RESOLVENTS OF ELLIPTIC BOUNDARY PROBLEMS ON CONIC MANIFOLDS

THOMAS KRAINER

Abstract. We prove the existence of sectors of minimal growth for realizations of boundary value problems on conic manifolds under natural ellipticity conditions. Special attention is devoted to the clarification of the analytic structure of the resolvent.

1. Introduction

The present article is a continuation of the investigation of resolvents of elliptic operators on conic manifolds from [8, 9] to the case of manifolds with boundary and realizations of operators under boundary conditions. Our principal focus of interest are resolvents of boundary value problems satisfying a parameter-dependent ellipticity condition that resembles the Shapiro-Lopatinsky condition.

While the study of operators on a conic manifold without boundary is mainly motivated by questions from spectral theory and geometric analysis, the analysis of boundary value problems has in addition a wide range of applications in partial differential equations. In particular, results about the structure and growth of resolvents of operators with respect to the spectral parameter have immediate consequences as regards the existence, uniqueness, and maximal regularity of solutions to parabolic linear and semilinear equations. Hence this paper naturally belongs to the study of partial differential equations in nonsmooth domains, a subject which due to its importance in models from applications has recently attracted increased attention (see [14], [16], [21], [22], [23], [24], to mention only a few). Our results, in particular, give a fairly complete picture about the existence of sectors of minimal growth for L^2-based realizations of general elliptic boundary value problems in domains with cone-like singularities on the boundary. They seem to be new also when specialized to second order equations under Dirichlet, Neumann, or oblique derivative boundary conditions.

We begin this paper with a brief discussion on manifolds with boundary and conical singularities. We recall the definitions and some properties of totally characteristic and cone differential operators, and give a short description of their symbols. We consider boundary value problems for both of these operator classes and give the suitable definition of parameter-dependent ellipticity (called b- and c-ellipticity). Our primary focus, however, are cone operators.

In case of boundary value problems for totally characteristic operators, there is just one realization of an elliptic operator, and the b-ellipticity with parameter in a

2000 *Mathematics Subject Classification.* Primary: 35J40; Secondary: 58J50, 58J05, 35J70, 35K35.

Key words and phrases. Resolvents, boundary value problems, manifolds with conical singularities, spectral theory, elliptic and parabolic equations, parametrices.

sector $\Lambda \subset \mathbb{C}$ already implies that Λ is a sector of minimal growth for this realization. The situation of boundary value problems for cone operators is completely different. There is a variety of domains for every boundary condition, cf. [4], [10], [17]. Each of these domains can be characterized by the asymptotic behavior of its elements near the singularities, and c-ellipticity with parameter in a sector Λ is not sufficient to insure that Λ is a sector of minimal growth for a given domain (c-ellipticity just entails Fredholm solvability). Similar to the boundaryless case in [8, 9], it turns out that an additional spectral condition needs to be required for a model boundary value problem on the model cone with an associated domain. The model boundary value problem is obtained by "freezing the coefficients" at the singularities, and the additional spectral condition can be regarded as a kind of Shapiro-Lopatinsky condition, but associated with the "singular boundary", coming from blowing up the conical points. The arbitrariness in the choice of a domain is the source of substantial difficulties which cannot be overcome merely by considering spaces with weights, a technique that is widely used in the literature since Kondratiev's seminal paper [15].

In the Sections 4–6 we discuss the domains of realizations of parameter-dependent elliptic boundary value problems for cone operators, the associated model boundary value problem, and the link between these two.

An important component of this paper is the parametrix construction that we perform in Section 7. Technically, our approach makes use of Boutet de Monvel's calculus away from the singularities (cf. [11], [31]), several aspects of pseudodifferential boundary value problems on conic manifolds without parameters (cf. [29]), and, near the singularities, we employ in addition some techniques from the edge-calculus as studied in the monograph [14].

Finally, in Section 8, we use our parametrix to describe the pseudodifferential structure of the resolvent and prove in Theorem 8.1 the existence of sectors of minimal growth for realizations of boundary value problems for cone operators under natural ellipticity conditions, the main result of this note:

Theorem: *Let A be a cone operator of order $m > 0$, and T be a vector of boundary conditions for A. Assume that the associated boundary value problem is c-elliptic with parameter in the closed sector $\Lambda \subset \mathbb{C}$, see Section 4.*

Let $\mathcal{D}_{\min}(A_T) \subset \mathcal{D}(A_T) \subset \mathcal{D}_{\max}(A_T)$ be any domain of A under the boundary condition $Tu = 0$ in a weighted L^2_b-space on the manifold, and let $\mathcal{D}_{\wedge,\min}(A_{\wedge,T_\wedge}) \subset \mathcal{D}_\wedge(A_{\wedge,T_\wedge}) \subset \mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge})$ be the associated domain to $\mathcal{D}(A_T)$ for the model operator $A_\wedge$ under the boundary condition $T_\wedge u = 0$ on the model cone as described in the Sections 5 and 6.

If Λ is a sector of minimal growth for $A_\wedge$ with domain $\mathcal{D}_\wedge(A_{\wedge,T_\wedge})$, then Λ is a sector of minimal growth for the operator A with domain $\mathcal{D}(A_T)$ (see Definition 1.1 below). Moreover, the resolvent of A with this domain can be written in the form

$$\big(A_{\mathcal{D}} - \lambda\big)^{-1} = \mathcal{B}_T(\lambda) + \big(A_{\mathcal{D}} - \lambda\big)^{-1}\Pi_T(\lambda),$$

where $\mathcal{B}_T(\lambda)$ is a parameter-dependent parametrix of $A - \lambda$ taking values in the minimal domain $\mathcal{D}_{\min}(A_T)$, and $\Pi_T(\lambda)$ is a regularizing projection operator onto a (finite dimensional) complement of the range of $A - \lambda$ on $\mathcal{D}_{\min}(A_T)$ (see Section 7).

The idea for proving this theorem is the construction of an invertible abstract reference problem (Theorem 7.21) that is used to reduce the resolvent constructions to the finite dimensional contribution beyond the minimal domain. In this sense, we perform a reduction to the "singular boundary", and the resulting operator family plays technically a similar role as, e.g., the Dirichlet-to-Neumann map in regular boundary value problems. However, a canonical reference domain for the operator A with boundary condition $Tu = 0$, which could be regarded as a "Dirichlet extension" with respect to the singular boundary, does not exist.

Compared to the boundaryless case in [8, 9], managing the boundary conditions requires special care and forces setting up a much more elaborated machinery. By consistently working with full operator matrices (with domains that are associated with the inhomogeneous boundary value problem), we are able to transfer some methods from the boundaryless situation. This also makes it possible to concentrate on the essential singular part of the problem when coming to the resolvent constructions in Section 8.

Acknowledgement: This article emerged from a collaboration with Juan Gil (Penn State Altoona) and Gerardo Mendoza (Temple University, Philadelphia). The author wishes to express his gratitude for many invaluable discussions on the subject of this paper.

Throughout this article, let

$$\Lambda = \{z = re^{i\theta};\ r \geq 0,\ |\theta - \theta_0| \leq \varepsilon\}$$

with $\theta_0 \in \mathbb{R}$ and $\varepsilon > 0$ be a closed sector in $\mathbb{C}$.

Definition 1.1. Let H be a Hilbert space and $A : \mathcal{D}(A) \subset H \to H$ be unbounded, densely defined, and closed, i.e. $\mathcal{D}(A)$ is complete in the graph norm

$$\|u\|_A = \|u\|_H + \|Au\|_H. \tag{1.2}$$

We call Λ a sector of minimal growth for the operator A (with domain $\mathcal{D}(A)$) if

$$A - \lambda : \mathcal{D}(A) \to H$$

is bijective for large $|\lambda| > 0$ in Λ, and if the following equivalent norm estimates for the resolvent $(A - \lambda)^{-1} : H \to \mathcal{D}(A)$ are satisfied:

i) $\|(A - \lambda)^{-1}\|_{\mathscr{L}(H)} = O(|\lambda|^{-1})$ as $|\lambda| \to \infty$.
ii) $\|(A - \lambda)^{-1}\|_{\mathscr{L}(H,\mathcal{D}(A))} = O(1)$ as $|\lambda| \to \infty$.

2. Manifolds with boundary and conical singularities

Definition 2.1. A compact conic manifold with boundary is a second countable compact Hausdorff topological space M_{sing} such that there exists a finite subset S with the following properties:

a) $M_{\mathrm{sing}} \setminus S$ is a smooth manifold with boundary.
b) Every $s \in S$ has a neighborhood $U(s) \subseteq M_{\mathrm{sing}}$ which is homeomorphic to a neighborhood $\tilde{U}$ of
$$\tilde{s} = (\{0\} \times \overline{Y})/(\{0\} \times \overline{Y})$$
in $(\overline{\mathbb{R}}_+ \times \overline{Y})/(\{0\} \times \overline{Y})$, where $\overline{Y}$ is a compact smooth manifold with boundary, and the homeomorphism restricts to a diffeomorphism $U(s)\backslash\{s\} \cong \tilde{U}\backslash\{\tilde{s}\}$.

The set S is the singular set in M_{sing}, the elements of S are called conical points.

As in the boundaryless case, analysis on conic manifolds with boundary is performed away from the conical points. Consequently, by eventually passing to

M_{sing}/S, we can and will assume henceforth that M_{sing} has only one conical point s. Note that the manifold $\overline{Y}$ in Definition 2.1 is not assumed to be connected.

It is evident from the definition that N_{sing}, the boundary of M_{sing}, is a compact conic manifold without boundary and conical point s.

Definition 2.2. A cone structure on M_{sing} is a maximal atlas consisting of a differential structure for the smooth manifold with boundary $M_{\mathrm{sing}} \setminus \{s\}$, as well as coordinate neighborhoods of the conical point s of the form $U(s)$ from Definition 2.1, where away from s the coordinate changes are C^∞-diffeomorphisms, and the coordinate change

$$\mathbb{R}_+ \times \overline{Y} \supseteq \tilde{U}\setminus\{\tilde{s}\} \cong \tilde{V}\setminus\{\tilde{s}\} \subseteq \mathbb{R}_+ \times \overline{Y}$$

of any two charts near the conical point s extends to a C^∞-mapping

$$((-\infty,0] \times \overline{Y}) \cup (\tilde{U}\setminus\{\tilde{s}\}) \longrightarrow ((-\infty,0] \times \overline{Y}) \cup (\tilde{V}\setminus\{\tilde{s}\})$$

between these open subsets of $\mathbb{R} \times \overline{Y}$. Note, in particular, that by continuity the cocycle property of coordinate changes near s is preserved up to the origin, i.e., up to $\{0\} \times \overline{Y}$.

Any cone structure on M_{sing} gives rise to a unique cone structure on the boundary N_{sing}. We will always assume that a cone structure on M_{sing} is fixed, and the boundary will be given the induced cone structure.

Let $\overline{M}$ be the compact space obtained from M_{sing} by blowing up the conical point s to $\overline{Y}$. Note that $\overline{M}$ and the canonical embedding $\overline{Y} \hookrightarrow \overline{M}$ are invariants of the cone structure, and each local chart of M_{sing} near the conical point s gives rise to a collar neighborhood of $\overline{Y}$ in $\overline{M}$.

We have a canonical identification $\overline{M}/\overline{Y} \cong M_{\mathrm{sing}}$ as compact conic manifolds with boundary. The double

$$2\overline{M} = \overline{M} \cup_{\overline{Y}} \overline{M}$$

is a compact smooth manifold with boundary, where the C^∞-structure is inherited from collar neighborhoods of $\overline{Y}$ in $\overline{M}$. Evidently, $\overline{N} = \partial(2\overline{M}) \cap \overline{M}$ is the blow-up of N_{sing}, a compact manifold with boundary $\partial\overline{Y}$.

Let us fix a defining function $x \in C^\infty(2\overline{M})$ for $\overline{Y}$ with $x > 0$ on $2\overline{M}\setminus\overline{Y}$.

Definition 2.3. By $\mathrm{Diff}_b^m(\overline{M})$ we denote the restrictions to $\overline{M}$ of the m-th order differential operators on $2\overline{M}$ which are totally characteristic with respect to $\overline{Y}$. Thus $\mathrm{Diff}_b^*(\overline{M})$ is the enveloping algebra generated by the restrictions to $\overline{M}$ of the vector fields on $2\overline{M}$ which are tangent to $\overline{Y}$ and $C^\infty(2\overline{M})$.

Observe that $\overline{N}\setminus\partial\overline{Y}$, the regular part of the boundary of $\overline{M}$, is not necessarily characteristic for the elements of $\mathrm{Diff}_b^*(\overline{M})$.

Correspondingly, let $\mathrm{Diff}_b^m(\overline{M};E,F)$ be the space of m-th order totally characteristic differential operators acting in sections of the bundles E and F. Note that we consider here complex vector bundles that are restrictions of smooth bundles on $2\overline{M}$ to $\overline{M}$.

The operators $A \in x^{-m}\,\mathrm{Diff}_b^m(\overline{M};E,F)$ are the cone operators of order m. If (x,y) are local coordinates near $p \in \overline{Y} \subset \overline{M}$ with $x \geq 0$ on $\overline{M}$ and $x = 0$ on $\overline{Y}$, then A takes the form

$$A = x^{-m} \sum_{k+|\alpha|\leq m} a_{k,\alpha}(x,y) D_y^\alpha (xD_x)^k$$

with coefficients $a_{k,\alpha}(x,y)$ that are smooth up to $x = 0$.

Totally characteristic operators $A \in \operatorname{Diff}_b^m(\overline{M}; E, F)$, $m \in \mathbb{N}_0$, have an invariant b-principal symbol on the compressed cotangent bundle ${}^bT^*\overline{M}$, see [25, 26]. Recall that ${}^bT\overline{M}$ is the bundle on $\overline{M}$ whose smooth sections are the restrictions of the vector fields on $2\overline{M}$ to $\overline{M}$ which are tangent to $\overline{Y}$. The compressed cotangent bundle ${}^bT^*\overline{M}$ is the dual of ${}^bT\overline{M}$.

In [8] the c-cotangent bundle was introduced, and it was proved that cone operators have invariantly defined principal symbols there. Consequently, with an operator $A \in x^{-m} \operatorname{Diff}_b^m(\overline{M}; E, F)$, $m \in \mathbb{N}_0$, we associate its c-principal symbol ${}^c\boldsymbol{\sigma}(A)$ on ${}^cT^*\overline{M}$, a section of the bundle $\operatorname{Hom}({}^c\pi^*E, {}^c\pi^*F)$. Here ${}^c\pi : {}^cT^*\overline{M} \to \overline{M}$ is the canonical projection map.

The definition of the c-cotangent bundle ${}^cT^*\overline{M}$ is similar to the b-construction, and its space of smooth sections are the restrictions of 1-forms from $2\overline{M}$ to $\overline{M}$ which are, over $\overline{Y}$, sections of the conormal bundle of $\overline{Y}$ in $T^*(2\overline{M})$.

There is a second principal symbol associated with an operator A, the b- or c-principal boundary symbol ${}^b\boldsymbol{\sigma}_\partial(A)$ or ${}^c\boldsymbol{\sigma}_\partial(A)$, respectively.

Definition 2.4. Let $A \in x^{-m} \operatorname{Diff}_b^m(\overline{M}; E, F)$, and let $U(p) \subset 2\overline{M}$ be a small neighborhood of the point $p \in \partial(2\overline{M}) \cap \overline{M}$ on the boundary of $\overline{M}$, and consider local coordinates (z', z_n) centered at p with $z_n \geq 0$ and $z_n = 0$ on $\partial(2\overline{M})$. Let ${}^c\boldsymbol{\sigma}(A)(z', z_n; \zeta', \zeta_n)$ be the local representation of the c-principal symbol in these coordinates, a $(N_- \times N_+)$-matrix function with $N_- = \dim E$ and $N_+ = \dim F$.

The operator family

$$ {}^c\boldsymbol{\sigma}_\partial(A)(z'; \zeta') = {}^c\boldsymbol{\sigma}(A)(z', 0; \zeta', D_{z_n}) : \mathscr{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^{N_-} \to \mathscr{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^{N_+} \tag{2.5} $$

is then a local representation of the c-principal boundary symbol ${}^c\boldsymbol{\sigma}_\partial(A)$ of A.

It is more tedious than hard to see that the c-principal boundary symbol is invariantly defined on ${}^cT^*\overline{N}$, a section of the bundle $\operatorname{Hom}({}^c\mathscr{S}_+ \otimes {}^c\pi^*E, {}^c\mathscr{S}_+ \otimes {}^c\pi^*F)$. Here ${}^c\mathscr{S}_+$ is a bundle on ${}^cT^*\overline{N}$ with fiber $\mathscr{S}(\overline{\mathbb{R}}_+)$ which comes up canonically when changing to a different local representation (2.5) near p.

Analogously, with a totally characteristic operator $A \in \operatorname{Diff}_b^m(\overline{M}; E, F)$ we associate the b-principal boundary symbol ${}^b\boldsymbol{\sigma}_\partial(A)$, a section of the bundle $\operatorname{Hom}({}^b\mathscr{S}_+ \otimes {}^b\pi^*E, {}^b\mathscr{S}_+ \otimes {}^b\pi^*F)$ over ${}^bT^*\overline{N}$.

Definition 2.6. Let $0 \in \Omega \subset \mathbb{C}$ be a conical subset. The operator family $A - \lambda \in \operatorname{Diff}_b^m(\overline{M}; E)$, $\lambda \in \Omega$, is called b-elliptic with parameter, if $\operatorname{spec}({}^b\boldsymbol{\sigma}(A)(z, \zeta)) \cap \Omega = \emptyset$ for $(z, \zeta) \in {}^bT^*\overline{M} \backslash 0$.

Analogously, for a cone operator $A \in x^{-m} \operatorname{Diff}_b^m(\overline{M}; E)$ the family $A - \lambda$ is called c-elliptic with parameter $\lambda \in \Omega$ iff $\operatorname{spec}({}^c\boldsymbol{\sigma}(A)(z, \zeta)) \cap \Omega = \emptyset$ for $(z, \zeta) \in {}^cT^*\overline{M} \backslash 0$.

Obviously, $A - \lambda \in x^{-m} \operatorname{Diff}_b^m(\overline{M}; E)$ is c-elliptic with parameter if and only if $x^m A - \lambda \in \operatorname{Diff}_b^m(\overline{M}; E)$ is b-elliptic with parameter, and, if $\Omega = \{0\}$, b- and c-ellipticity with parameter reduces to ordinary b- and c-ellipticity.

3. Boundary problems for totally characteristic operators

In this section we consider boundary value problems for totally characteristic operators $A \in \operatorname{Diff}_b^m(\overline{M}; E)$, $m \in \mathbb{N}$.

We assume henceforth that $A - \lambda$ is b-elliptic with parameter in the sector $\Lambda \subset \mathbb{C}$.

Proposition 3.1. *The b-principal boundary symbol*

$$ {}^b\boldsymbol{\sigma}_\partial(A)(z', \zeta') - \lambda : {}^b\mathscr{S}_+ \otimes {}^b\pi^*E \to {}^b\mathscr{S}_+ \otimes {}^b\pi^*E $$

is surjective and has finite dimensional kernel for all $(z',\zeta',\lambda)\in\big({}^bT^*\overline{N}\times\Lambda\big)\backslash 0$.

Consequently, the kernels form a vector bundle $\mathcal{K}$ *on* $\big({}^bT^*\overline{N}\times\Lambda\big)\backslash 0$.

For a sufficiently smooth section u of a bundle F on $\overline{M}\backslash\overline{Y}$ we denote by γu its restriction to the regular part $\overline{N}\backslash\partial\overline{Y}$ of the boundary, which gives rise to the restriction operator γ.

Now let $B_j\in\mathrm{Diff}_b^{m_j}(\overline{M};E,F_j)$, $m_j<m$, be totally characteristic, $j=1,\dots,K$. We consider the family of boundary value problems

$$\begin{aligned}(A-\lambda)u&=f\quad\text{in }\mathring{\overline{M}}=\overline{M}\backslash\overline{Y},\\ Tu&=g\quad\text{on }\mathring{\overline{N}}=\overline{N}\backslash\partial\overline{Y}\end{aligned}\tag{3.2}$$

for the operator A, where $T=(\gamma B_1,\dots,\gamma B_K)^t$.

Definition 3.3. The boundary value problem (3.2) is called b-elliptic with parameter $\lambda\in\Lambda$ if

$$\begin{pmatrix}({}^b\gamma_0\otimes I_{{}^b\pi^*F_1})\,{}^b\boldsymbol{\sigma}_\partial(B_1)(z',\zeta')\\ \vdots\\ ({}^b\gamma_0\otimes I_{{}^b\pi^*F_K})\,{}^b\boldsymbol{\sigma}_\partial(B_K)(z',\zeta')\end{pmatrix}:\mathcal{K}_{(z',\zeta',\lambda)}\to\bigoplus_{j=1}^K{}^b\pi^*F_j$$

is bijective for all $(z',\zeta',\lambda)\in\big({}^bT^*\overline{N}\times\Lambda\big)\backslash 0$. Here ${}^b\gamma_0:{}^b\mathscr{S}_+\to\mathbb{C}$ is the canonical evaluation map at zero.

Note that this notion of b-ellipticity is the appropriate version of the Shapiro-Lopatinsky condition for families of totally characteristic boundary problems.

Let $\mathfrak{m}>0$ be a b-density on $2\overline{M}$, i.e. $x\mathfrak{m}$ is a smooth everywhere positive density. Let $L^2_b(\overline{M};E)$ be the L^2-space of sections of the bundle E on $\overline{M}$ with respect to $\mathfrak{m}$ and a Hermitian inner product on E.

For $s\in\mathbb{N}_0$ let $H^s_b(\overline{M};E)$ be the Sobolev space of all L^2_b-sections u such that $Cu\in L^2_b$ for all totally characteristic operators C of order $\le s$, and let $H^s_{b,0}(\overline{M};E)$ be the closure of all C_0^∞-sections of E in $H^s_b(\overline{M};E)$. Note that the C_0^∞-sections here are supported away from the boundary $\overline{N}\cup\overline{Y}$.

For $s\in-\mathbb{N}$ let $H^s_b(\overline{M};E)$ be the dual of $H^{-s}_{b,0}(\overline{M};E)$, and analogously let $H^s_{b,0}(\overline{M};E)$ be the dual of $H^{-s}_b(\overline{M};E)$ with respect to the sesquilinear pairing induced by the L^2_b-inner product. For arbitrary real s we define $H^s_{b,0}(\overline{M};E)$ and $H^s_b(\overline{M};E)$ by interpolation. Analogously to the boundaryless case, we also consider weighted spaces $x^\mu H^s_b$ for arbitrary weights $\mu\in\mathbb{R}$.

For every $s>-1/2$ we consider the boundary value problem (3.2) as a family of continuous operators

$$\begin{pmatrix}A-\lambda\\ T\end{pmatrix}:x^\mu H^{s+m}_b(\overline{M};E)\to\begin{array}{c}x^\mu H^s_b(\overline{M};E)\\ \oplus\\ \bigoplus\limits_{j=1}^K x^\mu H_b^{s+m-m_j-1/2}(\overline{N};F_j)\end{array}.\tag{3.4}$$

Correspondingly, we consider the realization of A under the boundary condition $Tu=0$, i.e. the unbounded operator

$$A_T:\mathcal{D}^s(A_T)\subset x^\mu H^s_b(\overline{M};E)\to x^\mu H^s_b(\overline{M};E)$$

with domain $\mathcal{D}^s(A_T)=\{u\in x^\mu H^{s+m}_b(\overline{M};E);\ Tu=0\}$ that acts like A. We sometimes also say that this is the H^{s+m}-realization (or just H-realization) of A

in order to emphasize that we assume apriori smoothness of order $s+m$ in $\overset{\circ}{\overline{M}}$ as it is custom in boundary value problems.

It is a part of Theorem 3.5 below that the boundary condition

$$T : x^\mu H_b^{s+m}(\overline{M};E) \to \bigoplus_{j=1}^{K} x^\mu H_b^{s+m-m_j-1/2}(\overline{N};F_j)$$

is surjective for $s > -1/2$. In particular, the operator (3.4) is invertible for some $\lambda \in \Lambda$ if and only if $\lambda \notin \operatorname{spec}(A_T)$.

Theorem 3.5. *Let* (3.2) *be b-elliptic with parameter* $\lambda \in \Lambda$*, and let* $\mu \in \mathbb{R}$*. There exists* $R = R(\mu) > 0$ *such that for* $\lambda \in \Lambda$*,* $|\lambda| \geq R$*, the operator* (3.4) *is invertible for all* $s > -1/2$*.*

Moreover, we have $\|(A_T - \lambda)^{-1}\| = O(|\lambda|^{-1})$ *as* $|\lambda| \to \infty$ *for the* $\mathscr{L}(x^\mu L_b^2)$*-norm of the resolvent of* A *with domain* $\mathcal{D}(A_T) = \mathcal{D}^0(A_T) \subset x^\mu L_b^2(\overline{M};E)$*, and the norm of the resolvent in* $\mathscr{L}(x^\mu H_b^s)$ *of realizations in Sobolev spaces of higher regularity is polynomially bounded as* $|\lambda| \to \infty$*.*

As A with domain $\mathcal{D}(A_T)$ is closed, we thus obtain that b-ellipticity with parameter implies that Λ is a sector of minimal growth for A.

The proof of Theorem 3.5 follows by constructing a parametrix of (3.2) within a Boutet de Monvel's calculus of parameter-dependent boundary value problems of totally characteristic pseudodifferential operators (a suitable modification of the arguments given in Section 7 up to Proposition 7.8 will do).

As our interest in this article lies in resolvents and spectral properties of boundary value problems for cone operators we will not pursue this here.

Despite of the many similarities between cone operators and totally characteristic operators, the spectral theory for cone operators is much more complicated than the spectral theory of totally characteristic operators. This is underscored by a comparison of Theorem 3.5 and Theorem 8.1.

Assuming parameter-dependent ellipticity, for every weight $\mu \in \mathbb{R}$ there is only one H-realization of a totally characteristic operator A under the boundary condition $Tu = 0$, and this realization is well-behaved for large parameter values as Theorem 3.5 shows. In contrast, for every weight $\mu \in \mathbb{R}$ there are many H-realizations of cone operators, and the spectrum of every such realization could be $\mathbb{C}$ (see also [8, 9] for a discussion of the boundaryless case).

For later purposes, we recall the notion of the conormal symbol associated with a totally characteristic boundary value problem on $\overline{M}$:

If u is a smooth section of E on $2\overline{M}$ that vanishes on $\overline{Y}$, then also Pu vanishes on $\overline{Y}$ for every $P \in \operatorname{Diff}_b^*(\overline{M};E,F)$. Consequently, if v is a section of E on $\overline{Y}$ and u is any extension of v, then $(Pu)|_{\overline{Y}}$ does not depend on the choice of the extension. Thus, associated with P there is a differential operator

$$\hat{P}(0) : C^\infty(\overline{Y};E) \to C^\infty(\overline{Y};F)$$

of the same order as P. Since $\mathbb{C} \ni \sigma \mapsto x^{-i\sigma} P x^{i\sigma}$ is a family of totally characteristic operators, we so obtain the operator valued polynomial

$$\mathbb{C} \ni \sigma \mapsto \hat{P}(\sigma) \in \operatorname{Diff}^*(\overline{Y};E,F),$$

the conormal symbol associated with P. If P is part of a boundary condition γP, we associate with this condition its conormal symbol

$$\mathbb{C} \ni \sigma \mapsto \hat{\gamma}\hat{P}(\sigma),$$

where $\hat{\gamma}v$ denotes the restriction of the section v on $\overline{Y}$ to the boundary $\partial\overline{Y}$. In this way we obtain for each $\lambda \in \Lambda$ the conormal symbol of the boundary value problem (3.2), a family of boundary value problems

$$h(\sigma,\lambda) : C^\infty(\overline{Y};E) \to \begin{array}{c} C^\infty(\overline{Y};E) \\ \oplus \\ \bigoplus\limits_{j=1}^{K} C^\infty(\partial\overline{Y};F_j) \end{array} \tag{3.6}$$

on $\overline{Y}$ depending holomorphically on $\sigma \in \mathbb{C}$ and $\lambda \in \Lambda$.

Provided that (3.2) is b-elliptic with parameter $\lambda \in \Lambda$, the conormal symbol (3.6) is for every $\sigma \in \mathbb{C}$ and $\lambda \in \Lambda$ an elliptic boundary value problem on $\overline{Y}$, which is even elliptic with parameter $(\sigma,\lambda) \in \{\Im(\sigma) = \alpha\} \times \Lambda$ for every fixed $\alpha \in \mathbb{R}$.

4. Realizations of boundary problems for cone operators

Let $A \in x^{-m}\operatorname{Diff}_b^m(\overline{M};E)$, $m \in \mathbb{N}$, be a cone operator such that $A-\lambda$ is c-elliptic with parameter in the sector $\Lambda \subset \mathbb{C}$.

Analogously to the case of totally characteristic operators, we then know that the c-principal boundary symbol

$${}^c\boldsymbol{\sigma}_\partial(A)(z',\zeta') - \lambda : {}^c\mathscr{S}_+ \otimes {}^c\pi^*E \to {}^c\mathscr{S}_+ \otimes {}^c\pi^*E$$

is surjective and has finite dimensional kernel for all $(z',\zeta',\lambda) \in \big({}^cT^*\overline{N} \times \Lambda\big)\backslash 0$. Let $\mathcal{K}$ be the bundle of kernels on $\big({}^cT^*\overline{N} \times \Lambda\big)\backslash 0$.

Let $B_j \in x^{-m_j}\operatorname{Diff}_b^{m_j}(\overline{M};E,F_j)$, $m_j < m$, be cone operators, $j = 1,\ldots,K$, and consider the family of boundary value problems

$$\begin{aligned} (A-\lambda)u &= f \quad \text{in } \overset{\circ}{\overline{M}} = \overline{M}\backslash\overline{Y}, \\ Tu &= g \quad \text{on } \overset{\circ}{\overline{N}} = \overline{N}\backslash\partial\overline{Y} \end{aligned} \tag{4.1}$$

for the operator A, where $T = (\gamma B_1,\ldots,\gamma B_K)^t$.

Definition 4.2. The boundary value problem (4.1) is called c-elliptic with parameter $\lambda \in \Lambda$ if

$$\begin{pmatrix} ({}^c\gamma_0 \otimes I_{{}^c\pi^*F_1})\, {}^c\boldsymbol{\sigma}_\partial(B_1)(z',\zeta') \\ \vdots \\ ({}^c\gamma_0 \otimes I_{{}^c\pi^*F_K})\, {}^c\boldsymbol{\sigma}_\partial(B_K)(z',\zeta') \end{pmatrix} : \mathcal{K}_{(z',\zeta',\lambda)} \to \bigoplus_{j=1}^{K} {}^c\pi^*F_j$$

is bijective for all $(z',\zeta',\lambda) \in \big({}^cT^*\overline{N} \times \Lambda\big)\backslash 0$, where ${}^c\gamma_0 : {}^c\mathscr{S}_+ \to \mathbb{C}$ is evaluation at zero.

Similar to the case of totally characteristic operators, c-ellipticity is the appropriate version of the Shapiro-Lopatinsky condition for cone operators.

Lemma 4.3. *The boundary value problem* (4.1) *with cone operators A and B_j, $j = 1,\ldots,K$, is c-elliptic with parameter $\lambda \in \Lambda$ if and only if the boundary value* *problem*

$$\begin{aligned} ((x^m A) - \lambda)u &= f \quad \text{in } \mathring{\overline{M}}, \\ \gamma(x^{m_j} B_j)u &= g_j \quad \text{on } \mathring{\overline{N}},\ j = 1, \ldots, K \end{aligned}$$

is b-elliptic with parameter $\lambda \in \Lambda$ *in the sense of Definition 3.3.*

Our primary concern is to investigate the spectral properties of c-elliptic boundary value problems under the assumption of parameter-dependent ellipticity, i.e. we investigate the operator family

$$\begin{pmatrix} A - \lambda \\ T \end{pmatrix} : \mathcal{D}^s \begin{pmatrix} A \\ T \end{pmatrix} \subset x^\mu H_b^s(\overline{M}; E) \to \begin{array}{c} x^\mu H_b^s(\overline{M}; E) \\ \oplus \\ \bigoplus\limits_{j=1}^{K} x^{\mu+m-m_j} H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \end{array} \tag{4.4}$$

for $s > -1/2$ and some weight $\mu \in \mathbb{R}$, as well as the behavior of the associated family of unbounded operators

$$A - \lambda : \mathcal{D}^s(A_T) \subset x^\mu H_b^s(\overline{M}; E) \to x^\mu H_b^s(\overline{M}; E) \tag{4.5}$$

with domain $\mathcal{D}^s(A_T) = \mathcal{D}^s\binom{A}{T} \cap \ker T$. Of particular interest is of course the case $s = 0$, i.e. the $x^\mu L_b^2$-realization of the operator A under the boundary condition $Tu = 0$.

The domain in (4.4) can be any intermediate space

$$\mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} \subset \mathcal{D}^s\begin{pmatrix} A \\ T \end{pmatrix} \subset \mathcal{D}^s_{\max}\begin{pmatrix} A \\ T \end{pmatrix}$$

of the minimal and maximal $x^\mu H_b^{s+m}$-domains

$$\mathcal{D}^s_{\max}\begin{pmatrix} A \\ T \end{pmatrix} = \left\{ u \in x^\mu H_b^{s+m}(\overline{M}; E);\ \begin{array}{c} Au \in x^\mu H_b^s(\overline{M}; E), \\ \gamma B_j u \in x^{\mu+m-m_j} H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \\ \text{for } j = 1, \ldots, K \end{array} \right\},$$

$$\mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} = \mathcal{D}^s_{\max}\begin{pmatrix} A \\ T \end{pmatrix} \cap \bigcap_{\varepsilon > 0} x^{\mu+m-\varepsilon} H_b^{s+m}(\overline{M}; E).$$

As conjugation of (4.4) with the weight function x^δ for any $\delta \in \mathbb{R}$ gives rise to a unitary equivalent parameter-dependent c-elliptic boundary value problem of the form (4.1) in the corresponding shifted function spaces, we can without loss of generality base all our investigations on the weight $\mu = -m/2$. Moreover, we usually write $\mathcal{D}^s = \mathcal{D}^s\binom{A}{T}$ as well as $\mathcal{D} = \mathcal{D}^0\binom{A}{T}$.

A discussion of domains and adjoints of c-elliptic boundary value problems and normal boundary conditions is given in [4], generalizing previous results in [10] in the boundaryless case. In contrast to the mere c-elliptic case, our situation of parameter-dependent c-ellipticity makes it possible to avoid a technical discussion of the issue of normality.

The next proposition follows analogous to the boundaryless case from a corresponding analysis in the Mellin image using the conormal symbols of (4.1), see [17], [10]. The proof of the Fredholmness in part iv) follows by employing a standard parametrix (without parameters) of elliptic boundary value problems on the cone, see, e.g., [29], [4].

Proposition 4.6. *Assume that* (4.1) *is c-elliptic with parameter in some closed sector* $\Lambda \subset \mathbb{C}$.

i) $\mathcal{D}^s_{\max}$ *is complete in the norm*

$$\|u\|_{A_T} = \|u\|_{x^{-m/2}H_b^{s+m}} + \|Au\|_{x^{-m/2}H_b^s} + \sum_{j=1}^K \|\gamma B_j u\|_{x^{m/2-m_j}H_b^{s+m-m_j-1/2}}, \quad (4.7)$$

and $\mathcal{D}^s_{\min} \subset \mathcal{D}^s_{\max}$ *is a closed subspace of finite codimension.*

ii) We have

$$x^{m/2}H_b^{s+m} \hookrightarrow \mathcal{D}^s_{\min} \hookrightarrow \mathcal{D}^s_{\max} \hookrightarrow x^{-m/2+\varepsilon}H_b^{s+m}$$

for some $\varepsilon > 0$ *with continuous embeddings. In particular, the embedding* $\mathcal{D}^s_{\max} \hookrightarrow x^{-m/2}H_b^s$ *is compact.*

iii) $C_0^\infty(\overset{\circ}{\overline{M}}; E) \subset \mathcal{D}^s_{\min}$ *is a dense subspace.*

iv) For every $\lambda \in \mathbb{C}$ *the boundary value problem* (4.4) *is Fredholm with index independent of* λ *and* $s > -1/2$*, and we have the following relative index formula*

$$\operatorname{ind}\begin{pmatrix} A-\lambda \\ T \end{pmatrix}_{\mathcal{D}^s} = \operatorname{ind}\begin{pmatrix} A \\ T \end{pmatrix}_{\mathcal{D}^s_{\min}} + \dim \mathcal{D}^s/\mathcal{D}^s_{\min}. \quad (4.8)$$

Here the subscripts refer to the corresponding domains.

The quotient $\mathcal{D}^s_{\max}/\mathcal{D}^s_{\min}$ is actually independent of $s > -1/2$ and can be identified with a space of singular functions. We will come back to this in Section 6 soon (see also [9], Section 6).

From Lemma 4.3 and Theorem 3.5 we obtain that the boundary condition

$$T : x^{m/2}H_b^{s+m}(\overline{M}; E) \to \bigoplus_{j=1}^K x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N}; F_j)$$

is surjective, and necessarily so is its extension to $\mathcal{D}^s$ by Proposition 4.6. Consequently, for every $\lambda \in \mathbb{C}$,

$$A_T - \lambda : \mathcal{D}^s(A_T) \to x^{-m/2}H_b^s(\overline{M}; E)$$

is Fredholm with index

$$\operatorname{ind}(A-\lambda)_{\mathcal{D}^s(A_T)} = \operatorname{ind} A_{\mathcal{D}^s(A_T)} = \operatorname{ind}\begin{pmatrix} A \\ T \end{pmatrix}_{\mathcal{D}^s},$$

and a necessary condition for A with domain $\mathcal{D}(A_T)$ to admit nonempty resolvent set is that $\operatorname{ind}(A_{T,\min}) \le 0$ and $\operatorname{ind}(A_{T,\max}) \ge 0$ (in [8] such issues are discussed from a fairly abstract perspective, and many of the results therefore apply also to the situation under study in this article). Moreover, (4.4) is invertible for some λ if and only if (4.5) is bijective, i.e. if and only if $\lambda \notin \operatorname{spec}(A_T)$.

Let us formulate an immediate consequence of these observations (note, in particular, that this constitutes a substantial difference to the totally characteristic case):

Corollary 4.9. *Let* (4.1) *be c-elliptic with parameter in the closed sector* $\Lambda \subset \mathbb{C}$*. Then either the spectrum of the operator* $A_T : \mathcal{D}^s(A_T) \subset x^{-m/2}H_b^s(\overline{M}; E) \to x^{-m/2}H_b^s(\overline{M}; E)$ *is discrete or it is all of* $\mathbb{C}$*, and a necessary condition for the spectrum to be discrete is that* $\operatorname{ind} A_{\mathcal{D}^s(A_T)} = 0$.

Lemma 4.10. *The mapping $\mathcal{D}^s\binom{A}{T} \to \mathcal{D}^s(A_T) = \mathcal{D}^s\binom{A}{T} \cap \ker T$ is a bijection of the lattice of intermediate spaces*

$$\mathcal{D}^s_{\min}\begin{pmatrix}A\\T\end{pmatrix} \subset \mathcal{D}^s\begin{pmatrix}A\\T\end{pmatrix} \subset \mathcal{D}^s_{\max}\begin{pmatrix}A\\T\end{pmatrix}$$

onto the lattice of intermediate spaces

$$\mathcal{D}^s_{\min}(A_T) \subset \mathcal{D}^s(A_T) \subset \mathcal{D}^s_{\max}(A_T),$$

where

$$\mathcal{D}^s_{\min}(A_T) = \Big\{u \in \bigcap_{\varepsilon>0} x^{m/2-\varepsilon} H_b^{s+m}(\overline{M};E);\ Au \in x^{-m/2} H_b^s(\overline{M};E) \text{ and } Tu = 0\Big\},$$

$$\mathcal{D}^s_{\max}(A_T) = \big\{u \in x^{-m/2} H_b^{s+m}(\overline{M};E);\ Au \in x^{-m/2} H_b^s(\overline{M};E) \text{ and } Tu = 0\big\}.$$

We have $\dim \mathcal{D}^s_{\max}\binom{A}{T}/\mathcal{D}^s_{\min}\binom{A}{T} = \dim \mathcal{D}^s_{\max}(A_T)/\mathcal{D}^s_{\min}(A_T)$*. More precisely, given a basis*

$$s_j + \mathcal{D}^s_{\min}\begin{pmatrix}A\\T\end{pmatrix}, \quad j = 1,\ldots,M,$$

of $\mathcal{D}^s_{\max}\binom{A}{T}/\mathcal{D}^s_{\min}\binom{A}{T}$, we pick $u_j \in x^{m/2} H_b^{s+m}(\overline{M};E)$ with $Ts_j = Tu_j$ and obtain in this way a basis

$$(s_j - u_j) + \mathcal{D}^s_{\min}(A_T), \quad j = 1,\ldots,M,$$

of $\mathcal{D}^s_{\max}(A_T)/\mathcal{D}^s_{\min}(A_T)$.

As already mentioned, the s_j in Lemma 4.10 can be chosen to be singular functions (see also Section 6), and the domains $\mathcal{D}^s\binom{A}{T}$ as well as the corresponding domains $\mathcal{D}^s(A_T)$ are thus characterized in terms of a specified asymptotic behavior near $\overline{Y}$.

Proposition 4.11. *If $A_T - \lambda : \mathcal{D}^s(A_T) \to x^{-m/2} H_b^s(\overline{M};E)$ is invertible for some $\lambda \in \mathbb{C}$ and some domain $\mathcal{D}^s(A_T)$, then A is closed in the functional analytic sense for every domain $\mathcal{D}^s_{\min}(A_T) \subset \mathcal{D}^s \subset \mathcal{D}^s_{\max}(A_T)$, i.e. $\mathcal{D}^s_{\max}(A_T)$ is complete in the graph norm*

$$\|u\|_A = \|u\|_{x^{-m/2}H_b^s} + \|Au\|_{x^{-m/2}H_b^s}.$$

Proof. Let $(u_k)_k \subset \mathcal{D}^s_{\max}(A_T)$ be such that $u_k \to u$ in $x^{-m/2}H_b^s$ and $Au_k \to v$ in $x^{-m/2}H_b^s$. Consequently, $(A-\lambda)u_k \to v - \lambda u$ in $x^{-m/2}H_b^s$, and by the closed graph theorem the inverse

$$(A-\lambda)^{-1} : x^{-m/2} H_b^s \to \mathcal{D}^s(A_T)$$

is continuous, where $\mathcal{D}^s(A_T)$ is endowed with (4.7). Thus there exists a convergent sequence $(v_k)_k \subset \mathcal{D}^s(A_T) \subset \mathcal{D}^s_{\max}(A_T)$ with $(A-\lambda)v_k = (A-\lambda)u_k \to v - \lambda u$ as $k \to \infty$. Let $\ker(A-\lambda) \subset \mathcal{D}^s_{\max}(A_T)$ be the eigenspace of A with domain $\mathcal{D}^s_{\max}(A_T)$ associated with the eigenvalue λ. As $A - \lambda : \mathcal{D}^s_{\max}(A_T) \to x^{-m/2}H_b^s$ is Fredholm, this eigenspace is finite dimensional, and so the norm (4.7) and the $x^{-m/2}H_b^s$-norm are equivalent on this space. Consequently, the sequence $(u_k - v_k)_k \subset \ker(A-\lambda)$ is convergent with respect to (4.7), and thus $u_k = v_k + (u_k - v_k)$ is also convergent in $\mathcal{D}^s_{\max}(A_T)$ with respect to (4.7), and the limit necessarily coincides with the $x^{-m/2}H_b^s$-limit u. ∎

5. The associated boundary value problem on the model cone

For convenience, we fix from now on a collar neighborhood $U_{\overline{Y}} = \overline{Y} \times [0,1)$ of $\overline{Y}$ in $\overline{M}$. Let x be such that in this neighborhood it coincides with the projection to $[0,1)$, and the b-density $\mathfrak{m}$ be such that its pull-back equals $dy \otimes \frac{dx}{x}$. In this neighborhood, the vector bundles E and F_j, $j = 1,\ldots,K$, are isomorphic to the pull-backs of their restrictions to $\overline{Y}$, and we also fix such isomorphisms.

Every cone operator $B \in x^{-m}\operatorname{Diff}_b^m(\overline{M};E,F)$ can be written in the form

$$B = x^{-m}\sum_{k=0}^{N-1} P_k x^k + x^{N-m}\tilde{P}_N, \tag{5.1}$$

where $N \in \mathbb{N}$ is arbitrary, $\tilde{P}_N \in \operatorname{Diff}_b^m(\overline{M};E,F)$, and the $P_k \in \operatorname{Diff}_b^m(\overline{M};E,F)$ have coefficients independent of x near $\overline{Y}$.

Recall that an operator $P \in \operatorname{Diff}_b^m(\overline{M};E,F)$ is said to have coefficients independent of x near $\overline{Y}$, or simply constant coefficients, if

$$\nabla_{x\partial_x}P(u) = P(\nabla_{x\partial_x}u)$$

for any smooth section u of E supported in $U_{\overline{Y}}$. Here ∇ denotes a Hermitian connection on E or F, respectively.

Let $\overline{Y}^\wedge = \overline{\mathbb{R}}_+ \times \overline{Y}$ be the model cone, and correspondingly let $(\partial\overline{Y})^\wedge = \overline{\mathbb{R}}_+ \times (\partial\overline{Y})$ be the model cone associated with the boundary.

With $B \in x^{-m}\operatorname{Diff}_b^m(\overline{M};E,F)$ we associate on $\overline{Y}^\wedge$ the model operator $B_\wedge = x^{-m}P_0$, where P_0 is the constant term in the expansion (5.1). Moreover, if B is part of a boundary condition, we let $\gamma_\wedge B_\wedge$ be the model boundary condition associated with γB, where for every sufficiently smooth section u on $\overline{Y}^\wedge \setminus \overline{Y}$ we denote by $\gamma_\wedge u$ its restriction to the regular part of the boundary $(\partial\overline{Y})^\wedge \setminus \partial\overline{Y}$.

Consequently, for the family of boundary value problems (4.1) for the operator A there is the following associated family of model boundary value problems

$$\begin{aligned} (A_\wedge - \lambda)u &= f \quad \text{in } \mathring{\overline{Y}}^\wedge = \overline{Y}^\wedge \setminus \overline{Y}, \\ T_\wedge u &= g \quad \text{on } \partial\mathring{\overline{Y}}^\wedge = (\partial\overline{Y})^\wedge \setminus \partial\overline{Y} \end{aligned} \tag{5.2}$$

for $A_\wedge$ on the model cone $\overline{Y}^\wedge$, where $T_\wedge = (\gamma_\wedge B_{1,\wedge},\ldots,\gamma_\wedge B_{K,\wedge})^t$.

The problem (5.2) is naturally realized in the scale of $\mathcal{K}^{s,\alpha}$-spaces on the model cone. We briefly recall the definition of these spaces (see, e.g., [14]):

Definition 5.3. Let $\mathbb{D} \subset S^{n-1}$ be an embedded $(n-1)$-dimensional ball (with boundary). Let $H^s_{\mathrm{cone}}(\overline{Y}^\wedge;E)$ be the space of H^s_{loc}-distributions u such that given any coordinate patch Ω on $\overline{Y}$ diffeomorphic to an open subset of $\mathbb{D} \subset S^{n-1}$, and given any function $\varphi \in C_0^\infty(\Omega)$, we have $(1-\omega)\varphi u \in H^s(\mathbb{D}^\wedge;E)$, where $\mathbb{D}^\wedge = \mathbb{R}_+ \times \mathbb{D} \subset \mathbb{R}^n$ is regarded as the cone in $\mathbb{R}^n \setminus \{0\}$ over $\mathbb{D}$ in polar coordinates, and the Sobolev space on $\mathbb{D}^\wedge$ is the space of H^s-distributions in $\mathbb{R}^n$ restricted to that cone.

Correspondingly, we have the space $H^s_{\mathrm{cone}}((\partial\overline{Y})^\wedge;F)$ that is defined in the same way via regarding $(\partial\overline{Y})^\wedge$ (locally) as a cone in $\mathbb{R}^{n-1}$.

Here and in the sequel, $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ denotes a cut-off function near zero, i.e. ω is supported near the origin with $\omega \equiv 1$ near zero, and we consider ω a function either on $U_{\overline{Y}}$ or on $\overline{Y}^\wedge$ (or on $(\partial\overline{Y})^\wedge$) which depends only on the variable x.

For $s,\alpha\in\mathbb{R}$ we define $\mathcal{K}^{s,\alpha}(\overline{Y}^\wedge;E)$ as the space of distributions u such that

$$\omega u\in x^\alpha H_b^s(\overline{Y}^\wedge;E) \text{ and } (1-\omega)u\in x^{(n-m)/2}H^s_{\mathrm{cone}}(\overline{Y}^\wedge;E),$$

and $\mathcal{K}^{s,\alpha}((\partial\overline{Y})^\wedge;F)$ as the space of all u with

$$\omega u\in x^\alpha H_b^s((\partial\overline{Y})^\wedge;F) \text{ and } (1-\omega)u\in x^{(n-1-m)/2}H^s_{\mathrm{cone}}((\partial\overline{Y})^\wedge;F).$$

Obviously, the $\mathcal{K}^{s,\alpha}$-spaces have natural Hilbert space structures. Note, in particular, that $\mathcal{K}^{0,-m/2}=x^{-m/2}L_b^2$, and the $x^{-m/2}L_b^2$-inner product serves as the reference inner product on the model cone.

For $s>-\frac12$ the model boundary value problem (5.2) is considered as

$$\begin{pmatrix}A_\wedge-\lambda\\ T_\wedge\end{pmatrix}:\mathcal{D}^s_\wedge\begin{pmatrix}A_\wedge\\ T_\wedge\end{pmatrix}\subset\mathcal{K}^{s,-m/2}(\overline{Y}^\wedge;E)\to\begin{array}{c}\mathcal{K}^{s,-m/2}(\overline{Y}^\wedge;E)\\ \oplus\\ \bigoplus\limits_{j=1}^K\mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge;F_j)\end{array}\tag{5.4}$$

with $\mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge}\subset\mathcal{D}^s_\wedge\binom{A_\wedge}{T_\wedge}\subset\mathcal{D}^s_{\wedge,\max}\binom{A_\wedge}{T_\wedge}$, where

$$\mathcal{D}^s_{\wedge,\max}\begin{pmatrix}A_\wedge\\ T_\wedge\end{pmatrix}=\Big\{u\in\mathcal{K}^{s+m,-m/2}(\overline{Y}^\wedge;E);\ A_\wedge u\in\mathcal{K}^{s,-m/2}(\overline{Y}^\wedge;E),$$
$$\gamma_\wedge B_{j,\wedge}u\in\mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge;F_j) \text{ for } j=1,\ldots,K\Big\},$$
$$\mathcal{D}^s_{\wedge,\min}\begin{pmatrix}A_\wedge\\ T_\wedge\end{pmatrix}=\mathcal{D}^s_{\wedge,\max}\begin{pmatrix}A_\wedge\\ T_\wedge\end{pmatrix}\cap\bigcap_{\varepsilon>0}\mathcal{K}^{s+m,m/2-\varepsilon}(\overline{Y}^\wedge;E).$$

Analogously to Proposition 4.6 we have:

Proposition 5.5. *Let* (4.1) *be c-elliptic with parameter in the closed sector* Λ.

i) $\mathcal{D}^s_{\wedge,\max}\binom{A_\wedge}{T_\wedge}$ *is complete in the norm*

$$\|u\|=\|u\|_{\mathcal{K}^{s+m,-m/2}}+\|A_\wedge u\|_{\mathcal{K}^{s,-m/2}}+\sum_{j=1}^K\|\gamma_\wedge B_{j,\wedge}u\|_{\mathcal{K}^{s+m-m_j-1/2,m/2-m_j}},$$

and $\mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge}$ *is a closed subspace of finite codimension.*

ii) We have

$$\mathcal{K}^{s+m,m/2}\hookrightarrow\mathcal{D}^s_{\wedge,\min}\hookrightarrow\mathcal{D}^s_{\wedge,\max}\hookrightarrow\mathcal{K}^{s+m,-m/2+\varepsilon}$$

for some $\varepsilon>0$ *with continuous embeddings.*

The quotient $\mathcal{D}^s_{\wedge,\max}/\mathcal{D}^s_{\wedge,\min}$ is actually independent of $s>-\frac12$ and can be described in terms of singular functions as in the boundaryless case, see Section 6.

Notation 5.6. For functions φ,ψ we write $\varphi\prec\psi$ if $\psi\equiv1$ in a neighborhood of the support of φ.

Lemma 5.7. *Let* (4.1) *be c-elliptic with parameter in* Λ. *Then the model boundary condition*

$$T_\wedge:\mathcal{K}^{s+m,m/2}(\overline{Y}^\wedge;E)\to\bigoplus_{j=1}^K\mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge;F_j)$$

is surjective for every $s>-\frac12$, *and necessarily so is its extension to* $\mathcal{D}^s_\wedge\binom{A_\wedge}{T_\wedge}$ *by Proposition 5.5.*

Proof. We consider the b-elliptic boundary value problem

$$\begin{aligned} \big((x^m A) - \lambda\big)u &= f \quad \text{in } \overset{\circ}{\overline{M}}, \\ \gamma(x^{m_j} B_j)u &= g_j \quad \text{on } \overset{\circ}{\overline{N}}, \quad j = 1, \ldots, K. \end{aligned} \tag{5.8}$$

Let $h(\sigma, \lambda)$ be the conormal symbol of (5.8). Thus $h(\sigma, \lambda)$ is for every $\sigma \in \mathbb{C}$ and every $\lambda \in \Lambda$ an elliptic boundary value problem on $\overline{Y}$, and the b-ellipticity with parameter $\lambda \in \Lambda$ of (5.8) implies that $h(\sigma, \lambda)$ is elliptic with parameter $(\sigma, \lambda) \in \{\Im(\sigma) = -m/2\} \times \Lambda$. Consequently, $h(\sigma, \lambda)$ has a parameter-dependent parametrix in Boutet de Monvel's calculus on $\overline{Y}$ which is an inverse of $h(\sigma, \lambda)$ for all $\Im(\sigma) = -m/2$ and $|\lambda| > R$ sufficiently large.

Let $k(\sigma, \lambda)$ be the row matrix of potential operators in this parametrix, and define

$$K_1(\lambda) : C_0^\infty\Big(\mathbb{R}_+ \times (\partial\overline{Y}), \bigoplus_{j=1}^K F_j\Big) \to C^\infty(\mathbb{R}_+ \times \overline{Y}; E),$$

$$\big(K_1(\lambda)u\big)(x) := \frac{1}{2\pi} \int\limits_{\Im(\sigma) = -m/2} \int\limits_0^\infty \Big(\frac{x}{x'}\Big)^{i\sigma} k(\sigma, \lambda) \begin{pmatrix} x'^{m_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x'^{m_K} \end{pmatrix} u(x') \, \frac{dx'}{x'} \, d\sigma$$

for $u \in C_0^\infty\Big(\mathbb{R}_+ \times (\partial\overline{Y}), \bigoplus_{j=1}^K F_j\Big) \cong C_0^\infty\Big(\mathbb{R}_+, \bigoplus_{j=1}^K C^\infty(\partial\overline{Y}; F_j)\Big)$.

We now have $T_\wedge K_1(\lambda) = I$ for $|\lambda| > R$, and by continuity this identity holds on $\bigoplus_{j=1}^K x^{m/2-m_j} H_b^{s+m-m_j-1/2}((\partial\overline{Y})^\wedge; F_j)$. For cut-off functions $\omega \prec \tilde{\omega}$ near zero we thus have $T_\wedge \tilde{\omega} K_1(\lambda)\omega = \omega + R_1(\lambda)$ on $\bigoplus_{j=1}^K \mathcal{K}^{s+m-m_j-1/2, m/2-m_j}$ with a term $R_1(\lambda)$ which decreases rapidly in the norm as $|\lambda| \to \infty$.

On the other hand, the c-ellipticity with parameter $\lambda \in \Lambda$ of (4.1) implies that the boundary value problem (5.2) is away from $x = 0$ elliptic with parameter $\lambda \in \Lambda$ on the cone $\overline{Y}^\wedge$ as $x \to \infty$ in the sense that the triple of homogeneous principal symbols and boundary symbols associated with the covariables and parameter as well as the variables (as $x \to \infty$) is invertible – the noncompact end $x \to \infty$ on $\overline{Y}^\wedge$ is here a conical exit to infinity, not a cylindrical end, and parameter-dependent ellipticity and parametrices of (classical) boundary value problems on manifolds with conical exits to infinity are well investigated in the literature (see, e.g., [14]).

Consequently, there exists a parameter-dependent parametrix of $\binom{A_\wedge - \lambda}{T_\wedge}$ in the SG-calculus of boundary value problems (near infinity), and for the row matrix $K_2(\lambda)$ of potential operators of this parametrix and a suitable cut-off function $\hat{\omega} \prec \omega$ we have $T_\wedge(1 - \hat{\omega})K_2(\lambda)(1 - \omega) = (1 - \omega) + R_2(\lambda)$ on $\bigoplus_{j=1}^K \mathcal{K}^{s+m-m_j-1/2, m/2-m_j}$ with a term $R_2(\lambda)$ which decreases rapidly in the norm as $|\lambda| \to \infty$.

Thus for $K(\lambda) := \tilde{\omega} K_1(\lambda)\omega + (1 - \hat{\omega})K_2(\lambda)(1 - \omega)$ we have $T_\wedge K(\lambda) = I + \tilde{R}(\lambda)$ on $\bigoplus_{j=1}^K \mathcal{K}^{s+m-m_j-1/2, m/2-m_j}$, and for $|\lambda| > 0$ sufficiently large $I + \tilde{R}(\lambda)$ is invertible. $\square$

According to Lemma 5.7 it makes again sense to associate with the boundary value problem (5.4) a corresponding family of unbounded operators

$$A_\wedge - \lambda : \mathcal{D}^s_\wedge(A_{\wedge,T_\wedge}) \subset \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E) \to \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E), \tag{5.9}$$

where $\mathcal{D}^s_\wedge(A_{\wedge,T_\wedge}) = \mathcal{D}^s_\wedge\binom{A_\wedge}{T_\wedge} \cap \ker T_\wedge$.

Then (5.4) is invertible for some $\lambda \in \mathbb{C}$ if and only if (5.9) is invertible, i.e. if and only if $\lambda \notin \operatorname{spec}(A_{\wedge,T_\wedge})$. The analogue of Lemma 4.10 is true also for the associated problem on the model cone.

Proposition 5.10. *Let* (4.1) *be c-elliptic with parameter* $\lambda \in \Lambda$. *Then, for* $\lambda \neq 0$, *the operator* (5.4) *is Fredholm for every intermediate domain* $\mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge} \subset \mathcal{D}^s_\wedge\binom{A_\wedge}{T_\wedge} \subset \mathcal{D}^s_{\wedge,\max}\binom{A_\wedge}{T_\wedge}$ *with index independent of* $s > -\frac{1}{2}$. *More precisely, we have*

$$\begin{aligned}
\operatorname{ind}\begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix}_{\mathcal{D}^s_\wedge} &= \operatorname{ind}\begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix}_{\mathcal{D}^s_{\wedge,\min}} + \dim \mathcal{D}^s_\wedge/\mathcal{D}^s_{\wedge,\min} \\
&= \operatorname{ind}\begin{pmatrix} A \\ T \end{pmatrix}_{\mathcal{D}^s_{\min}} + \dim \mathcal{D}^s_\wedge/\mathcal{D}^s_{\wedge,\min} \\
&= \operatorname{ind}(A_{T,\min}) + \dim \mathcal{D}^s_\wedge/\mathcal{D}^s_{\wedge,\min},
\end{aligned}$$

and correspondingly the operator $A_{\wedge,T_\wedge} - \lambda : \mathcal{D}^s_\wedge(A_{\wedge,T_\wedge}) \to \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E)$ *is Fredholm for* $\lambda \neq 0$ *with the same index*

$$\begin{aligned}
\operatorname{ind}(A_{\wedge,T_\wedge} - \lambda)_{\mathcal{D}^s_\wedge} &= \operatorname{ind}(A_{\wedge,T_\wedge} - \lambda)_{\mathcal{D}^s_{\wedge,\min}} + \dim \mathcal{D}^s_\wedge/\mathcal{D}^s_{\wedge,\min} \\
&= \operatorname{ind}(A_{T,\min}) + \dim \mathcal{D}^s_\wedge/\mathcal{D}^s_{\wedge,\min}.
\end{aligned}$$

Proof. The Fredholmness follows from the parametrix construction in Section 7, the index formula is then elementary except for the assertion that

$$\operatorname{ind}\begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix}_{\mathcal{D}^s_{\wedge,\min}} = \operatorname{ind}\begin{pmatrix} A \\ T \end{pmatrix}_{\mathcal{D}^s_{\min}}.$$

Under the assumption that $\binom{A_\wedge - \lambda}{T_\wedge}$ is injective on $\mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge}$, this equality is a by-product of Theorem 7.21. However, the general case also follows by the same methods that lead to Theorem 7.21 by possibly enlarging the matrices of additional abstract conditions. □

Proposition 5.11. *If* $A_\wedge - \lambda : \mathcal{D}^s_\wedge(A_{\wedge,T_\wedge}) \to \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E)$ *is invertible for some* $\lambda \in \Lambda$ *and some domain* $\mathcal{D}^s_\wedge(A_{\wedge,T_\wedge})$, *then* $A_\wedge$ *is closed in the functional analytic sense for every domain* $\mathcal{D}^s_{\wedge,\min}(A_{\wedge,T_\wedge}) \subset \mathcal{D}^s_\wedge \subset \mathcal{D}^s_{\wedge,\max}(A_{\wedge,T_\wedge})$, *i.e.* $\mathcal{D}^s_{\wedge,\max}(A_{\wedge,T_\wedge})$ *is complete in the graph norm*

$$\|u\|_{A_\wedge} = \|u\|_{\mathcal{K}^{s,-m/2}} + \|A_\wedge u\|_{\mathcal{K}^{s,-m/2}}.$$

Proof. The proof follows along the lines of the proof of Proposition 4.11, noting that without loss of generality we may assume $\lambda \neq 0$, and hence

$$A_\wedge - \lambda : \mathcal{D}^s_{\wedge,\max}(A_{\wedge,T_\wedge}) \to \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E)$$

is Fredholm according to Proposition 5.10. □

Definition 5.12. i) For $\varrho > 0$ we define the normalized dilation group action for sections on $\overline{Y}^\wedge$ and $(\partial\overline{Y})^\wedge$ via

$$(\kappa_\varrho u)(x, y) = \varrho^{m/2} u(\varrho x, y).$$

κ_ϱ is a strongly continuous group action on the $\mathcal{K}^{s,\alpha}$-spaces, and the normalization factor $\varrho^{m/2}$ makes it an isometry on $x^{-m/2}L^2_b$.

ii) A family of operators $A(\lambda)$ defined on a κ-invariant space of distributions on the model cone is called κ-homogeneous of degree ν if

$$A(\varrho^m\lambda) = \varrho^\nu \kappa_\varrho A(\lambda)\kappa_\varrho^{-1}$$

for every $\varrho > 0$.

It is known that the dilation group action and the notion of κ-homogeneity play an important role when dealing with parameter-dependent cone operators, and they are systematically employed in Schulze's edge pseudodifferential calculus.

Observe that $\mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge}$ and $\mathcal{D}^s_{\wedge,\max}\binom{A_\wedge}{T_\wedge}$ as well as the associated domains $\mathcal{D}^s_{\wedge,\min}(A_{\wedge,T_\wedge})$ and $\mathcal{D}^s_{\wedge,\max}(A_{\wedge,T_\wedge})$ of the unbounded operator $A_\wedge$ under the boundary condition $T_\wedge u = 0$ are κ-invariant. This follows immediately from the κ-homogeneity

$$\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} = \left(\begin{array}{c|ccc} \varrho^m & 0 & \cdots & 0 \\ \hline 0 & \varrho^{m_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varrho^{m_K} \end{array}\right) \kappa_\varrho \begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} \kappa_\varrho^{-1}, \quad \varrho > 0, \tag{5.13}$$

of $\binom{A_\wedge}{T_\wedge}$. Moreover, this κ-homogeneity makes it possible to get a fairly complete picture of what it means to be a sector of minimal growth for realizations of the operator $A_\wedge$ under the boundary condition $T_\wedge u = 0$ as the following Proposition 5.14 shows. Note that the case of κ-invariant domains is particularly simple. In view of the characterization of the domains in terms of singular functions given in Section 6, the estimate (5.17) below can be regarded as a condition about the asymptotics of solutions of (5.2) as $|\lambda| \to \infty$.

Proposition 5.14. *Let* (4.1) *be c-elliptic with parameter $\lambda \in \Lambda$. Then the following are equivalent:*

i) *Λ is a sector of minimal growth for the operator $A_\wedge$ with domain $\mathcal{D}_\wedge(A_{\wedge,T_\wedge}) \subset x^{-m/2}L^2_b$.*

ii) *$A_\wedge - \lambda : \mathcal{D}_\wedge(A_{\wedge,T_\wedge}) \to x^{-m/2}L^2_b$ is invertible for large $\lambda \in \Lambda$, and the inverse satisfies the estimate*

$$\|\kappa^{-1}_{|\lambda|^{1/m}}(A_\wedge - \lambda)^{-1}\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\wedge,\max})} = O(|\lambda|^{-1})$$

as $|\lambda| \to \infty$.

iii)

$$\begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix} : \mathcal{D}_\wedge \begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} \to \begin{array}{c} x^{-m/2}L^2_b \\ \oplus \\ \bigoplus\limits_{j=1}^{K} \mathcal{K}^{m-m_j-1/2,m/2-m_j} \end{array} \tag{5.15}$$

is invertible for large $\lambda \in \Lambda$, and

$$\|\kappa^{-1}_{|\lambda|^{1/m}} \begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix}^{-1} \kappa_{|\lambda|^{1/m}}\| = O\begin{pmatrix} |\lambda|^{-1} & |\lambda|^{-m_1/m} & \cdots & |\lambda|^{-m_K/m} \end{pmatrix} \tag{5.16}$$

as $|\lambda| \to \infty$, where the bounds are to be understood componentwise (with values in $\mathcal{D}_{\wedge,\max}\binom{A_\wedge}{T_\wedge}$).

iv) (5.15) *is bijective for large* $\lambda \in \Lambda$, *and*

$$\|\kappa^{-1}_{|\lambda|^{1/m}} q_\wedge \begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix}^{-1} \kappa_{|\lambda|^{1/m}}\| = O\begin{pmatrix} |\lambda|^{-1} & |\lambda|^{-m_1/m} & \cdots & |\lambda|^{-m_K/m} \end{pmatrix} \tag{5.17}$$

as $|\lambda| \to \infty$, *where the bounds are to be understood componentwise with values in the quotient* $\mathcal{D}_{\wedge,\max}\binom{A_\wedge}{T_\wedge}/\mathcal{D}_{\wedge,\min}\binom{A_\wedge}{T_\wedge}$. *Here* $q_\wedge : \mathcal{D}_{\wedge,\max} \to \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ *denotes the canonical projection.*

Note that the group action κ_ϱ *descends to the quotient because both* $\mathcal{D}_{\wedge,\max}$ *and* $\mathcal{D}_{\wedge,\min}$ *are* κ*-invariant.*

If the domain $\mathcal{D}_\wedge(A_{\wedge,T_\wedge})$ *is* κ*-invariant, i)–iv) are equivalent to*

v) $A_\wedge - \lambda : \mathcal{D}_\wedge(A_{\wedge,T_\wedge}) \to x^{-m/2}L^2_b$ *is bijective for all* $\lambda \in \Lambda$ *with* $|\lambda| = 1$.

Proof. ii) $\Rightarrow$ *i)* follows immediately because the group action κ_ϱ, $\varrho > 0$, is unitary on $x^{-m/2}L^2_b$.

i) $\Rightarrow$ *ii)*: Note first that $\mathcal{D}^s_{\wedge,\max}$ is by assumption complete in the graph norm, see Proposition 5.11. Consequently, as κ_ϱ is an isometry on $x^{-m/2}L^2_b$, we only have to prove that

$$\|A_\wedge \kappa^{-1}_{|\lambda|^{1/m}}(A_\wedge - \lambda)^{-1}\|_{\mathscr{L}(x^{-m/2}L^2_b)} = O(|\lambda|^{-1})$$

as $|\lambda| \to \infty$. From the κ-homogeneity of $A_\wedge$ we obtain $A_\wedge \kappa^{-1}_{|\lambda|^{1/m}} = |\lambda|^{-1}\kappa^{-1}_{|\lambda|^{1/m}}A_\wedge$, and thus the desired estimate follows from the boundedness of the operator family $A_\wedge(A_\wedge - \lambda)^{-1}$ in $\mathscr{L}(x^{-m/2}L^2_b)$ (as $|\lambda| \to \infty$), which is part of our present assumption i).

ii) $\Rightarrow$ *iii)*: From the invertibility of $A_\wedge - \lambda : \mathcal{D}_\wedge \to x^{-m/2}L^2_b$ for large $|\lambda| > 0$ and the surjectivity of the boundary condition $T_\wedge$ (see Lemma 5.7) we obtain that (5.15) is invertible for large λ. Let

$$\begin{pmatrix} P_\wedge(\lambda) & K_\wedge(\lambda) \end{pmatrix} = \begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix}^{-1} : \begin{array}{c} x^{-m/2}L^2_b \\ \oplus \\ \bigoplus_{j=1}^K \mathcal{K}^{m-m_j-1/2,m/2-m_j} \end{array} \to \mathcal{D}_\wedge\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix}$$

be the inverse. The norm estimate in ii) implies the asserted norm estimate for $P_\wedge(\lambda) = (A_{\wedge,T_\wedge} - \lambda)^{-1}$ in iii) as $|\lambda| \to \infty$, noting that κ_ϱ is an isometry on $x^{-m/2}L^2_b$.

Let $\tilde{K} : \bigoplus_{j=1}^K \mathcal{K}^{m-m_j-1/2,m/2-m_j} \to \mathcal{K}^{m,m/2}(\overline{Y}^\wedge; E)$ be any right inverse of $T_\wedge$ according to Lemma 5.7, and define

$$\tilde{K}(\varrho) := \kappa_\varrho \tilde{K} \kappa_\varrho^{-1} \begin{pmatrix} \varrho^{-m_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varrho^{-m_K} \end{pmatrix}$$

for $\varrho > 0$. In view of the κ-homogeneity of $T_\wedge$, see (5.13), we conclude that $\tilde{K}(\varrho)$ is a right inverse of $T_\wedge$ for every $\varrho > 0$.

From $P_\wedge(\lambda)(A_\wedge - \lambda) + K_\wedge(\lambda)T_\wedge = 1$ we get for large $|\lambda| > 0$

$$K_\wedge(\lambda) = K_\wedge(\lambda)T_\wedge\tilde{K}(|\lambda|^{1/m}) = \tilde{K}(|\lambda|^{1/m}) - P_\wedge(\lambda)(A_\wedge - \lambda)\tilde{K}(|\lambda|^{1/m}).$$

Using the κ-homogeneity of $A_\wedge - \lambda$ we obtain by conjugation with the group action that $\kappa^{-1}_{|\lambda|^{1/m}} K_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}$ equals

$$\Big(1 - \big(\kappa^{-1}_{|\lambda|^{1/m}} P_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\big)|\lambda|\Big(A_\wedge - \frac{\lambda}{|\lambda|}\Big)\Big)\tilde{K}\begin{pmatrix} |\lambda|^{-m_1/m} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |\lambda|^{-m_K/m} \end{pmatrix},$$

and thus the asserted norm estimate in iii) holds for $K_\wedge(\lambda)$.

iii) $\Rightarrow$ *ii)* and *iii)* $\Rightarrow$ *iv)* are immediate.

iv) $\Rightarrow$ *ii)*: We just have to worry about the norm estimate. Let $\mathcal{B}_{T,\wedge}(\lambda) : x^{-m/2}L^2_b \to \mathcal{D}_{\wedge,\min}$ be the principal component of the interior part of the parametrix $\mathcal{B}(\lambda)$ from Theorem 7.21. Then

$$1 - \mathcal{B}_{T,\wedge}(\lambda)(A_\wedge - \lambda) \equiv 0 \quad \text{on } \mathcal{D}_{\wedge,\min}(A_{\wedge,T_\wedge})$$

for $\lambda \in \Lambda \setminus \{0\}$, and consequently the operator descends to

$$1 - \mathcal{B}_{T,\wedge}(\lambda)(A_\wedge - \lambda) : \mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge})/\mathcal{D}_{\wedge,\min}(A_{\wedge,T_\wedge}) \to \mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge}).$$

We may write

$$\big(A_{\wedge,\mathcal{D}_\wedge} - \lambda\big)^{-1} = \mathcal{B}_{T,\wedge}(\lambda) + \big(1 - \mathcal{B}_{T,\wedge}(\lambda)(A_\wedge - \lambda)\big)q_\wedge\big(A_{\wedge,\mathcal{D}_\wedge} - \lambda\big)^{-1}$$

as operators $x^{-m/2}L^2_b \to \mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge})$. By κ-homogeneity,

$$\begin{aligned}\kappa^{-1}_{|\lambda|^{1/m}}\big(A_{\wedge,\mathcal{D}_\wedge} - \lambda\big)^{-1} &= |\lambda|^{-1}\mathcal{B}_{T,\wedge}\Big(\frac{\lambda}{|\lambda|}\Big)\kappa^{-1}_{|\lambda|^{1/m}} \\ &\quad + \Big(1 - \mathcal{B}_{T,\wedge}\Big(\frac{\lambda}{|\lambda|}\Big)\Big(A_\wedge - \frac{\lambda}{|\lambda|}\Big)\Big)\Big(\kappa^{-1}_{|\lambda|^{1/m}} q_\wedge\big(A_{\wedge,\mathcal{D}_\wedge} - \lambda\big)^{-1}\Big),\end{aligned}$$

and so the norm estimate in ii) follows. Recall that the group action κ_ϱ is unitary on $x^{-m/2}L^2_b$.

If the domain $\mathcal{D}_\wedge(A_{\wedge,T_\wedge})$ is κ-invariant, then the invertibility of

$$A_\wedge - \lambda : \mathcal{D}_\wedge(A_{\wedge,T_\wedge}) \to x^{-m/2}L^2_b \tag{5.18}$$

for large $\lambda \in \Lambda$ is by means of the κ-homogeneity

$$A_\wedge - \varrho^m\lambda = \varrho^m\kappa_\varrho\big(A_\wedge - \lambda\big)\kappa_\varrho^{-1}$$

equivalent to the invertiblity of (5.18) for all $\lambda \in \Lambda \setminus \{0\}$ or, equivalently, only for $\lambda \in \Lambda$ with $|\lambda| = 1$. Moreover, from the identity

$$\kappa^{-1}_{|\lambda|^{1/m}}(A_\wedge - \lambda)^{-1}\kappa_{|\lambda|^{1/m}} = |\lambda|^{-1}\Big(A_\wedge - \frac{\lambda}{|\lambda|}\Big)^{-1} : x^{-m/2}L^2_b \to \mathcal{D}_\wedge(A_{\wedge,T_\wedge})$$

for $\lambda \neq 0$ we automatically obtain the norm estimate in ii), and consequently the equivalence of i)–iv) and v) is proved. □

6. Domains, associated domains, and singular functions

In this section we give a description of domains of the realizations of A and $A_\wedge$ under the boundary condition $Tu = 0$ and $T_\wedge u = 0$, respectively, in terms of singular functions, i.e. the domains are characterized by the asymptotic behavior of their elements near the "singular boundary" $\overline{Y}$.

Moreover, we explicitly construct an isomorphism

$$\theta : \mathcal{D}_{\max}/\mathcal{D}_{\min} \to \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$$

that will be used to associate with a domain $\mathcal{D}$ of A under the boundary condition $Tu = 0$ a corresponding domain $\mathcal{D}_\wedge$ of $A_\wedge$ under the boundary condition $T_\wedge u = 0$ via

$$\theta\big(\mathcal{D}/\mathcal{D}_{\min}\big) = \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}. \tag{6.1}$$

The ellipticity condition for the resolvent constructions for the operator A_T with domain $\mathcal{D}(A_T)$ in Section 8 then involves a spectral condition on the model operator $A_{\wedge,T_\wedge}$ with the associated domain $\mathcal{D}(A_{\wedge,T_\wedge})$ to $\mathcal{D}(A_T)$ according to (6.1). As the boundaryless case shows, a condition of such type is to be expected also in this more advanced situation.

Our approach to consider domains with inhomogeneous boundary conditions $\mathcal{D}\binom{A}{T}$ as well as $\mathcal{D}\binom{A_\wedge}{T_\wedge}$ makes it possible to transfer the methods from [8] and [9].

According to (5.1) we write (near $\overline{Y}$)

$$A \equiv x^{-m}\sum_{k=0}^{m-1} A_k x^k \qquad \mathrm{mod}\ \operatorname{Diff}_b^m(\overline{M};E), \tag{6.2}$$

$$B_j \equiv x^{-m_j}\sum_{k=0}^{m-1} B_{j,k} x^k \quad \mathrm{mod}\ x^{m-m_j}\operatorname{Diff}_b^{m_j}(\overline{M};E,F_j), \quad j=1,\dots,K, \tag{6.3}$$

with totally characteristic operators $A_k, B_{j,k}$ with coefficients independent of x near $\overline{Y}$, and therefore they can be regarded also as operators acting in sections on the model cone $\overline{Y}^\wedge$. Thus

$$\begin{pmatrix} A \\ \gamma B_1 \\ \vdots \\ \gamma B_K \end{pmatrix} = \begin{pmatrix} x^{-m} & 0 & \cdots & 0 \\ 0 & x^{-m_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^{-m_K} \end{pmatrix} \sum_{k=0}^{m-1} \begin{pmatrix} A_k \\ \gamma B_{1,k} \\ \vdots \\ \gamma B_{K,k} \end{pmatrix} x^k + \tilde{R}, \tag{6.4}$$

and set $\mathcal{A}_k = \begin{pmatrix} A_k & \gamma B_{1,k} & \cdots & \gamma B_{K,k} \end{pmatrix}^t$, $k = 0,\dots,m-1$. Let

$$\hat{\mathcal{A}}_k(\sigma) : H^{s+m}(\overline{Y};E) \to \begin{array}{c} H^s(\overline{Y};E) \\ \oplus \\ \bigoplus\limits_{j=1}^{K} H^{s+m-m_j-1/2}(\partial\overline{Y};F_j) \end{array}, \quad s > -\frac{1}{2}, \tag{6.5}$$

$\sigma \in \mathbb{C}$, be the conormal symbol of $\mathcal{A}_k$, see Section 3. From our standing assumption that (4.1) is c-elliptic (with parameter $\lambda \in \Lambda$), we obtain that the leading term $\hat{\mathcal{A}}_0(\sigma)$ is a holomorphic Fredholm family in (6.5) which has a finitely meromorphic inverse $\hat{\mathcal{A}}_0^{-1}(\sigma)$.

In the sequel, we make use of the following notion of Mellin transform for sections u on $\overline{M}$ or $\overline{Y}^\wedge$, respectively, which employs apriori a cut-off near $\overline{Y}$:

Fix a cut-off function $\omega \in C_0^\infty([0,1))$ near zero, i.e. ω is real valued and supported near the origin with $\omega \equiv 1$ near zero. As usual, we regard ω as a function on $\overline{M}$ supported in the collar neighborhood $U_{\overline{Y}} \subset \overline{M}$, or on $\overline{Y}^\wedge$. Then the Mellin transform of a section $u \in C_0^\infty(\mathring{\overline{M}};E)$ is defined to be the entire function $\hat{u} : \mathbb{C} \to C^\infty(\overline{Y};E|_{\overline{Y}})$ such that for any $v \in C^\infty(\overline{Y};E|_{\overline{Y}})$

$$\big(x^{-i\sigma}\omega u, \pi_{\overline{Y}}^* v\big)_{L_b^2(\overline{M};E)} = \big(\hat{u}(\sigma), v\big)_{L^2(\overline{Y};E|_{\overline{Y}})}, \tag{6.6}$$

where $\pi^*_{\overline{Y}} v$ is the section of E over $U_{\overline{Y}}$ obtained by parallel transport of v along the fibers of the projection $\pi_{\overline{Y}}$. The Mellin transform of sections $u \in C_0^\infty(\mathring{\overline{Y}}^\wedge; E)$ is defined in the same way, but the pairing in (6.6) is the inner product in $L^2_b(\overline{Y}^\wedge; E)$ (where, as before, we identify the bundle $E \to \overline{Y}^\wedge$ with the pull-back $\pi^*_{\overline{Y}} E|_{\overline{Y}}$).

The Mellin transform extends to the spaces $x^\alpha H^s_b$ and $\mathcal{K}^{s,\alpha}$ in such a way that $\hat{u}(\sigma)$ is a holomorphic $H^s(\overline{Y}; E)$-valued function in $\{\Im(\sigma) > -\alpha\}$ with well known integrability conditions along lines parallel to the real axis.

In the same way we also define the Mellin transform for sections on the boundary $\overline{N}$, and on the model cone $(\partial\overline{Y})^\wedge$ associated with the boundary, respectively.

Let

$$\operatorname{spec}_b \begin{pmatrix} A \\ T \end{pmatrix} = \{\sigma \in \mathbb{C};\ \hat{\mathcal{A}}_0(\sigma) \text{ is not invertible}\} \subset \mathbb{C}$$

be the boundary spectrum of $\binom{A}{T}$. Then

$$\operatorname{spec}_b \begin{pmatrix} A \\ T \end{pmatrix} \cap \{\sigma \in \mathbb{C};\ \alpha < \Im(\sigma) < \beta\}$$

is finite for all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and let

$$\Sigma := \operatorname{spec}_b \begin{pmatrix} A \\ T \end{pmatrix} \cap \{\sigma \in \mathbb{C};\ -m/2 < \Im(\sigma) < m/2\} \tag{6.7}$$

be the part of the boundary spectrum in the critical strip that is associated with realizations of A and $A_\wedge$ in $x^{-m/2}L^2_b$ under the boundary condition $Tu = 0$ and $T_\wedge u = 0$, respectively.

For $\sigma_0 \in \Sigma$ let $\tilde{\mathcal{E}}_{\wedge,\sigma_0}$ be the space of all singular functions of the form

$$q = \Big(\sum_{k=0}^{m_{\sigma_0}} c_{\sigma_0,k}(y) \log^k x\Big) x^{i\sigma_0} \in C^\infty(\mathring{\overline{Y}}^\wedge; E),$$

where $c_{\sigma_0,k} \in C^\infty(\overline{Y}; E)$ and $m_{\sigma_0} \in \mathbb{N}_0$, such that $\mathcal{A}_0 q = 0$. Using the Mellin transform, this is equivalent to the holomorphicity of $\hat{\mathcal{A}}_0(\sigma)\hat{q}(\sigma)$ on the whole complex plane, and as the inverse $\hat{\mathcal{A}}_0^{-1}(\sigma)$ is finitely meromorphic (with regularizing principal parts of Laurent expansions) we see that the space $\tilde{\mathcal{E}}_{\wedge,\sigma_0}$ is finite dimensional.

We set

$$\tilde{\mathcal{E}}_{\wedge,\max} = \bigoplus_{\sigma_0 \in \Sigma} \tilde{\mathcal{E}}_{\wedge,\sigma_0} \subset C^\infty(\mathring{\overline{Y}}^\wedge; E).$$

Let $u \in \mathcal{D}^s_{\wedge,\max}\binom{A_\wedge}{T_\wedge}$. By Mellin transform and the definition of the maximal domain, we thus obtain that $\hat{\mathcal{A}}_0(\sigma)\hat{u}(\sigma)$ is the Mellin transform of a vector of functions

$$v \in \begin{array}{c} \mathcal{K}^{s,m/2}(\overline{Y}^\wedge; E) \\ \oplus \\ \bigoplus_{j=1}^{K} \mathcal{K}^{s+m-m_j-1/2,m/2}((\partial\overline{Y})^\wedge; F_j) \end{array}.$$

In particular, $\hat{\mathcal{A}}_0(\sigma)\hat{u}(\sigma)$ is holomorphic in $\{\Im(\sigma) > -m/2\}$, and by the meromorphic structure of $\hat{\mathcal{A}}_0^{-1}(\sigma)$ we see that there is a singular function $q \in \tilde{\mathcal{E}}_{\wedge,\max}$ such that $\hat{u}(\sigma) - \hat{q}(\sigma)$ is holomorphic in the critical strip $\{\sigma \in \mathbb{C};\ -m/2 < \Im(\sigma) < m/2\}$. Consequently, $u - \omega q \in \mathcal{D}^s_{\wedge,\max}\binom{A_\wedge}{T_\wedge}$ with holomorphic Mellin transform, and thus

$u - \omega q \in \mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge}$. Note that the minimal domain as a subspace of the maximal domain is characterized by the property that the Mellin transforms of its elements are holomorphic in the critical strip.

Let us summarize this in the following proposition:

Proposition 6.8. *Every class*

$$u + \mathcal{D}^s_{\wedge,\min}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} \in \mathcal{D}^s_{\wedge,\max}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} / \mathcal{D}^s_{\wedge,\min}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix}$$

contains a representative of the form ωq with $q \in \tilde{\mathcal{E}}_{\wedge,\max}$, and the singular function q is uniquely determined by its class modulo $\mathcal{D}^s_{\wedge,\min}$.

In this way we obtain an isomorphism

$$\mathcal{D}^s_{\wedge,\max}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} / \mathcal{D}^s_{\wedge,\min}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} \cong \tilde{\mathcal{E}}_{\wedge,\max},$$

and the quotient $\mathcal{D}^s_{\wedge,\max}/\mathcal{D}^s_{\wedge,\min}$ is independent of $s > -\frac{1}{2}$.

Consequently, specifying a domain $\mathcal{D}^s_{\wedge,\min} \subset \mathcal{D}^s_\wedge \subset \mathcal{D}^s_{\wedge,\max}$ is equivalent to specifying a subspace of $\tilde{\mathcal{E}}_{\wedge,\max}$ of admissible conormal asymptotics for the elements $u \in \mathcal{D}^s_\wedge$ near $\overline{Y}$.

In view of

$$\mathcal{D}^s_{\wedge,\max}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} / \mathcal{D}^s_{\wedge,\min}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} \cong \mathcal{D}^s_{\wedge,\max}(A_{\wedge,T_\wedge}) / \mathcal{D}^s_{\wedge,\min}(A_{\wedge,T_\wedge}),$$

see also Lemma 4.10, we also obtain

$$\mathcal{D}^s_{\wedge,\max}(A_{\wedge,T_\wedge}) / \mathcal{D}^s_{\wedge,\min}(A_{\wedge,T_\wedge}) \cong \tilde{\mathcal{E}}_{\wedge,\max},$$

and the domains of the unbounded operator $A_\wedge$ under the boundary condition $T_\wedge u = 0$ are characterized in terms of the asymptotics near $\overline{Y}$.

Now let $u \in \mathcal{D}^s_{\max}\binom{A}{T}$. Then we obtain analogously to the case of the model cone that $\sum\limits_{k=0}^{m-1} \hat{\mathcal{A}}_k(\sigma)\hat{u}(\sigma + ik)$ is the Mellin transform of a vector of functions

$$v \in \begin{matrix} x^{m/2} H^s_b(\overline{M}; E) \\ \oplus \\ \bigoplus\limits_{j=1}^{K} x^{m/2} H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \end{matrix},$$

and consequently is holomorphic in $\{\Im(\sigma) > -m/2\}$. By inductively arguing for the strips $\{m/2 - k < \Im(\sigma) < m/2\}$, $k = 1, \ldots, m$, using thereby the meromorphic structure of the inverse $\hat{\mathcal{A}}_0^{-1}(\sigma)$ and the apriori holomorphicity of $\hat{u}(\sigma)$ in $\{\Im(\sigma) > m/2\}$, we conclude that $\hat{u}(\sigma)$ has a meromorphic extension to the critical strip $\{-m/2 < \Im(\sigma) < m/2\}$, and there exists a singular function of the form

$$q = \sum_{-\frac{m}{2} < \Im(\sigma) < \frac{m}{2}} \Big(\sum_{k=0}^{m_\sigma} c_{\sigma,k}(y) \log^k x \Big) x^{i\sigma} \tag{6.9}$$

with $c_{\sigma,k} \in C^\infty(\overline{Y}; E)$, $m_\sigma \in \mathbb{N}_0$, such that $\hat{u}(\sigma) - \hat{q}(\sigma)$ is holomorphic in this strip. Note that the sum in (6.9) is actually only a finite sum. Consequently, as also $\omega q \in \mathcal{D}^s_{\max}\binom{A}{T}$, we conclude that $u - \omega q \in \mathcal{D}^s_{\min}\binom{A}{T}$. We hence obtain an isomorphism

$$\mathcal{D}^s_{\max}\begin{pmatrix} A \\ T \end{pmatrix} / \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} \cong \tilde{\mathcal{E}}_{\max}$$

to a finite dimensional space of singular functions $\tilde{\mathcal{E}}_{\max} \subset C^\infty(\overset{\circ}{\overline{Y}}{}^\wedge; E)$ similar to the case of the model operator in Proposition 6.8.

Let us be more precise about the structure of the space $\tilde{\mathcal{E}}_{\max}$ of singular functions: We may write

$$\tilde{\mathcal{E}}_{\max} = \bigoplus_{\sigma_0 \in \Sigma} \tilde{\mathcal{E}}_{\sigma_0},$$

and the elements $q \in \tilde{\mathcal{E}}_{\sigma_0}$ are of the form

$$q = \sum_{\vartheta=0}^{N(\sigma_0)} \Big(\sum_{k=0}^{m_{\sigma_0 - i\vartheta}} c_{\sigma_0 - i\vartheta, k}(y) \log^k x \Big) x^{i(\sigma_0 - i\vartheta)}$$

with $c_{\sigma_0 - i\vartheta, k} \in C^\infty(\overline{Y}; E)$, $m_{\sigma_0 - i\vartheta} \in \mathbb{N}_0$, and $N(\sigma_0) \in \mathbb{N}_0$ the largest integer such that $\Im(\sigma_0) - N(\sigma_0) > -m/2$.

More precisely, there is an isomorphism

$$\theta : \tilde{\mathcal{E}}_{\max} \to \tilde{\mathcal{E}}_{\wedge,\max}$$

that was already mentioned in the introduction of this section, which restricts to isomorphisms $\theta|_{\tilde{\mathcal{E}}_{\sigma_0}} : \tilde{\mathcal{E}}_{\sigma_0} \to \tilde{\mathcal{E}}_{\wedge,\sigma_0}$. The inverse $\theta^{-1}|_{\tilde{\mathcal{E}}_{\sigma_0}}$ is of the form

$$\theta^{-1}|_{\tilde{\mathcal{E}}_{\sigma_0}} = \sum_{k=0}^{N(\sigma_0)} \mathrm{e}_{\sigma_0,k} : \tilde{\mathcal{E}}_{\wedge,\sigma_0} \to \tilde{\mathcal{E}}_{\sigma_0},$$

where the $\mathrm{e}_{\sigma_0,k} : \tilde{\mathcal{E}}_{\wedge,\sigma_0} \to C^\infty(\overset{\circ}{\overline{Y}}{}^\wedge; E)$ are inductively defined as follows:

- $\mathrm{e}_{\sigma_0,0} = I$, the identity map.
- Given $\mathrm{e}_{\sigma_0,0}, \ldots, \mathrm{e}_{\sigma_0,\vartheta-1}$ for some $\vartheta \in \{1, \ldots, N(\sigma_0) - 1\}$, we define $\mathrm{e}_{\sigma_0,\vartheta}(\psi)$ for $\psi \in \tilde{\mathcal{E}}_{\wedge,\sigma_0}$ to be the unique singular function of the form

  $$\Big(\sum_{k=0}^{m_{\sigma_0 - i\vartheta}} c_{\sigma_0 - i\vartheta, k}(y) \log^k x \Big) x^{i(\sigma_0 - i\vartheta)}$$

  such that

  $$(\mathrm{e}_{\sigma_0,\vartheta}(\psi))^\wedge(\sigma) + \hat{\mathcal{A}}_0(\sigma)^{-1} \Big(\sum_{k=1}^{\vartheta} \hat{\mathcal{A}}_k(\sigma) \mathrm{s}_{\sigma_0 - i\vartheta}(\mathrm{e}_{\sigma_0,\vartheta-k}(\psi))^\wedge(\sigma + ik) \Big)$$

  is holomorphic at $\sigma = \sigma_0 - i\vartheta$, where $(\mathrm{e}_{\sigma_0,\vartheta-k}(\psi))^\wedge(\sigma)$ is the Mellin transform of the function $\mathrm{e}_{\sigma_0,\vartheta-k}(\psi)$, and $\mathrm{s}_{\sigma_0 - i\vartheta}(\mathrm{e}_{\sigma_0,\vartheta-k}(\psi))^\wedge(\sigma + ik)$ is the singular part of its Laurent expansion at $\sigma_0 - i\vartheta$. Recall that our notion of Mellin transform involves apriori a cut-off near $\overline{Y}$, and so $(\mathrm{e}_{\sigma_0,\vartheta-k}(\psi))^\wedge(\sigma)$ is meromorphic in $\mathbb{C}$ with only one pole at $\sigma_0 - i(\vartheta - k)$.

It is of interest to note that this construction yields

$$\sum_{k=0}^{\vartheta} \big(\mathcal{A}_k x^k\big)(\mathrm{e}_{\sigma_0,\vartheta-k}(\psi)) = 0$$

for every $\psi \in \tilde{\mathcal{E}}_{\wedge,\sigma_0}$ and every $\vartheta = 0, \dots, N(\sigma_0)$.

In conclusion, every space $\tilde{\mathcal{E}}_{\sigma_0}$ consists indeed of singular functions of the form

$$q = \sum_{\vartheta=0}^{N(\sigma_0)} \Big(\sum_{k=0}^{m_{\sigma_0 - i\vartheta}} c_{\sigma_0 - i\vartheta,k}(y) \log^k x \Big) x^{i(\sigma_0 - i\vartheta)},$$

and we have

$$\theta q = \Big(\sum_{k=0}^{m_{\sigma_0}} c_{\sigma_0,k}(y) \log^k x \Big) x^{i\sigma_0}. \tag{6.10}$$

It is more tedious than hard to verify that this furnishes an isomorphism $\theta : \tilde{\mathcal{E}}_{\max} \to \tilde{\mathcal{E}}_{\wedge,\max}$ as desired (see also [8] for further information in the boundaryless context).

Let us summarize the above in the following proposition:

Proposition 6.11. *i) There is a natural isomorphism*

$$\mathcal{D}^s_{\max}\begin{pmatrix} A \\ T \end{pmatrix} / \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} \cong \tilde{\mathcal{E}}_{\max}, \quad u + \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} \mapsto q,$$

that is characterized by the property that $u - \omega q \in \mathcal{D}^s_{\min}\binom{A}{T}$*, where* $\omega \in C_0^\infty([0,1))$ *is any cut-off function near zero.*

Consequently, the quotient $\mathcal{D}^s_{\max}/\mathcal{D}^s_{\min}$ *is independent of* $s > -\frac{1}{2}$*, and its elements are characterized by their asymptotic behavior near* $\overline{Y}$*.*

ii) By Lemma 4.10,

$$\mathcal{D}^s_{\max}\begin{pmatrix} A \\ T \end{pmatrix} / \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} \cong \mathcal{D}^s_{\max}(A_T)/\mathcal{D}^s_{\min}(A_T),$$

and consequently also the quotient $\mathcal{D}^s_{\max}(A_T)/\mathcal{D}^s_{\min}(A_T)$ *of the maximal and minimal domains of the unbounded operator* A *under the boundary condition* $Tu = 0$ *is characterized by the conormal asymptotics in* $\tilde{\mathcal{E}}_{\max}$*.*

iii) There is a natural isomorphism $\theta : \tilde{\mathcal{E}}_{\max} \to \tilde{\mathcal{E}}_{\wedge,\max}$ *that by i), ii), and Proposition 6.8 gives rise to isomorphisms*

$$\theta : \begin{cases} \mathcal{D}^s_{\max}\binom{A}{T}/\mathcal{D}^s_{\min}\binom{A}{T} \to \mathcal{D}^s_{\wedge,\max}\binom{A_\wedge}{T_\wedge}/\mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge}, \\ \mathcal{D}^s_{\max}(A_T)/\mathcal{D}^s_{\min}(A_T) \to \mathcal{D}^s_{\wedge,\max}(A_{\wedge,T_\wedge})/\mathcal{D}^s_{\wedge,\min}(A_{\wedge,T_\wedge}). \end{cases}$$

For a domain $\mathcal{D}^s_{\min} \subset \mathcal{D}^s \subset \mathcal{D}^s_{\max}$ *we therefore have an associated domain*

$$\mathcal{D}^s_{\wedge,\min} \subset \theta(\mathcal{D}^s) = \mathcal{D}^s_\wedge \subset \mathcal{D}^s_{\wedge,\max}$$

via $\theta\big(\mathcal{D}^s/\mathcal{D}^s_{\min}\big) = \mathcal{D}^s_\wedge/\mathcal{D}^s_{\wedge,\min}$*.*

7. Parametrix construction

Let $\mathcal{A}(\lambda)$ denote the boundary value problem (4.1). Our goal in this section is the construction of a parametrix under the assumption that $\mathcal{A}(\lambda)$ is c-elliptic with parameter $\lambda \in \Lambda$, and that the model operator

$$\mathcal{A}_\wedge(\lambda) = \begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix} : \mathcal{D}^s_{\wedge,\min}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} \to \begin{array}{c} \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E) \\ \oplus \\ \bigoplus_{j=1}^{K} \mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge; F_j) \end{array}$$

is injective for some $s > -\frac{1}{2}$ and all $\lambda \in \Lambda \setminus \{0\}$.

More precisely, we will construct a parametrix

$$\mathcal{B}(\lambda) = \begin{pmatrix} \mathcal{B}_T(\lambda) & K(\lambda) \end{pmatrix} : \begin{array}{c} x^{-m/2}H_b^s(\overline{M};E) \\ \oplus \\ \bigoplus_{j=1}^{K} x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N};F_j) \end{array} \to \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix}$$

such that

$$\mathcal{B}(\lambda)\mathcal{A}(\lambda) - 1 : \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} \to \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix}$$

is regularizing and compactly supported in $\lambda \in \Lambda$. In particular, for λ sufficiently large, the boundary value problem

$$\mathcal{A}(\lambda) : \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} \to \begin{array}{c} x^{-m/2}H_b^s(\overline{M};E) \\ \oplus \\ \bigoplus_{j=1}^{K} x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N};F_j) \end{array} \tag{7.1}$$

is injective and $\mathcal{B}(\lambda)$ is a left inverse. Moreover, the regularizing remainder

$$\Pi(\lambda) = 1 - \mathcal{A}(\lambda)\mathcal{B}(\lambda)$$

is a finite dimensional projection to a complement of the range of (7.1).

For the actual construction of this parametrix we employ some ideas from pseudodifferential operator theory of Shapiro-Lopatinsky elliptic edge-degenerate boundary value problems, the central topic of the monograph [14].

Choose local coordinates on $\overline{Y}$ centered at zero, and let $(0,1)\times\Omega$ be corresponding coordinates in the collar neighborhood $U_{\overline{Y}} \subset \overline{M}$ of $\overline{Y}$. In these coordinates, the operator $A - \lambda$ takes the form

$$A - \lambda = x^{-m}\Big(\sum_{k+|\alpha|\le m} a_{k,\alpha}(x,y)D_y^\alpha(xD_x)^k - x^m\lambda\Big),$$

and thus its complete symbol $\tilde{a}(x,y,\xi,\eta,\lambda)$ is given by

$$\tilde{a}(x,y,\xi,\eta,\lambda) = x^{-m}a(x,y,x\xi,\eta,x^m\lambda)$$

with a symbol $a(x,y,\xi,\eta,\lambda)$ that is smooth up to $x=0$. The c-ellipticity with parameter $\lambda\in\Lambda$ of $A-\lambda$ is equivalent to the invertibility of the principal component $a_{(m)}(x,y,\xi,\eta,\lambda)$ for all covectors (ξ,η,λ) different from zero, and all $(x,y)\in[0,1)\times\Omega$ (up to $x=0$). Note that the principal component $a_{(m)}$ is (anisotropic) homogeneous, i.e.

$$a_{(m)}(x,y,\varrho\xi,\varrho\eta,\varrho^m\lambda) = \varrho^m a_{(m)}(x,y,\xi,\eta,\lambda)$$

for $\varrho > 0$.

Assume for a moment that $\Omega \subset \mathbb{R}^{n-1}$ corresponds to an interior chart on $\overline{Y}$. Then the parametrix construction from Section 5 in [9] implies that there exists a symbol $p(x,y,\xi,\eta,\lambda)$ with the following properties:

i) p is smooth in all variables up to $x=0$.
ii) We have
$$\big|\partial^\alpha_{(x,y)}\partial^\beta_{(\xi,\eta)}\partial^\gamma_\lambda p(x,y,\xi,\eta,\lambda)\big| = O\big((1+|\xi|+|\eta|+|\lambda|^{1/m})^{-m-|\beta|-m|\gamma|}\big)$$
as $|(\xi,\eta,\lambda)|\to\infty$, locally uniformly for $(x,y)\in[0,1)\times\Omega$.

iii) p is a classical symbol, i.e. it admits an asymptotic expansion

$$p \sim \sum_{j=0}^{\infty} \chi(\xi,\eta,\lambda) p_{(-m-j)}(x,y,\xi,\eta,\lambda),$$

where $\chi \in C^\infty(\mathbb{R} \times \mathbb{R}^{n-1} \times \Lambda)$ is a function such that $\chi \equiv 0$ near the origin and $\chi \equiv 1$ for large $|(\xi,\eta,\lambda)|$, and the components $p_{(-m-j)}$ are anisotropic homogeneous, i.e. we have

$$p_{(-m-j)}(x,y,\varrho\xi,\varrho\eta,\varrho^m\lambda) = \varrho^{-m-j} p_{(-m-j)}(x,y,\xi,\eta,\lambda)$$

for $\varrho > 0$.

iv) $(A-\lambda)\mathrm{Op}(x^m p(x,y,x\xi,\eta,x^m\lambda)) - 1$ and $\mathrm{Op}(x^m p(x,y,x\xi,\eta,x^m\lambda))(A-\lambda) - 1$ are parameter-dependent smoothing pseudodifferential operators on $(0,1)\times\Omega$, where $\mathrm{Op}(x^m p(x,y,x\xi,\eta,x^m\lambda))$ denotes the standard Kohn-Nirenberg quantized pseudodifferential operator in $(0,1)\times\Omega$ with symbol $x^m p(x,y,x\xi,\eta,x^m\lambda)$.

Now let $\Omega \subset \overline{\mathbb{R}}^{n-1}_+$ correspond to a boundary chart on $\overline{Y}$. We slightly extend Ω as well as $\tilde{a}$ over the boundary $\mathbb{R}^{n-2} \subset \overline{\mathbb{R}}^{n-1}_+$ such that the structure of the complete symbol $\tilde{a}$ of $A-\lambda$ and the c-ellipticity with parameter remains preserved (by possibly shrinking Ω to a relatively compact chart, this is always possible).

Now we can use the beforementioned results about the existence of a parametrix in the extended domain $(0,1)\times\Omega'$, where the symbol p in addition has the transmission property with respect to the boundary $\mathbb{R}^{n-2} \subset \overline{\mathbb{R}}^{n-1}_+$.

Passing, as is usual in pseudodifferential boundary value problems, to

$$\mathrm{Op}^+(x^m p(x,y,x\xi,\eta,x^m\lambda)) = r^+ \mathrm{Op}(x^m p(x,y,x\xi,\eta,x^m\lambda)) e^+,$$

where e^+ denotes the operator of extension by zero from the original domain $(0,1)\times\Omega$ to the extended domain $(0,1)\times\Omega'$ and r^+ denotes restriction, we obtain that

$$\mathrm{Op}^+(x^m p(x,y,x\xi,\eta,x^m\lambda))(A-\lambda) = 1 + G_0(\lambda) + G_{-\infty}(\lambda),$$

where $G_{-\infty}(\lambda)$ is a parameter-dependent regularizing singular Green operator in Boutet de Monvel's calculus, and $G_0(\lambda)$ is a parameter-dependent singular Green operator in Boutet de Monvel's calculus of order zero whose boundary symbol has the form

$$\tilde{g}(x,y',\xi,\eta',\lambda) = g(x,y',x\xi,\eta',x^m\lambda)$$

with an (anisotropic) parameter-dependent singular Green symbol $g(x,y',\xi,\eta',\lambda)$ of order zero. The structure of the composition

$$(A-\lambda)\mathrm{Op}^+(x^m p(x,y,x\xi,\eta,x^m\lambda))$$

is the same.

By combining the standard parametrix constructions on a manifold with boundary away from $\overline{Y}$ with the above considerations, we arrive at the following

Lemma 7.2. *$A-\lambda$ has a parametrix $P^+(\lambda)$ of order $-m$ and type zero in the parameter-dependent Boutet de Monvel's calculus $\mathcal{B}^{-m,0}(\mathring{\overline{M}};\Lambda)$ on $\mathring{\overline{M}} = \overline{M} \setminus \overline{Y}$. When restricted to the collar neighborhood $U_{\overline{Y}} \cong (0,1)\times\overline{Y}$, this parametrix takes the form*

$$P^+(\lambda)u(x) = \frac{1}{2\pi}\iint e^{i(x-x')\xi}\tilde{p}(x,\xi,\lambda)u(x')\,dx'\,d\xi + C(\lambda)u(x)$$

for $u \in C_0^\infty((0,1), C^\infty(\overline{Y}; E))$, *where* $C(\lambda) \in \mathcal{B}^{-\infty,0}((0,1)\times\overline{Y};\Lambda)$ *is a parameter-dependent regularizing singular Green operator of type zero on* $(0,1)\times\overline{Y}$, *and* $\tilde{p}(x,\xi,\lambda) = x^m p(x, x\xi, x^m\lambda)$ *with a symbol*

$$p(x,\xi,\lambda) \in C^\infty([0,1), \mathcal{B}^{-m,0}(\overline{Y}; \mathbb{R}\times\Lambda)),$$

i.e. p *is a smooth function in* $x \in [0,1)$ *taking values in the space of operators of order* $-m$ *and type zero in the parameter-dependent Boutet de Monvel's calculus on* $\overline{Y}$ *(depending on the isotropic parameter* ξ *and the anisotropic parameter* λ*).*

The remainders $(A-\lambda)P^+(\lambda)-1$ *and* $P^+(\lambda)(A-\lambda)-1$ *are parameter-dependent singular Green operators in Boutet de Monvel's calculus on* $\mathring{\overline{M}}$ *of order zero and appropriate types (given by the standard type formula for the composition of operators). When restricted to* $U_{\overline{Y}}$*, they take the form*

$$G_0(\lambda)u(x) = \frac{1}{2\pi}\iint e^{i(x-x')\xi}\tilde{g}(x,\xi,\lambda)u(x')\,dx'\,d\xi$$

modulo parameter-dependent regularizing singular Green operators on $(0,1)\times\overline{Y}$, *where* $\tilde{g}(x,\xi,\lambda) = g(x, x\xi, x^m\lambda)$ *with a symbol*

$$g(x,\xi,\lambda) \in C^\infty([0,1), \mathcal{B}_G^{0,d}(\overline{Y}; \mathbb{R}\times\Lambda)),$$

i.e. g *is smooth in* $x \in [0,1)$ *taking values in the space of parameter-dependent singular Green operators in Boutet de Monvel's calculus on* $\overline{Y}$ *of order zero and type* d, *where* $d = 0$ *or* $d = m$, *respectively, and* $\xi \in \mathbb{R}$ *is again the isotropic parameter, while* $\lambda \in \Lambda$ *is the anisotropic parameter.*

Observe, in particular, that $P^+(\lambda)$ *has a well-defined homogeneous principal c-symbol* ${}^c\boldsymbol{\sigma}(P^+)(z,\zeta,\lambda)$ *on* $\big({}^cT^*\overline{M}\times\Lambda\big)\setminus 0$, *as well as a principal c-boundary symbol, which is a (twisted) homogeneous section*

$${}^c\boldsymbol{\sigma}_\partial(P^+)(z',\zeta',\lambda) : {}^c\mathscr{S}_+ \otimes {}^c\pi^*E \to {}^c\mathscr{S}_+ \otimes {}^c\pi^*E$$

on $\big({}^cT^*\overline{N}\times\Lambda\big)\setminus 0$.

Proposition 7.3. *There exists a matrix of parameter-dependent generalized singular Green operators*

$$\mathcal{G}_1(\lambda) = \begin{pmatrix} G_1(\lambda) & K_1(\lambda) & \cdots & K_K(\lambda)\end{pmatrix} : \begin{array}{c} C_0^\infty(\mathring{\overline{M}}; E) \\ \oplus \\ \bigoplus\limits_{j=1}^{K} C_0^\infty(\mathring{\overline{N}}; F_j)\end{array} \to C^\infty(\mathring{\overline{M}}; E)$$

in Boutet de Monvel's calculus of orders $-m$, $-m_1-\frac{1}{2}$, $\ldots$, $-m_K-\frac{1}{2}$ *and type zero in* $\mathcal{B}_G^*(\mathring{\overline{M}};\Lambda)$, *such that its restriction to the collar neighborhood* $U_{\overline{Y}}$ *of* $\overline{Y}$ *is (modulo a regularizing parameter-dependent generalized singular Green operator) of the form*

$$\mathcal{G}_1(\lambda)\begin{pmatrix} u \\ v_1 \\ \vdots \\ v_K\end{pmatrix}(x) = \frac{1}{2\pi}\iint e^{i(x-x')\xi}\tilde{g}(x,\xi,\lambda)\begin{pmatrix} u(x') \\ v_1(x') \\ \vdots \\ v_K(x')\end{pmatrix}dx'\,d\xi$$

for $u \in C_0^\infty((0,1), C^\infty(\overline{Y}; E))$ *and* $v_j \in C_0^\infty((0,1), C^\infty(\partial\overline{Y}; F_j))$, $j = 1,\ldots,K$, *where*

$$\tilde{g}(x,\xi,\lambda) = \begin{pmatrix} x^m g(x,x\xi,x^m\lambda) & x^{m_1}k_1(x,x\xi,x^m\lambda) & \cdots & x^{m_K}k_K(x,x\xi,x^m\lambda)\end{pmatrix},$$

and $g(x,\xi,\lambda)$ as well as the $k_j(x,\xi,\lambda)$, $j=1,\dots,K$, are smooth with respect to $x\in[0,1)$ taking values in the parameter-dependent generalized singular Green operators of orders $-m$ and $-m_j-\frac{1}{2}$, $j=1,\dots,K$, and type zero in Boutet de Monvel's calculus on $\overline{Y}$ (depending on the isotropic parameter $\xi\in\mathbb{R}$ and the anisotropic parameter $\lambda\in\Lambda$).

The operator family

$$\mathcal{B}_1(\lambda)=\begin{pmatrix}P^+(\lambda)+G_1(\lambda) & K_1(\lambda) & \cdots & K_K(\lambda)\end{pmatrix}:\begin{array}{c}C_0^\infty(\overset{\circ}{\overline{M}};E)\\ \oplus\\ \bigoplus\limits_{j=1}^{K}C_0^\infty(\overset{\circ}{\overline{N}};F_j)\end{array}\to C^\infty(\overset{\circ}{\overline{M}};E)$$

is a parameter-dependent parametrix in Boutet de Monvel's calculus of the boundary value problem (4.1) *in the sense that the remainders*

$$\begin{pmatrix}A-\lambda\\ T\end{pmatrix}\mathcal{B}_1(\lambda)-1,\quad \mathcal{B}_1(\lambda)\begin{pmatrix}A-\lambda\\ T\end{pmatrix}-1\in\mathcal{B}^{-\infty}(\overset{\circ}{\overline{M}};\Lambda)$$

are parameter-dependent regularizing generalized singular Green operators in Boutet de Monvel's calculus on $\overset{\circ}{\overline{M}}$.

Proof. For any $d\in\mathbb{N}_0$ we consider the space $x^{-\varrho}\mathcal{B}_G^{\mu,d}$ of parameter-dependent generalized singular Green operators in Boutet de Monvel's calculus on $\overset{\circ}{\overline{M}}$ which consist of matrix entries of the following form:

- *Operators in the interior*:
$$G(\lambda):C_0^\infty(\overset{\circ}{\overline{M}};E)\to C^\infty(\overset{\circ}{\overline{M}};E)$$
is a parameter-dependent singular Green operator of order μ and type d in Boutet de Monvel's calculus on $\overset{\circ}{\overline{M}}$, and when restricted to $U_{\overline{Y}}$ it takes the form
$$G(\lambda)u(x)=\frac{1}{2\pi}\iint e^{i(x-x')\xi}\tilde{g}(x,\xi,\lambda)u(x')\,dx'\,d\xi$$
(modulo a regularizing parameter-dependent singular Green operator of type d in $U_{\overline{Y}}$) with a symbol $\tilde{g}(x,\xi,\lambda)=x^{-\varrho}g(x,x\xi,x^m\lambda)$, where $g(x,\xi,\lambda)$ is smooth with respect to $x\in[0,1)$ taking values in the space $\mathcal{B}_G^{\mu,d}(\overline{Y};\mathbb{R}\times\Lambda)$ of parameter-dependent singular Green operators of order μ and type d in Boutet de Monvel's calculus on $\overline{Y}$, depending on the isotropic parameter $\xi\in\mathbb{R}$ and the anisotropic parameter $\lambda\in\Lambda$.
- *Trace operators*:
$$T(\lambda):C_0^\infty(\overset{\circ}{\overline{M}};E)\to C^\infty(\overset{\circ}{\overline{N}};F)$$
is a parameter-dependent trace operator of order μ and type d in Boutet de Monvel's calculus on $\overset{\circ}{\overline{M}}$, and when restricted to $U_{\overline{Y}}$ it takes the form
$$T(\lambda)u(x)=\frac{1}{2\pi}\iint e^{i(x-x')\xi}\tilde{t}(x,\xi,\lambda)u(x')\,dx'\,d\xi$$
(modulo a regularizing parameter-dependent trace operator of type d in $U_{\overline{Y}}$) with a symbol $\tilde{t}(x,\xi,\lambda)=x^{-\varrho+\frac{1}{2}}t(x,x\xi,x^m\lambda)$, where $t(x,\xi,\lambda)$ is smooth with respect to $x\in[0,1)$ taking values in the space of parameter-dependent trace operators of order μ and type d in Boutet de Monvel's calculus on $\overline{Y}$.

- *Potential operators*:

$$K(\lambda) : C_0^\infty(\overset{\circ}{\overline{N}}; F) \to C^\infty(\overset{\circ}{\overline{M}}; E)$$

is a parameter-dependent potential operator of order μ in Boutet de Monvel's calculus on $\overset{\circ}{\overline{M}}$, and when restricted to the collar neighborhood it takes the form

$$K(\lambda)v(x) = \frac{1}{2\pi} \iint e^{i(x-x')\xi} \tilde{k}(x,\xi,\lambda) v(x')\, dx'\, d\xi$$

(modulo a regularizing parameter-dependent potential operator in $U_{\overline{Y}}$) with a symbol $\tilde{k}(x,\xi,\lambda) = x^{-\varrho-\frac{1}{2}} k(x, x\xi, x^m\lambda)$, where $k(x,\xi,\lambda)$ is smooth with respect to $x \in [0,1)$ taking values in the space of parameter-dependent potential operators of order μ in Boutet de Monvel's calculus on $\overline{Y}$.
- *Operators on the boundary*:

$$Q(\lambda) : C_0^\infty(\overset{\circ}{\overline{N}}; F_1) \to C^\infty(\overset{\circ}{\overline{N}}; F_2)$$

is a parameter-dependent pseudodifferential operator of order μ on $\overset{\circ}{\overline{N}}$, and when restricted to the collar neighborhood it takes the form

$$Q(\lambda)v(x) = \frac{1}{2\pi} \iint e^{i(x-x')\xi} \tilde{q}(x,\xi,\lambda) v(x')\, dx'\, d\xi$$

(modulo a regularizing parameter-dependent pseudodifferential operator in $(0,1) \times \partial\overline{Y}$) with a symbol $\tilde{q}(x,\xi,\lambda) = x^{-\varrho} q(x, x\xi, x^m\lambda)$, where $q(x,\xi,\lambda)$ is smooth with respect to $x \in [0,1)$ taking values in the space of parameter-dependent pseudodifferential operators on $\partial\overline{Y}$ of order μ (where, as in all the other cases, $\xi \in \mathbb{R}$ is the isotropic parameter, and $\lambda \in \Lambda$ is the anisotropic parameter).

Observe that every $\mathcal{G}(\lambda) \in x^{-\mu}\mathcal{B}_G^{\mu,d}$ has a well defined principal c-boundary symbol ${}^c\boldsymbol{\sigma}_\partial(G)(z',\zeta',\lambda)$ on $\big({}^cT^*\overline{N} \times \Lambda\big) \setminus 0$, which is (twisted) homogeneous of degree μ, and we have in a canonical way a split exact sequence

$$0 \longrightarrow x^{-\mu}\mathcal{B}_G^{\mu-1,d} \longrightarrow x^{-\mu}\mathcal{B}_G^{\mu,d} \underset{{}^c\boldsymbol{\sigma}_\partial}{\longrightarrow} {}^c\Sigma \longrightarrow 0. \tag{7.4}$$

Moreover, by standard arguments in the calculus of pseudodifferential operators with (twisted) operator-valued symbols, we see that $x^{-\varrho}\mathcal{B}_G^{*,d}$ is asymptotically complete, i.e. asymptotic summation is possible within the class. Recall that the boundary symbolic calculus in Boutet de Monvel's algebra can be formulated in terms of twisted operator-valued symbols, where the function spaces in the normal direction are equipped with suitable dilation group actions. The principal boundary symbols are twisted homogeneous, i.e. homogeneous up to conjugation with the groups (cf. [31]).

Notice that the assertion of the proposition about the structure of $\mathcal{G}_1(\lambda)$ just means that $G_1(\lambda) \in x^m\mathcal{B}_G^{-m,0}$, while $K_j(\lambda) \in x^{m_j+\frac{1}{2}}\mathcal{B}_G^{-m_j-\frac{1}{2},0}$ for $j = 1,\dots,K$. Moreover, for the boundary conditions in (4.1) we find $\gamma B_j \in x^{-m_j-\frac{1}{2}}\mathcal{B}_G^{m_j+\frac{1}{2},m}$, $j = 1,\dots,K$.

Let

$$ {}^c\boldsymbol{\sigma}_\partial(\mathcal{A})(z',\zeta',\lambda) = \begin{pmatrix} {}^c\boldsymbol{\sigma}_\partial(A)(z',\zeta') - \lambda \\ ({}^c\gamma_0 \otimes I_{{}^c\pi^*F_1})\,{}^c\boldsymbol{\sigma}_\partial(B_1)(z',\zeta') \\ \vdots \\ ({}^c\gamma_0 \otimes I_{{}^c\pi^*F_K})\,{}^c\boldsymbol{\sigma}_\partial(B_K)(z',\zeta') \end{pmatrix} : {}^c\mathscr{S}_+ \otimes {}^c\pi^*E \to \begin{array}{c} {}^c\mathscr{S}_+ \otimes {}^c\pi^*E \\ \oplus \\ \bigoplus_{j=1}^K {}^c\pi^*F_j \end{array} $$

be the principal c-boundary symbol of (4.1) on $\big({}^cT^*\overline{N} \times \Lambda\big) \setminus 0$. By choosing a smooth positive definite metric on ${}^cT^*\overline{N}$ we can consider ${}^c\boldsymbol{\sigma}_\partial(\mathcal{A})(z',\zeta',\lambda)$ for $\big(|\zeta'|^{2m} + |\lambda|^2\big)^{1/2m} = 1$ only, and by (twisted) homogeneity we still recover the full information. As (4.1) is assumed to be c-elliptic with parameter $\lambda \in \Lambda$ we obtain that ${}^c\boldsymbol{\sigma}_\partial(\mathcal{A})(z',\zeta',\lambda)$ is invertible, and as we consider this function now only on a compact sphere bundle the standard arguments in Boutet de Monvel's calculus can be applied to show that its inverse is of the form

$$ {}^c\boldsymbol{\sigma}_\partial(\mathcal{A})(z',\zeta',\lambda)^{-1} = \big({}^c\boldsymbol{\sigma}_\partial(P^+)(z',\zeta',\lambda) \quad 0\big) + {}^c\boldsymbol{\sigma}_\partial(\mathcal{G}_1)(z',\zeta',\lambda), $$

where ${}^c\boldsymbol{\sigma}_\partial(\mathcal{G}_1)(z',\zeta',\lambda)$ already is the principal c-boundary symbol of the parameter-dependent singular Green operator $\mathcal{G}_1(\lambda)$ of the assertion of the proposition.

In order to find $\mathcal{G}_1(\lambda)$, we first use (for each entry) the split exactness of (7.4) to obtain a matrix $\mathcal{G}_2(\lambda)$ of generalized parameter-dependent singular Green operators with ${}^c\boldsymbol{\sigma}_\partial(\mathcal{G}_1)(z',\zeta',\lambda) = {}^c\boldsymbol{\sigma}_\partial(\mathcal{G}_2)(z',\zeta',\lambda)$, and set $\mathcal{B}'(\lambda) = \big(P^+(\lambda) \quad 0\big) + \mathcal{G}_2(\lambda)$. The composition rules imply that $\mathcal{B}'(\lambda)\mathcal{A}(\lambda) = 1 + \mathcal{G}'(\lambda)$ with a parameter-dependent singular Green operator $\mathcal{G}'(\lambda) \in x^0\mathcal{B}_G^{-1,m}$. The standard formal Neumann series argument now shows that there exists $\mathcal{G}(\lambda) \in x^0\mathcal{B}_G^{-1,m}$ (properly supported, uniformly for $\lambda \in \Lambda$) such that with

$$ \mathcal{B}_1(\lambda) = (1 + \mathcal{G}(\lambda))\mathcal{B}'(\lambda) = \big(P^+(\lambda) \quad 0\big) + \mathcal{G}_1(\lambda) $$

we have that $\mathcal{B}_1(\lambda)\mathcal{A}(\lambda) - 1$ is a parameter-dependent regularizing singular Green operator in Boutet de Monvel's calculus on $\overline{\mathring{M}}$ of type (at most) m.

A right parametrix is obtained in the same way. Note that the composition

$$ \mathcal{A}(\lambda)\mathcal{B}'(\lambda) - 1 = \left(\begin{array}{c|ccc} G_{0,0}(\lambda) & G_{0,1}(\lambda) & \cdots & G_{0,K}(\lambda) \\ \hline G_{1,0}(\lambda) & G_{1,1}(\lambda) & \cdots & G_{1,K}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ G_{K,0}(\lambda) & G_{K,1}(\lambda) & \cdots & G_{K,K}(\lambda) \end{array}\right) $$

with $G_{i,j}(\lambda) \in x^{-(n_i-n_j)}\mathcal{B}_G^{n_i-n_j-1,0}$, where $n_0 = m$ and $n_j = m_j + \frac{1}{2}$, $j = 1,\ldots,K$, i.e. the matrix $\big(G_{i,j}(\lambda)\big)_{i,j}$ is of order -1 with respect to an order convention of Douglis-Nirenberg type (see also the proof of Lemma 7.13). Consequently, the formal Neumann series argument also applies in this situation (with Douglis-Nirenberg order convention), and the proposition is proved. □

Modulo a regularizing parameter-dependent generalized singular Green operator of type zero, the restriction of the parametrix $\mathcal{B}_1(\lambda)$ from Proposition 7.3 to $U_{\overline{Y}}$ can also be written in the form

$$ \mathcal{B}_1(\lambda) \begin{pmatrix} u \\ v_1 \\ \vdots \\ v_K \end{pmatrix}(x) = \frac{1}{2\pi} \iint e^{i(x-x')\xi}\tilde{b}(x,\xi,\lambda) \begin{pmatrix} {x'}^m u(x') \\ {x'}^{m_1} v_1(x') \\ \vdots \\ {x'}^{m_K} v_K(x') \end{pmatrix} dx'\, d\xi \tag{7.5} $$

for $u \in C_0^\infty((0,1), C^\infty(\overline{Y}; E))$ and $v_j \in C_0^\infty((0,1), C^\infty(\partial\overline{Y}; F_j))$, $j = 1, \ldots, K$, where $\tilde{b}(x,\xi,\lambda) = b(x, x\xi, x^m\lambda)$,

$$b(x,\xi,\lambda) = \begin{pmatrix}(p'+g')(x,\xi,\lambda) & k_1'(x,\xi,\lambda) & \cdots & k_K'(x,\xi,\lambda)\end{pmatrix},$$

and the entries of $b(x,\xi,\lambda)$ are smooth with respect to $x \in [0,1)$ taking values in the parameter-dependent Boutet de Monvel's calculus on $\overline{Y}$ of type zero and corresponding orders. By Mellin quantization (see, e.g., [14]), the operator (7.5) has a representation

$$\mathcal{Q}(\lambda)\begin{pmatrix}u\\ v_1\\ \vdots\\ v_K\end{pmatrix}(x) = \frac{1}{2\pi}\int\limits_{\Im(\sigma)=-m/2}\int\limits_{(0,1)}\Big(\frac{x}{x'}\Big)^{i\sigma} h(x,\sigma,x^m\lambda)\begin{pmatrix}{x'}^m u(x')\\ {x'}^{m_1} v_1(x')\\ \vdots\\ {x'}^{m_K} v_K(x')\end{pmatrix}\frac{dx'}{x'}\,d\sigma$$

modulo a regularizing parameter-dependent generalized singular Green operator in Boutet de Monvel's calculus, where the Mellin symbol $h(x,\sigma,\lambda)$ is given by the formula

$$h(x,\sigma,\lambda) = \frac{1}{2\pi}\iint e^{-i(r-1)\xi} r^{i\sigma}\varphi(r) b(x,\xi,\lambda)\,dr\,d\xi \tag{7.6}$$

for $r, \xi \in \mathbb{R}$, $\sigma \in \mathbb{C}$, and $\varphi \in C_0^\infty(\mathbb{R}_+)$ is a function such that $\varphi \equiv 1$ near $r = 1$.

Pick cut-off functions $\omega, \tilde{\omega}, \hat{\omega} \in C_0^\infty([0,1))$ near zero with $\hat{\omega} \prec \omega \prec \tilde{\omega}$, and consider these functions as functions on $\overline{M}$ (or $\overline{N}$) supported in $U_{\overline{Y}}$. With the parametrix $\mathcal{B}_1(\lambda)$ from Proposition 7.3 we define

$$\mathcal{B}_2(\lambda) = \omega\mathcal{Q}(\lambda)\tilde{\omega} + (1-\omega)\mathcal{B}_1(\lambda)(1-\hat{\omega}) : \begin{array}{c} C_0^\infty(\mathring{\overline{M}}; E)\\ \oplus\\ \bigoplus\limits_{j=1}^K C_0^\infty(\mathring{\overline{N}}; F_j)\end{array} \to C^\infty(\mathring{\overline{M}}; E). \tag{7.7}$$

By construction, we obtain the following

Proposition 7.8. *$\mathcal{B}_2(\lambda)$ is a parameter-dependent parametrix in Boutet de Monvel's calculus of* (4.1) *which is properly supported, uniformly with respect to $\lambda \in \Lambda$.*

We have $\mathcal{B}_2(\lambda) - \mathcal{B}_1(\lambda) \in \mathcal{B}^{-\infty,0}(\mathring{\overline{M}}; \Lambda)$, and

$$\mathcal{B}_2(\lambda) : \begin{array}{c} x^{-m/2}H_b^s(\overline{M}; E)\\ \oplus\\ \bigoplus\limits_{j=1}^K x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N}; F_j)\end{array} \to x^{m/2}H_b^{s+m}(\overline{M}; E) \hookrightarrow \mathcal{D}^s_{\min}\begin{pmatrix}A\\ T\end{pmatrix}$$

is continuous for all $s > -\frac{1}{2}$.

In order to further refine the parametrix $\mathcal{B}_2(\lambda)$, we first recall the notion of operator-valued symbols on the sector Λ (general information about such symbol classes can be found in [30, 31]):

Definition 7.9. Let $\mathbf{H}$ and $\tilde{\mathbf{H}}$ be Hilbert spaces endowed with strongly continuous groups of isomorphisms $\{\kappa_\varrho\}$ and $\{\tilde{\kappa}_\varrho\}$, $\varrho > 0$, respectively.

A function $g \in C^\infty(\Lambda, \mathscr{L}(\mathbf{H}, \tilde{\mathbf{H}}))$ is called an operator-valued symbol of order $\mu \in \mathbb{R}$, if for all multi-indices $\alpha \in \mathbb{N}_0^2$

$$\big\|\tilde{\kappa}^{-1}_{[\lambda]^{1/m}}\partial_\lambda^\alpha g(\lambda)\kappa_{[\lambda]^{1/m}}\big\|_{\mathscr{L}(\mathbf{H},\tilde{\mathbf{H}})} = O\big(|\lambda|^{\mu/m-|\alpha|}\big) \tag{7.10}$$

as $|\lambda| \to \infty$, where $[\cdot]$ is a smooth function on $\mathbb{C}$ with $[\lambda] > 1$ for all $\lambda \in \mathbb{C}$, and $[\lambda] = |\lambda|$ for $|\lambda| > 2$.

Moreover, for every $j \in \mathbb{N}_0$ there should exist (twisted) homogeneous (or κ-homogeneous) components $g_{(\mu-j)} \in C^\infty(\Lambda \setminus \{0\}, \mathscr{L}(\mathbf{H}, \tilde{\mathbf{H}}))$, i.e.

$$g_{(\mu-j)}(\varrho^m \lambda) = \varrho^{\mu-j} \tilde{\kappa}_\varrho g_{(\mu-j)}(\lambda) \kappa_\varrho^{-1}$$

for $\varrho > 0$, such that for some excision function $\chi \in C^\infty(\mathbb{C})$ ($\chi \equiv 0$ near zero and $\chi \equiv 1$ near infinity) and all $N \in \mathbb{N}_0$ the symbol estimates (7.10) hold for $g(\lambda) - \sum\limits_{j=0}^{N-1} \chi(\lambda) g_{(\mu-j)}(\lambda)$ in the place of $g(\lambda)$, and μ replaced by $\mu - N$. We sometimes write

$$g(\lambda) \sim \sum_{j=0}^{\infty} g_{(\mu-j)}(\lambda),$$

and $g_\wedge(\lambda) := g_{(\mu)}(\lambda)$ is called the principal component of $g(\lambda)$.

We call the operator-valued symbol $g(\lambda)$ compact, if in all conditions above we may replace the space of all bounded operators $\mathscr{L}(\mathbf{H}, \tilde{\mathbf{H}})$ by the ideal of compact operators $\mathcal{K}(\mathbf{H}, \tilde{\mathbf{H}})$.

In the considerations below the Hilbert spaces $\mathbf{H}$ and $\tilde{\mathbf{H}}$ will either be function spaces on the model cone $\overline{Y}^\wedge$ and $(\partial\overline{Y})^\wedge$, or just $\mathbb{C}^N$, and the group action is either the normalized dilation κ_ϱ from Definition 5.12 on the function spaces, or the trivial group action ($\tilde{\kappa}_\varrho \equiv I$) on $\mathbb{C}^N$.

The following Definition 7.11 of generalized Green remainders is essential for understanding the structure of remainders of the parametrix construction, and for further necessary refinement of the parametrix itself.

Let ∂_+ denote normal differentiation on $2\overline{M}$ (near the boundary $\partial(2\overline{M})$), where "normal" refers to some Riemannian metric, smooth up to $\partial(2\overline{M})$, that coincides near $\overline{Y}$ with $dx^2 + dy^2$.

Definition 7.11. Let $d \in \mathbb{N}_0$ and $\mu \in \mathbb{R}$. An operator family

$$\mathcal{G}(\lambda) = \sum_{j=0}^{d} \mathcal{G}_j(\lambda) \begin{pmatrix} \partial_+ & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^j : \begin{matrix} C_0^\infty(\mathring{\overline{M}}; E) \\ \oplus \\ C_0^\infty(\mathring{\overline{N}}; F) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} C^\infty(\mathring{\overline{M}}; E) \\ \oplus \\ C^\infty(\mathring{\overline{N}}; F') \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix} \tag{7.12}$$

is called a generalized Green remainder of order μ and type d in the scales $\begin{pmatrix} x^\alpha H_b^s \\ x^\beta H_b^s \\ \mathbb{C}^{N_-} \end{pmatrix}$ to $\begin{pmatrix} x^{\alpha'} H_b^s \\ x^{\beta'} H_b^s \\ \mathbb{C}^{N_+} \end{pmatrix}$ if for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty([0,1))$ near zero the following holds:

i) For all $j = 0, \dots, d$

$$(1-\omega)\mathcal{G}_j(\lambda),\ \mathcal{G}_j(\lambda)(1-\tilde{\omega}) \in \bigcap_{s,t\in\mathbb{R}} \mathscr{S}\left(\Lambda, \mathcal{K}\begin{pmatrix} x^{\alpha}H^s_{b,0}(\overline{M};E) & x^{\alpha'}H^t_b(\overline{M};E) \\ \oplus & \oplus \\ x^{\beta}H^s_b(\overline{N};F) \quad , & x^{\beta'}H^t_b(\overline{N};F') \\ \oplus & \oplus \\ \mathbb{C}^{N_-} & \mathbb{C}^{N_+} \end{pmatrix}\right).$$

ii) For all $j = 0, \dots, d$,

$$g_j(\lambda) = \omega\mathcal{G}_j(\lambda)\tilde{\omega} : \begin{matrix} C_0^\infty(\overset{\circ}{\overline{Y}}{}^\wedge;E) \\ \oplus \\ C_0^\infty(\partial\overset{\circ}{\overline{Y}}{}^\wedge;F) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} C^\infty(\overset{\circ}{\overline{Y}}{}^\wedge;E) \\ \oplus \\ C^\infty(\partial\overset{\circ}{\overline{Y}}{}^\wedge;F') \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}$$

is a Green symbol, i.e. a compact operator-valued symbol

$$\begin{matrix} {}_\delta\mathcal{K}_0^{s,\alpha}(\overline{Y}^\wedge;E) \\ \oplus \\ {}_\delta\mathcal{K}^{s,\beta}((\partial\overline{Y})^\wedge;F) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} {}_{\delta'}\mathcal{K}^{t,\alpha'}(\overline{Y}^\wedge;E) \\ \oplus \\ {}_{\delta'}\mathcal{K}^{t,\beta'}((\partial\overline{Y})^\wedge;F') \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}$$

of order $\mu \in \mathbb{R}$ for all $s, t, \delta, \delta' \in \mathbb{R}$.

Here, for $s, \delta, \alpha \in \mathbb{R}$, we write ${}_\delta\mathcal{K}^{s,\alpha} = \omega\mathcal{K}^{s,\alpha} + (1-\omega)x^{-\delta}\mathcal{K}^{s,\alpha}$, as well as ${}_\delta\mathcal{K}_0^{s,\alpha} = \left({}_{-\delta}\mathcal{K}^{-s,-\alpha-m}\right)'$, where the dual space is to be understood with respect to the pairing induced by the $x^{-m/2}L^2_b$-inner product.

Correspondingly, the operator family (7.12) is called a generalized Green remainder of order μ and type d in the scales $\begin{pmatrix} x^\alpha H^s_b \\ x^\beta H^s_b \\ \mathbb{C}^{N_-}\end{pmatrix}$ to $\begin{pmatrix} \mathcal{D}^s_{\min} \\ x^{\beta'} H^s_b \\ \mathbb{C}^{N_+}\end{pmatrix}$ if for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty([0,1))$ near zero the following holds:

iii) For all $j = 0, \dots, d$

$$(1-\omega)\mathcal{G}_j(\lambda),\ \mathcal{G}_j(\lambda)(1-\tilde{\omega}) \in \bigcap_{\substack{s\in\mathbb{R} \\ t>-\frac{1}{2}}} \mathscr{S}\left(\Lambda, \mathcal{K}\begin{pmatrix} x^{\alpha}H^s_{b,0}(\overline{M};E) & \mathcal{D}^t_{\min}\binom{A}{T} \\ \oplus & \oplus \\ x^{\beta}H^s_b(\overline{N};F) \quad , & x^{\beta'}H^t_b(\overline{N};F') \\ \oplus & \oplus \\ \mathbb{C}^{N_-} & \mathbb{C}^{N_+} \end{pmatrix}\right).$$

iv) For all $j = 0, \dots, d$, $g_j(\lambda) = \omega\mathcal{G}_j(\lambda)\tilde{\omega}$ is a compact operator-valued symbol

$$\begin{matrix} {}_\delta\mathcal{K}_0^{s,\alpha}(\overline{Y}^\wedge;E) \\ \oplus \\ {}_\delta\mathcal{K}^{s,\beta}((\partial\overline{Y})^\wedge;F) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} {}_{\delta'}\mathcal{D}^t_{\wedge,\min}\binom{A_\wedge}{T_\wedge} \\ \oplus \\ {}_{\delta'}\mathcal{K}^{t,\beta'}((\partial\overline{Y})^\wedge;F') \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}$$

of order μ for all $s,t,\delta,\delta' \in \mathbb{R}$, $t > -\frac{1}{2}$, where analogously to the above

$$_{\delta'}\mathcal{D}^t_{\wedge,\min}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} = \omega \mathcal{D}^t_{\wedge,\min}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} + (1-\omega)x^{-\delta'}\mathcal{D}^t_{\wedge,\min}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix}.$$

In ii) and iv), the property of being an operator-valued symbol always refers to the group action $\begin{pmatrix} \kappa_\varrho & 0 & 0 \\ 0 & \kappa_\varrho & 0 \\ 0 & 0 & I \end{pmatrix}$, i.e. we consider the normalized dilation group κ_ϱ from Definition 5.12 on the function spaces, and the trivial group action on $\mathbb{C}^{N_\pm}$. Moreover, multiplication of the $\mathcal{G}_j(\lambda)$ by the cut-off function ω (or $\tilde{\omega}$) above is to be understood as multiplication by the diagonal matrix $\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & I \end{pmatrix}$, while 1 always is the identity matrix. In particular, $1-\omega$ is to be understood as multiplication by the matrix $\begin{pmatrix} 1-\omega & 0 & 0 \\ 0 & 1-\omega & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

It is needless to say that these definitions also apply to each entry of the matrix $\mathcal{G}(\lambda)$ separately (which corresponds, e.g., to $N_\pm = 0$ or $F = 0$, $F' = 0$). For $N_- = N_+ = 0$, every generalized Green remainder of order μ and type d is an element of $\mathcal{B}^{-\infty,d}(\mathring{\overline{M}};\Lambda)$, the class of regularizing parameter-dependent generalized singular Green operators in Boutet de Monvel's calculus on $\mathring{\overline{M}}$ of type d, i.e. we pass to a specific class of admissible remainders here. Both meanings of "Green" should not be mixed up, and it will always be clear from the context which notion applies.

It is not hard to prove that every generalized Green remainder $\mathcal{G}(\lambda)$ of order μ has an associated sequence $\mathcal{G}_{(\mu-j)}(\lambda)$ of (twisted) homogeneous components of order $\mu - j$, $j \in \mathbb{N}_0$ — namely the components of the operator-valued symbol $\omega\mathcal{G}(\lambda)\tilde{\omega}$ — and these components are unique, i.e. they do not depend on the choice of cut-off functions $\omega, \tilde{\omega} \in C_0^\infty([0,1))$ (the argument is similar to Lemma 5.19 in [9]). Consequently, we write

$$\mathcal{G}(\lambda) \sim \sum_{j=0}^{\infty} \mathcal{G}_{(\mu-j)}(\lambda),$$

and call $\mathcal{G}_\wedge(\lambda) := \mathcal{G}_{(\mu)}(\lambda)$ the principal component of $\mathcal{G}(\lambda)$. Moreover, the intersection of all generalized Green remainders of order μ and fixed type d consists of the so called regularizing generalized Green remainders of type d.

Observe, in particular, that the operator

$$\mathcal{G}(\lambda) : \begin{array}{c} x^\alpha H_b^s(\overline{M};E) \\ \oplus \\ x^\beta H_b^s(\overline{N};F) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \rightarrow \begin{array}{c} x^{\alpha'} H_b^t(\overline{M};E) \\ \oplus \\ x^{\beta'} H_b^t(\overline{N};F') \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

is compact for all $\lambda \in \Lambda$ and all $s,t \in \mathbb{R}$, $s > d - \frac{1}{2}$, where $\mathcal{G}(\lambda)$ is any generalized Green remainder of type $d \in \mathbb{N}_0$ in the scales $\begin{pmatrix} x^{\alpha}H_b^s \\ x^{\beta}H_b^s \\ \mathbb{C}^{N_-} \end{pmatrix}$ to $\begin{pmatrix} x^{\alpha'}H_b^s \\ x^{\beta'}H_b^s \\ \mathbb{C}^{N_+} \end{pmatrix}$. Analogously,

$$\mathcal{G}(\lambda) : \begin{array}{c} x^{\alpha}H_b^s(\overline{M};E) \\ \oplus \\ x^{\beta}H_b^s(\overline{N};F) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} \mathcal{D}^t_{\min}\binom{A}{T} \\ \oplus \\ x^{\beta'}H_b^t(\overline{N};F') \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

is compact for all $\lambda \in \Lambda$ and all $s > d - \frac{1}{2}$, $t > -\frac{1}{2}$, for any generalized Green remainder $\mathcal{G}(\lambda)$ of type $d \in \mathbb{N}_0$ in the scales $\begin{pmatrix} x^{\alpha}H_b^s \\ x^{\beta}H_b^s \\ \mathbb{C}^{N_-} \end{pmatrix}$ to $\begin{pmatrix} \mathcal{D}^s_{\min} \\ x^{\beta'}H_b^s \\ \mathbb{C}^{N_+} \end{pmatrix}$.

It is easy to see from the definition that the generalized Green remainders form an algebra, i.e. the composition $\mathcal{G}_1(\lambda)\mathcal{G}_2(\lambda)$ of generalized Green remainders $\mathcal{G}_j(\lambda)$ of orders μ_j and types d_j, $j = 1,2$, is a generalized Green remainder of order $\mu_1+\mu_2$ and type d_2, and the principal component of the composition equals the product of the principal components of the factors (here it is of course assumed that the scales fit together such that the composition makes sense).

Lemma 7.13. *Let*

$$\mathcal{G}(\lambda) = \left(\begin{array}{c|cccc|c} \mathcal{G}_{0,0}(\lambda) & \mathcal{G}_{0,1}(\lambda) & \cdots & \mathcal{G}_{0,K}(\lambda) & \mathcal{G}_{0,K+1}(\lambda) \\ \hline \mathcal{G}_{1,0}(\lambda) & \mathcal{G}_{1,1}(\lambda) & \cdots & \mathcal{G}_{1,K}(\lambda) & \mathcal{G}_{1,K+1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{G}_{K,0}(\lambda) & \mathcal{G}_{K,1}(\lambda) & \cdots & \mathcal{G}_{K,K}(\lambda) & \mathcal{G}_{K,K+1}(\lambda) \\ \hline \mathcal{G}_{K+1,0}(\lambda) & \mathcal{G}_{K+1,1}(\lambda) & \cdots & \mathcal{G}_{K+1,K}(\lambda) & \mathcal{G}_{K+1,K+1}(\lambda) \end{array}\right)$$

be a matrix of generalized Green remainders of fixed type $d \in \mathbb{N}_0$, and let $n_i - n_j$ be the order of $\mathcal{G}_{i,j}(\lambda)$, $i,j = 0,\ldots,K+1$. Here $\mathcal{G}(\lambda)$ is an operator

$$\begin{array}{c} x^{-m/2}H_b^s(\overline{M};E) \\ \oplus \\ \bigoplus_{j=1}^{K} x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N};F_j) \\ \oplus \\ \mathbb{C}^N \end{array} \to \begin{array}{c} x^{-m/2}H_b^s(\overline{M};E) \\ \oplus \\ \bigoplus_{j=1}^{K} x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N};F_j) \\ \oplus \\ \mathbb{C}^N \end{array}$$

for $s > d - \frac{1}{2}$, and the $\mathcal{G}_{i,j}(\lambda)$ are assumed to be Green in the corresponding scales of spaces.

Let $\mathcal{G}_\wedge(\lambda)$ be the matrix of principal parts of $\mathcal{G}(\lambda)$, an operator family in the spaces

$$\begin{array}{c} \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge;E) \\ \oplus \\ \bigoplus_{j=1}^{K} \mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge;F_j) \\ \oplus \\ \mathbb{C}^N \end{array} \to \begin{array}{c} \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge;E) \\ \oplus \\ \bigoplus_{j=1}^{K} \mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge;F_j) \\ \oplus \\ \mathbb{C}^N \end{array}$$

for $s > d-\frac{1}{2}$ and $\lambda \in \Lambda\setminus\{0\}$, and assume that $1+\mathcal{G}_\wedge(\lambda)$ is invertible for $\lambda \in \Lambda\setminus\{0\}$.

Then $1+\mathcal{G}(\lambda)$ is invertible for large $\lambda \in \Lambda$, and there exists a matrix $\tilde{\mathcal{G}}(\lambda) = \big(\tilde{\mathcal{G}}_{i,j}(\lambda)\big)$ of generalized Green remainders $\tilde{\mathcal{G}}_{i,j}(\lambda)$ of the same type $d \in \mathbb{N}_0$ and order $n_i - n_j$, $i,j = 0,\ldots,K+1$, such that $\big(1+\mathcal{G}(\lambda)\big)^{-1} = 1+\tilde{\mathcal{G}}(\lambda)$ for $|\lambda| > 0$ sufficiently large.

Proof. Note that the matrix $\mathcal{G}(\lambda)$ is of order zero with respect to an order convention of Douglis-Nirenberg type: A matrix $\big(\mathcal{G}_{i,j}(\lambda)\big)$ is to be considered of order $\mu \in \mathbb{R}$ if $\mathcal{G}_{i,j}(\lambda)$ has order $n_i - n_j + \mu$, and correspondingly a matrix $\mathcal{G}_\wedge(\lambda)$ is κ-homogeneous of Douglis-Nirenberg order μ if

$$\mathcal{G}_\wedge(\varrho^m\lambda) = \varrho^\mu \begin{pmatrix} \varrho^{n_0}\kappa_\varrho & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \varrho^{n_K}\kappa_\varrho & 0 \\ 0 & \cdots & 0 & \varrho^{n_{K+1}} \end{pmatrix} \mathcal{G}_\wedge(\lambda) \begin{pmatrix} \varrho^{n_0}\kappa_\varrho & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \varrho^{n_K}\kappa_\varrho & 0 \\ 0 & \cdots & 0 & \varrho^{n_{K+1}} \end{pmatrix}^{-1}$$

for $\varrho > 0$, where in our situation $\mu = 0$.

Therefore we see that the inverse of $1+\mathcal{G}_\wedge(\lambda)$ is of the form $1+\tilde{\mathcal{G}}_\wedge(\lambda)$, where $\tilde{\mathcal{G}}_\wedge(\lambda)$ is κ-homogeneous of (Douglis-Nirenberg) order zero, and from the identity

$$\big(1+\mathcal{G}_\wedge(\lambda)\big)^{-1} = 1 - \mathcal{G}_\wedge(\lambda) + \mathcal{G}_\wedge(\lambda)\big(1+\mathcal{G}_\wedge(\lambda)\big)^{-1}\mathcal{G}_\wedge(\lambda)$$

we see that $\tilde{\mathcal{G}}_\wedge(\lambda)$ is a principal Green symbol of type d and Douglis-Nirenberg order zero.

With a cut-off function $\omega \in C_0^\infty([0,1))$ and a function $\chi \in C^\infty(\mathbb{C})$ with $\chi \equiv 0$ near zero and $\chi \equiv 1$ near infinity define

$$\mathcal{G}'(\lambda) := \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \chi(\lambda)\tilde{\mathcal{G}}_\wedge(\lambda) \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{G}'(\lambda)$ is a generalized Green remainder of Douglis-Nirenberg order zero and type d, and

$$\big(1+\mathcal{G}(\lambda)\big)\big(1+\mathcal{G}'(\lambda)\big) - 1,\ \big(1+\mathcal{G}'(\lambda)\big)\big(1+\mathcal{G}(\lambda)\big) - 1$$

are generalized Green remainders of Douglis-Nirenberg order -1 and type d.

As the classes of generalized Green remainders are asymptotically complete, a standard formal Neumann series argument now shows that $1+\mathcal{G}(\lambda)$ has an inverse of the asserted form modulo regularizing generalized Green remainders of type d, and as these are rapidly decreasing in the norm as $|\lambda| \to \infty$ the assertion of the lemma regarding the invertibility of $1+\mathcal{G}(\lambda)$ for large λ follows.

Finally, it remains to note that if $\hat{\mathcal{G}}(\lambda)$ is a regularizing generalized Green remainder of type d, the inverse of $1+\hat{\mathcal{G}}(\lambda)$ for large λ is of the form

$$\big(1+\hat{\mathcal{G}}(\lambda)\big)^{-1} = 1 - \hat{\mathcal{G}}(\lambda) + \hat{\mathcal{G}}(\lambda)\chi(\lambda)\big(1+\hat{\mathcal{G}}(\lambda)\big)^{-1}\hat{\mathcal{G}}(\lambda),$$

where $\chi \in C^\infty(\mathbb{C})$ is an excision function as above, and

$$-\hat{\mathcal{G}}(\lambda) + \hat{\mathcal{G}}(\lambda)\chi(\lambda)\big(1+\hat{\mathcal{G}}(\lambda)\big)^{-1}\hat{\mathcal{G}}(\lambda)$$

is obviously a regularizing generalized Green remainder of type d. □

Let $\hat{\mathcal{A}}_0(\sigma)$ be the conormal symbol of (4.1). The c-ellipticity implies that the inverse $\hat{\mathcal{A}}_0^{-1}(\sigma)$ is a finitely meromorphic Fredholm function on $\mathbb{C}$, and there exists a sufficiently small $\varepsilon_0 > 0$ such that $\hat{\mathcal{A}}_0(\sigma)$ is invertible in

$$\{\sigma \in \mathbb{C};\ -m/2 - \varepsilon_0 < \Im\sigma < -m/2 + \varepsilon_0,\ \Im\sigma \neq -m/2\}.$$

Define

$$h_0(\sigma) = \hat{\mathcal{A}}_0^{-1}(\sigma) - h(0,\sigma,0), \tag{7.14}$$

where h is the holomorphic Mellin symbol from (7.6). Then $h_0(\sigma)$ is finitely meromorphic in $\mathbb{C}$ taking values in $\mathcal{B}^{-\infty,0}(\overline{Y})$, and it is rapidly decreasing as $|\Re\sigma| \to \infty$, uniformly for $\Im\sigma$ in compact intervals (this is subject to the composition behavior of cone pseudodifferential operators without parameters; a proof can be found in [29]). Moreover, the set

$$\{\sigma \in \mathbb{C};\ -m/2 - \varepsilon_0 < \Im\sigma < -m/2 + \varepsilon_0,\ \Im\sigma \neq -m/2\}$$

is free of poles of $h_0(\sigma)$.

Let $\omega \in C_0^\infty([0,1))$ be a cut-off function and $0 < \varepsilon < \varepsilon_0$. Define

$$m(x,x',\sigma,\lambda) := \omega(x[\lambda]^{1/m})h_0(\sigma)\omega(x'[\lambda]^{1/m}) \begin{pmatrix} {x'}^m & 0 & \cdots & 0 \\ 0 & {x'}^{m_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {x'}^{m_K} \end{pmatrix},$$

$$m_\wedge(x,x',\sigma,\lambda) := \omega(x|\lambda|^{1/m})h_0(\sigma)\omega(x'|\lambda|^{1/m}) \begin{pmatrix} {x'}^m & 0 & \cdots & 0 \\ 0 & {x'}^{m_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {x'}^{m_K} \end{pmatrix},$$

and associated operators

$$M(\lambda) : \begin{matrix} C_0^\infty(\mathring{\overline{M}};E) \\ \oplus \\ \bigoplus_{j=1}^K C_0^\infty(\mathring{\overline{N}};F_j) \end{matrix} \to C^\infty(\mathring{\overline{M}};E), \tag{7.15}$$

$$M_\wedge(\lambda) : \begin{matrix} C_0^\infty(\mathring{\overline{Y}}^\wedge;E) \\ \oplus \\ \bigoplus_{j=1}^K C_0^\infty(\partial\mathring{\overline{Y}}^\wedge;F_j) \end{matrix} \to C^\infty(\mathring{\overline{Y}}^\wedge;E) \tag{7.16}$$

via

$$\begin{pmatrix} u \\ v_1 \\ \vdots \\ v_K \end{pmatrix} \mapsto \Bigg(\frac{1}{2\pi}\int\limits_{\Im\sigma=-m/2+\varepsilon}\int\limits_{\mathbb{R}_+}\Big(\frac{x}{x'}\Big)^{i\sigma} m_{(\wedge)}(x,x',\sigma,\lambda)\begin{pmatrix} u(x') \\ v_1(x') \\ \vdots \\ v_K(x') \end{pmatrix}\frac{dx'}{x'}\,d\sigma\Bigg).$$

$M(\lambda)$ is a regularizing parameter-dependent generalized singular Green operator in Boutet de Monvel's calculus of type zero, and since the function $\omega(x[\lambda]^{1/m})$ is supported in the collar $[0,1)\times\overline{Y}$, $M(\lambda)$ can be regarded as an operator both on the manifold and the model cone. Observe, moreover, that the components of the matrix $M_\wedge(\lambda)$ are κ-homogeneous of degrees $-m$, $-m_1$, …, $-m_K$.

We define a refinement of the parameter-dependent parametrix $\mathcal{B}_2(\lambda)$ of (4.1) from Proposition 7.8 via

$$\mathcal{B}_3(\lambda) := \mathcal{B}_2(\lambda) + M(\lambda) : \begin{array}{c} C_0^\infty(\overset{\circ}{\overline{M}}; E) \\ \oplus \\ \bigoplus_{j=1}^K C_0^\infty(\overset{\circ}{\overline{N}}; F_j) \end{array} \to C^\infty(\overset{\circ}{\overline{M}}; E), \tag{7.17}$$

and correspondingly let

$$\mathcal{B}_{3,\wedge}(\lambda) := \mathcal{B}_{2,\wedge}(\lambda) + M_\wedge(\lambda), \tag{7.18}$$

$\lambda \in \Lambda \setminus \{0\}$, be the principal part of $\mathcal{B}_3(\lambda)$, where

$$\mathcal{B}_{2,\wedge}(\lambda) \begin{pmatrix} u \\ v_1 \\ \vdots \\ v_K \end{pmatrix}(x) = \frac{1}{2\pi} \int_{\Im(\sigma)=-m/2} \int_{\mathbb{R}_+} \Big(\frac{x}{x'}\Big)^{i\sigma} h(0,\sigma,x^m\lambda) \begin{pmatrix} {x'}^m u(x') \\ {x'}^{m_1} v_1(x') \\ \vdots \\ {x'}^{m_K} v_K(x') \end{pmatrix} \frac{dx'}{x'}\, d\sigma.$$

Proposition 7.19. *Let*

$$\mathcal{G}(\lambda) = \big(\mathcal{G}_{i,j}(\lambda)\big)_{\substack{i=0,\ldots,K+1 \\ j=0,1}} : \begin{array}{c} \mathcal{D}^s_{\min}\binom{A}{T} \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} x^{-m/2} H_b^s(\overline{M}; E) \\ \oplus \\ \bigoplus_{j=1}^K x^{m/2-m_j} H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

be a matrix of generalized Green remainders, where $\mathcal{G}_{i,0}(\lambda) \equiv 0$ and $\mathcal{G}_{i,1}(\lambda)$ has order m_i, $i = 0,\ldots,K+1$, with $m_0 = m$ and arbitrary $m_{K+1} \in \mathbb{R}$.

Moreover, let

$$\tilde{\mathcal{G}}(\lambda) = \big(\tilde{\mathcal{G}}_{i,j}(\lambda)\big)_{\substack{i=0,1 \\ j=0,\ldots,K+1}} : \begin{array}{c} x^{-m/2} H_b^s(\overline{M}; E) \\ \oplus \\ \bigoplus_{j=1}^K x^{m/2-m_j} H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \\ \oplus \\ \mathbb{C}^{N_+} \end{array} \to \begin{array}{c} \mathcal{D}^s_{\min}\binom{A}{T} \\ \oplus \\ \mathbb{C}^{N_-} \end{array}$$

be a matrix of generalized Green remainders of type zero, where $\tilde{\mathcal{G}}_{i,j}(\lambda)$ has order $-m_j$, $i = 0,1$, $j = 0,\ldots,K+1$.

Then

$$\left[\left(\begin{array}{c|c} A-\lambda & 0 \\ \hline T & 0 \\ \hline 0 & 0 \end{array}\right) + \mathcal{G}(\lambda)\right] \cdot \left[\left(\begin{array}{cc} \mathcal{B}_3(\lambda) & 0 \\ \hline 0 & 0 \end{array}\right) + \tilde{\mathcal{G}}(\lambda)\right] = 1 + \hat{\mathcal{G}}(\lambda),$$

$$\left[\left(\begin{array}{c|c} A_\wedge-\lambda & 0 \\ \hline T_\wedge & 0 \\ \hline 0 & 0 \end{array}\right) + \mathcal{G}_\wedge(\lambda)\right] \cdot \left[\left(\begin{array}{cc} \mathcal{B}_{3,\wedge}(\lambda) & 0 \\ \hline 0 & 0 \end{array}\right) + \tilde{\mathcal{G}}_\wedge(\lambda)\right] = 1 + \hat{\mathcal{G}}_\wedge(\lambda),$$

where $\hat{\mathcal{G}}(\lambda) = \big(\hat{\mathcal{G}}_{i,j}(\lambda)\big)_{i,j=0,\ldots,K+1}$ is a matrix of generalized Green remainders of type zero, and $\hat{\mathcal{G}}_{i,j}(\lambda)$ has order $m_i - m_j$.

Proof. The proof of this proposition amounts in understanding the structure of the following compositions:

i) $(A-\lambda)\tilde{\mathcal{G}}(\lambda)$ and $\gamma B_j\tilde{\mathcal{G}}(\lambda)$, $j=1,\dots,K$.
ii) $\mathcal{G}(\lambda)\mathcal{B}_3(\lambda)$.
iii) $\binom{A-\lambda}{T}\mathcal{B}_3(\lambda)$.

In i) and ii), $\mathcal{G}(\lambda)$ and $\tilde{\mathcal{G}}(\lambda)$ are appropriate (matrices of) generalized Green remainders. Using the identity

$$\mathcal{D}^s_{\min}\begin{pmatrix}A\\T\end{pmatrix}=\mathcal{D}^s_{\max}\begin{pmatrix}A\\T\end{pmatrix}\cap\bigcap_{\varepsilon>0}x^{m/2-\varepsilon}H_b^{s+m}(\overline{M};E),$$

as well as (anisotropic modifications of) the results about the structure and composition behavior of parameter-dependent pseudodifferential cone operators in the edge symbolic calculus in scales of Sobolev spaces from [14], we can employ here the same strategy as in the boundaryless case, see [9]:

- Using the expansions (6.2), (6.3), and (6.4) and a similar expansion for $\mathcal{B}_2(\lambda)$ (Taylor expansion of the symbol $h(x,\sigma,\lambda)$ from (7.6) in $x=0$), an inspection of the proof of Lemma 5.20 in [9] reveals that the analogue of this lemma also holds in our present situation, i.e. the compositions i) and ii) above result in Green remainder terms as asserted, and the principal components satisfy the desired multiplicative identity. Note, moreover, that each component of $M(\lambda)$ gives rise to an operator-valued symbol taking values in $\mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge}$.
- Composition iii) is of the form "1 + Green", and the proof of this follows similar to the corresponding result for the boundaryless case, see Theorem 5.24 in [9]:

  The composition behavior of parameter-dependent cone operators in Sobolev spaces implies

$$\begin{pmatrix}A-\lambda\\T\end{pmatrix}\mathcal{B}_2(\lambda)=1+\tilde{M}(\lambda)+\mathcal{G}'(\lambda),$$

$$\tilde{M}(\lambda)\begin{pmatrix}u\\v_1\\\vdots\\v_K\end{pmatrix}(x)=\frac{1}{2\pi}\int\limits_{\Im\sigma=-m/2}\int\limits_{\mathbb{R}_+}\Big(\frac{x}{x'}\Big)^{i\sigma}\tilde{m}(x,x',\sigma,\lambda)\begin{pmatrix}u(x')\\v_1(x')\\\vdots\\v_K(x')\end{pmatrix}\frac{dx'}{x'}\,d\sigma,$$

  where $\mathcal{G}'(\lambda)$ is a matrix of generalized Green remainders, and $\tilde{m}(x,x',\sigma,\lambda)$ equals

$$-\begin{pmatrix}x^{-m}&0&\cdots&0\\0&x^{-m_1}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&x^{-m_K}\end{pmatrix}\omega\big(x[\lambda]^{1/m}\big)\hat{\mathcal{A}}_0(\sigma)h_0(\sigma)\omega\big(x'[\lambda]^{1/m}\big)\begin{pmatrix}x'^{m}&0&\cdots&0\\0&x'^{m_1}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&x'^{m_K}\end{pmatrix}$$

  with $h_0(\sigma)$ from (7.14). The composition $\binom{A-\lambda}{T}M(\lambda)$ compensates the Mellin term $\tilde{M}(\lambda)$, and the remainder is therefore Green.

$\square$

Remark 7.20. In the situation of Proposition 7.19, we also have that

$$\left[\begin{pmatrix} \mathcal{B}_3(\lambda) & 0 \\ \hline 0 & 0 \end{pmatrix} + \tilde{\mathcal{G}}(\lambda)\right] \cdot \left[\begin{pmatrix} A-\lambda & 0 \\ \hline T & 0 \\ \hline 0 & 0 \end{pmatrix} + \mathcal{G}(\lambda)\right] = 1 + \check{\mathcal{G}}(\lambda),$$

where $\check{\mathcal{G}}(\lambda) = \big(\check{\mathcal{G}}_{i,j}(\lambda)\big)_{i,j=0,1}$, and $\check{\mathcal{G}}_{0,0}(\lambda) \in \mathcal{B}^{-\infty,m}(\overline{\mathring{M}};\Lambda)$ is a regularizing parameter-dependent singular Green operator in Boutet de Monvel's calculus on $\overline{\mathring{M}}$. In addition, we have the following properties (see also Lemma 5.20, Proposition 5.22, and Theorem 5.24 in [9]):

For all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty([0,1))$ near zero the following holds:

i)

$$\begin{pmatrix} 1-\omega & 0 \\ 0 & 0 \end{pmatrix} \check{\mathcal{G}}(\lambda),\ \check{\mathcal{G}}(\lambda) \begin{pmatrix} 1-\tilde{\omega} & 0 \\ 0 & 0 \end{pmatrix} \in \bigcap_{s,t>-\frac{1}{2}} \mathscr{S}\left(\Lambda, \mathcal{K}\left(\begin{matrix} \mathcal{D}^s_{\min}\binom{A}{T} \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix}, \begin{matrix} \mathcal{D}^t_{\min}\binom{A}{T} \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix}\right)\right).$$

ii) $g(\lambda) = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \check{\mathcal{G}}(\lambda) \begin{pmatrix} \tilde{\omega} & 0 \\ 0 & 1 \end{pmatrix}$ is a compact operator-valued symbol

$$\begin{matrix} {}_{\delta}\mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge} \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \quad \longrightarrow \quad \begin{matrix} {}_{\delta'}\mathcal{D}^t_{\wedge,\min}\binom{A_\wedge}{T_\wedge} \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix}$$

of order zero for all $\delta, \delta' \in \mathbb{R}$, $s,t > -\frac{1}{2}$.

The reader surely noticed that in Definition 7.11 of generalized Green remainders the case of operators whose domain is the $\mathcal{D}_{\min}$-scale was excluded. For the purposes of this paper, we can consider the abovementioned properties as a definition for generalized Green remainders of type m (and order zero). However, if one is interested in parametrices for parabolic equations with time-dependent coefficients on conic manifolds, this definition will not describe in an appropriate way the structure of the symbol kernels.

For $\check{\mathcal{G}}_\wedge(\lambda) := g_\wedge(\lambda)$, where $g(\lambda)$ is the operator-valued symbol in ii) above, we also have

$$\left[\begin{pmatrix} \mathcal{B}_{3,\wedge}(\lambda) & 0 \\ \hline 0 & 0 \end{pmatrix} + \tilde{\mathcal{G}}_\wedge(\lambda)\right] \cdot \left[\begin{pmatrix} A_\wedge-\lambda & 0 \\ \hline T_\wedge & 0 \\ \hline 0 & 0 \end{pmatrix} + \mathcal{G}_\wedge(\lambda)\right] = 1 + \check{\mathcal{G}}_\wedge(\lambda)$$

for $\lambda \in \Lambda \setminus \{0\}$.

Observe, in particular, that the parametrix $\mathcal{B}_3(\lambda)$ is a Fredholm inverse of the boundary value problem (4.1) by Proposition 7.19 and Remark 7.20, and the principal part $\mathcal{B}_{3,\wedge}(\lambda)$ is a Fredholm inverse of the associated problem on the model cone $\begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix}$ for $\lambda \in \Lambda \setminus \{0\}$.

The following Theorem 7.21 deals with the final refinement of the parametrix, and constitutes the main result as regards the parametrix construction of (4.1).

Theorem 7.21. *Assume that* (4.1) *is c-elliptic with parameter* $\lambda \in \Lambda$*, and assume that the model boundary value problem*

$$\mathcal{A}_\wedge(\lambda) = \begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix} : \mathcal{D}^s_{\wedge,\min}\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} \to \begin{array}{c} \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E) \\ \oplus \\ \bigoplus_{j=1}^K \mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge; F_j) \end{array} \tag{7.22}$$

is injective for some $s > -\frac{1}{2}$ *and all* $\lambda \in \Lambda \setminus \{0\}$*. Recall that, by* κ*-homogeneity, the injectivity needs to be required for* $|\lambda| = 1$ *only, and the injectivity of* (7.22) *is equivalent to the injectivity of*

$$A_{\wedge,T_\wedge} - \lambda : \mathcal{D}^s_{\wedge,\min}(A_{\wedge,T_\wedge}) \to \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E) \tag{7.23}$$

for all $\lambda \in \Lambda \setminus \{0\}$ *(or* $|\lambda| = 1$*).*

a) There exists a generalized Green remainder $\mathcal{K}_0(\lambda) : \mathbb{C}^{d''} \to x^{-m/2}H^s_b(\overline{M}; E)$ *of order* $m_0 = m$*, where* $d'' = -\operatorname{ind}\binom{A}{T}_{\mathcal{D}^s_{\min}}$*, such that*

$$\left(\begin{array}{c|c} A-\lambda & \mathcal{K}_0(\lambda) \\ \hline \gamma B_1 & 0 \\ \vdots & \vdots \\ \gamma B_K & 0 \end{array}\right) : \begin{array}{c} \mathcal{D}^s_{\min}\binom{A}{T} \\ \oplus \\ \mathbb{C}^{d''} \end{array} \to \begin{array}{c} x^{-m/2}H^s_b(\overline{M}; E) \\ \oplus \\ \bigoplus_{j=1}^K x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \end{array} \tag{7.24}$$

is invertible for all $s > -\frac{1}{2}$ *and* $\lambda \in \Lambda$ *with* $|\lambda| > 0$ *sufficiently large.*

Moreover, there exists a matrix

$$\left(\begin{array}{c|ccc} \mathcal{G}_0(\lambda) & \mathcal{G}_1(\lambda) & \cdots & \mathcal{G}_K(\lambda) \\ \hline \mathcal{T}_0(\lambda) & \mathcal{T}_1(\lambda) & \cdots & \mathcal{T}_K(\lambda) \end{array}\right) : \begin{array}{c} x^{-m/2}H^s_b(\overline{M}; E) \\ \oplus \\ \bigoplus_{j=1}^K x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \end{array} \to \begin{array}{c} \mathcal{D}^s_{\min}\binom{A}{T} \\ \oplus \\ \mathbb{C}^{d''} \end{array}$$

of generalized Green remainders $\mathcal{G}_j(\lambda)$ *and* $\mathcal{T}_j(\lambda)$ *of orders* $-m_j$ *and type zero, such that the inverse of* (7.24) *is of the form*

$$\left(\begin{array}{c} \mathcal{B}_3(\lambda) + \mathcal{G}(\lambda) \\ \hline \mathcal{T}(\lambda) \end{array}\right) : \left[\begin{array}{c} x^{-m/2}H^s_b(\overline{M}; E) \\ \oplus \\ \bigoplus_{j=1}^K x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \end{array}\right] \to \begin{array}{c} \mathcal{D}^s_{\min}\binom{A}{T} \\ \oplus \\ \mathbb{C}^{d''} \end{array}, \tag{7.25}$$

where $\mathcal{B}_3(\lambda)$ *is the parametrix from* (7.17)*, and*

$$\begin{aligned} \mathcal{G}(\lambda) &= \begin{pmatrix} \mathcal{G}_0(\lambda) & \mathcal{G}_1(\lambda) & \cdots & \mathcal{G}_K(\lambda) \end{pmatrix}, \\ \mathcal{T}(\lambda) &= \begin{pmatrix} \mathcal{T}_0(\lambda) & \mathcal{T}_1(\lambda) & \cdots & \mathcal{T}_K(\lambda) \end{pmatrix}. \end{aligned}$$

In particular, the boundary value problem

$$\mathcal{A}(\lambda) = \left(\begin{array}{c} A-\lambda \\ \hline \gamma B_1 \\ \vdots \\ \gamma B_K \end{array}\right) : \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} \to \begin{array}{c} x^{-m/2}H^s_b(\overline{M}; E) \\ \oplus \\ \bigoplus_{j=1}^K x^{m/2-m_j}H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \end{array} \tag{7.26}$$

is injective for large $\lambda \in \Lambda$*, and the parametrix* $\mathcal{B}(\lambda) := \mathcal{B}_3(\lambda) + \mathcal{G}(\lambda)$ *is a left inverse.*

b) Let

$$\Pi(\lambda) := 1 - \mathcal{A}(\lambda)\mathcal{B}(\lambda) = \left(\begin{array}{c|ccc} \Pi_{0,0}(\lambda) & \Pi_{0,1}(\lambda) & \cdots & \Pi_{0,K}(\lambda) \\ \hline \Pi_{1,0}(\lambda) & \Pi_{1,1}(\lambda) & \cdots & \Pi_{1,K}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_{K,0}(\lambda) & \Pi_{K,1}(\lambda) & \cdots & \Pi_{K,K}(\lambda) \end{array}\right).$$

Then $\Pi_{i,j}(\lambda)$ is a generalized Green remainder of order $m_i - m_j$ and type zero, and $\Pi(\lambda)$ is for large λ a finite-dimensional projection onto a complement of the range of (7.26). *Whenever*

$$\mathcal{A}_{\mathcal{D}}(\lambda_0) = \begin{pmatrix} A - \lambda_0 \\ \hline \gamma B_1 \\ \vdots \\ \gamma B_K \end{pmatrix} : \mathcal{D}^s\begin{pmatrix} A \\ T \end{pmatrix} \to \begin{array}{c} x^{-m/2} H_b^s(\overline{M}; E) \\ \oplus \\ \bigoplus_{j=1}^{K} x^{m/2-m_j} H_b^{s+m-m_j-1/2}(\overline{N}; F_j) \end{array}$$

is invertible for some $\lambda_0 \in \Lambda$ and some domain $\mathcal{D}^s_{\min}\binom{A}{T} \subset \mathcal{D}^s\binom{A}{T} \subset \mathcal{D}^s_{\max}\binom{A}{T}$, the inverse $\mathcal{A}_{\mathcal{D}}(\lambda_0)^{-1}$ can be written in the form $\mathcal{B}(\lambda_0) + \mathcal{A}_{\mathcal{D}}(\lambda_0)^{-1}\Pi(\lambda_0)$.

c) Let $\mathcal{B}_T(\lambda) : x^{-m/2}H_b^s(\overline{M}; E) \to \mathcal{D}^s_{\min}\binom{A}{T}$ be the interior part of the parametrix $\mathcal{B}(\lambda)$. Then, for large $|\lambda| > 0$,

$$\mathcal{B}_T(\lambda) : x^{-m/2}H_b^s(\overline{M}; E) \to \mathcal{D}^s_{\min}(A_T) = \mathcal{D}^s_{\min}\begin{pmatrix} A \\ T \end{pmatrix} \cap \ker T,$$

and $\mathcal{B}_T(\lambda)$ is a left inverse of

$$A_T - \lambda : \mathcal{D}^s_{\min}(A_T) \to x^{-m/2}H_b^s(\overline{M}; E). \tag{7.27}$$

The operator $\Pi_T(\lambda) := \Pi_{0,0}(\lambda) = 1-(A_T-\lambda)\mathcal{B}_T(\lambda)$ is a generalized Green remainder of order and type zero, and for large λ a (finite-dimensional) projection onto a complement of the range of (7.27). *Whenever*

$$A_T - \lambda_0 : \mathcal{D}^s(A_T) \to x^{-m/2}H_b^s(\overline{M}; E)$$

is invertible for some $\lambda_0 \in \Lambda$ with $|\lambda_0| > 0$ sufficiently large and some domain $\mathcal{D}^s_{\min}(A_T) \subset \mathcal{D}^s(A_T) \subset \mathcal{D}^s_{\max}(A_T)$, the resolvent can be written as

$$(A_T - \lambda_0)^{-1} = B_T(\lambda_0) + (A_T - \lambda_0)^{-1}\Pi_T(\lambda_0).$$

d) The principal component

$$\begin{pmatrix} A_\wedge - \lambda & \mathcal{K}_{0,\wedge}(\lambda) \\ T_\wedge & 0 \end{pmatrix} : \begin{array}{c} \mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge} \\ \oplus \\ \mathbb{C}^{d''} \end{array} \to \begin{array}{c} \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E) \\ \oplus \\ \bigoplus_{j=1}^{K} \mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge; F_j) \end{array} \tag{7.28}$$

of (7.24) *is invertible for all $\lambda \in \Lambda \setminus \{0\}$, and the principal component*

$$\begin{pmatrix} \mathcal{B}_{3,\wedge}(\lambda) + \mathcal{G}_\wedge(\lambda) \\ \hline \mathcal{T}_\wedge(\lambda) \end{pmatrix} : \left[\begin{array}{c} \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E) \\ \oplus \\ \bigoplus_{j=1}^{K} \mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge; F_j) \end{array}\right] \to \begin{array}{c} \mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge} \\ \oplus \\ \mathbb{C}^{d''} \end{array} \tag{7.29}$$

of (7.25) *is the inverse.*

Proof. Let $X = \Lambda \cap S^1$, and consider the operator family $A_{\wedge,T_\wedge} - \lambda$ from (7.23) as a smooth Fredholm function on X. By well known results about Fredholm families on compact spaces and a density argument (see also the appendix of [9]), there exists a function

$$\mathcal{K}_{0,\wedge}(\lambda) \in C^\infty(X) \otimes (\mathbb{C}^{d''})^* \otimes C_0^\infty(\overline{\mathring{Y}}^\wedge; E)$$

such that

$$\begin{pmatrix} A_\wedge - \lambda & \mathcal{K}_{0,\wedge}(\lambda) \end{pmatrix} : \begin{array}{c} \mathcal{D}^s_{\wedge,\min}(A_{\wedge,T_\wedge}) \\ \oplus \\ \mathbb{C}^{d''} \end{array} \to \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E) \tag{7.30}$$

is invertible for $\lambda \in X$, and so is the extension of (7.30) to $\Lambda \setminus \{0\}$ by (twisted) homogeneity of degree m (we will use the same notation $\mathcal{K}_{0,\wedge}(\lambda)$). A simple calculation now shows that also (7.28) is invertible for $\lambda \in \Lambda \setminus \{0\}$ for this choice of $\mathcal{K}_{0,\wedge}(\lambda)$.

As $\operatorname{ind} \mathcal{A}_\wedge(\lambda) = -\operatorname{ind} \mathcal{B}_{3,\wedge}(\lambda)$, the same abstract results about Fredholm families on compact spaces as applied before and extension by (twisted) homogeneity now imply the existence of a matrix $\mathcal{C}_\wedge(\lambda)$ of principal Green symbols of type zero and suitable $N_-, N_+ \in \mathbb{N}_0$,

$$N_+ - N_- = \operatorname{ind} \mathcal{B}_{3,\wedge}(\lambda) = -d'',$$

such that

$$\begin{pmatrix} \mathcal{B}_{3,\wedge}(\lambda) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{C}_\wedge(\lambda) : \begin{array}{c} \mathcal{K}^{s,-m/2}(\overline{Y}^\wedge; E) \\ \oplus \\ \bigoplus_{j=1}^{K} \mathcal{K}^{s+m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge; F_j) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} \mathcal{D}^s_{\wedge,\min}\binom{A_\wedge}{T_\wedge} \\ \oplus \\ \mathbb{C}^{N_+} \end{array} \tag{7.31}$$

is invertible for $\lambda \in \Lambda \setminus \{0\}$. By possibly enlarging N_- and N_+ and the matrix $\mathcal{C}_\wedge(\lambda)$, we may assume that $N_+ \geq d''$, and by possibly augmenting the matrix (7.28) by an invertible lower right corner (if $N_+ > d''$), we can multiply (7.28) and (7.31). The product is an invertible matrix of the form $1 + \mathcal{C}'_\wedge(\lambda)$, where $\mathcal{C}'_\wedge(\lambda)$ is a matrix of principal Green symbols. As

$$\big(1 + \mathcal{C}'_\wedge(\lambda)\big)^{-1} = 1 + \mathcal{C}''_\wedge(\lambda)$$

with a matrix $\mathcal{C}''_\wedge(\lambda)$ of principal Green symbols, we conclude that the inverse of (7.28) is indeed of the form (7.29) for suitable matrices $\mathcal{G}_\wedge(\lambda)$ and $\mathcal{T}_\wedge(\lambda)$ of principal Green symbols of the asserted order and type zero. Here we used the fact that the matrices of the form "$1 + \text{Green}$" are spectrally invariant, see the proof of Lemma 7.13.

Choose a cut-off function $\omega \in C_0^\infty([0,1))$ near zero, and an excision function $\chi \in C^\infty(\mathbb{C})$, i.e. $\chi \equiv 0$ near zero, and $\chi \equiv 1$ near infinity. Define $\mathcal{K}_0(\lambda) = \omega\chi(\lambda)\mathcal{K}_{0,\wedge}(\lambda)$, and $\mathcal{G}'(\lambda) = \omega\chi(\lambda)\mathcal{G}_\wedge(\lambda)\omega$, $\mathcal{T}'(\lambda) = \chi(\lambda)\mathcal{T}_\wedge(\lambda)\omega$. Then $\mathcal{K}_0(\lambda)$, $\mathcal{G}'(\lambda)$, and $\mathcal{T}'(\lambda)$ are matrices of generalized Green remainders, and

$$\left(\begin{array}{c|c} A - \lambda & \mathcal{K}_0(\lambda) \\ \hline T & 0 \end{array}\right) \left(\begin{array}{c} \mathcal{B}_3(\lambda) + \mathcal{G}'(\lambda) \\ \hline \mathcal{T}'(\lambda) \end{array}\right) = 1 + \mathcal{G}''(\lambda),$$

where $\mathcal{G}''(\lambda) = \big(\mathcal{G}''_{i,j}(\lambda)\big)_{i,j=0,\ldots,K}$ is a matrix of generalized Green remainders of type zero, and $\mathcal{G}''_{i,j}(\lambda)$ has order $m_i - m_j$ (where $m_0 = m$). Moreover, by construction we have the situation of Lemma 7.13 for $1+\mathcal{G}''(\lambda)$, and thus $\begin{pmatrix} A-\lambda & \mathcal{K}_0(\lambda) \\ T & 0 \end{pmatrix}$ is invertible from the right for $\lambda \in \Lambda$ with $|\lambda| > 0$ sufficiently large, and the right inverse is of the form (7.25). Hence both a) and d) will be proved if we show that $\begin{pmatrix} A-\lambda & \mathcal{K}_0(\lambda) \\ T & 0 \end{pmatrix}$ is also invertible from the left.

To this end, note that

$$\left(\begin{array}{c} \mathcal{B}_3(\lambda)+\mathcal{G}'(\lambda) \\ \hline \mathcal{T}'(\lambda) \end{array}\right)\left(\begin{array}{c|c} A-\lambda & \mathcal{K}_0(\lambda) \\ \hline T & 0 \end{array}\right) = 1 + \check{\mathcal{G}}(\lambda)$$

with an operator which satisfies the conditions of Remark 7.20. Moreover, by construction $\check{\mathcal{G}}_\wedge(\lambda) = 0$, where $\check{\mathcal{G}}_\wedge(\lambda)$ is the principal part of $\check{\mathcal{G}}(\lambda)$, and thus

$$\Big(\sum_{j=0}^{N}(-1)^j\check{\mathcal{G}}(\lambda)^j\Big)\big(1+\check{\mathcal{G}}(\lambda)\big) = 1 + \check{\mathcal{G}}_{N+1}(\lambda),$$

and for $N > 0$ sufficiently large the operator norm of $\check{\mathcal{G}}_{N+1}(\lambda)$ in $\mathscr{L}\begin{pmatrix} \mathcal{D}^s_{\min}\binom{A}{T} \\ \oplus \\ \mathbb{C}^{d''} \end{pmatrix}$ is tending to zero as $|\lambda| \to \infty$. This shows that (7.24) is invertible from the left and completes the proof of a) and d). Note that b) and c) follow immediately from a) by simple algebraic calculations. $\square$

8. Resolvents

The final section is devoted to the main theorem of this article:

Theorem 8.1. *Let* (4.1) *be c-elliptic with parameter in the closed sector* $\Lambda \subset \mathbb{C}$, *and consider the unbounded operator* A *in* $x^{-m/2}L^2_b(\overline{M};E)$ *under the boundary condition* $Tu = 0$ *on some intermediate domain* $\mathcal{D}_{\min}(A_T) \subset \mathcal{D}(A_T) \subset \mathcal{D}_{\max}(A_T)$.

Let $\mathcal{D}_\wedge(A_{\wedge,T_\wedge}) = \theta(\mathcal{D}(A_T))$ *be the associated domain for the model operator* $A_\wedge$ *under the boundary condition* $T_\wedge u = 0$ *according to Proposition 6.11, and assume that* Λ *is a sector of minimal growth for* $A_\wedge$ *with this domain, i.e.*

$$A_\wedge - \lambda : \mathcal{D}_\wedge(A_{\wedge,T_\wedge}) \to x^{-m/2}L^2_b(\overline{Y}^\wedge;E)$$

is invertible for $\lambda \in \Lambda$ *with* $|\lambda| > 0$ *sufficiently large, and the resolvent satisfies the norm estimate*

$$\|\big(A_{\wedge,\mathcal{D}_\wedge} - \lambda\big)^{-1}\|_{\mathscr{L}(x^{-m/2}L^2_b)} = O(|\lambda|^{-1})$$

as $|\lambda| \to \infty$.

Then Λ *is a sector of minimal growth for the operator* A *in* $x^{-m/2}L^2_b(\overline{M};E)$ *with domain* $\mathcal{D}(A_T)$, *and for large* $\lambda \in \Lambda$ *the resolvent can be written in the form*

$$\big(A_{\mathcal{D}} - \lambda\big)^{-1} = \mathcal{B}_T(\lambda) + \big(A_{\mathcal{D}} - \lambda\big)^{-1}\Pi_T(\lambda) \tag{8.2}$$

with the parametrix $\mathcal{B}_T(\lambda)$ *and projection* $\Pi_T(\lambda)$ *onto a complement of the range of* $A_{\min} - \lambda$ *from Theorem 7.21.*

The resolvent condition on $A_\wedge$ from Theorem 8.1 is an analogue of the Shapiro-Lopatinsky condition and is associated with the "singular boundary" $\overline{Y}$ of $\overline{M}$ (see Proposition 5.14 for a discussion of this assumption).

With the preparations from the previous sections, we are able to follow the same idea as in the boundaryless case in [9].

Let

$$\theta : \tilde{\mathcal{E}}_{\max} = \bigoplus_{\sigma_0 \in \Sigma} \tilde{\mathcal{E}}_{\sigma_0} \to \bigoplus_{\sigma_0 \in \Sigma} \tilde{\mathcal{E}}_{\wedge,\sigma_0} = \tilde{\mathcal{E}}_{\wedge,\max}$$

be the isomorphism of the spaces of singular functions $\tilde{\mathcal{E}}_{\max} \cong \mathcal{D}_{\max}/\mathcal{D}_{\min}$ and $\tilde{\mathcal{E}}_{\wedge,\max} \cong \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ that was constructed in Section 6. Recall that Σ is the part of the boundary spectrum of $\binom{A}{T}$ in $\{\sigma \in \mathbb{C};\ -m/2 < \Im(\sigma) < m/2\}$, and for $\sigma_0 \in \Sigma$ let $N(\sigma_0) \in \mathbb{N}_0$ be the largest integer such that $\Im(\sigma_0) - N(\sigma_0) > -m/2$.

The normalized dilation group κ_ϱ respects the space $\tilde{\mathcal{E}}_{\wedge,\max}$, i.e. $\kappa_\varrho : \tilde{\mathcal{E}}_{\wedge,\max} \to \tilde{\mathcal{E}}_{\wedge,\max}$ for $\varrho > 0$. Consequently, we can define a group action $\tilde{\kappa}_\varrho$ on $\tilde{\mathcal{E}}_{\max}$ via

$$\tilde{\kappa}_\varrho = \theta^{-1}\kappa_\varrho\theta : \tilde{\mathcal{E}}_{\max} \to \tilde{\mathcal{E}}_{\max}.$$

We may write $\tilde{\kappa}_\varrho = \kappa_\varrho L_\varrho$, where

$$L_\varrho = \kappa_\varrho^{-1}\theta^{-1}\kappa_\varrho\theta : \tilde{\mathcal{E}}_{\max} \to C^\infty(\overset{\circ}{\overline{Y}}{}^\wedge; E)$$

is the direct sum of the operators $L_\varrho|_{\tilde{\mathcal{E}}_{\sigma_0}}$ which act as follows:

For $\tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}$ we have

$$L_\varrho \tilde{u} = \sum_{\vartheta=0}^{N(\sigma_0)} \varrho^{-\vartheta} \mathrm{e}_{\sigma_0,\vartheta}(\varrho)(\theta\tilde{u}), \tag{8.3}$$

where $\mathrm{e}_{\sigma_0,\vartheta}(\varrho)$ is defined as

$$\mathrm{e}_{\sigma_0,\vartheta}(\varrho) = \varrho^{\vartheta}\kappa_\varrho^{-1}\mathrm{e}_{\sigma_0,\vartheta}\kappa_\varrho : \tilde{\mathcal{E}}_{\wedge,\sigma_0} \to C^\infty(\overset{\circ}{\overline{Y}}{}^\wedge; E)$$

with the operators $\mathrm{e}_{\sigma_0,\vartheta}$ from Section 6. In particular, $\mathrm{e}_{\sigma_0,0}(\varrho)(\tilde{u}) = \tilde{u}$ for all $\varrho \in \mathbb{R}_+$ and $\tilde{u} \in \tilde{\mathcal{E}}_{\wedge,\sigma_0}$.

Lemma 8.4.

i) For every $\psi \in \tilde{\mathcal{E}}_{\wedge,\sigma_0}$ and every $\vartheta \in \{0,\dots,N(\sigma_0)\}$ there exists a polynomial $q_\vartheta(y,\log x,\log\varrho)$ in $(\log x,\log\varrho)$ with coefficients in $C^\infty(\overline{Y};E)$ such that

$$\mathrm{e}_{\sigma_0,\vartheta}(\varrho)(\psi) = q_\vartheta(y,\log x,\log\varrho)x^{i(\sigma_0 - i\vartheta)}, \tag{8.5}$$

and the degree of q_ϑ with respect to $(\log x,\log\varrho)$ is bounded by some $\mu \in \mathbb{N}_0$ which is independent of $\sigma_0 \in \Sigma$, $\psi \in \tilde{\mathcal{E}}_{\wedge,\sigma_0}$, and $\vartheta \in \{0,\dots,N(\sigma_0)\}$.

ii) Let $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ be any cut-off function near the origin, i.e., $\omega = 1$ near zero and $\omega = 0$ near infinity. Then the operator family

$$\omega\big(L_\varrho - \theta\big) : \tilde{\mathcal{E}}_{\max} \to \mathcal{K}^{\infty,-m/2}(\overline{Y}^\wedge; E)$$

satisfies for every $s \in \mathbb{R}$ the norm estimate

$$\|\omega\big(L_\varrho - \theta\big)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},\mathcal{K}^{s,-m/2})} = O(\varrho^{-1}\log^\mu\varrho) \quad \text{as } \varrho \to \infty,$$

where $\mu \in \mathbb{N}_0$ is the bound for the degrees of the polynomials q_ϑ in i).

Proof. The proof is literally the same as in Lemma 6.18 from [9]. □

Lemma 8.6. *Fix a cut-off function $\omega \in C_0^\infty([0,1))$ near 0. For $\varrho > 1$ consider the operator family*

$$\tilde{K}(\varrho) = \omega_\varrho \tilde{\kappa}_\varrho : \tilde{\mathcal{E}}_{\max} \to \mathcal{D}^\infty_{\max}\binom{A}{T} = \bigcap_{t > -\frac{1}{2}} \mathcal{D}^t_{\max}\binom{A}{T},$$

where $\omega_\varrho(x) = \omega(\varrho x)$. If $q : \mathcal{D}_{\max}\binom{A}{T} \to \tilde{\mathcal{E}}_{\max}$ is the canonical projection, then

$$q \circ \tilde{K}(\varrho) = \tilde{\kappa}_\varrho,$$

and we have the following norm estimates as $\varrho \to \infty$:

$$\|\tilde{K}(\varrho)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max}, x^{-m/2}L^2_b)} = O(1), \tag{8.7}$$

$$\|\kappa_\varrho^{-1} A \tilde{K}(\varrho)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max}, x^{-m/2}L^2_b)} = O(\varrho^m), \tag{8.8}$$

$$\|\kappa_\varrho^{-1} \gamma B_\ell \tilde{K}(\varrho)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max}, \mathcal{K}^{m-m_\ell-1/2, m/2-m_\ell})} = O(\varrho^{m_\ell}), \quad \ell = 1, \ldots, K. \tag{8.9}$$

Note that $\tilde{K}(\varrho)\tilde{u}$ is supported in $(0, \varrho^{-1}] \times \overline{Y} \subset U_{\overline{Y}}$ for all $\tilde{u} \in \tilde{\mathcal{E}}_{\max}$, and thus it makes sense to apply the group action κ_ϱ^{-1} in the estimates (8.8) *and* (8.9).

Proof. That $\tilde{K}(\varrho)$ is a lift of $\tilde{\kappa}_\varrho$ to $\mathcal{D}^\infty_{\max}\binom{A}{T}$ is evident from the definition. In order to show the norm estimates, it is sufficient to consider for each $\sigma_0 \in \Sigma$ the restriction

$$\tilde{K}_{\sigma_0}(\varrho) = \tilde{K}(\varrho)|_{\tilde{\mathcal{E}}_{\sigma_0}} : \tilde{\mathcal{E}}_{\sigma_0} \to \mathcal{D}^\infty_{\max}\binom{A}{T}$$

and prove the estimates for this operator. Recall that $\tilde{\kappa}_\varrho = \kappa_\varrho L_\varrho$ so that for $\tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}$ we have $\tilde{K}_{\sigma_0}(\varrho)\tilde{u} = \kappa_\varrho(\omega L_\varrho \tilde{u})$.

The norm estimates (8.7) and (8.8) follow in the same way as the corresponding assertion in the boundaryless case, see Lemma 6.20 in [9]. The same method of proof also gives (8.9); for sake of completeness, we give a proof of this estimate below, i.e. we prove that there exists a constant $C > 0$, independent of $\tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}$ and $\varrho \geq 1$, such that

$$\|\kappa_\varrho^{-1} \gamma B_\ell(\kappa_\varrho(\omega L_\varrho \tilde{u}))\|_{\mathcal{K}^{m-m_\ell-1/2, m/2-m_\ell}} \leq C \varrho^{m_\ell} \|\omega \tilde{u}\|_{\mathcal{D}_{\max}}.$$

To this end we split B_ℓ near $\overline{Y}$ as in (6.3), i.e.

$$B_\ell \equiv x^{-m_\ell} \sum_{k=0}^{m-1} B_{\ell,k} x^k + \tilde{B}_{\ell,m}$$

with totally characteristic operators $B_{\ell,k} \in \mathrm{Diff}_b^{m_\ell}(\overline{Y}^\wedge; E, F_\ell)$ with coefficients independent of x, and $\tilde{B}_{\ell,m} \in x^{m-m_\ell}\,\mathrm{Diff}_b^{m_\ell}(\overline{Y}^\wedge; E, F_\ell)$. As we are working exclusively near $\overline{Y}$, we may without loss of generality assume that the coefficients of $\tilde{B}_{\ell,m}$ vanish near infinity.

By (8.3) we obtain

$$\begin{aligned}
&\kappa_\varrho^{-1}\gamma B_\ell(\kappa_\varrho(\omega L_\varrho\tilde u)) \\
&\quad= \kappa_\varrho^{-1}\Big(x^{-m_\ell}\sum_{k=0}^{m-1}\gamma B_{\ell,k}x^k\Big)\kappa_\varrho\big(\omega L_\varrho\tilde u\big)+\kappa_\varrho^{-1}\gamma\tilde B_{\ell,m}\kappa_\varrho\big(\omega L_\varrho\tilde u\big) \\
&\quad= \varrho^{m_\ell}\Big(x^{-m_\ell}\sum_{k=0}^{m-1}\varrho^{-k}\gamma B_{\ell,k}x^k\Big)\Big(\omega\sum_{j=0}^{N(\sigma_0)}\varrho^{-j}\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde u)\Big)+\kappa_\varrho^{-1}\gamma\tilde B_{\ell,m}\kappa_\varrho\big(\omega L_\varrho\tilde u\big) \\
&\quad= \sum_{\vartheta=0}^{2m-2}\varrho^{m_\ell-\vartheta}\Big(x^{-m_\ell}\sum_{\substack{k+j=\vartheta\\0\le k,j\le m-1}}\big(\gamma B_{\ell,k}x^k\big)\big(\omega\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde u)\big)\Big)+\kappa_\varrho^{-1}\gamma\tilde B_{\ell,m}\kappa_\varrho\big(\omega L_\varrho\tilde u\big)
\end{aligned}\tag{8.10}$$

with the convention that $\mathrm{e}_{\sigma_0,j}(\varrho)=0$ for $j>N(\sigma_0)$.

For every $\vartheta\in\{0,\ldots,2m-2\}$ we consider the family of linear maps

$$\begin{aligned}
\tilde u\mapsto x^{-m_\ell}&\sum_{\substack{k+j=\vartheta\\0\le k,j\le m-1}}\big(\gamma B_{\ell,k}x^k\big)\big(\omega\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde u)\big) \\
&: \tilde{\mathcal E}_{\sigma_0}\to\mathcal K^{m-m_\ell-1/2,m/2-m_\ell}((\partial\overline{Y})^\wedge;F_\ell).
\end{aligned}\tag{8.11}$$

We will prove that (8.11) is well-defined, i.e., every $\tilde u\in\tilde{\mathcal E}_{\sigma_0}$ is indeed mapped into $\mathcal K^{m-m_\ell-1/2,m/2-m_\ell}$, and that the norms are bounded by a constant times $\log^\mu\varrho$ as $\varrho\to\infty$ with μ as in Lemma 8.4. Thus for every $\vartheta\in\{0,\ldots,2m-2\}$ we have

$$\begin{aligned}
\Big\|\varrho^{m_\ell-\vartheta}\Big(x^{-m_\ell}&\sum_{\substack{k+j=\vartheta\\0\le k,j\le m-1}}\big(\gamma B_{\ell,k}x^k\big)\big(\omega\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde u)\big)\Big)\Big\|_{\mathcal K^{m-m_\ell-1/2,m/2-m_\ell}} \\
&\le \mathrm{const}\cdot\big(\varrho^{m_\ell-\vartheta}\log^\mu\varrho\big)\|\omega\tilde u\|_{\mathcal D_{\max}},
\end{aligned}$$

while for $\vartheta=0$,

$$\varrho^{m_\ell}x^{-m_\ell}\gamma B_{\ell,0}\omega\mathrm{e}_{\sigma_0,0}(\varrho)(\theta\tilde u)=\varrho^{m_\ell}\gamma_\wedge B_{\ell,\wedge}\omega(\theta\tilde u),\tag{8.12}$$

so for this term we have a norm estimate without log.

Let $\tilde\omega\in C_0^\infty(\overline{\mathbb R}_+)$ be a cut-off function near 0 with $\omega\prec\tilde\omega$. Then there exist suitable $\varphi,\tilde\varphi\in C_0^\infty(\mathbb R_+)$ such that for all $\tilde u\in\tilde{\mathcal E}_{\sigma_0}$,

$$\begin{aligned}
&x^{-m_\ell}\sum_{\substack{k+j=\vartheta\\0\le k,j\le m-1}}\big(\gamma B_{\ell,k}x^k\big)\big(\omega\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde u)\big) \\
&\quad=\tilde\omega x^{-m_\ell}\sum_{\substack{k+j=\vartheta\\0\le k,j\le m-1}}\big(\gamma B_{\ell,k}x^k\big)\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde u)+\tilde\varphi x^{-m_\ell}\sum_{\substack{k+j=\vartheta\\0\le k,j\le m-1}}\big(\gamma B_{\ell,k}x^k\big)\varphi\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde u).
\end{aligned}\tag{8.13}$$

According to Lemma 8.4 the second sum in (8.13) is a polynomial in $\log\varrho$ of degree at most μ with coefficients in $C_0^\infty(\partial\overset{\circ}{\overline{Y}}{}^\wedge;F_\ell)$. As both $\kappa_\varrho^{-1}\gamma B_\ell(\kappa_\varrho(\omega L_\varrho\tilde u))$ and $\kappa_\varrho^{-1}\gamma\tilde B_{\ell,m}(\kappa_\varrho(\omega L_\varrho\tilde u))$ belong to $\mathcal K^{m-m_\ell-1/2,m/2-m_\ell}$, we get from the equations (8.10) and (8.13) that necessarily

$$x^{-m_\ell}\sum_{\substack{k+j=\vartheta\\0\le k,j\le m-1}}\big(\gamma B_{\ell,k}x^k\big)\big(\omega\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde u)\big)\in\mathcal K^{m-m_\ell-1/2,m/2-m_\ell}$$

for all $\varrho \in \mathbb{R}_+$ and all $\tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}$, and, moreover, that

$$\tilde{\omega} x^{-m_\ell} \sum_{\substack{k+j=\vartheta \\ 0\le k,j\le m-1}} \big(\gamma B_{\ell,k} x^k\big) \mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u}) = 0$$

for $\Im(\sigma_0) - \vartheta \geq -m/2$ because these functions are of the form

$$\tilde{\omega}\Big(\sum_{\nu} c_{\sigma_0 - i(\vartheta - m_\ell),\nu}(y') \log^\nu x\Big) x^{i(\sigma_0 - i(\vartheta - m_\ell))}.$$

For $\Im(\sigma_0) - \vartheta < -m/2$ every single summand $\tilde{\omega} x^{-m_\ell}\big(\gamma B_{\ell,k} x^k\big) \mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})$ belongs to the space $\mathcal{K}^{m-m_\ell-1/2,m/2-m_\ell}$, and by Lemma 8.4 is a polynomial in $\log\varrho$ of degree at most μ with coefficients in $\mathcal{K}^{m-m_\ell-1/2,m/2-m_\ell}$.

Summing up, we have shown that for every $\tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}$ the function

$$x^{-m_\ell} \sum_{\substack{k+j=\vartheta \\ 0\le k,j\le m-1}} \big(\gamma B_{\ell,k} x^k\big)\big(\omega \mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})\big)$$

is a polynomial in $\log\varrho$ of degree at most μ with coefficients in $\mathcal{K}^{m-m_\ell-1/2,m/2-m_\ell}$, and from the Banach-Steinhaus theorem we now obtain the desired norm estimates for the family of maps (8.11).

On the other hand,

$$\|\kappa_\varrho^{-1}\gamma\tilde{B}_{\ell,m}\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big)\|_{\mathcal{K}^{m-m_\ell-1/2,m/2-m_\ell}} \leq$$
$$\|\kappa_\varrho^{-1}\gamma\tilde{B}_{\ell,m}\kappa_\varrho\|_{\mathscr{L}(\mathcal{K}^{m,-m/2},\mathcal{K}^{m-m_\ell-1/2,m/2-m_\ell})}\|\omega L_\varrho\tilde{u}\|_{\mathcal{K}^{m,-m/2}}.$$

Lemma 8.4 implies $\|\omega L_\varrho\tilde{u}\|_{\mathcal{K}^{m,-m/2}} \leq \mathrm{const}\|\omega\tilde{u}\|_{\mathcal{D}_{\max}}$, and so

$$\|\kappa_\varrho^{-1}\gamma\tilde{B}_{\ell,m}\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big)\|_{\mathcal{K}^{m-m_\ell-1/2,m/2-m_\ell}} \leq \mathrm{const}\|\omega\tilde{u}\|_{\mathcal{D}_{\max}}$$

since $\|\kappa_\varrho^{-1}\gamma\tilde{B}_{\ell,m}\kappa_\varrho\| = O(\varrho^{m_\ell - m})$ as $\varrho\to\infty$. Thus (8.9) is proved. $\square$

Proof of Theorem 8.1. Fix some complement $\mathcal{E}_{\max}$ of $\mathcal{D}_{\min}\binom{A}{T}$ in $\mathcal{D}_{\max}\binom{A}{T}$ and let $\mathcal{E} \subset \mathcal{E}_{\max}$ be a subspace such that $\mathcal{D}\binom{A}{T} = \mathcal{D}_{\min}\binom{A}{T} \oplus \mathcal{E}$. With respect to this decomposition we may write the boundary value problem (4.1) as

$$\mathcal{A}_{\mathcal{D}}(\lambda) = \begin{pmatrix} A-\lambda \\ T\end{pmatrix} = \begin{pmatrix}\mathcal{A}_{\mathcal{D}_{\min}}(\lambda) & \mathcal{A}(\lambda)|_{\mathcal{E}}\end{pmatrix} = \begin{pmatrix} (A-\lambda)|_{\mathcal{D}_{\min}} & (A-\lambda)|_{\mathcal{E}} \\ T|_{\mathcal{D}_{\min}} & T|_{\mathcal{E}}\end{pmatrix}$$
$$: \begin{matrix}\mathcal{D}_{\min}\binom{A}{T} \\ \oplus \\ \mathcal{E}\end{matrix} \to \begin{matrix} x^{-m/2}L^2_b(\overline{M};E) \\ \oplus \\ \bigoplus_{j=1}^K x^{m/2-m_j}H_b^{m-m_j-1/2}(\overline{N};F_j)\end{matrix}.$$

Let $d'' = \dim\mathcal{E}$. Under the assumptions of Theorem 8.1 we may apply Theorem 7.21 and obtain the existence of a parametrix $\mathcal{B}(\lambda)$ and a generalized Green remainder $\mathcal{K}(\lambda) = \begin{pmatrix}\mathcal{K}_0(\lambda) \\ 0\end{pmatrix}$ of order m such that

$$\begin{pmatrix} (A-\lambda)|_{\mathcal{D}_{\min}} & \mathcal{K}_0(\lambda) \\ T|_{\mathcal{D}_{\min}} & 0\end{pmatrix} : \begin{matrix}\mathcal{D}_{\min}\binom{A}{T} \\ \oplus \\ \mathbb{C}^{d''}\end{matrix} \to \begin{matrix} x^{-m/2}L^2_b(\overline{M};E) \\ \oplus \\ \bigoplus_{j=1}^K x^{m/2-m_j}H_b^{m-m_j-1/2}(\overline{N};F_j)\end{matrix}$$

is invertible for $\lambda \in \Lambda$ sufficiently large with inverse

$$\begin{pmatrix} (A-\lambda)|_{\mathcal{D}_{\min}} & \mathcal{K}_0(\lambda) \\ T|_{\mathcal{D}_{\min}} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{B}(\lambda) \\ \hline \mathcal{T}(\lambda) \end{pmatrix}, \tag{8.14}$$

where $\mathcal{T}(\lambda) = \begin{pmatrix} \mathcal{T}_0(\lambda) & \cdots & \mathcal{T}_K(\lambda) \end{pmatrix}$ is a matrix of generalized Green remainders of orders $-m, -m_1, \ldots, -m_K$ and type zero. Since

$$I = \begin{pmatrix} \mathcal{B}(\lambda) \\ \mathcal{T}(\lambda) \end{pmatrix} \begin{pmatrix} \mathcal{A}_{\mathcal{D}_{\min}}(\lambda) & \mathcal{K}(\lambda) \end{pmatrix} = \begin{pmatrix} \mathcal{B}(\lambda)\mathcal{A}_{\mathcal{D}_{\min}}(\lambda) & \mathcal{B}(\lambda)\mathcal{K}(\lambda) \\ \mathcal{T}(\lambda)\mathcal{A}_{\mathcal{D}_{\min}}(\lambda) & \mathcal{T}(\lambda)\mathcal{K}(\lambda) \end{pmatrix},$$

we have $\mathcal{B}(\lambda)\mathcal{A}_{\mathcal{D}_{\min}}(\lambda) = 1$, $\mathcal{T}(\lambda)\mathcal{A}_{\mathcal{D}_{\min}}(\lambda) = 0$, and $\mathcal{T}(\lambda)\mathcal{K}(\lambda) = 1$. Then

$$\begin{pmatrix} \mathcal{B}(\lambda) \\ \mathcal{T}(\lambda) \end{pmatrix} \begin{pmatrix} \mathcal{A}_{\mathcal{D}_{\min}}(\lambda) & \mathcal{A}(\lambda)|_{\mathcal{E}} \end{pmatrix} = \begin{pmatrix} 1 & \mathcal{B}(\lambda)\mathcal{A}(\lambda)|_{\mathcal{E}} \\ 0 & \mathcal{T}(\lambda)\mathcal{A}(\lambda)|_{\mathcal{E}} \end{pmatrix} \tag{8.15}$$

which implies that $\begin{pmatrix} \mathcal{A}_{\mathcal{D}_{\min}}(\lambda) & \mathcal{A}(\lambda)|_{\mathcal{E}} \end{pmatrix}$ is invertible if and only if

$$F(\lambda) = \mathcal{T}(\lambda)\mathcal{A}(\lambda) = \mathcal{T}(\lambda)\begin{pmatrix} A-\lambda \\ T \end{pmatrix} : \mathcal{E} \to \mathbb{C}^{d''} \tag{8.16}$$

is invertible. Moreover, we get the explicit representation

$$\begin{pmatrix} A-\lambda \\ T \end{pmatrix}^{-1} = \mathcal{A}_{\mathcal{D}}(\lambda)^{-1} = \mathcal{B}(\lambda) + \big(1 - \mathcal{B}(\lambda)\mathcal{A}(\lambda)\big)F(\lambda)^{-1}\mathcal{T}(\lambda), \tag{8.17}$$

and (8.2) follows from Theorem 7.21.

As $F(\lambda) = \mathcal{T}(\lambda)\mathcal{A}(\lambda)$ and $1 - \mathcal{B}(\lambda)\mathcal{A}(\lambda)$ vanish on $\mathcal{D}_{\min}\binom{A}{T}$ for large λ, they descend to operators $F(\lambda) : \tilde{\mathcal{E}}_{\max} \to \mathbb{C}^{d''}$ and $1 - \mathcal{B}(\lambda)\mathcal{A}(\lambda) : \tilde{\mathcal{E}}_{\max} \to \mathcal{D}_{\max}\binom{A}{T}$. If $\tilde{\mathcal{E}} = \mathcal{D}\binom{A}{T}/\mathcal{D}_{\min}\binom{A}{T}$, then the invertibility of (8.16) is equivalent to the invertibility of

$$F(\lambda) : \tilde{\mathcal{E}} \to \mathbb{C}^{d''},$$

and in this case, (8.17) still makes sense in this context.

Let $q : \mathcal{D}_{\max}\binom{A}{T} \to \tilde{\mathcal{E}}_{\max}$ be the canonical projection. The inverses $\mathcal{A}_{\mathcal{D}}(\lambda)^{-1}$ and $F(\lambda)^{-1} : \mathbb{C}^{d''} \to \tilde{\mathcal{E}} \subset \tilde{\mathcal{E}}_{\max}$ are related by the formulas

$$F(\lambda)^{-1} = q\mathcal{A}_{\mathcal{D}}(\lambda)^{-1}\mathcal{K}(\lambda) : \mathbb{C}^{d''} \to \tilde{\mathcal{E}}_{\max},$$

$$q\mathcal{A}_{\mathcal{D}}(\lambda)^{-1} = F(\lambda)^{-1}\mathcal{T}(\lambda) : \begin{array}{c} x^{-m/2}L^2_b(\overline{M};E) \\ \oplus \\ \bigoplus\limits_{j=1}^{K} x^{m/2-m_j}H_b^{m-m_j-1/2}(\overline{N};F_j) \end{array} \to \tilde{\mathcal{E}}_{\max}$$

in view of $\mathcal{T}(\lambda)\mathcal{K}(\lambda) = 1$.

Under the assumptions of Theorem 8.1 we prove that $F(\lambda) : \tilde{\mathcal{E}} \to \mathbb{C}^{d''}$ is invertible for large $\lambda \in \Lambda$, and that the inverse satisfies the norm estimate

$$\|\tilde{\kappa}^{-1}_{[\lambda]^{1/m}}F(\lambda)^{-1}\|_{\mathscr{L}(\mathbb{C}^{d''},\tilde{\mathcal{E}}_{\max})} = O(1) \ \text{ as } |\lambda| \to \infty. \tag{8.18}$$

Observe that the parametrix construction from Theorem 7.21 gives the relation

$$\begin{pmatrix} (A_\wedge-\lambda)|_{\mathcal{D}_{\wedge,\min}} & \mathcal{K}_{0,\wedge}(\lambda) \\ T_\wedge|_{\mathcal{D}_{\wedge,\min}} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{B}_\wedge(\lambda) \\ \mathcal{T}_\wedge(\lambda) \end{pmatrix}$$

for the κ-homogeneous principal parts of (8.14). Thus with the same reasoning as above we conclude that

$$\mathcal{A}_{\wedge,\mathcal{D}_\wedge}(\lambda) = \begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix} : \mathcal{D}_\wedge\begin{pmatrix} A_\wedge \\ T_\wedge \end{pmatrix} \to \begin{array}{c} x^{-m/2}L^2_b(\overline{Y}^\wedge;E) \\ \oplus \\ \bigoplus_{j=1}^K \mathcal{K}^{m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge;F_j) \end{array}$$

is invertible if and only if the restriction of the induced operator

$$F_\wedge(\lambda) = \mathcal{T}_\wedge(\lambda)\mathcal{A}_\wedge(\lambda) = \mathcal{T}_\wedge(\lambda)\begin{pmatrix} A_\wedge - \lambda \\ T_\wedge \end{pmatrix} : \tilde{\mathcal{E}}_{\wedge,\max} \to \mathbb{C}^{d''}$$

to $\tilde{\mathcal{E}}_\wedge = \mathcal{D}_\wedge\binom{A_\wedge}{T_\wedge}/\mathcal{D}_{\wedge,\min}\binom{A_\wedge}{T_\wedge}$ is invertible. Let $q_\wedge : \mathcal{D}_{\wedge,\max}\binom{A_\wedge}{T_\wedge} \to \tilde{\mathcal{E}}_{\wedge,\max}$ be the canonical projection. From the relations

$$F_\wedge(\lambda)^{-1} = q_\wedge\mathcal{A}_{\wedge,\mathcal{D}_\wedge}(\lambda)^{-1}\mathcal{K}_\wedge(\lambda) : \mathbb{C}^{d''} \to \tilde{\mathcal{E}}_{\wedge,\max},$$

$$q_\wedge\mathcal{A}_{\wedge,\mathcal{D}_\wedge}(\lambda)^{-1} = F_\wedge(\lambda)^{-1}\mathcal{T}_\wedge(\lambda) : \begin{array}{c} x^{-m/2}L^2_b(\overline{Y}^\wedge;E) \\ \oplus \\ \bigoplus_{j=1}^K \mathcal{K}^{m-m_j-1/2,m/2-m_j}((\partial\overline{Y})^\wedge;F_j) \end{array} \to \tilde{\mathcal{E}}_{\wedge,\max},$$

and Proposition 5.14, we deduce that our assumption about the resolvent of $A_\wedge$ under the boundary condition $T_\wedge u = 0$ is equivalent to the invertibility of $\mathcal{A}_{\wedge,\mathcal{D}_\wedge}(\lambda)$ with domain $\mathcal{D}_\wedge\binom{A_\wedge}{T_\wedge}$, where $\mathcal{D}_\wedge = \theta(\mathcal{D})$ is the associated domain to $\mathcal{D}$, and

$$\|\kappa^{-1}_{|\lambda|^{1/m}}F_\wedge(\lambda)^{-1}\|_{\mathscr{L}(C^{d''},\tilde{\mathcal{E}}_{\wedge,\max})} = O(1) \quad \text{as } |\lambda| \to \infty. \tag{8.19}$$

Note that $\|\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{K}_{0,\wedge}(\lambda)\| = O(|\lambda|)$ and $\|\mathcal{T}_{k,\wedge}(\lambda)\kappa_{|\lambda|^{1/m}}\| = O(|\lambda|^{-m_k/m})$ as $|\lambda| \to \infty$ for $k = 0,\dots,m_K$, where $m_0 = m$.

Write the operator $F(\lambda)\theta^{-1}F_\wedge(\lambda)^{-1} : \mathbb{C}^{d''} \to \mathbb{C}^{d''}$ as

$$F(\lambda)\theta^{-1}F_\wedge(\lambda)^{-1} = 1 + \big(F(\lambda) - F_\wedge(\lambda)\theta\big)\tilde{\kappa}_{|\lambda|^{1/m}}\theta^{-1}\kappa^{-1}_{|\lambda|^{1/m}}F_\wedge(\lambda)^{-1},$$

and let

$$R(\lambda) = \big(F(\lambda) - F_\wedge(\lambda)\theta\big)\tilde{\kappa}_{|\lambda|^{1/m}}\theta^{-1}\kappa^{-1}_{|\lambda|^{1/m}}F_\wedge(\lambda)^{-1}.$$

We will prove in Lemma 8.24 that

$$\|(F(\lambda) - F_\wedge(\lambda)\theta)\tilde{\kappa}_{|\lambda|^{1/m}}\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},\mathbb{C}^{d''})} \to 0 \quad \text{as } |\lambda| \to \infty.$$

Thus together with (8.19) we obtain that $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$. Hence $1+R(\lambda)$ is invertible for large $|\lambda| > 0$, and the inverse is of the form $1+\tilde{R}(\lambda)$ with $\|\tilde{R}(\lambda)\| \to 0$ as $|\lambda| \to \infty$. This shows that $F(\lambda) : \tilde{\mathcal{E}} \to \mathbb{C}^{d''}$ is invertible from the right for large λ, and by (8.19) the right inverse $\theta^{-1}F_\wedge(\lambda)^{-1}(1+\tilde{R}(\lambda))$ satisfies the norm estimate (8.18). Since

$$\dim\tilde{\mathcal{E}} = \dim\tilde{\mathcal{E}}_\wedge = d'',$$

we conclude that $F(\lambda)$ is also injective, and so the invertibility of $F(\lambda)$ is proved. In particular, the operator $\mathcal{A}_{\mathcal{D}}(\lambda)$ is invertible for large λ, and consequently also

$$A_{\mathcal{D}} - \lambda : \mathcal{D}(A_T) \to x^{-m/2}L^2_b(\overline{M};E)$$

is invertible for large $\lambda \in \Lambda$. It remains to show the resolvent estimates for $\big(A_{\mathcal{D}} - \lambda\big)^{-1}$ as $|\lambda| \to \infty$, see Definition 1.1.

In order to prove these estimates we make use of the family $\tilde{K}(\varrho)$ from Lemma 8.6 and the representation (8.17). We may write

$$\begin{aligned}(A_{\mathcal{D}}-\lambda)^{-1} &= \mathcal{B}_T(\lambda) + (1-\mathcal{B}(\lambda)\mathcal{A}(\lambda))\tilde{K}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\mathcal{T}_0(\lambda)\\ &= \mathcal{B}_T(\lambda) + \tilde{K}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\mathcal{T}_0(\lambda)\\ &\qquad - \mathcal{B}(\lambda)\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\mathcal{T}_0(\lambda).\end{aligned}$$

By construction of the parametrix we have $\|\mathcal{B}_T(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} = O(|\lambda|^{-1})$ as $|\lambda| \to \infty$. In view of $\|\mathcal{T}_0(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathbb{C}^{d''})} = O(|\lambda|^{-1})$ and (8.18) we further obtain

$$\|\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\mathcal{T}_0(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\tilde{\mathcal{E}}_{\max})} = O(|\lambda|^{-1}) \text{ as } |\lambda| \to \infty,$$

and consequently, using (8.7), we get

$$\|\tilde{K}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\mathcal{T}_0(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} = O(|\lambda|^{-1}) \text{ as } |\lambda| \to \infty.$$

On the other hand, by (8.18) and the estimates (8.7)–(8.9) we have

$$\|\kappa^{-1}_{|\lambda|^{1/m}}(A-\lambda)\tilde{K}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\|_{\mathscr{L}(\mathbb{C}^{d''},x^{-m/2}L^2_b)} = O(|\lambda|), \tag{8.20}$$

$$\|\kappa^{-1}_{|\lambda|^{1/m}}\gamma B_j\tilde{K}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\|_{\mathscr{L}(\mathbb{C}^{d''},\mathcal{K}^{m-m_j-1/2,m/2-m_j})} = O(|\lambda|^{m_j/m}) \tag{8.21}$$

as $|\lambda| \to \infty$ for all $j = 1,\ldots,K$.

Let $\tilde{\omega} \in C_0^\infty([0,1))$ be any cut-off function near zero. For large λ we may write

$$\mathcal{B}(\lambda)\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\mathcal{T}_0(\lambda) = \mathcal{B}(\lambda)\tilde{\omega}\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\mathcal{T}_0(\lambda),$$

and for the components of the parametrix

$$\mathcal{B}(\lambda) = \begin{pmatrix}\mathcal{B}_T(\lambda) & \mathcal{B}_1(\lambda) & \cdots & \mathcal{B}_K(\lambda)\end{pmatrix}$$

we have by construction the norm estimates

$$\|\mathcal{B}_T(\lambda)\tilde{\omega}\kappa_{|\lambda|^{1/m}}\|_{\mathscr{L}(x^{-m/2}L^2_b(\overline{Y}^\wedge;E),x^{-m/2}L^2_b(\overline{M};E))} = O(|\lambda|^{-1}), \tag{8.22}$$

$$\|\mathcal{B}_j(\lambda)\tilde{\omega}\kappa_{|\lambda|^{1/m}}\|_{\mathscr{L}(\mathcal{K}^{m-m_j-1/2,m/2-m_j},x^{-m/2}L^2_b(\overline{M};E))} = O(|\lambda|^{-m_j/m}) \tag{8.23}$$

as $|\lambda| \to \infty$ for $j = 1,\ldots,K$ (the operator family $g(\lambda) = \omega'\mathcal{B}_j(\lambda)\tilde{\omega}$ satisfies the estimate (7.10) with $\mu = -m_j$ in $\mathcal{K}^{m-m_j-1/2,m/2-m_j} \to \mathcal{D}_{\wedge,\min}$ with respect to the normalized dilation group action κ_ϱ on both spaces).

In view of (8.20)–(8.23) and $\|\mathcal{T}_0(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathbb{C}^{d''})} = O(|\lambda|^{-1})$ we now conclude that, as $|\lambda| \to \infty$,

$$\|\mathcal{B}(\lambda)\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\mathcal{T}_0(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} = O(|\lambda|^{-1}).$$

Summing up, we obtain

$$\|(A_{\mathcal{D}}-\lambda)^{-1}\|_{\mathscr{L}(x^{-m/2}L^2_b)} = O(|\lambda|^{-1}) \text{ as } |\lambda| \to \infty$$

as desired. □

The following lemma completes the proof of Theorem 8.1.

Lemma 8.24. *With the notation of the proof of Theorem 8.1, let*

$$F(\lambda) = \mathcal{T}(\lambda)\mathcal{A}(\lambda) : \tilde{\mathcal{E}}_{\max} \to \mathbb{C}^{d''},$$

$$F_\wedge(\lambda) = \mathcal{T}_\wedge(\lambda)\mathcal{A}_\wedge(\lambda) : \tilde{\mathcal{E}}_{\wedge,\max} \to \mathbb{C}^{d''}.$$

Then

$$\|(F(\lambda) - F_\wedge(\lambda)\theta)\tilde{\kappa}_{|\lambda|^{1/m}}\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},\mathbb{C}^{d''})} \to 0 \quad \text{as } |\lambda| \to \infty. \tag{8.25}$$

Proof. For proving (8.25) it is sufficient to consider the restrictions

$$(F(\lambda) - F_\wedge(\lambda)\theta)\tilde{\kappa}_{|\lambda|^{1/m}} : \tilde{\mathcal{E}}_{\sigma_0} \to \mathbb{C}^{d''}$$

for all $\sigma_0 \in \Sigma$. First of all, observe that

$$\begin{aligned} F(\lambda)\tilde{\kappa}_{|\lambda|^{1/m}} &= \mathcal{T}(\lambda)\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m}), \quad \text{and} \\ F_\wedge(\lambda)\theta\tilde{\kappa}_{|\lambda|^{1/m}} &= F_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\theta = \mathcal{T}_\wedge(\lambda)\mathcal{A}_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\omega\theta \end{aligned}$$

with the operator family $\tilde{K}(\varrho) = \omega(\varrho x)\tilde{\kappa}_\varrho$ from Lemma 8.6. If $\omega_0 \in C_0^\infty([0,1))$ is a cut-off function near zero with $\omega \prec \omega_0$, then

$$\begin{aligned} (F(\lambda) - F_\wedge(\lambda)\theta)\tilde{\kappa}_{|\lambda|^{1/m}} &= \mathcal{T}(\lambda)\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m}) - \mathcal{T}_\wedge(\lambda)\mathcal{A}_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\omega\theta \\ &= \mathcal{T}(\lambda)\omega_0\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m}) - \mathcal{T}_\wedge(\lambda)\omega_0\mathcal{A}_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\omega\theta \\ &= \mathcal{T}(\lambda)\omega_0\Big(\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m}) - \mathcal{A}_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\omega\theta\Big) \\ &\qquad + (\mathcal{T}(\lambda) - \mathcal{T}_\wedge(\lambda))\omega_0\mathcal{A}_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\omega\theta. \end{aligned}$$

Now

$$(\mathcal{T}(\lambda) - \mathcal{T}_\wedge(\lambda))\omega_0\mathcal{A}_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\omega\theta = \Big((\mathcal{T}(\lambda) - \mathcal{T}_\wedge(\lambda))\omega_0\kappa_{|\lambda|^{1/m}}\Big)\Big(\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{A}_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\omega\theta\Big),$$

and consequently this term is $o(1)$ as $|\lambda| \to \infty$. Recall that

$$\|(\mathcal{T}_k(\lambda) - \mathcal{T}_{k,\wedge}(\lambda))\omega_0\kappa_{|\lambda|^{1/m}}\| = O(|\lambda|^{-(m_k-1)/m})$$

for $k = 0, \ldots, K$, where $m_0 = m$. On the other hand, we have

$$\begin{aligned} \mathcal{T}(\lambda)\omega_0\Big(\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m}) - \mathcal{A}_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\omega\theta\Big) = \\ \Big(\mathcal{T}(\lambda)\omega_0\kappa_{|\lambda|^{1/m}}\Big)\Big(\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{A}(\lambda)\tilde{K}(|\lambda|^{1/m}) - \kappa^{-1}_{|\lambda|^{1/m}}\mathcal{A}_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\omega\theta\Big). \end{aligned}$$

By (8.10) and (8.12) in Lemma 8.6 (see also Lemma 6.20 in [9]) and Lemma 8.4 it follows that each summand in this matrix multiplication is $o(1)$ in the norm as $|\lambda| \to \infty$, and so the lemma follows. $\square$

Finally, we want to point out that under the assumptions of Theorem 8.1 we get the existence of the resolvent of A with polynomial bounds for the norm also for realizations in Sobolev spaces of arbitrary smoothness $s > -\frac{1}{2}$. The proof follows along the lines of this section. The advantage in this case is that we need not be as precise with the bounds as for the case of $x^{-m/2}L^2_b$-realizations.

Theorem 8.26. *Let (4.1) be c-elliptic with parameter in the closed sector $\Lambda \subset \mathbb{C}$, and consider the unbounded operator A in $x^{-m/2}H^s_b(\overline{M};E)$ under the boundary condition $Tu = 0$ on some intermediate domain $\mathcal{D}^s_{\min}(A_T) \subset \mathcal{D}^s(A_T) \subset \mathcal{D}^s_{\max}(A_T)$, where $s > -\frac{1}{2}$.*

Let $\mathcal{D}_\wedge(A_{\wedge,T_\wedge}) = \theta(\mathcal{D}(A_T))$ be the associated domain for the model operator $A_\wedge$ under the boundary condition $T_\wedge u = 0$ in $x^{-m/2}L^2_b(\overline{Y}^\wedge;E)$ according to Proposition 6.11, and assume that Λ is a sector of minimal growth for $A_\wedge$ with this domain.

Then

$$A_{\mathcal{D}^s} - \lambda : \mathcal{D}^s(A_T) \to x^{-m/2}H^s_b(\overline{M};E)$$

is invertible for large $\lambda \in \Lambda$, and the resolvent can be written in the form (8.2). Moreover, there exists $M(s) \in \mathbb{R}$ such that

$$\|\big(A_{\mathcal{D}^s} - \lambda\big)^{-1}\|_{\mathscr{L}(x^{-m/2}H_b^s)} = O(|\lambda|^{M(s)})$$

as $|\lambda| \to \infty$.

References

[1] M. Agranovich and M. Vishik, *Elliptic problems with a parameter and parabolic problems of general type*, Russ. Math. Surveys **19** (1963), 53–159.

[2] B. Ammann, R. Lauter, and V. Nistor, *Pseudodifferential operators on manifolds with a Lie structure at infinity*, preprint 2003 (math.AP/0304044 at arXiv.org).

[3] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[4] S. Coriasco, J. Seiler, and E. Schrohe, *Realizations of differential operators on conic manifolds with boundary*, preprint 2004 (math.AP/0401395 at arXiv.org).

[5] M. Dauge, *Elliptic boundary value problems on corner domains*, vol. 1341 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1988.

[6] G. Eskin, *Boundary value problems for elliptic pseudodifferential equations*, Translations of Mathematical Monographs, vol. 52. American Mathematical Society, Providence, R.I., 1981.

[7] J. Gil, *Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators*, Math. Nachr. **250** (2003), 25–57.

[8] J. Gil, T. Krainer, and G. Mendoza, *Geometry and spectra of closed extensions of elliptic cone operators*, preprint 2004 (math.AP/0410178 at arXiv.org).

[9] J. Gil, T. Krainer, and G. Mendoza, *Resolvents of elliptic cone operators*, preprint 2004 (math.AP/0410176 at arXiv.org).

[10] J. Gil and G. Mendoza, *Adjoints of elliptic cone operators*, Amer. J. Math. **125** (2003), no. 2, 357–408.

[11] G. Grubb, *Functional calculus of pseudodifferential boundary problems*, 2nd ed., Progress in Mathematics, vol. 65. Birkhäuser, Basel, 1996.

[12] G. Grubb and R. Seeley, *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems*, Invent. Math. **121** (1995), no. 3, 481–529.

[13] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130** (1995), no. 1, 161–219.

[14] D. Kapanadze and B.-W. Schulze, *Crack Theory and Edge Singularities*, Mathematics and its Applications, vol. 561, Kluwer Academic Publishers, Dordrecht-Boston-London, 2003.

[15] V. Kondrat'ev, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Mosc. Math. Soc. **16** (1967), 227–313.

[16] V. Kozlov, V. Maz'ya, and J. Roßmann, *Spectral problems associated with corner singularities of solutions to elliptic equations*, vol. 85 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2001.

[17] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Math. vol 136, B.G. Teubner, Stuttgart, Leipzig, 1997.

[18] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Grundlehren der mathematischen Wissenschaften, vol. 181–183. Springer-Verlag, New York-Heidelberg, 1972, 1973.

[19] P. Loya, *On the resolvent of differential operators on conic manifolds*, Comm. Anal. Geom. **10** (2002), no. 5, 877–934.

[20] V. Maz'ya, S. Nazarov, and B. Plamenevsky, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains I, II*, vol. 111 and 112 of *Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, 2000.

[21] D. Mitrea, M. Mitrea, and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, vol. 713 of *Memoirs of the American Mathematical Society.* American Mathematical Society, Providence, RI, 2001.

[22] M. Mitrea and V. Nistor, *Boundary value problems and layer potentials on manifolds with cylindrical ends*, preprint 2004 (math.AP/0410186 at arXiv.org).

[23] M. Mitrea, M. Taylor, and A. Vasy, *Lipschitz domains, domains with corners, and the Hodge Laplacian*, preprint 2004 (math.AP/0408438 at arXiv.org).

[24] V. Nistor, *Singular integral operators on non-compact manifolds and analysis on polyhedral domains*, preprint 2004 (math.AP/0402322 at arXiv.org).

[25] R. Melrose, *Transformation of boundary value problems*, Acta Math. **147** (1981), no. 3-4, 149–236.

[26] ———, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, A K Peters, Ltd., Wellesley, MA, 1993.

[27] R. Melrose and G. Mendoza, *Elliptic operators of totally characteristic type*, MSRI Preprint, 1983.

[28] E. Schrohe, *Fréchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance*, Math. Nachr. **199** (1999), 145–185.

[29] E. Schrohe and B.-W. Schulze, *Boundary value problems in Boutet de Monvel's algebra for manifolds with conical singularities I,II*, Math. Top., vol. 5 and 8, Akademie Verlag, Berlin, (1994, 1995), pp. 97–209, 70–205.

[30] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, Studies in Mathematics and its Applications, vol. 24. North-Holland Publishing Co., Amsterdam, 1991.

[31] ———, *Boundary value problems and singular pseudo-differential operators*, Pure and Applied Mathematics. John Wiley & Sons, Ltd., Chichester, 1998.

[32] R. Seeley, *The resolvent of an elliptic boundary problem*, Amer. J. Math. **91** (1969), 889–920.

[33] ———, *Topics in Pseudo-Differential Operators*, CIME Conference of Pseudo-Differential Operators 1968, Edizione Cremonese, Roma, 1969, pp. 169–305.

Institut für Mathematik, Universität Potsdam, Postfach 60 15 53, D-14415 Potsdam, Germany

E-mail address: krainer@math.uni-potsdam.de