Conormal symbols of mixed elliptic problems with singular interfaces

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Abstract. Mixed elliptic problems are characterised by conditions that have a discontinuity on an interface of the boundary of codimension 1. The case of a smooth interface is treated in [3]; the investigation there refers to additional interface conditions and parametrices in standard Sobolev spaces. The present paper studies a necessary structure for the case of interfaces with conical singularities, namely, corner conormal symbols of such operators. These may be interpreted as families of mixed elliptic problems on a manifold with smooth interface. We mainly focus on second order operators and additional interface conditions that are holomorphic in an extra parameter. In particular, for the case of the Zaremba problem we explicitly obtain the number of potential conditions in this context. The inverses of conormal symbols are meromorphic families of pseudo-differential mixed problems referring to a smooth interface. Pointwise they can be computed along the lines [3].

Keywords: Corner boundary value problems, mixed elliptic problems, interfaces with conical singularities, Zaremba problem.

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Contents

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Introduction

This paper studies the conormal symbolic structure of mixed elliptic problems when the interface has conical singularities. The situation is as follows. Let X be the closure of an open bounded set $G \subset \mathbb{R}^d$ with boundary Y, and let Y be subdivided into subsets $Y_\pm$, such that $Y = Y_- \cup Y_+, Y_\pm \subset Y$ closed, where $Z := Y_+ \cap Y_-$ is a submanifold of Y with conical singularity v and $Z_{\mathrm{reg}} := Z \setminus \{v\}$ of codimension 1 in Y. Let A be an elliptic differential operator of second order in $G = \operatorname{int} X$ with smooth coefficients up to the boundary, and consider the mixed boundary value problem

$$Au = f \quad \text{in} \quad G, \quad T_\pm u = g_\pm \quad \text{on} \quad \operatorname{int} Y_\pm, \tag{1}$$

with boundary operators $T_\pm := \mathrm{r}_\pm B_\pm$ for differential operators $B_\pm$ of order $\mu_\pm$, given in a neighbourhood of $Y_\pm$ in $\mathbb{R}^d$, satisfying the Shapiro-Lopatinskij condition, uniformly up to Z from the respective sides. Here $\mathrm{r}_\pm$ denotes the operators of restriction to $\operatorname{int} Y_\pm$. A well known case is the so called Zaremba problem for the Laplacian $A = \Delta$ with Dirichlet/Neumann conditions on the minus/plus side of the boundary. The problem is now to understand the regularity of solutions and parametrices of the operator $\mathcal{A} = {}^{\mathrm{t}}(A \quad T_- \quad T_+)$ in suitable weighted Sobolev spaces, both near Z_{reg} and v (for simplicity, we consider one conical singularity; the case with finitely many such points is similar). This problem has been studied in [5], based on earlier papers [1] for the case of smooth Z (with regularity in weighted edge Sobolev spaces) and [3] in standard Sobolev spaces. In [1] the interface Z is regarded as a smooth edge on the boundary of X, in [5] the regular part Z_{reg} of Z is also a smooth edge, but v plays the role of a corner point, and we established the elliptic regularity of solutions in weighted corner Sobolev spaces. The operators in this case are described by a principal symbolic hierarchy $\sigma = (\sigma_\psi, \sigma_\partial, \sigma_\wedge, \sigma_{\mathrm{c}})$ with σ_ψ being the standard homogeneous principal symbol of $\mathcal{A}$, σ_∂ the pair of boundary symbols on the $\pm$ sides of the boundary, $\sigma_\wedge$ the edge symbol on Z_{reg} and σ_{c} the corner conormal symbol. There are two weights $(\gamma, \delta) \in \mathbb{R}^2$ in the corner Sobolev spaces, and the ellipticity with respect to $\sigma_\wedge$ refers to the 'cone weight' γ, that of σ_{c} to the corner weight δ. In the Zaremba case we proved that for a suitable set of admissible weights γ the ellipticity with respect to σ_{c} is satisfied for all δ, except for a discrete set of exceptional weights. The complete answer employed extra interface conditions on Z_{reg}, depending on γ, the number of which was also computed.

The present paper is aimed at studying the meromorphic corner conormal symbolic structure in more detail. In some parameter-dependent cases we can organise the extra interface data in such a way that the conormal symbol is bijective on a prescribed weight line.

1 Mixed problems in an infinite cylinder

In contrast to the notation in the introduction we now slightly change the context and consider mixed elliptic problems in an infinite cylinder.

Let $N = \overline{\Omega}$ be the closure of a smooth bounded domain in $\mathbb{R}^m$ and let $M := 2N$ denote double (obtained by gluing together two copies $N_\pm$ of N along the common boundary to a closed compact C^∞ manifold; we then identify N with N_+). Let $H^s(\mathbb{R} \times M)$ denote the cylindrical Sobolev space on $\mathbb{R} \times M$ of smoothness $s \in \mathbb{R}$. Let us briefly recall the definition. The space $L^\mu_{\mathrm{cl}}(M; \mathbb{R}^l_\lambda)$ of parameter-dependent (with parameter $\lambda \in \mathbb{R}^l$) pseudo-differential operators on M of order μ contains an element $R^\mu(\lambda)$ which is parameter-dependent elliptic and induces isomorphisms $R^\mu(\lambda) : H^s(M) \to H^{s-\mu}(M)$ for all $s \in \mathbb{R}, \lambda \in \mathbb{R}^l$. Then $H^s(\mathbb{R} \times M)$ is defined as the closure of $C_0^\infty(\mathbb{R}, C^\infty(M))$ with respect to the norm

$$\Big\{ \int_{\mathbb{R}} ||R^s(\tau)(Fu)(\tau)||^2_{L^2(M)} d\tau \Big\}^{\frac{1}{2}}$$

where $R^s(\tau) \in L^s_{\mathrm{cl}}(M; \mathbb{R}_\tau)$ is an order reducing element as mentioned before, and F is the Fourier transform in $t \in \mathbb{R}$. The space $L^2(M)$ refers to a fixed Riemannian metric on M. Moreover, let

$$H^s(\mathbb{R} \times \operatorname{int} N) := \{u|_{\mathbb{R} \times \operatorname{int} N} : u \in H^s(\mathbb{R} \times 2N)\}.$$

In order to investigate conormal symbols in a corner situation we now study mixed elliptic problems in an infinite cylinder. In order to avoid too complicated notation we assume $X, Y = \partial X, Y_\pm$ and Z as before, but now assume Z to be a C^∞ submanifold of Y of codimension 1, $Y_- \cup Y_+ = Y, Y_- \cap Y_+ = Z$. According to the above definition we have the cylindrical Sobolev spaces $H^s(\mathbb{R} \times \operatorname{int} X), H^s(\mathbb{R} \times \operatorname{int} Y_\pm), H^s(\mathbb{R} \times Z)$. Let us set

$$H^{s,\delta}(\mathbb{R} \times \operatorname{int} X) := e^{-t\delta} H^s(\mathbb{R} \times \operatorname{int} X)$$

and

$$H^{s,\delta}(\mathbb{R} \times \operatorname{int} Y_\pm) := e^{-t\delta} H^s(\mathbb{R} \times \operatorname{int} Y_\pm).$$

Let

$$A = \sum_{|\alpha| \le 2} a_\alpha(t, x) D^\alpha_{t,x}$$

be an elliptic differential operator of second order with coefficients $a_\alpha \in C^\infty(\mathbb{R} \times X)$. Moreover, let $T_\pm := \mathrm{r}_\pm B_\pm$ be boundary operators with $\mathrm{r}_\pm$ being the restriction to $\mathbb{R} \times \operatorname{int} Y_\pm$ and

$$B_\pm = \sum_{|\beta| \le \mu_\pm} b_{\beta,\pm}(t,x) D^\beta_{t,x}$$

differential operators with coefficients $b_{\beta,\pm} \in C^\infty(\mathbb{R} \times U_\pm)$, where $U_\pm$ are open neighbourhoods of $Y_\pm$. We assume that the boundary operators $T_\pm$ are elliptic on $Y_\pm$ with respect to A, i.e., satisfy the Shapiro-Lopatinskij condition uniformly up to Z from the respective $\pm$-sides. Under suitable assumptions on the coefficients for $|t| \to \infty$ we then obtain continuous operators

$$\mathcal{A} = \begin{pmatrix} A \\ T_- \\ T_+ \end{pmatrix} : H^{s,\delta}(\mathbb{R} \times \operatorname{int} X) \to \begin{array}{c} H^{s-2,\delta-2}(\mathbb{R} \times \operatorname{int} X) \\ \oplus \\ H^{s-\mu_- - \frac{1}{2}, \delta-\mu_-}(\mathbb{R} \times \operatorname{int} Y_-) \\ \oplus \\ H^{s-\mu_+ - \frac{1}{2}, \delta-\mu_+}(\mathbb{R} \times \operatorname{int} Y_+) \end{array} \tag{2}$$

for every fixed choice of δ and for all $s \in \mathbb{R}, s > \max\{\mu_\pm + \frac{1}{2}\}$. Let us compare (2) with a mixed boundary value problem on the infinite stretched cone $(\operatorname{int} X)^\wedge = \mathbb{R}_+ \times \operatorname{int} X \ni (r,x)$ that we obtain by substituting the diffeomorphism $\mathbb{R} \to \mathbb{R}_+, t \to e^{-t} = r$. To this end we introduce the space

$$\mathcal{H}^{s,\gamma}(M^\wedge), \;\; s,\gamma \in \mathbb{R}, \;\; M^\wedge := \mathbb{R}_+ \times M \ni (r,x),$$

for a closed compact C^∞ manifold M of dimension d as the completion of the space $C_0^\infty(\mathbb{R}_+, C^\infty(M))$ with respect to the norm
$\{\frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} ||R^s(\operatorname{Im} \boldsymbol{w}) M_{r \to \boldsymbol{w}} u(\boldsymbol{w})||^2_{L^2(M)} d\boldsymbol{w}\}^{\frac{1}{2}}$, where $M_{r\to\boldsymbol{w}}$ is the Mellin transform $M_{r\to\boldsymbol{w}} u(\boldsymbol{w}) = \int_0^\infty r^{\boldsymbol{w}-1} u(r) dr$ on $u(r) \in C_0^\infty(\mathbb{R}_+, C^\infty(M))$ (which is holomorphic in $\boldsymbol{w}$), $\Gamma_\beta := \{\boldsymbol{w} \in \mathbb{C} : \operatorname{Re} \boldsymbol{w} = \beta\}$, and $R^s(\tau) \in L^s_{\mathrm{cl}}(M; \mathbb{R}_\tau)$ is an order reducing family of order s. If X is a compact C^∞ manifold with C^∞ boundary ∂X we define

$$\mathcal{H}^{s,\gamma}((\operatorname{int} X)^\wedge) := \{u|_{(\operatorname{int} X)^\wedge} : u \in \mathcal{H}^{s,\gamma}((2X)^\wedge)\},$$

where $2X$ is the double of X, obtained by gluing together two copies $X_\pm$ along the common boundary ∂X, with X_+ being identified with X. Then $v(t,x) \to u(r,x)$, defined by $u(e^{-t},x) = v(t,x)$, induces an isomorphism

$$H^{s,\delta}(\mathbb{R} \times X) \to \mathcal{H}^{s,\gamma}(X^\wedge)$$

for $\gamma = \delta + \frac{d+1}{2}$, $d = \dim X$.

The operator A then takes the form

$$\boldsymbol{A} := r^{-2} \sum_{j=0}^{2} \boldsymbol{a}_j(r)(-r\partial_r)^j,$$

i.e., is a Fuchs type differential operator on the infinite stretched cone $(\operatorname{int} X)^\wedge = \mathbb{R}_+ \times \operatorname{int} X$, with coefficients $\boldsymbol{a}_j \in C^\infty(\overline{\mathbb{R}}_+ \times \operatorname{Diff}^{2-j}(X))$. Moreover, the boundary operators are transformed into

$$\boldsymbol{T}_\pm := \mathrm{r}_\pm \boldsymbol{B}_\pm \quad \text{for} \quad \boldsymbol{B}_\pm := r^{-\mu_\pm} \sum_{k=0}^{\mu_\pm} \boldsymbol{b}_{k,\pm}(r)(-r\partial_r)^k$$

with coefficients $\boldsymbol{b}_{k,\pm} \in C^\infty(\overline{\mathbb{R}}_+, \operatorname{Diff}^{\mu_\pm - k}(U_\pm))$ for $U_\pm$ as above. Assuming the coefficients $\boldsymbol{a}_j$ and $\boldsymbol{b}_{k,\pm}$ to be independent of r for large r, obtain continuous operators

$$\boldsymbol{\mathcal{A}} := \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{T}_- \\ \boldsymbol{T}_+ \end{pmatrix} : \mathcal{H}^{s,\gamma}((\operatorname{int} X)^\wedge) \to \begin{array}{c} \mathcal{H}^{s-2,\gamma-2}((\operatorname{int} X)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\mu_- - \frac{1}{2}, \gamma - \mu_- - \frac{1}{2}}((\operatorname{int} Y_-)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\mu_+ - \frac{1}{2}, \gamma - \mu_+ - \frac{1}{2}}((\operatorname{int} Y_+)^\wedge) \end{array} \qquad (3)$$

for all $s \in \mathbb{R}, s > \max\{\mu_\pm + \frac{1}{2}\}$. Here and in future δ is fixed and chosen below in a suitable way.

The operator (3) represents a mixed boundary value problem in a cone $(\operatorname{int} X)^\wedge$ with a subdivision of $\partial X^\wedge$ into $Y_\pm^\wedge$, where now the interface $Z^\wedge$ (written in stretched form) has conical singularities (in this description at $r = 0$). According to the calculus of operators on a manifold with conical points the operator (3) has a conormal symbol, defined as the operator family

$$\sigma_{\mathrm{c}}(\boldsymbol{\mathcal{A}})(\boldsymbol{w}) := \begin{pmatrix} \sigma_{\mathrm{c}}(\boldsymbol{A}) \\ \sigma_{\mathrm{c}}(\boldsymbol{T}_-) \\ \sigma_{\mathrm{c}}(\boldsymbol{T}_+) \end{pmatrix} (\boldsymbol{w}) : H^s(\operatorname{int} X) \to \begin{array}{c} H^{s-2}(\operatorname{int} X) \\ \oplus \\ H^{s-\mu_- - \frac{1}{2}}(\operatorname{int} Y_-) \\ \oplus \\ H^{s-\mu_+ - \frac{1}{2}}(\operatorname{int} Y_+) \end{array}$$

holomorphic in $\boldsymbol{w} \in \boldsymbol{C}$, where

$$\sigma_{\mathrm{c}}(\boldsymbol{A})(\boldsymbol{w}) = \sum_{j=0}^{2} \boldsymbol{a}_j(0)\boldsymbol{w}^j, \quad \sigma_{\mathrm{c}}(\boldsymbol{T}_\pm)(\boldsymbol{w}) = \mathrm{r}_\pm \sum_{k=0}^{\mu_\pm} \boldsymbol{b}_{k,\pm}(0)\boldsymbol{w}^k.$$

2 Reduction to the boundary

Let us consider another boundary operator

$$\boldsymbol{T} = \mathrm{r}\boldsymbol{B} \quad \text{for} \quad \boldsymbol{B} = r^{-\mu}\sum_{k=0}^{\mu} \boldsymbol{b}_k(r)(-r\partial_r)^k$$

with coefficients $\boldsymbol{b}_k \in C^\infty(\overline{\mathbb{R}}_+ \times \mathrm{Diff}^{\mu-k}(U))$ for a neighbourhood U of Y. Let us assume, as above, that $\boldsymbol{a}_j, \boldsymbol{b}_k$ are independent of r for large r. Then the operator

$$\boldsymbol{\mathcal{D}} := \begin{pmatrix} \sum_{j=0}^{2} \boldsymbol{a}_j(r)(-r\partial_r)^j \\ \mathrm{r}\sum_{k=0}^{\mu} \boldsymbol{b}_k(r)(-r\partial_r)^k \end{pmatrix} : \mathcal{H}^{s,\gamma}((\mathrm{int}\, X)^\wedge) \to \begin{array}{c} \mathcal{H}^{s-2,\gamma}((\mathrm{int}\, X)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\mu-\frac{1}{2},\gamma}(Y^\wedge) \end{array} \tag{4}$$

for all $\gamma, s \in \mathbb{R}, s > \mu + \frac{1}{2}$, represents a boundary value problem on $X^\wedge$. Assume that $\boldsymbol{T}$ satisfies the Shapiro-Lopatinskij condition with respect to $\boldsymbol{A}$ (in the Fuchs type sense, cf. [6]). Then the conormal symbol

$$\sigma_\mathrm{c}(\boldsymbol{\mathcal{D}})(\boldsymbol{w}) := \begin{pmatrix} \sum_{j=0}^{2} \boldsymbol{a}_j(0)\boldsymbol{w}^j \\ \mathrm{r}\sum_{k=0}^{\mu} \boldsymbol{b}_k(0)\boldsymbol{w}^k \end{pmatrix} \tag{5}$$

is a holomorphic (operator-valued) function in $\boldsymbol{w} \in \boldsymbol{C}$ and defines a parameter-dependent elliptic family of boundary value problems on X with the parameter $\tau = \mathrm{Im}\, \boldsymbol{w}$. There is then a countable set $\boldsymbol{D} \subset \boldsymbol{C}$ with finite intersection $\boldsymbol{D} \cap \{\boldsymbol{w} \in \boldsymbol{C} : c \leq \mathrm{Re}\, \boldsymbol{w} \leq c'\}$ for every $c \leq c'$ such that the operators (5) define isomorphisms

$$\sigma_\mathrm{c}(\boldsymbol{\mathcal{D}})(\boldsymbol{w}) : H^s(\mathrm{int}\, X) \to \begin{array}{c} H^{s-2}(\mathrm{int}\, X) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}(Y) \end{array}$$

for all $\boldsymbol{w} \in \boldsymbol{C} \setminus \boldsymbol{D}$ and all sufficiently large $s \in \mathbb{R}$. Since the main purpose of our investigation is to determine admissible corner weights of mixed problems from now on for simplicity we assume the coefficients a_j and b_k to be independent of r. We then have

$$\boldsymbol{\mathcal{D}} = \mathrm{op}_M^{\gamma-\frac{d}{2}}(\sigma_\mathrm{c}(\boldsymbol{\mathcal{D}}))$$

defines isomorphisms (4) for all $\gamma \in \mathbb{R}$ such that $\Gamma_{\frac{d+1}{2}-\gamma} \cap \boldsymbol{D} = \emptyset$ and all sufficiently large $s \in \mathbb{R}$. Here, $\Gamma_\beta := \{\boldsymbol{w} \in \boldsymbol{C} : \boldsymbol{w} = \beta + i\tau, \tau \in \mathbb{R}\}$.

We have

$$\mathrm{op}_M^{\gamma-\frac{d}{2}}(\sigma_\mathrm{c}(\boldsymbol{\mathcal{D}})) = \frac{1}{2\pi i}\int_{\Gamma_{\frac{d+1}{2}-\gamma}} \int_0^\infty \left(\frac{r}{r'}\right)^{-\boldsymbol{w}} \sigma_\mathrm{c}(\boldsymbol{\mathcal{D}})(\boldsymbol{w})u(r')\frac{dr'}{r'}d\boldsymbol{w}$$

$$= \frac{1}{2\pi} r^{-\frac{d+1}{2}+\gamma} \int_{-\infty}^{\infty}\int_0^\infty \left(\frac{r}{r'}\right)^{-i\tau} \sigma_\mathrm{c}(\boldsymbol{\mathcal{D}})(\frac{d+1}{2} - \gamma + i\tau)(r')^{\frac{d+1}{2}-\gamma}u(r')\frac{dr'}{r'}d\tau.$$

As noted before the transformation $u(r,x) \to u(e^{-t},x)$ induces an isomorphism $\mathcal{H}^{s,\frac{d+1}{2}}(X^\wedge) \to H^s(\mathbb{R} \times \operatorname{int} X)$ for all $s \in \mathbb{R}$. Hence it follows an operator

$$\mathcal{D} := \operatorname{op}^\delta(\mathfrak{d}) = F^{-1}\mathfrak{d}(w)F : H^{s,\delta}(\mathbb{R} \times \operatorname{int} X) \to \begin{array}{c} H^{s-2,\delta}(\mathbb{R} \times \operatorname{int} X) \\ \oplus \\ H^{s-\mu-\frac{1}{2},\delta}(\mathbb{R} \times Y), \end{array} \tag{6}$$

where $\mathfrak{d}(w) := {}^{\mathrm{t}}(\mathfrak{e}(w) \quad \mathfrak{t}(w)) = \sigma_{\mathrm{c}}(\boldsymbol{\mathcal{D}})(\boldsymbol{w}), \boldsymbol{w} = iw$, that is an isomorphism for all $\delta \in \mathbb{R}$ such that $I_\delta \cap D = \emptyset$. Here $I_\beta := \{w \in \mathbb{C} : \operatorname{Im} w = \beta\}, \beta \in \mathbb{R}$, and $D = \{w \in \mathbb{C} : iw \in \boldsymbol{D}\}$.

If M is a compact C^∞ manifold by $L^\mu_{\mathrm{cl}}(M;\mathbb{R}^l)$ we denote the space of all classical pseudo-differential operators of order $\mu \in \mathbb{R}$ on M depending on a parameter $\lambda \in \mathbb{R}$. Moreover, if F is a Fréchet space and $U \subseteq \mathbb{C}$ an open set by $\mathcal{A}(U,F)$ we denote the space of all holomorphic functions in U with values in F.

We now employ the fact that for every constants $c \le c'$ there exists a holomorphic operator function

$$\mathfrak{r}(w) \in \mathcal{A}(\mathbb{C}, L^{s-\mu-\frac{1}{2}}_{\mathrm{cl}}(Y))$$

such that

$$\mathfrak{r}(\tau + i\beta) \in L^{s-\mu-\frac{1}{2}}_{\mathrm{cl}}(Y;\mathbb{R}_\tau)$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals, and

$$\mathfrak{r}(\tau + i\beta) : H^{s-\mu-\frac{1}{2}}(Y) \to L^2(Y)$$

is a family of isomorphisms for all $\tau \in \mathbb{R}$ and all $c \le \beta \le c'$ ($s \in \mathbb{R}$ is now fixed). Then, in particular, we obtain an isomorphism

$$\operatorname{op}^\delta(\mathfrak{r}) : H^{s-\mu-\frac{1}{2},\delta}(\mathbb{R} \times Y) \to H^{0,\delta}(\mathbb{R} \times Y),$$

$\delta \in \mathbb{R}$. We will choose $\mathfrak{r}(w)$ as follows.

Let $\alpha \in \mathbb{R}$ (which plays the role of $s - \mu - \frac{1}{2}$), and fix a collar neighbourhood $\cong [-1,1] \times Z$ of the interface $Z \subset Y$. Choose local coordinates $(n,z) \in [-1,1] \times U$ for an open set $U \subset \mathbb{R}^{d-2}$ with covariables $(\nu,\zeta) \in \mathbb{R}^{d-1}$, and form a symbol of the following kind:

$$p^\alpha_-(n,\nu,\zeta,\lambda) := \left(f\left(\frac{\nu}{C\langle\zeta,\lambda\rangle}\right) \langle\zeta,\lambda\rangle - i\nu \right)^{\alpha\omega(n)} \langle\nu,\zeta,\lambda\rangle^{\alpha(1-\omega(n))}. \tag{7}$$

Here $\omega \in C^\infty_0(-1,1)$ is a real-valued function, $0 \le \omega \le 1$, that is equal to 1 in a neighbourhood of the origin, $\lambda \in \mathbb{R}^l$, and $f(\nu) \in \mathcal{S}(\mathbb{R})$ is a function such that $f(0) = 1$ and $\operatorname{supp} F^{-1}f \subset \mathbb{R}_-$ (with the Fourier transform on the n-axis). We then have $p^\alpha_-(n,\nu,\zeta,\lambda) \in S^\alpha_{\mathrm{cl}}(\mathbb{R} \times \mathbb{R}^{d-1+l}_{\nu,\zeta,\lambda})$, and p^α_- is elliptic with respect to the covariable (ν, ζ, λ) when $C > 0$ is chosen sufficiently large. On Y we now define a parameter-dependent elliptic operator $\mathfrak{p}_-^\alpha(\lambda) \in L^\alpha_{\mathrm{cl}}(Y; \mathbb{R}^l)$ taking $p_-^\alpha(n, \nu, \zeta, \lambda)$ as local amplitude functions in the collar neighbourhood of Z and $\langle \eta, \lambda \rangle^\alpha$ outside that neighbourhood, with η being the covariable on Y. The precise (standard) construction in terms of an open covering of Y by charts, a subordinate partition unity, etc., may be found in [3]. In similar manner, starting from $p_+^\alpha(n, \nu, \zeta, \lambda)$, defined as the complex conjugate of (7), we obtain a family $\mathfrak{p}_+^\alpha(\lambda) \in L^\alpha_{\mathrm{cl}}(Y; \mathbb{R}^l)$.

By virtue of the specific properties of the symbol (7) in a neighbourhood of Z we have the following results.

Let $\mathrm{e}_+^s : H^s(\mathrm{int}\, Y_+) \to H^s(Y)$ denote a continuous operator such that $\mathrm{r}_+\mathrm{e}_+^s = \mathrm{id}_{H^s(\mathrm{int}\, Y_+)}$. Moreover, for $s > -\frac{1}{2}$ we consider $\mathrm{e}_+ : H^s(\mathrm{int}\, Y_+) \to H^{\min(s,0)}(Y)$, the operator of extension by 0 to the opposide side of Y.

Theorem 2.1 *There is a constant $M > 0$ such that the operators*

$$\mathfrak{p}_-^\alpha(\lambda) : H^s(\mathrm{int}\, Y) \to H^{s-\alpha}(Y)$$

and

$$\mathrm{r}_+\mathfrak{p}_-^\alpha(\lambda)\mathrm{e}_+^s : H^s(\mathrm{int}\, Y_+) \to H^{s-\alpha}(\mathrm{int}\, Y_+)$$

are isomorphisms for all $\lambda \in \mathbb{R}^l, |\lambda| \geq M$. For $s > -\frac{1}{2}$ also

$$\mathrm{r}_+\mathfrak{p}_-^\alpha(\lambda)\mathrm{e}_+ : H^s(\mathrm{int}\, Y_+) \to H^{s-\alpha}(\mathrm{int}\, Y_+)$$

is a family of isomorphisms for all $\lambda \in \mathbb{R}^l, |\lambda| \geq M$. We then have $(\mathrm{r}_+\mathfrak{p}_-^\alpha(\lambda)\mathrm{e}_+)^{-1} = \mathrm{r}_+(\mathfrak{p}_-^\alpha)^{-1}(\lambda)\mathrm{e}_+$. An analogous result holds for $\mathfrak{p}_+(\lambda)$ when we interchange $+$ and $-$ signs.

Let $L^\alpha_{\mathrm{cl}}(Y; \mathbb{C} \times \mathbb{R}^l)$ denote the space of all $h(w, \lambda) \in \mathcal{A}(\mathbb{C}, L^\alpha_{\mathrm{cl}}(Y; \mathbb{R}^l_\lambda))$ such that

$$h(\tau + i\beta, \lambda) \in L^\alpha_{\mathrm{cl}}(Y; \mathbb{R}^{1+l}_{\tau,\lambda})$$

for all $\beta \in \mathbb{R}$, uniformly in compact β-intervals. Let us now replace the parameter λ by $(\tau, \lambda) \in \mathbb{R}^{1+l}$ and consider the corresponding families $\mathfrak{p}_\pm^\alpha(\tau, \lambda)$. Choose a $\psi(b) \in C_0^\infty(\mathbb{R})$ such that $\psi(b) \equiv 1$ in a neighbourhood of $b = 0$. Then, setting

$$\mathfrak{r}_\pm^\alpha(w, \lambda) := \int_{\mathbb{R}} e^{-iwb} \Big\{ \int_{\mathbb{R}} \psi(b) e^{i\tau b} \mathfrak{p}_\pm^\alpha(\tau, \lambda) đ\tau \Big\} db \tag{8}$$

we obtain an operator function in $L^\alpha_{\mathrm{cl}}(Y; \mathbb{C} \times \mathbb{R}^l)$.

Theorem 2.2 *For every constants $c \leq c'$ there exists an $M > 0$ such that*

$$\mathfrak{r}_\pm^\alpha(w, \lambda) : H^s(Y) \to H^{s-\alpha}(Y)$$

and

$$\mathrm{r}_+\mathfrak{r}_-^\alpha(w,\lambda)\mathrm{e}_+^s : H^s(\mathrm{int}\,Y_+) \to H^{s-\alpha}(\mathrm{int}\,Y_+),$$

$$\mathrm{r}_-\mathfrak{r}_+^\alpha(w,\lambda)\mathrm{e}_-^s : H^s(\mathrm{int}\,Y_-) \to H^{s-\alpha}(\mathrm{int}\,Y_-,)$$

are isomorphisms for all $c \le \mathrm{Im}\,w \le c'$ *and all* $\lambda \in \mathbb{R}^l, |\lambda| \ge M$. *For* $s > -\frac{1}{2}$ *also*

$$\mathrm{r}_+\mathfrak{r}_-^\alpha(w,\lambda)\mathrm{e}_+ : H^s(\mathrm{int}\,Y_+) \to H^{s-\alpha}(\mathrm{int}\,Y_+),$$

$$\mathrm{r}_-\mathfrak{r}_+^\alpha(w,\lambda)\mathrm{e}_- : H^s(\mathrm{int}\,Y_-) \to H^{s-\alpha}(\mathrm{int}\,Y_-)$$

are families of isomorphisms for those w *and* λ.

A proof may be found in [4].

In the following we also use the notation $\mathrm{e}_\pm^s$ and $\mathrm{e}_\pm$ for the corresponding extension operators $H^{s,\delta}(\mathbb{R}\times(\mathrm{int}\,Y_\pm)) \to H^{s,\delta}(\mathbb{R}\times Y), s\in\mathbb{R}$, and $H^{s,\delta}(\mathbb{R}\times(\mathrm{int}\,Y_\pm)) \to H^{\min(s,0),\delta}(\mathbb{R}\times Y), s\in\mathbb{R}, s>-\frac{1}{2}$, respectively, for any $\delta\in\mathbb{R}$.

Corollary 2.3 *Let* $\mathfrak{r}_-(w,\lambda)$ *denote the operator function of Theorem 2.2. Then*

$$\mathrm{op}^\delta(\mathfrak{r}_-^\alpha)(\lambda) : H^{s,\delta}(\mathbb{R}\times Y) \to H^{s-\alpha,\delta}(\mathbb{R}\times Y),$$

$$\mathrm{r}_+\mathrm{op}^\delta(\mathfrak{r}_-^\alpha)(\lambda)\mathrm{e}_+^s : H^{s,\delta}(\mathbb{R}\times\mathrm{int}\,Y_+) \to H^{s-\alpha,\delta}(\mathbb{R}\times\mathrm{int}\,Y_+)$$

for $\delta, s\in\mathbb{R}$ *and*

$$\mathrm{r}_+\mathrm{op}^\delta(\mathfrak{r}_-^\alpha)(\lambda)\mathrm{e}_+ : H^{s,\delta}(\mathbb{R}\times\mathrm{int}\,Y_+) \to H^{s-\alpha,\delta}(\mathbb{R}\times\mathrm{int}\,Y_+)$$

for $\delta, s\in\mathbb{R}, s>-\frac{1}{2}$, *are isomorphisms for all* $|\lambda|\ge M$ *for a suitable* $M>0$. *Analogous relations hold when we interchange* $+$ *and* $-$ *signs.*

We now fix a $\lambda_1\in\mathbb{R}^l, |\lambda_1|>M$, and set $\mathfrak{r}_\pm^\alpha(w) := \mathfrak{r}_\pm^\alpha(w,\lambda_1)$. It is known that there is a meromorphic inverse $(\mathfrak{r}_\pm^\alpha)^{-1}(w)$, and we then have

$$\mathrm{op}^\delta((\mathfrak{r}_\pm^\alpha)^{-1}) = (\mathrm{op}^\delta(\mathfrak{r}_\pm^\alpha))^{-1}.$$

Similarly, the operators $\mathrm{r}_+\mathrm{op}^\delta(\mathfrak{r}_-^\alpha)\mathrm{e}_+^s$, $\mathrm{r}_+\mathrm{op}^\delta(\mathfrak{r}_-^\alpha)\mathrm{e}_+$ and $\mathrm{r}_-\mathrm{op}^\delta(\mathfrak{r}_+^\alpha)\mathrm{e}_-^s$, $\mathrm{r}_-\mathrm{op}^\delta(\mathfrak{r}_+^\alpha)\mathrm{e}_-$ can be inverted.

From the operator (6) we now pass to a reduction of orders to 0 on the boundary. As above we write $\mathfrak{d}(w) := {}^{\mathrm{t}}(\mathfrak{e}(w) \quad \mathfrak{t}(w))$ and form

$$\mathrm{diag}\,(1,\mathrm{op}^\delta(\mathfrak{r}_+^\alpha))\mathrm{op}^\delta(\mathfrak{d}) = \mathrm{op}^\delta\begin{pmatrix}\mathfrak{e}\\ \mathfrak{r}_+^\alpha\mathfrak{t}\end{pmatrix} : H^{s,\delta}(\mathbb{R}\times\mathrm{int}\,X) \to \begin{array}{c} H^{s-2,\delta}(\mathbb{R}\times\mathrm{int}\,X)\\ \oplus\\ L^{2,\delta}(\mathbb{R}\times Y)\end{array}$$

where $\alpha = s-\mu-\frac{1}{2}$. The order reduction with the $+$ operator is taken for convenience; we could take any other order reduction as well.

With the restriction operators $\mathrm{r}_\pm$ to $\mathbb{R}\times \operatorname{int} Y_\pm$ and the extensions $\mathrm{e}_\pm$ by zero to $\mathbb{R}\times \operatorname{int} Y$ we have an isomorphism

$$(\mathrm{e}_- \quad \mathrm{e}_+): \begin{array}{c} L^{2,\delta}(\mathbb{R}\times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_+) \end{array} \to L^{2,\delta}(\mathbb{R}\times Y)$$

with the inverse ${}^{\mathrm{t}}(\mathrm{r}_- \quad \mathrm{r}_+)$.

Similarly as in the calculus of pseudo-differential boundary value problems the operator function $\mathfrak{d}(w)$ has a meromorphic inverse $\mathfrak{d}^{-1}(w) =: (\mathfrak{g}(w) \quad \mathfrak{k}(w))$.

Remark 2.4 *It is known that the Laurent coefficients of $\mathfrak{d}^{-1}$ are smoothing operators of finite rank, more precisely, smoothing in the calculus of boundary value problems on X with the transmission property at Y.*

Let us now form the operator

$$\begin{aligned} \mathcal{L} &:= (\mathrm{op}^\delta(\mathfrak{g}) \quad \mathrm{op}^\delta(\mathfrak{k}(\mathfrak{r}_+^\alpha)^{-1})\mathrm{e}_- \quad \mathrm{op}^\delta(\mathfrak{k}(\mathfrak{r}_+^\alpha)^{-1})\mathrm{e}_+) \\ &: \begin{array}{c} H^{s-2,\delta}(\mathbb{R}\times \operatorname{int} X) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_+) \end{array} \to H^{s,\delta}(\mathbb{R}\times \operatorname{int} X) \end{aligned}$$

which is an isomorphism (recall that $\alpha = s-\mu-\frac{1}{2}$). Multiplying $\mathcal{L}$ from the left by $\mathcal{A}$, cf. the formula (2), we obtain the operator

$$\mathcal{A}\mathcal{L}: \begin{array}{c} H^{s-2,\delta}(\mathbb{R}\times \operatorname{int} X) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R}\times Y_+) \end{array} \to \begin{array}{c} H^{s-2,\delta}(\mathbb{R}\times \operatorname{int} X) \\ \oplus \\ H^{s-\mu_--\frac{1}{2},\delta}(\mathbb{R}\times \operatorname{int} Y_-) \\ \oplus \\ H^{s-\mu_+-\frac{1}{2},\delta}(\mathbb{R}\times \operatorname{int} Y_+) \end{array}. \tag{9}$$

By virtue of $\mathcal{D}\mathcal{D}^{-1} = \operatorname{diag}(1,1)$ we obtain the operator $\mathcal{A}\mathcal{L}$ in the form

$$\mathcal{A}\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 \\ T_-G & T_-KR^{-1}\mathrm{e}_- & T_-KR^{-1}\mathrm{e}_+ \\ T_+G & T_+KR^{-1}\mathrm{e}_- & T_+KR^{-1}\mathrm{e}_+ \end{pmatrix}$$

where we employ the abbreviation $G := \mathrm{op}^\delta(\mathfrak{g})$, $K := \mathrm{op}^\delta(\mathfrak{k})$, $R := \mathrm{op}^\delta(\mathfrak{r}_+^\alpha)$, $\alpha = s-\mu-\frac{1}{2}$.

We also want to reduce the Sobolev spaces on the $\mathbb{R}\times \operatorname{int} Y_\mp$ on the right of (9) to zero. To this end we take the elements $\mathfrak{r}_\pm^{\alpha_\mp}(w)$, $\alpha_\mp = s-\mu_\mp-\frac{1}{2}$.

Set

$$R_- := \mathrm{r}_-\mathrm{op}^\delta(\mathfrak{r}_+^{\alpha_-})\mathrm{e}_-, \quad R_+ := \mathrm{r}_+\mathrm{op}^\delta(\mathfrak{r}_-^{\alpha_+})\mathrm{e}_+$$

for $s > \max\{\mu_+, \mu_-\}$. Setting $\mathcal{R} := \mathrm{diag}\,(1, R_-, R_+)$, and multiplying (9) from the left by $\mathcal{R}$ we get an operator

$$\mathcal{A}_0 := \mathcal{R}\mathcal{A}\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 \\ R_-T_-G & R_-T_-KR^{-1}\mathrm{e}_- & R_-T_-KR^{-1}\mathrm{e}_+ \\ R_+T_+G & R_+T_+KR^{-1}\mathrm{e}_- & R_+T_+KR^{-1}\mathrm{e}_+ \end{pmatrix}$$

with the 2×2 lower right corner

$$\begin{pmatrix} R_-T_-KR^{-1}\mathrm{e}_- & R_-T_-KR^{-1}\mathrm{e}_+ \\ R_+T_+KR^{-1}\mathrm{e}_- & R_+T_+KR^{-1}\mathrm{e}_+ \end{pmatrix} : \begin{array}{c} L^{2,\delta}(\mathbb{R} \times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R} \times Y_+) \end{array} \to \begin{array}{c} L^{2,\delta}(\mathbb{R} \times Y_-) \\ \oplus \\ L^{2,\delta}(\mathbb{R} \times Y_+) \end{array}. \tag{10}$$

The latter operator represents the reduction of our mixed problem to the boundary, combined with suitable reductions of orders.

3 Ellipticity with interface conditions

We assume that the boundary condition T_- is the restriction of T to $\mathrm{int}\, Y_-$, that means, $\mu = \mu_-$, or $\alpha = \alpha_-$. In that case, since the order reducing operators R and R_- are connected by the relation $R_- = \mathrm{r}_- R \mathrm{e}_-$, we obtain

$$R_-T_-KR^{-1}\mathrm{e}_- = \mathrm{id}_{L^{2,\delta}(\mathbb{R}\times Y_-)} \tag{11}$$

and

$$R_-T_-KR^{-1}\mathrm{e}_+ = 0. \tag{12}$$

In fact, from $T = \mathrm{r}B$ for a differential operator B in a neighbourhood of Y it follows that $T_- = \mathrm{r}_-\mathrm{r}B$ which implies that $\mathrm{r}BK = 1$ and

$$R_-T_-KR^{-1}\mathrm{e}_- = \mathrm{r}_- R \mathrm{e}_- \mathrm{r}_- R^{-1}\mathrm{e}_-,$$

i.e., we obtain (11). Moreover, (12) is equal to $\mathrm{r}_- R\mathrm{e}_-\mathrm{r}_- R^{-1}\mathrm{e}_+$ which vanishes because of $\mathrm{r}_- R^{-1}\mathrm{e}_+ = \mathrm{r}_-\mathrm{op}^\delta(\mathfrak{r}_+^{-\alpha})\mathrm{e}_+ = 0$. Thus the operator (10) is a triangular matrix with the lower right corner

$$F := R_+T_+KR^{-1}\mathrm{e}_+ : L^{2,\delta}(\mathbb{R} \times Y_+) \to L^{2,\delta}(\mathbb{R} \times Y_+). \tag{13}$$

The operator (13) can be written in the form

$$F = \mathrm{op}^\delta(\mathfrak{f})$$

for a meromorphic operator family

$$\mathfrak{f}(w) = \mathrm{r}_+ \mathfrak{r}_-^{\alpha_+}(w)\mathrm{e}_+ \mathfrak{t}_+(w)\mathfrak{k}(w)\mathfrak{r}_+^{-\alpha_-}(w)\mathrm{e}_+ : L^2(Y_+) \to L^2(Y_+).$$

The operators $\mathfrak{f}(w)$ are parameter-dependent elliptic of order zero, with the parameter $\mathrm{Re}\, w = \tau \in \mathbb{R}$. The homogeneous principal boundary symbol $\sigma_\partial(\mathfrak{f})(z,\tau,\zeta)$ is a family of continuous operators

$$\sigma_\partial(\mathfrak{f})(z,\tau,\zeta) : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \tag{14}$$

independent of the choice of δ and homogeneous in the sense

$$\sigma_\partial(\mathfrak{f})(z,\lambda\tau,\lambda\zeta) = \sigma_\partial(\mathfrak{f})(z,\tau,\zeta)$$

for every $\lambda \in \mathbb{R}_+, (\tau,\zeta) \neq 0$.

By construction the operator family $f(w)$ depends on $s \in \mathbb{R}$. We now assume $s \in \mathbb{R}$ to be chosen in such a way that (14) is a family of Fredholm operators for all $(\tau,\zeta) \neq 0$. The criterion for that is that the subordinate conormal symbol has no zeros on the line $\Gamma_{\frac{1}{2}}$. This property will be checked in our concrete example below. In the case of the Fredholm property we have a K-theoretic index element

$$\mathrm{ind}_{\boldsymbol{S}^*Z}\sigma_\partial(\mathfrak{f}) \in K(\boldsymbol{S}^*Z);$$

here $\boldsymbol{S}^*Z$ is defined as the compact space $\{(z,\tau,\zeta) \in \mathbb{R} \times T^*Z : |\tau,\zeta| = 1\}$ with the canonical projection $\pi_1 : \boldsymbol{S}^*Z \to Z$. Another condition to be imposed is

$$\mathrm{ind}_{\boldsymbol{S}^*Z}\sigma_\partial(\mathfrak{f}) \in \pi_1^* K(Z).$$

There is a block matrix family of isomorphisms

$$\begin{pmatrix} \sigma_\partial(\mathfrak{f})(z,\tau,\zeta) & \sigma_\partial(\mathfrak{k})(z,\tau,\zeta) \\ \sigma_\partial(\mathfrak{t})(z,\tau,\zeta) & \sigma_\partial(\mathfrak{q})(z,\tau,\zeta) \end{pmatrix} : \pi_1^* \begin{pmatrix} L^2(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi_1^* \begin{pmatrix} L^2(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix}$$

for suitable $J_\pm \in \mathrm{Vect}(Z)$ between the corresponding pull backs with respect to π_1.

We now choose a system of charts $\chi_j : U_j \to \mathbb{R}^{d-2}$ on Z for an open covering $(U_j)_{j=1,\dots,N}$ of Z. Let $(\varphi_j)_{j=1,\dots,N}$ be a subordinate partition of unity and $(\psi_j)_{j=1,\dots,N}$ a system of functions $\psi_j \in C_0^\infty(U_j)$ such that $\psi_j \equiv 1$ on $\mathrm{supp}\,\varphi_j$ for all j. Moreover, let $\sigma, \tilde{\sigma} \in C^\infty(Y_+)$ be supported in a collar neighbourhood of Z, $\tilde{\sigma} \equiv 1$ in a neighbourhood of $\mathrm{supp}\,\sigma$, and $\sigma \equiv 1$ in a neighbourhood of Z. We then define the operator family

$$\sum_{j=1}^{N} \begin{pmatrix} \sigma\varphi_j(\chi_j^* \times \mathrm{id}) & 0 \\ 0 & \varphi_j\chi_j^* \end{pmatrix} \mathrm{Op}_z(g_j)(\tau) \begin{pmatrix} (\chi_j^* \times \mathrm{id})^{-1}\tilde{\sigma}\psi_j & 0 \\ 0 & (\chi_j^*)^{-1}\psi_j \end{pmatrix} \tag{15}$$

where $g_j(z,\tau,\zeta)$ is given by $\chi(\tau,\zeta)\begin{pmatrix} 0 & \sigma_\partial(\mathfrak{k}) \\ \sigma_\partial(\mathfrak{t}) & \sigma_\partial(\mathfrak{q}) \end{pmatrix}(z,\tau,\zeta)$ in local coordinates with respect to the charts $\chi_j : U_j \to \mathbb{R}^{d-2}$ and $\chi_j \times \mathrm{id} : U_j \times [0,1) \to$

$\mathbb{R}^{d-2} \times \overline{\mathbb{R}}_+$ on Z and in a collar neighbourhood of Z with the normal variable in $[0,1)$.

Now (15) is a block matrix family of operators

$$g(\tau) := \begin{pmatrix} 0 & g_{12} \\ g_{21} & g_{22} \end{pmatrix}(\tau) : \begin{array}{c} L^2(Y_+) \\ \oplus \\ L^2(Z, J_-) \end{array} \longrightarrow \begin{array}{c} L^2(Y_+) \\ \oplus \\ L^2(Z, J_+) \end{array}.$$

Moreover,

$$\begin{pmatrix} \mathfrak{f}(\tau + i\delta) & g_{12}(\tau) \\ g_{21}(\tau) & g_{22}(\tau) \end{pmatrix} \tag{16}$$

is a family of Fredholm operators which defines isomorphisms for $|\tau| > C$ for some constant $C > 0$. Similarly as (8) we now pass to a holomorphic operator function

$$\mathfrak{g}(w) := \int_{\mathbb{R}} e^{-iwb} \Big\{ \int \psi(b) e^{-i\tau b} g(\tau) đ\tau \Big\} db$$

for a $\psi \in C_0^\infty(\mathbb{R})$ that is equal to 1 near the origin (clearly, the integrals may be carried out for the entries separately). We then have $\mathfrak{g}(w) = (\mathfrak{g}_{ij}(w))_{i,j=1,2}$ with $\mathfrak{g}_{11}(w) = 0$. This gives us a family of operators

$$\begin{pmatrix} \mathfrak{f}(w) & \mathfrak{g}_{12}(w) \\ \mathfrak{g}_{21}(w) & \mathfrak{g}_{22}(w) \end{pmatrix} : \begin{array}{c} L^2(Y_+) \\ \oplus \\ L^2(Z, J_-) \end{array} \longrightarrow \begin{array}{c} L^2(Y_+) \\ \oplus \\ L^2(Z, J_+) \end{array} \tag{17}$$

which is meromorphic in $w \in \mathbb{C}$.

Proposition 3.1 *There is a discrete set $M \subset \mathbb{R}$ such that* (17) *is a family of isomorphisms for all $w = \tau + i\delta, \tau \in \mathbb{R}, \delta \in \mathbb{R} \setminus M$.*

Proof. The family (17) is parameter-dependent elliptic in the class of boundary value problems on Y_+ (of order zero and without the transmission property at Z), cf. [2], [8], with parameter $\tau = \mathrm{Re}\, w$. The meromorphy is clear by construction; $\mathfrak{g}_{ij}(w)$ are even holomorphic for all i, j. Let us assume for the moment that also $\mathfrak{f}(w)$ is holomorphic in the complex plane. The operators (16) are parameter-dependent elliptic and the principal parameter-dependent interior and boundary symbols are independent of δ. The same is true of (17), i.e., (17) is Fredholm for every $w \in \mathbb{C}$ and a holomorphic operator function. Moreover, there is a constant $c > 0$ such that (16) are isomorphisms for all $|\tau| > c$. Thus our operator function satisfies a well known condition on holomorphic Fredholm families which are isomorphic for at least one value of the complex parameter. This gives us the invertibility for all w with $\mathrm{Im}\, w$ outside some discrete set.

In the case that $\mathfrak{f}(w)$ is meromorphic we can argue in a similar manner when we take into account that the Laurent coefficients are smoothing and of finite rank, cf. Remark 2.4. □

We have (17) as a meromorphic operator function which is invertible for all $w \in \mathbb{C} \setminus N$ for some discrete set $N \subset \mathbb{C}$ such that $N \cap \{c \leq \operatorname{Im} w \leq c'\}$ is finite for every $c \leq c'$. Our next objective is to pass from the symbol $\mathfrak{a}(w) : H^s(\operatorname{int} X) \to \tilde{H}^{s-2}(\operatorname{int} X)$ for $\tilde{H}^{s-2}(\operatorname{int} X) := H^{s-2}(\operatorname{int} X) \oplus H^{s-\mu_- -\frac{1}{2}}(\operatorname{int} Y_-) \oplus H^{s-\mu_+ -\frac{1}{2}}(\operatorname{int} Y_+)$ to an operator function $\tilde{\mathfrak{a}}(w)$ by adding extra entries of trace and potential type such that $\tilde{\mathfrak{a}}(w) : H^s(\operatorname{int} X) \oplus L^2(Z, J_-) \to \tilde{H}^{s-2}(\operatorname{int} X) \oplus L^2(Z, J_+)$ are meromorphic and invertible in such a sense. To this end we form the block matrix operator family

$$\begin{pmatrix} 1 & 0 & 0 \\ \mathfrak{m}(w) & \mathfrak{f}(w) & \mathfrak{g}_{12}(w) \\ 0 & \mathfrak{g}_{21}(w) & \mathfrak{g}_{22}(w) \end{pmatrix} : \begin{matrix} L^2(Y_-) \\ \oplus \\ L^2(Y_+) \\ \oplus \\ L^2(Z, J_-) \end{matrix} \to \begin{matrix} L^2(Y_-) \\ \oplus \\ L^2(Y_+) \\ \oplus \\ L^2(Z, J_+) \end{matrix} \tag{18}$$

for the meromorphic operator function $\mathfrak{m}(w) := \mathrm{r}_+ \mathfrak{r}_-^{\alpha_+}(w) \mathrm{e}_+ \mathfrak{t}_+(w) \mathfrak{k}(w) \mathfrak{r}_+^{-\alpha_-}(w) \mathrm{e}_-$ which has the property

$$\operatorname{op}^\delta(\mathfrak{m}) = R_+ T_+ K R^{-1} \mathrm{e}_-.$$

Moreover, for $\mathfrak{n}_\pm(w) := \mathrm{r}_\pm \mathfrak{r}_\mp^{\alpha_\pm}(w) \mathrm{e}_\pm \mathfrak{t}_\pm(w) \mathfrak{g}(w)$ we have

$$\operatorname{op}^\delta(\mathfrak{n}_\pm) = R_\pm T_\pm G.$$

Setting

$$\tilde{\mathfrak{a}}_0(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathfrak{n}_-(w) & 1 & 0 & 0 \\ \mathfrak{n}_+(w) & \mathfrak{m}(w) & \mathfrak{f}(w) & \mathfrak{g}_{12}(w) \\ 0 & 0 & \mathfrak{g}_{21}(w) & \mathfrak{g}_{22}(w) \end{pmatrix} : \begin{matrix} H^{s-2}(\operatorname{int} X) \\ \oplus \\ L^2(Y_-) \\ \oplus \\ L^2(Y_+) \\ \oplus \\ L^2(Z, J_-) \end{matrix} \to \begin{matrix} H^{s-2}(\operatorname{int} X) \\ \oplus \\ L^2(Y_-) \\ \oplus \\ L^2(Y_+) \\ \oplus \\ L^2(Z, J_+) \end{matrix}$$

gives us an operator $\tilde{\mathcal{A}}_0 = \operatorname{op}^\delta(\tilde{\mathfrak{a}}_0)$ that has $\mathcal{A}_0$ as the upper left corner.

Setting $\mathfrak{l}(w) := \operatorname{diag}\left(\mathfrak{g}(w), \mathfrak{k}(w)(\mathfrak{r}_+^\alpha(w))^{-1}\mathrm{e}_-, \mathfrak{k}(w)(\mathfrak{r}_-^\alpha(w))^{-1}\mathrm{e}_+\right)$ and $\mathfrak{r}(w) := \operatorname{diag}\left(1, \mathrm{r}_- \mathfrak{r}_+^{\alpha_-}(w)\mathrm{e}_-, \mathrm{r}_+ \mathfrak{r}_-^{\alpha_+}(w)\mathrm{e}_+\right)$ we have $\mathcal{L} = \operatorname{op}^\delta(\mathfrak{l})$ and $\mathcal{R} = \operatorname{op}^\delta(\mathfrak{r})$. Moreover, let

$$\tilde{\mathfrak{l}}(w) := \operatorname{diag}\left(\mathfrak{l}(w), \mathrm{id}_{L^2(Z,J_-)}\right)$$

and

$$\tilde{\mathfrak{r}}(w) = \operatorname{diag}\left(\mathfrak{r}(w), \mathrm{id}_{L^2(Z,J_+)}\right).$$

We then obtain an operator function

$$\tilde{\mathfrak{a}}(w) := \tilde{\mathfrak{r}}^{-1}(w)\tilde{\mathfrak{a}}_0(w)\tilde{\mathfrak{l}}^{-1}(w) : \begin{array}{c} H^s(\operatorname{int} X) \\ \oplus \\ L^2(Z, J_-) \end{array} \rightarrow \begin{array}{c} H^{s-2}(\operatorname{int} X) \\ \oplus \\ H^{s-\mu_- - \frac{1}{2}}(\operatorname{int} Y_-) \\ \oplus \\ H^{s-\mu_+ - \frac{1}{2}}(\operatorname{int} Y_+) \\ \oplus \\ L^2(Z, J_+) \end{array}. \tag{19}$$

Remark 3.2 *We have*

$$\tilde{\mathfrak{a}}(w) = \begin{pmatrix} \mathfrak{a}(w) & \mathfrak{k}_Z(w) \\ \mathfrak{t}_Z(w) & \mathfrak{q}_Z(w) \end{pmatrix}$$

where $\mathfrak{a}(w)$ is the symbol of the original mixed problem (2)*. The other entries play the role of trace, potential, etc., symbols with respect to the interface Z.*

Theorem 3.3 *There is a discrete set $N \subset \mathbb{C}, N \cap \{c \leq \operatorname{Im} w \leq c'\}$ finite for every $c \leq c'$, such that $\tilde{\mathfrak{a}}(w)$ is a family of isomorphisms for all $w \in \mathbb{C} \backslash N$.*

Proof. From Proposition 3.1 we have an operator function of the asserted kind. Thus (18) as well as $\tilde{\mathfrak{a}}_0(w)$ also have this property. Finally, $\tilde{\mathfrak{a}}(w)$ itself is as desired, since the factors at $\tilde{\mathfrak{a}}_0(w)$ on the left hand side of (19) preserve this structure. □

Corollary 3.4 *The operator*

$$\tilde{\mathcal{A}} := \operatorname{op}^\delta(\tilde{\mathfrak{a}}) : \begin{array}{c} H^s(\mathbb{R} \times \operatorname{int} X) \\ \oplus \\ L^2(\mathbb{R} \times Z, J_-) \end{array} \rightarrow \begin{array}{c} H^{s-2}(\mathbb{R} \times \operatorname{int} X) \\ \oplus \\ H^{s-\mu_- - \frac{1}{2}}(\mathbb{R} \times \operatorname{int} Y_-) \\ \oplus \\ H^{s-\mu_+ - \frac{1}{2}}(\mathbb{R} \times \operatorname{int} Y_+) \\ \oplus \\ L^2(\mathbb{R} \times Z, J_+) \end{array}$$

is an isomorphism for all $\delta \in \mathbb{R}$ such that $I_\delta \cap N = \emptyset$.

4 The Zaremba problem

Let us consider the Zaremba problem

$$\mathcal{A} = \begin{pmatrix} \Delta \\ T_- \\ T_+ \end{pmatrix} : H^{s,\delta}(\mathbb{R} \times \operatorname{int} X) \rightarrow \begin{array}{c} H^{s-2,\delta-2}(\mathbb{R} \times \operatorname{int} X) \\ \oplus \\ H^{s-\frac{1}{2},\delta}(\mathbb{R} \times \operatorname{int} Y_-) \\ \oplus \\ H^{s-\frac{3}{2},\delta-1}(\mathbb{R} \times \operatorname{int} Y_+) \end{array}$$

on a cylinder $\mathbb{R}\times X$ for the Laplace operator Δ with Dirichlet and Neumann conditions on Y_- and Y_+, respectively, where $X := \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| \le 1\}$ and $\mathbb{R}^2$ identified with the complex plane, $Y_- := \{x = e^{i\phi} : 0 \le \phi \le \alpha\}, Y_+ := \{x = e^{i\phi} : \alpha \le \phi \le 2\pi\}$ for some $0 < \alpha < 2\pi$. We have

$$\begin{aligned} &\Delta v = e^{2t}\{\partial_t^2 v - \partial_t v + \Delta_X v\},\\ &T_- v = v(t, e^{i\phi})|_{0\le\phi\le\alpha},\\ &T_+ v = \rho^{-1}\partial_\phi v(t, e^{i\phi})|_{\alpha\le\phi\le 2\pi}, \end{aligned} \tag{20}$$

$v(t,x) \in H^{s,\delta}(\mathbb{R}\times \operatorname{int} X)$, where ρ is the exterior normal direction to Y. We have

$$\mathcal{A} = \begin{pmatrix} A \\ T_- \\ T_+ \end{pmatrix} : H^{s,\delta}(\mathbb{R}\times\operatorname{int} X) \to \begin{matrix} H^{s-2,\delta-2}(\mathbb{R}\times\operatorname{int} X) \\ \oplus \\ H^{s-\frac{1}{2},\delta}(\mathbb{R}\times\operatorname{int} Y_-) \\ \oplus \\ H^{s-\frac{1}{2},\delta}(\mathbb{R}\times\operatorname{int} Y_+) \end{matrix} \ ,$$

for every fixed δ and all $s \in \mathbb{R}, s > \frac{3}{2}$. After the diffeomorphism

$$H^{s,\delta}(\mathbb{R}\times X) \to \mathcal{H}^{s,\gamma}(X^\wedge), \quad \mathbb{R}\to\mathbb{R}_+, \quad e^{-t}\to r,$$

$\gamma = \delta + \frac{3}{2}$, the operators in (20) take the form

$$\begin{aligned} &\boldsymbol{\Delta} u = r^{-2}\{(r\partial_r)^2 u + r\partial_r u + \Delta_X u\},\\ &\boldsymbol{T}_- u = u(r, e^{i\phi})|_{0\le\phi\le\alpha},\\ &\boldsymbol{T}_+ u = \rho^{-1}\partial_\phi u(r, e^{i\phi})|_{\alpha\le\phi\le 2\pi}, \end{aligned}$$

and we get the continuous operators

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} \boldsymbol{\Delta} \\ \boldsymbol{T}_- \\ \boldsymbol{T}_+ \end{pmatrix} : \mathcal{H}^{s,\gamma}((\operatorname{int} X)^\wedge) \to \begin{matrix} \mathcal{H}^{s-2,\gamma-2}((\operatorname{int} X)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma}((\operatorname{int} Y_-)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\frac{3}{2},\gamma}((\operatorname{int} Y_+)^\wedge) \end{matrix} \ ,$$

$v(t,x) = u(e^{-t}, x)$, for every fixed γ and for all $s \in \mathbb{R}, s > \frac{3}{2}$. The corresponding conormal symbols have the form

$$\sigma_{\mathrm{c}}(\boldsymbol{\Delta})(\boldsymbol{w})u = \boldsymbol{w}^2 u - \boldsymbol{w}u + \Delta_X u, \ \ \sigma_{\mathrm{c}}(\boldsymbol{T}_-)u = u(e^{i\varphi})|_{0\le\varphi\le\alpha}, \ \ \sigma_{\mathrm{c}}(\boldsymbol{T}_+)u = \rho^{-1}\partial_\phi u|_{\alpha\le\varphi\le 2\pi},$$

$u \in H^s(X)$.

Let us take as another boundary operator $\mathrm{r}u := \boldsymbol{T}u = u(r, e^{i\phi})$ which represents the Dirichlet condition on Y. Then we have

$$\boldsymbol{\mathcal{D}} = \begin{pmatrix} (r\partial_r)^2 + r\partial_r + \Delta_X \\ \mathrm{r} \end{pmatrix} : \mathcal{H}^{s,\gamma}((\operatorname{int} X)^\wedge) \to \begin{matrix} \mathcal{H}^{s-2,\gamma-2}((\operatorname{int} X)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y^\wedge) \end{matrix}$$

for all $s, \gamma \in \mathbb{R}, s > \frac{1}{2}$, and

$$\sigma_{\mathrm{c}}(\boldsymbol{\mathcal{D}})(\boldsymbol{w})u = \begin{pmatrix} \boldsymbol{w}^2 u - \boldsymbol{w}u + \Delta_X u \\ \mathrm{r}u \end{pmatrix},$$

$u \in H^s(X)$. We have

$$\boldsymbol{\mathcal{D}} = \mathrm{op}_M^{\gamma-1}(\sigma_{\mathrm{c}}(\boldsymbol{\mathcal{D}}))$$

and

$$\mathcal{D} = \mathrm{op}^{\delta}(\mathfrak{d})$$

for $\mathfrak{d}(w) = {}^{\mathrm{t}}(\mathfrak{e}(w) \quad \mathfrak{t}(w))$, where $\mathfrak{e}(w) = -w^2 - iw + \Delta_X, \mathfrak{t}(w) = \mathrm{r}$. According to [7, Section 11.1] the symbol $\mathfrak{d}(w)$ defines isomorphisms

$$H^s(\mathrm{int}\, X) \to \begin{matrix} H^{s-2}(\mathrm{int}\, X) \\ \oplus \\ H^{s-\frac{1}{2}}(Y) \end{matrix}$$

for all $w = \tau + i\delta, \delta \in [-1, 0]$. Let us fix such a δ.

In this case we have $\alpha = \alpha_- = s - \frac{1}{2}, \alpha_+ = s - \frac{3}{2}$ as order reduction operators we take

$$\mathfrak{r}_-^{s-\frac{3}{2}}(w) = (f(\frac{\nu}{C\langle\tau\rangle})\langle\tau\rangle - i\nu)^{s-\frac{3}{2}}, \quad \mathfrak{r}_+^{-s+\frac{1}{2}}(w) = \overline{(f(\frac{\nu}{C\langle\tau\rangle})\langle\tau\rangle - i\nu)^{-s+\frac{1}{2}}},$$

$w = \tau + i\delta$.

The corresponding family (14) is a family of Fredholm operators for all $(\tau, \zeta) \neq 0$ if $s \notin \mathbb{Z} + \frac{1}{2}$, cf. [3, Proposition 3.1].

In our example we have

$$\sigma_\partial(f)(\tau) = \mathrm{r}^+ \mathrm{op}(b)(\tau)\mathrm{e}^+ \tag{21}$$

for $b(\nu, \tau) = (f(\frac{\nu}{C|\tau|})|\tau| - i\nu)^{s-\frac{3}{2}} |\nu, \tau| \overline{(f(\frac{\nu}{C|\tau|})|\tau| - i\nu)^{-s+\frac{1}{2}}}$.

According to the result from [3, Section 3.1] we have that the operator (21) is

(i) bijective for $\frac{1}{2} < s < \frac{1}{2}$,

(ii) surjective for $\frac{1}{2} < s + j < \frac{3}{2}, j \in \mathbb{N}$, where $\dim\ker \sigma_\partial(f)(\tau) = j$,

(iii) injective for $\frac{1}{2} < s + j < \frac{3}{2}, -j \in \mathbb{N}$, where $\dim\mathrm{coker}\, \sigma_\partial(f)(\tau) = -j$.

Then there is a family of isomorphisms

$$(\sigma_\partial(\mathfrak{f})(\tau) \quad \sigma_\partial(\mathfrak{k})(\tau)) : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ Z \times \mathbb{C}^{j_-} \end{matrix} \to L^2(\mathbb{R}_+)$$

for $j_- := [s - \frac{1}{2}]$.

Remark 4.1 *In problems of Zaremba type, given as meromorphic families of conormal symbols there is also a parameter-dependent variant, i.e., where w is replaced by (w, λ) and $(\mathrm{Re}\, w, \lambda) \in \mathbb{R}^{1+l}$ is the parameter. The construction of extra conditions (here of potential type) can also be performed λ-depending. Then for every weight δ there is a λ such that $\tilde{\mathfrak{a}}(w, \lambda)$, the parameter-dependent version of $\tilde{\mathfrak{a}}(w)$, is a family of isomorphisms* (19) *for all* $\mathrm{Re}\, w = \delta$, *provided that $|\lambda|$ is chosen sufficiently large.*

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