GREEN OPERATORS IN THE EDGE CALCULUS

B.-W. SCHULZE AND A. VOLPATO

ABSTRACT. Green operators on manifolds with edges are known to be an ingredient of parametrices of elliptic (edge-degenerate) operators. They play a similar role as corresponding operators in boundary value problems. Close to edge singularities the Green operators have a very complex asymptotic behaviour. We give a new characterisation of Green edge symbols in terms of kernels with discrete and continuous asymptotics in the axial variable of local model cones.

CONTENTS

INTRODUCTION

This paper is aimed at studying the structure of so called Green operators on a manifold W with edges (with or without boundary). Green operators as well as trace and potential operators appear in problems of mechanics, for instance, in crack theory, cf. the monograph [7], or the paper [8] on boundary contact problems with interfaces having singularities. Another category of problems with edges are mixed boundary value problems, e.g., the Zaremba problem, where boundary conditions have a jump along an interface on the boundary, cf. [3]. In all these

problems we have to expect asymptotics of solutions. The machinary is to construct parametrices and to state elliptic regularity of solutions in weighted Sobolev spaces with asymptotics; then the Green operators are just the crucial 'remainders' which encode the asymptotic information. It is therefore of interest to characterise Green operators in a manageable form, despite on the fact that they may appear in a form which does not reveal at once their nature.

Another important aspect is that boundary (or edge) conditions of trace and potential type can be subsumed under the concept of Green operators. Moreover, the behaviour of potentials of surface distributions and various kinds, or jump relations may be understood in terms of properties of Green operators, cf. [6]. The composition of a trace and a potential operator is of Green type; the composition in the other way gives us a pseudo-differential operator on the boundary (or edge). This is a background of the well known reduction of problems to the boundary (or edge). Moreover, as noted before, crack problems can be interpreted as edge problems when we assume smoothness of the crack boundary (otherwise, such problems belong to a higher singular calculus). The crack boundary in this case is the edge, and the model cone of local wedges is the slit normal plane to the crack boundary, cf. [7].

Locally a manifold with edges is modelled on a wedge $X^\triangle \times \Omega$ for an open set $\Omega \subseteq \mathbb{R}^q$ and a model cone $X^\triangle := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$, with a base X that is assumed to be a compact C^∞ manifold (with or without boundary). A special case of such a wedge is the 'half-space' $\overline{\mathbb{R}}_+ \times \Omega$ which corresponds to $\dim X = 0$. As is well known, parametrices of classical elliptic boundary value problems on a (say, compact) C^∞ manifold W with boundary, contain Green's function as an ingredient of solution operators (or parametrices). For instance, if we write the solution u of the Dirichlet problem $\Delta u = f$, $u|_{\partial W} = 0$ (with Δ being a Laplace operator on W) in the form $u = Pf$, then we have $P = G + E$ where E is a fundamental solution (or a parametrix) of Δ and G a Green operator in our sense, cf. [1].

Similarly, the Green operators in the edge calculus are pseudo-differential along the edge, cf. [4], [12]. Their symbols (in the framework of twisted homogeneity) take values in so called Green operators on the model cone of local wedges. Green operators in the cone algebra with discrete asymptotics admit useful kernel characterisations, cf. [15]. They can be employed to characterise Green operators of the edge calculus locally by integral expressions of a specific kind, cf. [14] for the case of discrete asymptotics. The relevance of such representations lies in the fact that they are transparent and concise, although, as noted before, the Green operators of the edge calculus appear in various operations as remainders that may be of an extremely complex structure.

Let us stress that Green operators are a typical 'pseudo-differential' effect in the analysis on configurations with singularities. In fact, if we ask a calculus which is able to express parametrices of elliptic differential boundary problems in a domain with conical singularities or edges, cf. [9] or [10], then the answer may be produced in terms of pseudo-differential and Green operators, cf. [7] or [12].

In order to illustrate the origin of asymptotic data in solutions, consider, for instance, an edge-degenerate differential operator A on a manifold W with edge Y which is locally near Y in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$ of the form

$$(0.0.1) \qquad A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y) \, (-r\partial_r)^j \, (rD_y)^\alpha$$

with $\mathrm{Diff}^{\mu-(j+|\alpha|)}(X)$-valued coefficients $a_{j\alpha}$, smooth up to $r = 0$. Ellipticity of A means (apart from the standard ellipticity on $W \setminus Y$) that

$$\sum_{j+|\alpha| \leq \mu} a_{j\alpha}(0, y) \, (-i\widetilde{\varrho})^j \, \widetilde{\eta}^\alpha$$

is parameter-dependent elliptic on X with parameters $(\widetilde{\varrho}, \widetilde{\eta}) \in \mathbb{R}^{1+q}$, for every $y \in \Omega$. Elliptic regularity of solutions to the equation $Au = f$ near Y in weighted edge Sobolev spaces can be obtained by constructing a left parametrix P such that $PA = 1 + G$, where G is just a Green operator, that is smoothing in $W \setminus Y$, but with a specific singular kernel near Y. The parametrix P is essentially a Mellin pseudo-differential operator in the axial variable r. The singularities are determined by the non-bijectivity points $z \in \mathbb{C}$ of the operator family

$$(0.0.2) \qquad \sigma_M(A)(y, z) := \sum_{j=0}^{\mu} a_{j0}(0, y) z^j : H^s(X) \to H^{s-\mu}(X)$$

in Sobolev spaces on X (for simplicity, we assume here that X is a closed C^∞ manifold). More precisely, the pointwise inverses $\sigma_M(A)^{-1}(y, z)$ form a y-dependent family of meromorphic operator functions. These appear as symbols of Mellin operators along the (stretched) model cone $X^\wedge := \mathbb{R}_+ \times X$ which are then involved in operator-valued symbols depending on $(y, \eta) \in T^*Y \setminus 0$. The poles of $\sigma_M(A)^{-1}(y, z)$ including multiplicities and Laurent coefficients may depend on y; they constitute the asymptotic type of the Mellin operators and finally contribute to the singular kernels of Green operators. In this paper we do not develop all elements of the asymptotic properties of parametrices of elliptic edge degenerate operators; more details may be found in [12]. Let us only mention here, that the y-depending asymptotic types motivate the concept of continuous asymptotics of weighted Sobolev distributions on a cone which is necessary to express the mapping properties of Green operators. The present paper is organised as follows.

In Chapter 1 we first formulate Green operators and Green edge symbols modelled on spaces with constant discrete asymptotics. The main feature is the pseudo-differential structure along the edge $\mathbb{R}^q$ with symbols that operate along the infinite (stretched) model cone $X^\wedge$ of the local wedges. The symbolic estimates are based on the action of a strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ on the parameter spaces on $X^\wedge$ with asymptotics for $r \to 0$. For the analysis later on it is essential to represent those spaces in a convenient manner, as projective limits of Hilbert spaces that are preserved under the action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$.

Chapter 2 starts with a simple class of Green symbols with discrete asymptotics, represented in integral form (2.1.2) with kernels on the infinite cone, tensorised with scalar symbols on the edge. The main result is that in fact every Green operator

is of that form (Theorem 2.1.1). We then formulate an analogous result for the trace and potential symbols of the edge calculus. Compared with [14] our starting point is more general; we do not employ projective tensor products of spaces with asymptotics but a much weaker assumption. This allows us to obtain analogous results also for Green operators with continuous asymptotics.

Chapter 3 starts with a brief description of spaces with continuous asymptotics, motivated by parameter-dependent discrete asymptotics, where poles and multiplicities of Mellin transformed functions may vary along the edge variable y. The Green operators on the model cone $X^\wedge$ then belong to the tensor product of corresponding spaces with continuous asymptotics. By similar methods as in Chapter 1 we then obtain a general integral representation of the corresponding operator-valued Green edge (as well as of trace and potential) symbols.

Chapter 4 studies Green operators with (discrete or continuous) asymptotics on a configuration with edges.

1. Edge operators of Green type

1.1. **Asymptotics near conical singularities.** Green operators in the context of classical boundary value problems are (locally in a collar neighbourhood of the boundary) pseudo-differential operators along the boundary with symbols acting as operators normal to the boundary. More precisely, the values of the symbols are operators G in $L^2(\mathbb{R}_+)$ such that $G, G^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+)(= \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+})$ are continuous (this concerns the so called type zero; otherwise the operators are combined with differentiations transversal to the boundary). In the generalisation to the case of a manifold with edges we replace the inner normal $\overline{\mathbb{R}}_+$ by a non-trivial model cone $X^\triangle := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ belonging to corresponding local wedges, and the spaces $L^2(\mathbb{R}_+)$ and $\mathcal{S}(\overline{\mathbb{R}}_+)$ by weighted spaces $\mathcal{K}^{0,\gamma}(X^\wedge)$ and $\mathcal{S}_{\mathcal{P}}^\beta(X^\wedge)$, respectively, on the open stretched cone $X^\wedge := \mathbb{R}_+ \times X \ni (r, x)$ with a certain behaviour for $r \to 0$, encoded by a so called asymptotic type $\mathcal{P}$. In simplest cases asymptotics will have the form

$$(1.1.1) \qquad u(r, x) \sim \sum_j \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r \qquad \text{for} \quad r \to 0$$

with a sequence of triples $\mathcal{P} := \{(p_j, m_j, L_j)\}_{j=0,1,\ldots,N}$, $N \in \mathbb{N} \cup \{\infty\}$, $p_j \in \mathbb{C}$, $m_j \in \mathbb{N}$, and finite-dimensional subspaces $L_j \subset C^\infty(X)$, such that $c_{jk} \in L_j$ for all $0 \le k \le m_j$, and all j. For $\dim X = 0$ we have, in particular, a natural identification of $\mathcal{S}(\overline{\mathbb{R}}_+)$ with $\mathcal{S}_T^0(\mathbb{R}_+)$ for the Taylor asymptotic type $T = \{(-j, 0)\}_{j=0,1,\ldots}$ (the spaces L_j disappear in this case). Let us now pass to the precise definitions.

We say that $\mathcal{P}$ is associated with weight data (γ, Θ) for a weight $\gamma \in \mathbb{R}$ and $\Theta = (\vartheta, 0]$ for some $-\infty \le \vartheta < 0$ if the set $\pi_{\mathbb{C}}\mathcal{P} = \{p_j\}_{0 \le j \le N}$ is contained in the strip $\left\{ \frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} z < \frac{n+1}{2} - \gamma \right\}, n = \dim X$, $\pi_{\mathbb{C}}\mathcal{P}$ finite for finite ϑ, and $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ when $\vartheta = -\infty$ and $N = \infty$.

In this paper a cut-off function on the half-axis is any $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ equal to 1 near 0.

Given $\mathcal{P}$ associated with (γ, Θ) for a finite weight interval Θ we set

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) := \Big\{ \sum_{j=0}^{N} \sum_{k=0}^{m_j} \omega(r) c_{jk} r^{-p_j} \log^k r : c_{jk} \in L_j \text{ for } 0 \leq k \leq m_j, \, 0 \leq j \leq N \Big\}.$$

Moreover, let $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$, denote the subspace of all $u(r,x) \in r^{\gamma - \frac{n}{2}} L^2(X^\wedge)$ (with L^2 referring to $drdx$) such that

$$(r\partial_r)^k D_x^\alpha u(r,x) \in r^{\gamma - \frac{n}{2}} L^2(X^\wedge)$$

for every $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, $k + |\alpha| \leq s$; here $D_x^\alpha := v_1^{\alpha_1} \cdot \ldots \cdot v_n^{\alpha_n}$ means the differentiation with arbitrary vector fields v_j on X. In particular, we have $\mathcal{H}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(X^\wedge)$. Then duality and interpolation give us a definition of $\mathcal{H}^{s,\gamma}(X^\wedge)$ for arbitrary $s, \gamma \in \mathbb{R}$.

There is another useful scale of weighted spaces on $X^\wedge$ defined by

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega f + (1-\omega)g : f \in \mathcal{H}^{s,\gamma}(X^\wedge), \, g \in H_{\mathrm{cone}}^s(X^\wedge)\}$$

for some cut-off function ω; here $H_{\mathrm{cone}}^s(X^\wedge)$ is defined to be the subspace of all $g \in H_{\mathrm{loc}}^s(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}$ such that for every chart $\chi : \mathcal{U} \to B$ on X to $B := \{x \in \mathbb{R}^n : |x| < 1\}$ and every $\varphi \in C_0^\infty(\mathcal{U})$ we have

$$(1-\omega)\varphi g \in (\beta \circ (1 \times \chi))^* H^s(\mathbb{R}^{n+1})|_\Gamma$$

for $\Gamma := \big\{(r, \widetilde{x}) \in \mathbb{R}^{n+1} : r \in \mathbb{R}_+, \, \widetilde{x}/r \in B\big\}$, and $\beta : \mathbb{R}_+ \times B \to \Gamma$, $\beta(r,x) := (r, rx)$, $(1 \times \chi)(r, .) := (r, \chi(.))$.

The spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ can be endowed with scalar products such that they are Hilbert spaces in a natural way; in particular, $\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(X^\wedge)$.

For a finite weight interval $\Theta = (\vartheta, 0]$ we set

$$(1.1.2) \qquad \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) := \varprojlim_{N \in \mathbb{N}} \mathcal{K}^{s, \gamma - \vartheta - \frac{1}{N+1}}(X^\wedge)$$

which is a Fréchet space in the projective limit topology, and

$$\mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge),$$

as a direct sum, for every asymptotic type $\mathcal{P}$ which is associated with the weight data (γ, Θ). For purposes below for every $N \in \mathbb{N}$ and for $\gamma = 0$, we now form the spaces

$$(1.1.3) \qquad\qquad B^N := \langle r \rangle^{-N} \mathcal{K}^{N,0}(X^\wedge)$$

and

$$(1.1.4) \qquad A_{\mathcal{P}}^N := \langle r \rangle^{-N} \mathcal{K}^{N, -\vartheta - \frac{1}{N+1}}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge) = w^{-\vartheta - \frac{1}{N+1}} B^N + \mathcal{E}_{\mathcal{P}}(X^\wedge);$$

here, $w(r) := 1 + (r-1)\omega(r)$.

These are Hilbert spaces in a natural way, and we set

$$\mathcal{S}^0(X^\wedge) := \varprojlim_{N \in \mathbb{N}} B^N, \quad \mathcal{S}_{\mathcal{P}}^0(X^\wedge) := \varprojlim_{N \in \mathbb{N}} A_{\mathcal{P}}^N.$$

More generally, we can form

$$\mathcal{S}^{\gamma}(X^{\wedge}) := w^{\gamma}\mathcal{S}^{0}(X^{\wedge}),$$

and

$$\mathcal{S}^{\gamma}_{T^{-\gamma}\mathcal{P}}(X^{\wedge}) := w^{\gamma}\mathcal{S}^{0}_{\mathcal{P}}(X^{\wedge}),$$

respectively, where $T^{-\gamma}\mathcal{P} := \{(p_j - \gamma, m_j, L_j)\}_{j=0,\ldots,N}$.

Remark 1.1.1.   (i) *There are canonical continuous embeddings*

$$(1.1.5) \qquad\qquad A^N_{\mathcal{P}} \hookrightarrow A^{N-1}_{\mathcal{P}}, \quad B^N \hookrightarrow B^{N-1}$$

   for all $N \geq 1$;

   (ii) *let us set*

$$(1.1.6) \qquad\qquad (\kappa_{\lambda}u)(r,x) := \lambda^{\frac{n+1}{2}}u(\lambda r, x),$$

   $\lambda \in \mathbb{R}_+$. *Then we obtain strongly continuous groups of isomorphisms*

$$\kappa_{\lambda} : A^N_{\mathcal{P}} \to A^N_{\mathcal{P}} \quad \text{as well as} \quad \kappa_{\lambda} : B^N \to B^N$$

   for every $N \in \mathbb{N}$ (recall that $\{\kappa_{\lambda}\}_{\lambda \in \mathbb{R}_+}$ is called to be strongly continuous on a Banach space B if $\lambda \mapsto \kappa_{\lambda}b$ represents a continuous function $\mathbb{R}_+ \to B$ for every $b \in B$).

1.2. **Green edge symbols.** A manifold with edges (without boundary) is a topological space W with a subspace Y (the edge) such that $W \setminus Y$ and Y are C^{∞} manifolds, and W is the quotient space of a so called stretched manifold $\mathbb{W}$ with respect to an equivalence relation defined as follows. $\mathbb{W}$ is a C^{∞} manifold with boundary $\partial\mathbb{W}$, and $\partial\mathbb{W}$ is a smooth fibre bundle over Y with fibre X; here X is a closed compact C^{∞} manifold. If $\pi : \partial\mathbb{W} \longrightarrow Y$ denotes the canonical projection, for $w, w' \in \mathbb{W}$ we write $w \sim w'$ if and only if $\pi w = \pi w'$ for $w, w' \in \partial\mathbb{W}$, or $w = w'$ for $w, w' \notin \partial\mathbb{W}$. Then $W := \mathbb{W}/\sim$. Representing a collar neighbourhood of $\partial\mathbb{W}$ in the form $\overline{\mathbb{R}}_+ \times \partial\mathbb{W}$ we obtain a cylinder bundle over Y with fibres $\overline{\mathbb{R}}_+ \times X$. Locally over Y we then have trivialisations $\overline{\mathbb{R}}_+ \times X \times \Omega$ for open $\Omega \subseteq \mathbb{R}^q$, $q = \dim Y$, with a splitting of variables (r, x, y). As noted in the beginning we formulate things for $r > 0$, i.e., consider open stretched wedges $X^{\wedge} \times \Omega$ in the local descriptions.

As noted before Green symbols are particular operator-valued symbols within the framework of 'twisted homogeneity'. Homogeneity in that sense means the following. Let E be a Hilbert space equipped with a strongly continuous group $\{\kappa_{\lambda}\}_{\lambda \in \mathbb{R}_+}$ of isomorphisms $\kappa_{\lambda} : E \to E$, $\lambda \in \mathbb{R}_+$, such that $\kappa_{\lambda}\kappa_{\rho} = \kappa_{\lambda\rho}$ for all $\lambda, \rho \in \mathbb{R}_+$ (in such a case we simply say that E is endowed with a group action). If $\widetilde{E}$ is another Hilbert space with group action $\{\widetilde{\kappa}_{\lambda}\}_{\lambda \in \mathbb{R}_+}$, a C^{∞} function $a_{(\mu)}(y, \eta)$ in $\Omega \times (\mathbb{R}^q \setminus \{0\})$, $\Omega \subseteq \mathbb{R}^q$ open, with values in $\mathcal{L}(E, \widetilde{E})$ is called homogeneous of order $\mu \in \mathbb{R}$ if

$$a_{(\mu)}(y, \lambda\eta) = \lambda^{\mu}\widetilde{\kappa}_{\lambda}a_{(\mu)}(y, \eta)\kappa_{\lambda}^{-1}$$

for all $\lambda \in \mathbb{R}_+$.

Let $\chi(\eta)$ be an excision function, i.e., any $\chi \in C^{\infty}(\mathbb{R}^q)$ that vanishes near $\eta = 0$ and is equal to 1 for $|\eta| \geq C$ for some $C > 0$. Then, if $a_{(\mu)}(y, \eta)$ is homogeneous of

order μ, the function $a(y, \eta) := \chi(\eta) a_{(\mu)}(y, \eta)$ is an element of the space of classical operator-valued symbols

$$(1.2.1) \qquad\qquad S_{\mathrm{cl}}^{\mu}(\Omega \times \mathbb{R}^q; E, \widetilde{E}).$$

Let us give the definition of $S^{\mu}(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ without subscript 'cl'. This space consists of the set of all C^{∞} functions $a(y, \eta)$ in $\Omega \times \mathbb{R}^q$ with values in $\mathcal{L}(E, \widetilde{E})$ such that

$$(1.2.2) \qquad\qquad \sup_{\substack{y \in K \\ \eta \in \mathbb{R}^q}} \langle \eta \rangle^{|\beta| - \mu} \left\| \widetilde{\kappa}_{\langle \eta \rangle}^{-1} \left\{ D_y^{\alpha} D_{\eta}^{\beta} a(y, \eta) \right\} \kappa_{\langle \eta \rangle} \right\|_{\mathcal{L}(E, \widetilde{E})}$$

is finite for every $K \subset\subset \Omega$ and every $\alpha, \beta \in \mathbb{N}^q$; here $\langle \eta \rangle := (1 + |\eta|^2)^{\frac{1}{2}}$. Moreover, we denote by $S^{-\infty}(\Omega \times \mathbb{R}^q; E, \widetilde{E}) := \bigcap_{\mu \in \mathbb{R}} S^{\mu}(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ the space of symbols of infinite order.

Symbols of that kind form a Fréchet space with the expressions (1.2.2) as seminorms. They are 'twisted' analogues of Hörmander's symbol spaces from the scalar case (i.e., when $E = \widetilde{E} = \mathbb{C}$, $\kappa_{\lambda} = \widetilde{\kappa}_{\lambda} = \mathrm{id}_{\mathbb{C}}$ for all $\lambda \in \mathbb{R}_+$). Standard manipulations known from the scalar case also make sense in analogous form in the operator-valued case. In particular, we can form asymptotic sums of sequences $a_j(y, \eta)$ of symbols the order of which tend to $-\infty$ as $j \to \infty$. Now (1.2.1) is defined as the subspace of $a(y, \eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ which admit asymptotic expansions into symbols of the kind $\chi(\eta) a_{(\mu - j)}(y, \eta)$, $j \in \mathbb{N}$, where $a_{(\mu - j)}(y, \eta)$ is homogeneous in the above sense, of order $\mu - j$ and $\chi(\eta)$ an excision function.

In order to define Green symbols we need a slight generalisation to the case of Fréchet spaces. We say, that a Fréchet space E, written as the projective limit of a sequence of Hilbert spaces E^j, $j \in \mathbb{N}$, with continuous embedding $E^{j+1} \hookrightarrow E^j \hookrightarrow \ldots \hookrightarrow E^0$ for all j, is endowed with a group action $\{\kappa_{\lambda}\}_{\lambda \in \mathbb{R}_+}$, if $\{\kappa_{\lambda}\}_{\lambda \in \mathbb{R}_+}$ is a group action on E^0 and $\{\kappa_{\lambda}|_{E^j}\}_{\lambda \in \mathbb{R}_+}$ defines a group action on E^j for every j.

Symbol spaces of the kind (1.2.1) will be applied for the spaces

$$(1.2.3) \qquad\qquad \mathcal{K}^{s, \gamma}(X^{\wedge}), \quad \mathcal{S}_{\mathcal{P}}^{\beta}(X^{\wedge})$$

for some discrete asymptotic type $\mathcal{P}$ (associated to weight data (β, Θ), cf. Section 1.1). The spaces (1.2.3) will be considered with the group action (1.1.6).

Definition 1.2.1. *An operator function* $g(y, \eta) \in C^{\infty}(\Omega \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{0, \gamma}(X^{\wedge}), \mathcal{K}^{0, \beta}(X^{\wedge})))$ *is said to be a Green symbol of order* $\mu \in \mathbb{R}$, *with (discrete) asymptotic types* $\mathcal{P}$ *and* $\mathcal{Q}$ *(associated with the weight data* (β, Θ) *and* $(-\gamma, \Theta)$, *respectively) if* $g(y, \eta)$ *induces symbols*

$$(1.2.4) \qquad g(y, \eta) \in S_{\mathrm{cl}}^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s, \gamma}(X^{\wedge}), \mathcal{S}_{\mathcal{P}}^{\beta}(X^{\wedge}))$$

and

$$(1.2.5) \qquad g^*(y, \eta) \in S_{\mathrm{cl}}^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s, -\beta}(X^{\wedge}), \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^{\wedge}))$$

for all $s \in \mathbb{R}$. *Here* g^* *denotes the* (y, η)-*wise formal adjoint with respect to the respective sesquilinear pairings*

$$\mathcal{K}^{s, \beta}(X^{\wedge}) \times \mathcal{K}^{-s, -\beta}(X^{\wedge}) \to \mathbb{C}$$

induced by the $\mathcal{K}^{0,0}(X^\wedge)$ scalar product, for arbitrary $s, \beta \in \mathbb{R}$.

Remark 1.2.2. *Observe that the Green operators of type 0 in the calculus of classical (pseudo-differential) boundary value problems are operators with special such symbols. In this case it suffices to replace $\mathcal{K}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,-\beta}(X^\wedge)$ by $L^2(\mathbb{R}_+)$ (with $\mathbb{R}_+$ being the inner normal to the boundary in consideration) and $\mathcal{S}_\mathcal{P}^\beta(X^\wedge)$ and $\mathcal{S}_\mathcal{Q}^{-\gamma}(X^\wedge)$ by $\mathcal{S}(\overline{\mathbb{R}}_+)$, cf. [13].*

Remark 1.2.3. *The conditions (1.2.4) and (1.2.5) are slightly stronger than necessary. It suffices to require them for $s = 0$; however, this is not the main point of our consideration. What we can see immediately is that it suffices to require the conditions (1.2.4) and (1.2.5) for all $s \in \mathbb{Z}$ owed by the interpolation property of the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ in s. It follows that the space of Green symbols of order μ and fixed $\mathcal{P}, \mathcal{Q}$ is a Fréchet space.*

From the Green symbols which are known from the calculus of operators on a manifold with edges we know in fact more, namely, that the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,-\beta}(X^\wedge)$ may even be replaced by $\langle r \rangle^j \mathcal{K}^{s,\gamma}(X^\wedge)$ and $\langle r \rangle^j \mathcal{K}^{s,-\beta}(X^\wedge)$, respectively, for arbitrary $j \in \mathbb{N}$. Therefore, we start with that property. In that case it is known that the kernels of the homogeneous components $g_{(\mu-j)}$ are C^∞ functions of $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$ with values in the space

$$\left\{ \mathcal{S}_\mathcal{P}^\beta(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}^{-\gamma}(X^\wedge) \right\} \cap \left\{ \mathcal{S}^\beta(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}_{\overline{\mathcal{Q}}}^{-\gamma}(X^\wedge) \right\}$$

where $\mathcal{S}^\beta(X^\wedge) = \varprojlim_{j \in \mathbb{N}} \langle r \rangle^{-j} \mathcal{K}^{\infty,\beta}(X^\wedge)$ for any $\beta \in \mathbb{R}$. Here $\overline{\mathcal{Q}} := \left\{ (\overline{q}_j, n_j, \overline{L}_j) \right\}_j$ when $\mathcal{Q} = \{ (q_j, n_j, L_j) \}_j$, and $\widehat{\otimes}_\pi$ denotes the (completed) projective tensor product between the respective Fréchet spaces. In this conclusion we employ the fact that when an operator $g : H \to F$ is continuous from a Hilbert space H to a nuclear Fréchet space F (written as $\varprojlim_{j \in \mathbb{N}} F_j$ for Hilbert spaces F_j with nuclear embeddings $F_{j+1} \hookrightarrow F_j$ for all j), the operator g has a kernel in $\varprojlim_{j \in \mathbb{N}} F_j \otimes_H H^* = \varprojlim_{j \in \mathbb{N}} F_j \widehat{\otimes}_\pi H^* = F \widehat{\otimes}_\pi H^*$, cf. [5].

2. INTEGRAL REPRESENTATIONS

2.1. **Green symbols with discrete asymptotics.** Let $f(r, x, r', x'; y, \eta)$ be a function in the space

$$(2.1.1) \quad \left\{ \mathcal{S}_\mathcal{P}^\beta(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}^{-\gamma}(X^\wedge) \right\} \cap \left\{ \mathcal{S}^\beta(X^\wedge) \widehat{\otimes}_\pi \mathcal{S}_{\overline{\mathcal{Q}}}^{-\gamma}(X^\wedge) \right\} \widehat{\otimes}_\pi S_{\mathrm{cl}}^{\mu+n+1}(\Omega \times \mathbb{R}^q).$$

and let $\eta \mapsto [\eta]$ denote any strictly positive C^∞ function in $\mathbb{R}^q$ such that $[\eta] = |\eta|$ for $|\eta| > C$ for some constant $C > 0$. Form the operator function

$$(2.1.2) \qquad g(y, \eta) u(r, x) := \int_X \int_0^\infty f(r[\eta], x, r'[\eta], x'; y, \eta) u(r', x') (r')^n \, dr' \, dx'.$$

Then we get a Green symbols of order μ in the sense of Definition 1.2.1. For purposes below we set

$$(2.1.3) \qquad\qquad\qquad g_f(y, \eta) := g(y, \eta).$$

Theorem 2.1.1. *Every Green symbol $g(y,\eta)$ of order μ as in Definition 1.2.1 has a representation of the form (2.1.2) for an element $f(r,x,r',x';y,\eta)$ in the space (2.1.1).*

Proof. For convenience we consider a Green symbol with constant coefficients, i.e., $g = g(\eta)$ (the straightforward generalisation of arguments to the y-dependent case will be omitted). First observe that a simple composition of g with suitable powers in r and r', allows us to consider the case $\beta = \gamma = 0$. Moreover, without loss of generality we may assume $\mu = 0$ (it suffices to replace g by $[\eta]^{-\mu}g$). In other words we start with $g \in S_{\text{cl}}^0(\mathbb{R}^q; \mathcal{K}^{s,0}(X^\wedge), \mathcal{S}_{\mathcal{P}}^0(X^\wedge))$ with the homogeneous components $g_{(-j)}(\eta)$, $j \in \mathbb{N}$. We use the fact that the series

$$(2.1.4) \qquad \widetilde{g}_l(\eta) := \sum_{j=l}^\infty \chi(\frac{\eta}{c_j})g_{(-j)}(\eta)$$

converges in $S_{\text{cl}}^{-l}(\mathbb{R}^q; \mathcal{K}^{s,0}(X^\wedge), \mathcal{S}_{\mathcal{P}}^0(X^\wedge))$ for every $l \in \mathbb{N}$. Here $\chi(\eta)$ is any excision function in $\mathbb{R}^q$, and c_j are constants tending to ∞ sufficiently fast. Then $g(\eta) - \widetilde{g}(\eta)$, for $\widetilde{g}(\eta) := \widetilde{g}_0(\eta)$, is of order $-\infty$ in the sense of the first part of Definition 1.2.1. In a similar manner we can proceed with the formal adjoint and choose, if necessary, the constants c_j once again larger, such that $g^*(\eta) - \widetilde{g}^*(\eta)$ is of order $-\infty$ in the sense of the second part of Definition 1.2.1.

Setting

$$(2.1.5) \qquad \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge) := \{\mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\pi \mathcal{S}^0(X^\wedge)\} \cap \{\mathcal{S}^0(X^\wedge)\widehat{\otimes}_\pi \mathcal{S}_{\mathcal{Q}}^0(X^\wedge)\},$$

the components $g_{(-j)}(\eta)$ can be identified with an η-dependent kernel function of the form $|\eta|^{n+1-j} e_{(-j)}(r|\eta|,x,r'|\eta|,x';\frac{\eta}{|\eta|})$, $\eta \neq 0$, for $e_{(-j)}(r|\eta|,x,r'|\eta|,x';\frac{\eta}{|\eta|}) \in C^\infty(S_\eta^{q-1}, \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge))$, with S^{q-1} being the unit sphere in $\mathbb{R}^q$, such that

$$g_{(-j)}(\eta)u(r,x) =$$

$$(2.1.6)$$
$$|\eta|^{n+1-j} \int_X \int_0^\infty e_{(-j)}(r|\eta|,x,r'|\eta|,x';\frac{\eta}{|\eta|})u(r',x')(r')^n dr'dx'.$$

If E is a Fréchet space with the countable semi-norm system $(p_k)_{k\in\mathbb{N}}$ we denote by $S^\mu(\mathbb{R}^q, E)$ the set of all $a \in C^\infty(\mathbb{R}^q, E)$ such that

$$\sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{-\mu+|\alpha|} p_k(D_\eta^\alpha a) < \infty$$

for all $\alpha \in \mathbb{N}^q$, $k \in \mathbb{N}$. There is then the subspace $S_{\text{cl}}^\mu(\mathbb{R}^q, E)$ of classical E-valued symbols in terms of asymptotic expansions of elements $\chi(\eta)a_{(\mu-j)}(\eta)$ with homogeneous components $a_{(\mu-j)}(\eta) \in C^\infty(\mathbb{R}^q \setminus \{0\}, E)$ of order $\mu - j$.

Setting

$$(2.1.7) \qquad h_j(r,x,r',x';\eta) := \chi(\frac{\eta}{c_j})[\eta]^{n+1-j}e_{(-j)}(r,x,r',x';\frac{\eta}{|\eta|})$$

we obtain elements

$$(2.1.8) \qquad h_j \in S_{\text{cl}}^{n+1-j}(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge)).$$

Choosing the constants $c_j > 0$ increasing sufficiently fast as $j \to \infty$ we obtain convergence of $a_l(\eta) := \sum_{j=l}^{\infty} h_j(\eta)$ in $S_{\mathrm{cl}}^{n+1-l}\big(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge)\big)$ for every $l \in \mathbb{N}$. Clearly we can take the same constants as in (2.1.4); it suffices to take the maximums of both choices. Note that $h_j(r, x, r', x'; \eta)$ may be replaced by

$$h_j(r[\eta], x, r'[\eta], x'; \eta) = \chi(\frac{\eta}{c_j})\,|\eta|^{n+1-j}\,e_{(-j)}(r\,|\eta|\,, x, r'\,|\eta|\,, x'; \frac{\eta}{|\eta|})$$

when we choose c_0 sufficiently large and $c_j > c_0$ for all $j \geq 1$. According to (2.1.3) we obtain associated Green symbols $g_{h_j}(\eta)$, and $\sum_{j=l}^{\infty} g_{h_j}(\eta)$ converges to $g_{a_l}(\eta)$ in the Fréchet space of Green symbols of order $-l$ for the given fixed $\mathcal{P}$, $\mathcal{Q}$; this holds for every $l \in \mathbb{N}$. Thus it follows that $c(\eta) := g(\eta) - g_{a_0}(\eta)$ is a Green symbol of order $-\infty$. It remains to prove that there is an $m(r, x, r', x'; \eta) \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge)\big)$ such that $c(\eta) = g_m(\eta)$. The Green symbol $c(\eta)$ is of order $-\infty$; then there is a

$$(2.1.9) \qquad k(r, x, r', x'; \eta) \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge)\big)$$

such that

$$c(\eta)u(r, x) = \int_X \int_0^\infty k(r, x, r', x'; \eta)u(r', x')(r')^n dr' dx'.$$

In Lemma 2.1.2 below we will show that

$$(2.1.10) \qquad k(\frac{r}{[\eta]}, x, \frac{r'}{[\eta]}, x'; \eta) =: m(r, x, r', x'; \eta) \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\Gamma \mathcal{S}_{\mathcal{Q}}^0(X^\wedge)\big).$$

Then we obviously obtain $c(\eta) = g_m(\eta)$. $\qquad\square$

Lemma 2.1.2. *We have* (2.1.9) $\Rightarrow$ (2.1.10).

Proof. The proof is elementary though voluminous. Therefore, we only describe the typical steps. By virtue of (2.1.5) it suffices to show that

$$(2.1.11) \qquad k \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\pi \mathcal{S}^0(X^\wedge)\big) \Rightarrow m \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\pi \mathcal{S}^0(X^\wedge)\big)$$

and a similar relation for Schwartz functions with values in the second space of (2.1.5). Let us consider, for instance, the case (2.1.11). We now observe that

$$\mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\pi \mathcal{S}^0(X^\wedge) = \varprojlim_{N \in \mathbb{N}} A^N \otimes_H B^N$$

for the spaces $A^N := A_{\mathcal{P}}^N$ and B^N, cf. Section 1.1, with $\otimes_H$ being the Hilbert tensor product. Then we have

$$\mathcal{S}\big(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^0(X^\wedge)\widehat{\otimes}_\pi \mathcal{S}^0(X^\wedge)\big) = \varprojlim_{N \in \mathbb{N}} \mathcal{S}\big(\mathbb{R}^q, A^N \otimes_H B^N\big).$$

As the semi-norm system for this space we can take

$$(2.1.12) \qquad \sup_{\eta \in \mathbb{R}^q} \left\| \langle \eta \rangle^l D_\eta^\beta k(\frac{r}{[\eta]}, x, \frac{r'}{[\eta]}, x'; \eta) \right\|_{A^N \otimes_H B^N}$$

for all $l, N \in \mathbb{N}$, $\beta \in \mathbb{N}^q$.

It suffices to show that for every l, β, N there are finitely many triples (l', β', N') such that

$$(2.1.13) \qquad \sup_{\eta \in \mathbb{R}^q} \left\| \langle \eta \rangle^{l'} D_\eta^{\beta'} k(\frac{r}{[\eta]}, x, \frac{r'}{[\eta]}, x'; \eta) \right\|_{A^{N'} \otimes_H B^{N'}} < \infty$$

for all those (l', β', N') implies that $(2.1.12)$ is finite.

Let us look at the case $q = 1$ and $n = \dim X = 0$; the general case is completely analogous. For $\beta = 0$ we use the fact that when $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is a strongly continuous group of isomorphisms on a Hilbert space E, there are constants $c, M > 0$ such that

$$(2.1.14) \qquad \|\kappa_\lambda\|_{\mathcal{L}(E, \widetilde{E})} \le c \left(\max(\lambda, \lambda^{-1}) \right)^M$$

for all $\lambda \in \mathbb{R}_+$. From (a slight modification of) Remark 1.1.1 we know that $u(r, x) \mapsto u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$, induces strongly continuous groups of isomorphisms on the spaces A^N and B^N for all $N \in \mathbb{N}$. Then $(2.1.14)$ yields estimates of the kind

$$(2.1.15) \qquad \left\| k(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta) \right\|_{A^N \otimes_H B^N} \le c \langle \eta \rangle^M \| k(r, r'; \eta) \|_{A^N \otimes_H B^N}$$

for all η, for suitable constants $c, M > 0$, for all $k \in A^N \otimes_H B^N$. This gives us immediately the conclusion $(2.1.13) \Rightarrow (2.1.12)$ with $\beta' = 0$ and $l' = l + M$.

Let us now check the case $\beta = 1$. In this case we obtain

$$\frac{d}{d\eta} k(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta) = \left(\{ \varphi r \partial_r + \varphi r' \partial_{r'} + \partial_\eta \} k \right) (\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta)$$

with a uniformly bounded function $\varphi(\eta)$ and $\partial/\partial\eta$ denoting the derivative in the third variable. Then

$$
\begin{aligned}
\left\| \langle \eta \rangle^l \left(\frac{d}{d\eta} k \right) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N} \le \ & c \left\| \langle \eta \rangle^l (r \partial_r k) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N} \\
& + c \left\| \langle \eta \rangle^l (r' \partial_{r'} k) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N} \\
& + \left\| \langle \eta \rangle^l (\partial_\eta k) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta \right) \right\|_{A^N \otimes_H B^N}
\end{aligned}
$$

with some $c > 0$. The operator $r \partial_r$ is continuous in the sense

$$(2.1.16) \qquad r \partial_r : A^N \to A^{N-1}, \quad B^N \to B^{N-1}$$

for every $N \ge 1$. In combination with the estimates $(2.1.15)$ this implies

$$\left\| \langle\eta\rangle^l \left(\frac{d}{d\eta}k\right) \left(\frac{r}{[\eta]}, \frac{r'}{[\eta]}; \eta\right) \right\|_{A^N \otimes_H B^N} \leq c \left\| \langle\eta\rangle^{l+M} k\left(r, r'; \eta\right) \right\|_{A^{N-1} \otimes_H B^N}$$

$$+ \; c \left\| \langle\eta\rangle^{l+M} k\left(r, r'; \eta\right) \right\|_{A^N \otimes_H B^{N-1}}$$

$$+ \; \left\| \langle\eta\rangle^{l+M} \left(\partial_\eta k\right)\left(r, r'; \eta\right) \right\|_{A^N \otimes_H B^N}.$$

The desired estimate then follows from Remark 1.1.1 (i). In an analogous manner we can treat the semi-norms with higher η-derivatives. $\qquad\square$

Remark 2.1.3. *Theorem 2.1.1 remains true in analogous form if we replace $[\eta]$ in the formula (2.1.2) by any other strictly positive C^∞ function $p(\eta)$ such that $c[\eta] \leq p(\eta) \leq c'[\eta]$ for all η, with suitable constants $0 < c < c'$. In particular, we may take $p(\eta) = \langle\eta\rangle$.*

2.2. **Trace and potential symbols with discrete asymptotics.** The Definition 1.2.1 at page 7 can be generalised to 2×2 block matrix-valued functions $g(y, \eta) \in S_{\mathrm{cl}}^\mu\left(\Omega \times \mathbb{R}^q; \mathcal{K}^{0,\gamma}(X^\wedge) \oplus \mathbb{C}, \mathcal{K}^{0,\beta}(X^\wedge) \oplus \mathbb{C}\right)$ such that

$$g(y, \eta) \in S_{\mathrm{cl}}^\mu\left(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}, \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge) \oplus \mathbb{C}\right)$$

and

$$g^*(y, \eta) \in S_{\mathrm{cl}}^\mu\left(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\beta}(X^\wedge) \oplus \mathbb{C}, \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge) \oplus \mathbb{C}\right)$$

for all $s \in \mathbb{R}$, with suitable g-dependent discrete asymptotic types $\mathcal{P}, \mathcal{Q}$ (the pointwise adjoints refer to corresponding sesquilinear pairings induced by the scalar product of $\mathcal{K}^{0,0}(X^\wedge) \oplus \mathbb{C}$. In $\mathbb{C}$ we always assume the trivial group action, i.e., κ_λ acts as the identity for all $\lambda \in \mathbb{R}_+$. Writing $g(y, \eta) = (g_{ij}(y, \eta))_{i,j=1,2}$, we call $g_{21}(y, \eta)$ a trace symbol and $g_{12}(y, \eta)$ a potential symbol of order $\mu \in \mathbb{R}$ (of the edge calculus). Clearly $g_{22}(y, \eta)$ is nothing other than a classical scalar symbol of order μ.

Examples of trace and potential symbols may be obtained by functions in

$$(2.2.1) \qquad \mathcal{S}_{\overline{\mathcal{Q}}}^{-\gamma}(X^\wedge) \widehat{\otimes}_\pi S_{\mathrm{cl}}^{\mu+\frac{n+1}{2}}(\Omega \times \mathbb{R}^q) \ni f_{21}(r', x'; y, \eta)$$

and

$$(2.2.2) \qquad \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge) \widehat{\otimes}_\pi S_{\mathrm{cl}}^{\mu+\frac{n+1}{2}}(\Omega \times \mathbb{R}^q) \ni f_{12}(r, x; y, \eta)$$

respectively. The symbols themselves are obtained by integral representations of the kind

$$(2.2.3) \qquad g_{21}(y, \eta)u = \int_X \int_0^\infty f_{21}(r'[\eta], x'; y, \eta)u(r', x')(r')^n dr' dx',$$

$u(y, \eta) \in \mathcal{K}^{s,\gamma}(X^\wedge)$, and

$$(2.2.4) \qquad g_{12}(y, \eta)c(r, x) = cf_{12}(r[\eta], x; y, \eta),$$

$c \in \mathbb{C}$, respectively.

Theorem 2.2.1.   (i) *Every trace symbol $g_{21}(y, \eta)$ can be written in the form (2.2.3) for an element (2.2.1);*

(ii) *every potential symbol* $g_{12}(y, \eta)$ *can be written in the form* (2.2.4) *for an element* (2.2.2).

The proof employs analogous arguments as those for Theorem 2.1.1.

3. Continuous asymptotics

3.1. **Continuous asymptotics and Green symbols.** In Section 1.1 we have formulated spaces $\mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge)$ with discrete asymptotics of type $\mathcal{P}$ for $r \to 0$, cf. the formula (1.1.1) at page 4. As is known, cf. [12], asymptotics of that form can also be written as

$$(3.1.1) \qquad u(r,x) \sim \sum_j \langle \zeta_j, r^{-z} \rangle$$

where ζ_j are $C^\infty(X)$-valued analytic functionals carried by the points $p_j \in \mathbb{C}$ which are of finite order (in fact, derivatives of the Dirac distribution at p_j of order $m_j + 1$ in the notation of the formula (1.1.1)).

For an open $\mathcal{U} \subseteq \mathbb{C}$ and for a Fréchet space E, we denote by $\mathcal{A}(\mathcal{U}, E) = \mathcal{A}(\mathcal{U}) \widehat{\otimes}_\pi E$ the space of all holomorphic E-valued functions in $\mathcal{U}$. Moreover, let $\mathcal{A}'(K, C^\infty(X))$ $(= \mathcal{A}'(K) \widehat{\otimes}_\pi C^\infty(X))$ denote the space of all analytic functionals carried by a compact set $K \subset \mathbb{C}$. From generalities on analytic functionals it follows that every $\zeta \in \mathcal{A}'(K, C^\infty(X))$ can be represented in the form

$$(3.1.2) \qquad \zeta : h \mapsto \frac{1}{2\pi i} \int_C f(z) h(z) dz$$

for some $f \in \mathcal{A}(\mathbb{C} \setminus K, C^\infty(X))$, where C is a C^∞ curve counter clockwise surrounding K (such that the winding number with respect to every $z \in K$ is equal to 1). In other words, to express (3.1.1) it suffices to represent ζ_j by a meromorphic function with a pole at p_j of order $m_j + 1$ and Laurent coefficients belonging to the space L_j (cf. the notation in Section 1.1).

Now an element $u(r,x) \in \mathcal{K}^{s,\gamma}(X^\wedge)$ is said to have continuous asymptotics (first in a finite weight strip Θ) if there is an element $\zeta \in \mathcal{A}'(K, C^\infty(X))$ for a suitable compact $K \subset \left\{ z : \operatorname{Re} z < \frac{n+1}{2} - \gamma \right\}$ such that

$$u(r,x) = \omega(r) \langle \zeta, r^{-z} \rangle + u_\Theta(r,x)$$

for some $u_\Theta \in \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$.

In order to unify notation in connection with discrete or continuous asymptotics we consider the space

$$(3.1.3) \qquad \left\{ \omega(r) \langle \zeta, r^{-z} \rangle : \zeta \in \mathcal{A}'(K, C^\infty(X)) \right\}.$$

The quotient space of (3.1.3) with respect to the equivalence relation $u \sim v \Leftrightarrow u - v \in \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$ is called a continuous asymptotic type $\mathcal{P}$, associated with weight data (γ, Θ). The cut-off function ω is fixed, but the quotient space is independent of ω. Denoting the space (3.1.3) by $\mathcal{E}_{\mathcal{P}}(X^\wedge)$ we then define

$$(3.1.4) \qquad \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge)$$

in the Fréchet topology of the non-direct sum. To recall the terminology, the non-direct sum of two Fréchet spaces E and F (embedded in a Hausdorff topological

vector space) is defined as $E + F := \{e + f : e \in E, \ f \in F\}$ endowed with the Fréchet topology from $E + F \simeq E \oplus F/\Delta$, where $\Delta := \{(e, -e) : e \in E \cap F\}$. Note that $\mathcal{P}$ only depends on the set $K \cap \{\operatorname{Re} z > \frac{n+1}{2} - \gamma + \vartheta\}$. For a generalisation to infinite weight intervals we define the system $\mathcal{V}$ of closed subsets $V \subset \mathbb{C}$ such that $V \cap \{c \leq \operatorname{Re} z \leq c'\}$ is compact for every $c \leq c'$. Then for every $V \in \mathcal{V}$ contained in $\{\operatorname{Re} z < \frac{n+1}{2} - \gamma\}$ we can consider $V_\vartheta := V \cap \{\operatorname{Re} z \geq \frac{n+1}{2} - \gamma + \vartheta - 1\}$ and the associated continuous asymptotic type $\mathcal{P}_\vartheta$. We then have continuous embedding

$$\mathcal{K}^{s;\gamma}_{\mathcal{P}_\vartheta}(X^\wedge) \hookrightarrow \mathcal{K}^{s;\gamma}_{\mathcal{P}_{\vartheta'}}(X^\wedge)$$

for every $\vartheta < \vartheta'$, and we then set $\mathcal{K}^{s;\gamma}_{\mathcal{P}}(X^\wedge) := \varprojlim_{\vartheta \in \mathbb{R}_-} \mathcal{K}^{s;\gamma}_{\mathcal{P}_\vartheta}(X^\wedge)$ in the Fréchet topology of the projective limit; the subscript $\mathcal{P}$ incorporates a continuous asymptotic type associated with (γ, Θ) for $\Theta = (-\infty, 0]$, and stands for an equivalence class of sequences $\{\mathcal{P}_\vartheta\}$ where ϑ runs over any monotonely decreasing sequence of negative reals, tending to $-\infty$. The equivalence relation just means the equality of the respective projective limits. The set V is called a carrier set of the asymptotic type $\mathcal{P}$ (when $\Theta = +\infty$, otherwise $V \cap \{\operatorname{Re} z \geq \frac{n+1}{2} - \gamma + \vartheta\}$ is called the carrier of the corresponding $\mathcal{P}$ when Θ is finite).

We do not need the sets V in full generality. Let us content ourselves with those V that are convex in imaginary direction, i.e., $z_0, z_1 \in V$ and $\operatorname{Re} z_0 = \operatorname{Re} z_1$ imply $\lambda z_0 + (1 - \lambda) z_1 \in V$ for all $0 \leq \lambda \leq 1$. There is then an obvious one-to-one correspondence between such V contained in $\{\operatorname{Re} z < \frac{n+1}{2} - \gamma\}$ and associated continuous asymptotic types by the above construction.

If $\mathcal{P}$ is a continuous asymptotic type, we set

$$\mathcal{S}^\gamma_{\mathcal{P}}(X^\wedge) := \varprojlim_{N \in \mathbb{N}} \langle r \rangle^{-N} \mathcal{K}^{\infty,\gamma}_{\mathcal{P}}(X^\wedge)$$

which is a nuclear Fréchet space in the topology of the projective limit.

Let us make some remarks about the motivation of continuous asymptotics. As noted in the introduction the elliptic regularity of solutions to elliptic equations $Au = f$ on a wedge $X^\wedge \times \Omega$, $\Omega \subseteq \mathbb{R}^q$ open, and A edge-degenerate of the form (0.0.1), contains a statement on asymptotics of $u(r, x, y)$ for $r \to 0$, even if we are considering C^∞ functions on $X^\wedge \times \Omega$. Similarly as (1.1.1) the asymptotics have the form

$$(3.1.5) \qquad u(r, x, y) \sim \sum_j \sum_{k=0}^{m_j(y)} c_{jk}(x, y) r^{-p_j(y)} \log^k r \qquad \text{for} \quad r \to 0,$$

where the exponents $-p_j(y)$ and the numbers $m_j(y)$ are determined by those points $z \in \mathbb{C}$ where the operators (0.0.2) are not bijective, cf. [11]. These points (as well as the $m_j(y)$) may depend on y in a very irregular way. This may happen even for $\dim X = 0$. The inverse of (0.0.2) is then a family of meromorphic functions, and the main ingredients of the parametrices P of A are Mellin operators with such symbols. Applying $\mathcal{P}$ to functions (say, with compact support with respect to $r \in \mathbb{R}_+$) gives us functions $u(r, x, y)$ of a behaviour like (3.1.5). If we consider the Mellin transform $(M(\omega u))(z, x, y)$ (for any cut-off function $\omega(r)$ on the half-axis) we obtain a family of meromorphic functions in the complex plane the poles and multiplicities

of which inherit the corresponding behaviour of the Mellin symbols. Now the parameter-dependent asymptotics of u can be interpreted in terms of functions $\zeta(y) \in C^\infty\left(\Omega, \mathcal{A}'(K, C^\infty(X))\right)$ for suitable compact K, such that $\zeta(y)$ is pointwise discrete and of finite order but of the above mentioned irregular behaviour. Here 'pointwise discrete' means that $\langle \zeta(y), r^{-z} \rangle$ has the form (3.1.5) for certain $p_j \in K$, $m_j \in \mathbb{N}$ for every $y \in \Omega$.

Proposition 3.1.1. *For every continuous asymptotic type $\mathcal{P}$ associated with weight data (γ, Θ) there is a scale of Hilbert spaces $A_\mathcal{P}^N$, $N \in \mathbb{N}$, with nuclear embedding $A_\mathcal{P}^N \hookrightarrow A_\mathcal{P}^{N-1}$ for every $N \geq 1$ such that*

$$\mathcal{S}_\mathcal{P}^\gamma(X^\wedge) = \varprojlim_{N \in \mathbb{N}} A_\mathcal{P}^N.$$

The spaces $A_\mathcal{P}^N$ can be chosen as continuously embedded subspaces of $\mathcal{K}^{0,\gamma}(X^\wedge)$ such that (1.1.6) *induces a strongly continuous group of isomorphisms*

$$\kappa_\lambda : A_\mathcal{P}^N \to A_\mathcal{P}^N$$

for every $N \in \mathbb{N}$.

Proof. We first assume the weight interval Θ to be finite. Similarly as (3.1.4) we write $\mathcal{S}_\mathcal{P}^\gamma(X^\wedge)$ as a (non-direct) sum of Fréchet spaces, namely

$$\mathcal{S}_\mathcal{P}^\gamma(X^\wedge) = \varprojlim_{N \in \mathbb{N}} \langle r \rangle^{-N} \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge) + \mathcal{E}_\mathcal{P}(X^\wedge).$$

We then consider the spaces

$$(3.1.6) \qquad \widetilde{A}_\mathcal{P}^N := \langle r \rangle^{-N} \mathcal{K}^{N,\gamma-\vartheta-\frac{1}{N+1}}(X^\wedge) + \mathcal{E}_\mathcal{P}^N(X^\wedge).$$

The meaning of the first summand is clear, cf. also the formula (1.1.2); so it remains to define $\mathcal{E}_\mathcal{P}^N(X^\wedge)$. Recall that the space (3.1.3) may be described in terms of a compact set $K := V \cap \left\{ \operatorname{Re} z \geq \frac{n+1}{2} - \gamma + \vartheta - 1 \right\}$ for a set $V \subset \left\{ \operatorname{Re} z < \frac{n+1}{2} - \gamma \right\}$ of the above mentioned kind. Choose any C^∞ curve $C_N \subset \mathbb{C} \setminus K$ counter clockwise surrounding K, such that the winding number with respect to any $z \in K$ is equal to 1 and $\operatorname{dist}(z, K) \leq \frac{1}{N}$ (it is well known that such curves always exist). Then the weighted Mellin transforms $\left(M_{\gamma-\frac{n}{2}} u\right)|_{C_N}$ belong to $C^\infty(C_N)$ (concerning notation around the weighted Mellin transform, cf. Section 3.2 below). By $\mathcal{E}_\mathcal{P}^N(X^\wedge)$ we then denote the completion of $\left\{ \left(M_{\gamma-\frac{n}{2}} u\right)|_{C_N} : u \in \mathcal{E}_\mathcal{P}(X^\wedge) \right\}$ in the norm of $H^N(C_N)$, the Sobolev space of smoothness N on the curve C_N. This is a Hilbert space, and we have $\mathcal{E}_\mathcal{P}(X^\wedge) = \varprojlim_{N \in \mathbb{N}} \mathcal{E}_\mathcal{P}^N(X^\wedge)$.

It is now clear that the space (3.1.6) is nuclearly embedded into a corresponding space of analogous structure belonging to $N' < N$ when $N - N'$ is sufficiently large. This allows us to find a sequence of N_j, $j \in \mathbb{N}$, with $N_{j+1} > N_j$ such that, if we set $A_\mathcal{P}^j := \widetilde{A}_\mathcal{P}^{N_j}$, we have nuclearity of $A_\mathcal{P}^{j+1} \hookrightarrow A_\mathcal{P}^j$ for all j. We then have

$$\mathcal{S}_\mathcal{P}^\gamma(X^\wedge) = \varprojlim_{j \in \mathbb{N}} A_\mathcal{P}^j.$$

It remains to note that the group action $\kappa_\lambda : u(r, .) \mapsto \lambda^{\frac{n+1}{2}} u(\lambda r, .)$ restricts to a group action on the space $A_\mathcal{P}^j$ for every j. It suffices to check that for the spaces (3.1.6). The factor $\lambda^{\frac{n+1}{2}}$ is not essential, so we have to look at the influence of rescaling to the space $\mathcal{E}_\mathcal{P}^N(X^\wedge)$. By definition we restrict the Mellin transform to

the curve C_N and measure the result in $H^N(C_N)$. The Mellin transform of the rescaled function is obtained by multiplying the original one by λ^{-z}. The continuous dependence of the $H^N(C_N)$-norm on $\lambda \in \mathbb{R}_+$ is then obvious. For the infinite weight interval $\Theta = (-\infty, 0]$ we first write $\mathcal{S}_{\mathcal{P}}^{\gamma}(X^{\wedge}) = \varprojlim_{m \in \mathbb{N}} \mathcal{S}_{\mathcal{P}_{\vartheta_m}}^{\gamma}(X^{\wedge})$ for a sequence of finite $\vartheta_m < 0$ tending to $-\infty$ and form the spaces $A_{\mathcal{P}_{\vartheta_m}}^N$ for every ϑ of this sequence, such that $\mathcal{S}_{\mathcal{P}_{\vartheta_M}}^{\gamma} = \varprojlim_{N \in \mathbb{N}} A_{\mathcal{P}_{\vartheta_M}}^N$. Then we can set $A_{\mathcal{P}}^N := A_{\mathcal{P}_{\vartheta_N}}^N$. $\qquad\square$

Remark 3.1.2. Definition 1.2.1 *has an immediate generalisation to Green symbols with continuous asymptotic types $\mathcal{P}$ and $\mathcal{Q}$, associated with weight data (β, Θ) and $(-\gamma, \Theta)$, respectively.*

3.2. **Mellin symbols.** As noted in the beginning Green operators on a manifold with conical singularities belong to the algebra of cone pseudo-differential operators. Technically they appear as remainders in some typical operations with so called smoothing Mellin operators, also defined in terms of asymptotic data. Let us first recall some basic notation. By M we denote the Mellin transform on $\mathbb{R}_+$, i.e., $Mu(z) := \int_0^{\infty} r^{z-1} u(r) dr$, first for $u \in C_0^{\infty}(\mathbb{R}_+)$ and then extended to more general function and distribution spaces, also vector-valued ones. Let $\Gamma_{\beta} := \{z \in \mathbb{C} : \operatorname{Re} z = \beta\}$ for any $\beta \in \mathbb{R}$, then the map $M_{\gamma} : u \mapsto Mu(z)|_{\Gamma_{\frac{1}{2}-\gamma}}$, $u \in C_0^{\infty}(\mathbb{R}_+)$, is called the weighted Mellin transform with weight γ. It is known to induce an isomorphism $M_{\gamma} : r^{\gamma} L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\gamma})$. With M_{γ} we can associate Mellin (pseudo-differential) operators

$$\operatorname{op}_M^{\gamma}(f)u(r) := M_{\gamma}^{-1}\{f(z)(M_{\gamma}u)(z)\}$$

for symbols $f(z) \in S^{\mu}(\Gamma_{\frac{1}{2}-\gamma})$ (here, $S^{\mu}(\mathbb{R})$ is Hörmander's classical $S_{1,0}^{\mu}$ space of symbols of order $\mu \in \mathbb{R}$ (with constant coefficients), and $\Gamma_{\frac{1}{2}-\gamma}$ in place or $\mathbb{R}$ gives us a corresponding space with the covariable $\operatorname{Im} z$ for $z \in \Gamma_{\frac{1}{2}-\gamma}$). More generally, we can also form Mellin operators with operator-valued amplitude functions $f(r, r', z) \in C^{\infty}\left(\mathbb{R}_+ \times \mathbb{R}_+, L^{\mu}(X; \Gamma_{\frac{1}{2}-\gamma})\right)$; here $L^{\mu}(X; \mathbb{R})$ is the (Fréchet) space of parameter-dependent pseudo-differential operators on a C^{∞} manifold X of order μ with parameter on $\mathbb{R}$. We can also talk about the parameter $z \in \Gamma_{\frac{1}{2}-\gamma}$ which explains our notation.

Operators of that kind on an infinite stretched cone $X^{\wedge}$ occur as the values of operator-valued symbols in the calculus of operators on a wedge. In this connection the Mellin symbols depend on edge variables and covariables, and the mapping properties refer to asymptotic data for $r \to 0$. In this connection it is typical that the Mellin amplitude functions are not only defined on $\Gamma_{\frac{1}{2}-\gamma}$ but in the complex z-plane, up to a subset V which encodes asymptotic properties, similarly as in the context of functions with (discrete or continuous) asymptotics. We want to give a definition and then observe the way how Green operators are induced by Mellin operators with asymptotics.

From now on, we assume sets $V \in \mathcal{V}$ to be convex in imaginary direction. A V-excision function is any $\chi \in C^{\infty}(\mathbb{C})$ such that $\chi(z) = 0$ when $\operatorname{dist}(z, V) < \varepsilon_0$,

$\chi(z) = 1$ for $\operatorname{dist}(z, V) > \varepsilon_1$ for certain $0 < \varepsilon_0 < \varepsilon_1$.

By $M_V^{-\infty}(X)$ we denote the space of all $f(z) \in \mathcal{A}\left(\mathbb{C} \setminus V, L^{-\infty}(X)\right)$ such that

$$\chi(z)f(z)|_{\Gamma_\beta} \in \mathcal{S}\left(\Gamma_\beta, L^{-\infty}(X)\right)$$

for every V-excision function χ and every real β, uniformly in compact β-intervals. Moreover, let $M_{\mathcal{O}}^\mu(X)$, $\mu \in \mathbb{R}$, denote the space of all $h(z) \in \mathcal{A}\left(\mathbb{C}, L_{\mathrm{cl}}^\mu(X)\right)$ such that

$$h(z)|_{\Gamma_\beta} \in L_{\mathrm{cl}}^\mu(X; \Gamma_\beta)$$

for every real β, uniformly in compact β-intervals.

The spaces $M_V^{-\infty}(X)$ and $M_{\mathcal{O}}^\mu(X)$ are nuclear Fréchet spaces in a natural way.

Let us set

$$M_V^\mu(X) := M_{\mathcal{O}}^\mu(X) + M_V^{-\infty}(X)$$

in the Fréchet topology of the non-direct sum. Then, for every $f(r, r', z)$ belonging to the space $C^\infty\left(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_V^\mu(X)\right)$, we can form associated weighted Mellin operators $\operatorname{op}_M^\beta(f)$, for every weight $\beta \in \mathbb{R}$ such that $V \cap \Gamma_{\frac{1}{2}-\beta} = \emptyset$.

Theorem 3.2.1. *For every* $f(r, r', z) \in C^\infty\left(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_V^\mu(X)\right)$, $V \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$, *the operator* $\omega\operatorname{op}_M^{\gamma-\frac{n}{2}}(f)\widetilde{\omega}$ *(with cut-off functions* $\omega(r), \widetilde{\omega}(r)$) *induces continuous operators*

$$\omega\operatorname{op}_M^{\gamma-\frac{n}{2}}(f)\widetilde{\omega} : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma}(X^\wedge)$$

and

$$\omega\operatorname{op}_M^{\gamma-\frac{n}{2}}(f)\widetilde{\omega} : \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge) \to \mathcal{K}_{\mathcal{Q}}^{s-\mu,\gamma}(X^\wedge)$$

for every $s \in \mathbb{R}$ *and every continuous asymptotic type* $\mathcal{P}$ *with some resulting continuous asymptotic type* $\mathcal{Q}$, *associated with the weight data* (γ, Θ) *for every* $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$.

This result is known, cf. [12]. Recall that the main idea of the continuity in spaces with continuous asymptotics is to characterise the Mellin transforms of ωu as holomorphic functions outside the union of V and the carrier set of the asymptotic type $\mathcal{P}$; then we obtain another carrier set which just determines the asymptotic type $\mathcal{Q}$.

Mellin operators as in Theorem 3.2.1 belong to the ingredients of parametrices of elliptic (pseudo-differential) operators on manifolds with conical singularities (modelled on $X^\wedge$) or edges (modelled on $X^\wedge \times \Omega$ for some open set $\Omega \subseteq \mathbb{R}^q$). These operators are combined with other operators of the calculus.

Green operators in the set-up of conical singularities appear in the following manner. Consider an element $f \in M_V^{-\infty}(X)$, let $j > 0$, $\mu \in \mathbb{R}$, and let $\gamma - j \leq \beta, \delta \leq \gamma$ for some reals β, δ, such that

$$V \cap \Gamma_{\frac{n+1}{2}-\beta} = V \cap \Gamma_{\frac{n+1}{2}-\delta} = \emptyset.$$

The following result is known; for completeness we give a proof here.

Proposition 3.2.2. *For the operator*

$$g := \omega r^{-\mu+j} \mathrm{op}_M^{\beta-\frac{n}{2}}(f)\widetilde{\omega} - \omega r^{-\mu+j} \mathrm{op}_M^{\delta-\frac{n}{2}}(f)\widetilde{\omega}$$

there are continuous asymptotic types $\mathcal{P}$ and $\mathcal{Q}$ associated with the weight data $(\gamma - \mu, \Theta)$ and $(-\gamma, \Theta)$, respectively, such that

$$(3.2.1) \qquad\qquad g : \langle r \rangle^l \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{S}_{\mathcal{P}}^{\gamma-\mu}(X^\wedge),$$

and

$$(3.2.2) \qquad\qquad g : \langle r \rangle^l \mathcal{K}^{s,-\gamma+\mu}(X^\wedge) \to \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge)$$

are continuous operators for all $s \in \mathbb{R}$, $l \in \mathbb{N}$.

Proof. Let us check the mapping property (3.2.1); the property (3.2.2) can be verified in an analogous manner by passing to formal adjoints of the involved Mellin operators, using the fact that they are of analogous type with resulting 'adjoint' Mellin symbols, etc., cf. [12]. By virtue of the fact that the operators contain cut-off functions we immediately see that the factors $\langle r \rangle^l$ are harmless; so we may look at the case $l = 0$. In addition it suffices to assume $\mu = 0$. The operator g is then continuous as a map $\mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$ because of the assumed weight conditions. We have

$$(3.2.3) \qquad gu(r) = \frac{1}{2\pi i} r^j \int_{\Gamma_{\frac{n+1}{2}-\beta}} r^{-z} f(z) M\widetilde{\omega}u(z)\,dz - \frac{1}{2\pi i} r^j \int_{\Gamma_{\frac{n+1}{2}-\delta}} r^{-z} f(z) M\widetilde{\omega}u(z)\,dz.$$

Let, for instance, $\beta \leq \delta$. Observe that $M\widetilde{\omega}u(z)$ for $u \in \mathcal{K}^{s,\gamma}(X^\wedge)$ is holomorphic in $\mathrm{Re}\, z \geq \frac{n+1}{2} - \gamma$. Thus, because of the position of $\Gamma_{\frac{n+1}{2}-\beta}$ and $\Gamma_{\frac{n+1}{2}-\delta}$ on the right of $\Gamma_{\frac{n+1}{2}-\gamma}$ we can replace the difference of integrals (3.2.3) as an integration over a closed curve C counter clockwise surrounding the compact set $K := V \cap \left\{ \frac{n+1}{2} - \beta < \mathrm{Re}\, z < \frac{n+1}{2} - \delta \right\}$. The function $\widetilde{f}(z) := f(z)M\widetilde{\omega}u(z)$ is holomorphic in the strip outside K. Hence (3.2.3) takes the form (3.1.2) for $h(z) = r^{-z}$, up to the factor $\omega(r)r^j$. We thus obtain altogether $gu(r) = \omega(r)r^j \langle \zeta, r^{-z} \rangle$ for $\zeta \in \mathcal{A}'(K, C^\infty(X))$ which gives us the mapping property (3.2.1), where the asymptotic type $\mathcal{P}$ is represented by the compact set K, cf. the notation in connection with (3.1.3). $\qquad\square$

Let us consider what are called Mellin edge symbols. Such symbols are finite linear combinations of operator families of the form

$$(3.2.4) \qquad m(y, \eta) := \omega(r[\eta]) r^{-\mu+j} \mathrm{op}_M^{\gamma_j-\frac{n}{2}}(f_{j\alpha})(y) \eta^\alpha \widetilde{\omega}(r[\eta])$$

for cut-off functions $\omega, \widetilde{\omega}$, and $f_{j\alpha}(y) \in C^\infty\big(\Omega, M_V^{-\infty}(X)\big)$ for a set $V \in \mathcal{V}$ such that $V \cap \Gamma_{\frac{n+1}{2}-\gamma_j} = \emptyset$, $\Omega \subseteq \mathbb{R}^q$ open. In such expressions we have $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^q$, $|\alpha| \leq j$, and the weights $\gamma_j \in \mathbb{R}$ are assumed to satisfy the condition

$$(3.2.5) \qquad\qquad \gamma - j \leq \gamma_j \leq \gamma$$

for every $j \in \mathbb{N}$.

Then (3.2.4) is a C^∞ family of continuous operators

$$m(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{S}^{\gamma-\mu}(X^\wedge),$$

cf. Section 1.1. We have, in fact, more, namely

$$m(y, \eta) \in S_{\mathrm{cl}}^\mu\big(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^{\gamma-\mu}(X^\wedge)\big)$$

for every $s \in \mathbb{R}$, cf. notation (1.2.1), and

$$m(y,\eta) \in S_{\mathrm{cl}}^{\mu}\left(\Omega \times \mathbb{R}^q; \mathcal{K}_{\mathcal{P}}^{s;\gamma}(X^{\wedge}), \mathcal{S}_{\mathcal{Q}}^{\gamma-\mu}(X^{\wedge})\right)$$

for every continuous asymptotic type $\mathcal{P}$ with some resulting continuous asymptotic type $\mathcal{Q}$ (associated with the weight data (γ, Θ) and $(\gamma - \mu, \Theta)$, respectively). Moreover, the pointwise formal adjoint $m^*(y,\eta)$ (cf. also Definition 1.2.1), has a similar structure as (3.2.4), i.e., we have

$$m^*(y,\eta) \in S_{\mathrm{cl}}^{\mu}\left(\Omega \times \mathbb{R}^q; \mathcal{K}_{(\mathcal{R})}^{s,-\gamma+\mu}, \mathcal{S}_{(\mathcal{S})}^{-\gamma}(X^{\wedge})\right)$$

for all $s \in \mathbb{R}$, where the subscripts mean without or with the corresponding continuous asymptotic types.

There are now several essential operations in the edge symbolic calculus which produce Green symbols in the sense of Remark 3.1.2. More precisely, we obtain Green symbols $g(y,\eta)$ of the kind

$$(3.2.6) \qquad g(y,\eta) \in S_{\mathrm{cl}}^{\mu}\left(\Omega \times \mathbb{R}^q; \langle r \rangle^l \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{S}_{\mathcal{P}}^{\gamma-\mu}(X^{\wedge})\right),$$

such that

$$(3.2.7) \qquad g^*(y,\eta) \in S_{\mathrm{cl}}^{\mu}\left(\Omega \times \mathbb{R}^q; \langle r \rangle^l \mathcal{K}^{s,-\gamma+\mu}(X^{\wedge}), \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^{\wedge})\right)$$

for suitable continuous asymptotic types $\mathcal{P}$ and $\mathcal{Q}$ associated with the weight data $(\gamma - \mu, \Theta)$ and $(-\gamma, \Theta)$, respectively, for all $l \in \mathbb{N}$, $s \in \mathbb{R}$.

Remark 3.2.3. *An element $m(y,\eta)$ of the form* (3.2.4) *is a Green symbol for every $j > -\vartheta$, where $\Theta = (\vartheta, 0]$ is the finite weight strip which plays the role in the continuous analogue of* Definition 1.2.1.

Another point concerns the fact that there may be different choices of γ_j (when $j > 0$, otherwise for $j = 0$ we have $\gamma_0 = \gamma$) such that (3.2.5) holds. Let $\widetilde{\gamma}_j$ denote any other choice. Then we have the following result:

Remark 3.2.4. *Let $j > 0$, and let γ_j and $\widetilde{\gamma}_j$ denote different weights satisfying* (3.2.5). *Consider the operators* (3.2.4) *for both weights, e.g.*

$$\widetilde{m}(y,\eta) := \omega(r[\eta]) r^{-\mu+j} \mathrm{op}_M^{\widetilde{\gamma}_j - \frac{n}{2}}(f_{j\alpha})(y) \eta^{\alpha} \widetilde{\omega}(r[\eta]).$$

Then we have $m(y,\eta) = \widetilde{m}(y,\eta)$ modulo a Green symbol with continuous asymptotics, cf. Remark 3.1.2.

3.3. Integral representations in the continuous case. Let $f(r, x, r', x'; y, \eta)$ be a function in the space

$$(3.3.1) \quad \left\{\mathcal{S}_{\mathcal{P}}^{\gamma-\mu}(X^{\wedge}) \widehat{\otimes}_{\pi} \mathcal{S}^{-\gamma}(X^{\wedge})\right\} \cap \left\{\mathcal{S}^{\gamma-\mu}(X^{\wedge}) \widehat{\otimes}_{\pi} \mathcal{S}_{\overline{\mathcal{Q}}}^{-\gamma}(X^{\wedge})\right\} \widehat{\otimes}_{\pi} S_{\mathrm{cl}}^{\mu+n+1}(\Omega \times \mathbb{R}^q),$$

now for continuous asymptotic types $\mathcal{P}$ and $\mathcal{Q}$. Then the integral representation

$$(3.3.2) \qquad g(y,\eta)u(r,x) := \int_X \int_0^{\infty} f(r[\eta], x, r'[\eta], x'; y, \eta) u(r', x')(r')^n \, dr' \, dx',$$

$n = \dim X$, gives us special Green symbols with the properties (3.2.6) and (3.2.7) for all $l \in \mathbb{N}$, $s \in \mathbb{R}$.

Theorem 3.3.1. *Let $g(y,\eta)$ satisfy the conditions* (3.2.6) *and* (3.2.7) *for all $l \in \mathbb{N}$, $s \in \mathbb{R}$. Then there is an $f(r, x, r', x'; y, \eta)$ in the space* (3.3.1) *such that the integral representation* (3.3.2) *holds.*

Proof. The proof employs analogous steps as that of Theorem 2.1.1; so we only discuss the main ideas. For simplicity we omit again the y-variable and write the Green symbol $g(\eta)$ as an asymptotic sum of the kind (2.1.4), modulo a Green symbol of order $-\infty$. For the homogeneous components we take the integral representation (2.1.6) for all j, then form the functions (2.1.7) and obtain the symbols (2.1.8). This yields the corresponding analogue of $g_{a_0}(\eta)$ which is of the desired integral form, modulo a Green symbol of order $-\infty$, given by a kernel like (2.1.9). It then remains to show the analogue of Lemma 2.1.2 for the case with continuous asymptotics. The proof of that is a purely technical (but elementary) construction in terms of the scales of spaces A^N and B^N. The spaces B^N are the same as before, while the A^N are constructed in Proposition 3.1.1. The main new aspect to be employed in the proof is the first of the relations (2.1.16). In the present case we have to look at (3.1.6). The first summand is as in (1.1.4), and it remains to observe that $-r\partial_r$ transforms the space $\mathcal{E}_{\mathcal{P}}^N(X^\wedge)$ to $\mathcal{E}_{\mathcal{P}}^{N-1}(X^\wedge)$, modulo a flat contribution which is absorbed by the first summand in (3.1.6). Applying $-r\partial_r$ to the second factor of $\omega(r)\langle\zeta, r^{-z}\rangle \in \mathcal{E}_{\mathcal{P}}^N(X^\wedge)$, cf. (3.1.3), we obtain $\omega(r)\langle\zeta, zr^{-z}\rangle$; thus we remain in the space $\mathcal{E}_{\mathcal{P}}(X^\wedge)$ and hence, from the continuity of $\mathcal{E}_{\mathcal{P}}(X^\wedge) \to \mathcal{E}_{\mathcal{P}}(X^\wedge)$, $\omega(r)\langle\zeta, r^{-z}\rangle \mapsto \omega(r)\langle\zeta, zr^{-z}\rangle$ and the definition of $\mathcal{E}_{\mathcal{P}}^N(X^\wedge)$, we immediately obtain the desired relation, i.e., $-r\partial_r : \mathcal{E}_{\mathcal{P}}^N(X^\wedge) \to \mathcal{E}_{\mathcal{P}}^{N-1}(X^\wedge)$. The other element of the proof are very close to the ones of Lemma 2.1.2 and will be omitted. $\square$

Analogously to the discrete case, cf. Section 2.2, we can consider 2×2 block matrix-valued functions $g(y,\eta) \in S_{\mathrm{cl}}^\mu\big(\Omega \times \mathbb{R}^q; \mathcal{K}^{0,\gamma}(X^\wedge) \oplus \mathbb{C}, \mathcal{K}^{0,\beta}(X^\wedge) \oplus \mathbb{C}\big)$ such that

$$(3.3.3) \qquad g(y,\eta) \in S_{\mathrm{cl}}^\mu\big(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}, \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge) \oplus \mathbb{C}\big)$$

and

$$(3.3.4) \qquad g^*(y,\eta) \in S_{\mathrm{cl}}^\mu\big(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\beta}(X^\wedge) \oplus \mathbb{C}, \mathcal{S}_{\mathcal{Q}}^{-\gamma}(X^\wedge) \oplus \mathbb{C}\big)$$

for all $s \in \mathbb{R}$, with suitable g-dependent continuous asymptotic types $\mathcal{P}, \mathcal{Q}$. Let $g(y,\eta) = (g_{ij}(y,\eta))_{i,j=1,2}$; then we call $g_{21}(y,\eta)$ a trace symbol and $g_{12}(y,\eta)$ a potential symbol of order $\mu \in \mathbb{R}$, while $g_{22}(y,\eta)$ is nothing other than a classical scalar symbol (of order μ).

Let $f_{21} \in \mathcal{S}_{\overline{\mathcal{Q}}}^{-\gamma}(X^\wedge)\widehat{\otimes}_\pi S_{\mathrm{cl}}^{\mu+\frac{n+1}{2}}(\Omega \times \mathbb{R}^q)$ and $f_{12} \in \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge)\widehat{\otimes}_\pi S_{\mathrm{cl}}^{\mu+\frac{n+1}{2}}(\Omega \times \mathbb{R}^q)$, and consider the integral representations

$$(3.3.5) \qquad g_{21}(y,\eta)u = \int_X \int_0^\infty f_{21}(r'[\eta], x'; y, \eta)u(r', x')(r')^n dr' dx',$$

$u(r,x) \in \mathcal{K}^{s,\gamma}(X^\wedge)$, and

$$(3.3.6) \qquad g_{12}(y,\eta)c(r,x) = cf_{12}(r[\eta], x; y, \eta),$$

$c \in \mathbb{C}$. Then we have (trace and potential) symbols satisfying the mapping properties (3.3.3) and (3.3.4).

Theorem 3.3.2.   (i) *Every trace symbol $g_{21}(y,\eta)$ can be written in the form* (3.3.5) *for an element* $f_{21}(r', x'; y, \eta) \in \mathcal{S}_{\overline{\mathcal{Q}}}^{-\gamma}(X^\wedge)\widehat{\otimes}_\pi S_{\mathrm{cl}}^{\mu+\frac{n+1}{2}}(\Omega \times \mathbb{R}^q)$;

    (ii) *every potential symbol $g_{12}(y,\eta)$ can be written in the form* (3.3.6) *for an element* $f_{12}(r, x; y, \eta) \in \mathcal{S}_{\mathcal{P}}^\beta(X^\wedge)\widehat{\otimes}_\pi S_{\mathrm{cl}}^{\mu+\frac{n+1}{2}}(\Omega \times \mathbb{R}^q)$.

This can be proved by analogous arguments as for Theorem 3.3.1.

Remark 3.3.3. (*i*) *Green symbols in the sense of block matrices* (3.3.3) *can be composed within the respective spaces of Green symbols* (*with discrete or continuous asymptotics*) *and the homogeneous principal components behave multiplicatively.*

(*ii*) *The classes of Green symbols* (*with discrete or continuous asymptotics*) *are closed under asymptotic summation when the involved asymptotic types are the same for all the summands.*

Let us conclude this section with a few intuitive remarks on the nature of continuous asymptotics which give rise to some 'unexpected' examples of Green, trace, or potential operators in that context. First note that when $\omega(r)$ is any fixed cut-off function and $z \in \mathbb{C}$, $\operatorname{Re} z < \frac{n+1}{2} - \gamma$, then we have $\omega(r)r^{-z}c(x) \in \mathcal{K}^{\infty,\gamma}(X^\wedge)$ for any $c \in C^\infty(X)$. Recall that such functions may be interpreted as singular functions of the discrete cone asymptotics for $r \to 0$ with exponent $-z$. Now if $\zeta(y) \in C^\infty\big(\Omega, \mathcal{A}'(K, C^\infty(X))\big)$ is any family of analytic functionals carried by a compact set $K \subset \big\{\operatorname{Re} z < \frac{n+1}{2} - \gamma\big\}$, the function

$$(3.3.7) \qquad\qquad \omega(r)\langle \zeta(y), r^{-z}\rangle$$

may be interpreted as the linear superposition of singular functions for the discrete asymptotics with the (y-dependent) density $\zeta(y)$. Such 'densities' may be organised as follows. Choose an arbitrary function $f(y, z) \in C^\infty(\Omega \times (\mathbb{C} \setminus K), C^\infty(X))$, holomorphic in $z \in \mathbb{C} \setminus K$ that extends for every $y \in \Omega$ to a certain ($C^\infty(X)$-valued) meromorphic function in $z \in \mathbb{C}$ with poles $p_j(y) \in K$, $j = 1, \ldots, N(y)$, of multiplicities $m_j(y) + 1$. These poles including multiplicities may be not constant in y. Then, setting

$$\zeta(y) : h(z) \mapsto \frac{1}{2\pi i}\int_C h(z)f(y, z)dz$$

where $C \subset \big\{\operatorname{Re} z < \frac{n+1}{2} - \gamma\big\}$ is a compact (say, C^∞) curve counter clockwise surrounding the set K such that C has the winding number 1 with respect to every point of K, we obtain an element $\zeta(y) \in C^\infty\big(\Omega, \mathcal{A}'(K, C^\infty(X))\big)$ such that (3.3.7) has discrete asymptotics of the kind (3.1.5) for every fixed y. Now

$$\mathbb{C} \ni c \mapsto c \cdot \omega(r[\eta])[\eta]^{\mu + \frac{n+1}{2}}\langle \zeta(y), (r[\eta])^{-z}\rangle$$

defines a potential symbol of order μ in the frame of continuous asymptotics which just produces functions with pointwise (in y) discrete but branching asymptotics. In a similar manner we can organise trace symbols which reflect such asymptotics as well as more general Green symbols of that kind. Constructions of that kind may also be found in [12].

4. Green operators

4.1. **Green operators on a manifold with edges.** Let W be a compact manifold with edge Y, locally near any $y \in Y$ modelled on $X^\triangle \times \mathbb{R}^q$, where X is a closed compact C^∞ manifold. Recall that transition functions between (open) stretched wedges $\mathbb{R}_+ \times X \times \mathbb{R}^q \ni (r, x, y)$ are assumed to be C^∞ up to $r = 0$. In addition we choose the global atlas by such singular charts near Y in such a way that the transition functions are constant with respect to r for $0 < r < \varepsilon$ for some $\varepsilon > 0$. By

$\mathbb{W}$ we denote the stretched manifold associated with W, cf. Section 1.2.

We consider the weighted edge Sobolev space $\mathcal{W}^{s,\gamma}(\mathbb{W})$ that is defined as the subspace of all $u \in H^s_{\mathrm{loc}}(\mathrm{int}\mathbb{W})$ which locally near Y in the coordinates (r,x,y) belong to $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$. Here $\mathcal{W}^s(\mathbb{R}^q, E)$ for a Hilbert space E with group action κ_λ is the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm

$$\left\{ \int \langle \eta \rangle^{2s} \left\| \kappa_{\langle\eta\rangle}^{-1} \hat{u}(\eta) \right\|_E^2 d\eta \right\}^{\frac{1}{2}},$$

with $\hat{u}(\eta)$ being the Fourier transform of u in $\mathbb{R}^q$. In a similar manner we define $\mathcal{W}^s(\mathbb{R}^q, E)$ for a Fréchet space E which is the projective limit of Hilbert spaces E^j with group actions, with continuous embeddings $E^{j+1} \hookrightarrow E^j \hookrightarrow \cdots \hookrightarrow E^0$ for all $j \in \mathbb{N}$, such that the group action on E^j is the restriction of the one on E^0 for every j.

This allows us to define subspaces

(4.1.1) $\mathcal{W}^s(\mathbb{R}^q, \mathcal{S}_{\mathcal{P}}^\gamma(X^\wedge))$

of $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ for any (discrete or continuous) asymptotic type $\mathcal{P}$, using the fact that $\mathcal{S}_{\mathcal{P}}^\gamma(X^\wedge)$ is a Fréchet space with group action induced by κ_λ on $\mathcal{K}^{s,\gamma}(X^\wedge)$, $s \in \mathbb{R}$. Globally on $\mathbb{W}$ we then define $\mathcal{W}_{\mathcal{P}}^{s,\gamma}(\mathbb{W})$ to be the subspace of $\mathcal{W}^{s,\gamma}(\mathbb{W})$ locally near the edge described by (4.1.1).

A Green operator G (of the type of an upper left corner) with (discrete or continuous) asymptotics is an operator that is locally near Y in stretched coordinates (r,x,y) of the form

$$\mathrm{Op}_y(g)u(y) = \int_{\mathbb{R}^q} \int_{\mathbb{R}^q} e^{i(y-y')\eta} g(y,\eta) u(y') dy' \bar{d}\eta,$$

for a Green symbol $g(y,\eta)$ of order μ, modulo a global smoothing operator. The latter category of operators is characterised by the property to define a continuous map $\mathcal{W}^{s,\gamma}(\mathbb{W}) \to \mathcal{W}_{\mathcal{P}}^{\infty,\gamma-\mu}(\mathbb{W})$ for some asymptotic type $\mathcal{P}$ and a similar property of the formal adjoint. The global definition of Green operators is justified by the following remark.

Remark 4.1.1. *With symbols $g(y,\eta) \in S_{(\mathrm{cl})}^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ in the sense of the notation in* Section 1.2 *we can form associated operators*

$$\mathrm{Op}_y(g)u(y) = \int_{\mathbb{R}^q} \int_\Omega e^{i(y-y')\eta} g(y,\eta) u(y') dy' \bar{d}\eta,$$

$\bar{d}\eta := (2\pi)^{-q} d\eta$. *In particular, if $g(y,\eta)$ is a Green symbol of the kind* (3.2.6), *then for every $\varphi(r) \in C_0^\infty(\mathbb{R}_+\times)$ the operators $\varphi\mathrm{Op}_y(g)$ and $\mathrm{Op}_y(g)\varphi$ are smoothing on $\mathbb{R}_+ \times \mathbb{R}^q \times X$. In particular, we see that the singularities of Green operators are concentrated on the boundary $\{0\} \times \Omega \times X$ as is expected in analogy to a corresponding behaviour of Green operators in classical boundary value problems.*

In fact, let us first note that the operators of multiplication $\mathcal{M}_\varphi$ by φ generate (non-classical) symbols $\mathcal{M}_\varphi \in S^0(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge))$ for every $\gamma \in \mathbb{R}$. Moreover, the multiplication by $r^{-N}\varphi$ for any $N \in \mathbb{N}$ is of similar behaviour. Then,

$$\varphi g(y,\eta) = r^{-N}\varphi r^N g(y,\eta) = r^{-N}\varphi[\eta]^{-N}(r[\eta])^N g(y,\eta).$$

Since the order of $(r[\eta])^N g(y, \eta)$ is the same as that of $g(y, \eta)$ and the multiplication by $[\eta]^{-N}$ gives rise to an order shift by $-N$ we obtain that $\varphi g(y, \eta)$ is an operator-valued symbol of order $-\infty$, and hence $\varphi \mathrm{Op}_y(g) = \mathrm{Op}_y(\varphi g)$ is smoothing.

Observe that a Green operator G on $\mathbb{W}$ induces continuous operators

$$\mathcal{W}^{s,\gamma}(\mathbb{W}) \to \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathcal{P}}(\mathbb{W})$$

for every $s \in \mathbb{R}$, where $\mathcal{P}$ is a (discrete or continuous) asymptotic type associated with G. This is a consequence of general continuity on Sobolev spaces.

Remark 4.1.2. *Green operators on a (stretched) manifold $\mathbb{W}$ with edges form an algebra, and the composition is compatible with the local symbolic structure; in particular, the homogeneous principal symbols (in the sense of twisted homogeneity) behave multiplicatively.*

4.2. **Green operators with parameters.** The concept of operator-valued symbols as in Section 1.2 has a parameter-dependent analogue, when we replace the covariable $\eta \in \mathbb{R}^q$ by $(\eta, \lambda) \in \mathbb{R}^q \times \mathbb{R}^l$ and require the symbolic estimates with respect to (η, λ). In particular, we obtain a generalisation of Definition 1.2.1 to the λ-dependent case, cf. also Remark 3.1.2.

The construction of the preceding section then gives us parameter-dependent families of Green operators. According to the iterative concept of building up pseudo-differential calculi on manifolds with higher (polyhedral) singularities we may employ such parameter-dependent families as (operator-valued) symbols of a next generation of operators, for instance, on the infinite (stretched) cone $\mathbb{R}_+ \times \mathbb{W}$ with base $\mathbb{W}$. Constructions in that sense may be found in the paper [2], in particular, a number of kernel cut-off results for such operator functions.

Kernel cut-offs can be organised on the level of symbols. In order to illustrate the effects we want to consider Green symbols with discrete asymptotics as in Definition 1.2.1 which belong to spaces of the kind

$$(4.2.1) \qquad S^\mu_{\mathrm{cl}}\left(\Omega \times \mathbb{R}^q \times \mathbb{R}^l; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^\beta_{\mathcal{P}}(X^\wedge)\right)$$

such that $g^*(y, \eta, \lambda)$ belongs to corresponding analogue of the space in (1.2.5). For $l = 1$ we also write Γ_δ instead of $\mathbb{R}$ when λ is involved in the form $z = \delta + i\lambda$ for some $\delta \in \mathbb{R}$. Moreover, let

$$S^\mu_{\mathrm{cl}}\left(\Omega \times \mathbb{R}^q \times \mathbb{C}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^\beta_{\mathcal{P}}(X^\wedge)\right)$$

denote the space of all $g(y, \eta, z)$ which are holomorphic in $z \in \mathbb{C}$ such that $g(y, \eta, \delta + i\lambda)$ belongs to (4.2.1) (for $l = 1$) for every $\delta \in \mathbb{R}$, uniformly in compact δ-intervals, and where $g^*(y, \eta, z)$ satisfies an analogous condition.

Theorem 4.2.1. *For every $\delta \in \mathbb{R}$ there is a continuous map*

$$S^\mu_{\mathrm{cl}}\left(\Omega \times \mathbb{R}^q \times \Gamma_\delta; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^\beta_{\mathcal{P}}(X^\wedge)\right) \to S^\mu_{\mathrm{cl}}\left(\Omega \times \mathbb{R}^q \times \mathbb{C}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^\beta_{\mathcal{P}}(X^\wedge)\right),$$

$g(y, \eta, \delta + i\lambda) \mapsto h(y, \eta, \delta + i\lambda)$, *such that*

$$g(y, \eta, z) - h(y, \eta, z)|_{\Omega \times \mathbb{R}^q \times \Gamma_\delta} \in S^{-\infty}\left(\Omega \times \mathbb{R}^q \times \Gamma_\delta; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^\beta_{\mathcal{P}}(X^\wedge)\right)$$

for every $s \in \mathbb{R}$ (*and such that the pointwise formal adjoint have an analogous property*).

Theorem 4.2.1 can be proved by applying a kernel cut-off argument as used in an analogous context in [2]. A similar result holds for Green symbols with continuous asymptotics.

According to Theorem 2.1.1 the holomorphic symbol $h(y, \eta, z)$ has a family of integral kernels

$$(4.2.2) \qquad f_{\operatorname{Re} z}(r[\eta, \lambda], x, r'[\eta, \lambda], x', y, \eta, \lambda),$$

$\lambda = \operatorname{Im} z$, via a representation of the form (2.1.2). This is valid for every fixed $\operatorname{Re} z$; however, the holomorphic dependence of (4.2.2) on z is by no means obvious. In other words, kernel cut-off constructions which produce a holomorphy in a complex covariable are better applied to the symbols in their original definition rather than their integral kernels.

REFERENCES

[1] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[2] G. De Donno and B.-W. Schulze, *Meromorphic symbolic structures for boundary value problems on manifolds with edges*, Preprint 2003/10, Institut für Mathematik, Potsdam, 2003, Math. Nachr. (to appear).

[3] N. Dines, G. Harutjunjan, and B.-W. Schulze, *The Zaremba problem in edge Sobolev spaces*, Preprint 2003/13, Institut für Mathematik, Potsdam, 2003.

[4] Ju. V. Egorov and B.-W. Schulze, *Pseudo-differential operators, singularities, applications*, Operator Theory, Advances and Applications, vol. 93, Birkhäuser Verlag, Basel, 1997.

[5] H. Jarchow, *Locally convex spaces*, B.G. Teubner, Stuttgart, 1981.

[6] D. Kapanadze and B.-W. Schulze, *Asymptotics of potentials in the edge calculus*, Preprint 2003/05, Institut für Mathematik, Potsdam, 2003.

[7] D. Kapanadze and B.-W. Schulze, *Crack theory and edge singularities*, Kluwer Academic Publ., Dordrecht, 2003.

[8] D. Kapanadze and B.-W. Schulze, *Boundary - contact problems for domains with conical singularities*, Preprint 2004/11, Institut für Mathematik, Potsdam, 2004, J.Diff.Equ. (to appear).

[9] V.A. Kondratyev, *Boundary value problems for elliptic equations in domains with conical points*, Trudy Mosk. Mat. Obshch. **16** (1967), 209–292.

[10] V.G. Maz'ja and J. Rossmann, *Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten*, Math. Nachr. **138** (1988), 27–53.

[11] B.-W. Schulze, *Regularity with continuous and branching asymptotics for elliptic operators on manifolds with edges*, Integral Equations Operator Theory **11** (1988), 557–602.

[12] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.

[13] B.-W. Schulze, *The variable discrete asymptotics of solutions of singular boundary value problems*, Operator Theory: Advances and Applications, vol. 57, Birkhäuser Verlag, Basel, 1992, pp. 271–279.

[14] B.-W. Schulze and N.N. Tarkhanov, *Green pseudodifferential operators on a manifold with edges*, Comm. Partial Differential Equations **23**, 1-2 (1998), 171–200.

[15] J. Seiler, *The cone algebra and a kernel characterization of Green operators*, Advances in Partial Differential Equations (Approaches to Singular Analysis) (J. Gil, D. Grieser, and Lesch M., eds.), Oper. Theory Adv. Appl., Birkhäuser, Basel, 2001, pp. 1–29.