Xi \times \mathbb{R}^q; (\gamma + \nu, \theta + \nu), (\gamma - \mu, \theta - \mu)\right)$.

Proof. For the formal computations we first omit the arguments and simply write

$$a = \omega_1 A_0\omega_0 + (1 - \omega_1)A_1(1 - \omega_2), \quad b = \omega_1 B_0\omega_0 + (1 - \omega_1)B_1(1 - \omega_2).$$

Then we obtain

$$ab = S + R_1 + R_2 \tag{32}$$

for

$$S := \omega_1 A_0\omega_1 B_0\omega_0 + (1 - \omega_1)A_1(1 - \omega_1)B_1(1 - \omega_2),$$

$$R_1 := \omega_1 A_0(\omega_0 - \omega_1)B_1(1 - \omega_2), \quad R_2 := (1 - \omega_1)A_1(\omega_1 - \omega_2)B_0\omega_0.$$

$\square$

Here and in the sequel we systematically apply the relation $\omega_2 \prec \omega_1 \prec \omega_0$ which implies $\omega_i\omega_j = \omega_j$ and $(1 - \omega_i)(1 - \omega_j) = 1 - \omega_i$ for $i < j$. We then have

$$R_1 = \omega_2 A_0(\omega_0 - \omega_1)B_1(1 - \omega_2) + (\omega_1 - \omega_2)A_0(\omega_0 - \omega_1)B_1(1 - \omega_2) = G_1 + \widetilde{R}_1$$

for

$$\begin{aligned} G_1 &:= \omega_2 A_0(\omega_0 - \omega_1)B_1(1 - \omega_2) + (\omega_1 - \omega_2)A_0(\omega_0 - \omega_1)B_1(1 - \omega_0), & (33) \\ \widetilde{R}_1 &:= (\omega_1 - \omega_2)A_0(\omega_0 - \omega_1)B_1(\omega_0 - \omega_2). \end{aligned}$$

The operator family G_1 is smoothing because of pseudo-locality; in fact, we have $\omega_2(\omega_0 - \omega_1) = 0$, and $(\omega_1 - \omega_2)(1 - \omega_0) = \omega_1 - \omega_2 - \omega_1\omega_0 + \omega_2\omega_0 = 0$.

For similar reasons also the remainders G_k below will be smoothing. More precisely, we shall see that these are flat Green symbols.

Moreover,

$$R_2 = (1 - \omega_0)A_1(\omega_1 - \omega_2)B_0\omega_0 + (\omega_0 - \omega_1)A_1(\omega_1 - \omega_2)B_0\omega_0 = G_2 + \widetilde{R}_2$$

for

$$\begin{aligned} G_2 &= (1 - \omega_0)A_1(\omega_1 - \omega_2)B_0\omega_0 + (\omega_0 - \omega_1)A_1(\omega_1 - \omega_2)B_0\omega_2, & (34) \\ \widetilde{R}_2 &= (\omega_0 - \omega_1)A_1(\omega_1 - \omega_2)B_0(\omega_0 - \omega_2). \end{aligned}$$

For S we write $S = P + T_1 + T_2$ for

$$P = \omega_1 A_0 \omega_0 B_0 \omega_0 + (1 - \omega_1)A_1(1 - \omega_2)B_1(1 - \omega_2), \tag{35}$$

$$T_1 = \omega_1 A_0(\omega_1 - \omega_0)B_0\omega_0, \quad T_2 = (1 - \omega_1)A_1(\omega_2 - \omega_1)B_1(1 - \omega_2).$$

Reformulating these expressions gives us

$$T_1 = (\omega_1 - \omega_2)A_0(\omega_1 - \omega_0)B_0\omega_0 + \omega_2 A_0(\omega_1 - \omega_0)B_0\omega_0 = G_3 + \widetilde{T}_1$$

for

$$\begin{aligned} G_3 &= (\omega_1 - \omega_2)A_0(\omega_1 - \omega_0)B_0\omega_2 + \omega_2 A_0(\omega_1 - \omega_0)B_0\omega_0, & (36) \\ \widetilde{T}_1 &= (\omega_1 - \omega_2)A_0(\omega_1 - \omega_0)B_0(\omega_0 - \omega_2). \end{aligned}$$

Moreover,

$$T_2 = (1 - \omega_0)A_1(\omega_2 - \omega_1)B_1(1 - \omega_2) + (\omega_0 - \omega_1)A_1(\omega_2 - \omega_1)B_1(1 - \omega_2) = G_4 + \widetilde{T}_2$$

for

$$\begin{aligned} G_4 &= (1 - \omega_0)A_1(\omega_2 - \omega_1)B_1(1 - \omega_2) + (\omega_0 - \omega_1)A_1(\omega_2 - \omega_1)B_1(1 - \omega_0), & (37) \\ \widetilde{T}_2 &= (\omega_0 - \omega_1)A_1(\omega_2 - \omega_1)B_1(\omega_0 - \omega_2). \end{aligned}$$

We now obtain altogether

$$ab = P + T_1 + T_2 + R_1 + R_2 \tag{38}$$

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and $T_1 + T_2 + R_1 + R_2 = \widetilde{T}_1 + \widetilde{T}_2 + \widetilde{R}_1 + \widetilde{R}_2 + \sum_{k=1}^4 G_k$, where

$$
\begin{aligned}
\widetilde{T}_1 &+ \widetilde{T}_2 + \widetilde{R}_1 + \widetilde{R}_2 \\
&= (\omega_1 - \omega_2) A_0 (\omega_1 - \omega_0) B_0 (\omega_0 - \omega_2) + (\omega_0 - \omega_1) A_1 (\omega_2 - \omega_1) B_1 (\omega_0 - \omega_2) \\
&+ (\omega_1 - \omega_2) A_0 (\omega_0 - \omega_1) B_1 (\omega_0 - \omega_2) + (\omega_0 - \omega_1) A_1 (\omega_1 - \omega_2) B_0 (\omega_0 - \omega_2) \\
&= (\omega_1 - \omega_2) A_0 (\omega_1 - \omega_0) B_0 (\omega_0 - \omega_2) + (\omega_0 - \omega_1) A_0 (\omega_2 - \omega_1) B_0 (\omega_0 - \omega_2) \\
&+ (\omega_1 - \omega_2) A_0 (\omega_0 - \omega_1) B_0 (\omega_0 - \omega_2) + (\omega_0 - \omega_1) A_0 (\omega_1 - \omega_2) B_0 (\omega_0 - \omega_2) + G_5 \\
&= G_5.
\end{aligned}
\tag{39}
$$

The operator $G_5 := L_1 + L_2 + L_3$ is the smoothing remainder which arises as a sum of expressions, where we replaced A_1 and B_1 by A_0 and B_0, respectively, taking into account that A_0 and B_0 are Mellin quantisations of A_1 and B_1, respectively, namely,

$$
\begin{aligned}
L_1 &:= (\omega_0 - \omega_1) A_1 (\omega_2 - \omega_1) B_1 (\omega_0 - \omega_2) - (\omega_0 - \omega_1) A_0 (\omega_2 - \omega_1) B_0 (\omega_0 - \omega_2), \\
L_2 &:= (\omega_1 - \omega_2) A_0 (\omega_0 - \omega_1) B_1 (\omega_0 - \omega_2) - (\omega_1 - \omega_2) A_0 (\omega_0 - \omega_1) B_0 (\omega_0 - \omega_2), \\
L_3 &:= (\omega_0 - \omega_1) A_1 (\omega_1 - \omega_2) B_0 (\omega_0 - \omega_2) - (\omega_0 - \omega_1) A_0 (\omega_1 - \omega_2) B_0 (\omega_0 - \omega_2).
\end{aligned}
$$

The composition ab is characterised in the desired manner if we show (apart from the verification that all G_k are flat Green symbols, cf. Section 3.2) that P itself, given by the expression (35), is of a form as the right hand side of (31). This is the result of the following two lemmas.

Lemma 2.5. *Let $p(t, z, \tau, \zeta)$ be given by (23) (which is the operator function involved in $a(z, \zeta)$ via (26), (27), (28)), and let, analogously,*

$$
q(t, z, \tau, \zeta) := \widetilde{q}(t, z, t\tau, t\zeta) \quad \text{with} \quad \widetilde{q}(t, z, \widetilde{\tau}, \widetilde{\zeta}) \in C^\infty\big(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^\mu(W, \boldsymbol{g}; \mathbb{R}_{\widetilde{\tau}, \widetilde{\zeta}}^{1+p})\big)
$$

(which is involved in $b(z, \zeta)$). Then there is an

$$
r(t, z, \tau, \zeta) := \widetilde{r}(t, z, t\tau, t\zeta) \quad \text{with} \quad \widetilde{r}(t, z, \widetilde{\tau}, \widetilde{\zeta}) \in C^\infty\big(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^{\mu+\nu}(W, \boldsymbol{g}; \mathbb{R}_{\widetilde{\tau}, \widetilde{\zeta}}^{1+p})\big)
$$

which is the Leibniz product of p and q in the sense

$$
r(t, z, \tau, \zeta) \sim \sum_k \frac{1}{k!} \partial_\tau^k \big(t^{-\nu} p(t, z, \tau, \zeta)\big) D_t^k t^{-\mu} \sigma_0(t) \widetilde{\sigma}_1(t) q(t, z, \tau, \zeta)
$$

such that

$$
\begin{aligned}
(1 &- \omega_1(t[\zeta])) A_1(z, \zeta)(1 - \omega_2(t'[\zeta])) B_1(z, \zeta)(1 - \omega_2(t''[\zeta])) \\
&= \sigma_1(t) t^{-(\mu+\nu)} (1 - \omega_1(t[\zeta])) \omega(t[\zeta], t'[\zeta]) \operatorname{op}_t(r)(z, \zeta)(1 - \omega_2(t'[\zeta])) \sigma_0(t') + g(z, \zeta)
\end{aligned}
$$

for a flat Green symbol $g(z, \zeta)$ of order $\mu + \nu$.

The proof is simple after the technique of [4]; details will be omitted.

Lemma 2.6. *The operator*

$$
\omega_1(t[\zeta]) A_0(z, \zeta)(\omega_0(t'[\zeta]) - 1) B_0(z, \zeta)(z, \zeta) \omega_0(t''[\zeta])
$$

is a flat Green symbol of order $\mu + \nu$.

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The technique of proving Lemma 2.6 is the same as that for the remainders G_k to be characterised in Section 3.2.

The proof of the following observation is a straightforward consequence of the fact that Mellin operators with symbols can be composed within this category of operators, cf., similarly, [8, Proposition 4.3].

Lemma 2.7. *Let $h(t, z, w, \zeta)$ be the (holomorphic in w) Mellin symbol (25) related to $p(t, z, \tau, \zeta)$ (via Theorem 2.1), and, analogously, $f(t, z, w, \zeta)$ the Mellin symbol related to $q(t, z, \tau, \zeta)$. Then there is a (holomorphic in w) Mellin symbol $l(t, z, w, \zeta)$ related to $r(t, z, \tau, \zeta)$, cf. the notation in Lemma 2.5, such that, when we set*

$$c_0(z, \zeta) := t^{-(\mu+\nu)} \omega_1(t[\zeta]) \operatorname{op}_M^{\theta+\nu-\frac{d}{2}}(l)(z, \zeta)\omega_0(t'[\zeta]),$$

we have

$$\omega_1(t[\zeta])A_0(z, \zeta)B_0(z, \zeta)\omega_0(t'[\zeta]) = c_0(z, \zeta) + g(z, \zeta)$$

for a flat Green symbol $g(z, \zeta)$.

3 Parameter-dependent operators on an infinite cone

3.1 A result on Mellin quantization

In this section we prove Theorem 2.1 on the Mellin quantization. Let us first formulate a theorem which can be obtained in a similar manner as an analogous result of [7], namely,

Theorem 3.1. *Let $\widetilde{\boldsymbol{a}}(t, y, \widetilde{\tau}, \eta, \widetilde{\zeta}) \in \boldsymbol{R}^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q+p}_{\widetilde{\tau}, \eta, \widetilde{\zeta}}, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu)$, and set $\boldsymbol{a}(t, y, \tau, \eta, \zeta) := \widetilde{\boldsymbol{a}}(t, y, t\tau, \eta, t\zeta)$. Moreover, let $\varphi \in C_0^\infty(\mathbb{R}_+)$ be a function such that $\varphi \equiv 1$ near 1. Then there exists an $\widetilde{h}(t, y, w, \eta, \widetilde{\zeta}) \in \boldsymbol{R}^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{C}_w \times \mathbb{R}^{q+p}_{\eta, \widetilde{\zeta}}, \boldsymbol{g})$ such that for $\boldsymbol{h}(t, y, w, \eta, \zeta) := \widetilde{\boldsymbol{h}}(t, y, w, \eta, t\zeta)$ we have*

$$\operatorname{op}_{t,y}(\varphi(t'/t)\boldsymbol{a})(\zeta) = \operatorname{op}_M^{\frac{1}{2}}(\operatorname{op}_y(\boldsymbol{h}))(\zeta)$$

for all $\zeta \in \mathbb{R}^p$.

Proof of Theorem 2.1. Let us first recall from [8] that the simpler variant of Theorem 2.1 for the case of a closed compact C^∞ manifold M rather than W is the following. Let $p_{\text{int}}(t, z, \tau, \zeta) = \widetilde{p}_{\text{int}}(t, z, t\tau, t\zeta)$ for a $\widetilde{p}_{\text{int}}(t, z, \widetilde{\tau}, \widetilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, L^\mu_{\text{cl}}(M; \mathbb{R}^{1+p}_{\widetilde{\tau}, \widetilde{\zeta}}))$, and let $\varphi \in C_0^\infty(\mathbb{R}_+)$ be a function which is equal to 1 near 1. Then there exists an $h_{\text{int}}(t, z, w, \zeta) = \widetilde{h}_{\text{int}}(t, z, w, t\zeta)$ with $\widetilde{h}_{\text{int}}(t, z, w, \widetilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, L^\mu_{\text{cl}}(M; \mathbb{C} \times \mathbb{R}^p))$ such that

$$\operatorname{op}_t(p_{\text{int}})(z, \zeta) - \operatorname{op}_M^{\frac{1}{2}}(h_{\text{int}})(t, \zeta) = \operatorname{op}_t(r_{\text{int}})(z, \zeta) \tag{40}$$

for $r_{\text{int}}(t, t', z, \tau, \zeta) := (1 - \varphi(t'/t))p_{\text{int}}(t, z, \tau, \zeta)$, where

$$\operatorname{op}_t(\varphi(t'/t)p_{\text{int}})(z, \zeta) = \operatorname{op}_M^{\frac{1}{2}}(h_{\text{int}})(z, \zeta). \tag{41}$$

The result is based on a corresponding variant of Theorem 3.1 for scalar amplitude functions rather than operator-valued ones. Applying this technique we see that $\widetilde{h}$ may be found in the form

$$\widetilde{h}(t, y, w, \eta, \widetilde{\zeta}) = t^w \operatorname{op}_M^{\frac{1}{2}}(\varphi(t'/t)\widetilde{f})(y, \eta, \widetilde{\zeta})(t')^{-w},$$

where

$$\begin{aligned}
\widetilde{f}(t, t', y, i\tau, \eta, \widetilde{\zeta}) &= M(t, t')t'\widetilde{a}(t, y, -M(t, t')t\tau, \eta, \widetilde{\zeta}) \\
&\in \boldsymbol{R}^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_y^q \times \Gamma_0 \times \mathbb{R}^{q+p}, \boldsymbol{g})
\end{aligned}$$

and $M(t, t') := (\log t - \log t')(t - t')^{-1}$ (recall that the latter function belongs to $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ and is strictly positive for $t, t' \in \mathbb{R}_+$). The variable $z \in \Xi$ will not play any essential role, so we consider the z-independent case.

By Definition 1.8 the operator function $\widetilde{p}(t, \widetilde{\tau}, \widetilde{\zeta})$ has the form

$$\widetilde{p}(t, \widetilde{\tau}, \widetilde{\zeta}) = \widetilde{p}_{\mathrm{edge}}(t, \widetilde{\tau}, \widetilde{\zeta}) + \widetilde{p}_{\mathrm{int}}(t, \widetilde{\tau}, \widetilde{\zeta}) + \widetilde{g}(t, \widetilde{\tau}, \widetilde{\zeta}),$$

cf. the formula (22), which gives us

$$p(t, \tau, \zeta) = p_{\mathrm{edge}}(t, \tau, \zeta) + p_{\mathrm{int}}(t, \tau, \zeta) + g(t, \tau, \zeta) \tag{42}$$

when we pass to $p_{\mathrm{edge}}(t, \tau, \zeta) := \widetilde{p}_{\mathrm{edge}}(t, t\tau, t\zeta)$, etc. In this notation we may interpret $\widetilde{p}_{\mathrm{int}}(t, \widetilde{\tau}, \widetilde{\zeta})$ as an element of $C^\infty(\overline{\mathbb{R}}_+, L_{\mathrm{cl}}^\mu(2\mathbb{W}; \mathbb{R}_{\widetilde{\tau}, \widetilde{\zeta}}^{1+p}))$, and we can apply the formula (40). Let us now write $\operatorname{op}_t(p)(\zeta) = \operatorname{op}_t(\varphi(t'/t)p)(\zeta) + \operatorname{op}_t(r)(\zeta)$ where $r(t, t', \tau, \zeta)$ is given in Theorem 2.1. By integration by parts for every $N \in \mathbb{N}$ we can write $\operatorname{op}_t(r)(\zeta) = \operatorname{op}_t(r_N)(\zeta)$, where $r_N(t, t', \tau, \eta) = ((1 - \varphi(t'/t))(t'/t - 1)^{-N} D_\tau^N \widetilde{p}(t, t\tau, t\zeta))$, which belongs to $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{Y}^{\mu-N}(W, \boldsymbol{g}; \mathbb{R}^{1+p}))$. This gives us the desired characterisation of $\operatorname{op}_t(r)(\zeta)$.

Let $\{U_1, \ldots, N_N\}$ be an open covering of Y and fix charts $\theta_j : U_j \to \mathbb{R}^q$, $j = 1, \ldots, N$. Moreover, let $\{\varphi_1, \ldots, \varphi_N\}$ be a subordinate partition of unity, and $\{\psi_1, \ldots, \psi_N\}$ a system of functions $\psi_j \in C_0^\infty(U_j)$ such that $\varphi_j \psi_j = \varphi_j$ for all j. Then the operator function $\widetilde{p}(t, t\tau, t\zeta)$ can be written as

$$\widetilde{p}(t, \widetilde{\tau}, \widetilde{\zeta}) = \sum_{j=1}^N \varphi_j \widetilde{p}_{\mathrm{edge},j}(t, \widetilde{\tau}, \widetilde{\zeta})\psi_j + \widetilde{p}_{\mathrm{int}}(t, \widetilde{\tau}, \widetilde{\zeta}) + \widetilde{p}_\infty(t, \widetilde{\tau}, \widetilde{\zeta})$$

with a remainder $\widetilde{p}_\infty(t, \widetilde{\tau}, \widetilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{Y}^{-\infty}(W, \boldsymbol{g}; \mathbb{R}_{\widetilde{\tau}, \widetilde{\zeta}}^{1+p}))$, cf. Definition 1.8 (iii). This gives us, using (41),

$$\begin{aligned}
\operatorname{op}_t(\varphi(t'/t)p)(\zeta) &= \sum_{j=1}^N \varphi_j \operatorname{op}_t[\varphi(t'/t)(\theta_j^{-1})_* \operatorname{op}_y(\boldsymbol{a}_j)](\zeta)\psi_j \\
&+ \operatorname{op}_M^{\frac{1}{2}}(h_{\mathrm{int}})(\zeta) + \operatorname{op}_t(\varphi(t'/t)p_\infty)(\zeta).
\end{aligned}$$

Here $\boldsymbol{a}_j(t, y, \widetilde{\tau}, \eta, \widetilde{\zeta}) = \widetilde{\boldsymbol{a}}_j(t, y, t\tau, \eta, t\zeta)$ for corresponding $\widetilde{\boldsymbol{a}}_j(t, y, \widetilde{\tau}, \eta, \widetilde{\zeta}) \in \boldsymbol{R}^\mu(\overline{\mathbb{R}}_+ \times \mathbb{R}_y^q \times \mathbb{R}_{\widetilde{\tau}, \eta, \widetilde{\zeta}}^{1+q+p}, \boldsymbol{g})$, $j = 1, \ldots, N$. From Theorem 3.1 we obtain Mellin symbols $\widetilde{h}_j(t, y, w, \eta, \widetilde{\zeta}) \in$

$\boldsymbol{R}^{\mu}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q \times \mathbb{C}_w \times \mathbb{R}^{q+p}_{\eta,\zeta})$ such that $\mathrm{op}_t[\varphi(t/t')(\theta_j^{-1})_* \, \mathrm{op}_y(a_j)](\zeta) = \mathrm{op}_M^{\frac{1}{2}}(H_j)(\zeta)$ for $H_j(t,w,\zeta) = (\theta_j^{-1})_* \, \mathrm{op}_y(h_j)(t,w,\zeta)$ for every $\zeta \in \mathbb{R}^p$. Setting

$$h_{\mathrm{edge}}(t,w,\zeta) = \sum_{j=1}^{N} \varphi_j H_j(t,w,\zeta) \psi_j$$

it follows that

$$\mathrm{op}_t(\varphi(t'/t)p)(\zeta) = \mathrm{op}_M^{\frac{1}{2}}(h_{\mathrm{edge}} + h_{\mathrm{int}})(\zeta) + \mathrm{op}_t(\varphi(t'/t)p_\infty)(\zeta).$$

It remains to consider $\mathrm{op}_t(\varphi(t'/t)p_\infty)(\zeta)$. By construction we have

$$p_\infty(t,\tau,\zeta) = \widetilde{p}_\infty(t,t\tau,t\zeta)$$

for some $\widetilde{p}_\infty(t,\widetilde{\tau},\widetilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{Y}^{-\infty}(W,\boldsymbol{g};\mathbb{R}^{1+p}_{\widetilde{\tau},\widetilde{\zeta}}))$. We obtain that there is an

$$\widetilde{h}_\infty(t,w,\widetilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{Y}^{-\infty}(W,\boldsymbol{g};\mathbb{C}_w \times \mathbb{R}^p_{\widetilde{\zeta}}))$$

such that

$$\mathrm{op}_t(\varphi(t'/t)p_\infty)(\zeta) = \mathrm{op}_M^{\frac{1}{2}}(h_\infty)(\zeta) \tag{43}$$

for $h_\infty(t,w,\zeta) = \widetilde{h}_\infty(t,w,t\zeta)$. However, the way to show (43) is nearly precisely the same as the corresponding part of the proof of [8, Theorem 3.2]. The point is that the variables in the kernels of smoothing operators along W are not touchend by the corresponding Mellin reformulation of $\mathrm{op}_t(\varphi(t'/t)p_\infty)(\zeta)$; so the method is essentially reduced to the one-dimensional case $\mathbb{R}_+ \ni t$ with parameter. $\qquad\square$

3.2   Green remainders in compositions

We now characterise the remainders in the composition of corner symbols of Section 2.2, namely (33), (34), (36), (37), (39). They consist again of several summands, and we only look at some typical terms. The remaining ones can be treated in an analogous manner. For convenience, we will ignore the dependence on z which is not the essential point. The first summand of G_1 is the operator function

$$G(\zeta) := \omega_2(t[\zeta])A_0(\zeta)(\omega_0(t'[\zeta]) - (\omega_1(t'[\zeta]))B_1(\zeta)(1 - \omega_2(t''[\zeta])).$$

Let $\varphi \in C_0^\infty(\mathbb{R}_+)$ be a function such that $\varphi \equiv 1$ on $\mathrm{supp}(\omega_0 - \omega_1)$. Then G can be written as the composition of

$$C(\zeta) := \omega_2(t[\zeta])A_0(\zeta)(\omega_0(t'[\zeta]) - (\omega_1(t'[\zeta]))$$

and

$$D(\zeta) := \varphi(t[\zeta])B_1(\zeta)(1 - \omega_2(t'[\zeta])).$$

As noted in Section 2.2, we have $\omega_2 \equiv 0$ on $\mathrm{supp}(\omega_0 - \omega_1)$; so the smoothing effect in the composition comes from $C(\zeta)$. In order to verify that $G(\zeta)$ has the properties of Definition 1.6 (ii) we first show that

$$G(\zeta) \in C^\infty(\mathbb{R}^p_\zeta, \mathcal{L}(\mathcal{K}^{s,(\gamma+\nu,\theta+\nu)}(W^\wedge), \mathcal{S}^{(\gamma-\mu+\varepsilon,\theta-\mu+\varepsilon)}(W^\wedge))) \tag{44}$$

for every $s \in \mathbb{R}$ and a certain $\varepsilon > 0$. The considerations for the pointwise adjoints are analogous and left to the reader. For (44) we first need that

$$G(\zeta) : \mathcal{K}^{s,(\gamma+\nu,\theta+\nu)}(W^\wedge) \to \mathcal{S}^{(\gamma-\mu+\varepsilon,\theta-\mu+\varepsilon)}(W^\wedge)$$

for every ζ. This will be a consequence of the continuity of the operators

$$D(\zeta) : \mathcal{K}^{s,(\gamma+\nu,\theta+\nu)}(W^\wedge) \to \mathcal{K}^{s-\nu(\gamma,\theta+\delta)}(W^\wedge) \qquad (45)$$

for every $\delta > 0$, and of

$$C(\zeta) : \mathcal{K}^{s-\nu,(\gamma,\theta+\delta)}(W^\wedge) \to \mathcal{S}^{(\gamma-\mu+\varepsilon,\theta-\mu+\varepsilon)}(W^\wedge) \qquad (46)$$

for some $\varepsilon > 0$. The continuity of (45) is a consequence of the arguments in the proof of [4, Theorem 3.8]. What concerns $C(\zeta)$, the factors ω_2 on the left and $\omega_0 - \omega_1$ on the right have an excision effect for the distributional kernel off the diagonal with respect to $(t, t') \in \mathbb{R}_+ \times \mathbb{R}_+$. This allows us to repeatedly apply integrations by part under the Mellin oscillatory integral with respect to the t-variable which defines $A_0(\zeta)$, cf. the formula (29). Integrations by parts have the effect that the Mellin symbol h is to be differentiated with respect to the Mellin covariable w as often as we want. We then see that $C(\zeta)$ takes values in $\mathcal{Y}^{-\infty}(W^\wedge, (\gamma + \nu, \gamma); \mathbb{R}_\zeta^p)$ which are just smoothing Green operators in the edge calculus on $W^\wedge$. As such they improve the r-weight of argument functions by an $\varepsilon > 0$. At the same time, since h is holomorphic in w, the flatness in t remains preserved, i.e., the t-weight in the image under (46) can be taken as any $0 < \varepsilon \leq \delta$. Finally, the Schwartz property in t for $t \to \infty$ is a consequence of the fact that $C(\zeta)$ contains the factor ω_2 from the left. The smoothness of (44) in $\zeta \in \mathbb{R}^q$ is straightforward. What remains is to show that $G(\zeta)$ is a classical symbol of order $\mu + \nu$. Assuming for the moment that $h(t, w, \zeta) = \widetilde{h}(w, t\zeta)$ and, similarly, $q(t, \tau, \zeta) = \widetilde{q}(t\tau, t\zeta)$ (with obvious natation) we have

$$G(\lambda\zeta) = \lambda^{\mu+\nu} \kappa_\lambda G(\zeta) \kappa_\lambda^{-1}$$

for all $\lambda \geq 1$, and $|\zeta| > R$ for some $R > 0$ large enough. This shows in this case that $G(\zeta)$ is classical in ζ, as desired, cf. the formula (19). In the general case, i.e., when $\widetilde{h}(t, w, \widetilde{\zeta})$ and $\widetilde{q}(t, \widetilde{\tau}, \widetilde{\zeta})$ explicitly depend on t (smooth up to $t = 0$), then we can apply a tensor-product argument analogously as in the proof of [4, Theorem 3.8]. This shows that the first summand of (33) is a Green symbol in the edge calculus of second generation. To treat the second summand of G_1 which is of the form

$$G := (\omega_1 - \omega_2) A_0 (\omega_0 - \omega_1) B_1 (1 - \omega_0)$$

we choose another cut-off function ω_3 with the property $\omega_2 \prec \omega_1 \prec \omega_3 \prec \omega_0$ (this is always possible). This allows us to write

$$G = H + L$$

for $H = (\omega_1 - \omega_2) A_0 (1 - \omega_3)(\omega_0 - \omega_1) B_1 (1 - \omega_0)$, $L = (\omega_1 - \omega_2) A_0 \omega_3 (\omega_0 - \omega_1) B_1 (1 - \omega_0)$. We can write $H = CD$ for

$$C := (\omega_1 - \omega_2) A_0 (1 - \omega_3)\varphi, \ D := (\omega_0 - \omega_1) B_1 (1 - \omega_0)$$

for any $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi \equiv 1$ on $\mathrm{supp}(\omega_0 - \omega_1)$. Since $1 - \omega_3 \equiv 0$ on $\mathrm{supp}(\omega_1 - \omega_2)$, the factor C is smoothing. Herefore H can be treated in a similar manner as (44). Moreover, we can write

$$L = \widetilde{C}\widetilde{D}$$

for $\widetilde{C} := (\omega_1 - \omega_2)A_0(\omega_0 - \omega_1)$, $\widetilde{D} := \varphi \omega_3 B_1(1 - \omega_0)$, for any $\varphi \in C_0^\infty(\mathbb{R}_+)$ as before. Since $1 - \omega_0 \equiv 0$ on $\mathrm{supp}\,\omega_3$, the factor $\widetilde{D}$ is smoothing, and thus we can treat L again along the lines of (44). In other words, we have characterised G_1 as a Green symbol in the sense of Definition 1.6 (ii).

Let us now pass to G_2. The first summand of (34) can be written as a product of

$$C := (1 - \omega_0)A_1(\omega_1 - \omega_2), D := \varphi B_0 \omega_0,$$

for a $\varphi \in C_0^\infty(\mathbb{R}_+)$ which is equal to 1 on $\mathrm{supp}(\omega_1 - \omega_2)$. For similar reasons as before C is smoothing, so we can argue in an analogous manner as for (44). To treat the second summand of (34) we choose a cut-off function ω_3 such that $\omega_2 \prec \omega_3 \prec \omega_1 \prec \omega_0$ and write

$$(\omega_0 - \omega_1)A_1(\omega_1 - \omega_2)B_0\omega_2 = H + L$$

for $H := (\omega_0 - \omega_1)A_1\omega_3(\omega_1 - \omega_2)B_0\omega_2$, $L := (\omega_0 - \omega_1)A_1(1 - \omega_3)(\omega_1 - \omega_2)B_0\omega_2$.

In H the factor $(\omega_0 - \omega_1)A_1\omega_3$ is smoothing and in L the factor $(1 - \omega_3)B_0\omega_2$. So the desired characterisation of the terms can be obtained by the same scheme as before. The terms G_3 and G_4 can also be treated in that manner. So it remains to consider G_5. This is obviously of the form

$$
\begin{aligned}
G_5 &= (\omega_0 - \omega_1)A_1(\omega_2 - \omega_1)(B_1 - B_0)(\omega_0 - \omega_2) \\
&+ (\omega_1 - \omega_2)A_0(\omega_0 - \omega_1)(B_1 - B_0)(\omega_0 - \omega_2).
\end{aligned}
$$

The terms are all concentrated on a compact subset of $\mathbb{R}_+$. The operator $B_1 - B_0$ is smoothing, since B_0 is the Mellin quantisation of B_1. This completes the characterisation of the Green remainders in the composition formula.

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