Pseudodifferential subspaces and their applications in elliptic theory

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Abstract

The aim of this paper is to explain the notion of subspace defined by means of pseudodifferential projection and give its applications in elliptic theory. Such subspaces are indispensable in the theory of well-posed boundary value problems for an arbitrary elliptic operator, including the Dirac operator, which has no classical boundary value problems. Pseudodifferential subspaces can be used to compute the fractional part of the spectral Atiyah–Patodi–Singer eta invariant, when it defines a homotopy invariant (Gilkey's problem). Finally, we explain how pseudodifferential subspaces can be used to give an analytic realization of the topological K-group with finite coefficients in terms of elliptic operators. It turns out that all three applications are based on a theory of elliptic operators on closed manifolds acting in subspaces.

Keywords: elliptic operator, boundary value problem, pseudodifferential subspace, dimension functional, η-invariant, index, mod$n$-index, parity condition

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Introduction

Pseudodifferential subspaces and boundary value problems. A subspace (of some space of functions) is said to be *pseudodifferential* if it is determined by a projection that is a pseudodifferential operator.

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The notion of pseudodifferential projections (and subspaces) goes back to the work of Hardy, who defined the celebrated Hardy space as the range of a pseudodifferential projection in the space of square integrable functions on the circle; since then, pseudodifferential subspaces were used in the theory of Toeplitz operators (Gohberg–Krein [37]), in the proof of the finiteness theorem for classical boundary value problems (Calderón, Seeley [26, 67, 69]), in the construction of asymptotics of eigenvalues (Birman–Solomyak [14]) and in other places.

Subspaces also have significant applications in studies related to topology. Let us give some examples. In the paper of Wojciechowski [73] (an extended account of the results can be found in [18]) it was shown that the space of projections that differ from a fixed (say, pseudodifferential) projection by a compact operator is a classifying space for K-theory. This result is similar to the Atiyah–Jänich theorem [6], which gives a realization of the classifying space for K-theory in terms of Fredholm operators. Subspaces defined by pseudodifferential projections served as a prototype for the definition by Kasparov [42] and Brown–Douglas–Fillmore [25] of the odd analytic K-homology.

Pseudodifferential subspaces are also important in modern elliptic theory. Indeed, it is well known that by no means all elliptic operators on a manifold with boundary have well-defined (Fredholm) boundary value problems. In other words, for a general elliptic operator one cannot impose boundary conditions such that the so-called Shapiro–Lopatinskii condition is satisfied (e.g., see [41]). Unfortunately, the class of operators for which such conditions do not exist includes many important operators such as Cauchy–Riemann, Dirac, signature operators and others. The following question emerges naturally: is there a natural extension of elliptic theory which enables one to define Fredholm boundary value problems for geometric operators?

The answer to this question is contained in this paper. Let us explain it here in a few words. The simplest examples like the Cauchy–Riemann operator show that although these operators do not have Fredholm problems in Sobolev spaces, Fredholm problems do exist if the boundary values belong to some *subspaces* of the Sobolev spaces, e.g., the Hardy space (for the Cauchy–Riemann operator).

Important examples of well-defined boundary value problems were defined for Dirac operators in the series of papers by Atiyah, Patodi, and Singer [2, 3, 4] on spectral asymmetry. In particular, it was shown that for a suitable choice of the subspace the boundary value problem has the Fredholm property and its index was calculated. Actually, a closer look shows that pseudodifferential subspaces appear already in classical boundary value problems as the so-called Calderón–Seeley subspaces. In this case, the Shapiro–Lopatinskii (Atiyah–Bott) condition requires this subspace to be isomorphic to a Sobolev space. Of course, this is a very restrictive condition. Calderón–Seeley subspaces rarely satisfy it. On the other hand, in the framework of pseudodifferential subspaces this restrictive condition is absent and the existence of well-defined boundary value problems for arbitrary operators is a trivial statement: it suffices to take the Calderón–Seeley subspace as the space of boundary values of the problem.

Homotopy invariants of pseudodifferential subspaces and the Atiyah–Patodi–Singer η-invariant. The question of finding homotopy invariants of pseudodifferential subspaces is very important and interesting. It turns out that such invariants can be obtained from suitable index formulas for elliptic operators acting in subspaces. For sufficiently large classes of operators, the index is the sum of contributions of the principal symbol and of the subspaces. More precisely, there are index formulas [64, 65]

$$\operatorname{ind}(D, \widehat{L}_1, \widehat{L}_2) = f(\sigma) + d(\widehat{L}_1) - d(\widehat{L}_2), \tag{1}$$

where the triple $D, \widehat{L}_1, \widehat{L}_2$ defines an elliptic operator in subspaces $\widehat{L}_1$ and $\widehat{L}_2$, σ is the principal symbol of D, d is the dimension functional defined on the class of pseudodifferential subspaces, and f is a functional on the set of principal symbols of elliptic operators. If the subspaces $\widehat{L}_{1,2}$ satisfy the so-called parity condition (see below), $d(\widehat{L})$ is equal to the η-invariant of a self-adjoint operator having $\widehat{L}$ as its positive spectral subspace.

Let us note that the functional d is not a homotopy invariant of the *principal symbol* of the projection defining the subspace. However, one can show that the fractional part of this functional is homotopy invariant. Thus we have the problem of computing this fractional part in topological terms.

The interest in this problem originates from the fact that, as we mentioned earlier, if the parity condition is satisfied, then the functional d coincides with the η-invariant of Atiyah, Patodi, and Singer. The computation of the fractional part of the η-invariant has important applications (see below).

It is well known that the η-invariant of an elliptic self-adjoint operator is only a spectral invariant. However, in some classes of operators its fractional part defines a homotopy invariant. If the η-invariant is a homotopy invariant, one obtains the problem of computing it in topological terms. The first computation of the η-invariant was made in [3, 4], where operators with coefficients in flat bundles were considered.

Another interesting class of operators with homotopy invariant η-invariants was found by P. Gilkey [35]. This class consists of differential operators with the parity of their order opposite to the parity of the dimension of the manifold (*Gilkey's parity condition*). The homotopy invariant fractional part of the η-invariant is computed in this case in terms of subspaces (see [60], [62]). Such computations have important applications in geometry. Here we confine ourselves to a brief survey of results directly related to the present paper. For other aspects of the η-invariant, we refer the reader to remarkable surveys [70], [15], [52], [58].

For example, in the theory of pin^c bordisms (the distinctive feature of this theory is that the bordism group has elements of all arbitrary large orders 2^n) there is a natural question: what numerical invariants can be used to detect torsion elements of high order? It turns out that the answer can be given in terms of the η-invariant. The point is that odd-dimensional pin^c-manifolds bear the natural Dirac operator [33]. The fractional part of the η-invariant of this operator gives a (fractional) genus of pin^c-manifolds. It is proved in [12] that the Stiefel–Whitney numbers and this fractional genus classify pin^c-manifolds up to bordism.

We would like to note that formulas like (1) hold for many more operators than those specified by the parity condition. However, the invariants appearing in such formulas may not coincide with the η-invariant. Their determination is a very interesting question. For instance, an analog of the index formula (1) for the Dirac operator (its positive spectral subspace does not satisfy Gilkey's condition) involves the Kreck–Stolz invariant [45] and the Eells–Kuiper invariant [29] (see section 3 of the present paper). Let us recall that the former distinguishes homotopy classes of positive scalar curvature metrics, while the latter deals with the 28 exotic 7-spheres of Milnor.

Pseudodifferential subspaces and mod n **index theory**. The K-group of the cotangent bundle T^*X of a closed manifold classifies elliptic operators on X up to stable homotopy. It follows, in particular, that any element of the K-group $K_c(T^*X)$ can be realized as an elliptic operator on X. Does the same statement hold for the K-groups with finite coefficients $\mathbb{Z}_n$? We show (see also [60]) that the answer is "yes": on a smooth manifold, the elements of the K-group with coefficients are realized by pseudodifferential operators in subspaces.

We would like to mention that a similar problem appeared earlier. For instance, in the theory of spectral boundary value problems or b-pseudodifferential operators, Freed and Melrose studied so-called "modulo n" index theory in [30, 31], where manifolds with $\mathbb{Z}_n$-singularities appear as a geometric model [55, 71, 50]. Topological aspects of such manifolds were also studied. Mironov [49] defined the product of $\mathbb{Z}_n$-manifolds, which is again a $\mathbb{Z}_n$-manifold (the classification problem for products in the smooth situation is studied, for example, in [22]). From this point of view, the following problem suggested by Buchstaber is of interest: when is the index (modulo n) of elliptic operators multiplicative under Mironov's product? We note that for the signature operator the answer is "yes" [50].

Outline of the paper. In the first section, we explain the theory of elliptic operators in subspaces of Sobolev space on a closed manifold without boundary. Here we give two important results. The first concerns the necessary and sufficient conditions for the decomposition of the index of an elliptic operator into the sum of homotopy invariant contributions of the principal symbol and the subspaces in which it acts. The second result is the index formula for elliptic operators in pseudodifferential subspaces. Let us note one important fact: the index of an elliptic operator in subspaces is not determined by the principal symbol of the operator, but also depends on the subspaces.[1] In the second section, we discuss the theory of boundary value problems for elliptic operators in subspaces. We show how pseudodifferential subspaces appear in classical boundary value problems (i.e., boundary value satisfying the Shapiro–Lopatinskii condition), Atiyah–Patodi–Singer spectral boundary value problems [2], and boundary value problems for general elliptic operators [66]. We give index formulas for general operators and for specific operators. The third section explains the application of pseudodifferential subspaces to the problem of computation of the fractional part of the η-invariant for the case in which the latter has the homotopy invariance property. We give formulas for the η-invariant

[1] In fact, even the *complete symbols* of the operator and the projections defining the subspaces are insufficient to determine the index.

in terms of Poincaré duality in K-theory. Finally, in Section 4 we explain index theory "modulo n" of Freed and Melrose on $\mathbb{Z}_n$-manifolds and index theory "modulo n" on manifolds without boundary.

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1 Elliptic theory in subspaces on a closed manifold

In this section, we introduce elliptic operators acting in subspaces on a closed manifolds. We use this theory (which is very simple from the analytic point of view) to illustrate the main topological aspects of index theory in subspaces.

1.1 Statement of problems in subspaces

Subspaces and symbols. Let E be a complex vector bundle over a closed manifold M.

Definition 1 [13] A linear subspace $\widehat{L} \subset C^\infty(M, E)$ is said to be *pseudodifferential* if it can be represented as the range

$$\widehat{L} = \operatorname{Im} P$$

of a projection $P : C^\infty(M, E) \to C^\infty(M, E)$, $P^2 = P$, that is a classical (see [43]) pseudodifferential operator of order zero.

Just as pseudodifferential operators are distinguished in the set of all linear operators by the property that they have symbols (which is a function on the cotangent bundle of the manifold), pseudodifferential subspaces also have symbols.

Definition 2 *The symbol L of a pseudodifferential subspace $\widehat{L}$* is the vector bundle

$$L = \operatorname{Im} \sigma(P) \subset \pi^* E, \quad L \in \operatorname{Vect}(S^* M)$$

over the cosphere bundle S^*M, defined as the range of the principal symbol of P. Here $\pi : S^*M \to M$ is the natural projection.

The symbol of a subspace does not depend on the choice of a projection.

Example 1 *The Hardy space $\widehat{\mathcal{H}} \subset C^\infty(\mathbb{S}^1)$* of boundary values of holomorphic functions in the unit disc $D^2 \subset \mathbb{C}$ is pseudodifferential. Indeed, the orthogonal projection P onto the Hardy space is a pseudodifferential operator of order zero with principal symbol equal to (e.g., see [56])

$$\sigma(P)(\varphi, \xi) = \begin{cases} 1, & \xi = 1, \\ 0, & \xi = -1, \end{cases} \qquad (\varphi, \xi) \in S^*\mathbb{S}^1 = \mathbb{S}^1_+ \sqcup \mathbb{S}^1_-. \tag{2}$$

This gives us

$$\mathcal{H}(\varphi, \xi) = \operatorname{Im} \sigma(P)(\varphi, \xi) = \begin{cases} \mathbb{C}, & \text{if } \xi = 1, \\ 0, & \text{if } \xi = -1. \end{cases}$$

To put this another way, the symbol is one-dimensional on the first component of the cosphere bundle and zero-dimensional on the second component. However, in higher dimensions the space of boundary values of holomorphic functions is no longer pseudodifferential. The projections defining such subspaces are called *Szegö projections*. The notion of symbol of such operators and subspaces requires more subtle techniques (see [23]) and goes beyond the scope of the present paper.

Example 2 The space of sections of a vector bundle E is defined by the identity projection, and the symbol coincides with the pullback of E to S^*M.

Subspaces and self-adjoint operators. There is a convenient way to construct subspaces starting from self-adjoint elliptic operators. If A is an elliptic self-adjoint operator of order ≥ 0 on M, then the *nonnegative spectral subspace* denoted by $\widehat{L}_+(A)$ is the subspace generated by eigenvectors of A with nonnegative eigenvalues.

For example, the nonnegative spectral subspace of $-id/d\varphi$ on the circle is the Hardy space. It turns out that in the general case the spectral subspace is pseudodifferential and its symbol can be identified easily.

Proposition 1 *The symbol of the spectral subspace is equal to*

$$L_+(A) = L_+(\sigma(A)), \tag{3}$$

where $L_+(\sigma(A)) \in \mathrm{Vect}(S^*M)$ *is the subbundle in* π^*E *generated by eigenvectors of* $\sigma(A)$ *with positive eigenvalues.*

Formula (3) can be obtained if we rewrite the projection $\Pi_+(A)$ defining $\widehat{L}_+(A)$ as

$$\Pi_+(A) = \frac{|A| + A}{2|A|}, \quad |A| = \sqrt{A^2}$$

(we assume that A is invertible). By a theorem of Seeley [68], the symbol of the absolute value of an operator is equal to the absolute value of the symbol. Hence $\Pi_+(A)$ is a pseudodifferential operator with symbol

$$\sigma(\Pi_+(A)) = \frac{|\sigma(A)| + \sigma(A)}{2|\sigma(A)|} = \Pi_+(\sigma(A)).$$

This implies (3). □

Example 3 The space of closed forms of degree k on a compact manifold M without boundary is pseudodifferential, since it is the spectral subspace of the elliptic self-adjoint operator $d\delta - \delta d$ of order two (δ is the adjoint of the exterior derivative d).

It follows from Proposition 1 that an arbitrary smooth subbundle $L \subset \pi^*E$ is the symbol of a pseudodifferential subspace [13]. To prove this, it suffices to define an elliptic self-adjoint operator A with $L_+(\sigma(A)) = L$. This is obviously possible.

The infinite Grassmannian and the relative index of subspaces. We point out that there are many subspaces with the same symbol. For example, we do not change the symbol if we add any finite-dimensional subspace to a given subspace.

It is useful to make a comparison with the usual operators. Here the space of operators with a given symbol is contractible. In the case of subspaces, the corresponding space has a nontrivial topology. In more detail, let us fix the symbol L of a subspace. Let Gr_L be the (infinite) *Grassmannian* of subspaces with symbol equal to L.

Theorem 1 (Wojciechowski [73]) *Suppose that* $0 < \dim L < \dim E$. *Then the Grassmannian* Gr_L *is a classifying space for* K*-theory; i.e., the set of homotopy classes of maps* $[X, \mathrm{Gr}_L]$ *is isomorphic to the group* $K(X)$ *for any compact space* X.

The classifying map can be given explicitly in terms of one very important invariant of subspaces [24]. *The relative index of a pair of subspaces* $\widehat{L}_{1,2}$ with the same principal symbol is the index of the following Fredholm operator:

$$\mathrm{ind}(\widehat{L}_1, \widehat{L}_2) \stackrel{def}{=} \mathrm{ind}(P_{\widehat{L}_2} : \widehat{L}_1 \to \widehat{L}_2) \in \mathbb{Z},$$

where $P_{\widehat{L}_2}$ is the orthogonal projection onto $\widehat{L}_2$. The relative index is sometimes referred to as the relative dimension of subspaces, since if one of the subspaces is inside another, then it coincides with the corresponding codimension.

Now we use the relative index to give an explicit formula for the isomorphism in the theorem of Wojciechowski: the map takes a family $\{\widehat{L}_x\}_{x\in X}$ of subspaces to the relative index $\mathrm{ind}(\widehat{L}_x, \widehat{L}) \in K(X)$ with some given subspace $\widehat{L}$. It follows from this theorem that the Grassmannian has countably many connected components and two subspaces are homotopic if and only if their relative index is zero.

Operators in subspaces.

Definition 3 [64] *A pseudodifferential operator of order* m *in subspaces* is a triple $(D, \widehat{L}_1, \widehat{L}_2)$, where

$$D : \widehat{L}_1 \longrightarrow \widehat{L}_2$$

is a linear operator acting between pseudodifferential subspaces. We assume that D is a restriction of a pseudodifferential operator of order m acting in the ambient spaces of sections.

For operators in subspaces, it is easy to prove most of the analytic facts of elliptic theory, such as the symbolic calculus, ellipticity, smoothness of solutions and so on.

Definition 4 *The symbol of an operator in subspaces* is the vector bundle homomorphism

$$\sigma(D) : L_1 \longrightarrow L_2. \tag{4}$$

The symbol is well defined, since the condition $D\widehat{L}_1 \subset \widehat{L}_2$ can be restated in terms of the projections defining $\widehat{L}_{1,2}$ as $P_2DP_1 = DP_1$. If we consider the symbols of operators, the latter equality gives (4). Note finally that an arbitrary homomorphism (4) is the symbol of some operator in subspaces.

Elliptic operators.

Definition 5 An operator in subspaces is *elliptic* if its principal symbol (4) is an isomorphism.

Theorem 2 *An elliptic operator D of order m has the Fredholm property as an operator*

$$D : H^s(M, E_1) \supset \overline{\widehat{L}_1} \longrightarrow \overline{\widehat{L}_2} \subset H^{s-m}(M, E_2),$$

in the closures of the subspaces $\widehat{L}_{1,2} \subset C^\infty(M, E_{1,2})$ in the Sobolev norm.

To prove the theorem, it suffices to take as a regularizer an arbitrary operator from $\widehat{L}_2$ to $\widehat{L}_1$ with symbol $\sigma(D)^{-1} : L_2 \longrightarrow L_1$. The desired properties of the regularizer follow from the standard composition formulas. □

The index does not depend on the Sobolev smoothness exponent s and is denoted by $\operatorname{ind}(D, \widehat{L}_1, \widehat{L}_2)$.

As opposed to analytical aspects, the topological aspects of index theory in subspaces have new effects compared with the Atiyah–Singer theory. We will describe these effects in the next section.

1.2 Index decompositions and dimensions of infinite-dimensional subspaces

The most important property of the index of pseudodifferential operators is its homotopy invariance, i.e., constancy for continuous deformations of operators. For operators in subspaces, the index remains constant also for deformations of the subspaces.

Proposition 2 *For a continuous family of Fredholm operators*

$$D_t : \operatorname{Im} P_t \longrightarrow \operatorname{Im} Q_t, \qquad t \in [0,1], \quad \operatorname{Im} P_t \in H_1, \operatorname{Im} Q_t \in H_2$$

in subspaces $\operatorname{Im} P_t, \operatorname{Im} Q_t$ of some fixed Hilbert spaces, the index remains constant. By continuity we mean the continuity of the family $D_t : H_1 \to H_2$ and continuity of the families P_t, Q_t.

Proof of this proposition can be obtained if we reduce our family to a family in fixed spaces. A reduction can be done by virtue of the following well-known fact: for a continuous family of projections, there exists a continuous family of invertible operators U_t realizing the equivalence of projections $P_t = U_tP_0U_t^{-1}$ (e.g., see [17]). □

As soon as the index is homotopy invariant, we arrive at the index problem: the index has to be computed in topological terms. However, unlike the index of the usual elliptic operators in sections of vector bundles, the index of operators in subspaces is not determined by the principal symbol of the operator. For example, all finite-dimensional operators have zero principal symbol, but their index can be any number. A closer look at the problem shows that the index is determined if we fix the the principal symbol and the subspaces

$$\operatorname{ind}\left(D, \widehat{L}_1, \widehat{L}_2\right) = f\left(\sigma\left(D\right), \widehat{L}_1, \widehat{L}_2\right).$$

Index decomposition problem. Thus there are two sorts of contributions to the index: of the finite-dimensional data of the problem (the principal symbol) and infinite-dimensional (the subspaces). A natural question arises: is the index equal to the sum

$$\operatorname{ind}\left(D, \widehat{L}_1, \widehat{L}_2\right) = f_1\left(\sigma\left(D\right)\right) + f_2\left(\widehat{L}_1, \widehat{L}_2\right), \tag{5}$$

of these two contributions? If such a decomposition of the index is possible, then how to obtain the corresponding index formula? Since the index is a homotopy invariant, we will also require that both contributions are homotopy invariant.

Let us analyze the index decomposition (5). We first make an obvious remark. If the subspaces were of finite dimension, then the index would be equal to zero plus the difference of dimensions of the spaces. This observation enables us to give the following important reformulation of the index decomposition problem.

Dimension functionals.

Definition 6 A homotopy invariant functional d on the set of subspaces is a *dimension functional* if it has the following property: for two subspaces with equal symbols,

$$d(\widehat{L}_1) - d(\widehat{L}_2) = \operatorname{ind}(\widehat{L}_1, \widehat{L}_2).$$

Remark 1 Usually dimension functionals are defined in terms of trace functionals. Namely, if $T : A \longrightarrow \mathbb{C}$ is a trace functional (this means that T is linear and $T(ab) = T(ba)$) on an operator algebra A that extends the usual operator trace on finite-rank operators. Then a dimension functional of a subspace $\widehat{L} = \operatorname{Im} P$ defined as the range of projection $P \in A$ is defined as $d(\widehat{L}) := T(P)$. Such extensions of the operator trace were studied by Kontsevich-Vishik [44] for algebras of pseudodifferential operators on smooth manifolds. We would also like to refer the reader to [38, 39, 46, 48, 51, 54] for some of the studies of traces on more general operator algebras and applications.

Lemma 1 *There exists an index decomposition* (5) *for operators in subspaces* $D : \widehat{L} \to C^\infty(M, F)$ *if and only if there exists a dimension functional on the set of subspaces.*

For the proof, it suffices to show that the difference $\operatorname{ind}(D, \widehat{L}) - d(\widehat{L})$ does not depend on the choice of the subspaces and is a homotopy invariant of the symbol. This is proved using the logarithmic property of the index in subspaces: if we take an elliptic operator and replace a subspace by a different subspace with the same principal symbol, then the index is changed by the relative index of subspaces. □

The assumption in the lemma that one of the subspaces is the space of vector bundle sections does not restrict generality, since an arbitrary operator $D:\widehat{L}_1\to\widehat{L}_2$ can be reduced to such a form by adding the identity operator in the orthogonal complement $\widehat{L}_2^{\perp}$.

Obstruction to index decomposition. As a rule, pseudodifferential subspaces are infinite-dimensional (in the usual sense). Hence it is no wonder that there is an obstruction to the existence of dimension functionals. It is most convenient to describe this obstruction using self-adjoint operators.

Suppose that the desired dimension functional exists. Consider a family of elliptic self-adjoint operators $A_t, t\in[0,1]$. Let us examine what happens with the corresponding family of spectral subspaces $\widehat{L}_+(A_t)$. This family may have discontinuities for smooth variations of the parameter: if some eigenvalue changes its sign, then the spectral subspace changes by a jump (a finite-dimensional subspace is either added to it if the sign changes from minus to plus, or subtracted in the opposite case.Thus the value of the dimension functional of spectral subspaces has to change by the algebraic number of eigenvalues of the family that cross zero during the homotopy:

$$d(\widehat{L}_+(A_1))-d(\widehat{L}_+(A_0))=\left\{\begin{array}{c}\text{algebraic number of eigenvalues}\\ \text{crossing zero during the homotopy}\end{array}\right\}. \tag{6}$$

It turns out that there exist *periodic* homotopies of operators ($A_0=A_1$) for which the number on the right-hand side in (6) is nonzero (simple examples can be found in [59]). Thus we arrive at a contradiction. This shows that a universal dimension functional does not exist.

In other words, to define a dimension functional, one cannot consider the entire Grassmannian; rather one has to search for a dimension functional on some smaller classes of subspaces. It is not hard to give a criterion for the existence of such decompositions. Before we formulate the corresponding result exactly, let us introduce one notion appearing in this criterion.

Spectral flow [4]. Let $A_t, t\in[0,1]$ be a continuous family of elliptic self-adjoint operators. Then the number on the right-hand side of (6) is called the *spectral flow* of the family and denoted by $\operatorname{sf}\{A_\tau\}_{\tau\in[0,1]}$.

Note that this definition makes sense only in the case of general position, when the graph of the spectrum of the family is transversal to the line $\lambda=0$. A well-defined formula for the spectral flow can be obtained if we put the objects in general position (see [47], [57]). In our situation, this can be done explicitly: we take a small perturbation of the straight line $\lambda=0$ that makes it a broken line, see Fig. 1, with alternating horizontal and vertical segments. We only assume that the horizontal segments do not meet the spectrum of the family.

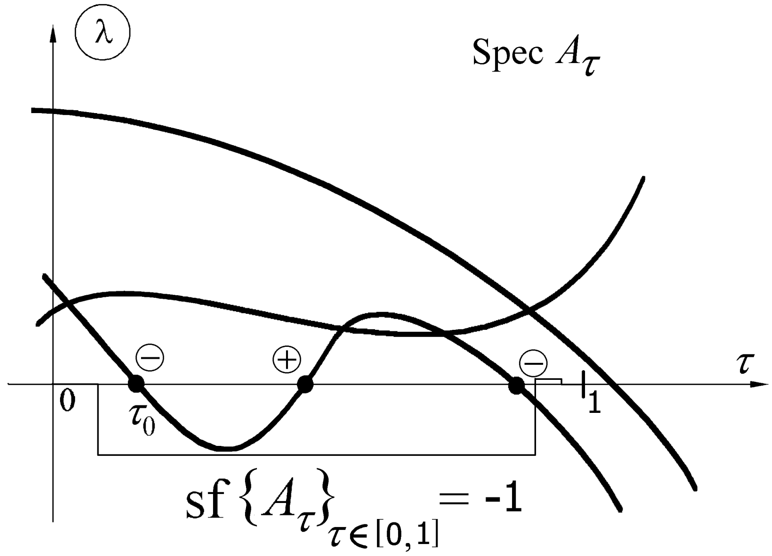

Figure 1: Spectral flow.

Denote the coordinates of vertices of our broken line as $\{(\tau_i,\lambda_i)\}_{i=0,N}$. Let us use this broken line to compute[2] the spectral flow as the net number of eigenvalues passing the broken line from below. In terms of relative indices, the corresponding formula is the sum over vertices

$$\operatorname{sf}\{A_\tau\}_{\tau\in[0,1]}=\sum_{i=1}^{N-1}\operatorname{ind}(\operatorname{Im}\Pi_{\lambda_i}(A_{\tau_i}),\operatorname{Im}\Pi_{\lambda_{i-1}}(A_{\tau_i})), \tag{7}$$

where $\Pi_\lambda(\cdot)$ is the spectral projection of a self-adjoint operator corresponding to eigenvalues greater than or equal to λ. One can show that the spectral flow is well defined; i.e., this number does not depend on the choice of a broken line.

Using this formula as the definition of the spectral flow, it is not hard to obtain Eq. (6). Let us now state the necessary and sufficient conditions of the existence of index decompositions.

Criterion of index decompositions [59]. Let us fix a subspace Σ in the space of symbols of all pseudodifferential subspaces on M, and let Gr_Σ be the Grassmannian of all pseudodifferential subspaces with symbols in Σ.

Theorem 3 *There exists a dimension functional on the Grassmannian* Gr_Σ *if and only if for every periodic family* $\{A_\tau\}_{\tau\in\mathbb{S}^1}$ *of elliptic self-adjoint operators one has*

$$\operatorname{sf}\{A_\tau\}_{\tau\in\mathbb{S}^1}=0$$

whenever the symbols of the spectral projections of the operators A_t *belong to* Σ *for all* t.

[2] Or, speaking rigorously, define.

Sketch of proof. The *necessity* of the vanishing of the spectral flow follows from (6).

Sufficiency. For each connected component $\Sigma_\alpha \subset \Sigma$, let us choose one elliptic operator A_α with the symbol of the spectral subspace $L_+(A_\alpha)$ in Σ_α. We shall consider these operators as reference points; in particular, we define the dimension functional to be zero on them.

Let now A be an elliptic operator on M. Then its principal symbol is an element of some Σ_α. Hence there is a homotopy $\{A_t\}_{t\in[0,1]}$ between A and A_α. Now we can define the dimension functional to be equal to the spectral flow of the homotopy:

$$d\left(\widehat{L}_+(A)\right) \stackrel{\text{def}}{=} \text{sf}\{A_t\}_{t\in[0,1]}.$$

□

The assumptions of this theorem can be verified effectively. Indeed, the spectral flow of a periodic family $\mathcal{A} = \{A_t\}_{t\in\mathbb{S}^1}$ of elliptic self-adjoint operators on a closed manifold M is computed by the Atiyah–Patodi–Singer formula [4]

$$\text{sf}\,\{A_t\}_{t\in\mathbb{S}^1} = \left\langle \text{ch}L_+(\mathcal{A})\text{Td}\,(T^*M\otimes\mathbb{C})\,,\left[S^*M\times\mathbb{S}^1\right]\right\rangle. \qquad (8)$$

Here $\text{ch}L_+(\mathcal{A}) \in H^{ev}\left(S^*M\times\mathbb{S}^1\right)$ is the Chern character of the bundle

$$L_+(\mathcal{A}) \in \text{Vect}\left(S^*M\times\mathbb{S}^1\right)$$

defined by the principal symbol of the family, Td is the Todd class, and $\langle, [S^*M\times\mathbb{S}^1]\rangle$ stands for the pairing with the fundamental class.

Thus as a corollary we obtain the following criterion for the existence of index decompositions.

Theorem 4 (on index decompositions) *There exists an index decomposition for elliptic operators in subspaces of the Grassmannian* Gr_Σ *if and only if for an arbitrary periodic family of elliptic self-adjoint operators whose spectral subspaces have symbols in* Σ *the spectral flow is zero.*

Let us consider examples in which this condition is satisfied.

Example 4 (Gilkey's parity condition) Let Σ be the set of symbols of spectral subspaces of elliptic self-adjoint *differential* operators. The spectral flow of periodic families of elliptic operators from this class will be zero if the so-called *parity condition* is satisfied [35]:

$$\text{ord}A + \dim M \equiv 1 (\text{mod } 2).$$

For example, for first-order operators the spectral flow of a periodic family A_t is equal to the index of the *differential* operator $\partial/\partial t + A_t$ on the odd-dimensional manifold $M\times\mathbb{S}^1$. It is well known that such indices are trivial (e.g., see [56]).

Actually, the "differentiality" of operators in the parity condition has a geometric origin. Namely, the principal symbol of a differential operator of even order is invariant under the involution $\alpha : (x, \xi) \mapsto (x, -\xi)$. Therefore, the symbol of the spectral subspace is also invariant

$$\alpha^* L = L, \quad L \in \operatorname{Vect}(S^* M). \tag{9}$$

Such symbols are called *even*. Similarly, the symbols of spectral subspaces of odd-order differential operators are called *odd*. Odd symbols satisfy the condition

$$\alpha^* L \oplus L = \pi^* E,$$

where E stands for the ambient bundle ($L \subset \pi^* E$). In other words, the fibers of an odd symbol L are complementary subspaces at antipodal points of the cosphere bundle. This explains why the natural analog of Gilkey's parity condition for pseudodifferential operators requires that the symbol is even in odd dimensions and odd otherwise.

Let us restrict ourselves to these classes Σ (further examples and explicit index formulas will appear later in the paper, see also [59]).

1.3 Example. Index under Gilkey's parity condition

In this section, we obtain index decompositions for operators in even and odd subspaces. We first consider the even case.

Dimension of even subspaces. Let $\widehat{\mathrm{Even}}\,(M)$ be the set of even pseudodifferential subspaces on a manifold M. The Grothendieck group of the semigroup of homotopy classes of even subspaces is denoted by $K\big(\widehat{\mathrm{Even}}\,(M)\big)$.

Proposition 3 [64] *On an odd-dimensional manifold, one has*

$$(\mathbb{Z} \oplus K(M)) \otimes \mathbb{Z}\,[1/2] \simeq K\big(\widehat{\mathrm{Even}}\,(M)\big) \otimes \mathbb{Z}\,[1/2]\,. \tag{10}$$

Here $\mathbb{Z}\,[1/2]$ is the ring of dyadic numbers $k/2^n$, $k, n \in \mathbb{Z}$. The map takes each natural number k to a projection of rank k and each vector bundle $E \in \operatorname{Vect}(M)$ to a projection that defines E as a subbundle in a trivial bundle.

Corollary 1 *In odd dimensions, there exists a unique dimension functional* (*see Definition* 6)

$$d : \widehat{\mathrm{Even}}\,(M) \longrightarrow \mathbb{Z}\,[1/2]$$

that is additive and satisfies the normalization condition

$$d\,(C^\infty(M, E)) = 0. \tag{11}$$

The starting point of the proof is the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K(\widehat{\mathrm{Even}}(M)) \longrightarrow K(P^* M) \longrightarrow 0, \tag{12}$$

where $P^* X = S^* X / \mathbb{Z}_2$ is the projectivization of the cosphere bundle. The first map corresponds to the embedding of finite-dimensional subspaces in the even subspaces. The second is induced by the symbol map.

This sequence admits further simplification. Namely, the projection $P^*M \to M$ induces an isomorphism on K-groups modulo 2-torsion if $\dim M$ is odd [35]. Thus taking the tensor product of (12) by $\mathbb{Z}[1/2]$ (the product preserves the exactness!) we obtain the exact sequence

$$0 \longrightarrow \mathbb{Z}[1/2] \longrightarrow K(\widehat{\mathrm{Even}}(M)) \otimes \mathbb{Z}[1/2] \longrightarrow K(M) \otimes \mathbb{Z}[1/2] \longrightarrow 0.$$

The latter sequence is easy to split. The splitting map

$$K^0(M) \otimes \mathbb{Z}[1/2] \longrightarrow K\big(\widehat{\mathrm{Even}}(M)\big) \otimes \mathbb{Z}[1/2].$$

takes each vector bundle to the projection onto the space of its sections. The splitting gives the desired isomorphism (10). □

Index formula. To obtain an index formula for operators in even subspaces, it is also necessary to define a homotopy invariant of the principal symbol of the operator.

It turns out that the principal symbol of an elliptic operator in even subspaces defines the usual elliptic symbol, i.e., the symbol of elliptic operator in vector bundle sections. Indeed, for a symbol (L_1 and L_2 are even)

$$\sigma(D) : L_1 \to L_2,$$

the composition $\alpha^*(\sigma^{-1}(D))\,\sigma(D)$ takes L_1 to itself. Thus one defines the elliptic symbol

$$\alpha^*\left(\sigma^{-1}(D)\right)\sigma(D) \oplus 1 : \pi^*E \longrightarrow \pi^*E, \tag{13}$$

where we make use of the decomposition $\pi^*E = L_1 \oplus L_1^\perp$ into complementary bundles.

Theorem 5 [64] *The following index formula holds:*

$$\operatorname{ind}(D, \widehat{L}_1, \widehat{L}_2) = \frac{1}{2}\operatorname{ind}_t[\alpha^*\left(\sigma^{-1}(D)\right)\sigma(D) \oplus 1] + d(\widehat{L}_1) - d(\widehat{L}_2) \tag{14}$$

provided that the subspaces are even and the dimension of the manifold is an odd number. Here ind_t *is the topological index of Atiyah and Singer.*

Proof (sketch). 1) Let us take the contributions of the subspaces to the left-hand side of (14). Then we interpret the formula as an equality of two homotopy invariants of the principal symbol. Thus it is sufficient to verify the formula for one representative in each homotopy class of principal symbols. 2) The simplest representative can be obtained by Proposition 3. Namely, in geometric terms this proposition claims that the direct sum of 2^N copies of the symbol of the subspace is homotopic to the symbol lifted from the base. Such a homotopy can be lifted to a homotopy of operators in subspaces. 3) For an operator acting in spaces of vector bundle sections, both sides of (14) are computed by the Atiyah–Singer formula. They turn out to be equal. □

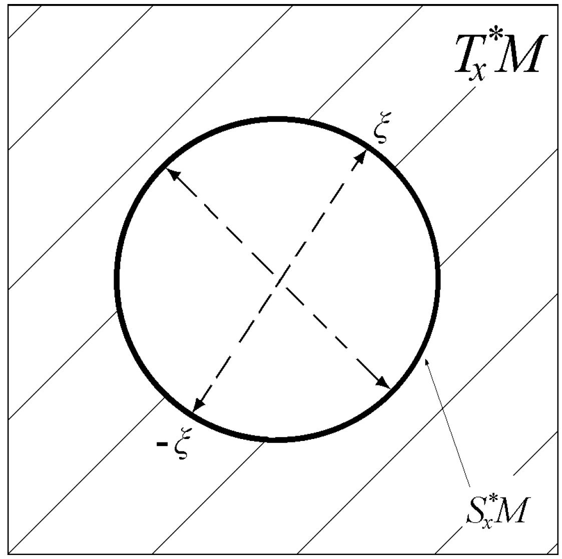

Figure 2: The blow-up ${}^pT^*M$ of the cotangent bundle

Remark 2 The contribution of the principal symbol to the index is of course computed by the Atiyah–Singer formula. However, there is a direct geometric realization of this contribution. Namely, consider the so-called *blow-up* ${}^pT^*M$ (e.g., see [19]) of the cotangent bundle T^*M along the zero section $M \subset T^*M$

$$ {}^pT^*M = \{(x,\gamma,\xi) \in P^*M \times T^*M | \, \xi \in \gamma\}. $$

In other words ${}^pT^*M$ is obtained from the cotangent bundle by two operations: we first delete a tubular neighborhood of the zero section and then identify antipodal points on the boundary (see Fig. 2).

The principal symbol of an operator in even subspaces defines a virtual vector bundle over the blow-up, and the contribution of the principal symbol to the index is expressed by the cohomological formula [64]

$$ \operatorname{ind}_t \left[\alpha^* \left(\sigma^{-1}(D)\right) \sigma(D) \oplus 1\right] = \langle \operatorname{ch}[\sigma(D)] \operatorname{Td}(T^*M \otimes \mathbb{C}), [{}^pT^*M]\rangle. $$

Thus at the cohomology level the only difference of this topological expression from the Atiyah–Singer formula is a different domain of integration.

Odd theory [65]. The main results of elliptic theory in even subspaces, like the dimension functional and the index formula, have analogs in elliptic theory in odd subspaces modulo some modifications: on an even-dimensional manifold, there exists a unique additive dimension functional of odd subspaces subject to the normalization

$$ d(\widehat{L}) + d(\widehat{L}^{\perp}) = 0, $$

where $\widehat{L}^\perp$ is the complementary bundle. The index formula in odd subspaces is (cf. (14))

$$\operatorname{ind}(D,\widehat{L}_1,\widehat{L}_2)=\frac{1}{2}\operatorname{ind}_t[\alpha^*\sigma(D)\oplus\sigma(D)]+d(\widehat{L}_1)-d(\widehat{L}_2).$$

Note that the proofs in the odd case are technically more complicated, since the symbols of odd subspaces cannot be interpreted as vector bundles over the projective space. For example, one has the following interesting fact.

Proposition 4 *The dimension of an odd bundle $L\subset\pi^*E$ over a manifold M of dimension n satisfies*

$$\left.\begin{array}{l} n=2k \\ n=2k+1 \end{array}\right\} \Rightarrow \dim L \text{ is divisible by } 2^{k-1}. \tag{15}$$

The proof is based on the well-known property of odd functions [34]: an odd function on $\mathbb{S}^n$ defines a section of the Hopf bundle

$$\gamma=\mathbb{S}^n\times\mathbb{C}/\left\{(x,t)\sim(-x,-t)\right\},$$

while an invertible vector-valued function defines a trivialization $l\gamma\simeq\mathbb{C}^l$. On the other hand, the Hopf bundle gives the generator of

$$K\left(\mathbb{RP}^{2k}\right)\simeq K\left(\mathbb{RP}^{2k+1}\right)\simeq\mathbb{Z}_{2^k}$$

(the Adams theorem). Therefore, $2^{\lfloor n/2\rfloor}$ divides $\dim E$, and we obtain the desired relation (15), since an odd vector bundle defined by the projection $p(\xi)$ gives us an invertible odd function

$$i\tau+\left(2p\left(\xi\right)-1\right)\left|\xi\right|.$$

□

2 Boundary value problems and subspaces

2.1 Classical boundary value problems

Let D

$$D:C^\infty\left(M,E\right)\longrightarrow C^\infty\left(M,F\right)$$

be an elliptic differential operator of order $m\geq 1$ on a manifold M with boundary $X=\partial M$. Such operators are never Fredholm: the kernel is infinite-dimensional. To define a Fredholm operator, D should be equipped with boundary conditions.

Boundary conditions. It is convenient to define the boundary conditions using the boundary operator

$$j_X^{m-1}:C^\infty\left(M,E\right)\longrightarrow C^\infty\left(X,E^m|_X\right),$$

which is defined in terms of the trivialization $X\times[0,1)\subset M$ of a neighborhood of the boundary with normal coordinate t. The operator

$$j_X^{m-1}u=\left(u|_X\,,-i\frac{\partial}{\partial t}u\Big|_X,...,\left(-i\frac{\partial}{\partial t}\right)^{m-1}u\Big|_X\right)$$

takes each function u to the value at the boundary of its jet in the normal direction.

Definition 7 *A classical boundary value problem* (see [41]) for a differential operator D is a system of equations of the form

$$\begin{cases} Du=f, & u\in H^s(M,E)\,, f\in H^{s-m}(M,F)\,,\\ Bj_X^{m-1}u=g, & g\in H^\sigma(X,G)\,,\end{cases}\tag{16}$$

where

$$B:\bigoplus_{k=0}^{m-1}H^{s-1/2-k}\left(X,\,E|_X\right)\longrightarrow H^\sigma(X,G)\tag{17}$$

is a pseudodifferential operator at the boundary; here we assume that the orders of the components $B_k:H^{s-1/2-k}\left(X,\,E|_X\right)\to H^\sigma(X,G)$ are $s-1/2-k-\sigma$.

If the smoothness exponent of the Sobolev space is sufficiently large, $s>m-1/2$, then the operator (D,B) is well defined.

Ellipticity of boundary value problems and the Calderón subspace.

The ellipticity condition for classical boundary value problems, known as the *Shapiro–Lopatinskii condition*, can easily be obtained with the use of the following result (see [67], [41]).

Theorem 6 (on the Calderón–Seeley subspace) *Let D be an elliptic differential operator on a manifold with boundary. Then the following assertions hold.*

1. *The cokernel of D is finite-dimensional.*

2. *The Calderón–Seeley space $j_X^{m-1}(\ker D)$ of jets at the boundary of the elements of the kernel is a pseudodifferential subspace. The boundary operator is Fredholm*

$$j_X^{m-1}:\ker D\longrightarrow j_X^{m-1}\ker D.$$

Denote the Calderón–Seeley subspace by $\widehat{L}_+(D)$. Its symbol $L_+(D)$ is a vector bundle over S^*X.

Definition 8 (*Shapiro–Lopatinskii condition*) A Boundary value problem is *elliptic* if the restriction of the principal symbol of the boundary condition B to the Calderón subspace is an isomorphism

$$\sigma(B):L_+(D)\longrightarrow\pi^*G.\tag{18}$$

In other words, the ellipticity of the boundary value problem is equivalent to the ellipticity of the boundary operator as an operator in subspaces. $\square$

Theorem 7 *A boundary value problem (D,B) for an elliptic operator D has the Fredholm property if and only if it is elliptic.*

This finiteness theorem follows directly from the properties of the Calderón subspace and the finiteness theorem for operators in subspaces.

The symbol of the Calderón–Seeley subspace can be computed easily. Let $(x,\xi')\in S^*X$ be a point on the cosphere bundle of the boundary. Let

$$L_+\left(D\right)_{x,\xi'}\subset E^m_x,$$

be the subspace of Cauchy data of solutions $u\left(t\right)$ of the ordinary differential equation

$$\sigma\left(D\right)\left(x,0,\xi',-i\frac{d}{dt}\right)u\left(t\right)=0,\qquad (x,\xi')\in S^*X$$

with constant coefficients on the half-line $\{t\geq 0\}$, that are bounded as $t\to+\infty$. Globally, this family of subspaces defines the smooth vector bundle

$$L_+\left(D\right)\subset \pi^*\left.E^m\right|_X,\qquad \pi: S^*X\to X.$$

It can be proved [67] that this bundle is the symbol of the Calderón–Seeley subspace.

Example 5 For the Laplace operator, the bundle $L_+(\Delta)$ coincides with the image of the diagonal embedding $\mathbb{C}\subset\mathbb{C}\oplus\mathbb{C}$. For the Cauchy–Riemann operator $\partial/\partial\overline{z}$ in the unit disk, L_+ is not constant:

$$L_+\left(\frac{\partial}{\partial\overline{z}}\right)\bigg|_{\mathbb{S}^1_+}\simeq\mathbb{C},\qquad L_+\left(\frac{\partial}{\partial\overline{z}}\right)\bigg|_{\mathbb{S}^1_-}=0,\ \text{where } S^*\mathbb{S}^1=\mathbb{S}^1_+\sqcup\mathbb{S}^1_-.$$

The Atiyah–Bott obstruction and index theorem for boundary value problems. The Shapiro–Lopatinskii condition (18) is a restrictive condition on the class of operators D, for which one can define elliptic boundary conditions. Indeed, if an elliptic boundary condition for D exists, then the bundle $L_+\left(D\right)\in\operatorname{Vect}\left(S^*X\right)$ is a pullback of some bundle on the base X. Such a pullback exists by no means for all operators (the simplest example for which the pullback does not exist is given by the Cauchy–Riemann operator).

The essence of this restriction was uncovered by Atiyah and Bott [7]. They showed that, up to a certain stabilization, the operators possessing elliptic boundary conditions are precisely those with symbols at the boundary homotopic to the symbols independent of the covariables. The situation can be represented by the following K-theory exact sequence:

$$\to K_c(T^*(M\setminus\partial M))\longrightarrow K_c(T^*M)\stackrel{\partial}{\longrightarrow}K_c(\partial T^*M)\to\ldots$$

Namely, elliptic symbols $\sigma(D)$ on M define elements of the group in the center (via the difference construction). On the other hand, the elements of the leftmost group correspond to symbols that are independent of the covariables in a neighborhood of the boundary. Thus the Atiyah–Bott result says that the existence of an elliptic boundary value problem is equivalent to the property that our element comes from $K_c(T^*(M \setminus \partial M))$, while the element obtained by the boundary map ∂ is the obstruction to the existence of elliptic boundary conditions (*the Atiyah–Bott obstruction*). Moreover, Atiyah and Bott showed that the choice of an elliptic boundary condition B explicitly determines some specific element in $K_c(T^*(M \setminus \partial M))$.

Let us note that there is a well-defined topological index map on $K_c(T^*(M \setminus \partial M))$, which, together with the Atiyah–Bott construction, gives an index formula for classical boundary value problems. The reader can find the proof of the index theorem for boundary value problems in [41].

Example 6 Consider the Euler operator

$$d + \delta : \Lambda^{ev}(M) \longrightarrow \Lambda^{odd}(M) \tag{19}$$

on a compact manifold with boundary. Here $\Lambda^{ev/odd}(M)$ are the spaces of even (odd) differential forms. As elliptic boundary conditions, we can take the absolute boundary conditions

$$j^*(*\omega) = g, \qquad j^* : \Lambda^{odd}(M) \to \Lambda^{odd}(X) \tag{20}$$

(where $*$ is the Hodge star operator). By Hodge theory on manifolds with boundary (e.g., see [36], [28]), the index of (19), (20) is equal to the Euler characteristic of M:

$$\operatorname{ind}(d + \delta, j^* *) = \chi(M).$$

However, the classical theory has one very significant drawback. Among the classical operators considered in index theory, only the Euler operator admits classical elliptic boundary value problems. The Dirac, Hirzebruch and Todd operators do not admit elliptic boundary conditions: even at a point $x \in M$ the principal symbol of these operators is a rational generator of $K_c(T_x^* M) \simeq \mathbb{Z}$ (e.g., see [56]) and hence is by no means homotopic to a constant symbol.

2.2 Spectral problems of Atiyah, Patodi, and Singer and general boundary value problems in subspaces

We saw in the previous section that many elliptic operators (e.g., Dirac and signature operator) do not have elliptic boundary conditions, since the Atiyah–Bott obstruction for these operators does not vanish. Since these operators are very important in applications, there naturally emerges a question of defining a class of elliptic boundary value problems for *general* elliptic operators, in particular those with a nontrivial Atiyah–Bott obstruction. Such a class of boundary value problems is naturally constructed using the following reasoning.

Recall that the ellipticity condition for a boundary value problem (D, B) requires the isomorphism

$$L_+(D) \stackrel{\sigma(B)}{\longrightarrow} \pi^* G \tag{21}$$

defined by the principal symbol of the boundary condition B. Meanwhile, the obstruction explained by the asymmetry of (21): the a priori general bundle $L_+(D)$ over S^*X must be isomorphic to a bundle of a very special form, i.e., a bundle lifted from X. Hence it is clear that the obstruction will disappear if we manage to make a generalization of the notion of boundary conditions such that we could insert an arbitrary vector bundle on S^*X into the right-hand side of (21). The simplest realization of this idea is given by the so-called spectral boundary value problems.

Atiyah–Patodi–Singer spectral boundary value problems [2]. Let D be an elliptic differential operator of order one. We shall assume that near the boundary it has a decomposition

$$D|_{\partial M\times[0,1)} = \gamma\left(\frac{\partial}{\partial t} + A\right),$$

where A is an elliptic self-adjoint operator on $X = \partial M$. The *spectral boundary value problem for* D is the system of equations

$$\begin{cases} Du = f, & u \in H^s(M, E), f \in H^{s-1}(M, F), \\ \Pi_+(A)\, u|_X = g, & g \in \operatorname{Im} \Pi_+(A). \end{cases} \tag{22}$$

This boundary value problem has the Fredholm property. The reader can prove the coincidence of the bundles $L_+(D)$ and $\operatorname{Im} \sigma(\Pi_+(A))$. Hence (21) is the identity map in this case. The statement of spectral problems for differential operators of any order can be found in [53].

General boundary value problems [18, 66]. For an elliptic operator D, consider the boundary value problems

$$\begin{cases} Du = f, & u \in H^s(M, E), f \in H^{s-m}(M, F), \\ Bj_X^{m-1} u = g, & g \in \operatorname{Im} P \subset H^\sigma(X, G), \end{cases} \tag{23}$$

which differ from classical boundary value problems (16) in the space of boundary data $\operatorname{Im} P$, which is a subspace of the Sobolev space at the boundary and is determined by a pseudodifferential projection P of order zero.

Definition 9 Boundary value problem (23) is said to be *elliptic* if the principal symbol of the operator of boundary conditions defines a vector bundle isomorphism

$$\sigma(B) : L_+(D) \to \operatorname{Im} \sigma(P),$$

i.e., the restriction of B to the Calderón subspace is an elliptic operator in subspaces.

The following finiteness theorem holds.

Theorem 8 *Boundary value problem* (23) *defines a Fredholm operator if and only if it is elliptic.*

The proof can be obtained from the theorem on the Calderón–Seeley subspace. □

Order reduction of boundary value problems. It is possible to reduce orders of boundary value problems. For classical boundary value problems, one can reduce the boundary value problem using order reduction to a pseudodifferential operator which is a multiplication operator near the boundary and does not require boundary conditions (see [41] or [61] for the description of the reduction procedure; note that the index of such zero-order operators is computed by the Atiyah–Singer formula [9], cf. [27]). For boundary value problems in subspaces, the same method enables one to reduce an arbitrary boundary value problem to a spectral problem for a first-order operator [61]. In addition, the pseudodifferential subspace of the spectral problem can be chosen to coincide with subspace of boundary data of the original problem. For this reason, we consider only spectral problems in the rest of this section.

2.3 Index of boundary value problems in subspaces

We have seen that subspaces are useful if we study analytical properties of spectral problems. In this section, we show that subspaces are also important in the study of topological aspects of these problems: many index formulas for operators in subspaces on closed manifolds (see Section 1) have natural analogs for boundary value problems. To save space, we will give only the formulations of the results.

The index of spectral boundary value problems is not determined by the principal symbol of the operator D. To have a definite index, we have to fix the principal symbol and the spectral subspace. It is impossible to decompose the index as a sum of homotopy invariant contributions of the symbol and the subspace. A decomposition exists if and only if for the class of spectral subspaces at the boundary there exists a dimension functional. Let us give two examples when explicit index formulas can be obtained.

2.4 Examples. The index of operators with parity condition. The index of the signature operator

The index of spectral problems in even subspaces. Consider spectral boundary value problems $(D, \Pi_+(A))$ on an even-dimensional manifold M and suppose additionally that the spectral subspace $\operatorname{Im}\Pi_+(A)$ is even. Finally, we assume that the principal symbol of A is an even function of the covariables.

It turns out that in this case $\sigma(D)$ has a natural continuation to the double of M. Recall that the double

$$2M = M\bigcup\nolimits_{\partial M} M$$

is obtained by gluing two copies of M along the boundary.

To construct the desired continuation, we consider two copies of the manifold. We take the symbol $\sigma(D)$ on the first copy and $\alpha^*\sigma(D)$ on the second copy. Here $\alpha : S^*M \longrightarrow$

S^*M is the antipodal involution of M. Near the boundary, the symbols $\sigma(D)$ and $\alpha^*\sigma(D)$ are

$$i\tau + a(x,\xi) \quad \text{and} \quad -i\tau + a(x,\xi).$$

It is clear that they are mapped one into another as we glue neighborhoods of the boundary:

$$x \to x, \quad t \to -t.$$

Thus the two symbols define an elliptic symbol $\sigma(D) \cup \alpha^*\sigma(D)$ on the double of M. This symbol defines the difference element

$$[\sigma(D) \cup \alpha^*\sigma(D)] \in K_c(T^*2M).$$

in the K-group with compact supports of the cotangent bundle of the double. We define the *topological index* of D to be half the usual topological index of the element on the double

$$\mathrm{ind}_t D \overset{def}{=} \frac{1}{2}\mathrm{ind}_t[\sigma(D) \cup \alpha^*\sigma(D)].$$

Theorem 9 [64] *For spectral boundary value problems in even subspaces, one has*

$$\mathrm{ind}(D, \Pi_+(A)) = \mathrm{ind}_t D - d(\mathrm{Im}\, \Pi_+(A)).$$

The proof is by analogy with the proof in the case of closed manifolds: one uses homotopies to reduce the spectral problem to the simplest form. In this case, the simplest spectral problem is a classical boundary value problem; i.e., its spectral subspace is the space of sections of a vector bundle. □

Remark 3 A similar index formula is valid for operators in odd subspaces. In this case, one defines the operator $D \cup \alpha^* D^{-1}$ on the double with symbol equal to $\alpha^*\sigma\left(D\right)^{-1}$ on the second copy of the manifold.

The index of the signature operator [2]. On a $4k$-dimensional oriented manifold M, consider the signature operator

$$d + d^* : \Lambda^+(M) \longrightarrow \Lambda^-(M),$$

where the $\Lambda^\pm(M)$ are subspaces of forms invariant under the involution

$$\alpha : \Lambda^*(M) \longrightarrow \Lambda^*(M), \quad \alpha|_{\Lambda^p(M)} = (-1)^{\frac{p(p-1)}{2}+k} * .$$

On the boundary of M, we have $\Lambda^\pm(M)|_{\partial M} \simeq \Lambda^*(\partial M)$. If we take a product metric in a neighborhood of the boundary, then the signature operator is equal to

$$\frac{\partial}{\partial t} + A$$

modulo vector bundle isomorphisms (see [2]). The *tangential signature operator* A acts on the boundary

$$A : \Lambda^* (\partial M) \longrightarrow \Lambda^* (\partial M), \quad A\omega = (-1)^{k+p} (d * - \varepsilon * d) \omega,$$

where for a form $\omega \in \Lambda^{2p} (\partial M)$ of even degree we have $\varepsilon = 1$, while for $\omega \in \Lambda^{2p-1} (\partial M)$ — $\varepsilon = -1$. This operator is elliptic and self-adjoint.

The index of the spectral boundary value problem can be computed by de Rham–Hodge theory:

$$\operatorname{ind} (d + d^*, \Pi_+) = \operatorname{sign} M - \dim H^* (\partial M)/2,$$

where $\operatorname{sign} M$ is the signature of a manifold with boundary.

We will obtain the index decomposition for the Dirac operator later in Section 3.8, since it involves a new invariant — the η-invariant of Atiyah, Patodi, and Singer.

3 The spectral η-invariant of Atiyah, Patodi, and Singer

3.1 Definition of the η-invariant

Let A be an elliptic self-adjoint operator of a positive order on a closed manifold M. Let us define the *spectral η-function*

$$\eta (A, s) = \sum_{\lambda_j \in \operatorname{Spec} A, \lambda_j \neq 0} \operatorname{sgn} \lambda_i |\lambda_i|^{-s} \equiv \operatorname{Tr} \left(A \left(A^2 \right)^{-s/2-1/2} \right).$$

It is analytic in the half-plane $\operatorname{Re} s > \dim M / \operatorname{ord} D$ (for these parameter values, the series is absolutely convergent).

Definition 10 [2] *The η-invariant* of the operator A is

$$\eta (A) = \frac{1}{2} (\eta (A, 0) + \dim \ker A) \in \mathbb{R}. \tag{24}$$

Remark 4 The spectral η-invariant can be understood as a kind of infinite-dimensional analog of the notion of signature of a quadratic form, since in finite dimensions a self-adjoint operator defines a quadratic form and the η-invariant of an invertible operator is equal to the signature modulo the factor 1/2.

Of course, for (24) to make sense, it is necessary to have the analytic continuation of the η-function to $s = 0$.

Theorem 10 [4],[32] *The η-function extends to a meromorphic function on the complex plane with possible poles at $s_j = \frac{\operatorname{ord} D - j}{\dim M}$, $j \in \mathbb{Z}_+$. At $s = 0$, the function is analytic.*

Let us note that the meromorphic continuation is a consequence of the expression of the η-function in terms of the ζ-function

$$\zeta(A,s)=\sum_{\lambda_j\in \operatorname{Spec} A}\lambda_i^{-s}$$

of positive operators:

$$\eta(A,s)=\frac{\zeta(A_+,s)-\zeta(A_-,s)}{2^s-1},\qquad \text{where } A_\pm=\frac{(3|A|\pm A)}{2}.$$

The meromorphic continuation for the ζ-function is well known (e.g., see [68]).

However, the analyticity of the η-function at the origin is more intricate. More precisely, the residue is equal to

$$\operatorname*{Res}_{s=0}\eta(A,s)=\frac{\zeta(A_+,0)-\zeta(A_-,0)}{\ln 2}. \tag{25}$$

The ζ-invariants in this formula can be expressed as integrals over M of some complicated expressions in the complete symbol of A. The integrand is in general nonzero! Nevertheless, Atiyah, Patodi, and Singer proved for odd-dimensional manifolds [4] and Gilkey [32] proved for even-dimensional manifolds that the residue is zero. Hence the η-function is holomorphic at the origin and the η-invariant is well defined.

Rather surprisingly, up to now there is no purely analytic proof of the analyticity of the η-function at the origin. The results cited earlier all rely on global topological methods. However, the triviality of the residue is proved by an explicit analytic computation for Dirac type operators in [16].

Example 7 On a circle of length 2π with coordinate φ, consider

$$A_t=-i\frac{d}{d\varphi}+t.$$

Here t is a real constant. Let us compute the η-invariant. The spectrum of A is the lattice $t+\mathbb{Z}$. Thus the η-invariant is a periodic function of t (with period 1). Assume that $0<t<1$. Gathering the eigenvalues in pairs, we obtain

$$\eta(A_t,s)=\sum_{n\geq 1}\left[(n+t)^{-s}-(n-t)^{-s}\right]+t^{-s}.$$

This series is absolutely convergent on the semiaxis $s>0$, and the limit as $s\to +0$ is $-2t+1$ (we use the Taylor expansion for the expression in the brackets); hence

$$\eta(A_t)=\frac{\eta(A_t,0)+\dim\ker A_t}{2}=\frac{1}{2}-\{t\},$$

where $\{\}\in[0,1)$ is the fractional part. Thus we see that for our smooth elliptic family A_t the family of η-invariants is only piecewise smooth. Moreover, the jumps (they are integral) happen as some eigenvalue of the operator changes its sign.

The behaviour of the η-invariant under deformations of the operator. In the last example, we observed the piecewise smooth variation of the η-invariant for smooth variation of operators. It turns out that the η-invariant has similar properties in the general case. More precisely, the following result holds.

Proposition 5 [4] *Let $A_t, t \in [0,1]$, be a smooth family of elliptic self-adjoint operators. Then the function $\eta(A_t)$ is piecewise smooth. It decomposes as the sum*

$$\eta(A_{t'}) - \eta(A_0) = \mathrm{sf}(A_t)_{t\in[0,t']} + \int_0^{t'} \omega(t_0)\,dt_0, \tag{26}$$

of a locally constant function, the spectral flow of Section 1.2, *and the smooth function*

$$\omega(t_0) = \left.\frac{d}{dt}\zeta(B_{t,t_0})\right|_{t=t_0} \in C^\infty[0,1],$$

where we use the ζ-invariant of the auxiliary family $B_{t,t_0} = |A_{t_0}| + P_{\ker A_{t_0}} + (t-t_0)\dot{A}_{t_0}$. Here $P_{\ker A}$ is the projection onto the kernel of A.

Proof (sketch). If the family is invertible, then one can easily write out the derivatives of the η- and ζ-functions:

$$\frac{d}{dt}\zeta(B_t, s) = -s\mathrm{Tr}\left(\dot{B}_t\, B_t^{-s-1}\right), \qquad \frac{d}{dt}\eta(A_t, s) = -s\mathrm{Tr}\left(\dot{A}_t\, \left(A_t^2\right)^{-\frac{1}{2}(s+1)}\right).$$

It is clear now that (26) holds for $s = t = 0$.

If the family is not invertible, then the decomposition (26) can be obtained making use of broken lines from the definition of spectral flow (see Fig. 1). This technique reduces us to the case of invertible families. □

Remark 5 (Singer) These properties motivate an interesting interpretation of the η-invariant, which is similar to the interpretation of index as the invariant labelling the connected components of the space of Fredholm operators. Consider the space of self-adjoint Fredholm operators. Atiyah and Singer [10] proved that this space consists of three connected components. Two components correspond to semibounded operators and are contractible. However, the third component (containing operators with spectrum unbounded in both directions) has a nontrivial topology. Let us denote it by $\mathcal{F}_s$. This space is a classifying space for odd K-theory:

$$[X, \mathcal{F}_s] \simeq K^1(X).$$

In particular, its first cohomology is $H^1(\mathcal{F}_s) \simeq \mathbb{Z}$. The generator of this group is given by the spectral flow of periodic families

$$[\mathrm{sf}] \in H^1(\mathcal{F}_s),$$

more precisely, the value of the cocycle sf on a loop $(A_t)_{t\in\mathbb{S}^1}$ is equal to the spectral flow along the loop. It turns out that the η-invariant provides a de Rham representative of this cohomology class (at least on the subspace of pseudodifferential operators). More precisely, let us define the 1-form*:* for the loop $(A_t)_{t\in[0,\varepsilon]}\subset\mathcal{F}_s$ in the space of *pseudodifferential operators,* we set

$$\omega(A_t)=\frac{d}{dt}\{\eta(A_t)\}\Big|_{t=0}.$$

Proposition 5 gives the equality of cohomology classes $-[\omega]$ and $[\mathrm{sf}]$, in other words, one has

$$-\int_{(A_t)_{t\in\mathbb{S}^1}}\omega=\mathrm{sf}\,(A_t)_{t\in\mathbb{S}^1}.$$

3.2 How to make η homotopy invariant?

The η-invariant for general operators is not homotopy invariant and takes arbitrary real values. However, for special classes of operators it is possible to define homotopy invariants using the η-invariant. To this end, it is necessary to require that both components in (26) are equal to zero. The triviality of the spectral flow sf can be achieved in two ways: either we consider only the fractional part of the η-invariant $\{\eta(A)\}\in\mathbb{R}/\mathbb{Z}$ (this is used in [4] when considering invariants of flat bundles, see also Section 3.3) or by requiring that the spectral flow is trivial for the operators considered (such situation appears for the signature operator or for the Dirac operator under the positive scalar curvature assumption, e.g., see [21]). To prove the vanishing of the smooth component of the variation, it is necessary to have a formula for the derivative of the ζ-function. R. Seeley [68] proved (see also [1] and [40]) that the value of the ζ-function at zero can be computed in terms of the principal symbol of the operator. Let us proceed to the formula. Let A be an elliptic self-adjoint nonnegative operator with complete symbol

$$\sigma(A)\sim a_m+a_{m-1}+a_{m-2}+\ldots.$$

Let us introduce the following recurrent family of symbols b_{-m-j}, $j\geq 0$:

$$\begin{aligned}&b_{-m-j}(x,\xi,\lambda)(a_m(x,\xi)-\lambda)+\\&+\sum_{\substack{k+l+|\alpha|=j,\\ l>0}}\frac{1}{\alpha!}(-i\partial_\xi)^\alpha b_{-m-k}(x,\xi,\lambda)(-i\partial_x)^\alpha a_{m-l}(x,\xi)=0.\end{aligned}\tag{27}$$

The symbols depend on auxiliary parameter λ. Then the ζ-invariant is

$$2\zeta(A)\overset{\text{def}}{=}\zeta(A,0)+\dim\ker A=\frac{1}{(2\pi)^{\dim M}\operatorname{ord}A}\int_{S^*M}dxd\xi\int_0^\infty b_{-\dim M-\operatorname{ord}A}(x,\xi,-\lambda)\,d\lambda.\tag{28}$$

Analyzing the symmetries of this formula, one can find a number of operator classes for which the derivative of the η-invariant is zero. Two such classes are considered in the next sections.

3.3 η-invariants and flat bundles

Recall that a vector bundle $\gamma \in \operatorname{Vect}(M)$ is *flat* if it is defined by locally constant transition functions. Consider an operator

$$A : C^\infty(M,E) \longrightarrow C^\infty(M,F).$$

Then we can define the *operator A with coefficients in the flat bundle*:

$$A \otimes 1_\gamma : C^\infty(M, E\otimes\gamma) \longrightarrow C^\infty(M, F\otimes\gamma).$$

It can be defined by patching together local expressions in coordinate charts for the direct sum of $\dim\gamma$ copies of A using the transition functions. To preserve the self-adjointness, one requires additionally that the transition functions for the flat bundle are unitary. Finally, if A is a pseudodifferential operator, then the operator with coefficients is well defined modulo infinitely smoothing operators.

Example 8 On the circle, the operator $-id/d\varphi + t$ is isomorphic to the operator $-id/d\varphi \otimes 1_\gamma$ with coefficients in γ, where the line bundle γ is defined by the transition function $e^{2\pi i t}$. The isomorphism

$$e^{-ti\varphi}\left(-i\frac{d}{d\varphi}\right)e^{it\varphi} = -i\frac{d}{d\varphi} + t$$

uses the trivialization $e^{it\varphi}$ of γ.

Proposition 6 [4] *The fractional part of the η-invariant is homotopy invariant in the class of direct sums*

$$A \otimes 1_\gamma \oplus (-\dim\gamma A)$$

with a given flat vector bundle γ.

To prove the proposition, one notes that $A\otimes 1_\gamma$ and $\dim\gamma A$ are locally isomorphic. Therefore, we obtain

$$\frac{d}{dt}\{\eta(A_t\otimes 1_\gamma)\} = \frac{d}{dt}\{n\eta(A_t)\}$$

by means of the locality of these derivatives, see (28). □

ρ-invariant [3]. Consider an oriented Riemannian manifold M of dimension $4k-1$. There is a self-adjoint Hirzebruch operator

$$A|_{\Lambda^{2p}(M)} = (-1)^{k+p}(d* - *d), \qquad A : \Lambda^{ev}(M) \longrightarrow \Lambda^{ev}(M).$$

In this case, the difference

$$\eta(A\otimes 1_\gamma) - \dim\gamma\eta(A) \in \mathbb{R}$$

defines a homotopy invariant. Indeed, by Hodge theory the kernels of A and $A\otimes 1_\gamma$ coincide with the corresponding cohomology of M (with a local coefficient system γ in the second case); hence their dimensions do not depend on the choice of metric on M. This difference is called the *ρ-invariant of manifold M and flat bundle γ.*

Operators with coefficients in flat bundles have been thoroughly studied already in the classical paper of Atiyah, Patodi, and Singer. Thus, in this paper, we recall only the index formula pertaining to this case.

The index formula in trivialized flat bundles [4]. Suppose that the flat bundle γ is trivial $\gamma \overset{\alpha}{\simeq} \mathbb{C}^n$ and A is an elliptic self-adjoint operator as above.

Then the triple (γ, α, A) defines an elliptic operator in subspaces:

$$\Pi_+(nA)(1 \otimes \alpha^*) : \operatorname{Im} \Pi_+(A \otimes 1_\gamma) \longrightarrow \operatorname{Im} \Pi_+(nA). \tag{29}$$

Let us fix the flat bundle with its trivialization and consider the index decomposition problem for operators (29) into the sum of contributions of the principal symbol of the operator and the contribution of subspaces. It is not difficult to see that the necessary condition for such decompositions (Theorem 4) is satisfied. Then we can take the difference of the η-invariants

$$\eta(A \otimes 1_\gamma) - n\eta(A)$$

as the contribution of the subspaces. This difference is called *the relative η-invariant.* The corresponding index theorem in trivialized flat bundles was obtained by Atiyah, Patodi, and Singer.

Theorem 11 *One has*

$$\begin{aligned} \operatorname{ind}(\Pi_+(nA)(1 \otimes \alpha^*) : \operatorname{Im} \Pi_+(A \otimes 1_\gamma) \longrightarrow \operatorname{Im} \Pi_+(nA)) = \\ \langle \operatorname{ch} L_+(A) \operatorname{ch}(\gamma, \alpha) Td(T^*M \otimes \mathbb{C}), [S^*M] \rangle + \eta(A \otimes 1_\gamma) - n\eta(A), \end{aligned} \tag{30}$$

where $\operatorname{ch}(\gamma, \alpha) \in H^{odd}(M, \mathbb{Q})$ *is the Chern character of the trivialized flat bundle.*

The proof uses the heat equation method.

As a corollary, let us take the fractional part of the index formula. Then we obtain an expression of the fractional part of the relative η-invariant in topological terms. For nontrivial flat bundles, the relative η-invariant was also computed in [4], but the formula in this case is written in K-theoretic terms and is less explicit.

3.4 η-invariant and parity conditions

One more class of examples of η-invariants without continuous component of the variation is related to parity conditions.

Theorem 12 [35] *The fractional part of the η-invariant of an elliptic self-adjoint differential operator A on a manifold M is invariant under homotopies if the following parity condition is satisfied:*

$$\operatorname{ord} A + \dim M \equiv 1 (\operatorname{mod} 2).$$

Idea of the proof. The homogeneous components of the complete symbol of a differential operator are polynomials. Hence they are even or odd with respect to the involution $\xi \mapsto -\xi$ acting on the covariables. An accurate account of this symmetry in (28) shows that the local expression for the derivative of the η-invariant is zero. □

The η-invariant as a dimension functional. It is clear that if the continuous component of the variation of the η-invariant is missing, then the η-invariant can be considered as a dimension functional (compare (6) with (26)).

Theorem 13 [64] *Let A be an elliptic self-adjoint differential operator of a positive order. Then the η-invariant is equal to the value of the dimension functional of Section* 1.3 *on the spectral subspace $\widehat{L}_+(A)$*

$$\eta(A) = d\left(\widehat{L}_+(A)\right)$$

provided that $\mathrm{ord} A + \dim M \equiv 1 (\mathrm{mod} 2)$*.*

To prove the theorem, it suffices to check the normalization condition.

This result shows that *we can substitute the η-invariant for the functional d in the index formulas of Section* 1.3 *provided that the pseudodifferential subspace is defined as the spectral subspace of a differential operator.*

Remark 6 To prove Theorem 13, one has to work with η-invariants in the broader context of *pseudodifferential operators*, for which the statement of Theorem 12 is true. We refer the reader to [64] for the precise statement of the parity condition for this case.

Computation of the fractional part of the η-invariant. If the parity condition is satisfied, then the fractional part $\{\eta(A)\}$ is topologically invariant and can be computed in topological terms. It turns out that this invariant strongly depends on the orientation bundle $\Lambda^n(M)$.

Theorem 14 [63] *The fractional part of twice the η-invariant is equal to the pairing*

$$\{2\eta(A)\} = \left\langle [\sigma(A)], 1 - [\Lambda^n(M)] \right\rangle \in \mathbb{Z}\left[\frac{1}{2}\right]/\mathbb{Z}$$

of the difference element of the operator with the orientation bundle $\Lambda^n(M)$, $n = \dim M$, where the brackets denote the (nondegenerate) Poincaré duality

$$\langle , \rangle : \ \mathrm{Tor} K_c^1(T^*M) \times \mathrm{Tor} K^0(M) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

in K-theory for the torsion subgroups.

Let us make a couple of remarks concerning this formula.

1. The computation is based on the following property of symbols of subspaces with parity conditions. For N sufficiently large, the sum $2^N L$ can be lifted from the base M. If we choose an isomorphism $\sigma : 2^N L \longrightarrow \pi^* F$, where $F \in \mathrm{Vect}(M)$, then the index formula

in subspaces expresses the fractional part of the η-invariant in terms of the index of the corresponding operator

$$\widehat{\sigma} : \widehat{L} \longrightarrow C^\infty(M, F),$$

as a *residue* modulo 2^N. Such an index-residue can be computed in K-theory with coefficients in the group $\mathbb{Z}_{2^N}$ (the corresponding index theory modulo n is discussed in Section 4.2). Finally, the expression in terms of Poincaré duality is none other than a short way of expressing the corresponding K-theoretical formula.

2. The orientation bundle appears naturally in the problem, since the involution $(x, \xi) \leftrightarrow (x, -\xi)$ acts on $K_c^*(T^*M)$ as a product with the element $(-1)^{\dim M}[\Lambda^n(M)]$ (see [62]).

Corollary 2 *If the parity condition is satisfied, then the η-invariant on an orientable manifold is half-integer. On a nonorientable manifold M of dimension $2k$ or $2k+1$, the following estimate of the denominator of the η-invariant holds:*

$$\left\{2^{k+1}\eta\left(A\right)\right\} = 0. \tag{31}$$

Indeed, the orientation bundle $\Lambda^n\left(M^n\right)$ has the structure group $\mathbb{Z}_2$. Hence it is induced by the canonical bundle over $\mathbb{RP}^n$. The reduced K-groups of the projective spaces are the torsion groups $\widetilde{K}\left(\mathbb{RP}^{2k}\right) \simeq \widetilde{K}\left(\mathbb{RP}^{2k+1}\right) \simeq \mathbb{Z}_{2^k}$. Hence

$$2^k\left(1 - \left[\Lambda^n\left(M^n\right)\right]\right) = 0.$$

Substituting this equality into the formula for the η-invariant, we obtain the desired assertion. □

Remark 7 The formula for the fractional part of the η-invariant can be rewritten, by analogy with the Atiyah–Singer formula, in terms of the direct image map

$$\{2\eta(A)\} = f_![\sigma(A)],$$

where $f : M \to \mathbb{RP}^{2N}$ is the map classifying the orientation bundle. Here we assume the identification $K^1(T^*\mathbb{RP}^{2n}) = \mathbb{Z}_{2^n} \subset \mathbb{Q}/\mathbb{Z}$.

Examples of first-order operators. We have seen that the properties of the η-invariant for operators with Gilkey's parity condition substantially depend on the properties of the manifold. In the orientable case, one can obtain a half-integral η-invariant at most. This possibility is easy to realize, e.g., by the operators $d + \delta$ on all forms:

$$\{\eta(d+\delta)\} = \left\{\frac{\chi(M)}{2}\right\}.$$

The computation is based on the fact that this operator is isomorphic to the matrix $\begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$ with the Euler operator $D = d + \delta$ acting from even to odd forms. The eigenvalues of this matrix are symmetric with respect to the origin. Therefore, the η-function is zero identically.

It turns out that on *nonorientable manifolds* there exist operators with arbitrary dyadic η-invariants. Examples of such operators were first constructed by P. Gilkey [33].

An operator on $\mathbb{RP}^{2n}$ with a very fractional η-invariant. Let us define a Dirac type operator on an even-dimensional real projective space $\mathbb{RP}^{2n}$. To this end, we consider a set of Hermitian Clifford $2^n \times 2^n$ matrices $e_0, e_1, \ldots, e_{2n}$:

$$e_k e_j + e_j e_k = 2\delta_{kj}.$$

For a vector $v = (v_0, \ldots, v_{2n}) \in \mathbb{R}^{2n+1}$, we define a linear operator

$$e(v) = \sum_{i=0}^{2n} v_i e_i : \mathbb{C}^{2^n} \longrightarrow \mathbb{C}^{2^n}.$$

It is invertible if $v \neq 0$. Consider the Hermitian symbol

$$\sigma(D)(x,\xi) = ie(x)e(\xi) : \mathbb{C}^{2^n} \longrightarrow \mathbb{C}^{2^n}$$

on the unit sphere $\mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$, where ξ is a tangent vector at $x \in \mathbb{S}^{2n}$. The symbol is invariant under the involution $(x,\xi) \to (-x,-\xi)$. Thus it defines a symbol on $\mathbb{RP}^{2n}$.

Theorem 15 [33]

$$\{\eta(D)\} = \frac{1}{2^{n+1}}. \tag{32}$$

For simplicity, we will only compute the denominator of the η-invariant.

The reduced K-group of $\mathbb{RP}^{2n}$ is a cyclic group $\widetilde{K}\left(\mathbb{RP}^{2n}\right) \simeq \mathbb{Z}_{2^n}$, and the generator is given by the orientation bundle

$$1 - \left[\Lambda^{2n}\left(\mathbb{RP}^{2n}\right)\right] \in \widetilde{K}\left(\mathbb{RP}^{2n}\right).$$

On the other hand, the symbol defines the generator of the isomorphic group

$$[\sigma(D)] \in K_c^1\left(T^*\mathbb{RP}^{2n}\right) = \operatorname{Tor} K_c^1\left(T^*\mathbb{RP}^{2n}\right) \simeq \mathbb{Z}_{2^n}.$$

Hence, by Poincaré duality for torsion groups (see above) the pairing of the generators is nonzero and has the largest possible denominator

$$\left\langle 2^{n-1}[\sigma(D)], 1 - \left[\Lambda^{2n}\left(\mathbb{RP}^{2n}\right)\right]\right\rangle = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z}.$$

It remains now to express the pairing in terms of the η-invariant. We have

$$\{2^n \eta(D)\} = \frac{1}{2}.$$

□

3.5 Examples of second-order operators with nontrivial η-invariants

The problem of nontriviality of the η-invariant for second-order operators was stated by P. Gilkey [35]. For a long time, the main difficulty of the problem was the absence of nontrivial elliptic operators of order two. There was essentially one nontrivial operator $d\delta - \delta d$ acting on differential forms. However, its η-invariant turned out to be integer-valued [64]. From a different point of view, this operator is generated by the operators of de Rham–Hodge theory and is in some sense an analog of the Euler operator. To obtain more interesting operators, one has to define the analog of the Dirac operator.

Such an operator was constructed in [63].

Example 9 We define a second-order differential operator $\mathcal{D}$ on $\mathbb{RP}^{2n} \times \mathbb{S}^1$. To this end, we denote the coordinates by x, φ, the dual coordinates by ξ, τ. On the cylinder $\mathbb{RP}^{2n} \times [0,\pi]$ we define

$$\mathcal{D}' = \begin{pmatrix} 2\sin\varphi\left(-i\frac{\partial}{\partial\varphi}\right)D - i\cos\varphi D & \triangle_x e^{-i\varphi} + \left(-i\frac{\partial}{\partial\varphi}\right)e^{i\varphi}\left(-i\frac{\partial}{\partial\varphi}\right) \\ \triangle_x e^{i\varphi} + \left(-i\frac{\partial}{\partial\varphi}\right)e^{-i\varphi}\left(-i\frac{\partial}{\partial\varphi}\right) & 2\sin\varphi\left(i\frac{\partial}{\partial\varphi}\right)D + i\cos\varphi D \end{pmatrix}, \qquad (33)$$

where D is the pin^c Dirac operator on the projective space (see previous section), and $\triangle_x = D^2$ is its Laplacian. The operator $\mathcal{D}'$ is symmetric and elliptic. The ellipticity follows from the following formula for the principal symbol

$$\sigma\left(\mathcal{D}'\right)^2(\xi,\tau) = \left(\xi^2+\tau^2\right)^2.$$

(In other words, the operator $\mathcal{D}'$ is the square root of the square of the Laplacian.) Let F be the vector bundle over the product $\mathbb{RP}^{2n} \times \mathbb{S}^1$, obtained by twisting the trivial bundle $\mathbb{C}^{2^n} \oplus \mathbb{C}^{2^n}$ with the matrix-valued function

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

defined on the base $\mathbb{RP}^{2n} \times \{0\}$. Then $\sigma(\mathcal{D})'$ can be considered as acting in F:

$$\sigma(\mathcal{D}') : \pi^*F \longrightarrow \pi^*F, \quad \pi : S^*(\mathbb{RP}^{2n} \times \mathbb{S}^1) \to \mathbb{RP}^{2n} \times \mathbb{S}^1.$$

Denote by

$$\mathcal{D} : C^\infty\left(\mathbb{RP}^{2n} \times \mathbb{S}^1, F\right) \longrightarrow C^\infty\left(\mathbb{RP}^{2n} \times \mathbb{S}^1, F\right)$$

the elliptic self-adjoint differential operator obtained by smoothing the coefficients of $\mathcal{D}'$.

The topological formulas for the η-invariant obtained earlier enables us to prove the following result, solving the problem of nontriviality of η-invariants for even-order operators.

Theorem 16 [63] *One has*

$$\{2\eta(\mathcal{D})\} = \frac{1}{2^{n-1}}.$$

The idea of the proof is to interpret the operator $\mathcal{D}$ as an exterior tensor product of an operator on the projective space by an elliptic operator on the circle. Then the η-invariant is also a product of the η-invariant on $\mathbb{RP}^n$ and the index on $\mathbb{S}^1$. Unfortunately, the operator itself does not have this product structure. But K-theoretically such a representation holds:

$$[\sigma(\mathcal{D})] = [\sigma(D)] \cdot [\sigma(D_1)] \in K_c^1\left(T^*\left(\mathbb{RP}^{2n} \times \mathbb{S}^1\right)\right) \tag{34}$$

with an elliptic pseudodifferential operator of index two

$$D_1 = \frac{1}{2}\left[e^{-i\varphi}(Q + |Q|) + e^{i\varphi}(|Q| - Q)\right], \quad Q = -i\frac{d}{d\varphi}$$

on $\mathbb{S}^1$. To obtain the theorem, it now suffices to substitute (34) in the formula for the η-invariant in terms of Poincaré duality and use the multiplicative property of the pairing. □

3.6 Applications to bordisms and embeddings of manifolds

η-invariants on pin^c-manifolds and bordisms.

The operator on the projective space constructed in Section 3.4 is a specialization of the Dirac operator on a (nonorientable) pin^c-manifold. The definition of this operator can be found in [33]. We note only that the group pin$^c(n)$ is defined in terms of the extension

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{pin}^c(n) \longrightarrow O(n) \times U(1) \longrightarrow 0.$$

This sequence defines a natural projection pin$^c(n) \to O(n)$. Finally, a manifold M of dimension n is a pin^c-*manifold* if its structure group is reduced to pin$^c(n)$.

On even-dimensional pin^c-manifolds, the Dirac operator, denoted by D, is self-adjoint. Therefore, on such manifolds the fractional topological invariant

$$\{\eta(D)\} \in \mathbb{Z}\left[\frac{1}{2}\right]/\mathbb{Z}$$

is well defined. Moreover, one can also show that this fractional part is invariant under bordisms of pin^c-manifolds.

Theorem 17 [12] pin^c-*manifolds M_1 and M_2 are bordant if and only if they have equal Stiefel–Whitney numbers and the fractional parts of the η-invariants*

$$\{\eta(D_{M_1})\} = \{\eta(D_{M_2})\}.$$

Note that the characteristic property of the theory of pin^c-bordisms is that the bordism group Ω^{pin^c} has nontrivial elements (represented by the projective spaces $\mathbb{RP}^{2k}$) of arbitrarily large order 2^k. To distinguish these elements, the fractional analytic invariant is indispensable.

Application to embeddings. A natural geometric setting when one can consider second-order operators of Section 3.5 was found by P. Gilkey [32]. Let us describe the situation in more detail.

Suppose N is a submanifold with trivial normal bundle in a closed manifold M. Then one can define an elliptic self-adjoint second-order operator that is concentrated in a neighborhood of the submanifold in the following sense. This operator is a sum of Laplacians outside a neighborhood of N. We shall consider for simplicity the codimension one case.

Let us introduce coordinates in a tubular neighborhood U of the boundary, x tangent to the submanifold, and $\varphi \in [0, 2\pi]$ normal to the submanifold. The dual coordinates are ξ, τ.

Consider the quadratic transformation

$$h(\tau, \xi) = \left(\tau^2 - \xi^2, \tau\xi\right) : S^*M|_U \longrightarrow S^*M|_U$$

over U. At a point $(x, \varphi) \in U$, this map is a two-sheeted covering of the sphere. It takes big circles passing through the North pole to big circles passing through the North pole being run through with double speed. Let us define the family of vector bundle homomorphisms

$$\Phi_\varphi : T^*M|_N \longrightarrow \mathbb{R} \oplus T^*M|_N$$

parametrized by $\varphi \in [0, 2\pi]$:

$$\Phi_\varphi(\tau, \xi) = \begin{cases} \left(\cos\varphi\left(\xi^2 + \tau^2\right), \sin\varphi h(\tau, \xi)\right), & \varphi \in [0, \pi], \\ \left(\cos\varphi\left(\xi^2 + \tau^2\right), \sin\varphi\left(\xi^2 + \tau^2\right), 0, \ldots\right), & \varphi \in [\pi, 2\pi]. \end{cases}$$

Suppose that $N \times [0, 2\pi]$ is equipped with a pin^c-structure. Consider the corresponding Clifford module

$$c : \mathrm{Cl}\left(\mathbb{R} \oplus T^*M|_N\right) \longrightarrow \mathrm{End}(E),$$

where $\mathrm{Cl}(V)$ is the bundle of Clifford algebras of a real vector bundle and E is the spinor bundle of $N \times [0, 2\pi]$. The symbol $\sigma(D)$ of order two is defined in a neighborhood of N as the composition

$$\sigma(D)(\varphi, \tau, \xi) \overset{\mathrm{def}}{=} c\left(\Phi_\varphi(\tau, \xi)\right).$$

On the boundary of the neighborhood, the symbol is

$$\sigma(D)(\varphi, \tau, \xi)|_{\varphi = 0, 2\pi} = c(1, 0, \ldots, 0)\left(\xi^2 + \tau^2\right).$$

It coincides with the direct sum of the symbols of Laplacians. Thus, $\sigma(D)$ extends outside U as the direct sum of symbols of Laplacians.

Second-order operators associated with submanifolds with trivial normal bundles enable one to construct some topological invariants.

Proposition 7 *Let M be a closed smooth manifold,* $\dim M = 2k + 1$. *A necessary condition for an embedding*

$$\mathbb{RP}^{2k} \subset M$$

of the projective space $\mathbb{RP}^{2k}$ *with trivial normal bundle is the surjectivity of the direct image map*

$$f_! : \mathrm{Tor} K_c^1 \left(T^* M\right) \longrightarrow \mathbb{Z}_{2^k} \subset \mathbb{Z}\left[1/2\right]/\,\mathbb{Z},$$

induced by the map $f : M \longrightarrow B\mathbb{Z}_2 = \mathbb{RP}^\infty$ *classifying the orientation bundle* $\Lambda^{2k+1}\left(M\right)$. *In particular,* $K_c^1\left(T^* M\right)$ *has to have nontrivial torsion elements of order* 2^k.

The proposition can be proved if one notes that on M we have a second-order operator with the η-invariant having denominator 2^{k+1}. On the other hand, the η-invariant is computed by the direct image map corresponding to the classifying space. □

3.7 The Atiyah–Patodi–Singer formula

An expression for the index of spectral boundary value problems was found in [2]. Namely, using the heat equation method [8], the relation

$$\mathrm{ind}(D, \Pi_+(A)) = \int_X a(D) - \eta(A) \tag{35}$$

was obtained for the index of a spectral boundary value problem on a manifold X for an elliptic operator of order one that has the decomposition $\partial/\partial t + A$ near the boundary. Here $a(D)$ is by definition the constant term in the local asymptotic expansion

$$\mathrm{tr}(e^{-tD^*D}(x,x)) - \mathrm{tr}(e^{-tDD^*}(x,x))$$

as $t \to 0$. It is defined (as in the case of operators on closed manifolds) as some algebraic expression in the coefficients of the operator and their derivatives. The second term is the η-*invariant* of the tangential operator A.

In the general case, the formula for $a(D)$ is extremely cumbersome. However, for the classical operators (Euler operator, signature operator, etc.) it is described by explicitly computable formulas. For example, if D is the signature operator with coefficients in a bundle E equipped with a connection, we have

$$a(D) = L(X)\mathrm{ch}E,$$

where $L(X) \in \Lambda^{ev}(X)$ stands for the Hirzebruch polynomial [56] in the Pontryagin forms of the Riemannian manifold and $\mathrm{ch}E \in \Lambda^{ev}(X)$ is the Chern character of the bundle computed in terms of the connection via Chern–Weil theory.

A similar expression for the form is valid for the remaining classical operators; one has only to substitute polynomials corresponding to the operators in place of the L-polynomial.

The Atiyah–Patodi–Singer formula has numerous applications ranging from algebraic geometry [5] to quantum field theory [72]. As an explanation of this phenomenon, M. Atiyah points out that for the signature operator the formula (35) relates three objects of entirely different nature: a *topological* invariant (the signature) on the left-hand side and a *metric* invariant (the integral of the Pontryagin forms) as well as the *spectral* η-invariant on the right-hand side.

3.8 The index decomposition of the Dirac operator (Kreck-Stolz invariant)

Consider the Dirac operator on a $4k$-dimensional manifold M. In this section, we obtain, following [45], a decomposition of the index of this operator. Strikingly enough, it turns out that the index defect can be defined using the signature operator! The decomposition is made under the assumption that the boundary has trivial Pontryagin classes.

Denote the Dirac operator by $\mathcal{D}$ and its tangential operator by $\mathcal{A}$. By Atiyah–Patodi–Singer theorem, the sum $\operatorname{ind}\mathcal{D} + \eta(\mathcal{A})$ is equal to the integral over the manifold with boundary of the $\widehat{A}$ polynomial in the Pontryagin forms

$$\int_M \widehat{A}(p).$$

Hence to construct an index decomposition we have to decompose this integral into a geometric invariant determined by the boundary and the remainder homotopy invariant term. Such a decomposition is obtained for all decomposable components of the $\widehat{A}$-polynomial (except the top component p_k!) by the following lemma.

Lemma 2 *Let α, β be positive degree forms on M whose restrictions to the boundary are exact. Then*

$$\int_M \alpha\wedge\beta = \int_{\partial M} \widehat{\alpha}\wedge\beta + \left\langle j^{-1}[\alpha]\cup j^{-1}[\beta], [M,\partial M]\right\rangle,$$

where $d\widehat{\alpha} = \alpha|_{\partial M}$, $j^{-1}[\alpha]$ is an arbitrary preimage of the cohomology class $[\alpha]\in H^*(M)$ under the restriction map $j : H^*(M,\partial M)\to H^*(M)$, and $\langle\cdot,[M,\partial M]\rangle$ is the pairing with the fundamental class. Moreover, the terms on the right-hand side of the relation do not depend on the choices.

The proof uses integration by parts. $\square$

Denote the first term in the formula of the lemma by

$$\int_{\partial M} d^{-1}(\alpha\wedge\beta) \stackrel{\text{def}}{=} \int_{\partial M} \widehat{\alpha}\wedge\beta.$$

It only remains now to decompose the integral of the top Pontryagin class. Here we make use of the signature operator: for this operator, the Atiyah–Patodi–Singer formula contains the integral of the L-class. In turn, the L-class also includes the top Pontryagin class. A standard computation shows that the sum $\widehat{A}(p) + a_k L(p)$, where $a_k = (2^{2k+1}(2^{2k-1}-1))^{-1}$ in degrees $\le 4k$, contains only products of Pontryagin classes of positive degrees, i.e., does not contain the top class p_k. For example, for an 8-manifold one has

$$\widehat{A}(M) = \frac{1}{5760}(7p_1^2 - 4p_2), \quad L(M) = \frac{1}{45}(7p_2 - (p_1)^2).$$

Further, by rewriting the sum $\operatorname{ind}\mathcal{D} + a_k \operatorname{ind} D$ by Atiyah–Patodi–Singer theorem, we obtain

$$\operatorname{ind}\mathcal{D} + \eta(\mathcal{A}) - a_k\eta(A) - \int_{\partial M} d^{-1}(\widehat{A} + a_k L)(p) = t(M),$$

where $t(M)$ denotes the following topological invariant of manifolds with boundary:

$$t(M) = \langle (\widehat{A} + a_k L)(j^{-1}p(M)), [M, \partial M] \rangle - a_k \operatorname{ind} D.$$

The contribution of the boundary is called the *Kreck–Stolz invariant* $s(\partial M, g)$ of the manifold ∂M with metric g.

Theorem 18 [45] *The index of the Dirac operator on a manifold with boundary having trivial Pontryagin classes has the decomposition*

$$\operatorname{ind} \mathcal{D} = t(M) + s(\partial M, g),$$

where the Kreck–Stolz invariant $s(\partial M, g)$ is a homotopy invariant of the metric in the class of metrics of positive scalar curvature.

4 Elliptic theory "modulo n"

Another field of applications of elliptic theory in subspaces concerns so-called theories with coefficients in finite groups $\mathbb{Z}_n$. The characteristic feature of such theories is that, for some reason, the index in such theories makes sense only as a residue.

In this section, we briefly discuss two versions of this theory: on $\mathbb{Z}_n$-manifolds and on closed manifolds.

4.1 The Freed–Melrose theory on $\mathbb{Z}_k$-manifolds

Definition 11 A $\mathbb{Z}_k$*-manifold* is a compact smooth manifold M with boundary ∂M, which is a disjoint union of k copies of some manifold X

$$\partial M = M_1 \sqcup ... \sqcup M_k, \quad M_i \overset{g_i}{\approx} X$$

with fixed diffeomorphisms g_i.

$\mathbb{Z}_k$-manifolds naturally define the singular spaces

$$\overline{M} = M / \{ M_i \overset{g_j^{-1} g_i}{\sim} M_j \}, \tag{36}$$

identifying points on the components of the boundary (see Fig. 3).

$\mathbb{Z}_k$-manifolds were introduced by Sullivan [71]. One of the motivations indicating the interest in this class of singular spaces is the fact that (in the orientable case) a singular manifold $\overline{M}$ carries a fundamental cycle in homology with coefficients $\mathbb{Z}_k$

$$\left[\overline{M}\right] \in H_m \left(\overline{M}, \mathbb{Z}_k\right), \qquad m = \dim M.$$

These singular manifolds were also used as a geometric realization of bordisms with coefficients in $\mathbb{Z}_k$. For further research in this direction, we refer the reader to [20].

On a $\mathbb{Z}_k$-manifold, we fix a collar neighborhood of the boundary

$$U_{\partial M} \approx [0, 1) \times X \times \{1, ..., k\}. \tag{37}$$

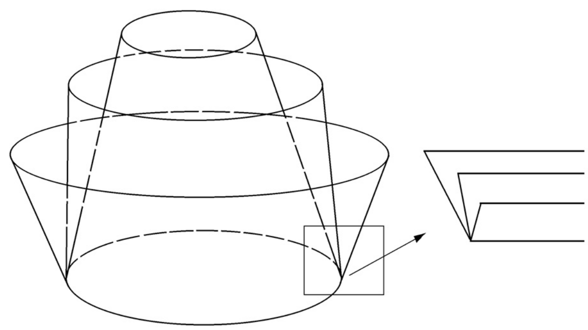

Figure 3: Manifold with singularities

Definition 12 *An operator on a* $\mathbb{Z}_k$*-manifold* is an operator D on M, which is invariant under the group of permutations of the components of the collar neighborhood (37) of the boundary.

We equip elliptic operators D on $\mathbb{Z}_k$-manifolds with spectral boundary conditions.

Proposition 8 *The index residue* $\operatorname{mod} k\text{-}\operatorname{ind}(D,\Pi_+(A))$ *is constant for homotopies of the operator* D.

Indeed, for a continuous homotopy $\{D_t\}_{t\in[0,1]}$ the change of the index is equal to the spectral flow of the family of tangential operators on the boundary

$$\operatorname{ind}(D_1,\Pi_+(A_1)) - \operatorname{ind}(D_0,\Pi_+(A_0)) = -\operatorname{sf}\{A_t\}_{t\in[0,1]}.$$

On the other hand, the family A_t at the boundary is by assumption the direct sum of k copies of some family on X. Therefore, the spectral flow is divisible by k. $\square$

This homotopy invariant index residue was computed in terms of the principal symbol by Freed and Melrose. Let us briefly recall their result.

Theorem of Freed and Melrose. The cotangent bundle T^*M is a noncompact $\mathbb{Z}_k$-manifold, and the principal symbol of operator D defines an element in the K-group

$$[\sigma(D)] \in K_c\left(\overline{T^*M}\right).$$

(Here we use identification (36).) It turns out that the direct image mapping in K-theory extends to the category of $\mathbb{Z}_k$-manifolds (the morphisms are by definition those embeddings which map boundary to boundary preserving the $\mathbb{Z}_k$-structure). More precisely, for an embedding $f: M \to N$ one has

$$f_! : K_c\left(\overline{T^*M}\right) \longrightarrow K_c\left(\overline{T^*N}\right).$$

On the other hand, one can construct a universal space for such embeddings (i.e., the space into which any $\mathbb{Z}_k$ manifold can be embedded). The universal space can be obtained from $\mathbb{R}^L$ by deleting k disjoint disks of a sufficiently small radius. We obtain the $\mathbb{Z}_k$-manifold M_k whose boundary is the union of k spheres (diffeomorphisms of spheres are given by parallel translations). It is easy to compute the K-group of the cotangent bundle of this space

$$K_c\big(\overline{T^*M_k}\big) \simeq \mathbb{Z}_k.$$

Freed and Melrose proved the following index theorem.

Theorem 19 [31] *One has*

$$\operatorname{mod} k\text{-ind}D = f_!\left[\sigma\left(D\right)\right],$$

where the direct image map $f_! : K_c\left(\overline{T^*M}\right) \longrightarrow K_c\big(\overline{T^*M_k}\big) \simeq \mathbb{Z}_k$ *is induced by an embedding* $f : M \longrightarrow M_k$.

The proof models the K-theoretic proof of the Atiyah–Singer theorem based on embeddings. The main part of the proof is the statement that the analytical index is preserved for embeddings, i.e., for an embedding of M in N the following diagram commutes

$$\begin{array}{ccc} K_c(\overline{T^*M}) & \xrightarrow{f_!} & K_c(\overline{T^*N}) \\ & \searrow \quad \swarrow & \\ & \mathbb{Z}_n. & \end{array}$$

4.2 Index modulo n on closed manifolds

Index-residues also arise on a closed manifold. Consider the following question: what objects of elliptic theory correspond to the elements of K-group $K_c(T^*M, \mathbb{Z}_n)$ with coefficients $\mathbb{Z}_n$?

The answer is given in terms of operators in subspaces

$$D : n\widehat{L} \longrightarrow C^\infty\left(M, F\right). \tag{38}$$

Leu us show how symbols of such operators define elements of the K-group with coefficients. To this end, we recall the definition of the latter.

K-theory with coefficients is defined as

$$K_c\left(T^*M, \mathbb{Z}_n\right) = K_c\left(T^*M \times \mathbb{M}_n, T^*M \times pt\right); \tag{39}$$

where $\mathbb{M}_n$ is the so-called Moore space of the group $\mathbb{Z}_n$. An explicit construction of this space can be found in [3]. We will only use the fact that the reduced K-groups of the Moore space is $\mathbb{Z}_n$ and generated by the difference $1 - [\gamma]$, where γ is a line bundle. We will also fix a trivialization

$$n\gamma \overset{\beta}{\simeq} \mathbb{C}^n.$$

Geometric construction of elements of K-groups with coefficients. It follows from definition (39) that elements of $K_c(T^*M,\mathbb{Z}_n)$ can be realized in terms of families of elliptic symbols[3] on M. The family is parametrized by the Moore space. It is easy to define such a family as a composition:

$$\begin{array}{l} C^\infty(M,F) \xrightarrow{D^{-1}} n\widehat{L}, \\ \qquad n\widehat{L} \xrightarrow{\beta^{-1}\otimes 1_{\widehat{L}}} \gamma\otimes n\widehat{L}, \\ \qquad\qquad \gamma\otimes n\widehat{L} \xrightarrow{1_\gamma\otimes D} \gamma\otimes C^\infty(M,F), \end{array} \tag{40}$$

where D^{-1} is an almost inverse and the last family is obtained by twisting with γ. The family of symbols corresponding to this composition defines the desired element in the K-group with coefficients. Denote it by

$$[\sigma(D)]\in K_c(T^*M,\mathbb{Z}_n).$$

In [60], it is shown that the K-group with coefficients is actually isomorphic to the group of stable homotopy classes of operators (38). Let us conclude this section with an index theorem.

Index theorem. Note that the index of operator (38) as a residue modulo n

$$\operatorname{mod} n\text{-ind}\, D\in\mathbb{Z}_n$$

is a homotopy invariant of the principal symbol of the operator. The following theorem gives an expression for this index in topological terms.

Theorem 20 *One has*

$$\operatorname{mod} n\text{-ind}\, D = p_![\sigma(D)], \tag{41}$$

where the direct image map $p_!: K(T^*M,\mathbb{Z}_n)\longrightarrow \widetilde{K}(pt,\mathbb{Z}_n)=\mathbb{Z}_n$ in K-theory with coefficients is induced by $p: M\longrightarrow pt$.

Let us apply the Atiyah–Singer index formula for families to compute the index of the composition (40). This formula expresses the index as the right-hand side of (41). On the other hand, the index of the composition can be computed directly as

$$\operatorname{ind} D([\gamma]-1)\in K(\mathbb{M}_n),$$

i.e., it coincides with the modulo n index of the operator in subspaces in the group $\widetilde{K}(pt,\mathbb{Z}_n)=\mathbb{Z}_n$. □

[3]Here we use the difference construction for families [11]. It associates element $[\sigma]\in K(T^*M\times X)$ with a family $\sigma(x)$, $x\in X$ of elliptic symbols on M parametrized by space X:

$$\sigma(x):\pi^*E\longrightarrow\pi^*F,\quad E,F\in\operatorname{Vect}(M\times X),\ \pi: S^*M\times X\to M\times X.$$

References

[1] M. S. Agranovich. Elliptic operators on closed manifolds. In *Partial differential equations. VI*, volume 63 of *Encycl. Math. Sci.*, pages 1–130, 1994.

[2] M. Atiyah, V. Patodi, and I. Singer. Spectral asymmetry and Riemannian geometry I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[3] M. Atiyah, V. Patodi, and I. Singer. Spectral asymmetry and Riemannian geometry II. *Math. Proc. Cambridge Philos. Soc.*, 78:405–432, 1976.

[4] M. Atiyah, V. Patodi, and I. Singer. Spectral asymmetry and Riemannian geometry III. *Math. Proc. Cambridge Philos. Soc.*, 79:71–99, 1976.

[5] M. F. Atiyah. The logarithm of the Dedekind η-function. *Math. Annalen*, 278:335–380, 1987.

[6] M. F. Atiyah. *K-Theory.* The Advanced Book Program. Addison–Wesley, Inc., second edition, 1989.

[7] M. F. Atiyah and R. Bott. The index problem for manifolds with boundary. In *Bombay Colloquium on Differential Analysis*, pages 175–186, Oxford, 1964. Oxford University Press.

[8] M. F. Atiyah, R. Bott, and V. K. Patodi. On the heat equation and the index theorem. *Invent. Math.*, 19:279–330, 1973.

[9] M. F. Atiyah and I. M. Singer. The index of elliptic operators I. *Ann. of Math.*, 87:484–530, 1968.

[10] M. F. Atiyah and I. M. Singer. Index theory for skew-adjoint Fredholm operators. *Publ. Math. IHES*, 37:5–26, 1969.

[11] M. F. Atiyah and I. M. Singer. The index of elliptic operators IV. *Ann. Math.*, 93:119–138, 1971.

[12] A. Bahri and P. Gilkey. The eta invariant, Pin^c bordism, and equivariant $Spin^c$ bordism for cyclic 2-groups. *Pacific Jour. Math.*, 128(1):1–24, 1987.

[13] M. S. Birman and M. Z. Solomyak. On the subspaces admitting a pseudodifferential projection. *Vestnik LGU*, 1:18–25, 1982.

[14] M. Sh. Birman and M. Z. Solomyak. The asymptotic behavior of the spectrum of variational problems on solutions of elliptic systems. *Zapiski LOMI*, 115:23–39, 1982. (in Russian).

[15] J.-M. Bismut. Local index theory, eta invariants and holomorphic torsion: A survey. In C. C. Hsiung et al., editors, *Surveys in differential geometry*, volume 3, pages 1–76, Cambridge, 1998. Harward University.

[16] J.-M. Bismut and D. S. Freed. The analysis of elliptic families. I. *Commun. Math. Phys.*, 106:159–176, 1986.

[17] B. Blackadar. *K-Theory for Operator Algebras*. Number 5 in Mathematical Sciences Research Institute Publications. Cambridge University Press, 1998. Second edition.

[18] B. Booß-Bavnbek and K. Wojciechowski. *Elliptic Boundary Problems for Dirac Operators*. Birkhäuser, Boston–Basel–Berlin, 1993.

[19] R. Bott and L. Tu. *Differential Forms in Algebraic Topology*, volume 82 of *Graduate Texts in Mathematics*. Springer–Verlag, Berlin–Heidelberg–New York, 1982.

[20] B. Botvinnik. Manifolds with singularities accepting a metric of positive scalar curvature. *Geom. Topol.*, 5:683–718, 2001.

[21] B. Botvinnik and P. Gilkey. The Gromov–Lawson–Rosenberg conjecture: The twisted case. *Houston J. Math.*, 23(1):143–160, 1997.

[22] B. I. Botvinnik, V. M. Bukhshtaber, S. P. Novikov, and S. A. Yuzvinskiĭ. Algebraic aspects of multiplication theory in complex cobordisms. *Uspekhi Mat. Nauk*, 55(4):5–24, 2000.

[23] L. Boutet de Monvel and V. Guillemin. *The spectral theory of Toeplitz operators*, volume 99 of *Ann. of Math. Studies*. Princeton University Press, Princeton, 1981.

[24] L. Brown, R. Douglas, and P. Fillmore. *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, volume 345 of *Lecture Notes in Math.* Springer Verlag, 1973.

[25] L. Brown, R. Douglas, and P. Fillmore. Extensions of C^*-algebras and K-homology. *Ann. Math. II*, 105:265–324, 1977.

[26] A. P. Calderón. Boundary value problems for elliptic equations. *Outlines of the Joint Soviet–American Symposium on Partial Differential Equations, Novosibirsk*, pages 303–304, 1963.

[27] C. Carvalho. A K-theory proof of the cobordism invariance of the index. math.KT/0408260, to appear in K-theory.

[28] A. A. Dezin. *Multidimensional Analysis and Discrete Models*. CRC–Press, Boca Raton, Florida, USA, 1995. English transl.: Nauka, Moscow, 1990.

[29] J. Eells Jr. and N.H. Kuiper. An invariant for certain smooth manifolds. *Ann. Mat. Pura Appl. (4)*, 60:93–110, 1962.

[30] D. Freed. $\mathbb{Z}/k$ manifolds and families of Dirac operators. *Invent. Math.*, 92(2):243–254, 1988.

[31] D. Freed and R. Melrose. A mod k index theorem. *Invent. Math.*, 107(2):283–299, 1992.

[32] P. B. Gilkey. The residue of the global eta function at the origin. *Adv. in Math.*, 40:290–307, 1981.

[33] P. B. Gilkey. The eta invariant for even dimensional Pin^c manifolds. *Adv. in Math.*, 58:243–284, 1985.

[34] P. B. Gilkey. The eta invariant and non-singular bilinear products on $\mathbf{R}^\mathbf{n}$. *Can. Math. Bull.*, 30:147–154, 1987.

[35] P. B. Gilkey. The eta invariant of even order operators. *Lecture Notes in Mathematics*, 1410:202–211, 1989.

[36] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1995.

[37] I. Ts. Gohberg and M. G. Krein. Systems of integral equations on the half-line with kernels depending on the difference of the arguments. *Uspekhi Matem. Nauk*, 13(2):3–72, 1958.

[38] G. Grubb. A resolvent approach to traces and zeta Laurent expansions. In *Spectral geometry of manifolds with boundary and decomposition of manifolds*, volume 366 of *Contemp. Math.*, pages 67–93. Amer. Math. Soc., Providence, RI, 2005.

[39] G. Grubb and E. Schrohe. Traces and quasi-traces on the Boutet de Monvel algebra. *Ann. Inst. Fourier (Grenoble)*, 54(5):1641–1696, 2004.

[40] G. Grubb and R. T. Seeley. Zeta and eta functions for Atiyah-Patodi-Singer operators. *J. Geom. Anal.*, 6(1):31–77, 1996.

[41] L. Hörmander. *The Analysis of Linear Partial Differential Operators. III.* Springer–Verlag, Berlin Heidelberg New York Tokyo, 1985.

[42] G. G. Kasparov. The generalized index of elliptic operators. *Funct. Anal. Appl.*, 7:238–240, 1973.

[43] J. Kohn and L. Nirenberg. An algebra of pseudo-differential operators. *Comm. Pure Appl. Math.*, 18:269–305, 1965.

[44] M. Kontsevich and S. Vishik. Geometry of determinants of elliptic operators. In *Functional analysis on the eve of the 21'st century*, volume 131 of *Progr. Math.*, pages 173–197. Birkhäuser, Boston, 1995.

[45] M. Kreck and S. Stolz. Nonconnected moduli spaces of positive sectional curvature metrics. *J. Amer. Math. Soc.*, 6(4):825–850, 1993.

[46] R. Lauter and S. Moroianu. An index formula on manifolds with fibered cusp ends. *J. of Geom. Anal.*, 15(2):261–283, 2005.

[47] R. Melrose. *The Atiyah–Patodi–Singer Index Theorem.* Research Notes in Mathematics. A. K. Peters, Boston, 1993.

[48] R. Melrose. The eta invariant and families of pseudodifferential operators. *Math. Research Letters*, 2(5):541–561, 1995.

[49] O.K. Mironov. Existence of multiplicative structures in the theory of cobordism with singularities. *Izv. Akad. Nauk SSR Ser. Mat.*, 39(5):1065–1092, 1975.

[50] J. Morgan and D. Sullivan. The transversality characteristic class and linking cycles in surgery theory. *Ann. of Math., II. Ser*, 99:463–544, 1974.

[51] S. Moroianu. Homology and residues of adiabatic pseudodifferential operators. *Nagoya Math. J.*, 175:171–221, 2004.

[52] W. Müller. Eta-invariant (some recent developments). *Sem. Bourbaki. Asterisque*, 227:335–364, 1994.

[53] V. Nazaikinskii, B.-W. Schulze, B. Sternin, and V. Shatalov. Spectral boundary value problems and elliptic equations on singular manifolds. *Differents. Uravnenija*, 34(5):695–708, 1998. English trans.: Differential Equations, **34**, N 5 (1998), pp 696–710.

[54] V. Nistor. Asymptotics and index for families invariant with respect to a bundle of Lie groups. *Rev. Roum. Math. Pures Appl.*, 47(4):451–483, 2002.

[55] S. P. Novikov. Pontrjagin classes, the fundamental group and some problems of stable algebra. In *Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham)*, pages 147–155. Springer, New York, 1970.

[56] R. S. Palais. *Seminar on the Atiyah–Singer index theorem.* Princeton Univ. Press, Princeton, NJ, 1965.

[57] J. Phillips. Self-adjoint Fredholm operators and spectral flow. *Canad. Math. Bull.*, 39(4):460–467, 1996.

[58] S. Rosenberg. Nonlocal invariants in index theory. *Bull. AMS*, 34(4):423–433, 1997.

[59] A. Savin, B.-W. Schulze, and B. Sternin. On invariant index formulas for spectral boundary value problems. *Differentsial'nye uravnenija*, 35(5):709–718, 1999.

[60] A. Savin, B.-W. Schulze, and B. Sternin. Elliptic Operators in Subspaces and the Eta Invariant. *K-theory*, 27(3):253–272, 2002.

[61] A. Savin and B. Sternin. To the problem of homotopy classification of the elliptic boundary value problems. *Doklady Mathematics*, 63(2):174–178, 2001.

[62] A. Savin and B. Sternin. The eta-invariant and Pontryagin duality in K-theory. *Math. Notes*, 71(2):245–261, 2002. arXiv: math/0006046.

[63] A. Savin and B. Sternin. The eta invariant and parity conditions. *Adv. in Math.*, 182(2):173–203, 2004.

[64] A. Yu. Savin and B. Yu. Sternin. Elliptic operators in even subspaces. *Matem. sbornik*, 190(8):125–160, 1999. English transl.: Sbornik: Mathematics **190**, N 8 (1999), p. 1195–1228; arXiv: math/9907027.

[65] A. Yu. Savin and B. Yu. Sternin. Elliptic operators in odd subspaces. *Matem. sbornik*, 191(8):89–112, 2000. English transl.: Sbornik: Mathematics **191**, N 8 (2000), arXiv: math/9907039.

[66] B.-W. Schulze, B. Sternin, and V. Shatalov. On general boundary value problems for elliptic equations. *Math. Sb.*, 189(10):145–160, 1998. English transl.: Sbornik: Mathematics **189**, N 10 (1998), p. 1573–1586.

[67] R. T. Seeley. Singular integrals and boundary value problems. *Am. J. Math.*, 88:781–809, 1966.

[68] R. T. Seeley. Complex powers of an elliptic operator. *Proc. Sympos. Pure Math.*, 10:288–307, 1967.

[69] R. T. Seeley. Topics in pseudodifferential operators. In L. Nirenberg, editor, *Pseudo-Differential Operators*, pages 167–305, Roma, 1969. C.I.M.E. Conference on pseudo-differential operators, Stresa 1968, Cremonese.

[70] I. M. Singer. Eigenvalues of the Laplacian and invariants of manifolds. In *Proceedings of the International Congress of Mathematicians*, pages 187–199, Vancouver, 1974.

[71] D. Sullivan. *Geometric Topology. Localization, Periodicity and Galois Symmetry.* MIT, Cambridge, Massachusets, 1970.

[72] E. Witten. Global gravitational anomalies. *Commun. Math. Phys.*, 100:197–229, 1985.

[73] K. Wojciechowski. A note on the space of pseudodifferential projections with the same principal symbol. *J. Operator Theory*, 15(2):207–216, 1986.