# Monotone Method for Nonlocal Systems of First Order *

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**Abstract**

In this paper, the monotone method is extended to the initial-boundary value problems of nonlocal PDE system of first order, both quasi-monotone and non-monotone. A comparison principle is established, and a monotone scheme is given.

**Key words**: System of nonlocal PDE of first order, comparison principle, monotone method

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## 1 Introduction

In this paper, we concerned with the following initial-boundary value problem of nonlocal PDE system of first order:

$$\begin{cases} \partial_t u_i + \partial_x (g_i(x) u_i) = F_i(t, x, u_1, \cdots, u_m, P_1(u_1(t, \cdot)), \cdots, P_m(u_m(t, \cdot))), & a < x < b,\ 0 < t < T, \\ g_i(a) u_i(t, a) = \int_a^b \beta_i(\xi) u_i(t, \xi) d\xi, & 0 < t < T, \\ u_i(0, x) = \phi_i(x), & a < x < b. \\ \qquad i = 1, \cdots, m \end{cases} \tag{1}$$

where $P_i(u_i(t, \cdot)) = \int_a^b u_i(t, x) dx$.

The problem arises in many applications to biology and chemistry(see [2, 3, 4, 5, 1]). For example, size structured populations dynamics i.e., population evolution with $m$ species where individuals are distinguished by size, can be formulated into (1)(for convenience, let $m = 2$) with

$$F_1 = -c_1(t, x, u_1, u_2, P_1(u_1(t, \cdot)), P_2(u_2(t, \cdot))) u_1,$$
$$F_2 = -c_2(t, x, u_1, u_2, P_1(u_1(t, \cdot)), P_2(u_2(t, \cdot))) u_2,$$

where $u_i, i = 1, 2$ are the populations in size $x$ and at time $t$ , $P_1(u_1(t, \cdot))$ and $P_2(u_2(t, \cdot))$ are total population at time $t$, $g_i$, $\beta_i$ and $c_i$ are growth rates, reproduction rates and mortality rates of the

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$i$th species, respectively. The mortality rates $c_1, i = 1, 2$ depend on the relationship between the species, e.g., for *predator-prey system*, they may read

$$c_1 = -\mu_1 - \frac{\gamma_1 P_2(u_2(t,\cdot))}{1+\alpha_1 P_1^{q_1}(u_1(t,\cdot))},$$
$$c_2 = -\mu_2 - \frac{\gamma_2 P_2(u_1(t,\cdot))}{1+\alpha_2 P_1^{q_2}(u_1(t,\cdot))},$$

where $u_1$ and $u_2$ denote the populations of predator and prey species, respectively, and $\mu_i, 1 = 1, 2$ are natural death rates, or for *cooperation population system*,

$$c_1 = -\mu_1 - \frac{\gamma_1 P_1(u_1(t,\cdot))P_2(u_2(t,\cdot))}{1+\alpha_1 P_2(u_1(t,\cdot))},$$
$$c_2 = -\mu_2 - \frac{\gamma_2 P_1(u_1(t,\cdot))P_2(u_2(t,\cdot))}{1+\alpha_2 P_1(u_1(t,\cdot))},$$

or for *competition population system*,

$$c_1 = -\mu_1 - \frac{\gamma_1 P_1(u_1(t,\cdot))P_2(u_2(t,\cdot))}{1+\alpha_1 P_1(u_1(t,\cdot))},$$
$$c_2 = -\mu_2 - \frac{\gamma_2 P_1(u_1(t,\cdot))P_2(u_2(t,\cdot))}{1+\alpha_2 P_2(u_1(t,\cdot))}.$$

The monotone method is a useful measure in studying differential equations. Over the past decades, many authors have successfully applied the monotone method to nonlinear differential equations, see[**?**, 6, **?**, **?**, **?**, **?**, **?**]. However, the monotone method is used to be only applicable to ODE, elliptic and parabolic PDE and systems, which depends on the maximum principle strongly. Several years ago, A.S. Ackleh and K. Deng developed a new monotone method to nonlocal PDE of first order, see [2, 3, 4]. They introduced a weak partial order in a competitive function space, and then proved the comparison principle. That is to say, the weak order implies the strong order. In this paper, our main goal is to extend the monotone method to the above problem by coupled solution technique.

The paper is organized as follows. In Section 2, the existence and uniqueness of the solution of (1) is shown. In Section 3, we introduce the concept of lower and upper solutions to quasi-monotone system and prove the comparison principle. In Section 4, we define two monotone sequences which convergent to the solution to (1). Section 5 and 6 devote to the non-monotone system, by introducing a new definition of lower and upper solutions and corresponding monotone sequences, the comparison principle and convergence result are proved.

## 2 Existence and uniqueness

Let $U = (u_1, \cdots, u_m)$, $U_0 = (\phi_1, \cdots, \phi_m)$, $P(U(t,\cdot)) = (P_1(u_1(t,\cdot)), \cdots, P_m(u_m(t,\cdot)))$(or $P(U)$ for short), $H = \Big(L^2(a,b)\Big)^m$ and $H^1 = \Big(H^1(a,b)\Big)^m$. We first impose that the following hypotheses.

H1) $g_i \in C^1[a,b]$, $g_i(x) > 0$, in $[a,b]$ and $g_i(b) = 0$, $i = 1, 2, \cdots, m$.

H2) $\beta_i \in C[a,b]$ and $\beta_i(x) \geq 0$, $i = 1, 2, \cdots, m$.

H3) $\phi_i(x) \in H^1(a,b), i = 1, 2, \cdots, m,\ \lim\limits_{x \to b^-} g_i(x)\phi_i(x) = 0$, and satisfy the compatibility condition

$$g_i(a)\phi_i(a) = \int_a^b \beta_i(x)\phi_i(x)dx.$$

H4) Let $\mathcal{F}(t,U) = (F_1(t,\cdot,U(t,\cdot),P(U(t,\cdot))), \cdots, F_m(t,\cdot,U(t,\cdot),P(U(t,\cdot))))$, then

$$\mathcal{F}(t,U) : [0,T] \times \Big(L^2(a,b)\Big)^m \to \Big((L^2(a,b)\Big)^m$$

is a continuously differentiable mapping.

Consider

$$\begin{cases} \dfrac{dU}{dt} = \mathcal{A}U + \mathcal{F}(t,U) \\ U(0) = U_0, \end{cases}$$

where the linear operator $\mathcal{A}(t,U) : [0,T] \times H \supset \text{dom}\mathcal{A} \to H$ is defined by

$$\mathcal{A} = \Big(-\frac{\partial}{\partial x}(g_1 u_1), \cdots, -\frac{\partial}{\partial x}(g_m u_m)\Big)$$

with

$$\text{dom}\mathcal{A} = \Big\{U \in H \Big| \quad U \in H^1, \lim_{x \to b^-} g_i u_i = 0, (g_i u_i)(a) = \int_a^b \beta_i(x) u_i(t,x)dx,\ i = 1, 2, \cdots, m.\Big\}$$

Making use of Lemmas 2.1-2.6 of [8], it is easy to prove that $\mathcal{A}$ is the infinitesimal generator of a $C_0-$semigroup $S(t)$.

**Theorem 1.** *Suppose that (H1)-(H4) hold. Then problem(1) has a unique solution for* $0 \leq t \leq T$.

The proof follows from Theorem 1.5 in §6.1 of [7].

## 3 Quasi-monotone systems

The problem (1) is called to be *quasi-monotone*, if every $F_i$ is monotone w.r.t. every $u_j, j \neq i$ and $P_j,\ j = 1, \cdots, m$. To define the upper solution and lower solution for the quasi-monotone systems, we first introduce the following notations.

Let

$$U(t,x) = (u_1(t,x), \cdots, u_m(t,x)),\ V(t,x) = (v_1(t,x), \cdots, v_m(t,x)) : D_T \to \mathbf{R}^m,$$

$$P(U) = (P_1(u_1(t,\cdot)), \cdots, P_m(u_m(t,\cdot))),$$

$$Q(V) = (Q_1(v_1(t,\cdot)), \cdots, Q_m(v_m(t,\cdot)));$$

$A_i, B_i \subset \{1, 2, \cdots, m\}$ be the first and second increasing index sets of function $F_i$, respectively, i.e., $F_i$ is increasing w.r.t $u_j$ if $j \in A_i$, and decreasing w.r.t. $u_j$ if $j \notin A_i$, and is increasing w.r.t $P_j$ if $j \in B_i$, and decreasing w.r.t. $P_j$ if $j \notin B_i$;

$$W(U, V, A_i) = (W_1(U, V, A_i), \cdots, W_m(U, V, A_i)),$$
$$Z(P, Q, B_i) = (Z_1(P, Q, B_i), \cdots, Z_m(P, Q, B_i)),$$

where

$$W_j(U, V, A_i) = \begin{cases} u_j(t, x), & j \in A_i, \\ v_j(t, x), & j \notin A_i, \end{cases}$$
$$Z_j(P, Q, B_i) = \begin{cases} P_j(u_j(t, \cdot)), & j \in B_i, \\ Q_j(v_j(t, \cdot)), & j \notin B_i. \end{cases} \tag{2}$$

For convenience, we might assume that $i \in A_i, i = 1, 2, \cdots$. Otherwise, we could transform it into such a case via taking $u_i = v_i e^{-M_i t}, i = 1, 2, \cdots, m$, where $M_i, i = 1, 2, \cdots, m$ are constants such that $M_i + \partial_{u_i} F_i \geq 0$. The existence of constants $M_i, i = 1, 2, \cdots, m$ is guarantied by the condition (H4).

**Definition 1.** *A couple of functions $U(t, x)$ and $V(t, x)$ are called a couple of lower-upper solutions of (1) on $D_T = (a, b) \times [0, T]$ if all the following hold:*

- *(i) $U(t, x), V(t, x) \in C(D_T) \cap L^\infty(D_T)$;*
- *(ii) $\underline{U}(0, x) \leq U_0(x) = (\phi_1(x), \cdots, \phi_m(x)) \leq V(0, x)$;*
- *(iii) for every $t \in (0, T)$ and every set of nonnegative test functions $\xi_i(t, x) \in C^1(\overline{D}_T)$, $i = 1, 2, \cdots, m$,*

$$\begin{aligned} \int_a^b u_i(t, x)\xi_i(t, x)dx \leq \; & \int_0^t \xi_i(\tau, a) \int_a^b \beta_i(x) u_i(\tau, x) dx d\tau + \int_a^b \phi_i(x)\xi_i(0, x)dx \\ & + \int_0^t \int_a^b [\partial_\tau \xi_i(\tau, x) + g_i(x)\partial_x \xi_i(\tau, x)] u_i(\tau, x) dx d\tau \\ & + \int_0^t \int_a^b F_i(\tau, x, W(U, V, A_i), Z(P, Q, B_i))\xi_i(\tau, x) dx d\tau, \end{aligned} \tag{3}$$

$$\begin{aligned} \int_a^b v_i(t, x)\xi_i(t, x)dx \geq \; & \int_0^t \xi_i(\tau, a) \int_a^b \beta_i(x) v_i(\tau, x) dx d\tau + \int_a^b \phi_i(x)\xi_i(0, x)dx \\ & + \int_0^t \int_a^b [\partial_\tau \xi_i(\tau, x) + g_i(x)\partial_x \xi_i(\tau, x)] v_i(\tau, x) dx d\tau \\ & + \int_0^t \int_a^b F_i(\tau, x, W(V, U, A_i), Z(Q, P, B_i))\xi_i(\tau, x) dx d\tau. \end{aligned} \tag{4}$$

Here, $U$ and $V$ are also called upper solution and lower solution to (1), respectively.

**Theorem 2(Comparison Principle).** *Suppose that the problem (1) is quasi-monotone satisfying the conditions (H1)-(H4). Let $U$ and $V$ be a pair of lower and upper solution of (1), then*

$$U \le V, \qquad \text{i.e.,} \qquad u_i(t,x) \le v_i(t,x),\ i=1,2,\cdots,m, \qquad \text{a.e. on } \overline{D}_T.$$

**Proof**: Let $w_i(t,x) = u_i(t,x) - v_i(t,x)$ , $i=1,2,\cdots$. Then

$$w_i(0,x) = u_i(0,x) - v_i(0,x) \le 0,\ i=1,2,\cdots,m, \text{ in } [a,b] \tag{5}$$

and, for any nonnegative test function $\xi_i(t,x) \in C^1(\bar{D}_T), i=1,2,\cdots,m$,

$$\begin{aligned} \int_a^b w_i(t,x)\xi_i(t,x)dx \le \quad & \int_a^b w_i(0,x)\xi_i(0,x)dx + \int_0^t \xi_i(\tau,a)\int_a^b \beta_i(x)w_i(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b [\partial_\tau \xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x)]w_i(\tau,x)dxd\tau \\ & + \sum_{j=1}^m \int_0^t\int_a^b \xi_j(\tau,x)A_{ij}(\tau,x)w_j(\tau,x)dxd\tau \\ & + \sum_{j=1}^m \int_0^t\int_a^b \xi_j(\tau,x)B_{ij}(\tau,x)\int_a^b w_j(\tau,y)dydxd\tau, \end{aligned} \tag{6}$$

where

$$\begin{aligned} A_{ij}(t,x) &= (-1)^{\sigma_1(ij)}\partial_{u_j}F_i(t,x,v_1,\cdots,v_{j-1},\theta_{ij}(t,x),u_{j+1},\cdots,u_m,Z(P,Q,B_i)), \\ B_{ij}(t,x) &= (-1)^{\sigma_2(ij)}\partial_{P_j}F_i(t,x,W(V,U,A_i),Q_1,\cdots,Q_{j-1},\eta_{ij}(t),P_{j+1},\cdots,P_m), \\ & \qquad i,j=1,2,\cdots,m, \end{aligned} \tag{7}$$

WITH $\theta_{ij}(t,x)$ between $u_j(t,x)$ and $v_j(t,x)$, $\eta_{ij}(t)$ between $P_j(u_j(t,\cdot))$ and $Q_j(v_j(t,\cdot))$, while

$$\sigma_1(ij) = \begin{cases} 0, & j \in A_i, \\ 1, & j \notin A_i, \end{cases} \qquad \sigma_2(ij) = \begin{cases} 0, & j \in B_i, \\ 1, & j \notin B_i. \end{cases}$$

Due to the monotonicity, we know that $A_{ij} \ge 0,\ \ B_{ij} \ge 0,\ i,j=1,2,\cdots,m$ on $D_T$. Then we find

$$\begin{aligned} & \int_0^t\int_a^b \xi_i(\tau,x)\left[\sum_{j=1}^m A_{ij}(\tau,x)w_j(\tau,x) + \sum_{j=1}^m B_{ij}(\tau,x)\int_a^b w_j(y,\tau)dy\right]dxd\tau \\ \le\ & \int_0^t\int_a^b \xi_i(\tau,x)\left[\sum_{j=1}^m A_{ij}(\tau,x)w_j^+(\tau,x) + \sum_{j=1}^m B_{ij}(\tau,x)\int_a^b w_j^+(y,\tau)dy\right]dxd\tau, \\ & \qquad i=1,2,\cdots,m. \end{aligned} \tag{8}$$

Consider

$$\begin{cases} \partial_\tau\xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x) = 0, & 0<\tau<t,\ a<x<b, \\ \xi_i(\tau,b) = 0, & 0<\tau<t, \\ \xi_i(t,x) = \chi_i(x), & a \le x \le b \end{cases} \qquad i=1,2,\cdots,m, \tag{9}$$

where $\chi_i(x) \in C_0^\infty(a,b)$ and $0 \le \chi_i(x) \le 1$.

The existence of $\xi_i(t,x) \in C^1(\overline{D}_T)$ follows from the fact that by $s = t - \tau$, (9) can be rewritten into

$$\begin{cases} \partial_s \xi_i(s,x) - g_i(x)\partial_x \xi_i(s,x) = 0, & 0 < s < t,\ a < x < b, \\ \xi_i(s,b) = 0, & 0 < s < t, \\ \xi_i(s,x) = \chi_i(x), & a \le x \le b \end{cases} \qquad i = 1,2,\cdots,m.$$

Clearly, $0 \le \xi_i \le 1, i = 1,2,\cdots,m$. Substituting such $\xi_i$ and (8) into (6) yields

$$\int_a^b w_i(t,x)\chi_i(x)dx \le \int_a^b w_i^+(0,x)\xi_i(0,x)dx + \int_0^t \int_a^b \left[\sum_{j=1}^m a_{ij} w_j^+(\tau,x)\right] dxd\tau, \tag{10}$$

$$i = 1,2,\cdots,m,$$

where

$$a_{ii} = \max_{\overline{D}_T}[\beta_i(x) + A_{ii}(t,x) + (b-a)B_{ii}(t,x)], \quad i = 1,2,\cdots,m,$$
$$a_{ij} = \max_{\overline{D}_T}[A_{ij}(t,x) + (b-a)B_{ij}(t,x)], \quad i,j = 1,2,\cdots,m,\ j \ne i,$$

which are all positive due to the hypotheses.

Since (10) holds for every $\chi_i, i = 1,2,\cdots,m$, we can choose two sequences $\{\chi_i^{(n)}\}, i = 1,2,\cdots,m$ on $[a,b]$ converging to, respectively,

$$\chi_i = \begin{cases} 1, & \text{if } w_i(t,x) > 0, \\ 0, & \text{otherwise} \end{cases} \quad i = 1,2,\cdots,m.$$

Thus

$$\int_a^b w_i^+(t,x)dx \le \int_0^t \int_a^b \sum_{j=1}^m a_{ij} w_j^+ dxd\tau, \quad i = 1,2,\cdots,m, \tag{11}$$

which by Gronwall's inequality leads to

$$\int_a^b w_i^+(t,x)dx = 0, \quad i = 1,2,\cdots,m.$$

The theorem is proved.

# 4 Convergence of monotone sequences for quasi-monotone systems

In this section, we construct two monotone sequences of upper and lower solutions for quasi-monotone system and show they converge to the solution.

Let $U_0(t,x) = U(t,x)$ and $V_0(t,x) = V(t,x))$ are a pair of upper and lower solutions of (1).

For $k = 1, 2, \cdots$, let $U_k(t,x) = (u_1^{(k)}(t,x), \cdots, u_m^{(k)}(t,x))$ and $V_k(t,x) = (v_1^{(k)}(t,x), \cdots, v_m^{(k)}(t,x))$ satisfy

$$\begin{cases} \partial_t u_i^{(k)} + \partial_x(g_i u_i^{(k)}) = F_i(t,x,W(U_{k-1},V_{k-1},A_i), Z(P_{k-1},Q_{k-1},B_i)) & (t,x) \in D_T, \\ g_i(a)u_i^{(k)}(t,a) = \int_a^b \beta_i(x) u_i^{(k)}(t,x)dx, & 0 < t < T, \\ u_i^{(k)}(0,x) = \phi_i(x), & a < x < b, \end{cases} \tag{12}$$

and

$$\begin{cases} \partial_t v_i^{(k)} + \partial_x(g_i v_i^{(k)}) = F_i(t,x,W(V_{k-1},U_{k-1},A_i), Z(Q_{k-1},P_{k-1},B_i)), & (t,x) \in D_T, \\ g_i(a)v_i^{(k)}(t,a) = \int_a^b \beta_i(x) v_i^{(k)}(t,x)dx, & 0 < t < T, \\ v_i^{(k)}(0,x) = \phi_i(x), & a < x < b, \end{cases} \tag{13}$$

$$i = 1, 2, \cdots, m.$$

where $P_k = P(U_k), Q_k = Q(V_k)$. The solvability of (12) and (13), which are nonlocal initial boundary value problems of nonhomogeneous linear equations, is guarantied by the theorem 1.

We first show that $U_0 \le U_1 \le V_1 \le V_0$.

Let $W(t,x) = (w_1(t,x), \cdots, w_m(t,x)) = U_0(t,x) - U_1(t,x)$. Then $w_i, i = 1, 2, \cdots, m$ satisfy (5-6) with $W(0,x) = 0$ and $A_{ij}(t,x) = B_{ij}(t,x) = 0, i,j = 1,2,\cdots,m$. Then, $U_0 \le U_1$. Similarly, it can be proved also that $V_1 \le V_0$.

By the monotonicity, we see that

$$F_i(t,x,W(U_0,V_0,A_i), Z(P_0,Q_0,B_i)) \le F_i(t,x,W(U_1,V_1,A_i), Z(P_1,Q_1,B_i))$$
$$F_i(t,x,W(V_0,U_0,A_i), Z(Q_0,P_0,B_i)) \ge F_i(t,x,W(V_1,U_1,A_i), Z(Q_1,P_1,B_i))$$
$$i = 1, 2, \cdots, m.$$

Hence, it is easy to see that $U_1$ and $V_1$ are also a couple of lower and upper solutions to (1). Then, by theorem 2,

$$U_1(t,x) \le V_1(t,x), \quad \text{on } D_T.$$

Thus, by induction, $U_k$ and $V_k$, $k = 1, 2, \cdots$ are couples of lower and upper solutions to (1), and satisfy that

$$U_0 \le U_1 \le U_2 \le \cdots \le U_k \le \cdots \le V_k \le \cdots \le V_2 \le V_1 \le V_0, \text{ in } \overline{D}_T.$$

**Theorem 3.** *Suppose that (H1)-(H4) hold and the system (1) is quasi-monotone. Furthermore, suppose that $U_0$ and $V_0$ are a pair of lower and upper solutions to (1). Then, there exist monotone* *sequences* $\{U_k(t,x)\}$ *and* $\{V_k(t,x)\}$ *which converge to the unique solution* $U(t,x)$ *uniformly for* $0 \le t \le T$. *Moreover, the order of convergence is linear.*

**Proof**: We first note that, from the pointwise convergence and the boundedness of the pair of lower and upper solutions $\{U_k(t,x)\}$ and $\{V_k(t,x)\}$, one can easily obtain the following convergence using the dominate convergence theorem

$$\int_0^T \|U_k(t,\cdot) - U_{k-1}(t,\cdot)\|dt \to 0, \qquad \int_0^T \|V_k(t,\cdot) - V_{k-1}(t,\cdot)\|dt \to 0, \qquad \text{as } k \to \infty \tag{14}$$

From the solution representation formula, we have

$$U(t) = S(t)\Phi + \int_0^t S(t-\tau)\mathcal{F}(\tau, U(\tau))d\tau, \tag{15}$$

$$U_k(t) = S(t)\Phi + \int_0^t S(t-\tau)\mathcal{B}(\tau, U_{k-1}(\tau), V_{k-1}(\tau))d\tau \tag{16}$$

and

$$V_k(t) = S(t)\Phi + \int_0^t S(t-\tau)\mathcal{B}(\tau, V_{k-1}(\tau), U_{k-1}(\tau))d\tau \tag{17}$$

where

$$\Phi = (\phi_1(\cdot), \cdots, \phi_m(\cdot)),$$

$$\mathcal{B}(t,U,V) = (F_1(t,\cdot, W(U,V,A_1), Z(P,Q,B_1)), \cdots, F_m(t,\cdot, W(U,V,A_m), Z(P,Q,B_m))).$$

Since $S(t)$ is a $C_0-$semigroup, there exist positive constants $M_0$ and $a$ such that

$$\|S(t)\| \le M_0 e^{at}.$$

By (H4), both $\mathcal{F}(t,U)$ and $\mathcal{B}(t,U,V)$ are continuous in $t$ and Lipschitz continuous in $U$ uniformly for $0 \le t \le T$ and $U_0 \le U, V \le V_0$, there is a positive number $M_1$ such that

$$\begin{aligned}
\|U_k(t) - U(t)\| &\le \int_0^t \|S(t-\tau)[\mathcal{B}(\tau, U_{k-1}(\tau), V_{k-1}(\tau)) - \mathcal{F}(\tau, U(\tau))]\|d\tau \\
&\le M_0 e^{at} \int_0^t M_1 \left(\|U_{k-1}(\tau) - U(\tau)\| + \|V_{k-1}(\tau) - U(\tau)\|\right) d\tau \\
&\le M_0 M_1 e^{aT} \int_0^t \left(\|U_k(\tau) - U(\tau)\| + \|V_k(\tau) - U(\tau)\|\right) d\tau \\
&\quad + M_0 M_1 e^{aT} \int_0^t \left(\|U_k(\tau) - U_{k-1}(\tau)\| + \|V_k(\tau) - V_{k-1}(\tau)\|\right) d\tau
\end{aligned} \tag{18}$$

$$\begin{aligned}
\|V_k(t) - U(t)\| &\le \int_0^t \|S(t-\tau)[\mathcal{B}(\tau, V_{k-1}(\tau), U_{k-1}(\tau)) - \mathcal{F}(\tau, U(\tau))]\|d\tau \\
&\le M_0 e^{at} \int_0^t M_1 \left(\|V_{k-1}(\tau) - U(\tau)\| + \|U_{k-1}(\tau) - U(\tau)\|\right) d\tau \\
&\le M_0 M_1 e^{aT} \int_0^t \left(\|V_k(\tau) - U(\tau)\| + \|U_k(\tau) - U(\tau)\|\right) d\tau \\
&\quad + M_0 M_1 e^{aT} \int_0^t \left(\|U_k(\tau) - U_{k-1}(\tau)\| + \|V_k(\tau) - V_{k-1}(\tau)\|\right) d\tau
\end{aligned} \tag{19}$$

Using Gronwall's inequality, we then obtain

$$\begin{aligned}
&\|U_k(t) - U(t)\| + \|V_k(t) - U(t)\| \\
\leq\ & M_0 M_1 e^{(aT + M_0 M_1 \exp(aT))t} \int_0^t (\|U_k(\tau) - U_{k-1}(\tau)]\| + \|V_k(\tau) - V_{k-1}(\tau)\|)\, d\tau \\
&\to 0, \text{ as } k \to \infty
\end{aligned}$$

As for the linear convergence, in view of (6-19), we have that

$$\begin{aligned}
&\|U_k(t) - U(t)\| + \|V_k(t) - U(t)\| \\
\leq\ & M_3 \int_0^t (\|U_k(\tau) - U(\tau)]\| + \|V_k(t) - U(\tau)\|)\, d\tau \\
&+ M_3 \int_0^t (\|U(\tau) - U_{k-1}(\tau)]\| + \|U(\tau) - V_{k-1}(\tau)\|)\, d\tau
\end{aligned}$$

where $M_3 = M_0 M_1 e^{aT}$. By Gronwall's inequality again, we finally obtain

$$\sup_{[0,T]} (\|U_k(t) - U(t)\| + \|V_k(t) - U(t)\|) \leq T M_3 e^{T M_3} \sup_{[0,T]} (\|U_{k-1}(t) - U(t)]\| + \|V_{k-1}(t) - U(t)\|)\,.$$

The proof is complete.

# 5 Non-quasi-monotone system

From the condition (H4), we know that there are positive numbers $l_{ij}, L_{ij}, i, j = 1, \cdots, m$ such that

$$\partial_{u_j} F_i + l_{ij} \geq 0, \quad \partial_{P_j} F_i + L_{ij} \geq 0, \quad i, j = 1, 2, \cdots, m$$

for all $u_1, \cdots, u_m, P_1, \cdots, P_m, \in \mathbf{R},\ (t, x) \in D_T$.

**Definition 2.** *A couple of functions $U(t,x)$ and $V(t,x)$ are called a couple of lower-upper solution of (1) on $D_T = (a,b) \times [0,T]$ if all the following hold:*

- *(i) $U(t,x), V(t,x) \in C(D_T) \cap L^\infty(D_T)$;*
- *(ii) $\underline{U}(0,x) \leq U_0(x) = (\phi_1(x), \cdots, \phi_m(x)) \leq V(0,x)$;*
- *(iii) for $t \in (0,T)$ and every set of nonnegative test functions $\xi_i(t,x) \in C^1(\overline{D}_T),\ i = 1, 2, \cdots, m$*

$$\begin{aligned}
\int_a^b u_i(t,x)\xi_i(t,x)dx \leq\ & \int_0^t \xi_i(\tau,a) \int_a^b \beta_i(x) u_i(\tau,x) dx d\tau + \int_a^b \phi_i(x)\xi_i(0,x)dx \\
&+ \int_0^t \int_a^b [\partial_\tau \xi_i(\tau,x) + g_i(x)\partial_x \xi_i(\tau,x)] u_i(\tau,x) dx d\tau \\
&+ \int_0^t \int_a^b F_i(\tau,x,U,P)\xi_i(\tau,x) dx d\tau \\
&+ \sum_{j=1}^m \int_0^t \int_a^b l_{ij}(u_j(\tau,x) - v_j(\tau,x))\xi_i(\tau,x) dx d\tau \\
&+ \sum_{j=1}^m \int_0^t \int_a^b L_{ij}(P_j(u_j(t,\cdot)) - Q_j(v_j(\tau,\cdot)))\xi_i(\tau,x) dx d\tau
\end{aligned} \tag{20}$$

$$
\begin{aligned}
\int_a^b v_i(t,x)\xi_i(t,x)dx \geq\ & \int_0^t \xi_i(\tau,a)\int_a^b \beta_i(x)v_i(\tau,x)dxd\tau + \int_a^b \phi_i(x)\xi_i(0,x)dx \\
&+\int_0^t\int_a^b [\partial_\tau \xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x)]v_i(\tau,x)dxd\tau \\
&+\int_0^t\int_a^b F_i(\tau,x,V,Q)\xi_i(\tau,x)dxd\tau \\
&+\sum_{j=1}^m \int_0^t\int_a^b l_{ij}(v_j(\tau,x)-u_j(\tau,x))\xi_i(\tau,x)dxd\tau \\
&+\sum_{j=1}^m \int_0^t\int_a^b L_{ij}(Q_j(v_j(t,\cdot)) - P_j(u_j(\tau,\cdot)))\xi_i(\tau,x)dxd\tau
\end{aligned}
\tag{21}
$$

**Theorem 4.** *Suppose that the conditions (H1), (H2) and (H4) hold. Let $(U,V)$ be a couple lower-upper solution of (1) defined as Definition 2. Then*

$$
U \leq V, \qquad \text{i.e.,} \qquad u_i(t,x) \leq v_i(t,x),\ i=1,2,\cdots,m, \qquad \text{a.e. on } \overline{D}_T
$$

**Proof**: Let $w_i(t,x) = u_i(t,x) - v_i(t,x)$ , $i=1,2,\cdots,m$. Then

$$
\begin{aligned}
\int_a^b w_i(t,x)\xi_i(t,x)dx \leq\ & \int_0^t \xi_i(\tau,a)\int_a^b \beta_i(x)w_i(\tau,x)dxd\tau \\
&+\int_0^t\int_a^b [\partial_\tau\xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x)]w_i(\tau,x)dxd\tau \\
&+\int_0^t\int_a^b \left(F_i(\tau,x,U,P) + 2\sum_{j=1}^m (l_{ij}u_j(\tau,x) + L_{ij}P_j(u_j(\tau,\cdot)))\right)\xi_i(\tau,x)dxd\tau \\
&-\int_0^t\int_a^b \left(F_i(\tau,x,V,Q) + 2\sum_{j=1}^m (l_{ij}v_j(\tau,x) + L_{ij}Q_j(v_j(\tau,\cdot)))\right)\xi_i(\tau,x)dxd\tau
\end{aligned}
\tag{22}
$$

By the condition (H4),

$$
\begin{aligned}
&\int_0^t\int_a^b \left(F_i(\tau,x,U,P) + 2\sum_{j=1}^m (l_{ij}u_j(\tau,x) + L_{ij}P_j(u_j(\tau,\cdot)))\right)\xi_i(\tau,x)dxd\tau \\
&-\int_0^t\int_a^b \left(F_i(\tau,x,V,Q) + 2\sum_{j=1}^m (l_{ij}v_j(\tau,x) + L_{ij}Q_j(v_j(\tau,\cdot)))\right)\xi_i(\tau,x)dxd\tau \\
=\ & \sum_{j=1}^m \int_0^t\int_a^b (F_i(\tau,x,v_1,\cdots,v_{j-1},u_j,u_{j+1},\cdots,u_m,P) - F_i(\tau,x,v_1,\cdots,v_{j-1},v_j,u_{j+1},\cdots,u_m,P) \\
&\quad + 2l_{ij}(u_j(\tau,x) - v_j(\tau,x)))\,\xi_i(\tau,x)dxd\tau \\
&+\sum_{j=1}^m \int_0^t\int_a^b (F_i(\tau,x,V,Q_1,\cdots,Q_{j-1},P_j,P_{j+1},\cdots,P_m) - F_i(\tau,x,V,Q_1,\cdots,Q_{j-1},Q_j,P_{j+1},\cdots,P_m) \\
&\quad + 2L_{ij}(P_j(u_j(\tau,\cdot)) - Q_j(v_j(\tau,\cdot))))\,\xi_i(\tau,x)dxd\tau \\
\leq\ & \int_0^t\int_a^b \left(\sum_{j=1}^m (A_{ij}w_j^+(\tau,x) + B_{ij}P_j(w_j^+(\tau,\cdot))\right)\xi_i(\tau,x)dxd\tau
\end{aligned}
\tag{23}
$$

where

$$A_{ij}=\max_{D_T}\ \left\{\partial_{u_j}F_i(\tau,x,v_1,\cdots,v_{j-1},\theta_{ij},u_{j+1},\cdots,u_m,P)+2l_{ij}\right\}$$
$$B_{ij}=\max_{D_T}\ \left\{\partial_{P_j}F_i(\tau,x,V,Q_1,\cdots,Q_{j-1},\eta_{ij},P_{j+1},\cdots,P_m)+2L_{ij}\right\}.$$

with $\theta_{ij}(t,x)$ between $u_j(t,x)$ and $v_j(t,x)$, $\eta_{ij}(t)$ between $P_j(u_j(t,\cdot))$ and $Q_j(v_j(t,\cdot))$.

Hence

$$\begin{aligned}\int_a^b w_i(t,x)\xi_i(t,x)dx\le\ &\int_0^t\xi_i(\tau,a)\int_a^b\beta_i(x)w_i(\tau,x)dxd\tau\\&+\int_0^t\int_a^b[\partial_\tau\xi_i(\tau,x)+g_i(x)\partial_x\xi_i(\tau,x)]w_i(\tau,x)dxd\tau\\&+\int_0^t\int_a^b\sum_{j=1}^m\left(A_{ij}w_j^+(\tau,x)+B_{ij}\int_a^b w_j^+(\tau,y)dy\right)\xi_i(\tau,x)dxd\tau\end{aligned}\tag{24}$$

Then, similar to the proof of of theorem 2, we can obtain the conclusion.

# 6 Convergence of monotone sequences for non-quasi-monotone system

For non-quasi-monotone system, we reconstruct monotone sequences of upper and lower solutions as follows:

$$\begin{cases}\partial_t u_i^{(k)}+\partial_x(g_iu_i^{(k)})=\ F_i(t,x,U_{k-1},P(U_{k-1}))-\sum_{j=1}^m l_{ij}(u_j^{(k)}-u_j^{(k-1)})\\\qquad-\sum_{j=1}^m L_{ij}(P_j(u_j^{(k)})-P_j(u_j^{(k-1)})), & (t,x)\in D_T,\\g_i(a)u_i^{(k)}(t,a)=\int_a^b\beta_i(x)u_i^{(k)}(t,x)dx, & 0<t<T,\\u_i^{(k)}(0,x)=\phi_i(x), & a<x<b,\end{cases}\tag{25}$$

and

$$\begin{cases}\partial_t v_i^{(k)}+\partial_x(g_iv_i^{(k)})=\ F_i(t,x,V_{k-1},Q(V_{k-1}))-\sum_{j=1}^m l_{ij}(v_j^{(k)}-v_j^{(k-1)})\\\qquad-\sum_{j=1}^m L_{ij}(Q_j(v_j^{(k)})-Q_j(v_j^{(k-1)})), & (t,x)\in D_T,\\g_i(a)v_i^{(k)}(t,a)=\int_a^b\beta_i(x)v_i^{(k)}(t,x)dx, & 0<t<T,\\v_i^{(k)}(0,x)=\phi_i(x), & a<x<b,\end{cases}\tag{26}$$

$$i=1,2,\cdots,m.$$

We show again that $U_0\le U_1\le V_1\le V_0$, which are defined above.

Let $W(t,x) = (w_1(t,x), \cdots, w_m(t,x)) = U_0(t,x) - U_1(t,x)$ again. Then

$$\begin{aligned} \int_a^b w_i(t,x)\xi_i(t,x)dx \le\ & \int_0^t \xi_i(\tau,a)\int_a^b \beta_i(x)w_i(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b [\partial_\tau \xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x)]w_i(\tau,x)dxd\tau \\ & - \sum_{j=1}^m \int_0^t\int_a^b l_{ij}(u_j^{(1)}(\tau,x) - v_j^{(0)}(\tau,x)\xi_i(\tau,x)dxd\tau \\ & - \sum_{j=1}^m \int_0^t\int_a^b L_{ij}(P_j(u_j^{(1)}(t,\cdot)) - Q_j(v_j^{(0)}(t,\cdot))\xi_i(\tau,x)dxd\tau \end{aligned} \tag{27}$$

By the fact that $u_j^{(0)}(t,x) \le v_j^{(0)}(t,x), (t,x) \in D_T$ and $P_j(u_j^{(0)}(t,\cdot)) \le Q_j(v_j^{(0)}(t,\cdot)), t \in [0,T]$,

$$\begin{aligned} \int_a^b w_i(t,x)\xi_i(t,x)dx \le\ & \int_0^t \xi_i(\tau,a)\int_a^b \beta_i(x)w_i(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b [\partial_\tau \xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x)]w_i(\tau,x)dxd\tau \\ & - \sum_{j=1}^m \int_0^t\int_a^b l_{ij}w_j(\tau,x)\xi_i(\tau,x)dxd\tau \\ & - \sum_{j=1}^m \int_0^t\int_a^b L_{ij}P_j(w_j(t,\cdot))\xi_i(\tau,x)dxd\tau, \end{aligned} \tag{28}$$

that is (5-6) with $w_i(0,x) = 0, i = 1,2,\cdots,m$, $A_{ij} = -l_{ij}$, $B_{ij} = -L_{ij}$, $i,j = 1,2,\cdots,m$. Then,$U_0 \le U_1$. Similarly, $V_0 \le V_1$.

To show $U_1 \le V_1$, let $W_i(t,x) = (w_1^{(i)}(t,x), \cdots, w_m^{(i)}(t,x)) = U_i(t,x) - V_i(t,x), i = 0,1$, then $W_0(t,x) \le 0$. Hence, for every set of positive function $\xi_i \in C^1(\overline{D_T}), i = 1,2,\cdots,m$

$$\begin{aligned} \int_a^b w_i^{(1)}(t,x)\xi_i(t,x)dx =\ & \int_0^t \xi_i(\tau,a)\int_a^b \beta_i(x)w_i^{(1)}(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b [\partial_\tau \xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x)]w_i^{(1)}(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b [F_i(\tau,x,U_0,P(U_0)) - F_i(\tau,x,V_0,Q(V_0))]\xi_i(\tau,x)dxd\tau \\ & - \sum_{j=1}^m \int_0^t\int_a^b l_{ij}(w_j^{(1)}(\tau,x) - w_j^{(0)}(\tau,x))\xi_i(\tau,x)dxd\tau \\ & - \sum_{j=1}^m \int_0^t\int_a^b L_{ij}(P_j(w_j^{(1)}(t,\cdot)) - P_j(w_j^{(0)}(\tau,\cdot)))\xi_i(\tau,x)dxd\tau \\ =\ & \int_0^t \xi_i(\tau,a)\int_a^b \beta_i(x)w_i^{(1)}(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b [\partial_\tau \xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x)]w_i^{(1)}(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b \sum_{j=1}^m [(A_{ij} + l_{ij})w_j^{(0)}(\tau,x) - l_{ij}w_j^{(1)}(\tau,x)]\xi_i(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b \sum_{j=1}^m [(B_{ij} + L_{ij})P_j(w_j^{(0)}(\tau,\cdot)) - l_{ij}P_jw_j^{(1)}(\tau,\cdot))]\xi_i(\tau,x)dxd\tau \end{aligned}$$

where

$$A_{ij} = \partial_{u_j} F_i(\tau, x, u_1^{(1)}, \cdots, u_{j-1}^{(1)}, \zeta_{ij}, u_{j+1}^{(0)}, \cdots, u_m^{(0)}, P(U_0))$$
$$B_{ij} = \partial_{P_j} F_i(\tau, x, U_1, P_1(u_1^{(1)}), \cdots, P_{j-1}(u_{j-1}^{(1)}), \eta_{ij}, P_{j+1}(u_{j+1}^{(0)}), \cdots, P_m(u_m^{(0)})) \tag{29}$$

with $\zeta_{ij}$ between $u_j^{(0)}$ and $u_j^{(1)}$, $\eta_{ij}$ between $P_j(u_j^{(0)})$ and $P_j(u_j^{(1)})$. Due to that $\partial_{u_j} F_i + l_{ij} \geq 0$, $\partial_{P_j} F_i + L_{ij} \geq 0$ and $u_j^{(0)} \leq u_j^{(1)}$ for all $i, j = 1, 2, \cdots, m$, we have that

$$\begin{aligned} \int_a^b w_i^{(1)}(t,x)\xi_i(t,x)dx \leq & \int_0^t \xi_i(\tau, a) \int_a^b \beta_i(x) w_i^{(1)}(\tau, x) dx d\tau \\ & + \int_0^t \int_a^b [\partial_\tau \xi_i(\tau, x) + g_i(x)\partial_x \xi_i(\tau, x)] w_i^{(1)}(\tau, x) dx d\tau \\ & - \int_0^t \int_a^b \sum_{j=1}^m l_{ij} w_j^{(1)}(\tau, x)\xi_i(\tau, x) dx d\tau \\ & - \int_0^t \int_a^b \sum_{j=1}^m l_{ij} P_j w_j^{(1)}(\tau, \cdot))\xi_i(\tau, x) dx d\tau \end{aligned}$$

that is (5-6) with $W_1(0, x) = 0$, $A_{ij} = -l_{ij}$ and $B_{ij} = -L_{ij}$,$i, j = 1, 2, \cdots, m$. Similar to the proof above, we see that $U_1 \leq V_1$.

Next, we show that $(U_1, V_1)$ is also a couple of lower-upper solution to (1). From the facts proved above, we know that

$$\begin{aligned} & F_i(t, x, U_0, P(U_0)) - F_i(t, x, U_1, P(U_1)) + \sum_{j=1}^m [l_{ij}(u_j^{(0)} - u_j^{(1)}) + L_{ij}(P_j(u_j^{(0)}) - P_j(u_j^{(1)}))] \\ = & \sum_{j=1}^m [(A_{ij} + l_{ij})(u_j^{(0)} - u_j^{(1)}) + (B_{ij} + L_{ij})(P_j(u_j^{(0)}) - P_j(u_j^{(1)}))] \leq 0 \end{aligned}$$

where $A_{ij}$ and $B_{ij}$ are given in (29), and

$$\sum_{j=1}^m [l_{ij}(v_j^{(1)} - u_j^{(1)}) + L_{ij}(Q_j(v_j^{(1)}) - P_j(u_j^{(1)}))] \leq 0.$$

Putting them into

$$\begin{aligned} \int_a^b u_i^{(1)}(t,x)\xi_i(t,x)dx = & \int_0^t \xi_i(\tau, a) \int_a^b \beta_i(x) u_i^{(1)}(\tau, x) dx d\tau + \int_a^b \phi_i(x)\xi_i(0, x) dx \\ & + \int_0^t \int_a^b [\partial_\tau \xi_i(\tau, x) + g_i(x)\partial_x \xi_i(\tau, x)] u_i^{(1)}(\tau, x) dx d\tau \\ & + \int_0^t \int_a^b F_i(\tau, x, U_0, P(U_0))\xi_i(\tau, x) dx d\tau \\ & - \sum_{j=1}^m \int_0^t \int_a^b l_{ij}(u_i^{(1)}(\tau, x) - u_i^{(0)}(\tau, x))\xi_i(\tau, x) dx d\tau \\ & - \sum_{j=1}^m \int_0^t \int_a^b L_{ij}(P_j(u_i^{(1)}(t, \cdot)) - P_j(u_i^{(0)}(\tau, \cdot)))\xi_i(\tau, x) dx d\tau, \end{aligned} \tag{30}$$

yields that

$$\begin{aligned}\int_a^b u_i^{(1)}(t,x)\xi_i(t,x)dx \leq\ & \int_0^t \xi_i(\tau,a)\int_a^b \beta_i(x)u_i^{(1)}(\tau,x)dxd\tau + \int_a^b \phi_i(x)\xi_i(0,x)dx \\ & + \int_0^t\int_a^b [\partial_\tau \xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x)]u_i^{(1)}(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b F_i(\tau,x,U_1,P(U_1))\xi_i(\tau,x)dxd\tau \\ & + \sum_{j=1}^m \int_0^t\int_a^b l_{ij}(u_i^{(1)}(\tau,x) - v_i^{(1)}(\tau,x))\xi_i(\tau,x)dxd\tau \\ & + \sum_{j=1}^m \int_0^t\int_a^b L_{ij}(P_j(u_i^{(1)}(t,\cdot)) - Q_j(v_i^{(1)}(\tau,\cdot)))\xi_i(\tau,x)dxd\tau.\end{aligned} \tag{31}$$

Similarly, we have also that

$$\begin{aligned}\int_a^b v_i^{(1)}(t,x)\xi_i(t,x)dx \geq\ & \int_0^t \xi_i(\tau,a)\int_a^b \beta_i(x)v_i^{(1)}(\tau,x)dxd\tau + \int_a^b \phi_i(x)\xi_i(0,x)dx \\ & + \int_0^t\int_a^b [\partial_\tau \xi_i(\tau,x) + g_i(x)\partial_x\xi_i(\tau,x)]v_i^{(1)}(\tau,x)dxd\tau \\ & + \int_0^t\int_a^b F_i(\tau,x,V_1,P(V_1))\xi_i(\tau,x)dxd\tau \\ & + \sum_{j=1}^m \int_0^t\int_a^b l_{ij}(v_i^{(1)}(\tau,x) - u_i^{(1)}(\tau,x))\xi_i(\tau,x)dxd\tau \\ & + \sum_{j=1}^m \int_0^t\int_a^b L_{ij}(Q_j(v_i^{(1)}(t,\cdot)) - P_j(u_i^{(1)}(\tau,\cdot)))\xi_i(\tau,x)dxd\tau.\end{aligned} \tag{32}$$

That implies that $(U_1, V_1)$ is a couple of lower-upper solution to (1).

Thus, by induction, $\{(U_k, V_k)\}_{k=0}^{\infty}$ is mixed-quasi-monotone sequence of coupled lower-upper solutions to (1) satisfying

$$U_0 \leq U_1 \leq U_2 \leq \cdots \leq U_k \leq \cdots \leq V_k \leq \cdots \leq V_2 \leq V_1 \leq V_0, \text{ in } \overline{D}_T.$$

**Theorem 5.** *Suppose that (H1)-(H4) hold. Furthermore , suppose that $(U_0, V_0)$ is a couple of lower-upper solution to (1). Then, there exist monotone sequences $\{U_k(t,x)\}$ and $\{V_k(t,x)\}$ which converge to the unique solution $U(t,x)$ uniformly for $0 \leq t \leq T$. Moreover, the order of convergence is linear.*

**Proof**: Similar to the proof of Theorem 3, we define the operator

$$\tilde{\mathcal{B}}(t, U_k, U_{k-1}) = \mathcal{F}(t, U_{k-1}) - \Lambda_1(U_k - U_{k-1}) - \Lambda_2(P(U_k) - P(U_{k-1})),$$

where $\Lambda_1 = (l_{ij})_{m\times m}, \Lambda_2 = (L_{ij})_{m\times m}$, then we have that

$$U_k(t) = S(t)\Phi + \int_0^t S(t-\tau)\tilde{\mathcal{B}}(\tau, U_k(\tau), U_{k-1}(\tau))d\tau. \tag{33}$$

where $S(t)$ is a $C_0-$semigroup mentioned above satisfying $\|S(t)\| \le M_0 e^{at}$. On the same time,

$$
\begin{aligned}
& \|\mathcal{B}(t, U_k, U_{k-1}) - \mathcal{F}(t, U)\| \\
= & \|\mathcal{F}(t, U_{k-1}) + \Lambda_1(U_k - U_{k-1}) + \Lambda_2(P(U_k) - P(U_{k-1})) - \mathcal{F}(t, U)\| \\
\le & M_1\|U_{k-1} - U\| + M_2\|U_k - U_{k-1}\|
\end{aligned}
$$

where $M_1$ is Lipshcitz constant of $\mathcal{F}$ and $M_2 = \|\Lambda_1\| + (b-a)\|\Lambda_2\|$. Thus,

$$
\begin{aligned}
\|U_k(t) - U(t)\| \le & \int_0^t \|S(t-\tau)[\mathcal{B}(\tau, U_k(\tau), U_{k-1}(\tau)) - \mathcal{F}(\tau, U(\tau))]\| d\tau \\
\le & M_1 M_0 e^{at} \int_0^t \|U_k(\tau) - U(\tau)\| d\tau \\
& + (M_1 + M_2) M_0 e^{at} \int_0^t \|U_k(\tau) - U_{k-1}(\tau)\| d\tau
\end{aligned} \tag{34}
$$

Using Gronwall's inequality, we then obtain

$$
\begin{aligned}
\|U_k(t) - U(t)\| \le & (M_1 + M_2) M_0 e^{(aT + M_0 M_1 \exp(aT))t} \int_0^t \|U_k(\tau) - U_{k-1}(\tau)]\| d\tau \\
& \to 0, \text{ as } k \to \infty.
\end{aligned}
$$

As for the linear convergence, in view of (**??**), we have that

$$
\|U_k(t) - U(t)\| \le M_3 \int_0^t \|U_k(\tau) - U(\tau)]\| d\tau + M_4 \int_0^t \|U_{k-1}(t) - U(\tau)\| d\tau
$$

where $M_3 = (2M_1 + M_2) M_0 e^{aT}$ and $M_4 = (M_1 + M_2) M_0 e^{aT}$. By Gronwall's inequality again, we finally obtain

$$
\sup_{[0,T]} \|U_k(t) - U(t)\| \le T M_4 e^{T M_3} \sup_{[0,T]} \|U_{k-1}(t) - U(t)]\|.
$$

It can be shown similarly that $\{V_k\}$ converges linearly to $U$.

The proof is complete.

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