The Zaremba Problem with Singular Interfaces as a Corner Boundary Value Problem

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Abstract. We study mixed boundary value problems for an elliptic operator A on a manifold X with boundary Y, i.e., $Au = f$ in $\operatorname{int} X, T_{\pm}u = g_{\pm}$ on $\operatorname{int} Y_{\pm}$, where Y is subdivided into subsets $Y_{\pm}$ with an interface Z and boundary conditions $T_{\pm}$ on $Y_{\pm}$ that are Shapiro-Lopatinskij elliptic up to Z from the respective sides. We assume that $Z \subset Y$ is a manifold with conical singularity v. As an example we consider the Zaremba problem, where A is the Laplacian and T_{-} Dirichlet, T_{+} Neumann conditions. The problem is treated as a corner boundary value problem near v which is the new point and the main difficulty in this paper. Outside v the problem belongs to the edge calculus as is shown in [3].
With a mixed problem we associate Fredholm operators in weighted corner Sobolev spaces with double weights, under suitable edge conditions along $Z \setminus \{v\}$ of trace and potential type. We construct parametrices within the calculus and establish the regularity of solutions.

Keywords: Zaremba problem, corner Sobolev spaces with double weights, pseudo-differential boundary value problems

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Contents

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Introduction

This paper is aimed at studying the Zaremba problem in a smooth bounded domain $G \subset \mathbb{R}^3$ when the interface, i.e., the subset Z of $Y := \partial G$ with the jump of the boundary conditions has conical singularities. The problem is represented by equations

$$Au = f \quad \text{in} \quad G, \quad T_\pm u = g_\pm \quad \text{on} \quad \operatorname{int} Y_\pm, \tag{1}$$

where A is an elliptic differential operator of second order in G with smooth coefficients up to the boundary; the boundary Y is written as $Y = Y_- \cup Y_+$ for closed subsets $Y_\pm$, and $Z := Y_- \cap Y_+$ is a curve with conical singularities. The operators $T_\pm$ which represent the boundary conditions are assumed to be of the form $T_\pm := \mathrm{r}_\pm B_\pm$ for differential operators $B_\pm$, given in an open neighbourhood of $Y_\pm$ with smooth coefficients, $\mathrm{r}_\pm$ denote the restriction operators to $\operatorname{int} Y_\pm$, and $T_\pm$ satisfy the Shapiro-Lopatinskij condition on $\operatorname{int} Y_\pm$ with respect to A, uniformly up to Z.

The Zaremba problem corresponds to the case $A = \Delta$ (the Laplacian) with Dirichlet and Neumann conditions on $\operatorname{int} Y_-$ and $\operatorname{int} Y_+$, respectively.

The present paper is focused on new effects close to the conical singularities of Z. Mixed problems for the case of smooth Z in weighted edge Sobolev spaces are studied in [3], based on an earlier paper [6] on the behaviour of mixed problems in standard Sobolev spaces.

We are interested in an approach which allows us to construct parametrices within a calculus of operators with a specific corner symbolic hierarchy which gives us elliptic regularity in certain corner Sobolev spaces with double weights close to the conical points Z_{sing} of Z. Another essential aspect will be to observe the role of edge weights along $Z_{\text{reg}} := Z \setminus Z_{\text{sing}}$ in connection with extra conditions on Z of trace and potential type that are of a similar origin as elliptic conditions in standard elliptic boundary value problems. To illustrate the structure, we first observe that (1) represents continuous operators

$$\mathcal{A} := \begin{pmatrix} A \\ T_- \\ T_+ \end{pmatrix} : H^s(G) \to \begin{matrix} H^{s-2}(G) \\ \oplus \\ H^{s-\frac{1}{2}}(\operatorname{int} Y_-) \\ \oplus \\ H^{s-\frac{3}{2}}(\operatorname{int} Y_+) \end{matrix} \tag{2}$$

(when A is of second order and the orders of $T_\pm$ as for the Zaremba problem), for all $s > \frac{3}{2}$, with $H^s(G), s \in \mathbb{R}$, being the standard Sobolev spaces and $H^s(\operatorname{int} Y_\pm) := H^s(Y)|_{\operatorname{int} Y_\pm}$. As already noted in [6] for the case of smooth Z, the operator (2) will not be Fredholm since $\operatorname{coker} \mathcal{A}$ is not of finite dimension. First we realise $\mathcal{A}$ as continuous operators

$$\mathcal{A} : \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X}) \to \tilde{\mathcal{V}}^{s-2,(\gamma-2,\delta-2)}(\mathbb{X}), \tag{3}$$

where the space on the right hand side is defined as

$$\mathcal{V}^{s-2,(\gamma-2,\delta-2)}(\mathbb{X}) \oplus \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{Y}_-) \oplus \mathcal{V}^{s-\frac{3}{2},(\gamma-\frac{3}{2},\delta-\frac{3}{2})}(\mathbb{Y}_+),$$

with weighted corner Sobolev spaces $\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X})$, $\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{Y}_\pm)$ with double weights $(\gamma, \delta) \in \mathbb{R}^2$, cf. Definition 2.3 (i), (ii) below. Here $\mathbb{X}$ and $\mathbb{Y}_\pm$ denote stretched versions of $X := \overline{G}$ and $Y_\pm$ obtained by introducing polar coordinates locally near the conical singularities of Z and interpreting Z_{reg} as an edge (cf. Section 1.3 for more details). We will show that, up to a discrete set of exceptional weights $\gamma \in \mathbb{R}$, the operator (3) can be filled up to a block matrix

$$\mathfrak{A} := \begin{pmatrix} \mathcal{A} & \mathcal{K} \\ \mathcal{T} & \mathcal{Q} \end{pmatrix} : \begin{matrix} \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X}) \\ \oplus \\ \mathcal{H}^{s-1,\delta-1}(Z, \mathbb{C}^{j_-}) \end{matrix} \to \begin{matrix} \tilde{\mathcal{V}}^{s-2,(\gamma-2,\delta-2)}(\mathbb{X}) \\ \oplus \\ \mathcal{H}^{s-3,\delta-3}(Z, \mathbb{C}^{j_+}) \end{matrix} \tag{4}$$

by additional operators of trace and potential type $\mathcal{T}$ and $\mathcal{K}$, respectively, and an operator $\mathcal{Q}$ in the cone algebra on Z (as a manifold with conical singularities), where $\mathcal{H}^{s,\delta}(Z, \mathbb{C}^j)$ are weighted cone Sobolev spaces, cf. Definition 2.3 (iii) below, such that (4) is Fredholm up to a discrete set of exceptional corner weights δ, cf. Theorem 3.13 below.

This paper is organised as follows.

In Chapter 1 we formulate the principal edge symbolic structure of mixed problems outside the conical singularity v of Z, where $Z_{\text{reg}} := Z \setminus \{v\}$ plays the role of an edge. Moreover, we introduce some necessary terminology on configurations with corners. We finally consider weighted edge spaces and observe the continuity properties of our operators in such spaces.

Chapter 2 gives an interpretation of mixed boundary value problems near $v \in Z$ as corner problems. We define corner Sobolev spaces with double weights and establish the continuity of the operators from the problems in those spaces. We then define corner amplitude functions which contain additional trace and potential data along the interface. At this moment we generalise the context and admit an arbitrary compact manifold N with boundary as the base of the local model cones of wedges outside v. We then introduce holomorphic corner Mellin symbols and smoothing Mellin symbols near a weight line. They will be crucial for the calculus of corner boundary value problems which we consider as problems in an infinite cone with a base that is a manifold with conical singularities and boundary. We establish the principal symbolic hierarchy of such operators, formulate ellipticity, and obtain parametrices within that corner calculus.

In Chapter 3 we specify the context and come back to the situation of the Zaremba problem. In contrast to Chapter 2 where the orders are assumed to be unified (which is adequate when we take into account the existence of order reductions within the calculus) we take for the Zaremba problem the ‘natural’ orders and corresponding modified notation. We then apply the results from Chapter 2 to study ellipticity with extra edge conditions, cf. Theorem 3.12, and establish parametrices and the Fredholm property, cf. Theorem 3.13.

Chapter 4 has the character of an appendix, where we present necessary material from the calculus of boundary value problems with conical and edge singularities.

Let us finally note that mixed elliptic and other singular boundary value problems have been studied by numerous authors before, cf. Zaremba [19], Vishik and Eskin [18], Zargaryan and Maz’ya [20]. A detailed bibliography is given in [9]. The pseudo-differential approach refers to the general ideas of the edge and corner calculus, developed in [14] and [15], cf. also [16], [17].

1 Singular mixed problems in edge representation

1.1 Basic observations

In order to organise the operators (3) and (4) which are connected with our mixed elliptic problems (1) we find it advisable to first ignore the conical singularities of Z, i.e., look at the smooth part of Z. Choose a chart on $X = \overline{G}$

$$\chi : U \to \overline{\mathbb{R}}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq 0\}$$

for a neighbourhood U of a point of Z in X that restricts to charts

$$\chi' : U' \to \mathbb{R}^2 = \{x \in \mathbb{R}^3 : x_3 = 0\}, \quad \chi'' : U'' \to \mathbb{R} = \{x \in \mathbb{R}^3 : x_3 = x_2 = 0\}$$

for $U' := U \cap \partial X, U'' := U \cap Z$. In other words the variable on Z is identified with $x_1 \in \mathbb{R}$ which we also call z.

In local coordinates we have

$$A = \sum_{|\alpha| \leq 2} a_\alpha(x) D_x^\alpha \tag{5}$$

with coefficients in $C^\infty(\overline{\mathbb{R}}^3_+)$. Let us introduce polar coordinates (r, ϕ) in $\overline{\mathbb{R}}^2_+ \setminus \{0\}$, where $\overline{\mathbb{R}}^2_+ = \{(x_2, x_3) \in \mathbb{R}^2 : x_3 \geq 0\}$. Then A takes the form of an edge-degenerate operator

$$A = r^{-2} \sum_{j+k \leq 2} a_{jk}(r, z)(-r\partial_r)^j (rD_z)^k \tag{6}$$

with coefficients $a_{jk} \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}, \mathrm{Diff}^{2-(j+k)}(S^1_+))$. Here $S^1_+ := \{\phi \in S^1 : 0 \leq \phi \leq \pi\}$, and $\mathrm{Diff}^\nu(M)$ for a C^∞ manifold M with smooth boundary denotes the space of all differential operators of order ν with smooth coefficients up to the boundary.

In a similar manner we reformulate the boundary operators $T_\pm$ which have the form $T_\pm : u \to \mathrm{r}_\pm B_\pm u$ for $\mathrm{r}_\pm f := f(x_1, x_2, 0)|_{x_2 \gtrless 0}$ and

$$B_\pm u = \sum_{|\alpha| \leq \mu_\pm} b_{\pm,\alpha}(x) D_x^\alpha \tag{7}$$

(in our case for $\mu_+ = 1, \mu_- = 0$) with smooth coefficients $b_{\pm,\alpha}$ in a neighbourhood of $\{(x_1, x_2, 0) : x_2 \gtrless 0\}$ in $\mathbb{R}^3$. In polar coordinates we then obtain

$$T_\pm = \mathrm{r}_\pm r^{-\mu_\pm} \sum_{j+k \leq \mu_\pm} b_{\pm,jk}(r, z)(-r\partial_r)^j (rD_z)^k \tag{8}$$

with coefficients $b_{\pm,jk} \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}, \mathrm{Diff}^{\mu_\pm-(j+k)}(S^1_+))$ where $\mathrm{r}_\pm u(r, \phi, z) := u(r, \phi_\pm, z)$ for $\phi_+ = 0, \phi_- = \pi$.

The choice of our formulations is motivated by the fact that our constructions easily extend to (elliptic) differential operators A of arbitrary order, together with mixed elliptic boundary conditions.

We want to associate with (6) and (8) continuous operators in weighted edge Sobolev spaces. To this end we form the operator family

$$\boldsymbol{a}(z, \zeta) := \begin{pmatrix} r^{-2} \sum_{j+k \leq 2} a_{jk}(r, z)(-r\partial_r)^j (r\zeta)^k \\ \mathrm{r}_- r^{-\mu_-} \sum_{j+k \leq \mu_-} b_{-,jk}(r, z)(-r\partial_r)^j (r\zeta)^k \\ \mathrm{r}_+ r^{-\mu_+} \sum_{j+k \leq \mu_+} b_{+,jk}(r, z)(-r\partial_r)^j (r\zeta)^k \end{pmatrix}. \tag{9}$$

For the case of the Zaremba problem (9) can be interpreted as a family of continuous maps $\boldsymbol{a}(z,\zeta):\mathcal{K}^{s,\gamma}((S^1_+)^\wedge)\to\tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge)$ for

$$\tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge):=\mathcal{K}^{s-2,\gamma-2}((S^1_+)^\wedge)\oplus\mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_-)\oplus\mathcal{K}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{R}_+), \tag{10}$$

$s>\frac{3}{2}$. Here, if N is a set, we write $N^\wedge:=\mathbb{R}_+\times N$, regarded as a stretched cone with base N; concerning the spaces, cf. the formula (76). In our situation it is adequate to assume that the coefficients $a_{jk}(r,z)$ and $b_{\pm,jk}(r,z)$ are independent of r for large r. In addition, since the calculations refer to local coordinates (and are later on combined with factors from a partition of unity) we assume the coefficients to be independent of z for large $|z|$.

Let $\mathcal{K}^{s,\gamma}((S^1_+)^\wedge)$ be endowed with the standard group action

$$\kappa^{(1)}_\lambda:\mathcal{K}^{s,\gamma}((S^1_+)^\wedge)\to\mathcal{K}^{s,\gamma}((S^1_+)^\wedge),\quad u(r,\phi)\to\lambda u(\lambda r,\phi),$$

$\lambda\in\mathbb{R}_+$, cf. (78), while in $\tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge)$ we take

$$\begin{aligned}&\boldsymbol{\kappa}_\lambda:\tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge)\to\tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge),\\&u(r,\phi)\oplus v_-(r)\oplus v_+(r)\to\kappa^{(1)}_\lambda u(r,\phi)\oplus\lambda^{-\frac{3}{2}}\kappa^{(0)}_\lambda v_-(r)\oplus\lambda^{-\frac{1}{2}}\kappa^{(0)}_\lambda v_+(r),\end{aligned}$$

$\lambda\in\mathbb{R}_+$. In the following results we employ notation from Section 4.2 below.

Proposition 1.1 *The operator family $\boldsymbol{a}(z,\zeta)$ represents a symbol*

$$\boldsymbol{a}(z,\zeta)\in S^2(\mathbb{R}_z\times\mathbb{R}_\zeta;\mathcal{K}^{s,\gamma}((S^1_+)^\wedge),\tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge))_{\kappa,\boldsymbol{\kappa}}$$

for $\kappa:=\{\kappa^{(1)}_\lambda\}_{\lambda\in\mathbb{R}_+}$, $\boldsymbol{\kappa}:=\{\boldsymbol{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, (cf. also the formula (82)).

The proof is elementary and will be omitted.

The following observation, known from [3], employs the abstract edge Sobolev spaces of Definition 4.3. For completeness we recall the short proof.

Proposition 1.2 *The operator $\mathcal{A}:={}^{\mathrm{t}}(A\quad T_-\quad T_+)$ given by (6), (8) induces continuous operators*

$$\mathcal{A}:\mathcal{W}^s(\mathbb{R},\mathcal{K}^{s,\gamma}((S^1_+)^\wedge))\to\tilde{\mathcal{W}}^{s-2}(\mathbb{R},\mathcal{K}^{s-2,\gamma-2}((S^1_+)^\wedge)) \tag{11}$$

for

$$\begin{aligned}\tilde{\mathcal{W}}^{s-2}(\mathbb{R},\mathcal{K}^{s-2,\gamma-2}((S^1_+)^\wedge))&:=\mathcal{W}^{s-2}(\mathbb{R},\mathcal{K}^{s-2,\gamma-2}((S^1_+)^\wedge))\\&\oplus\mathcal{W}^{s-\frac{1}{2}}(\mathbb{R},\mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_-))\oplus\mathcal{W}^{s-\frac{3}{2}}(\mathbb{R},\mathcal{K}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{R}_+))\end{aligned}$$

for every $s>\frac{3}{2}$ and $\gamma\in\mathbb{R}$.

Proof. From Proposition 1.1 together with Theorem 4.4 (cf. Section 4.2) we obtain the continuity of

$$\mathcal{A}=\mathrm{Op}_z(\boldsymbol{a}):\mathcal{W}^s(\mathbb{R},\mathcal{K}^{s,\gamma}((S^1_+)^\wedge))_\kappa\to\mathcal{W}^{s-2}(\mathbb{R},\tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge))_{\boldsymbol{\kappa}}$$

for every $s>\frac{3}{2}$ and $\gamma\in\mathbb{R}$ (cf. (84)). Then it suffices to observe that

$$\mathcal{W}^{s-2}(\mathbb{R},\tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge))_{\boldsymbol{\kappa}}=\tilde{\mathcal{W}}^{s-2}(\mathbb{R},\mathcal{K}^{s-2,\gamma-2}((S^1_+)^\wedge)).$$

□

1.2 Principal symbols and interface conditions

The operator (11) has a principal symbolic hierarchy

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A}))$$

which determines the ellipticity of $\mathcal{A}$ as an operator in the edge calculus with $\mathbb{R}$ being regarded as edge. The interior symbol $\sigma_\psi(\mathcal{A}) := \sigma_\psi(A)$ is nothing other than $\sigma_\psi(A)(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha$, the standard homogeneous principal symbol, $(x,\xi) \in T^*\overline{\mathbb{R}}^3_+ \setminus 0$. The boundary symbol $\sigma_\partial(\mathcal{A})$ is defined on $\mathbb{R}^2_\pm$ and consists of two components, namely, $\sigma_{\partial,\pm}(\mathcal{A})$. In local coordinates near any point of $\operatorname{int} Y_\pm$ we have $\sigma_{\partial,\pm}(\mathcal{A})(x_1,x_2,\xi_1,\xi_2) := \begin{pmatrix} \sigma_\partial(A)(x_1,x_2,\xi_1,\xi_2) \\ \sigma_\partial(T_\pm)(x_1,x_2,\xi_1,\xi_2) \end{pmatrix}$, where

$$\sigma_\partial(A)(x_1,x_2,\xi_1,\xi_2) := \sigma_\psi(A)(x_1,x_2,0,\xi_1,\xi_2,D_{x_3}),$$

$$\sigma_\partial(T_\pm)(x_1,x_2,\xi_1,\xi_2) := \mathrm{r}_\pm\sigma_\psi(B_\pm)(x_1,x_2,0,\xi_1,\xi_2,D_{x_3})$$

for $x_2 \gtrless 0, (\xi_1,\xi_2) \neq 0$. Clearly, on the minus side (for the Dirichlet condition) we simply have $\sigma_\partial(T_-)(x_1,x_2,\xi_1,\xi_2) : u \to u|_{x_3=0}$.

The edge symbol $\sigma_\wedge(\mathcal{A})$ has the form

$$\sigma_\wedge(\mathcal{A})(z,\zeta) := {}^{\mathrm{t}}(\sigma_\wedge(A)(z,\zeta) \quad \sigma_\wedge(T_-)(z,\zeta) \quad \sigma_\wedge(T_+)(z,\zeta))$$

for

$$\sigma_\wedge(A)(z,\zeta) := r^{-2}\sum_{j+k\le 2} a_{jk}(0,z)(-r\partial_r)^j(r\zeta)^k,$$

$$\sigma_\wedge(T_\pm)(z,\zeta) := \mathrm{r}_\pm r^{-\mu_\pm}\sum_{j+k\le\mu_\pm} b_{\pm,jk}(0,z)(-r\partial_r)^j(r\zeta)^k,$$

$(z,\zeta) \in T^*\mathbb{R}\backslash 0$, cf. the formula (9) (in the Zaremba case for $\mu_+ = 1, \mu_- = 0$). The edge symbol defines a family of continuous operators

$$\sigma_\wedge(\mathcal{A})(z,\zeta) : \mathcal{K}^{s,\gamma}((S^1_+)^\wedge) \to \tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge) \tag{12}$$

for all $s > \frac{3}{2}, \gamma \in \mathbb{R}$, and has the 'twisted homogeneity'

$$\sigma_\wedge(\mathcal{A})(z,\lambda\zeta) = \lambda^2\boldsymbol{\kappa}_\lambda\sigma_\wedge(\mathcal{A})(z,\zeta)\kappa^{(1)}_{\lambda^{-1}} \tag{13}$$

for all $\lambda \in \mathbb{R}_+$ with $\boldsymbol{\kappa}_\lambda$ as above. The realisation (12) of $\sigma_\wedge(\mathcal{A})(z,\zeta)$ in the spaces with weight $\gamma \in \mathbb{R}$ will now be denoted by $\sigma_\wedge(\mathcal{A}(\gamma))(z,\zeta)$.

Let us recall from [3] the following result:

Theorem 1.3 *For every $k \in \mathbb{Z}$ and $\gamma \in (\frac{1}{2}-k, \frac{3}{2}-k)$ the operators* (12) *are Fredholm, and we have*

$$\operatorname{ind}\sigma_\wedge(\mathcal{A}(\gamma))(z,\zeta) = k$$

for every $(z,\zeta) \in T^*\mathbb{R} \setminus 0$.

Remark 1.4 *For every* $(z,\zeta) \in T^*Z \setminus 0$ *and* $\gamma \notin \mathbb{Z} + \frac{1}{2}$ *the operator* (12) *belongs to the cone algebra of boundary value problems* (*with the transmission property*) *on the infinite* (*stretched*) *cone* $(S^1_+)^\wedge$, *and is elliptic with respect to the principal symbolic structure, i.e.,* $(\sigma_\psi, \sigma_\partial, \sigma_M)$, *the interior, the boundary and the conormal symbol in the cone algebra.*

As in [3] we are now able to fill up the Fredholm family (12) to a family of isomorphisms by additional entries of trace and potential type with respect to the interface. Their number is determined by the dimensions of kernels and cokernels.

First recall that by virtue of (13) we have

$$\operatorname{ind} \sigma_\wedge(\mathcal{A}(\gamma))(z, \frac{\zeta}{|\zeta|}) = \operatorname{ind} \sigma_\wedge(\mathcal{A}(\gamma))(z,\zeta) \tag{14}$$

for all $(z,\zeta) \in T^*\mathbb{R} \setminus 0$. Because z is of dimension 1 the unit cosphere bundle $S^*\mathbb{R}$ consists of two copies of $\mathbb{R}$, characterised by $(z,-1)$ and $(z,+1), z \in \mathbb{R}$. As we know from [3] the indices (14) on the plus and the minus side coincide.

Now we can choose dimensions $j_\pm(\gamma)$ for $\gamma \in (\frac{1}{2} - k, \frac{3}{2} - k)$ such that $j_+(\gamma) - j_-(\gamma) = k$ and isomorphisms

$$\begin{aligned} \sigma_\wedge(\mathfrak{a})(z,\zeta) \quad &:= \quad \begin{pmatrix} \sigma_\wedge(\mathcal{A}(\gamma))(z,\zeta) & \sigma_\wedge(\mathcal{K})(z,\zeta) \\ \sigma_\wedge(\mathcal{T})(z,\zeta) & \sigma_\wedge(\mathcal{Q})(z,\zeta) \end{pmatrix} \\ &: \quad \mathcal{K}^{s,\gamma}((S^1_+)^\wedge) \oplus \mathbb{C}^{j_-} \to \tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge) \oplus \mathbb{C}^{j_+} \end{aligned} \tag{15}$$

first for $|\zeta| = 1$ and then extended by homogeneity to all $\zeta \neq 0$ by the rule:

$$\sigma_\wedge(\mathfrak{a})(z, \lambda\zeta) = \lambda^2 \tilde{\chi}_\lambda \sigma_\wedge(\mathfrak{a})(z,\zeta) \chi_\lambda^{-1},$$

$\lambda \in \mathbb{R}_+$, where $\chi_\lambda := (\operatorname{diag}(\kappa^{(1)}_\lambda, \lambda \operatorname{id}_{\mathbb{C}^{j_-}})$ and $\tilde{\chi}_\lambda := \operatorname{diag}(\boldsymbol{\kappa}_\lambda, \lambda \operatorname{id}_{\mathbb{C}^{j_+}})$.

Because of the assumption that the upper left corner is independent of z for large $|z|$ we can (and will) choose the other entries of (15) also independent of z for large $|z|$.

Setting

$$\mathfrak{a}(z,\zeta) := \begin{pmatrix} \boldsymbol{a}(z,\zeta) & \boldsymbol{k}(z,\zeta) \\ \boldsymbol{t}(z,\zeta) & \boldsymbol{q}(z,\zeta) \end{pmatrix}$$

for $\boldsymbol{t}(z,\zeta) := \chi(\zeta)\sigma_\wedge(\mathcal{T})(z,\zeta)$, $\boldsymbol{k}(z,\zeta) := \chi(\zeta)\sigma_\wedge(\mathcal{K})(z,\zeta)$, $\boldsymbol{q}(z,\zeta)$ $:= \chi(\zeta)\sigma_\wedge(\mathcal{Q})(z,\zeta)$ for any fixed excision function $\chi(\zeta)$ we obtain an element

$$\mathfrak{a}(z,\zeta) \in S^2(\mathbb{R} \times \mathbb{R}; \mathcal{K}^{s,\gamma}((S^1_+)^\wedge) \oplus \mathbb{C}^{j_-}, \tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge) \oplus \mathbb{C}^{j_+})_{\chi,\tilde{\chi}}$$

and associated continuous operators

$$\operatorname{Op}(\mathfrak{a}) : \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma}((S^1_+)^\wedge) \oplus \mathbb{C}^{j_-})_\chi \to \mathcal{W}^{s-2}(\mathbb{R}, \tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge) \oplus \mathbb{C}^{j_+})_{\tilde{\chi}}. \tag{16}$$

for $\chi = \{\chi_\lambda\}_{\lambda \in \mathbb{R}_+}$ and $\tilde{\chi} = \{\tilde{\chi}_\lambda\}_{\lambda \in \mathbb{R}_+}$ with χ_λ and $\tilde{\chi}_\lambda$ as above.

1.3 Manifolds with corner and boundary

In order to describe the operator (4) in more detail we interpret X as a manifold with corner and boundary. Setting $X' := \partial X, X'' := Z, X''' := \{v\}$ we have a chain of strata $X \supset X' \supset X'' \supset X'''$, and X is the disjoint union of C^∞ manifolds $X = (X \setminus X') \cup (X' \setminus X'') \cup (X'' \setminus X''') \cup X'''$, namely, $X \setminus X' = \operatorname{int} X$, $X' \setminus X'' = \operatorname{int} Y_- \cup \operatorname{int} Y_+$, $X'' \setminus X''' = Z \setminus \{v\} =: Z_{\text{reg}}$, and $X''' = \{v\}$.

X itself is regarded as a manifold with corner v and boundary ∂X, $W := X \setminus \{v\}$ is a manifold with edge $Z \setminus \{v\}$ and boundary $\partial X \setminus \{v\}$, and $X \setminus Z$ is a C^∞ manifold with boundary $\partial X \setminus Z$.

The most specific aspects concern a neighbourhood U_0 of v which we identify with $\overline{\mathbb{R}}^3_+$, where v corresponds to the origin and Z to the union of two half-lines $L_k, k = 1, 2$, starting from the origin in $\mathbb{R}^2 = \partial\overline{\mathbb{R}}^3_+$, cf. also Remark 1.5 below. Setting $S^2_+ := \overline{\mathbb{R}}^3_+ \cap S^2$ we obtain an identification

$$\chi : U_0 \to (S^2_+)^\triangle \tag{17}$$

via polar coordinates in $\overline{\mathbb{R}}^3_+ \setminus \{0\}$ (here, $M^\triangle$ for a space M denotes the cone $(\overline{\mathbb{R}}_+ \times M)/(\{0\} \times M)$ with base M; the tip is represented by $\{0\} \times M$ in the quotient space). The closed half-sphere S^2_+ is regarded as a manifold with conical singularities v_1, v_2 on the boundary S^1, defined by $v_k = S^1 \cap L_k, k = 1, 2$. If S^2_+ is interpreted in that way, i.e., with conical singularities on the boundary, we will write in most cases $\mathbb{S}^2_+$ rather than S^2_+.

From (17) we obtain a map

$$\chi|_{U_0\setminus\{v\}} : U_0 \setminus \{v\} \to \mathbb{R}_+ \times \mathbb{S}^2_+ \tag{18}$$

which gives rise to a splitting of variables into $(t, \cdot)$ with $t \in \mathbb{R}_+$ being regarded as the corner axis variable. Two homeomorphisms $\chi : U_0 \to (\mathbb{S}^2_+)^\triangle$, $\tilde{\chi} : U_0 \to (\mathbb{S}^2_+)^\triangle$ of the kind (17) are called equivalent if

$$\tilde{\chi}|_{U_0\setminus\{v\}} \circ (\chi|_{U_0\setminus\{v\}})^{-1} : \mathbb{R}_+ \times \mathbb{S}^2_+ \to \mathbb{R}_+ \times \mathbb{S}^2_+$$

is the restriction to $\mathbb{R}_+ \times \mathbb{S}^2_+$ of an isomorphism $\mathbb{R} \times \mathbb{S}^2_+ \to \mathbb{R} \times \mathbb{S}^2_+$ in the category of manifolds with smooth edges and boundary (concerning this terminology in general, and the meaning of 'charts' in singular cases, cf. [2]). From $X \setminus \{v\}$ we pass to the stretched manifold $\mathbb{X}$ obtained by (invariantly) attaching $\{0\} \times \mathbb{S}^2_+$ to $X \setminus \{v\}$; in other words, $\mathbb{X}$ is locally near the 'former' corner v identified with a cylinder $\overline{\mathbb{R}}_+ \times \mathbb{S}^2_+$, and v itself is obtained by squeezing down $\mathbb{S}^2_+$ to a single point. In this connection the bottom $\mathbb{S}^2_+$ of the local cylinder that completes $X \setminus \{v\}$ to $\mathbb{X}$ will be called $\mathbb{X}_{\text{sing}}$, and we also set $\mathbb{X}_{\text{reg}} = \mathbb{X} \setminus \mathbb{X}_{\text{sing}}$ which is the same as $X \setminus \{v\}$. Thus there is a continuous map

$$\pi : \mathbb{X} \to X \tag{19}$$

which projects $\mathbb{X}_{\mathrm{sing}}$ to v and restricts to the identity map $\mathbb{X}_{\mathrm{reg}} \to X \setminus \{v\}$.

By a slight modification of these constructions also the subsets $Y_\pm \subset Y$ can be regarded as manifolds with corner $\{v\}$. In this case $Y_\pm \setminus \{v\}$ are manifolds with C^∞ boundary, and locally near v they are isomorphic to cones $(I_\pm)^\triangle$ for closed intervals $I_\pm$, regarded as manifolds with conical singularities which are the end points. More precisely, restricting (17) to $U_\pm := U_0 \cap Y_\pm$ we obtain homeomorphisms $\chi_\pm : U_\pm \to (I_\pm)^\triangle$ and then, analogously as (18), corresponding maps $\chi_\pm|_{U_\pm \setminus \{v\}} \to \mathbb{R}_+ \times I_\pm$ that are isomorphisms in the category of manifolds with C^∞ boundary.

Moreover, if $\tilde{\chi}_\pm : U_\pm \to (I_\pm)^\triangle$ are other homeomorphisms of that kind, then $\tilde{\chi}_\pm|_{U_\pm \setminus \{v\}} \circ (\chi_\pm|_{U_\pm \setminus \{v\}})^{-1} : \mathbb{R}_+ \times I_\pm \to \mathbb{R}_+ \times I_\pm$ are restrictions of isomorphisms $\mathbb{R} \times I_\pm \to \mathbb{R} \times I_\pm$ in the category of C^∞ manifolds with boundary to $\mathbb{R}_+ \times I_\pm$. There are then corresponding stretched manifolds $\mathbb{Y}_\pm$ that are locally near $\mathbb{Y}_{\pm,\mathrm{sing}}$ identified with $\overline{\mathbb{R}}_+ \times I_\pm$ (subscript 'sing' is of analogous meaning as before).

The space $W := X \setminus \{v\}$ as a manifold with edge Z_{reg} and boundary is locally near edge points modelled on $(S^1_+)^\triangle \times \mathbb{R}$ for $S^1_+ = \overline{\mathbb{R}}^2_+ \cap S^1$ with the (x_1, x_2) half-plane $\overline{\mathbb{R}}^2_+$ normal to Z. As we saw in the local descriptions in Sections 1.1, 1.2 it is convenient to pass to the stretched wedges

$$(S^1_+)^\wedge \times \mathbb{R} \ni (r, \phi, z). \tag{20}$$

They are the local models of the stretched manifold $\mathbb{W}$ associated with W. The global definition of $\mathbb{W}$ may be given in terms of the double $2W$ of W (obtained by gluing together two copies of W along the common boundary $\partial X \setminus \{v\}$); then $2W$ can be interpreted as a manifold with edge $Z_{\mathrm{reg}} := Z \setminus \{v\}$ and $(S^1)^\triangle$ as the local model cone rather than $(S^1_+)^\triangle$. The stretched manifold $2\mathbb{W}$ is then a C^∞ manifold with boundary $\partial(2\mathbb{W})$ which is an S^1-bundle over Z_{reg} (induced by the normal bundle of Z_{reg} and the Riemannian metric that we keep in mind).

We set $(2\mathbb{W})_{\mathrm{sing}} := \partial(2\mathbb{W})$, and

$$\mathbb{W}_{\mathrm{sing}} := (2\mathbb{W})_{\mathrm{sing}} \cap \mathbb{W}, \quad \mathbb{W}_{\mathrm{reg}} := \mathbb{W} \setminus \mathbb{W}_{\mathrm{sing}}.$$

By definition there is a canonical map $\pi : \mathbb{W} \to W$ which induces a diffeomorphism $\pi_{\mathrm{reg}} : \mathbb{W}_{\mathrm{reg}} \to X \setminus Z_{\mathrm{reg}}$.

The local weighted edge spaces $\mathcal{W}^{s,\gamma}(\mathbb{R} \times (S^1)^\wedge) := \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma}((S^1)^\wedge))$, cf. Section 4.2 below, can be pulled back to $2\mathbb{W}$ by means for the mappings $2V \setminus Z_{\mathrm{reg}} \to \mathbb{R} \times (S^1)^\wedge$ that belong to $2V$ near Z_{reg}, where $2V$ is the double of the neighbourhood V mentioned before. Then $\mathcal{W}^{s,\gamma}_{\mathrm{loc}}(2\mathbb{W})$ is defined as the space of locally finite sums of pull backs of elements of $\mathcal{W}^{s,\gamma}(\mathbb{R} \times (S^1)^\wedge)$ with bounded supports in r and z and elements belonging to $H^s_{\mathrm{loc}}(\mathrm{int}\,(2\mathbb{W}))$. Moreover, $\mathcal{W}^{s,\gamma}_{\mathrm{comp}}(2\mathbb{W})$ is the subspace of all elements that have compact support (which is admitted 'up to $\partial(2\mathbb{W})$'). Now we define

$$\mathcal{W}^{s,\gamma}_{\mathrm{comp(loc)}}(\mathbb{W}) := \{u|_{\mathrm{int}\,\mathbb{W}_{\mathrm{reg}}} : u \in \mathcal{W}^{s,\gamma}_{\mathrm{comp(loc)}}(2\mathbb{W})\}. \tag{21}$$

Moreover, we can treat the sets $Y_{\pm,\mathrm{reg}} := Y_\pm \setminus \{v\}$ as manifolds with edges Z_{reg} and then have the global weighted edge spaces $\mathcal{W}^{s,\gamma}_{\mathrm{comp(loc)}}(Y_{\pm,\mathrm{reg}})$ that are far from Z_{reg} modelled on $H^s_{\mathrm{comp(loc)}}(\mathrm{int}\, Y_\pm)$ and locally close to Z_{reg} by finite (locally finite) sums of pull backs from $\mathbb{R}_+ \times \mathbb{R}$ of elements of $\mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma}(\mathbb{R}_\pm))$ (with bounded supports in r and z). Finally, $\mathcal{W}^{s,\gamma}_{\mathrm{comp(loc)}}(Y_{\pm,\mathrm{reg}})$ is defined in a similar manner as the 'comp' ('loc') space on $\mathbb{W}$.

The mixed problem (1) will now be interpreted as a continuous operator $\mathcal{A} : \mathcal{W}^{s,\gamma}_{\mathrm{comp}}(\mathbb{W}) \to \tilde{\mathcal{W}}^{s-2,\gamma-2}_{\mathrm{comp}}(\mathbb{W})$ for

$$\tilde{\mathcal{W}}^{s-2,\gamma-2}_{\mathrm{comp}}(\mathbb{W}) := \mathcal{W}^{s-2,\gamma-2}_{\mathrm{comp}}(\mathbb{W}) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{\mathrm{comp}}(Y_{-,\mathrm{reg}}) \oplus \mathcal{W}^{s-\frac{3}{2},\gamma-\frac{3}{2}}_{\mathrm{comp}}(Y_{+,\mathrm{reg}}). \tag{22}$$

A similar continuity together with the trace and potential operators along Z_{reg} follows from (16). This is just the contribution to (4) outside any neighbourhood of v. The behaviour close to v is the main content of the following chapter.

Remark 1.5 *In general, the above local description of a neighbourhood U_0 of v as a corner $(\mathbb{S}^2_+)^\triangle \cong \overline{\mathbb{R}}^3_+$ such that $U_0 \cap \partial X$ corresponds to $\mathbb{R}^2$ with the half-lines Z_1, Z_2 from $Z \cap U_0$, cf.* (34) *below, is not possible in the frame of a smooth chart $U_0 \to \overline{\mathbb{R}}^3_+$ referring to X as a C^∞ manifold with boundary. We have to expect some singularity of the chart near v; otherwise we restrict the geometry of the conical singularity v of Z in a specific way. However, this is unimportant for our corner calculus, since we might assume from the very beginning X to have a 'regular' corner at v and $X \setminus \{v\}$ to be a 'regular' manifold with edges which is not smooth across $Z \setminus \{v\}$. The reason for that is that the only essential properties of our operators are their corner or edge-degenerate behaviour in local coordinates, and this is invariant under the relevant corner or edge charts in the general case.*

2 Corner operators in spaces with double weights

2.1 Transformation to a corner problem

The mixed elliptic problem (1) will now be studied in a neighbourhood of the conical point v, cf. Remark 1.5. We choose the correspondence between U_0 and $\overline{\mathbb{R}}^3_+$ via a 'singular' chart $\chi_0 : U_0 \to \overline{\mathbb{R}}^3_+$, cf. the discussion in connection with (18) and the considerations around (32) below. For simplicity, we assume that $\chi_0(v) = 0$ and $\chi_0(Z \cap U_0) = L_1 \cup L_2$ for $L_1 := \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 = x_3 = 0\}, L_2 := \{x \in \mathbb{R}^3 : x = 0,$ or $x_1 + ix_2 = te^{i\alpha}, t \in \mathbb{R}_+,$ for some $0 < \alpha < 2\pi, x_3 = 0\}$ (where we identify for the moment the (x_1, x_2) plane with $\mathbb{C}$). Let $S^2_+ := S^2 \cap \overline{\mathbb{R}}^3_+$. The operators (5) and (7) take the form

$$A = t^{-2}\sum_{j=0}^{2} a_j(t)(-t\partial_t)^j \quad \text{and} \quad B_\pm = t^{-\mu_\pm}\sum_{j=0}^{\mu_\pm} b_{\pm,j}(t)(-t\partial_t)^j$$

with $a_j \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{2-j}(S^2_+))$ and $b_{\pm,j} \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu_\pm - j}(S^2_+))$, respectively.

We now interpret the problem (1) as a corner problem in the Mellin calculus with corner axis variable $t \in \mathbb{R}_+$ and covariable $w \in \mathbb{C}$. In other words, we form the (t, w)-depending family of mixed problems on S^2_+ which is a manifold with boundary $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ which is subdivided into the intervals

$$I_+ = \{\vartheta \in S^1 : 0 \leq \vartheta \leq \alpha\}, \quad I_- = \{\vartheta \in S^1 : \alpha \leq \vartheta \leq 2\pi\}$$

mentioned in Section 1.3. Set

$$h(t, w) := {}^{\mathrm{t}}\Big(\sum_{j=0}^{2} a_j(t)w^j \quad \mathrm{r}_-\sum_{j=0}^{\mu_-} b_{-,j}(t)w^j \quad \mathrm{r}_+\sum_{j=0}^{\mu_+} b_{+,j}(t)w^j\Big), \tag{23}$$

where $\mathrm{r}_\pm$ denote the operators of restriction to $\mathrm{int}\, I_\pm$. Then our mixed problem takes the form $\mathcal{A}u = \mathrm{diag}\,(t^{-2}, t^{-\mu_-}, t^{-\mu_+})\mathrm{op}_M^{\delta-1}(h)u$ for

$$\mathcal{A} := {}^{\mathrm{t}}(A \quad T_- \quad T_+) \tag{24}$$

for any $\delta \in \mathbb{R}$. The task is to single out the adequate weighted corner Sobolev spaces such that the problem admits a parametrix which leaves compact remainders in the global calculus (together with the information of Section 1). The important observation is that (23) represents a family of mixed problems on S^2_+ with respect to the subdivision of the boundary $S^1 = I_- \cup I_+$.

Formally, we now proceed in a similar manner as in Section 1.1. Let v denote one of the points v_1, v_2, cf. Section 1.3, and choose a chart $U \to \overline{\mathbb{R}}^2_+$ on S^2_+ near v such that v is transformed to the origin of $\overline{\mathbb{R}}^2_+$ and $U \cap \partial S^2_+$ to $x_3 = 0$. It will also be convenient to consider the double $2U$ as a neighbourhood on S^2 and a corresponding chart $2U \to \mathbb{R}^2$. If (r, ϕ) are polar coordinates in $\mathbb{R}^2$ we then identify $U \setminus \{v\}$ with $\{(r, \phi) : r \in \mathbb{R}_+, 0 \leq \phi \leq \pi\}$ and $(U \setminus \{v\}) \cap \partial S^2_+$ with $\mathbb{R} \setminus \{0\}$, where $\{(r, \pi) : r \in \mathbb{R}_+\} \subset \mathbb{R} \setminus \{0\}$ corresponds to the Dirichlet side, also denoted by $\mathbb{R}_-$, and $\{(r, 0) : r \in \mathbb{R}_+\} \subset \mathbb{R} \setminus \{0\}$ to the Neumann side which is $\mathbb{R}_+$, or, conversely.

In order to unify some descriptions for the operators A and $B_\pm$ we replace for the moment the order by any $\mu \in \mathbb{N}$. Then A takes the form

$$A = t^{-\mu}\sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j \tag{25}$$

with coefficients $a_j(t) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(S^2_+))$. Writing the operators $a_j(t)$ locally in $U \subset S^2_+$ in polar coordinates $(r, \phi) \in \mathbb{R}_+ \times S^1_+$, we can insert $a_j(t) = r^{-\mu+j} \sum_{k=0}^{\mu-j} a_{jk}(r,t)(-r\partial_r)^k$ in (25) and then obtain

$$A = t^{-\mu} r^{-\mu} \sum_{j+k \leq \mu} a_{jk}(r,t)(-r\partial_r)^k(-rt\partial_t)^j$$

with coefficients $a_{jk}(r,t) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-(j+k)}(S^1_+))$. Here, for convenience, we assume that the coefficients $a_j(t)$ are independent of t for large t (which is automatically satisfied in the localised situations near $t = 0$).

Example 2.1 *We have* $\Delta = t^{-2}r^{-2}\{(rt\partial_t)^2 + (r\partial_r)^2 + r^2t\partial_t + \partial^2_\phi\}$ *for the Laplace operator in* $\mathbb{R}^3$.

In order to define the relevant corner Sobolev spaces we first consider the 'abstract' case, cf. also [15].

Definition 2.2 *Let E be a Hilbert space, endowed with a group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$, cf. Section 4.2. In addition, together with E we assume to be given a dimension m that will be specified in every concrete case. Then we define the abstract corner Sobolev space $\mathcal{V}^{s,\delta}(\mathbb{R}_+, E)$ (of smoothness $s \in \mathbb{R}$ and weight $\delta \in \mathbb{R}$) to be the completion of $C_0^\infty(\mathbb{R}_+, E)$ with respect to the norm* $\left\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{m+1}{2}-\delta}} \langle w\rangle^{2s} ||\kappa^{-1}_{\langle w\rangle}(M_{t\to w}u)(w)||^2_E dw\right\}^{\frac{1}{2}}$.

If we want to indicate m we also write $\mathcal{V}^{s,\delta}(\mathbb{R}_+, E)_{(m)}$. For $m = 0$ and $E = \mathbb{C}$ we have $\mathcal{V}^{s,\delta}(\mathbb{R}_+, E)_{(0)} = \mathcal{H}^{s,\delta}(\mathbb{R}_+)$, cf. the formula (74) for $\dim X = 0$. If we prescribe another m' in connection with the space E we have

$$\mathcal{V}^{s,\delta}(\mathbb{R}_+, E)_{(m)} = \mathcal{V}^{s,\delta+\frac{m'-m}{2}}(\mathbb{R}_+, E)_{(m')} \tag{26}$$

which is a consequence of the relation $\mathcal{V}^{s,\delta}(\mathbb{R}_+, E) = t^\delta \mathcal{V}^{s,0}(\mathbb{R}_+, E)$ for every $s, \delta \in \mathbb{R}$. In our concrete situation we insert the spaces

$$E = \mathcal{K}^{s,\gamma}((S^1_+)^\wedge) \quad \text{with} \quad m = 2 \tag{27}$$

and $\kappa^{(1)}_\lambda : u(r,\phi) \to \lambda u(\lambda r, \phi), \lambda \in \mathbb{R}_+$, and

$$E = \mathcal{K}^{s,\gamma}(\mathbb{R}_\pm) \quad \text{with} \quad m = 1 \tag{28}$$

and $\kappa^{(0)}_\lambda : v(r) \to \lambda^{\frac{1}{2}} v(\lambda r), \lambda \in \mathbb{R}_+$. We then obtain the spaces

$$\mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}((S^1_+)^\wedge)) =: \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{R}_+ \times (S^1_+)^\wedge) \tag{29}$$

and, similarly, $\mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}(\mathbb{R}_\pm)) =: \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{R}_+ \times \mathbb{R}_\pm)$; these are spaces of distributions $u(t,r,\phi)$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathrm{int}\, S^1_+$ and $v(t,r)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, respectively.

Note that the $\mathcal{V}^{s,\delta}$-spaces contain elements of the definition both of the $\mathcal{H}^{s,\delta}$-as well as of the $\mathcal{W}^s$-spaces, cf. the formulas (74), (77) and Definition 4.3. For instance, we have

$$\mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}((S^1_+)^\wedge)) \subset \mathcal{W}^s_{\text{loc}}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}((S^1_+)^\wedge)),$$

cf. the notation after Definition 4.3. Moreover, we have

$$\varphi(r,\phi)\mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}((S^1_+)^\wedge)) \subset \mathcal{H}^{s,\delta}(\mathbb{R}_+ \times M)$$

for every $\varphi(r,\phi) \in C_0^\infty(\mathbb{R}_+ \times \operatorname{int} S^1_+)$ and any 2-dimensional closed compact C^∞ manifold M such that $\operatorname{supp} \varphi$ with respect to ϕ is contained in a coordinate neighbourhood on M. Similar properties hold for the corner spaces with $\mathbb{R}_\pm$ instead of $(S^1_+)^\wedge$.

The latter compatibility properties allow us to form the global weighted corner Sobolev spaces on the stretched corners. On I_+ we choose a partition of unity consisting of functions $\{\omega_1, \omega_2\}$ such that $\omega_1 + \omega_2 = 1$, $\omega_1 \equiv 1$ near 0, $\omega_2 \equiv 1$ near α, and set

$$\begin{aligned}\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{R}_+ \times I_+) := \{\omega_1 g_1 + \omega_2 g_2 : g_1 \in \mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}(\mathbb{R}_+)),\\ g_2 \in \mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}(\mathbb{R}_-))\}\end{aligned}$$

when we identify a neighbourhood of $0\,(\alpha)$ in I_+ with $\overline{\mathbb{R}}_+\,(\overline{\mathbb{R}}_-)$.

In order to define corresponding corner spaces on X near v we identify a neighbourhood of $X \setminus \{v\}$ with $\mathbb{R}_+ \times \mathbb{S}^2_+$, where $\mathbb{S}^2_+$, as above, means the half-sphere S^2_+ in which the subdivision of $\partial S^2_+ = S^1$ into the intervals $I_\pm$ is kept in mind. For S^2_+ as a 'usual' compact C^∞ manifold with boundary we have the weighted Sobolev spaces $\mathcal{H}^{s,\delta}((S^2_+)^\wedge)$, cf. the formula (77). On S^1 we fix the points v_1, v_2 as conical points. Choose an open covering of S^2_+ by neighbourhoods $\{U_0, U_1, U_2\}$ such that $v_i \in U_i$ for $i = 1, 2, U_0 \cap (\{v_1\} \cup \{v_2\}) = \emptyset$, and U_i diffeomorphic to the half-plane through v_i that is orthogonal to the line containing v_i and the origin of $\mathbb{R}^3$. The construction of corner spaces (29) may be applied to $U_i \setminus \{v_i\} \cong \mathbb{R}_+ \times (S^1_+)^\wedge, i = 1, 2$. This gives us spaces that we denote by $\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{R}_+ \times (S^1_+)^\wedge)_{1,2}$.

Let us choose a partition of unity $\{\varphi_0, \varphi_1, \varphi_2\}$ on $\mathbb{S}^2_+$ where φ_i are supported by U_i and equal to 1 in a neighbourhood of v_i for $i = 1, 2$, and $\varphi_0 := 1 - (\varphi_1 + \varphi_2)$. Then we define

$$\begin{aligned}\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{R}_+ \times \mathbb{S}^2_+) := \{\varphi_0 u_0 + \varphi_1 u_1 + \varphi_2 u_2 : u_0 \in \mathcal{H}^{s,\delta}((S^2_+)^\wedge),\\ u_i \in \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{R}_+ \times (S^1_+)^\wedge)_i \quad \text{for} \quad i = 1, 2\}.\end{aligned} \tag{30}$$

In the latter expression the spaces for $i = 1, 2$ are combined with the pull backs under the charts $U_i \setminus \{v_i\} \cong \mathbb{R}_+ \times (S^1_+)^\wedge$.

We now define global corner Sobolev spaces $\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X})$ and $\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{Y}_\pm)$. First recall that in Section 1.3 we have fixed neighbourhoods U_0 of v and singular charts (18). Moreover, we have the spaces $\mathcal{W}^{s,\gamma}_{\mathrm{loc}}(\mathbb{W})$ for $W := X \setminus \{v\}$. In the following definition we tacitly identify $\mathcal{W}^{s,\gamma}_{\mathrm{loc}}(\mathbb{W})$ with a space of distributions on $X \setminus Z$.

Definition 2.3 *Let $s, \gamma, \delta \in \mathbb{R}$.*

(i) *We define*

$$\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X}) := \{u \in \mathcal{W}^{s,\gamma}_{\mathrm{loc}}(\mathbb{W}) : \omega u \in (\chi|_{U_0\setminus\{v\}})^* \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{R}_+ \times \mathbb{S}^2_+)\}$$

for any fixed $\omega \in C_0^\infty(U_0)$ which is equal to 1 *near the point v, cf.* (18).

(ii) *We set*

$$\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{Y}_\pm) := \{u \in \mathcal{W}^{s,\gamma}_{\mathrm{loc}}(Y_{\pm,\mathrm{reg}}) : \omega' u \in (\chi_\pm|_{U_\pm\setminus\{v\}})^* \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{R}_+ \times I_\pm)\}$$

for any fixed $\omega' \in C_0^\infty(U_0 \cap Y)$, which is equal to 1 *near the point v.*

(iii) *By $\mathcal{H}^{s,\delta}(Z)$ we define the set of all $u \in H^s_{\mathrm{loc}}(Z \setminus \{v\})$ such that $\omega_\pm u \in (\chi''_{0,\pm})^* \mathcal{H}^{s,\delta}(\mathbb{R}_\pm)$ for cut-off functions $\omega_\pm$ on $(\chi''_0)^{-1} L_\pm$, cf. the formulas* (33), (34) *below. $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$ for $s \in \mathbb{N}$ is the set of all $u(r) \in r^\delta L^2(\mathbb{R}_+)$ such that $(r\partial_r)^j u \in r^\delta L^2(\mathbb{R}_+)$ for all $0 \le j \le s$; for $-s \in \mathbb{N}$ we define the space by duality and for $s \in \mathbb{R}$ by interpolation. The definition on $\mathbb{R}_-$ is analogous.*

Let

$$\{U_0, U_1, \ldots, U_K, U_{K+1}, \ldots, U_L, U_{L+1}, \ldots, U_N\} \tag{31}$$

be an open covering of X by coordinate neighbourhoods with the following properties: U_0 is a neighbourhood of the corner point v, and $v \notin U_j$, $j > 0$; $U_j \cap Z \neq \emptyset$, $1 \le j \le K$; $U_j \cap Z = \emptyset, U_j \cap Y \neq \emptyset$, $K+1 \le j \le L$; $U_j \cap Y = \emptyset$, $L+1 \le j \le N$.

Moreover, we set $U'_j := U_j \cap Y, \quad 0 \le j \le L, \quad U''_j := U_j \cap Z, \quad 0 \le j \le K$.

Choose a partition of unity $\{\varphi_j\}_{0\le j\le N}$ subordinate to (31) and functions $\{\psi_j\}_{0\le j\le N}, \psi_j \in C_0^\infty(U_j)$, such that $\psi_j \equiv 1$ on $\operatorname{supp}\varphi_j$ for all $j = 0, \ldots, N$. Moreover, let $\varphi'_j := \varphi_j|_{U'_j}, \psi'_j := \psi_j|_{U'_j}, 0 \le j \le L$, $\varphi'_j = \psi'_j = 0, L+1 \le j \le N$, $\varphi''_j := \varphi_j|_{U''_j}, \psi''_j := \psi_j|_{U''_j}, 0 \le j \le K$, $\varphi''_j = \psi''_j = 0, K+1 \le j \le N$.

We now choose convenient charts on the open sets U_j, namely,

$$\chi_j : U_j \to \mathbb{R}^3, \quad L+1 \le j \le N, \quad \chi_j : U_j \to \overline{\mathbb{R}}^3_+, \quad 0 \le j \le L,$$

where $\chi'_j := \chi_j|_{U'_j} : U'_j \to \mathbb{R}^2_{x_1,x_2}$ are charts on $Y \setminus \{v\}$, $1 \le j \le L$, such that $\chi''_j := \chi_j|_{U''_j} : U''_j \to \mathbb{R}_{x_1}, 1 \le j \le K$.

Finally, $\chi_0 : U_0 \to \overline{\mathbb{R}}^3_+$ is assumed to be a homeomorphism, $\chi_0(v) = 0$, such that

$$\chi_0|_{U_0\setminus\{v\}} : U_0 \setminus \{v\} \to \overline{\mathbb{R}}^3_+ \setminus \{0\} \tag{32}$$

is an isomorphism in the category of manifolds with edge and boundary, and χ_0 restricts to homeomorphisms

$$\chi_0' := \chi_0|_{U_0'} : U_0' \to \mathbb{R}^2, \quad \chi_0'' := \chi_0|_{U_0''} : U_0'' \to L_1 \cup L_2, \tag{33}$$

where χ_0' is an isomorphism in the category of manifolds with corner (and without boundary) and χ_0'' an isomorphism between the respective manifolds with conical singularities. Then (32) restricts to homeomorphisms

$$\chi_0'|_{U_0'\setminus\{v\}} : U_0' \setminus \{v\} \to \mathbb{R}^2 \setminus \{0\},$$
$$\chi_{0,\pm}' := \chi_0'|_{(U_0'\setminus\{v\})\cap Y_\pm} : (U_0' \setminus \{v\}) \cap Y_\pm \to \mathbb{R}_+ \times I_\pm = I_\pm^\wedge,$$

that are also isomorphisms in the respective categories (especially, $\chi_{0,\pm}'$ are diffeomorphisms between C^∞ manifolds with boundary). Finally,

$$\chi_0''|_{U_0''\setminus\{v\}} : U_0'' \setminus \{v\} \to (L_1 \cup L_2) \setminus \{0\},$$
$$\chi_{0,k}'' := \chi_0''|_{(U_0''\setminus\{v\})\cap L_k} : (U_0'' \setminus \{v\}) \cap Z_k \to L_k \setminus \{0\} \tag{34}$$

for $Z_k := (\chi_0''|_{U_0''\setminus\{v\}})^{-1}(L_k \setminus \{0\}), k = 1, 2$.

The mixed problem (1), represented as a column matrix (24), can be decomposed in the form $\mathcal{A} = \sum_{j=0}^{N} {}^{\mathrm{t}}(\varphi_j A \psi_j \quad \varphi_{-,j}' T_- \psi_j \quad \varphi_{+,j}' T_+ \psi_j)$, where $\varphi_{\mp,j}' := \varphi_j'|_{Y_\mp}$. As is shown in [3] the operator

$$\sum_{j=1}^{N} {}^{\mathrm{t}}(\varphi_j A \psi_j \quad \varphi_{-,j}' T_- \psi_j \quad \varphi_{+,j}' T_+ \psi_j) : \mathcal{W}_{\mathrm{comp}}^{s,\gamma}(\mathbb{W}) \to \tilde{\mathcal{W}}_{\mathrm{comp}}^{s-2,\gamma-2}(\mathbb{W})$$

is continuous, cf. the formulas (21) and (22). For the continuity (3) it remains to show that ${}^{\mathrm{t}}(\varphi_0 A \psi_0 \quad \varphi_{-,0}' T_- \psi_0 \quad \varphi_{+,0}' T_+ \psi_0)$ is continuous in our spaces. This will follow from the considerations of Section 2.2 below.

Let us separately state the continuity of restriction operators:

Theorem 2.4 *The restrictions* $\mathrm{r}_\pm$ *from* $\mathbb{X}$ *to* $\mathbb{Y}_\pm$ *induce continuos operators* $\mathrm{r}_\pm : \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X}) \to \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{Y}_\pm)$ *for all* $s, \delta, \gamma \in \mathbb{R}$, $s > \frac{1}{2}$.

Proof. The elements $u \in \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X})$ can be written as $u = \sum_{j=0}^{N} \varphi_j u$. Let us consider, for instance, r_+ (the minus case is analogous). We then have $\mathrm{r}_+ u = \mathrm{r}_+(\varphi_0 u) + \sum_{j=1}^{K} \mathrm{r}_+(\varphi_j u) + \sum_{j=K+1}^{L} \mathrm{r}_+(\varphi_j u)$. From the properties of the restriction operator between standard Sobolev spaces we see that the operators $\mathrm{r}_+\varphi_j$ for $K+1 \leq j \leq L$ have the desired continuity property with a shift of smoothness s by $\frac{1}{2}$. Analogously, as is known of edge Sobolev spaces, the operators $\mathrm{r}_+\varphi_j$ for $1 \leq j \leq K$ are continuous between those spaces, with a shift of smoothness s and weight γ by $\frac{1}{2}$. Recall that the latter continuity comes from the fact that the restriction to the boundary between edge spaces is induced by $b : \mathcal{K}^{s,\gamma}((S_+^1)^\wedge) \to \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+)$ for $s > \frac{1}{2}$, and we have $b \in S_{\mathrm{cl}}^{\frac{1}{2}}(\mathbb{R}_\tau; \mathcal{K}^{s,\gamma}((S_+^1)^\wedge), \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+))$. Thus the restriction operator r_+ is equal to $\mathrm{Op}(b)$ taken along $\mathbb{R}_t$ (which is the local variable of Z). This gives us

$$\mathrm{r}_+\mathrm{Op}(b) : \mathcal{W}^s(\mathbb{R}, \mathcal{K}^{s,\gamma}((S^1_+)^\wedge)) \to \mathcal{W}^{s-\frac{1}{2}}(\mathbb{R}, \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+)). \tag{35}$$

The conclusion for $j = 0$ is analogous when we apply the formula (30) together with (29) and Definition 2.2. The change is that instead of (35) we take $\mathrm{r}_+ : \mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}((S^1_+)^\wedge)) \to \mathcal{V}^{s-\frac{1}{2},\delta-\frac{1}{2}}(\mathbb{R}_+, \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+))$. In this case the continuity of r_+ is a consequence of Theorem 2.5 below when we first interpret b as an element of $S^{\frac{1}{2}}(\Gamma_{\frac{m+1}{2}-\delta}; \mathcal{K}^{s,\gamma}((S^1_+)^\wedge), \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+))$ (independent of the covariable w) for $m = 2$ which yields

$$\mathrm{r}_+\mathrm{op}_M^{\delta-\frac{m}{2}}(b) : \mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}((S^1_+)^\wedge))_{(m)} \to \mathcal{V}^{s-\frac{1}{2},\delta}(\mathbb{R}_+, \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+))_{(m)}$$

and the space on the right coincides with $\mathcal{V}^{s-\frac{1}{2},\delta-\frac{1}{2}}(\mathbb{R}_+, \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+))_{(m')}$ for $m' = 1$, cf. the formulas (26), (27), (28). Finally, in order to show the continuity of $\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{R}_+ \times \mathbb{S}^2_+) \to \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{R}_+ \times \mathbb{R}_+)$, cf. the formula (30), we employ the continuity of the operator of restriction $\mathcal{H}^{s,\delta}((S^2_+)^\wedge) \to \mathcal{H}^{s-\frac{1}{2},\delta-\frac{1}{2}}((S^1)^\wedge)$. □

2.2 Continuity in weighted corner Sobolev spaces

We now consider Mellin (pseudo-differential) operators in weighted corner Sobolev spaces, first in the sense of Definition 2.2. Let E and $\tilde{E}$ be Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. According to Definition 4.1 we have the symbol spaces $S^\mu_{(\mathrm{cl})}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_\beta; E, \tilde{E}) \ni f(t, t', w)$; here $\tau := \mathrm{Im}\, w$ for $w \in \Gamma_\beta$ is treated as the covariable. By $S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Gamma_\beta; E, \tilde{E})$ we understand the subspace of symbols that are smooth in t, t' up to 0. For $\beta = \frac{1}{2} - \delta$ we define

$$\mathrm{op}^\delta_M(f)u(t) := \frac{1}{2\pi i}\int\int \left(\frac{t}{t'}\right)^{-(\frac{1}{2}-\delta+i\tau)} f(t, t', \frac{1}{2} - \delta + i\tau)u(t')\frac{dt'}{t'}đ\tau$$

which defines a continuous operator $C^\infty_0(\mathbb{R}_+, E) \to C^\infty(\mathbb{R}_+, \tilde{E})$.

Theorem 2.5 *Let $f(t, t', w) \in S^\mu(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Gamma_{\frac{m+1}{2}-\delta}; E, \tilde{E})$ (where the 'dimension number' m is the same for E and $\tilde{E}$, cf. Definition 2.2). Assume that $f(t, t', w)$ does not depend on t for $|t| > c$ and on t' for $|t'| > c$ for some $c > 0$. Then $\mathrm{op}_M^{\delta-\frac{m}{2}}(f)$ extends to a continuous operator*

$$\mathrm{op}_M^{\delta-\frac{m}{2}}(f) : \mathcal{V}^{s,\delta}(\mathbb{R}_+, E) \to \mathcal{V}^{s-\mu,\delta}(\mathbb{R}_+, \tilde{E})$$

for every $s \in \mathbb{R}$.

Proof. If the Mellin symbol f is independent of t, t', then the asserted continuity is an easy consequence of Definition 2.2, cf. also the second statement of Theorem 4.4 below. The operator norm tends to zero as soon as f tends to zero in the symbol space. In the general case we can write $f = f_0 + f_1$ for a (t, t')-independent symbol f_0 and a symbol f_1 with compact support in (t, t'). In that case we can apply a tensor product argument combined with the observation that the operator of multiplication by some $\varphi(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ defines a continuous operator in $\mathcal{V}^{s,\delta}(\mathbb{R}_+, E)$ and its norm tends to zero for $\varphi \to 0$ in $C_0^\infty(\overline{\mathbb{R}}_+)$ (cf. also the technique in [5, Section 1.2.2]). $\square$

As noted before in our case we have, for instance, $E = \mathcal{K}^{s,\gamma}((S^1_+)^\wedge)$ and $\tilde{E} = \mathcal{K}^{s-\mu,\gamma-\mu}((S^1_+)^\wedge)$ with the corresponding dimension number $m = 2$.

In order to construct concrete symbols which also play a role in parametrices of our corner boundary value problems we consider an element

$$\tilde{\boldsymbol{p}}(r, t, \tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathcal{B}^{\mu,\mathrm{d}}(N; \mathbb{R}^2_{\tilde{\boldsymbol{\rho}},\tilde{\boldsymbol{\tau}}})) \tag{36}$$

for some compact C^∞ manifold N with boundary, $n = \dim N$, cf. Definition 4.5. Later on we employ the parameters in the meaning

$$\tilde{\boldsymbol{\rho}} := r\rho, \quad \tilde{\boldsymbol{\tau}} := r\tilde{\tau} \quad \text{and} \quad \tilde{\tau} := t\tau.$$

For convenience, we assume that $\tilde{\boldsymbol{p}}$ is independent of t for large t. Let

$$\boldsymbol{p}(r, t, \rho, \tilde{\tau}) := \tilde{\boldsymbol{p}}(r, t, r\rho, r\tilde{\tau}) \quad \text{and} \quad \boldsymbol{h}(r, t, \boldsymbol{w}, \tilde{\tau}) := \tilde{\boldsymbol{h}}(r, t, \boldsymbol{w}, r\tilde{\tau})$$

for $\tilde{\boldsymbol{h}}(r, t, \boldsymbol{w}, \tilde{\boldsymbol{\tau}}) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathcal{B}^{\mu,\mathrm{d}}(N; \mathbb{C} \times \mathbb{R}_{\tilde{\boldsymbol{\tau}}}))$, cf. Definition 4.9 below, that we choose in such a way that

$$\mathrm{op}_r(\boldsymbol{p})(t, \tilde{\tau}) = \mathrm{op}^\beta_{\boldsymbol{M}}(\boldsymbol{h})(t, \tilde{\tau}) \mod C^\infty(\mathbb{R}_+, \mathcal{B}^{-\infty,\mathrm{d}}(N^\wedge; \mathbb{R}_{\tilde{\tau}}))$$

for every $\beta \in \mathbb{R}$. Here $\mathrm{op}^\beta_{\boldsymbol{M}}(\cdot)$ indicates the weighted Mellin action in $r \in \mathbb{R}_+$ (while $\mathrm{op}^\beta_M(\cdot)$ denotes the Mellin action in $t \in \mathbb{R}_+$). Moreover, setting

$$\boldsymbol{p}_0(r, t, \rho, \tilde{\tau}) := \tilde{\boldsymbol{p}}(0, t, r\rho, r\tilde{\tau}) \quad \text{and} \quad \boldsymbol{h}_0(r, t, \boldsymbol{w}, \tilde{\tau}) := \tilde{\boldsymbol{h}}(0, t, \boldsymbol{w}, r\tilde{\tau})$$

we have $\mathrm{op}_r(\boldsymbol{p}_0)(t, \tilde{\tau}) = \mathrm{op}^\beta_{\boldsymbol{M}}(\boldsymbol{h}_0)(t, \tilde{\tau}) \mod C^\infty(\mathbb{R}_+, \mathcal{B}^{-\infty,\mathrm{d}}(N^\wedge; \mathbb{R}_{\tilde{\tau}}))$.

Choose cut-off functions

$$\omega(r), \tilde{\omega}(r), \tilde{\tilde{\omega}}(r) \quad \text{and} \quad \sigma(r), \tilde{\sigma}(r), \tag{37}$$

and assume that $\tilde{\omega} \equiv 1$ on $\operatorname{supp}\omega$, $\omega \equiv 1$ on $\operatorname{supp}\tilde{\tilde{\omega}}$. According to the quantisations of the edge calculus we form

$$\begin{aligned} a(t, \tilde{\tau}) \;&:=\; \sigma(r) r^{-\mu} \{\omega(r[\tilde{\tau}]) \mathrm{op}^{\gamma - \frac{n}{2}}_{\boldsymbol{M}}(\boldsymbol{h})(t, \tilde{\tau}) \tilde{\omega}(r[\tilde{\tau}]) \\ &+\; (1 - \omega(r[\tilde{\tau}])) \mathrm{op}_r(\boldsymbol{p})(t, \tilde{\tau})(1 - \tilde{\tilde{\omega}}(r[\tilde{\tau}]))\} \tilde{\sigma}(r). \end{aligned} \tag{38}$$

We set

$$\begin{aligned}\sigma_\wedge(a)(t,\tilde\tau) \;:=\;& r^{-\mu}\{\omega(r|\tilde\tau|)\mathrm{op}_{\boldsymbol{M}}^{\gamma-\frac{n}{2}}(\boldsymbol{h}_0)(t,\tilde\tau)\tilde\omega(r|\tilde\tau|)\\ +\;& (1-\omega(r|\tilde\tau|))\mathrm{op}_r(\boldsymbol{p}_0)(t,\tilde\tau)(1-\tilde{\tilde\omega}(r|\tilde\tau|))\}.\end{aligned} \tag{39}$$

Let us set $\tilde{\mathcal{K}}^{s,\gamma}(N^\wedge) := \mathcal{K}^{s,\gamma}(N^\wedge) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial N)^\wedge)$ with the group action $\boldsymbol{\kappa} := \mathrm{diag}\,(\kappa_\lambda^{(n)}, \lambda^{\frac{1}{2}}\kappa_\lambda^{(n-1)})$.

Proposition 2.6 *For every* $s,\gamma\in\mathbb{R}$ *we have*

$$a(t,\tilde\tau) \in S^\mu(\overline{\mathbb{R}}_+\times\mathbb{R}; \tilde{\mathcal{K}}^{s,\gamma}(N^\wedge), \tilde{\mathcal{K}}^{s-\mu,\gamma-\mu}(N^\wedge))_{\boldsymbol{\kappa},\boldsymbol{\kappa}},$$

cf. the formula (82) *below.*

The technique of proving Proposition 2.6 may be found in [9].

Edge quantisations of the kind (38) for closed and compact N are given in [14]. The case with boundary is treated in [9] and [13]. Many useful details on the calculus are elaborated in Krainer [11].

Another important ingredient of our corner symbolic structure are the so called smoothing Mellin symbols and Green symbols. The smoothing Mellin symbols are of the form

$$m(t,\tilde\tau) := r^{-\mu}\omega(r[\tilde\tau])\mathrm{op}_{\boldsymbol{M}}^{\gamma-\frac{n}{2}}(\boldsymbol{f})(t)\tilde\omega(r[\tilde\tau]) \tag{40}$$

for an element $\boldsymbol{f}(t,\boldsymbol{w}) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{-\infty,\mathrm{d}}(N;\Gamma_{\frac{n+1}{2}-\gamma})_\varepsilon)$ for some $\varepsilon>0$ and cut-off functions $\omega,\tilde\omega$. Setting $\mathcal{E} := \tilde{\mathcal{K}}^{s,\gamma}(N^\wedge)$ and $\tilde{\mathcal{E}} := \tilde{\mathcal{K}}^{s-\mu,\gamma-\mu}(N^\wedge)$ we have

$$m(t,\tilde\tau) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\mathbb{R}; \mathcal{E},\tilde{\mathcal{E}})_{\boldsymbol{\kappa},\boldsymbol{\kappa}} \tag{41}$$

for all $s > \mathrm{d}-\frac{1}{2}$. We set

$$\sigma_\wedge(m)(t,\tilde\tau) := r^{-\mu}\omega(r|\tilde\tau|)\mathrm{op}_{\boldsymbol{M}}^{\gamma-\frac{n}{2}}(\boldsymbol{f})(t)\tilde\omega(r|\tilde\tau|).$$

Let us now pass to Green symbols, where we employ the spaces

$$\mathcal{E} := \tilde{\mathcal{K}}^{s,\gamma}(N^\wedge)\oplus\mathbb{C}^{j_-}, \tag{42}$$

$$\mathcal{S}_\varepsilon := \mathcal{S}_\varepsilon^{\gamma-\mu}(N^\wedge)\oplus\mathcal{S}_\varepsilon^{\gamma-\mu-\frac{1}{2}}((\partial N)^\wedge)\oplus\mathbb{C}^{j_+}, \tag{43}$$

cf. the notation in Section 4.1.

A Green symbol of order $\nu\in\mathbb{R}$ and type 0, associated with the weight data $\boldsymbol{g}_{\mathrm{cone}} = (\gamma,\gamma-\mu)$, is defined as an operator family

$$\mathfrak{g}(t,\tilde\tau) \in C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}, \mathcal{L}(\mathcal{E},\mathcal{S}_\varepsilon)), \tag{44}$$

$\varepsilon=\varepsilon(\mathfrak{g})>0, s>-\frac{1}{2}$, such that

$$\mathfrak{g}(t,\tilde\tau) \in S^\nu_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\mathbb{R}; \mathcal{E},\mathcal{S}_\varepsilon)_{\boldsymbol{\chi},\tilde{\boldsymbol{\chi}}} \tag{45}$$

for $\boldsymbol{\chi} := \boldsymbol{\kappa} \oplus \lambda^{\frac{n+1}{2}} \operatorname{id}_{\mathbb{C}^{j_-}}$, $\tilde{\boldsymbol{\chi}} := \boldsymbol{\kappa} \oplus \lambda^{\frac{n+1}{2}} \operatorname{id}_{\mathbb{C}^{j_+}}$; a similar property is required for $\mathfrak{g}^*(t, \tilde{\tau})$, the pointwise adjoint.

A family (44) is called a Green symbol of order $\nu \in \mathbb{R}$ and type $\mathrm{d} \in \mathbb{N}$ if it has the form $\mathfrak{g}(t, \tilde{\tau}) = \mathfrak{g}_0(t, \tilde{\tau}) + \sum_{j=1}^{\mathrm{d}} \mathfrak{g}_j(t, \tilde{\tau}) \operatorname{diag}(D^j, 0, 0)$ for a first order differential operator D on N which is defined in a neighbourhood of ∂N by the normal vector field to ∂N, and Green symbols $\mathfrak{g}_j(t, \tilde{\tau})$ of order ν and type 0 for $j = 0, \ldots, \mathrm{d}$.

Let $\mathfrak{R}_G^{\nu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$ denote the space of all such Green symbols. In particular, we have the space $\mathfrak{R}_G^{-\infty,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$.

For a $\mathfrak{g}(t, \tilde{\tau}) \in \mathfrak{R}_G^{\nu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$ let $\sigma_\wedge(\mathfrak{g})(t, \tilde{\tau})$ denote the matrix of homogeneous principal symbols.

Definition 2.7 *By $\mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$, $\mu \in \mathbb{Z}, \mathrm{d} \in \mathbb{N}$, we denote the space of all families of operators*

$$\mathfrak{a}(t, \tilde{\tau}) := \mathfrak{p}(t, \tilde{\tau}) + \mathfrak{m}(t, \tilde{\tau}) + \mathfrak{g}(t, \tilde{\tau}) : \mathcal{E} \to \tilde{\mathcal{E}}$$

for $\mathcal{E} := \tilde{\mathcal{K}}^{s,\gamma}(N^\wedge) \oplus \mathbb{C}^{j_-}$ as before, cf. the formula (42), *and*

$$\tilde{\mathcal{E}} := \tilde{\mathcal{K}}^{s-\mu,\gamma-\mu}(N^\wedge) \oplus \mathbb{C}^{j_+}, \tag{46}$$

$s > \mathrm{d} - \frac{1}{2}$, *such that* $\mathfrak{p}(t, \tilde{\tau}) = \begin{pmatrix} a(t, \tilde{\tau}) & 0 \\ 0 & 0 \end{pmatrix}$, $\mathfrak{m}(t, \tilde{\tau}) = \begin{pmatrix} m(t, \tilde{\tau}) & 0 \\ 0 & 0 \end{pmatrix}$, *with the 2×2 upper left corners* (38) *and* (40), *respectively, and an element $\mathfrak{g}(t, \tilde{\tau}) \in \mathfrak{R}_G^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$ (with $\varepsilon > 0$ being involved in the corresponding spaces).*

Let $\mathfrak{R}_{(G)}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+) := \bigcup_{\varepsilon > 0} \mathfrak{R}_{(G)}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$ (subscript '(G)' indicates Green or general operator families).

Moreover, $\mathfrak{R}_{(G)}^{\mu,\mathrm{d}}(\mathbb{R}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ denotes the corresponding elements which are independent of t.

The operator families $\mathfrak{a}(t, \tilde{\tau})$ have a principal symbolic structure $\sigma(\mathfrak{a}) := (\sigma_\psi(\mathfrak{a}), \sigma_\partial(\mathfrak{a}), \sigma_\wedge(\mathfrak{a}))$, consisting of the interior symbol $\sigma_\psi(\mathfrak{a})$, the boundary symbol $\sigma_\partial(\mathfrak{a})$, and the edge symbol $\sigma_\wedge(\mathfrak{a})$. The interior symbol is determined by the upper left corner $a(t, \tilde{\tau})$ of $\mathfrak{p}(t, \tilde{\tau})$, and defined as

$$\sigma_\psi(\mathfrak{a})(r, t, x, \rho, \tau, \xi) = r^{-\mu} \sigma_\psi(\tilde{\boldsymbol{p}})(r, t, x, \tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi)|_{\tilde{\boldsymbol{\rho}} = r\rho, \tilde{\boldsymbol{\tau}} = rt\tau} \tag{47}$$

where $\sigma_\psi(\tilde{\boldsymbol{p}})$ means the parameter-dependent homogeneous principal symbol of (36) of order μ, with the parameters $\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}$, in local coordinates x on N with the covariables ξ.

The boundary symbol is also determined by $\mathfrak{p}(t, \tilde{\tau})$, namely, as

$$\sigma_\partial(\mathfrak{a})(r, t, x', \rho, \tau, \xi') = r^{-\mu} \sigma_\partial(\tilde{\boldsymbol{p}})(r, t, x', \tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi')|_{\tilde{\boldsymbol{\rho}} = r\rho, \tilde{\boldsymbol{\tau}} = rt\tau} \tag{48}$$

with the parameter-dependent homogeneous principal boundary symbol of (36) of order μ, with the parameters $\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}$, in local coordinates x' on ∂N with the covariables ξ'.

The edge symbol is defined as

$$\sigma_\wedge(\mathfrak{a})(t,\tilde{\tau}) := \begin{pmatrix} \sigma_\wedge(a)(t,\tilde{\tau}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma_\wedge(m)(t,\tilde{\tau}) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\wedge(\mathfrak{g})(t,\tilde{\tau}),$$

$(t,\tilde{\tau}) \in \overline{\mathbb{R}}_+ \times (\mathbb{R}\setminus\{0\})$.

Theorem 2.8 *Let* $\mathfrak{a}(t,\tilde{\tau}) \in \mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_{\tilde{\tau}}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ *be independent of* t *for large* t, *and form* $\mathfrak{f}(t,w) := \mathfrak{a}(t,\tilde{\tau})$ *for* $\tilde{\tau} = \operatorname{Im} w, w \in \Gamma_{\frac{n+2}{2}-\delta}$. *Then* $\mathrm{op}_M^{\delta-\frac{n+1}{2}}(\mathfrak{f}) : \mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{E})_{(n+1)} \to \mathcal{V}^{s-\mu,\delta}(\mathbb{R}_+, \tilde{\mathcal{E}})_{(n+1)}$ *is continuous for every* $s > \mathrm{d} - \frac{1}{2}$.

Theorem 2.8 can be regarded as a Mellin version of the second continuity property of Theorem 4.4 below, also here applied to operator-valued symbols with twisted homogeneity, cf. Proposition 2.6 and the formulas (41), (45).

Remark 2.9 *According to the identity* (26) *the space* $\mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{E})_{(n+1)}$ *is equal to*

$$\mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}(N^\wedge))_{(n+1)} \oplus \mathcal{V}^{s,\delta-\frac{1}{2}}(\mathbb{R}_+, \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial N)^\wedge))_{(n)} \oplus \mathcal{V}^{s,\delta-\frac{n+1}{2}}(\mathbb{R}_+, \mathbb{C}^{j_-})_{(0)}.$$

Theorem 2.8 *can be modified to a continuity between spaces which are obtained by replacing* s *in the second and third component by* $s-\frac{1}{2}$ *and* $s-\frac{n+1}{2}$, *respectively. This will be the smoothness convention in our spaces in the applications below, cf. also the formula* (4) *where* $n = 1$.

2.3 Holomorphic corner symbols

We now turn to a category of Mellin amplitude functions in corner axis direction $t \in \mathbb{R}_+$ that are holomorphic in the complex Mellin covariable w.

Below $\mathcal{A}(D, E)$ for an open set $D \subseteq \mathbb{C}$ and a Fréchet space E denotes the space of all holomorphic functions in D with values in E.

Definition 2.10 *By* $\mathfrak{R}^{\mu,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ *for* $\mu \in \mathbb{Z}, \mathrm{d} \in \mathbb{N}$, *we denote the space of all* $\mathfrak{h}(w) \in \mathcal{A}(\mathbb{C}, \mathcal{L}(\mathcal{E}, \tilde{\mathcal{E}}))$ *with the spaces* (42), (46), $s > \mathrm{d} - \frac{1}{2}$, *such that* $\mathfrak{h}(\delta + i\tilde{\tau}) \in \mathfrak{R}^{\mu,\mathrm{d}}(\mathbb{R}_{\tilde{\tau}}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ *for every* $\delta \in \mathbb{R}$, *uniformly in compact intervals. In a similar way we define the space* $\mathfrak{R}_G^{\mu,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$.

Remark 2.11 *Let* $\mathfrak{R}^{-\infty,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$ *defined to be the space of all* $\mathfrak{c}(w) \in \mathcal{A}(\mathbb{C}, \mathcal{L}(\mathcal{E}, \mathcal{S}_\varepsilon))$ *with the above spaces* (42), (43), *such that* $\mathfrak{c}(\delta + i\tilde{\tau}) \in$

$\mathcal{S}(\mathbb{R}_{\tilde{\tau}}, \mathcal{L}(\mathcal{E}, \mathcal{S}_\varepsilon))$ *holds for every* $\delta \in \mathbb{R}$, *uniformly in compact intervals, and a similar condition holds for* $\mathfrak{c}^*(w)$.

The space $\mathfrak{R}^{-\infty,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$ *is Fréchet, and we set*

$$\begin{aligned}\mathfrak{R}^{-\infty,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon &:= C^\infty(\overline{\mathbb{R}}_+, \mathfrak{R}^{-\infty,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon),\\ \mathfrak{R}^{-\infty,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+) &:= \bigcup_{\varepsilon>0} \mathfrak{R}^{-\infty,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon.\end{aligned}$$

Then we have $\mathfrak{R}^{-\infty,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+) \subset \mathfrak{R}^{\mu,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$, but the left hand side does not coincide with $\bigcap_{\mu\in\mathbb{Z}} \mathfrak{R}^{\mu,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ since $\boldsymbol{g}_{\mathrm{cone}} = (\gamma, \gamma - \mu)$ is the same on both sides of the inclusion.

Remark 2.12 *Definition* 2.10 *can also be specified for a fixed* $\varepsilon > 0$ *as in Definition* 2.7 *before; then the spaces in Definition* 2.10 *are the union of all those spaces over* $\varepsilon > 0$. *In addition we can form*

$$\mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+) := C^\infty(\overline{\mathbb{R}}_+, \mathfrak{R}^{\mu,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)).$$

The operator families $\mathfrak{a}(t, \tilde{\tau})$ of Definition 2.7 belong to

$$S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{R}_{\tilde{\tau}}; \mathcal{E}, \tilde{\mathcal{E}})_{\boldsymbol{\chi},\tilde{\boldsymbol{\chi}}} \tag{49}$$

with respect to the spaces (42), (46) and the group actions $\boldsymbol{\chi}, \tilde{\boldsymbol{\chi}}$ mentioned before. In this situation we have kernel cut-off operations that produce holomorphic families. Kernel cut-off in corner degenerate situations for the case without boundary are studied in [7]. In the version of boundary value problems as is employed here the details may be found in [13] and [4]. The process in general is as follows.

Given a symbol $\mathfrak{a}(t, \tilde{\tau})$ in the space (49) with $\tilde{\tau}$ being interpreted as the covariable on $\Gamma_0 \subset \mathbb{C}$ we form $\mathfrak{k}(\mathfrak{a})(t, b) := \int_{\mathbb{R}} b^{-i\tilde{\tau}} \mathfrak{a}(t, \tilde{\tau}) đ\tilde{\tau}$. Then, for any $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi(b) = 1$ in a neighbourhood of $b = 1$ we set

$$\mathfrak{h}(t, w) := H(\psi)\mathfrak{a}(t, w) := \int_0^\infty b^{w-1}\psi(b)\mathfrak{k}(\mathfrak{a})(t, b)db.$$

We then obtain $\mathfrak{h}(t, w) \in \mathcal{A}(\mathbb{C}, C^\infty(\overline{\mathbb{R}}_+, \mathcal{L}(\mathcal{E}, \tilde{\mathcal{E}}))$ and $\mathfrak{h}(t, \delta+i\tilde{\tau})$ in the symbol space (49) for every $\delta \in \mathbb{R}$, uniformly in compact intervals.

Remark 2.13 *For* $\mu \in \mathbb{Z}, \mathrm{d} \in \mathbb{N}$, *we have* $\mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+) = \{\mathfrak{h}(t, w) + \mathfrak{c}(t, w) : \mathfrak{h}(t, w) = H(\psi)\mathfrak{a}(t, w)$ *for arbitrary* $\mathfrak{a}(t, \tilde{\tau}) \in \mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_{\tilde{\tau}}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ *and* $\mathfrak{c}(t, w) \in \mathfrak{R}^{-\infty,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)\}$.

Theorem 2.14 (i) *For every* $\tilde{\mathfrak{a}}(t, \tilde{\tau}) \in \mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_{\tilde{\tau}}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ *and every* $\beta \in \mathbb{R}$ *there exists an* $\mathfrak{h}(t, w) \in \mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+ \times \mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ *such that for* $\mathfrak{a}(t, \tau) := \tilde{\mathfrak{a}}(t, t\tau)$ *we have* $\mathrm{op}_t(\mathfrak{a}) = \mathrm{op}_M^{\beta-\frac{n+1}{2}}(\mathfrak{h})$ *modulo a smoothing operator as defined at the end of Section* 4.5 *below.*

(ii) *We have* $\sigma_\wedge(\tilde{\mathfrak{a}})(t,-\tau)=\sigma_\wedge(\mathfrak{h})(t,\delta+i\tau)$ *for* $\tilde{\mathfrak{a}}$ *and* $\mathfrak{h}$ *as in* (i) *for every* $\delta\in\mathbb{R}$.

The proof of Theorem 2.14 is based on the kernel cut-off technique combined with a Mellin operator convention as is developed in the present variant in [7], see also [4] and [13] for the case of boundary value problems.

Let us call $\tilde{\mathfrak{a}}(t,\tilde{\tau})\in\mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+\times\mathbb{R},\boldsymbol{g}_{\mathrm{cone}};j_-,j_+)$ elliptic if $\sigma_\psi(\tilde{\mathfrak{a}})$ is elliptic, i.e., $\sigma_\psi(\mathfrak{a})$ is non-vanishing on $T^*(\mathbb{R}_+\times\mathbb{R}_+\times N)\setminus 0$ and (in the notation (47)) $\sigma_\psi(\tilde{\boldsymbol{p}})(r,t,x,\tilde{\boldsymbol{\rho}},\tilde{\boldsymbol{\tau}},\xi)\neq 0$ for $(\tilde{\boldsymbol{\rho}},\tilde{\boldsymbol{\tau}},\xi)\neq 0$, up to $r=t=0$, and, moreover, if $\sigma_\partial(\mathfrak{a})$ is a family of isomorphisms in the sense of (87) below, parametrised by $(r,t,x',\rho,\tau,\xi')\in T^*(\mathbb{R}_+\times\mathbb{R}_+\times\partial N)\setminus 0$, and (in the notation (48)) $\sigma_\partial(\tilde{\boldsymbol{p}})(r,t,x',\tilde{\boldsymbol{\rho}},\tilde{\boldsymbol{\tau}},\xi')$ are isomorphisms for $(\tilde{\boldsymbol{\rho}},\tilde{\boldsymbol{\tau}},\xi')\neq 0$ up to $r=t=0$; finally, $\sigma_\wedge(\tilde{\mathfrak{a}})(t,\tilde{\tau}):\mathcal{E}\to\tilde{\mathcal{E}}$ is a family of isomorphisms for all $(t,\tilde{\tau})\in\overline{\mathbb{R}}_+\times(\mathbb{R}\setminus\{0\})$.

Remark 2.15 *Let* $\tilde{\mathfrak{a}}(t,\tilde{\tau})\in\mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+\times\mathbb{R},\boldsymbol{g}_{\mathrm{cone}};j_-,j_+)$ *be elliptic. Then also* $\mathfrak{h}(t,w)\in\mathfrak{M}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_+\times\mathbb{C},\boldsymbol{g}_{\mathrm{cone}};j_-,j_+)$*, associated with* $\tilde{\mathfrak{a}}$ *as in Theorem* 2.14*, is elliptic in the sense that* $\mathfrak{h}(t,\delta+i\tilde{\tau})$ *is elliptic in* $\mathfrak{R}^{\mu,\mathrm{d}}(\overline{\mathbb{R}}_t\times\mathbb{R}_{\tilde{\tau}},\boldsymbol{g}_{\mathrm{cone}};j_-,j_+)$ *for some* $\delta\in\mathbb{R}$. *This property is independent of* δ *and satisfied for all* $\delta\in\mathbb{R}$.

2.4 Corner boundary value problems

Let D be a manifold with conical singularity v and boundary; in particular, $D\setminus\{v\}$ is a C^∞ manifold with boundary. If D locally near v is written as $(\overline{\mathbb{R}}_+\times N)/(\{0\}\times N)$ for a compact C^∞ manifold N with smooth boundary, $n=\dim N$, we can pass to the double B which is a manifold with conical singularity v without boundary, locally near v identified with $(\overline{\mathbb{R}}_+\times 2N)/(\{0\}\times 2N)$, and we then have the stretched manifold $\mathbb{B}$ with $\mathbb{B}_{\mathrm{reg}}:=\mathbb{B}\setminus\partial\mathbb{B}$, $\mathbb{B}_{\mathrm{sing}}:=\partial\mathbb{B}$. In this case $\mathbb{B}$ can be written as the double $2\mathbb{D}$ of a space $\mathbb{D}$ which plays the role of the stretched manifold of D with subsets

$$\mathbb{D}_{\mathrm{reg}}:=\mathbb{D}\cap\mathbb{B}_{\mathrm{reg}},\quad \mathbb{D}_{\mathrm{sing}}:=\mathbb{D}\cap\mathbb{B}_{\mathrm{sing}}.$$

In particular, $\mathbb{D}_{\mathrm{reg}}$ is a C^∞ manifold with boundary $\cong\partial(D\setminus\{v\})$ and $\mathbb{D}_{\mathrm{sing}}$ is diffeomorphic to N, the base of the local cone near v. Observe that a stretched manifold $\mathbb{D}$ with conical singularities and boundary can also be doubled up by gluing together two copies $\mathbb{D}_-$ and $\mathbb{D}_+$ to a C^∞ manifold $\tilde{\mathbb{D}}$ with boundary by identifying $\mathbb{D}_{-,\mathrm{sing}}$ and $\mathbb{D}_{+,\mathrm{sing}}$ (observe that this operation has nothing to do with $2\mathbb{D}$, the double of $\mathbb{D}$ obtained by identifying $\mathbb{D}_-$ and $\mathbb{D}_+$ along $\partial(\mathbb{D}_{\mp,\mathrm{reg}})\cup\partial\mathbb{D}_{\mp,\mathrm{sing}}$).

We then have the spaces $\mathcal{V}^{s,\delta}(\mathbb{R}_+,\mathcal{K}^{s,\gamma}(N^\wedge))$ and $\mathcal{H}^{s,\delta}(\mathbb{R}_+\times\tilde{\mathbb{D}})$ for every $s,\gamma,\delta\in\mathbb{R}$. Let us fix a cut-off function $\omega\in C^\infty(\mathbb{B})$ which is equal to 1 in a collar neighbourhood of $\mathbb{B}_{\mathrm{sing}}$ and write ω also for the restriction to

$\mathbb{D}$. The function $1-\omega$ will also be interpreted as a function on $\tilde{\mathbb{D}}$ vanishing in a neighbourhood of $\mathbb{D}_-$.

Let us set

$$\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{D}^\wedge) := \{\omega v + (1-\omega)u : v \in \mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}(N^\wedge)), u \in \mathcal{H}^{s,\delta}(\mathbb{R}_+ \times \tilde{\mathbb{D}})\}.$$

In a similar manner we can form the spaces $\mathcal{V}^{s,(\gamma,\delta)}((\partial\mathbb{D})^\wedge)$, based on $\mathcal{V}^{s,\delta}(\mathbb{R}_+, \mathcal{K}^{s,\gamma}((\partial N)^\wedge))$ and $\mathcal{H}^{s,\delta}(\mathbb{R}_+ \times \partial\tilde{\mathbb{D}})$, respectively. Here $\partial\mathbb{D}$ is the stretched manifold belonging to $\partial(D \setminus \{v\}) \cup \{v\}$ which is a manifold with conical singularity v and without boundary. We then form

$$\tilde{\mathcal{V}}^{s,(\gamma,\delta)}(\mathbb{D}^\wedge) := \mathcal{V}^{s,(\gamma,\delta)}(\mathbb{D}^\wedge) \oplus \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}((\partial\mathbb{D})^\wedge).$$

This will be applied below to the case

$$\mathbb{D} = \mathbb{S}^2_+ \quad \text{and} \quad \partial\mathbb{D} = \mathbb{S}^1, \tag{50}$$

cf. Section 2.1. To avoid confusion let us stress once again that from the geometric point of view the conical points v_1, v_2 on ∂S^2_+ are only fictitious.

For the following definition we need operators C that are continuous as $C : \mathcal{H}^{s,\gamma}(\mathbb{D}) \oplus \mathcal{H}^{s',\gamma-\frac{1}{2}}(\partial\mathbb{D}) \oplus \mathbb{C}^{j_-} \to \mathcal{H}^{\infty,\gamma-\mu+\varepsilon}(\mathbb{D}) \oplus \mathcal{H}^{\infty,\gamma-\mu-\frac{1}{2}+\varepsilon}(\partial\mathbb{D}) \oplus \mathbb{C}^{j_+}$ for some $\varepsilon = \varepsilon(C) > 0$ and all $s, s' \in \mathbb{R}, s > -\frac{1}{2}$, such that the formal adjoint has analogous mapping properties. By $\mathcal{C}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$ for $\boldsymbol{g}_{\mathrm{cone}} = (\gamma, \gamma-\mu), \mathrm{d} \in \mathbb{N}$, we denote the space of all operators $C_0 + \sum_{j=1}^{\mathrm{d}} C_j \mathrm{diag}\,(D^j, 0, 0)$ where $C_j, 0 \le j \le \mathrm{d}$, are as described before, and D is a first order differential operator on $\mathbb{D}$ which is near $\partial\mathbb{D}_{\mathrm{reg}} \bigcup \partial\mathbb{D}_{\mathrm{sing}}$ equal to ∂_ν where ν is the normal coordinate to the boundary ∂N. The space $\mathcal{C}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon$ is Fréchet in a natural way.

Definition 2.16 (i) *By $\mathcal{M}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \Gamma_\beta)_\varepsilon$ we denote the set of all $f(w) \in \mathcal{A}(\beta-\varepsilon < \mathrm{Re}\, w < \beta+\varepsilon, \mathcal{C}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon)$ such that $f(\delta+i\tau) \in \mathcal{S}(\mathbb{R}_\tau, \mathcal{C}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)_\varepsilon)$ for every $\delta \in (\beta-\varepsilon, \beta+\varepsilon)$, uniformly in compact subintervals. Let $\mathcal{M}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \Gamma_\beta) := \bigcup_{\varepsilon>0} \mathcal{M}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \Gamma_\beta)_\varepsilon$. Moreover, we set*

$$\mathcal{M}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \mathbb{C}) := \bigcap_{\varepsilon>0} \mathcal{M}^{-\infty,\mathrm{d}}(\mathbb{D}; \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \Gamma_\beta)_\varepsilon$$

(which is, of course, independent of the choice of β).

(ii) *The space $\mathcal{M}^{\mu,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \mathbb{C})$, $\mu \in \mathbb{Z}, \mathrm{d} \in \mathbb{N}$, is defined to be the set of all operator functions $h(w) + f(w)$ where*

$$h(w) = \theta_{\mathrm{sing}} \mathfrak{h}(w) \tilde{\theta}_{\mathrm{sing}} + \theta_{\mathrm{reg}} \mathrm{diag}\,(h_{\mathrm{reg}}(w), 0) \tilde{\theta}_{\mathrm{reg}} \tag{51}$$

for arbitrary $\mathfrak{h}(w) \in \mathfrak{R}^{\mu,\mathrm{d}}(\mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+), h_{\mathrm{reg}}(w) \in \mathcal{B}^{\mu,\mathrm{d}}(\tilde{\mathbb{D}}; \mathbb{C})$, cf. Definition 4.9 (ii), and $f(w) \in \mathcal{M}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \mathbb{C})$ with a partition of unity $(\theta_{\mathrm{sing}}, \theta_{\mathrm{reg}})$ on $\mathbb{D}$, where $\theta_{\mathrm{sing}} \equiv 0$ outside a neighbourhood of $\mathbb{D}_{\mathrm{sing}}$ and $\theta_{\mathrm{sing}} \equiv 1$ near $\mathbb{D}_{\mathrm{sing}}$, moreover $\tilde{\theta}_{\mathrm{sing}}, \tilde{\theta}_{\mathrm{reg}}$ are C^∞ functions on $\mathbb{D}, \tilde{\theta}_{\mathrm{sing}}$ also supported in a neighbourhood of $\mathbb{D}_{\mathrm{sing}}$, $\mathrm{supp}\,\tilde{\theta}_{\mathrm{reg}} \cap \mathbb{D}_{\mathrm{sing}} = \emptyset$, $\tilde{\theta}_{\mathrm{sing}} \equiv 1$ on $\mathrm{supp}\,\theta_{\mathrm{sing}}$, $\tilde{\theta}_{\mathrm{reg}} \equiv 1$ on $\mathrm{supp}\,\theta_{\mathrm{reg}}$.

In the following by Φ, Ψ, etc., we denote diagonal matrices

$$\Phi := \operatorname{diag}(\varphi, \varphi, \varphi), \quad \Psi := \operatorname{diag}(\psi, \psi, \psi) \tag{52}$$

for arbitrary $\varphi, \psi \in C_0^\infty(\overline{\mathbb{R}}_+)$, acting as operators of multiplication.

By $\mathfrak{C}^{-\infty,0}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$ for $\boldsymbol{g} := (\gamma, \gamma-\mu; \delta, \delta-\mu)$ we denote the space of all $\mathfrak{G}$ that induce continuous operators

$$\Phi\mathfrak{G}\Psi : \begin{array}{c} \tilde{\mathcal{V}}^{s,(\gamma,\delta)}(\mathbb{D}^\wedge) \\ \oplus \\ \mathcal{H}^{s',\delta-\frac{n+1}{2}}(\mathbb{R}_+, \mathbb{C}^{j_-}) \end{array} \rightarrow \begin{array}{c} \tilde{\mathcal{V}}^{\infty,(\gamma-\mu+\varepsilon,\delta-\mu+\varepsilon)}(\mathbb{D}^\wedge) \\ \oplus \\ \mathcal{H}^{\infty,\delta-\mu-\frac{n+1}{2}+\varepsilon}(\mathbb{R}_+, \mathbb{C}^{j_+}) \end{array}$$

for some $\varepsilon = \varepsilon(\mathfrak{G}) > 0$ for every $s, s' \in \mathbb{R}, s > -\frac{1}{2}$, and for arbitrary Φ, Ψ of the kind (52); an analogous condition is required for the formal adjoint $\mathfrak{G}^*$ with weights $-\gamma+\mu, -\delta+\mu$ in the preimage and $-\gamma+\varepsilon, -\delta+\varepsilon$ in the image. Moreover, $\mathfrak{C}^{-\infty,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$ for $\mathrm{d} \in \mathbb{N}$ is defined to be the space of all $\mathfrak{G} = \mathfrak{G}_0 + \sum_{j=1}^{\mathrm{d}} \mathfrak{G}_j \operatorname{diag}(D^j, 0, 0)$ for arbitrary $\mathfrak{G}_j \in \mathfrak{C}^{-\infty,0}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$ and a first order differential operator D that is locally near ∂N equal to ∂_{x_n} where x_n is the normal coordinate to the boundary.

Definition 2.17 *The space of local corner operators*

$$\mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+), \quad \boldsymbol{g} = (\gamma, \gamma-\mu; \delta, \delta-\mu), \mu \in \mathbb{Z}, \mathrm{d} \in \mathbb{N},$$

is defined as the set of all operators $\mathfrak{A} = \mathfrak{L}_{\mathrm{corner}} + \mathfrak{G}$ with the following ingredients:

(i)

$$\mathfrak{L}_{\mathrm{corner}} = t^{-\mu} \mathrm{op}_M^{\delta-\frac{n+1}{2}}(h+f) \tag{53}$$

for a Mellin symbol $h(t,w) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}^{\mu,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \mathbb{C}))$ and $f(w) \in \mathcal{M}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \Gamma_{\frac{n+2}{2}-\delta})$;

(ii) $\mathfrak{G} \in \mathfrak{C}^{-\infty,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$.

Remark 2.18 *If we change the cut-off functions $\theta_{\mathrm{sing}}, \tilde{\theta}_{\mathrm{sing}}$ and $\theta_{\mathrm{reg}}, \tilde{\theta}_{\mathrm{reg}}$ in the formula* (51) *we obtain in* (53) *a remainder in $\mathfrak{C}^{-\infty,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$.*

If $f(t,w) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}^{-\infty,\mathrm{d}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+; \Gamma_{\frac{n+2}{2}-\delta}))$ vanishes at $t = 0$, then we have $t^{-\mu}\mathrm{op}_M^{\delta-\frac{n+1}{2}}(f) \in \mathfrak{C}^{-\infty,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$.

Remark 2.19 *For every $\varphi, \psi \in C_0^\infty(\overline{\mathbb{R}}_+)$ and $\mathfrak{A} \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$ we have $\Phi\mathfrak{A}, \mathfrak{A}\Psi \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$, cf. the formula* (52).

Theorem 2.20 *For every* $\mathfrak{A} \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$, $\boldsymbol{g} = (\gamma, \gamma-\mu; \delta, \delta-\mu)$, *we have continuous operators*

$$\Phi\mathfrak{A}\Psi : \begin{array}{c} \tilde{\mathcal{V}}^{s,(\gamma,\delta)}(\mathbb{D}^\wedge) \\ \oplus \\ \mathcal{H}^{s-\frac{n+1}{2},\delta-\frac{n+1}{2}}(\mathbb{R}_+, \mathbb{C}^{j_-}) \end{array} \to \begin{array}{c} \tilde{\mathcal{V}}^{s-\mu,(\gamma-\mu,\delta-\mu)}(\mathbb{D}^\wedge) \\ \oplus \\ \mathcal{H}^{s-\mu-\frac{n+1}{2},\delta-\mu-\frac{n+1}{2}}(\mathbb{R}_+, \mathbb{C}^{j_+}) \end{array}$$

for every $s \in \mathbb{R}, s > \mathrm{d} - \frac{1}{2}$, *and arbitrary* Φ, Ψ *as in* (52).

Proof. The result is an immediate consequence of Theorem 2.8 and of the mapping properties of the smoothing operators in Definition 2.17 (ii). □

Remark 2.21 *If the operator* $\mathfrak{A}$ *in Theorem* 2.20 *satisfies a suitable condition for large* t, *e.g., that* $h(t, w)$ *is independent of* t *for large* t *and the smoothing summand vanishes, then* $\mathfrak{A}$ *is continuous between the spaces without the factors* Φ *and* Ψ.

The operators $\mathfrak{A} \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$ have a principal symbolic hierarchy

$$\sigma(\mathfrak{A}) = (\sigma_\psi(\mathfrak{A}), \sigma_\partial(\mathfrak{A}), \sigma_\wedge(\mathfrak{A}), \sigma_\mathrm{c}(\mathfrak{A})) \tag{54}$$

which is as follows. Writing $\mathfrak{A} = (\mathfrak{A}_{ij})_{i,j=1,2,3}$ we have $\mathfrak{A}_{11}|_{\mathbb{R}_+\times(\mathrm{int}\,\mathbb{D}_{\mathrm{reg}})} \in L^\mu_{\mathrm{cl}}(\mathbb{R}_+\times(\mathrm{int}\,\mathbb{D}_{\mathrm{reg}}))$ with the homogeneous principal symbol $\sigma_\psi(\mathfrak{A}) := \sigma_\psi(\mathfrak{A}_{11})$ of order μ. Locally near $t = 0$ and $\mathbb{D}_{\mathrm{sing}}$ it has the form

$$\sigma_\psi(\mathfrak{A})(r, t, x, \rho, \tau, \xi) = t^{-\mu} r^{-\mu} \tilde{\boldsymbol{\sigma}}_\psi(\mathfrak{A})(r, t, x, r\rho, rt\tau, \xi)$$

for a function $\tilde{\boldsymbol{\sigma}}_\psi(\mathfrak{A})(r, t, x, \tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi)$ homogeneous of order μ in $(\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi) \neq 0$ and smooth up to $r = t = 0$.

Moreover, we have $(\mathfrak{A}_{ij})_{i,j=1,2} \in \mathcal{B}^{\mu,\mathrm{d}}(\mathbb{D}^\wedge_{\mathrm{reg}})$, and there is then a principal boundary symbol $\sigma_\partial(\mathfrak{A}) := \sigma_\partial((\mathfrak{A}_{ij})_{i,j=1,2})$ parametrised by $T^*(\partial\mathbb{D}^\wedge_{\mathrm{reg}}) \setminus 0$. Locally near $t = 0$ and $\partial\mathbb{D}^\wedge_{\mathrm{sing}}$ the boundary symbol is a family of maps

$$\sigma_\partial(\mathfrak{A})(r, t, x', \rho, \tau, \xi') = t^{-\mu} r^{-\mu} \tilde{\boldsymbol{\sigma}}_\partial(\mathfrak{A})(r, t, x', r\rho, rt\tau, \xi')$$

for a function $\tilde{\boldsymbol{\sigma}}_\partial(\mathfrak{A})(r, t, x', \tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi')$ which is κ_λ-homogeneous of order μ in $(\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi') \neq 0$ and smooth up to $r = t = 0$.

From the definition it follows that $\mathfrak{A} \in \mathfrak{Y}^{\mu,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ with $\mathbb{D}^\wedge = \mathbb{R}_+ \times \mathbb{D}$ being regarded as a (stretched) manifold with edge $\mathbb{R}_+ \ni t$, cf. Section 4.5, below. From the edge calculus the operator $\mathfrak{A}$ has a homogeneous principal edge symbol

$$\sigma_\wedge(\mathfrak{A})(t, \tau) : \tilde{\mathcal{K}}^{s,\gamma}(N^\wedge) \oplus \mathbb{C}^{j_-} \to \tilde{\mathcal{K}}^{s-\mu,\gamma-\mu}(N^\wedge) \oplus \mathbb{C}^{j_+} \tag{55}$$

for $N = \mathbb{D}_{\mathrm{sing}}$, $(t, \tau) \in T^*\mathbb{R}_+ \setminus 0$. In the present case there is an analogous operator function $\tilde{\sigma}_\wedge(\mathfrak{A})(t, \tilde{\tau})$ in $(t, \tilde{\tau}), \tilde{\tau} \neq 0$, smooth in t up to $t = 0$, such that

$$\sigma_\wedge(\mathfrak{A})(t, \tau) = t^{-\mu} \tilde{\sigma}_\wedge(\mathfrak{A})(t, t\tau).$$

Observe that when we form $\mathfrak{b}(t, -\tilde{\tau}) := \mathfrak{h}(t, \frac{n+2}{2} - \delta + i\tilde{\tau})$ we have $\sigma_\wedge(\mathfrak{b})(t, \tilde{\tau}) = \tilde{\sigma}_\wedge(\mathfrak{A})(t, \tilde{\tau})$.

Finally, to define the corner conormal symbol we set

$$\tilde{\mathcal{H}}^{s,\gamma}(\mathbb{D}) := \mathcal{H}^{s,\gamma}(\mathbb{D}) \oplus \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\partial\mathbb{D});$$

then

$$\sigma_{\mathrm{c}}(\mathfrak{A})(w) : \tilde{\mathcal{H}}^{s,\gamma}(\mathbb{D}) \oplus \mathbb{C}^{j_-} \to \tilde{\mathcal{H}}^{s-\mu,\gamma-\mu}(\mathbb{D}) \oplus \mathbb{C}^{j_+}, \tag{56}$$

$w \in \Gamma_{\frac{n+2}{2}-\delta}$, is defined by

$$\sigma_{\mathrm{c}}(\mathfrak{A})(w) = h(0, w) + f(w), \tag{57}$$

cf. the formula (53).

Theorem 2.22 *For every*

$$\mathfrak{A} \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{a}; j_0, j_+), \quad \mathfrak{B} \in \mathfrak{C}^{\nu,\mathrm{e}}(\mathbb{D}^\wedge, \boldsymbol{b}; j_-, j_0)$$

for $\mu, \nu \in \mathbb{Z}, \mathrm{d}, \mathrm{e} \in \mathbb{N}$, *and* $\boldsymbol{a} := (\gamma - \nu, \gamma - (\mu + \nu); \delta - \nu, \delta - (\mu + \nu))$, $\boldsymbol{b} := (\gamma, \gamma - \nu; \delta, \delta - \nu)$ *we have* $\mathfrak{A}\Phi\mathfrak{B}\Psi \in \mathfrak{C}^{\mu+\nu,\mathrm{h}}(\mathbb{D}^\wedge, \boldsymbol{a} \circ \boldsymbol{b}; j_-, j_+)$ *for* $\mathrm{h} = \max(\nu + \mathrm{d}, \mathrm{e}), \boldsymbol{a} \circ \boldsymbol{b} = (\gamma, \gamma - (\mu + \nu); \delta, \delta - (\mu + \nu))$ *and for every* Φ, Ψ *as in* (52)*, where*

$$\sigma(\mathfrak{A}\Phi\mathfrak{B}\Psi) = \sigma(\mathfrak{A}\Phi)\sigma(\mathfrak{B}\Psi)$$

with componentwise multiplication, and the rule for the conormal symbols

$$\sigma_{\mathrm{c}}(\mathfrak{A}\Phi\mathfrak{B}\Psi)(w) = \sigma_{\mathrm{c}}(\mathfrak{A}\Phi)(w - \nu)\sigma_{\mathrm{c}}(\mathfrak{B}\Psi)(w).$$

Theorem 2.22 states that the operator spaces of Definition 2.17 form (up to the localising factors Φ and Ψ) an algebra of boundary value problems on the stretched corner $\mathbb{D}^\wedge$. An analogous calculus for the case of corners without boundary is given in [15]. The technique of proving Theorem 2.22 is quite similar to that in the edge algebra of boundary value problems, cf. [9], here with respect to the weighted Mellin transform instead of the Fourier transform along the edge.

2.5 Ellipticity near the corner

Definition 2.23 *An operator* $\mathfrak{A}$ *as in Definition* 2.17 *is said to be elliptic with respect to the symbol* (54) *if*

(i) $\sigma_\psi(\mathfrak{A}) \neq 0$ *on* $T^*(\mathbb{R}_+ \times (\mathrm{int}\,\mathbb{D}_{\mathrm{reg}})) \setminus 0$ *and* $\tilde{\boldsymbol{\sigma}}_\psi(\mathfrak{A})(r, t, x, \tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi) \neq 0$ *for* $(\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi) \neq 0$, *up to* $r = t = 0$;

(ii) $\sigma_\partial(\mathfrak{A})$ *is bijective for all points on* $T^*(\partial\mathbb{D}^\wedge_{\mathrm{reg}})\setminus 0$, *and* $\tilde{\boldsymbol{\sigma}}_\partial(\mathfrak{A})(r, t, x', \tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi')$ *is bijective for* $(\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}, \xi') \neq 0$, *up to* $r = t = 0$;

(iii) $\sigma_\wedge(\mathfrak{A})(t,\tau)$ *defines isomorphisms* (55) *for all* $(t,\tau) \in T^*\mathbb{R}_+ \setminus 0$, *and* $\tilde{\sigma}_\wedge(\mathfrak{A})(t,\tilde{\tau})$ *defines analogous isomorphisms for* $\tilde{\tau} \neq 0$, *up to* $t = 0$;

(iv) (56) *is a family of isomorphisms for all* $w \in \Gamma_{\frac{n+2}{2}-\delta}$.

The conditions (ii)-(iv) of the latter definition are required for any $s \in \mathbb{R}, s > \max(\mu, \mathrm{d}) - \frac{1}{2}$; they are then independent of s.

Theorem 2.24 *An elliptic* $\mathfrak{A} \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{D}^\wedge, \boldsymbol{g}; j_-, j_+)$ *has a parametrix* $\mathfrak{P} \in \mathfrak{C}^{-\mu,(\mathrm{d}-\mu)^+}(\mathbb{D}^\wedge, \boldsymbol{g}^{-1}; j_+, j_-)$, $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma; \delta - \mu, \delta), \nu^+ := \max(\nu, 0)$, *in the following sense:*

$$\Phi\mathfrak{P}\tilde{\Phi}\mathfrak{A}\tilde{\tilde{\Phi}} = \Phi\mathfrak{I} \quad \text{and} \quad \Phi\mathfrak{A}\tilde{\Phi}\mathfrak{P}\tilde{\tilde{\Phi}} = \Phi\mathfrak{I}$$

modulo $\mathfrak{C}^{-\infty,\mathrm{d}_\mathrm{l}}(\mathbb{D}^\wedge, \boldsymbol{g}_\mathrm{l}; j_-, j_-)$ *and* $\mathfrak{C}^{-\infty,\mathrm{d}_\mathrm{r}}(\mathbb{D}^\wedge, \boldsymbol{g}_\mathrm{r}; j_+, j_+)$, *respectively, for* $\boldsymbol{g}_\mathrm{l} = (\gamma, \gamma; \delta, \delta), \mathrm{d}_\mathrm{l} = \max(\mu, \mathrm{d})$, $\boldsymbol{g}_\mathrm{r} = (\gamma-\mu, \gamma-\mu; \delta-\mu, \delta-\mu), \mathrm{d}_\mathrm{r} = (\mathrm{d}-\mu)^+$, *and arbitrary* $\Phi = \mathrm{diag}(\varphi, \varphi, \varphi)$, $\tilde{\Phi} = \mathrm{diag}(\tilde{\varphi}, \tilde{\varphi}, \tilde{\varphi})$, $\tilde{\tilde{\Phi}} = \mathrm{diag}(\tilde{\tilde{\varphi}}, \tilde{\tilde{\varphi}}, \tilde{\tilde{\varphi}})$ *with* $\varphi, \tilde{\varphi}, \tilde{\tilde{\varphi}} \in C_0^\infty(\mathbb{R}_+)$ *such that* $\tilde{\varphi} \equiv 1$ *on* $\mathrm{supp}\,\varphi$, $\tilde{\tilde{\varphi}} \equiv 1$ *on* $\mathrm{supp}\,\tilde{\varphi}$.

Proof. By virtue of Definition 2.17 (i) it suffices to assume that $\mathfrak{A}$ has the form (53). The ellipticity of $\mathfrak{L}_{\mathrm{corner}}$ gives us the existence of elements $h_{\mathrm{reg}}^{(-1)}(t,w) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{-\mu,\mathrm{d}_\mathrm{r}}(\tilde{\mathbb{D}}; \Gamma_{\frac{n+2}{2}-\delta}))$ and $\mathfrak{h}^{(-1)}(t,w) \in \mathfrak{R}^{-\mu,\mathrm{d}_\mathrm{r}}(\overline{\mathbb{R}}_+ \times \mathbb{C}, \boldsymbol{g}_{\mathrm{cone}}^{-1}; j_+, j_-)$ such that for

$$h^{[-1]}(t,w) := \theta_{\mathrm{sing}}\mathfrak{h}^{(-1)}(t,w)\tilde{\theta}_{\mathrm{sing}} + \theta_{\mathrm{reg}}\mathrm{diag}(h_{\mathrm{reg}}^{(-1)}(t,w), 0)\tilde{\theta}_{\mathrm{reg}}$$

we have

$$h^{[-1]}(t, w-\mu)h(t,w)|_{\Gamma_{\frac{n+2}{2}-\delta}} = 1 + f_0(t,w) \tag{58}$$

for some $f_0(t,w) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}^{-\infty,\mathrm{d}_\mathrm{l}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone,l}}; j_-, j_-; \Gamma_{\frac{n+2}{2}-\delta}))$, $\boldsymbol{g}_{\mathrm{cone,l}} = (\gamma, \gamma)$. Note that because of the holomorphy of the involved Mellin operator functions in $w \in \mathbb{C}$ this relation holds for all $\delta \in \mathbb{R}$, although we only employ it for the prescribed fixed corner weight δ. For the construction of the parametrix we also have to invert the principal conormal symbol (57). From (58) we obtain $h^{[-1]}(0, w-\mu)(h(0,w) + f(w)) = 1 + f_1(w)$ for some $f_1(w) \in \mathcal{M}^{-\infty,\mathrm{d}_\mathrm{l}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone,l}}; j_-, j_-; \Gamma_{\frac{n+2}{2}-\delta})$. We now use the fact that there exists an $f_2(w-\mu) \in \mathcal{M}^{-\infty,\mathrm{d}_\mathrm{l}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone,l}}; j_-, j_-; \Gamma_{\frac{n+2}{2}-\delta})$ such that $(1 + f_2(w-\mu))(1 + f_1(w)) = 1$. This gives us

$$(1 + f_2(w-\mu))h^{[-1]}(0, w-\mu)(h(0,w) + f(w)) = 1.$$

From $f_2(w-\mu)h^{[-1]}(0,w-\mu) =: g(w-\mu) \in \mathcal{M}^{-\infty,\mathrm{d}_\mathrm{r}}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}^{-1}; j_+, j_-; \Gamma_{\frac{n+2}{2}-\delta})$ we have $(h^{[-1]}(0, w-\mu) + g(w-\mu))(h(0,w) + f(w)) = 1$. It follows that

$\sigma_{\rm c}(\mathfrak{A})^{-1}(w) = h^{[-1]}(0, w-\mu) + g(w-\mu)$ for $w \in \Gamma_{\frac{n+2}{2}-\delta}$. We now employ the relation (58) which yields

$$h^{[-1]}(t, w-\mu) \natural h(t,w) - 1 =: h_1(t,w) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}^{-1,\mathrm{d}_1}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone},1}; j_-, j_-; \mathbb{C})).$$

Here $\natural$ denotes the Mellin-Leibniz product, cf. [12] for an analogous situation on $\mathbb{R}_+$. This allows us to construct an $h_2(t,w)$ in the same space as $h_1(t,w)$ with the property

$$(1+h_2(t,w)) \natural (1+h_1(t,w)) = 1 \mod C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}^{-\infty,\mathrm{d}_1}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone},1}; j_-, j_-; \mathbb{C})).$$

Thus, modulo a remainder of the same kind, $h^{(-1)}(t, w-\mu) \natural h(t,w) = 1$ for $h^{(-1)}(t, w-\mu) := (1 + h_2(t, w-\mu)) \natural h^{[-1]}(t, w-\mu)$ belonging to the space $C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}^{-\mu,(\mathrm{d}-\mu)^+}(\mathbb{D}, \boldsymbol{g}_{\mathrm{cone}}^{-1}; j_+, j_-; \mathbb{C}))$. Now $\mathfrak{P} = t^\mu \mathrm{op}_M^{\delta-\mu-\frac{n+1}{2}}(h^{(-1)} + g)$ is a parametrix as desired. $\square$

3 Corner-edge operators

3.1 Global corner boundary value problems

In this section we introduce a space of global boundary value problems on the (stretched) corner configuration $\mathbb{X}$, cf. Section 1.3. The most specific part comes from a (stretched) neighbourhood $(\mathbb{S}^2_+)^\wedge$ of the corner point $v \in Z$. The material on such corner operators is prepared in Section 2.5 for a general (stretched) manifold $\mathbb{D}$ with a conical singularity on the boundary. In the present case we have $\mathbb{S}^2_+ = \mathbb{D}$ with two conical points $v_1, v_2 \in S^1$. Recall that we then write $\mathbb{S}^1$ when we emphasise the presence of the conical points on the boundary of D, cf. also the formula (50). From Definition 2.17 we have the space $\mathfrak{C}^{\mu,\mathrm{d}}((\mathbb{S}^2_+)^\wedge, \boldsymbol{g}; j_-, j_+)$ of corner boundary value problems which are block matrices of operators $\mathfrak{A} = \begin{pmatrix} \mathcal{A} & \mathcal{K} \\ \mathcal{T} & \mathcal{Q} \end{pmatrix}$ that are (after a localisation by cut-off factors as in Theorem 2.20) continuous in the sense

$$\mathfrak{A} : \begin{array}{c} \tilde{\mathcal{V}}^{s,(\gamma,\delta)}((\mathbb{S}^2_+)^\wedge) \\ \oplus \\ \mathcal{H}^{s-1,\delta-1}(\mathbb{R}_+, \mathbb{C}^{j_-}) \end{array} \to \begin{array}{c} \tilde{\mathcal{V}}^{s-\mu,(\gamma-\mu,\delta-\mu)}((\mathbb{S}^2_+)^\wedge) \\ \oplus \\ \mathcal{H}^{s-\mu-1,\delta-\mu-1}(\mathbb{R}_+, \mathbb{C}^{j_+}) \end{array}, \tag{59}$$

$$\tilde{\mathcal{V}}^{s,(\gamma,\delta)}((\mathbb{S}^2_+)^\wedge) := \mathcal{V}^{s,(\gamma,\delta)}((\mathbb{S}^2_+)^\wedge) \oplus \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(I_-^\wedge) \oplus \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(I_+^\wedge).$$

For the following notation we define spaces $\tilde{\mathcal{V}}^{(s,s'),(\gamma,\delta)}((\mathbb{S}^2_+)^\wedge)$ in a similar manner as $\tilde{\mathcal{V}}^{s,(\gamma,\delta)}((\mathbb{S}^2_+)^\wedge)$ with the only difference that $s - \frac{1}{2}$ is replaced by $s' \in \mathbb{R}$. By $\mathfrak{C}^{-\infty,0}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)$ for $\boldsymbol{g} := (\gamma, \gamma-\mu; \delta, \delta-\mu)$ we denote the space of operators $\mathfrak{G}$ that are continuous as

$$\begin{aligned} \mathfrak{G} \quad : \quad & \tilde{\mathcal{V}}^{(s,s'),(\gamma,\delta)}(\mathbb{X}) \oplus \mathcal{H}^{s'',\delta-1}(Z, \mathbb{C}^{j_-}) \\ \to \quad & \tilde{\mathcal{V}}^{\infty,(\gamma-\mu+\varepsilon,\delta-\mu+\varepsilon)}(\mathbb{X}) \oplus \mathcal{H}^{\infty,\delta-\mu-1+\varepsilon}(Z, \mathbb{C}^{j_+}) \end{aligned} \tag{60}$$

for some $\varepsilon = \varepsilon(\mathfrak{G}) > 0$, for every $s, s', s'' \in \mathbb{R}$, $s > -\frac{1}{2}$; an analogous condition is required for the formal adjoint $\mathfrak{G}^*$, with weights $-\gamma + \mu, -\delta + \mu$ in the preimage and $-\gamma + \varepsilon, -\delta + \varepsilon$ in the image. Moreover, $\mathfrak{C}^{-\infty,\mathrm{d}}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)$ for $\mathrm{d} \in \mathbb{N}$ is defined to be the space of all operators

$$\mathfrak{G} = \mathfrak{G}_0 + \sum_{j=1}^{\mathrm{d}} \mathfrak{G}_j \mathrm{diag}\,(D^j, 0, 0, 0) \tag{61}$$

for arbitrary $\mathfrak{G}_j \in \mathfrak{C}^{-\infty,0}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)$ and a first order differential operator D that is locally near ∂X of the form ∂_{x_n} with x_n being the normal to the boundary.

The following definition will refer to localisations of operators on $\mathbb{X}$ near different singular strata.

Let us set (with the notation (31)) $\mathcal{U}_0 := U_0, \mathcal{U}_1 := \bigcup_{j=1}^{K} U_j, \mathcal{U}_2 := \bigcup_{j=K+1}^{N} U_j$ which form an open covering $\{\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2\}$ of X, and let $\{\varphi_0, \varphi_1, \varphi_2\}$ be a subordinate partition of unity. Moreover, let $\{\psi_0, \psi_1, \psi_2\}$ be C_0^∞ functions in the respective neighbourhoods such that $\psi_j \equiv 1$ on $\mathrm{supp}\,\varphi_j, j = 1, 2, 3$. The functions φ_j, ψ_j may also be regarded as functions on $\mathbb{X}$ when we identify them with the pull backs under the map (19). Moreover, we form 4×4 diagonal matrices

$$\Phi_{\mathrm{corner}}, \Psi_{\mathrm{corner}}, \quad \text{and} \quad \Phi_{\mathrm{edge}}, \Psi_{\mathrm{edge}}$$

defined by $\Phi_{\mathrm{corner}} := \mathrm{diag}\,(\varphi_0, \varphi_0|_{\mathrm{int}\,Y_-}, \varphi_0|_{\mathrm{int}\,Y_+}, \varphi_0|_Z)$ and, similarly, Ψ_{corner} in terms of ψ_0, furthermore, $\Phi_{\mathrm{edge}} := \mathrm{diag}\,(\varphi_1, \varphi_1|_{\mathrm{int}\,Y_-}, \varphi_1|_{\mathrm{int}\,Y_+}, \varphi_1|_Z)$ and, similarly, Ψ_{edge} in terms of ψ_1. Finally, we form $\Phi_{\mathrm{reg}} := \mathrm{diag}\,(\varphi_2, \varphi_2|_{\mathrm{int}\,Y_-}, \varphi_2|_{\mathrm{int}\,Y_+}, 0)$ and, similarly, Ψ_{reg} in terms of ψ_2.

Definition 3.1 *Let $\mathbb{X}$ be the (stretched) corner configuration as described in Section* 1.3. *Moreover, let $\mu \in \mathbb{Z}$, $\mathrm{d} \in \mathbb{N}$, and let $\boldsymbol{g} := (\gamma, \gamma - \mu; \delta, \delta - \mu)$ be weight data with a cone weight $\gamma \in \mathbb{R}$ and a corner weight $\delta \in \mathbb{R}$ (associated with the local axial variables $r \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$, respectively). Then*

$$\mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{X}, \boldsymbol{g}; j_-, j_+) \tag{62}$$

is defined to be the set of all operators

$$\mathfrak{A} = \mathfrak{A}_{\mathrm{corner}} + \mathfrak{A}_{\mathrm{edge}} + \mathfrak{A}_{\mathrm{reg}} + \mathfrak{G} \tag{63}$$

where the summands are as follows:

(i) $\mathfrak{A}_{\mathrm{corner}} := \Phi_{\mathrm{corner}} \mathfrak{L}_{\mathrm{corner}} \Psi_{\mathrm{corner}}$ *for* $\mathfrak{L}_{\mathrm{corner}} \in \mathfrak{C}^{\mu,\mathrm{d}}((\mathbb{S}^2_+)^\wedge, \boldsymbol{g}; j_-, j_+)$, *cf. Definition* 2.17 *for the case* $\mathbb{D} = \mathbb{S}^2_+$;

(ii) $\mathfrak{A}_{\mathrm{edge}} := \Phi_{\mathrm{edge}} \mathfrak{L}_{\mathrm{edge}} \Psi_{\mathrm{edge}}$ *for* $\mathfrak{L}_{\mathrm{edge}} \in \mathfrak{Y}^{\mu,\mathrm{d}}(\mathbb{U}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$, *cf. Section* 4.5 *below;*

(iii) $\mathfrak{A}_{\mathrm{reg}} := \Phi_{\mathrm{reg}}\mathfrak{L}_{\mathrm{reg}}\Psi_{\mathrm{reg}}$ *for* $\mathfrak{L}_{\mathrm{reg}} \in \mathcal{B}^{\mu,\mathrm{d}}(X \setminus Z)$;

(iv) $\mathfrak{G} \in \mathfrak{C}^{-\infty,\mathrm{d}}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)$.

Remark 3.2 *The properties* $(\mathrm{i})_{\mathrm{reg}}$, $(\mathrm{ii})_{\mathrm{reg}}$ *are not necessary for our calculus, but they simplify some constructions. If we only require* (i)-(iv) *we obtain an operator space which also admits ellipticity and the construction of parametrices. The condition* $(\mathrm{i})_{\mathrm{reg}}$ *for our singular Zaremba problem means that the Laplace operator itself is smooth across Z, while* $(\mathrm{ii})_{\mathrm{reg}}$ *says that the Dirichlet and Neumann conditions are also smoothly extendible to the whole boundary.*

Let $\tilde{\mathcal{V}}^{s,(\gamma,\delta)}(\mathbb{X})$ be defined as

$$\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X}) \oplus \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{Y}_-) \oplus \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{Y}_+),$$

cf. Definition 2.3.

Theorem 3.3 *Every* $\mathfrak{A} \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)$ *induces continuous operators*

$$\mathfrak{A} : \tilde{\mathcal{V}}^{s,(\gamma,\delta)}(\mathbb{X}) \oplus \mathcal{H}^{s-1,\delta-1}(Z, \mathbb{C}^{j_-}) \to \tilde{\mathcal{V}}^{s-\mu,(\gamma-\mu,\delta-\mu)}(\mathbb{X}) \oplus \mathcal{H}^{s-\mu-1,\delta-\mu-1}(Z, \mathbb{C}^{j_+}), \tag{64}$$

$s > \mathrm{d} - \frac{1}{2}$.

Proof. In order to prove (64) it suffices to consider the summands in (63) separately. The desired continuity of $\mathfrak{A}_{\mathrm{corner}}$ was given in Theorem 2.20. The continuity of $\mathfrak{A}_{\mathrm{edge}}$ is the same as in the edge calculus, cf. [9]. The operator $\mathfrak{A}_{\mathrm{reg}}$ corresponds to a standard pseudo-differential boundary value problem with the transmission property; thus it is continuous in standard Sobolev spaces, cf. the formula (86) below. The smoothing term $\mathfrak{G}$ is continuous by the properties (60), (61). $\square$

We now turn to the global principal symbolic structure of $\mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)$ which consists of 4 components, namely,

$$\sigma(\mathfrak{A}) := (\sigma_\psi(\mathfrak{A}), \sigma_\partial(\mathfrak{A}), \sigma_\wedge(\mathfrak{A}), \sigma_{\mathrm{c}}(\mathfrak{A})). \tag{65}$$

Writing $\mathfrak{A} = (\mathfrak{A}_{ij})_{i,j=1,\dots,4}$ from Definition 3.1 it follows that $\mathfrak{A}_{11} \in L^\mu_{\mathrm{cl}}(\mathrm{int}\, X)$. Thus we have $\sigma_\psi(\mathfrak{A}) := \sigma_\psi(\mathfrak{A}_{11})$, the standard homogeneous principal symbol of order μ as a function on $T^*(\mathrm{int}\, X) \setminus 0$.

Close to $Z_{\mathrm{reg}} = Z \setminus \{v\}$ in local coordinates (r, ϕ, z), cf. (20), and covariables (ρ, ϑ, ζ) we have

$$\sigma_\psi(\mathfrak{A})(r, \phi, z, \rho, \vartheta, \zeta) = r^{-\mu}\tilde{\sigma}_\psi(\mathfrak{A})(r, \phi, z, r\rho, \vartheta, r\zeta) \tag{66}$$

for a function $\tilde{\sigma}_\psi(\mathfrak{A})(r, \phi, z, \tilde{\rho}, \vartheta, \tilde{\zeta}), (\tilde{\rho}, \vartheta, \tilde{\zeta}) \neq 0$, smooth up to $r = 0$. Moreover, close to v near the branch $Z_k \subset Z, k = 1, 2$, in local coordinates $(r, \phi, t) \in \mathbb{R}_+ \times S^1_+ \times \mathbb{R}_+$ and covariables (ρ, ϑ, τ) we have

$$\sigma_\psi(\mathfrak{A})(r, \phi, t, \rho, \vartheta, \tau) = t^{-\mu} r^{-\mu} \tilde{\boldsymbol{\sigma}}_\psi(\mathfrak{A})(r, \phi, t, r\rho, \vartheta, rt\tau) \tag{67}$$

for a function $\tilde{\boldsymbol{\sigma}}_\psi(\mathfrak{A})(r, \phi, t, \tilde{\boldsymbol{\rho}}, \vartheta, \tilde{\boldsymbol{\tau}}), (\tilde{\boldsymbol{\rho}}, \vartheta, \tilde{\boldsymbol{\tau}}) \neq 0$, which is smooth up to $r = t = 0$. The second component of (65) which is the boundary symbol we employ that $(\mathfrak{A}_{ij})_{i,j=1,2,3}$ belongs to $\mathcal{B}^{\mu,\mathrm{d}}(X \setminus Z)$. The boundary of $X \setminus Z$ consists of $Y_\pm$; so there are two components, namely, $\sigma_\partial(\mathfrak{A}) := (\sigma_{\partial,-}(\mathfrak{A}), \sigma_{\partial,+}(\mathfrak{A}))$, where

$$\sigma_{\partial,-}(\mathfrak{A}) := \sigma_\partial((\mathfrak{A}_{ij})_{i,j=1,2}), \sigma_{\partial,+}(\mathfrak{A}) := \sigma_\partial((\mathfrak{A}_{ij})_{i,j=1,3}),$$
$$\sigma_{\partial,\mp}(\mathfrak{A}) : H^s(\mathbb{R}_+) \oplus \mathbb{C} \to H^{s-\mu}(\mathbb{R}_+) \oplus \mathbb{C}. \tag{68}$$

The latter boundary symbols are those of the boundary value problems $(\mathfrak{A}_{ij})_{i,j=1,2}$ on $\operatorname{int} Y_-$ and $(\mathfrak{A}_{ij})_{i,j=1,3}$ on $\operatorname{int} Y_+$ in the standard sense, i.e., operator families parametrised by $T^*(\operatorname{int} Y_\mp) \setminus 0$ and homogeneous of order μ.

Close to Z_{reg} in the variables $(r, \phi, z) \in \mathbb{R}_+ \times S^1_+ \times \mathbb{R}_+$ we have

$$\sigma_{\partial,\mp}(\mathfrak{A})(r, z, \rho, \zeta) = r^{-\mu} \tilde{\sigma}_{\partial,\mp}(\mathfrak{A})(r, z, r\rho, r\zeta)$$

for operator functions $\tilde{\sigma}_{\partial,\mp}(\mathfrak{A})(r, z, \tilde{\rho}, \tilde{\zeta}), (\tilde{\rho}, \tilde{\zeta}) \neq 0$, which are smooth up to $r = 0$. Moreover, close to v near the branch $Z_k \subset Z, k = 1, 2$, in the variables $(r, \phi, t) \in \mathbb{R}_+ \times S^1_+ \times \mathbb{R}_+$ we have

$$\sigma_{\partial,\mp}(\mathfrak{A})(r, t, \rho, \tau) = t^{-\mu} r^{-\mu} \tilde{\boldsymbol{\sigma}}_{\partial,\mp}(\mathfrak{A})(r, t, r\rho, rt\tau)$$

for operator functions $\tilde{\boldsymbol{\sigma}}_{\partial,\mp}(\mathfrak{A})(r, t, \tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}), (\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}) \neq 0$, which are smooth up to $r = t = 0$.

From Definition 3.1 (ii) we have the homogeneous principal edge symbol

$$\sigma_\wedge(\mathfrak{A})(z, \zeta) : \tilde{\mathcal{K}}^{s,\gamma}((S^1_+)^\wedge) \oplus \mathbb{C}^{j_-} \to \tilde{\mathcal{K}}^{s-\mu,\gamma-\mu}((S^1_+)^\wedge) \oplus \mathbb{C}^{j_+}, \tag{69}$$

$(z, \zeta) \in T^*(Z_{\mathrm{reg}}) \setminus 0$, where

$$\tilde{\mathcal{K}}^{s,\gamma}((S^1_+)^\wedge) := \mathcal{K}^{s,\gamma}((S^1_+)^\wedge) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_-) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+).$$

In a neighbourhood of $v \in Z$ in the coordinate $t \in L_\pm \setminus \{0\}$ with the covariable τ (i.e., (z, ζ) in (69) is replaced by (t, τ), cf. Definition 2.7) we employ $\sigma_\wedge(\mathfrak{A})(t, \tau)$ of the form $\sigma_\wedge(\mathfrak{A})(t, \tau) = t^{-\mu} \tilde{\sigma}_\wedge(\mathfrak{A})(t, t\tau)$ for an operator function

$$\tilde{\sigma}_\wedge(\mathfrak{A})(t, \tilde{\tau}) \tag{70}$$

between the spaces as in (69) which is smooth in t up to zero.

Finally, the principal conormal corner symbol of $\mathfrak{A}$ is a family of maps

$$\sigma_{\mathrm{c}}(\mathfrak{A})(w) : \tilde{\mathcal{H}}^{s,\gamma}(\mathbb{S}^2_+) \oplus \mathbb{C}^{j_-} \to \tilde{\mathcal{H}}^{s-\mu,\gamma-\mu}(\mathbb{S}^2_+) \oplus \mathbb{C}^{j_+}, \tag{71}$$

$w \in \Gamma_{\frac{3}{2}-\delta}$, cf. Definition 3.1 (i) and the formula (57); the spaces are defined by (79), (80) below.

Remark 3.4 *Let* $\mathfrak{A}$ *belong to* (62) *and assume that* $\sigma(\mathfrak{A}) = 0$. *Then* (64) *is a compact operator for every* $s > \mathrm{d} - \frac{1}{2}$.

Theorem 3.5 *For* $\mathfrak{A} \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{X}, \boldsymbol{a}; j_0, j_+)$, $\mathfrak{B} \in \mathfrak{C}^{\nu,\mathrm{e}}(\mathbb{X}, \boldsymbol{b}; j_-, j_0)$ *for* $\mu, \nu \in \mathbb{Z}, \mathrm{d}, \mathrm{e} \in \mathbb{N}$, *and* $\boldsymbol{a} := (\gamma-\nu, \gamma-(\mu+\nu); \delta-\nu, \delta-(\mu+\nu))$, $\boldsymbol{b} := (\gamma, \gamma-\nu; \delta, \delta-\nu)$, *we have* $\mathfrak{A}\mathfrak{B} \in \mathfrak{C}^{\mu+\nu,\mathrm{h}}(\mathbb{X}, \boldsymbol{a} \circ \boldsymbol{b}; j_-, j_+)$ *for* $\mathrm{h} = \max(\nu + \mathrm{d}, \mathrm{e}), \boldsymbol{a} \circ \boldsymbol{b} = (\gamma, \gamma - (\mu + \nu); \delta, \delta - (\mu + \nu))$, *where* $\sigma(\mathfrak{A}\mathfrak{B}) = \sigma(\mathfrak{A})\sigma(\mathfrak{B})$ *with componentwise multiplication, and the rule* $\sigma_{\mathrm{c}}(\mathfrak{A}\mathfrak{B})(w) = \sigma_{\mathrm{c}}(\mathfrak{A})(w - \nu)\sigma_{\mathrm{c}}(\mathfrak{B})(w)$ *for the conormal symbols.*

The proof of Theorem 3.5 is a combination of Theorem 2.22, specified to $\mathbb{D} = \mathbb{S}^2_+$, with the corresponding known composition behaviour in the edge algebra of boundary value problems outside $\{v\}$.

Remark 3.6 *Our applications will refer to a slightly modified definition of the operator space* $\mathfrak{C}^{\mu}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)$. *In Definition* 3.1 *we assumed, for simplicity, that the number of trace and potential conditions on* $Y_\pm \setminus Z$ *is equal to* 1. *In addition we assumed their orders to be* $\mu + \frac{1}{2}$ *and* $\mu - \frac{1}{2}$, *respectively. In the Zaremba problem we have, of course, only trace* (*no potential*) *conditions and order* $\frac{1}{2}$ *and* $\frac{3}{2}$, *respectively. For the expected parametrices in our calculus there are only potential* (*no trace*) *operators, etc. For that reason we will also employ the notation in a corresponding generalised sense, where* μ *is the order of the upper left corners, and the other orders are assumed to be known by the context, also the number of trace and potential entries referring to* $Y_\pm \setminus Z$. *In compositions we assume that rows and colums fit together. We then have a corresponding generalisation of Theorems* 3.3 *and* 3.5.

3.2 Ellipticity and parametrices

Definition 3.7 *An operator* $\mathfrak{A} \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)$ *is called elliptic, if*

(i) $\sigma_\psi(\mathfrak{A}) \neq 0$ *on* $T^*(\mathrm{int}\, X) \setminus 0$, *and near* Z_{reg} *we have* $\tilde{\sigma}_\psi(\mathfrak{A})(r, \phi, z, \tilde{\rho}, \vartheta, \tilde{\zeta}) \neq 0$, *for* $(\tilde{\rho}, \vartheta, \tilde{\zeta}) \neq 0$, *up to* $r = 0$, *and near* v *and* $Z_k, k = 1, 2$, *the function* $\tilde{\boldsymbol{\sigma}}_\psi(\mathfrak{A})(r, \phi, t, \tilde{\boldsymbol{\rho}}, \vartheta, \tilde{\boldsymbol{\tau}})$ *does not vanish for* $(\tilde{\boldsymbol{\rho}}, \vartheta, \tilde{\boldsymbol{\tau}}) \neq 0$ *up to* $r = t = 0$;

(ii) $\sigma_{\partial,\pm}(\mathfrak{A})$ *defines isomorphisms* (68) *for all points of* $T^*(\mathrm{int}\, Y_\pm) \setminus 0$, *and near* Z_{reg} *the mappings* $\tilde{\sigma}_{\partial,\pm}(\mathfrak{A})(r, z, \tilde{\rho}, \tilde{\zeta})$ *are isomorphisms for* $(\tilde{\rho}, \tilde{\zeta}) \neq 0$, *up to* $r = 0$, *and near* v *and* $Z_k, k = 1, 2$, *the mappings* $\tilde{\boldsymbol{\sigma}}_{\partial,\pm}(\mathfrak{A})(r, t, \tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}})$ *are isomorphisms for* $(\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{\tau}}) \neq 0$, *up to* $r = t = 0$;

(iii) $\sigma_\wedge(\mathfrak{A})$ *defines isomorphisms* (69) *for all* $(z, \zeta) \in T^*(Z_{\mathrm{reg}}) \setminus 0$, *and* $\tilde{\sigma}_\wedge(\mathfrak{A})(t, \tilde{\tau})$ *defines analogous isomorphisms for* $\tilde{\tau} \neq 0$ *up to* $t = 0$;

(iv) $\sigma_{\mathrm{c}}(\mathfrak{A})(w)$ *defines isomorphisms* (71) *for all* $w \in \Gamma_{\frac{3}{2}-\delta}$.

In the conditions (ii)-(iv) *we assume* $s > \max(\mu, \mathrm{d}) - \frac{1}{2}$.

Remark 3.8 *The operator spaces* $\mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)$ *contain subspaces with more regularity with respect to* σ_ψ *and* σ_∂. *Let* $\mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{X}, \boldsymbol{g}; j_-, j_+)_{\mathrm{reg}}$ *denote the class of all* $\mathfrak{A}$ *such that*

(i)$_{\mathrm{reg}}$ $\mathfrak{A}_{11}$ *is regular in the sense that there is a 'smooth' element* $A_{11} \in \mathcal{B}^{\mu,\mathrm{d}}(X)$ *such that* $A_{11}|_{X\setminus Z} - \mathfrak{A}_{11} \in \mathcal{B}^{-\infty,\mathrm{d}}(X \setminus Z)$;

(ii)$_{\mathrm{reg}}$ $(\mathfrak{A}_{ij})_{i,j=1,2}$ *is regular in the sense that there are 'smooth' elements* $\mathcal{A}_\pm \in \mathcal{B}^{\mu,\mathrm{d}}(X)$ *such that* $(\mathcal{A}_\pm - (\mathfrak{A}_{ij})_{i,j=1,2})|_{X\setminus Y_\mp} \in \mathcal{B}^{-\infty,\mathrm{d}}(X \setminus Y_\mp)$.

Then, if $\mathcal{A}_\pm$ *are elliptic as elements in that smooth calculus of boundary value problems with respect to* σ_ψ *and* σ_∂, *then the conditions* (i), (ii) *of Definition* 3.7 *with respect to* $\tilde{\sigma}_\psi, \tilde{\boldsymbol{\sigma}}_\psi$ *and* $\tilde{\sigma}_{\partial,\pm}, \tilde{\boldsymbol{\sigma}}_{\partial,\pm}$ *are automatically satisfied.*

Theorem 3.9 *Let* $\mathfrak{A} \in \mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{X}, \boldsymbol{g}; j_-, j_+), \boldsymbol{g} = (\gamma, \gamma - \mu; \delta, \delta - \mu)$, *be elliptic; then there exists a parametrix* $\mathfrak{P} \in \mathfrak{C}^{-\mu,(d-\mu)^+}(\mathbb{X}, \boldsymbol{g}^{-1}; j_+, j_-), \boldsymbol{g}^{-1} := (\gamma - \mu, \gamma; \delta - \mu, \delta)$, *i.e., we have*

$$\mathfrak{I} - \mathfrak{P}\mathfrak{A} \in \mathfrak{C}^{-\infty,\mathrm{d_l}}(\mathbb{X}, \boldsymbol{g}_{\mathrm{l}}; j_-, j_-), \quad \mathfrak{I} - \mathfrak{A}\mathfrak{P} \in \mathfrak{C}^{-\infty,\mathrm{d_r}}(\mathbb{X}, \boldsymbol{g}_{\mathrm{r}}; j_+, j_+) \qquad (72)$$

with the same meaning of $\boldsymbol{g}_{\mathrm{l}}, \mathrm{d_l}$, *etc., as in Theorem* 2.24.

Proof. We construct $\mathfrak{P}$ in the form $\mathfrak{P} = \mathfrak{P}_{\mathrm{corner}} + \mathfrak{P}_{\mathrm{edge}} + \mathfrak{P}_{\mathrm{reg}}$ where the summands have a similar meaning as in Definition 3.1. The ellipticity of $\mathfrak{A}$ implies that $\mathfrak{L}_{\mathrm{edge}} \in \mathfrak{Y}^{\mu,\mathrm{d}}(\mathbb{U}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ and $\mathfrak{L}_{\mathrm{reg}} \in \mathcal{B}^{\mu,\mathrm{d}}(X \setminus Z)$ are elliptic in the respective classes. Therefore, we have corresponding parametrices $\mathfrak{M}_{\mathrm{edge}} \in \mathfrak{Y}^{-\mu,(\mathrm{d}-\mu)^+}(\mathbb{U}, \boldsymbol{g}_{\mathrm{cone}}^{-1}; j_+, j_-)$ and $\mathfrak{M}_{\mathrm{reg}} \in \mathcal{B}^{-\mu,(\mathrm{d}-\mu)^+}(X \setminus Z)$, respectively. Concerning $\mathfrak{M}_{\mathrm{edge}}$ we refer, e.g., to [3]. The construction of $\mathfrak{M}_{\mathrm{reg}}$ is standard, cf. [1]. We then set $\mathfrak{P}_{\mathrm{edge}} := \Psi_{\mathrm{edge}}\mathfrak{M}_{\mathrm{edge}}\Phi_{\mathrm{edge}}$ and $\mathfrak{P}_{\mathrm{reg}} := \Psi_{\mathrm{reg}}\mathfrak{M}_{\mathrm{reg}}\Phi_{\mathrm{reg}}$. Thus the main step is to construct a parametrix $\mathfrak{M}_{\mathrm{corner}}$ of $\mathfrak{L}_{\mathrm{corner}}$, cf. the notation in Definition 3.1 (i), which is an immediate consequence of Theorem 2.24 for the case $\mathbb{D} = \mathbb{S}^2_+$ and in a slight modification (here, because of the two components $Y_\pm$ of $\partial(X \setminus Z)$ we have 4×4 matrices with a corresponding meaning of $\Phi, \tilde{\Phi}, \tilde{\tilde{\Phi}}$). We set $\mathfrak{P}_{\mathrm{corner}} := \Psi_{\mathrm{corner}}\mathfrak{M}_{\mathrm{corner}}\Phi_{\mathrm{corner}}$. □

Corollary 3.10 *Let* $\mathfrak{A}$ *be as in Theorem* 3.9, *then the associated operator* (64) *is Fredholm for every* $s > \max(\mu, \mathrm{d}) - \frac{1}{2}$.

In fact, the parametrix $\mathfrak{P}$ of $\mathfrak{A}$ has a principal symbolic hierarchy with the components inverse to those of (65) (up to a translation in w in the conormal symbol). Then, using the fact that $\mathfrak{P}\mathfrak{A}$ and $\mathfrak{A}\mathfrak{P}$ as well the identity operators belong to the corner calculus in the sense of Definition 3.1 for $\mu = 0$ and because the principal symbols are multiplicative under the operator composition, cf. Theorem 3.5, we see that the operators in (72) are compact, cf. Remark 3.4. This entails the Fredholm property.

Remark 3.11 *If $\mathfrak{A}$ is elliptic and belongs to $\mathfrak{C}^{\mu,\mathrm{d}}(\mathbb{X},\boldsymbol{g};j_-,j_+)_{\mathrm{reg}}$, it follows that $\mathfrak{P}\in\mathfrak{C}^{-\mu,(\mathrm{d}-\mu)^+}(\mathbb{X},\boldsymbol{g}^{-1};j_+,j_-)_{\mathrm{reg}}$.*

3.3 The singular Zaremba problem

In the following consideration we generalise the meaning of the notation in Definitions 2.17 and 3.1 by admitting different orders in the operators and spaces referring to the $\pm$ side of the boundary.

The assumptions in Definition 3.1 about entries in the 3×3 block matrices and their orders are made for convenience. We may admit modified operators as represented by the Zaremba problem (3) or the operator (4) with additional trace and potential data on $Z\setminus\{v\}$. In this case the potential operators in (2) do not occur. However, the corresponding version of Definition 3.7 is evident. The first two conditions only referring to the operator (2) are as follows:

(i)$_Z$ $\sigma_\psi(\mathfrak{A})(\xi)=-|\xi|^2\neq 0$ for $\xi\neq 0$;

(ii)$_Z$ $\sigma_{\partial,\pm}(\mathfrak{A})(y,\eta):H^s(\mathbb{R}_+)\to H^{s-2}(\mathbb{R}_+)\oplus\mathbb{C}$ is bijective for $T^*Y\setminus 0|_{Y_\pm}$.

This corresponds to the regular behaviour in the sense of Remark 3.8.

Concerning the existence of a block matrix $\mathfrak{A}$ in the sense of (4) we have the following result (notation is used here in the sense of Remark 3.6).

Theorem 3.12 *For every $\gamma\in(\frac{1}{2}-k,\frac{3}{2}-k), k\in\mathbb{Z}$, we find dimensions $j_\pm=j_\pm(\gamma)$ and elements $\mathcal{T},\mathcal{K}$ and $\mathcal{Q}$ in the corner operator space $\mathfrak{C}^{2,2}(\mathbb{X},\boldsymbol{g};j_-,j_+),\boldsymbol{g}=(\gamma,\gamma-2;\delta,\delta-2)$, of orders and weight shifts corresponding to the mapping property (4), for $j_+-j_-=k$, and for every $\delta\in\mathbb{R}$, such that*

(iii)$_Z$

$$\sigma_\wedge(\mathfrak{A})(z,\zeta):\mathcal{K}^{s,\gamma}((S^1_+)^\wedge)\oplus\mathbb{C}^{j_-}\to\tilde{\mathcal{K}}^{s-2,\gamma-2}((S^1_+)^\wedge)\ \oplus\ \mathbb{C}^{j_+}$$

is bijective and that also $\tilde{\sigma}_\wedge(\mathfrak{A})(t,\tilde{\tau})$ is bijective up to $t=0$, cf. the formula (10).

(iv)$_Z$ *There is a discrete set D of reals such that*

$$\sigma_{\mathrm{c}}(\mathfrak{A})(w):\mathcal{H}^{s,\gamma}(\mathbb{S}^2_+)\oplus\mathbb{C}^{j_-}\to\tilde{\mathcal{H}}^{s-2,\gamma-2}(\mathbb{S}^2_+)\oplus\mathbb{C}^{j_+}$$

for $\tilde{\mathcal{H}}^{s-2,\gamma-2}(\mathbb{S}^2_+):=\mathcal{H}^{s-2,\gamma-2}(\mathbb{S}^2_+)\oplus\mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(I_-)\oplus\mathcal{H}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(I_+)$ is bijective for all $w\in\Gamma_{\frac{3}{2}-\delta}$ and $\delta\in\mathbb{R}$ such that $D\cap\Gamma_{\frac{3}{2}-\delta}=\emptyset$.

Proof. The structure of the operator (2) was analysed in Sections 1.1 and 1.2. In particular, in Theorem 1.3 we established the Fredholm property of the principal edge symbol (12) for all $\gamma\in(\frac{1}{2}-k,\frac{3}{2}-k)$, with index k. In Section 1.2 we obtained extra entries on the level of edge symbols on $Z\setminus\{0\}$

that turn $\sigma_\wedge(\mathcal{A})(z,\zeta)$ to a family of isomorphisms (15), cf. also [3]. This construction also gives us the entries in such a way that in local coordinates under the chart (32) near in the variable t on $L_\pm$ and the covariable τ also (70) are isomorphisms up to $t=0$, for $\tilde{\tau}\neq 0$. From these homogeneous edge symbols in $(t,\tilde{\tau})$ we can construct an amplitude function $\tilde{\mathfrak{a}}(t,\tilde{\tau})\in \mathfrak{R}^{2,2}(\overline{\mathbb{R}}_+\times\mathbb{R}_{\tilde{\tau}};j_-,j_+)$, cf. Definition 2.7 and then pass via Theorem 2.14 to a holomorphic representative. Thus we have all ingredients to build up the operator $\mathfrak{A}$ which satisfies all ellipticity conditions (i)-(iv), including $(\mathrm{i})_Z$ and $(\mathrm{ii})_Z$. In particular, by Theorem 2.14 we have an exceptional set D of weights δ as in $(\mathrm{iv})_Z$. $\square$

Theorem 3.13 *For every $\gamma\in(\frac{1}{2}-k,\frac{3}{2}-k)$, $k\in\mathbb{Z}$, the operator $\mathcal{A}$ which represents the singular Zaremba problem*

$$\mathcal{A}:\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X})\to\tilde{\mathcal{V}}^{s-2,(\gamma-2,\delta-2)}(\mathbb{X})$$

(*cf. the notation in the formula* (3)) *can be completed by additional interface conditions on Z to an element $\mathfrak{A}\in\mathfrak{C}^{2,2}(\mathbb{X},\boldsymbol{g};j_-,j_+)$*

$$\mathfrak{A}:=\begin{pmatrix}\mathcal{A}&\mathcal{K}\\ \mathcal{T}&\mathcal{Q}\end{pmatrix}:\begin{array}{c}\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{X})\\ \oplus\\ \mathcal{H}^{s-1,\delta-1}(Z,\mathbb{C}^{j_-})\end{array}\to\begin{array}{c}\tilde{\mathcal{V}}^{s-2,(\gamma-2,\delta-2)}(\mathbb{X})\\ \oplus\\ \mathcal{H}^{s-3,\delta-3}(Z,\mathbb{C}^{j_+})\end{array},\qquad(73)$$

$j_+-j_-=k$, such that for a discrete set D of reals δ the operator (73) *is Fredholm for all $\delta\in\mathbb{R}\setminus D, s>\frac{3}{2}$. Moreover, $\mathfrak{A}$ has a parametrix $\mathfrak{P}$ of analogous structure as $\mathfrak{P}$ in Theorem* 3.9*, here with the corresponding modified orders.*

Proof. Theorem 3.13 is a special case of Theorem 3.9 and Corollary 3.10, up to an obvious modification of orders. $\square$

Using $\mathfrak{P}$ of the latter theorem as a left parametrix of $\mathfrak{A}$ and taking into account Theorem 3.3 we obtain the following elliptic regularity in weighted corner spaces:

Corollary 3.14 *Let $\mathfrak{A}$ be as in Theorem* 3.13*, and let $\mathfrak{A}u=f$ with f belonging to the space on the right of* (73)*, and $u\in\mathcal{V}^{-\infty,(\delta,\gamma)}(\mathbb{X})\oplus\mathcal{H}^{-\infty,\delta}(Z,\mathbb{C}^{j_-})$. Then we have $u\in\mathcal{V}^{s,(\delta,\gamma)}(\mathbb{X})\oplus\mathcal{H}^{s,\delta}(Z,\mathbb{C}^{j_-})$. A similar elliptic regularity is true of the corner operators of Section* 2.5.

In fact, it suffices to compose the equation $\mathfrak{A}u=f$ from the left with a parametrix $\mathfrak{P}$ which gives us $u=(\mathfrak{I}-\mathfrak{P}\mathfrak{A})u+\mathfrak{P}f$. We then obtain the desired property of u when we apply Theorem 3.3 to $\mathfrak{P}$ and the mapping properties of the smoothing remainder $\mathfrak{I}-\mathfrak{P}\mathfrak{A}$.

Remark 3.15 *As we saw, corner boundary value problems belong to a kind of cone calculus, where the base of the cone itself has conical singularities, and we then have two* (*local*) *axial variables $r,t\in\mathbb{R}_+$. Similarly* *as in Kondratyev's theory* [10] *we can ask the asymptotic properties of solutions close to the singularities. Then, for $r \to 0$ we have to expect 'edge asymptotics', and for $t \to 0$ corner asymptotics. The structure of such double asymptotics in the boundaryless case was analysed in* [15]*, see also* [16]*. Similar asymptotics should be established for the case with boundary, cf.* [4]*; this requires a refinement of the calculus with a corresponding characterisation of parametrices, together with the meromorphic structure of edge and corner conormal symbols.*

4 Elements of the edge and corner calculus

4.1 Weighted spaces and Mellin operators for conical singularities

Let M be a closed compact manifold, $n = \dim M$, and set $M^\wedge := \mathbb{R}_+ \times M \ni (r, x)$. Then the space

$$\mathcal{H}^{s,\gamma}(M^\wedge), \tag{74}$$

$s, \gamma \in \mathbb{R}$, denotes the completion of the space $C_0^\infty(\mathbb{R}_+, C^\infty(M))$ with respect to the norm $\left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} ||R^s(\operatorname{Im} z) M_{r\to z} u(z)||^2_{L^2(M)} dz \right\}^{\frac{1}{2}}$, where $M_{r\to z}$ is the Mellin transform $M_{r\to z}u(z) = \int_0^\infty r^{z-1}u(r)dr$ on $u(r) \in C_0^\infty(\mathbb{R}_+, C^\infty(M))$ (which is holomorphic in z), $\Gamma_\beta := \{z \in \mathbb{C} : \operatorname{Re} z = \beta\}$, and $R^s(\tau) \in L^s_{\mathrm{cl}}(M; \mathbb{R}_\tau)$ is an order reducing family of order s (we use the well known fact that for every $\mu \in \mathbb{R}$ there is a parameter-dependent elliptic element $R^\mu(\lambda) \in L^\mu_{\mathrm{cl}}(M; \mathbb{R}^l)$ which induces isomorphisms $R^\mu(\lambda) : H^s(M) \to H^{s-\mu}(M)$ for all $s \in \mathbb{R}, \lambda \in \mathbb{R}^l$).

In this paper a cut-off function on the half-axis is any real-valued element $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ that is equal to 1 in a neighbourhood of $r = 0$. We define

$$\mathcal{K}^{s,\gamma}(M^\wedge) := \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(M^\wedge), v \in H^s_{\mathrm{cone}}(M^\wedge)\}, \tag{75}$$

$s, \gamma \in \mathbb{R}$. Here $H^s_{\mathrm{cone}}(M^\wedge)$ denotes the subspace of all $v = \tilde{v}|_{M^\wedge}, \tilde{v} \in H^s_{\mathrm{loc}}(\mathbb{R} \times M)$, such that for every coordinate neighbourhood U on M, every diffeomorphism $\chi : U \to \tilde{U}$ to an open set $\tilde{U} \subset S^n, \chi(x) = \tilde{x}$, and every $\varphi \in C_0^\infty(U)$ the function $\varphi(\chi^{-1}(\tilde{x}))(1-\omega(r))v(r, \chi^{-1}(\tilde{x}))$ belongs to the space $H^s(\mathbb{R}^{n+1})$ (where $(r, \tilde{x})$ has the meaning of polar coordinates in $\mathbb{R}^{n+1} \setminus \{0\}$). If $n = 0$ we obtain the spaces $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$. The spaces (75) are Hilbert spaces, and for $s = \gamma = 0$ we have $\mathcal{K}^{0,0}(M^\wedge) = r^{-\frac{n}{2}} L^2(M^\wedge)$.

If N is a compact C^∞ manifold with C^∞ boundary ∂N we define

$$\mathcal{K}^{s,\gamma}(N^\wedge) := \{u|_{(\operatorname{int} N)^\wedge} : u \in \mathcal{K}^{s,\gamma}((2N)^\wedge)\}, \tag{76}$$

$$\mathcal{H}^{s,\gamma}(N^\wedge) := \{u|_{(\operatorname{int} N)^\wedge} : u \in \mathcal{H}^{s,\gamma}((2N)^\wedge)\}, \tag{77}$$

$s, \gamma \in \mathbb{R}$, where $2N$ is the double of N which is obtained by gluing together two copies $N_\pm$ along the common boundary ∂N, where we identify N_+ with

N. In particular, for $M := S^1$ we have the spaces $\mathcal{K}^{s,\gamma}((S^1)^\wedge)$, $\mathcal{K}^{s,\gamma}((S^1_{(\alpha,\beta)})^\wedge)$ for $S^1_{(\alpha,\beta)} = \{\phi \in S^1 : \alpha \leq \phi \leq \beta\}$. The spaces $\mathcal{K}^{s,\gamma}(M^\wedge)$ are endowed with a strongly continuous group of isomorphisms

$$\kappa^{(n)}_\lambda : \mathcal{K}^{s,\gamma}(M^\wedge) \to \mathcal{K}^{s,\gamma}(M^\wedge) \tag{78}$$

when we set $\kappa^{(n)}_\lambda u(r,x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x), \lambda \in \mathbb{R}_+$. This action will be called standard, while we also employ other group actions of the form $\lambda^m \kappa^{(n)}_\lambda$ for certain $m \in \mathbb{R}$ that depend on the context.

Finally, we set $\mathcal{S}^\gamma_\varepsilon(M^\wedge) := \varprojlim_{k\in\mathbb{N}} \langle r\rangle^{-k} \mathcal{K}^{k,\gamma+\varepsilon-(1+k)^{-1}}(M^\wedge)$, $\gamma \in \mathbb{R}$, $\varepsilon > 0$.

We now define weighted Sobolev spaces on a compact manifold D with conical singularities and boundary (on the boundary). First there is the double $B := 2D$ (obtained by gluing together two copies $D_\pm$ of D along the common boundary, with D being identified with D_+). Let us assume that there is only one conical singularity v (the case with finitely many conical singularities is completely analogous). With B we associate the stretched manifold $\mathbb{B}$ which is a C^∞ manifold with boundary. For $s, \gamma \in \mathbb{R}$ we then define $\mathcal{H}^{s,\gamma}(\mathbb{B}) := \{u \in H^s_{\text{loc}}(\text{int}\,\mathbb{B}) : \omega u \in \mathcal{H}^{s,\gamma}((\partial\mathbb{B})^\wedge)\}$ for any cut-off function $\omega(r)$ on the half-axis, $\text{supp}\,\omega \subset [0, 1-\varepsilon]$ for some $0 < \varepsilon < 1$, where we identify a collar neighbourhood of $\partial\mathbb{B}$ in $\mathbb{B}$ with $[0,1) \times \partial\mathbb{B}$. Using the canonical map $\pi : \mathbb{B} \to B$, where $\text{im}\,\partial\mathbb{B} = v$ and $\pi : \text{int}\,\mathbb{B} \to B \setminus \{v\}$ is a diffeomorphism, we define $\mathbb{D} := \pi^{-1}D$. Let $\mathbb{D}_{\text{sing}} := \mathbb{D} \cap \partial\mathbb{B}$, $\mathbb{D}_{\text{reg}} := \mathbb{D} \setminus \mathbb{D}_{\text{sing}}$; this is a C^∞ manifold with boundary. We then set

$$\mathcal{H}^{s,\gamma}(\mathbb{D}) := \{u|_{\text{int}\,\mathbb{D}_{\text{reg}}} : u \in \mathcal{H}^{s,\gamma}(\mathbb{B})\} \tag{79}$$

endowed with the quotient topology $\mathcal{H}^{s,\gamma}(\mathbb{B})/\sim$ under the equivalence relation $u \sim v \Leftrightarrow u|_{\text{int}\,\mathbb{D}_{\text{reg}}} = v|_{\text{int}\,\mathbb{D}_{\text{reg}}}$. We will employ this, in particular, to the case $D = S^2_+$ with two conical points v_1, v_2 on the boundary S^1. As before we have the associated stretched manifold $\mathbb{D} = \mathbb{S}^2_+$ and the spaces $\mathcal{H}^{s,\gamma}(\mathbb{S}^2_+)$, where (by notation) the weight γ is the same for v_1 and v_2. The boundary S^1 is subdivided into two intervals $I_\pm$ which are also interpreted as manifolds with two conical singularities v_1, v_2. The above definition then gives us the spaces $\mathcal{H}^{s,\gamma}(I_\pm)$ (with the same weight γ at the end points). Observe that the operators of restriction $u \to u|_{\text{int}\,I_\pm}, u \in C^\infty_0((\mathbb{S}^2_+)_{\text{reg}})$, extend to continuous operators $\text{r}_{\{\text{int}\,I_\pm\}} : \mathcal{H}^{s,\gamma}(\mathbb{S}^2_+) \to \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(I_\pm)$ for all $s > \frac{1}{2}$ and $\gamma \in \mathbb{R}$. In Section 3.1 we employed the spaces

$$\tilde{\mathcal{H}}^{s,\gamma}(\mathbb{S}^2_+) := \mathcal{H}^{s,\gamma}(\mathbb{S}^2_+) \oplus \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(I_-) \oplus \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(I_+). \tag{80}$$

4.2 Vector-valued symbols and abstract edge spaces

A Hilbert space E is said to be endowed with a group action $\kappa := \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, if κ is a group of isomorphisms $\kappa_\lambda : E \to E, \lambda \in \mathbb{R}_+$, such that $\kappa_\lambda \kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \mathbb{R}_+$, and strongly continuous in λ.

Let E and $\tilde{E}$ are Hilbert spaces endowed with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. Let $S^{(\mu)}(U\times(\mathbb{R}^q\setminus\{0\});E,\tilde{E})$ be the space of all $f(z,\zeta)\in C^\infty(U\times(\mathbb{R}^q\setminus\{0\}),\mathcal{L}(E,\tilde{E}))$ such that $f(z,\lambda\zeta)=\lambda^\mu\tilde{\kappa}_\lambda f(z,\zeta)\kappa_\lambda^{-1}$ for all $\lambda\in\mathbb{R}_+,(z,\zeta)\in U\times(\mathbb{R}^q\setminus\{0\})$.

Definition 4.1 (i) *The space $S^\mu(U\times\mathbb{R}^q;E,\tilde{E})$ for an open set $U\subseteq\mathbb{R}^p$ denotes the set of all $a(z,\zeta)\in C^\infty(U\times\mathbb{R}^q,\mathcal{L}(E,\tilde{E}))$ such that*

$$||\tilde{\kappa}^{-1}_{\langle\zeta\rangle}\{D^\alpha_z D^\beta_\zeta a(z,\zeta)\}\kappa_{\langle\zeta\rangle}||_{\mathcal{L}(E,\tilde{E})}\le c\langle\zeta\rangle^{\mu-|\beta|} \tag{81}$$

for all $\alpha\in\mathbb{N}^p,\beta\in\mathbb{N}^q$ and all $(z,\zeta)\in K\times\mathbb{R}^q$ for arbitrary $K\Subset U$, with constants $c=c(\alpha,\beta,K)>0$.

(ii) *The space $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\tilde{E})$ of classical symbols is defined to be the subspace of all $a(z,\zeta)\in S^\mu(U\times\mathbb{R}^q;E,\tilde{E})$ such that there are homogeneous components $a_{(\mu-j)}(z,\zeta)\in S^{(\mu-j)}(U\times(\mathbb{R}^q\setminus\{0\});E,\tilde{E}),j\in\mathbb{N}$, such that*

$$r_{N+1}(z,\zeta):=a(z,\zeta)-\chi(\zeta)\sum_{j=0}^N a_{(\mu-j)}(z,\zeta)\in S^{\mu-(N+1)}(U\times\mathbb{R}^q;E,\tilde{E})$$

for all $N\in\mathbb{N}$.

Note that when $\chi(\zeta)$ is an excision function, i.e., $\chi\in C^\infty(\mathbb{R}^q),\chi(\zeta)=0$ for $|\zeta|<c_0,\chi(\zeta)=1$ for $|\zeta|>c_1$ for certain $0<c_0<c_1$, then

$$\chi(\zeta)S^{(\mu)}(U\times(\mathbb{R}^q\setminus\{0\});E,\tilde{E})\subset S^\mu(U\times\mathbb{R}^q;E,\tilde{E}).$$

The space $S^\mu(U\times\mathbb{R}^q;E,\tilde{E})$ is Fréchet with the best constants in the estimates (81). Moreover, $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\tilde{E})$ is Fréchet in the topology of the projective limit under the maps of $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\tilde{E})$ to $S^{(\mu-j)}(U\times(\mathbb{R}^q\setminus\{0\});E,\tilde{E}),a(z,\zeta)\to a_{(\mu-j)}(z,\zeta),j\in\mathbb{N}$, and to $S^{\mu-(N+1)}(U\times\mathbb{R}^q;E,\tilde{E})$, $a(z,\zeta)\to r_{N+1}(z,\zeta),N\in\mathbb{N}$. If we talk about classical or general symbols we also write as subscript '(cl)'.

If $\tilde{E}$ is a Fréchet space such that $\tilde{E}=\varprojlim_{j\in\mathbb{N}}\tilde{E}^j$ for Hilbert spaces $\tilde{E}^j,j=1,2,\dots$, we define $S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E}):=\varprojlim_{j\in\mathbb{N}}S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E}^j)$.

If both E and $\tilde{E}$ are Fréchet spaces with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, we fix a function $\mathrm{r}:\mathbb{N}\to\mathbb{N}$ and set $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\tilde{E}):=\bigcup_{\mathrm{r}}S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})_{\mathrm{r}}$, where $S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})_{\mathrm{r}}$ is the projective limit of the spaces $S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E^{\mathrm{r}(j)},\tilde{E}^j)$.

Clearly the spaces of symbols depend on the choice of the group actions $\kappa=\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\tilde{\kappa}=\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively; usually they are known from the context. If we want to indicate this dependence we also write

$$S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})_{\kappa,\tilde{\kappa}}. \tag{82}$$

Example 4.2 *Let us write* $\mathcal{S}(\overline{\mathbb{R}}_+) := \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$ *as a projective limit of Hilbert spaces* $\mathcal{S}(\overline{\mathbb{R}}_+) = \varprojlim_{k\in\mathbb{N}} \tilde{E}^k$ *for* $\tilde{E}^k := \langle x_n\rangle^{-k} H^k(\mathbb{R}_+)$ *with the group action* $(\kappa_\lambda u)(x_n) = \lambda^{\frac{1}{2}} u(\lambda x_n), \lambda \in \mathbb{R}_+$. *Then we have the space of symbols*

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1} \times \mathbb{R}^l; L^2(\mathbb{R}_+) \oplus \mathbb{C}^m, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{m'}) \tag{83}$$

with the group actions $\mathrm{diag}\,(\kappa_\lambda, \mathrm{id})$ *on the respective direct sums.*

An element $g(x', \xi', \lambda) \in C^\infty(\Omega \times \mathbb{R}^{n-1+l}, \mathcal{L}(L^2(\mathbb{R}_+) \oplus \mathbb{C}^m, L^2(\mathbb{R}_+) \oplus \mathbb{C}^{m'}))$ *is said to be a Green symbol of order* μ *and type* 0 *(of the calculus of boundary value problems with the transmission property at* $x_n = 0$*) if* $g_0(x', \xi', \lambda) := \mathrm{diag}\,(1, \langle \xi', \lambda\rangle^{\frac{1}{2}}) g(x', \xi', \lambda) \mathrm{diag}\,(1, \langle \xi', \lambda\rangle^{-\frac{1}{2}})$ *and* $g_0^*(x', \xi', \lambda)$ *(with interchanged* m, m'*) belong to* (83)*, where '*' indicates pointwise adjoint in the sense* $(g(x', \xi', \lambda)u, v)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^{m'}} = (u, g^*(x', \xi', \lambda)v)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^m}$ *for all* $u \in L^2(\mathbb{R}_+) \oplus \mathbb{C}^m, v \in L^2(\mathbb{R}_+) \oplus \mathbb{C}^{m'}$. *An operator family* $g(x', \xi', \lambda)$ *is called a Green symbol of type* $\mathrm{d} \in \mathbb{N}$, *if it has the form*

$$g(x', \xi', \lambda) = g_0(x', \xi', \lambda) + \sum_{j=1}^{\mathrm{d}} g_j(x', \xi', \lambda) \mathrm{diag}\,(\partial^j_{x_n}, 0)$$

for Green symbols $g_j(x', \xi', \lambda)$ *of order* $\mu - j$ *and type* $0, j = 0, \ldots, \mathrm{d}$. *In this case we have* $g(x', \xi', \lambda) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1+l}; H^s(\mathbb{R}_+) \oplus \mathbb{C}^m, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{m'})$ *for every real* $s > \mathrm{d} - \frac{1}{2}$.

Definition 4.3 *Let* E *be a Hilbert space with group action* $\kappa := \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$. *The 'abstract' edge Sobolev space* $\mathcal{W}^s(\mathbb{R}^q, E)$ *of smoothness* $s \in \mathbb{R}$ *is the completion of* $\mathcal{S}(\mathbb{R}^q, E)$ *with respect to the norm* $\left\{ \int \langle\zeta\rangle^{2s} ||\kappa^{-1}_{\langle\zeta\rangle} Fu(\zeta)||^2_E d\zeta \right\}^{\frac{1}{2}}$, *where* $F_{z\to\zeta}$ *is the Fourier transform. If* $E = \varprojlim_{j\in\mathbb{N}} E^j$ *is Fréchet with a group action* $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ *we set* $\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j)$. *If necessary we write*

$$\mathcal{W}^s(\mathbb{R}^q, E)_\kappa \tag{84}$$

when we want to indicate the dependence of the space on the choice of κ.

Sobolev spaces of that kind, based on a group action in the parameter space E, have been introduced in [14]; concerning more functional analytic details, cf. [8].

Observe that $\mathcal{W}^s(\mathbb{R}^q, E) \subset \mathcal{S}'(\mathbb{R}^q, E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^q), E)$. Let $\Omega \subseteqq \mathbb{R}^q$ be an open set; then $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$ denotes the set of all $u \in \mathcal{D}'(\Omega, E)$ such that $\varphi u \in \mathcal{W}^s(\mathbb{R}^q, E)$ for $\varphi \in C^\infty_0(\Omega)$; moreover, $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ is defined as the subspace of all $u \in \mathcal{W}^s(\mathbb{R}^q, E)$ such that $\mathrm{supp}\, u \subset \Omega$ is compact.

Theorem 4.4 *For any* $a(z, z', \zeta) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^q; E, \tilde{E})$ *the associated pseudo-differential operator* $\operatorname{Op}(a) : C_0^\infty(\Omega, E) \to C^\infty(\Omega, \tilde{E})$ *extends to continuous operators* $\operatorname{Op}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E})$ *for all* $s \in \mathbb{R}$. *Moreover, if* $a = a(\zeta)$ *(i.e., is independent on* z*) we have continuous operators* $\operatorname{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{E})$ *for all* $s \in \mathbb{R}$.

4.3 Boundary value problems with the transmission property

Let N be a C^∞ manifold with boundary ∂N, not necessarily compact. With N we associate the double $2N$, obtained by gluing together two copies N_+ and N_- along the common boundary ∂N (with N being identified with N_+). We then obtain a C^∞ manifold $2N$.

On all smooth manifolds in consideration we fix Riemannian metrics; in the case of a C^∞ manifold N with boundary we choose a collar neighbourhood $\cong \partial N \times [0, 1) \in (x', x_n)$ and assume the Riemannian metric to be the product metric of a metric on ∂N and the standard metric on $[0, 1)$.

If M is a C^∞ manifold by $L^\mu_{(\mathrm{cl})}(M; \mathbb{R}^l)$ we denote the space of all (classical or general, indicated by '(cl)') pseudo-differential operators on M of order $\mu \in \mathbb{R}$ with parameter $\lambda \in \mathbb{R}^l$, i.e., the local (left-) symbols $a(x, \xi, \lambda)$ contain $(\xi, \lambda) \in \mathbb{R}^{n+l}$ as a covariable, $n = \dim M$, while $L^{-\infty}(M; \mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(M))$.

Let $L^\mu_{\mathrm{cl}}(2N; \mathbb{R}^l)_{\mathrm{tr}}$ for $\mu \in \mathbb{Z}$ denote the subspace of all $\tilde{A} \in L^\mu_{\mathrm{cl}}(2N; \mathbb{R}^l)$ the local symbols of which have the transmission property at ∂N. If e^+ denotes the extension by zero of distributions on $\operatorname{int} N$ (that belong locally to $H^s_{\mathrm{loc}}(2N)|_{\operatorname{int} N}$ for $s > -\frac{1}{2}$) by zero to the opposite side, and r^+ the operator of restriction of distributions on $2N$ to $\operatorname{int} N$, we can form the space of operators $L^\mu_{\mathrm{cl}}(N; \mathbb{R}^l)_{\mathrm{tr}} := \{A := \mathrm{r}^+ \tilde{A} \mathrm{e}^+ : \tilde{A} \in L^\mu_{\mathrm{cl}}(2N; \mathbb{R}^l)_{\mathrm{tr}}\}$.

An operator $C_0^\infty(\operatorname{int} N) \oplus C_0^\infty(\partial N, \mathbb{C}^{l_-}) \to C^\infty(\operatorname{int} N) \oplus C^\infty(\partial N, \mathbb{C}^{l_+})$ is called smoothing if the kernels of the entries (of the corresponding 2×2 block matrix) belong to $C^\infty(N \times N), C^\infty(N \times \partial N) \otimes \mathbb{C}^{j_-}, C^\infty(\partial N \times N) \otimes \mathbb{C}^{j_+}$ and $C^\infty(\partial N \times \partial N) \otimes \mathbb{C}^{j_+} \otimes \mathbb{C}^{j_-}$, respectively (with smoothness up to ∂N in the respective variables). The space of those operators is Fréchet, and we form $\mathcal{B}^{-\infty,0}(N; \mathbb{R}^l)$, defined as the space of all Schwartz functions in $\lambda \in \mathbb{R}^l$ with values in smoothing operators as explained before, for certain $j_-, j_+ \in \mathbb{N}$. More generally, $\mathcal{B}^{-\infty,\mathrm{d}}(N; \mathbb{R}^l)$ for any $\mathrm{d} \in \mathbb{N}$ is the space of all $\mathcal{C}(\lambda) := \mathcal{C}_0(\lambda) + \sum_{j=1}^{\mathrm{d}} \mathcal{C}_j(\lambda) \operatorname{diag}(D^j, 0)$ for arbitrary $\mathcal{C}_j \in \mathcal{B}^{-\infty,0}(N; \mathbb{R}^l)$ and any first order differential operator D on N that is equal to ∂_{x_n} in a tubular neighbourhood of ∂N.

All these notions have a straightforward generalisation to operators between distributional sections in smooth complex vector bundles on N and ∂N, respectively.

Let $(V_\iota)_{\iota \in \mathbb{N}}$ be a locally finite system of coordinate neighbourhoods on N near ∂N (i.e., $V'_\iota := V_\iota \cap \partial N \neq \emptyset$ for all $\iota \in \mathbb{N}$) such that V'_ι form an open covering by coordinate neighbourhoods of ∂N. Without loss of generality we assume $V_\iota = V'_\iota \times [0,1)$ with the above splitting of variables near ∂N.

Fix systems of functions $(\varphi_\iota)_{\iota\in\mathbb{N}}, (\psi_\iota)_{\iota\in\mathbb{N}}, \varphi_\iota, \psi_\iota \in C_0^\infty(V_\iota)$, such that $\sum_{\iota\in\mathbb{N}} \varphi_\iota \equiv 1$ in a neighbourhood of ∂N, and $\psi_\iota \equiv 1$ on $\operatorname{supp}\varphi_\iota$ for all ι. Set $\varphi'_\iota := \varphi_\iota|_{\partial N}$ and $\psi'_\iota := \psi_\iota|_{\partial N}$. Moreover, let $\omega \in C^\infty(N)$ be a function which is equal to 1 in a neighbourhood of ∂N and 0 outside $\partial N \times [0, \frac{1}{2})$. Consider charts $\chi_\iota : V'_\iota \to \Omega, \Omega \subseteq \mathbb{R}^{n-1}$ open. With symbols $g_\iota(x', \xi', \lambda)$ as in Example 4.2 we can associate operator families

$$\mathcal{G}_\iota(\lambda) := \operatorname{diag}(\omega\varphi_\iota, \varphi'_\iota)(\chi_\iota^{-1})_* \operatorname{Op}_{x'}(g_\iota)(\lambda) \operatorname{diag}(\omega\psi_\iota, \psi'_\iota), \tag{85}$$

where $\operatorname{Op}_{x'}(g_\iota)(\lambda)u(x') := \int\int e^{i(x'-\tilde{x}')\xi'} g_\iota(x', \xi', \lambda)u(\tilde{x}')d\tilde{x}' đ\xi'$ is interpreted as a map $C_0^\infty(\Omega \times \overline{\mathbb{R}}_+) \oplus C_0^\infty(\Omega, \mathbb{C}^{l_-}) \to C^\infty(\Omega \times \overline{\mathbb{R}}_+) \oplus C^\infty(\Omega, \mathbb{C}^{l_+})$. Then, if $\boldsymbol{v} := (l_-, l_+)$ denotes the pair of dimensions, by $\mathcal{B}_G^{\mu,\mathrm{d}}(N; \boldsymbol{v}; \mathbb{R}^l)$ for $\mu \in \mathbb{R}, \mathrm{d} \in \mathbb{N}$, we denote the space of all families $\mathcal{G}(\lambda) = \sum_{\iota\in\mathbb{N}} \mathcal{G}_\iota(\lambda) + \mathcal{C}(\lambda)$ for arbitrary $\mathcal{G}_\iota(\lambda)$ of the form (85) and $\mathcal{C}(\lambda) \in \mathcal{B}^{-\infty,\mathrm{d}}(N; \boldsymbol{v}; \mathbb{R}^l)$.

The sum over $\iota \in \mathbb{N}$ is locally finite. Moreover, if we form a similar operator as $\mathcal{G}(\lambda)$ with the same $g_\iota, \varphi_\iota, \psi_\iota$ but different ω, we obtain the original operator modulo $\mathcal{B}^{-\infty,\mathrm{d}}(N; \boldsymbol{v}; \mathbb{R}^l)$.

Definition 4.5 *The space* $\mathcal{B}^{\mu,\mathrm{d}}(N; \boldsymbol{v}; \mathbb{R}^l)$ *for any* $\mu \in \mathbb{Z}, \mathrm{d} \in \mathbb{N}, \boldsymbol{v} := (l_-, l_+)$ *is defined to be the set of all* $\mathcal{A}(\lambda) := \operatorname{diag}(A(\lambda), 0) + \mathcal{G}(\lambda)$ *for arbitrary* $A(\lambda) \in L_{\mathrm{cl}}^\mu(N; \mathbb{R}^l)_{\mathrm{tr}}$ *and* $\mathcal{G}(\lambda) \in \mathcal{B}_G^{\mu,\mathrm{d}}(N; \boldsymbol{v}; \mathbb{R}^l)$.

For $l = 0$ *we simply write* $\mathcal{B}^{\mu,\mathrm{d}}(N; \boldsymbol{v})$. *Moreover, we omit* $\boldsymbol{v}$ *when it is known from the context.*

Let us set

$$H^s_{(\mathrm{comp})}(\operatorname{int} N) := H^s_{\mathrm{comp}}(2N)|_{\operatorname{int} N}, \quad H^s_{(\mathrm{loc})}(\operatorname{int} N) := H^s_{\mathrm{loc}}(2N)|_{\operatorname{int} N}.$$

A standard property of the operators in $\mathcal{B}^{\mu,\mathrm{d}}(N; \boldsymbol{v}; \mathbb{R}^l)$ is the continuity

$$\begin{aligned} \mathcal{A} \quad : \quad & H^s_{(\mathrm{comp})}(\operatorname{int} N) \oplus H^{s-\frac{1}{2}}_{(\mathrm{comp})}(\partial N, \mathbb{C}^{l_-}) \\ \to \quad & H^{s-\mu}_{(\mathrm{loc})}(\operatorname{int} N) \oplus H^{s-\frac{1}{2}-\mu}_{(\mathrm{loc})}(\partial N, \mathbb{C}^{l_+}) \end{aligned} \tag{86}$$

for every $s > \mathrm{d} - \frac{1}{2}$ and every $\lambda \in \mathbb{R}^l$.

Next recall that operators $\mathcal{A} \in \mathcal{B}^{\mu,\mathrm{d}}(N; \boldsymbol{v}; \mathbb{R}^l)$ have parameter-dependent principal symbolic structure $\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$, where $\sigma_\psi(\mathcal{A}) := \sigma_\psi(A)$ is the (parameter-dependent) principal interior symbol, as a C^∞ function on $(T^*N \times \mathbb{R}^l) \setminus 0$ (where 0 indicates $(\xi, \lambda) = 0$) and $\sigma_\partial(\mathcal{A})$ the (parameter-dependent) principal boundary symbols as a family of continuous maps

$$\sigma_\partial(\mathcal{A})(x', \xi', \lambda) : H^s(\mathbb{R}_+) \oplus \mathbb{C}^{l_-} \to H^{s-\mu}(\mathbb{R}_+) \oplus \mathbb{C}^{l_+} \tag{87}$$

parametrised by $(x', \xi', \lambda) \in (T^*(\partial N) \times \mathbb{R}^l) \setminus 0$ (where 0 stands for $(\xi', \lambda) = 0$), $s > \mathrm{d} - \frac{1}{2}$.

Alternatively, we can also employ

$$\sigma_\partial(\mathcal{A})(x', \xi', \lambda) : \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{l_-} \to \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{l_+} \tag{88}$$

as the family of boundary symbols.

Definition 4.6 *An element $\mathcal{A}(\lambda) \in \mathcal{B}^{\mu,\mathrm{d}}(N; \boldsymbol{v}; \mathbb{R}^l)$ is said to be parameter-dependent elliptic if*

(i) $\sigma_\psi(\mathcal{A}) \neq 0$ *on* $(T^*N \times \mathbb{R}^l) \setminus 0$;

(ii) (88) *is an isomorphism* (*or, equivalently,* (87) *is an isomorphism for any* $s > \max(\mu, \mathrm{d}) - \frac{1}{2}$) *for every* $(x', \xi', \lambda) \in (T^*(\partial N) \times \mathbb{R}^l) \setminus 0$.

4.4 Operator families on an interval

Another situation when ∂N has several connected components is the case $N := I$ for an interval $I = [\alpha, \beta]$ on the real line. The operator families $\mathcal{A}(\lambda)$ of the space $\mathcal{B}^{\mu,\mathrm{d}}(I; \boldsymbol{v}; \mathbb{R}^l)$ then have the form

$$\mathcal{A}(\lambda) : H^s(I, \mathbb{C}^e) \oplus \mathbb{C}^{n_-} \oplus \mathbb{C}^{n_+} \to H^{s-\mu}(I, \mathbb{C}^{e'}) \oplus \mathbb{C}^{n'_-} \oplus \mathbb{C}^{n'_+},$$

continuous for $s > \mathrm{d} - \frac{1}{2}$; in this case $\boldsymbol{v} := (e, e'; n_-, n_+, n'_-, n'_+)$.

Since the latter case is basic for this exposition we want to formulate the classes of operator families for the case $e' = e = n_- = n_+ = n'_- = n'_+ = 1$ independently. The generalisation to arbitrary dimensions is then straightforward. Also for the case of different orders in the entries we can easily define corresponding operators if we first have formulated the operators for the same order μ in all entries. We will define the spaces $\mathcal{B}^{\mu,\mathrm{d}}(I; \mathbb{R}^l)$ for $\mu \in \mathbb{Z}, \mathrm{d} \in \mathbb{N}$ and $\mathcal{B}_G^{\mu,\mathrm{d}}(I; \mathbb{R}^l)$ for arbitrary $\mu \in \mathbb{R}$.

Let $\mathcal{B}_G^{-\infty,0}(I)$ defined to be the space of all 3×3 block matrix operators $g = (g_{ij})_{i,j=1,2,3} : H^s(I) \oplus \mathbb{C}^2 \to C^\infty(I) \oplus \mathbb{C}^2, s > -\frac{1}{2}$, where g_{11} is an integral operator with kernel in $C^\infty(I \times I)$, $g_{1j}c := f_{1j}(\phi)c$ for $j = 2, 3$, $c \in \mathbb{C}$, $g_{i1}u = \int_I f_{i1}(\phi)u(\phi)d\phi$ for $i = 2, 3$, with arbitrary functions $f_{1j}, f_{i1} \in C^\infty(I)$ for $i, j = 2, 3$, and an arbitrary 2×2 matrix $(g_{ij})_{i,j=2,3}$ with entries in $\mathbb{C}$. The components of $c = (c_\alpha, c_\beta) \in \mathbb{C}^2$ are related to the end points $\{\alpha\}$ and $\{\beta\}$ of the interval I. The space $\mathcal{B}_G^{-\infty,0}(I)$ is Fréchet in a natural way (as a direct sum of its 9 components), and we set $\mathcal{B}_G^{-\infty,0}(I; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, \mathcal{B}_G^{-\infty,0}(I))$. Moreover, let $\mathcal{B}_G^{-\infty,\mathrm{d}}(I; \mathbb{R}^l)$ for arbitrary $\mathrm{d} \in \mathbb{N}$ be the space of all operator families $g(\lambda) := g_0(\lambda) + \sum_{j=1}^{\mathrm{d}} G_j(\lambda)\mathrm{diag}(\partial_\phi^j, 0, 0)$ for arbitrary $g_j \in \mathcal{B}_G^{-\infty,0}(I; \mathbb{R}^l)$.

Let us now consider 2×2 block matrix symbols $f(\lambda)$ of the class

$$S_{\mathrm{cl}}^\mu(\mathbb{R}^l; L^2(\mathbb{R}_+) \oplus \mathbb{C},\ \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}), \tag{89}$$

where the group actions on $L^2(\mathbb{R}_+) \oplus \mathbb{C}$ or $\mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}$ are defined by $u(\phi) \oplus c \to \lambda^{\frac{1}{2}} u(\lambda\phi) \oplus c$, $\lambda \in \mathbb{R}_+$ such that the pointwise adjoint $f^*(\lambda)$ with respect to the $L^2(\mathbb{R}_+) \oplus \mathbb{C}$ scalar product belongs to the space (89).

With every such $f(\lambda)$ we can associate an operator family

$$a(\lambda) := \omega f(\lambda) \tilde{\omega} : H^s(I) \oplus \mathbb{C} \to C^\infty(I) \oplus \mathbb{C}, \tag{90}$$

$s > -\frac{1}{2}$, for any fixed choice of functions $\omega, \tilde{\omega} \in C^\infty(I)$ which are equal to 1 near $\phi = \alpha$ and vanish in a neighbourhood of the end point β. In a similar manner we can form operators

$$b(\lambda) := \chi_*(\omega f(\lambda) \tilde{\omega}) : H^s(I) \oplus \mathbb{C} \to C^\infty(I) \oplus \mathbb{C} \tag{91}$$

where $\chi : I \to I$ is the diffeomorphism defined by $\chi(\phi) := -\phi + \alpha + \beta$ which interchanges the role of α and β. In other words, the direct summands $\mathbb{C}$ in the spaces of (90) belong to the end point α, those in the spaces of (91) to the end point β. Writing (90) and (91) as block matrices with entries a_{ij} and b_{ij}, respectively, we now form

$$g(\lambda) := \begin{pmatrix} a_{11} + b_{11} & a_{12} & b_{12} \\ a_{21} & a_{22} & 0 \\ b_{21} & 0 & b_{22} \end{pmatrix} : H^s(I) \oplus \mathbb{C}^2 \to C^\infty(I) \oplus \mathbb{C}^2. \tag{92}$$

The space $\mathcal{B}_G^{\mu,\mathrm{d}}(I; \mathbb{R}^l)$ for $\mu \in \mathbb{R}$, $\mathrm{d} \in \mathbb{N}$ is defined to be the set of all operator functions $g_0(\lambda) + \sum_{j=1}^{\mathrm{d}} g_j(\lambda) \mathrm{diag}(\partial_\phi^j, 0, 0) + c(\lambda)$ for arbitrary $g_j(\lambda)$ of the kind (92), of order $\mu - j$, and $c(\lambda) \in \mathcal{B}_G^{-\infty,\mathrm{d}}(I; \mathbb{R}^l)$. Let $B_G^{\mu,\mathrm{d}}(I; \mathbb{R}^l)$ denote the space of upper left corners of elements of $\mathcal{B}_G^{\mu,\mathrm{d}}(I; \mathbb{R}^l)$.

Remark 4.7 *The space $\mathcal{B}_G^{\mu,\mathrm{d}}(I; \mathbb{R}^l)$ has a natural Fréchet topology. So we can form spaces of the kind $C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{B}_G^{\mu,\mathrm{d}}(I; \mathbb{R}^l)), U \subseteqq \mathbb{R}^p$ open, or $\mathcal{A}(D, \mathcal{B}_G^{\mu,\mathrm{d}}(I; \mathbb{R}^l))$ for an open set $D \subseteqq \mathbb{C}$..*

Let $S_{\mathrm{cl}}^\mu(I \times \mathbb{R}_\vartheta \times \mathbb{R}_\lambda^l)_{\mathrm{tr}}$ denote the space of all classical symbols of order $\mu \in \mathbb{Z}$ in the variable ϕ and covariables (ϑ, λ) (with ϑ being the covariable to ϕ). Recall that the transmission property at the end points of the interval I (for instance, at $\phi = \alpha$) of a symbol $a(\phi, \vartheta, \lambda)$ means that the homogeneous components $a_{(\mu-j)}(\phi, \vartheta, \lambda)$ of order $\mu - j$ satisfy the conditions

$$D_\phi^k D_\lambda^\gamma \{a_{(\mu-j)}(\phi, \vartheta, \lambda) - (-1)^{\mu-j} a_{(\mu-j)}(\phi, -\vartheta, -\lambda)\} = 0$$

on the set $\{(\phi, \vartheta, \lambda) : \phi = \alpha,\ \vartheta \in \mathbb{R} \setminus \{0\}, \lambda = 0\}$ for all $k \in \mathbb{N}$, $\gamma \in \mathbb{N}^l$, and all $j \in \mathbb{N}$. A similar condition is imposed at $\phi = \beta$.

Given an $a \in S_{\mathrm{cl}}^\mu(I \times \mathbb{R}_{\vartheta,\lambda}^{1+l})_{\mathrm{tr}}$ we set $\mathrm{op}^I(a)(\lambda) u(\phi) := \mathrm{r}\,\mathrm{op}(\tilde{a})(\lambda)\,\mathrm{e}u(\phi)$ where $\tilde{a}(\phi, \vartheta, \lambda) \in S_{\mathrm{cl}}^\mu(\mathbb{R} \times \mathbb{R}_{\vartheta,\lambda}^{1+l})$ is any symbol such that $a = \tilde{a}|_{I \times \mathbb{R}_{\vartheta,\lambda}^{1+l}}$; here e is the operator of extension by zero from I to $\mathbb{R} \setminus I$ and r the restriction to $\mathrm{int}\, I$. We then have continuous operators $\mathrm{op}^I(a)(\lambda) : H^s(I) \to H^{s-\mu}(I)$ for all reals $s > -\frac{1}{2}$ (clearly the operators do not depend on the choice of $\tilde{a}$).

Definition 4.8 *The space* $B^{\mu,\mathrm{d}}(I;\mathbb{R}^l_\lambda)$ *for* $\mu \in \mathbb{Z}$, $\mathrm{d} \in \mathbb{N}$, *is defined to be the set of all operator families of the form* $\mathrm{op}^I(a)(\lambda) + g(\lambda)$ *for arbitrary* $a \in S^\mu_{\mathrm{cl}}(I \times \mathbb{R}^{1+l}_{\vartheta,\lambda})_{\mathrm{tr}}$ *and* $g \in B^{\mu,\mathrm{d}}_G(I;\mathbb{R}^l)$.

Set $\mathcal{B}^{\mu,\mathrm{d}}(I;\mathbb{R}^l) := \{\mathrm{diag}\,(p,0,0) + g : p \in B^{\mu,\mathrm{d}}(I;\mathbb{R}^l), g \in \mathcal{B}^{\mu,\mathrm{d}}_G(I;\mathbb{R}^l)\}$. *In the case* $p \neq 0$ *we assume* $\mu \in \mathbb{Z}$, *otherwise we admit* $\mu \in \mathbb{R}$.

The space $\mathcal{B}^{\mu,\mathrm{d}}(I;\mathbb{R}^l)$ is Fréchet in a natural way.

4.5 Holomorphic Mellin symbols and edge quantisation

Definition 4.9 (i) *The space* $L^\mu_{\mathrm{cl}}(M;\mathbb{C} \times \mathbb{R}^l), l \in \mathbb{N}$, *for a* C^∞ *manifold* M, *is defined to be he set of all* $f(w,\lambda) \in \mathcal{A}(\mathbb{C}, L^\mu_{\mathrm{cl}}(M;\mathbb{R}^l))$ *such that* $f(\beta + i\tau, \lambda) \in L^\mu_{\mathrm{cl}}(M;\mathbb{R}^{l+1})$ *for every* $\beta \in \mathbb{R}$, *uniformly in compact* β*-intervals.*

(ii) *If* N *is a* C^∞ *manifold with boundary, the space* $\mathcal{B}^{\mu,\mathrm{d}}(N;\mathbb{C}\times\mathbb{R}^l), \mathrm{d}, l \in \mathbb{N}$, *is defined to be the set of all* $f(w,\lambda) \in \mathcal{A}(\mathbb{C}, \mathcal{B}^{\mu,\mathrm{d}}(N;\mathbb{R}^l))$ *such that* $f(\beta + i\tau, \lambda) \in \mathcal{B}^{\mu,\mathrm{d}}(N;\mathbb{R}^{l+1})$ *for every* $\beta \in \mathbb{R}$, *uniformly in compact* β*-intervals.*

(iii) *The space* $\mathcal{B}^{-\infty,\mathrm{d}}(N;\Gamma_\beta\times\mathbb{R}^l)_\varepsilon, \mathrm{d} \in \mathbb{N}$, *is the set of all* $f(w,\lambda) \in \mathcal{A}(\{\beta - \varepsilon < \mathrm{Re}\, z < \beta + \varepsilon\}, \mathcal{B}^{-\infty,\mathrm{d}}(N;\mathbb{R}^l))$ *for every* $\varepsilon > 0$ *such that* $f(\delta + i\tau, \lambda) \in \mathcal{S}(\mathbb{R}^{1+l}_{\tau,\lambda}, \mathcal{B}^{-\infty,\mathrm{d}}(N))$ *for every* $\delta \in (\beta - \varepsilon, \beta + \varepsilon)$, *uniformly in compact subintervals.*

Let us now consider a 'wedge' $U := N^\triangle \times T$ for a compact C^∞ manifold N with boundary, $n := \dim N$, and a C^∞ manifold $T, q := \dim T$, $N^\triangle = (\overline{\mathbb{R}}_+ \times N)/(\{0\} \times N)$. Moreover, let $\mathbb{U} := \overline{\mathbb{R}}_+ \times N \times T \ni (r,x,t)$ denote the associated stretched wedge. According to the general notation we have the spaces $\mathbb{U}_{\mathrm{reg}} := \mathbb{R}_+ \times N \times T$ and $\mathbb{U}_{\mathrm{sing}} := \{0\} \times N \times T$. In the following definitions we assume, for simplicity, the case that T is diffeomorphic to an open set in $\mathbb{R}^q$ (this is the situation we really need in Definition 3.1 for $q = 1$ and $T = Z \setminus \{v\}$); the general case is similar.

We form operator functions $\mathfrak{a}(t,\tau) \in S^\mu(T \times \mathbb{R}^q; E, \tilde{E})$ for the spaces

$$E := \mathcal{K}^{s,\gamma}(N^\wedge) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial N)^\wedge, \mathbb{C}^{l_-}) \oplus \mathbb{C}^{j_-}, \tag{93}$$

$$\tilde{E} := \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \oplus \mathcal{K}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}((\partial N)^\wedge, \mathbb{C}^{l_+}) \oplus \mathbb{C}^{j_+}$$

which have the following form:

$$\begin{aligned} & \mathfrak{a}(t,\tau) := \mathrm{diag}\,(\sigma,\sigma,0)\{r^{-\mu}\omega(r[\tau])\mathrm{op}_{\boldsymbol{M}}^{\gamma-\frac{n}{2}}(h)(t,\tau)\tilde{\omega}(r[\tau]) \\ + \quad & r^{-\mu}(1-\omega(r[\tau]))\mathrm{op}_r(p)(t,\tau)(1-\tilde{\tilde{\omega}}(r[\tau])) \\ + \quad & r^{-\mu}\omega(r[\tau])\mathrm{op}_{\boldsymbol{M}}^{\gamma-\frac{n}{2}}(f)(t)\tilde{\omega}(r[\tau])\}\mathrm{diag}\,(\tilde{\sigma},\tilde{\sigma},0) + \mathfrak{g}(t,\tau). \end{aligned}$$

Here $\omega, \tilde{\omega}, \tilde{\tilde{\omega}}$ and $\sigma, \tilde{\sigma}$ are cut-off functions as in (37). Moreover, we assume $p(r,t,\rho,\tau) := \tilde{p}(r,t,r\rho,r\tau), \quad h(r,t,w,\tau) := \tilde{h}(r,t,w,r\tau)$ for operator families

$$\tilde{p}(r,t,\xi,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T, \mathcal{B}^{\mu,\mathrm{d}}(N;\mathbb{R}^{1+q}_{\xi,\eta})), \tag{94}$$

$$\tilde{h}(r,t,w,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times T, \mathcal{B}^{\mu,\mathrm{d}}(N;\mathbb{C}_w \times \mathbb{R}^q_\eta)), \tag{95}$$

such that

$$\mathrm{op}_r(p)(t,\tau) = \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(t,\tau) \mod C^\infty(T, \mathcal{B}^{-\infty,\mathrm{d}}(N^\wedge;\mathbb{R}^q_\tau)) \tag{96}$$

(for every weight $\gamma \in \mathbb{R}$). For $p_0(r,t,\rho,\tau) := \tilde{p}(0,t,r\rho,r\tau), \quad h_0(r,t,w,\tau) := \tilde{h}(0,t,w,r\tau)$ we then also have a relation of the form (96). Furthermore, we assume

$$f(t,w) \in C^\infty(T, \mathcal{M}^{-\infty,\mathrm{d}}(N;\Gamma_{\frac{n+1}{2}-\gamma})_\varepsilon) \tag{97}$$

for some $\varepsilon = \varepsilon(f) > 0$. Here $\mathcal{M}^{-\infty,\mathrm{d}}(N;\Gamma_\beta)_\varepsilon$ denotes the space of all $f(w) \in \mathcal{A}(\{\beta-\varepsilon < \mathrm{Re}\, w < \beta+\varepsilon\}, \mathcal{B}^{-\infty,\mathrm{d}}(N))$ such that $f(\delta+i\rho) \in \mathcal{B}^{-\infty,\mathrm{d}}(N;\mathbb{R}_\rho)$ for every $\delta \in (\beta-\varepsilon, \beta+\varepsilon)$, uniformly in compact subintervals. Finally, we assume

$$\mathfrak{g}(t,\tau) \in S^\mu_{\mathrm{cl}}(T \times \mathbb{R}^q; E, \mathcal{S}_\varepsilon) \tag{98}$$

for some $\varepsilon = \varepsilon(\mathfrak{g}) > 0$, where E is as in (93) for arbitrary $s > \mathrm{d} - \frac{1}{2}$, and $\mathcal{S}_\varepsilon := \mathcal{S}^{\gamma-\mu}_\varepsilon(N^\wedge) \oplus \mathcal{S}^{\gamma-\mu-\frac{1}{2}}_\varepsilon((\partial N)^\wedge, \mathbb{C}^{l_+}) \oplus \mathbb{C}^{j_+}$, and the pointwise adjoint $\mathfrak{g}^*(t,\tau)$ satisfies a similar condition.

Now

$$\mathfrak{Y}^{\mu,\mathrm{d}}(\mathbb{U}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$$

denotes the set of all operators of the form

$$\mathrm{Op}_t(\mathfrak{a}) + \mathfrak{B} + \mathfrak{G}. \tag{99}$$

Here $\mathfrak{a}(t,\tau)$ are operator functions with arbitrary (94), (95) satisfying (96) and arbitrary (97), (98).

Moreover, $\mathfrak{B} := \mathrm{diag}\,(1-\sigma, 1-\sigma, 0)\mathcal{B}\,\mathrm{diag}\,(1-\tilde{\tilde{\sigma}}, 1-\tilde{\tilde{\sigma}}, 0)$ for a third cut-off function $\tilde{\tilde{\sigma}}(r)$ such that $\sigma \equiv 1$ on $\mathrm{supp}\,\tilde{\tilde{\sigma}}$ and arbitrary $\mathcal{B} \in \mathcal{B}^{\mu,\mathrm{d}}(\mathbb{R}_+ \times N \times T)$. Finally, $\mathfrak{G} := \mathfrak{G}_0 + \sum_{j=1}^{\mathrm{d}} \mathfrak{G}_j \mathrm{diag}\,(D^j, 0, 0)$ is smoothing; here D is a first order differential operator which is equal to ∂_ν near $\mathbb{R}_+ \times \partial N \times T$ with the normal variable ν to ∂N; the operators $\mathfrak{G}_j$ are defined by the mapping property

$$\begin{aligned} &\mathcal{W}^{s,\gamma}_{\mathrm{comp}}(N^\wedge \times T) \oplus \mathcal{W}^{s',\gamma-\frac{1}{2}}_{\mathrm{comp}}((\partial N)^\wedge \times T) \oplus H^{s''}_{\mathrm{comp}}(T, \mathbb{C}^{j_-}) \\ \to\ &\mathcal{W}^{\infty,\gamma-\mu+\varepsilon}_{\mathrm{loc}}(N^\wedge \times T) \oplus \mathcal{W}^{\infty,\gamma-\mu-\frac{1}{2}+\varepsilon}_{\mathrm{loc}}((\partial N)^\wedge \times T) \oplus H^\infty_{\mathrm{loc}}(T, \mathbb{C}^{j_+}) \end{aligned} \tag{100}$$

for every $s, s', s'' \in \mathbb{R}, s > \mathrm{d} - \frac{1}{2}$, and an $\varepsilon > 0$, depending on the operator, and a similar mapping property for the formal adjoint. In (100) the subscripts 'comp' and 'loc' at the edge spaces refer to the variable t, e.g., $\mathcal{W}^{s,\gamma}_{\mathrm{comp}}(N^\wedge, T) := \mathcal{W}^s_{\mathrm{comp}}(T, \mathcal{K}^{s,\gamma}(N^\wedge))$, etc.

Remark 4.10 *The space* $\mathfrak{Y}^{\mu,\mathrm{d}}(\mathbb{U}, \boldsymbol{g}_{\mathrm{cone}}; j_-, j_+)$ *is nothing other than the space of pseudo-differential edge boundary value problems on the* (*stretched*) *wedge* $\mathbb{U}$, *of order* μ *and type* d. *As such we have the material on the principal symbolic structure* $\sigma = (\sigma_\psi, \sigma_\partial, \sigma_\wedge)$, *ellipticity and parametrices, as is studied in* [9]. *For Definition* 3.1 (ii) *we only need the summand* $\mathrm{Op}_t(\mathfrak{a})$ *from* (99) *when the functions* $\Phi_{\mathrm{edge}}, \Psi_{\mathrm{edge}}$ *are chosen in such a way that* $\sigma \equiv 1$ *on* $\mathrm{supp}\,\varphi_1$, $\tilde{\sigma} \equiv 1$ *on* $\mathrm{supp}\,\psi_1$, *since the terms* $\mathfrak{B}$ *and* $\mathfrak{G}$ *are included in Definition* 3.1 (iii) *and* (iv), *respectively.*

References

[1] **L. Boutet de Monvel**: *Boundary problems for pseudo-differential operators.* Acta Math. **126** (1971), 11-51.

[2] **D. Calvo, C.-I. Martin, and B.-W. Schulze**: *Symbolic structures on corner manifolds.* Preprint 2004/10, Institut für Mathematik, Potsdam, 2004.

[3] **N. Dines, G. Harutyunyan, and B.-W. Schulze**: *The Zaremba Problem in Edge Sobolev Spaces.* Preprint 2003/13, Institut für Mathematik, Potsdam, 2003.

[4] **G. De Donno and B.-W. Schulze**: *Meromorphic symbolic structures for boundary value problems on manifolds with edges.* Preprint 2003/10, Institut für Mathematik, Potsdam, 2003. Math. Nachr. (to appear).

[5] **Ch. Dorschfeldt**: *Algebras of pseudo-differential operators near edge and corner singularities.* Vol. **102** of Math. Res. Wiley-VCH, Berlin, Weinheim, 1998.

[6] **G. Harutyunyan and B.-W. Schulze**: *Parametrices of mixed elliptic problems.* Math. Nachr. **277** (2004), 1-27.

[7] **G. Harutyunyan and B.-W. Schulze**: *The relative index for corner singularities.* Integral Equations Operator Theory (to appear).

[8] **T. Hirschmann**: *Functional analysis in cone and edge Sobolev spaces.* Ann. Global Anal. Geom., **8**, 2 (1990), 167-192.

[9] **D. Kapanadze and B.-W. Schulze**: *Crack theory and edge singularities.* Kluwer Academic Publ., Dordrecht, 2003.

[10] **V.A. Kondratyev**: *Boundary value problems for elliptic equations in domains with conical points.* Trudy Mosk. Mat. Obshch., 16:209-292, 1967.

[11] **T. Krainer**: *On the inverse of parabolic boundary value problems for large times.* Japan. J. Math., **30**, 1 (2004), 91-163.

[12] **J.E. Lewis and C. Parenti**: *Pseudo-differential operators of Mellin type.* Comm. Partial Differential Equations, **8**, 5 (1983), 447-544.

[13] **A. Oliaro and B.-W. Schulze**: *Parameter-dependent boundary value problems on manifolds with edges.* Int. Journal of Pure and Applied Math., **13**, 1 (2004), 39-100.

[14] **B.-W. Schulze**: *Pseudo-differential operators on manifolds with edges.* In Symposium "Partial Differential Equations", Holzhau 1988, Teubner-Texte zur Mathematik, Vol. **112**, Teubner, Leipzig, 1989, pp. 259-287.

[15] **B.-W. Schulze**: *The Mellin pseudo-differential calculus on manifolds with corners.* In Symposium: "Analysis in Domains and on Manifolds with Singularities", Breitenbrunn 1990, Teuber-Texte zur Mathematik, Vol. **131**, Teubner, Leipzig, 1992, pp. 208-289.

[16] **B.-W. Schulze**: *Operators with symbol hierarchies and iterated asymptotics.* Publications of RIMS, Kyoto University, **38**, 4 (2002), 735-802.

[17] **B.-W. Schulze**: *Crack theory with singularities at the boundary.* Pliska Stud. Math. Bulgar., **15** (2003), 21-66.

[18] **M. Vishik and G. Eskin**: *General problems with discontinuous boundary conditions.* (Russian) Dokl. Akad. Nauk SSSR **158**, 1 (1964), 25-28.

[19] **S. Zaremba**: *Sur un probleme mixte relatif a l'equation de Laplace.* Bull. de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles, Serie A (1910), 313-344.

[20] **S. Zargaryan and V. Maz'ya**: *Singularities of solutions of a system of equations of potential theory for Zaremba's problem.* (Russian) Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1** (1983), 43-48.