Harmonic Integrals on Domains with Edges

Nikolai Tarkhanov

Institut für Mathematik
Universität Potsdam
Postfach 60 15 53
14415 Potsdam
Germany

October 22, 2004

Abstract

We study the Neumann problem for the de Rham complex in a bounded domain of $\mathbb{R}^n$ with singularities on the boundary. The singularities may be general enough, varying from Lipschitz domains to domains with cuspidal edges on the boundary. Following Lopatinskii we reduce the Neumann problem to a singular integral equation of the boundary. The Fredholm solvability of this equation is then equivalent to the Fredholm property of the Neumann problem in suitable function spaces. The boundary integral equation is explicitly written and may be treated in diverse methods. This way we obtain, in particular, asymptotic expansions of harmonic forms near singularities of the boundary.

2000 Mathematics Subject Classification: primary: 35J25; secondary: 58A14, 35A27.

Key words: domains with singularities, de Rham complex, Neumann problem, Hodge theory.

Contents

1 Introduction

We testify two approaches to boundary value problems for second order elliptic equations in domains with singularities on the boundary. Namely, we study the Neumann problem for the de Rham complex in domains with edges on the boundary. This problem is known to be motivated by the Hodge theory on compact manifolds with boundary. The first approach consists of a calculus of pseudodifferential operators on a manifold with edges. It gives a criterion for the Fredholm property that consists of the unique solvability of a boundary value problem for a Bessel type ordinary differential equation in a semicylinder. The coefficients of the equation are operator-valued while the weighted Sobolev spaces under study are very delicate. The second approach reduces the Neumann problem to a pair of singular integral equations on the boundary which is a Lipschitz surface. The equations are written explicitly, and the symbol of the system is a triangular matrix. Hence the Fredholm property of the Neumann problem is equivalent to that of an explicit singular integral equation on the boundary. Thus, our treatment gives some suggestive evidence to the advantage of the latter approach in the case of second order elliptic equations.

2 Stratified spaces

In this paper we discuss boundary value problems for solutions of differential equations on manifolds with singularities of sufficiently general nature. As singularities we admit edges of diverse dimensions which meet each other at nonzero angles. The same sets are thought of as carriers of discontinuity of coefficients.

The manifolds with singularities in question are actually stratified spaces or, figuratively speaking, collections of differentiable manifolds which are suitably pasted together, cf. [Whi46]. We define them by induction in $n = 0, 1, \ldots$, n being the dimension.

Denote by $\mathcal{V}^0$ the class of finite collections of points endowed with the discrete topology. By a differentiable function on such a set is understood any function.

Suppose the classes $\mathcal{V}^0, \mathcal{V}^1, \ldots, \mathcal{V}^{n-1}$ have been already defined, and introduce $\mathcal{V}^n$. Let T be a compact Hausdorff topological space, and $\{U_i\}$ a finite open covering of T. We require that for each i there be a homeomorphism h_i of U_i onto a relatively compact subset of a product $\mathbb{R}^q \times \mathcal{C}(V)$, where $\mathcal{C}(V)$ is the topological cone over a space $V \in \mathcal{V}^{n-q-1}$. Away from the vertex $\mathcal{C}(V)$ just amounts to $\mathbb{R}_+ \times V$, which bears already a C^∞ structure by the very construction.

Given a point $p \in U_i$, write (y, z) for the projections of $h_i(p)$ to the factors of $\mathbb{R}^q \times \mathcal{C}(V)$. We can furthermore specify $z \in \mathcal{C}(V)$ as (r, x), where $r \geq 0$ and $x \in V$. In this way we reduce the definition of a C^∞ structure near the vertex of $\mathcal{C}(V)$, which is blown up to $\{0\} \times V$, to introducing a C^∞ structure at $r = 0$ on the half-axis $\bar{\mathbb{R}}_+$. This latter usually originates from the geometry of the cone which is given by a Riemannian metric $dr^2 + (\delta'(r))^{-2} dx^2$ of product type on $\mathbb{R}_+ \times V$, where δ is a C^∞ function on $\mathbb{R}_+$. As usual, $\mathbb{R}^q \times \mathcal{C}(V)$ is said to be a model wedge with an edge along $\mathbb{R}^q$. It is based on the cone $\mathcal{C}(V)$ whose cross-section or link is the singular space V. The case $q = 0$ is not excluded, hence any point singularity can be formally thought of as an edge of dimension 0. Every model wedge gives rise to a typical vector field $\delta'(r) D_x$ along the link V. The derivative D_x can in turn be degenerate, too, unless V is a smooth compact manifold.

In the sequel, the edges of diverse dimensions in T are denoted by Y, and $X = L_Y$ stands for the link of Y. We thus arrive at a finite partition Z of T into locally closed subspaces S, such that every piece $S \in Z$ is a smooth manifold without boundary in the induced topology, and if $R \cap \overline{S} \neq \emptyset$ for some $R, S \in Z$, then $R \subset \overline{S}$. In the latter case one writes $R < S$ and says that R is incident to S, or a boundary piece of S. The incidence relation is easily seen to be an order relation on the set of pieces of T. This justifies the designation $R < S$.

A compact manifold with boundary is a simplest example of a stratified space, the decomposition being $\partial M < M \setminus \partial M$. The boundary itself is an edge of codimension 1, with a link consisting of a single point. A compact manifold with a singular point on the boundary bears a natural stratification $P < \partial M \setminus P < M \setminus \partial M$. More generally, a compact manifold with a smooth edge on the boundary possesses a stratification $Y < \partial M \setminus Y < M \setminus \partial M$. Since the analysis of elliptic boundary value problems is local by the very nature, we restrict our discussion to this latter case. For a thorough treatment we refer the reader to [NP91].

3 Elliptic boundary value problems

Boundary value problems in domains with conical points on the boundary were studied quite thoroughly by Kondrat'ev [Kon67]. Boundary value problems with singularities of the boundary concentrated on a smooth q-dimensional manifold were studied from various points of view in [VE67], [Gru71], [Fei72], [MP77], [KO83], etc.

Let $\mathcal{D}$ be a relatively compact domain in $\mathbb{R}^n$ whose boundary contains an edge Y. This means, Y is a smooth q-dimensional submanifold of $\mathbb{R}^n$ without boundary and $\partial\mathcal{D} \setminus Y$ is a smooth hypersurface. Moreover, for each point $y \in Y$ there is a neighbourhood U in $\mathbb{R}^n$ and a diffeomorphism h of U onto an open set O in $\mathbb{R}^n$, such that $h(U \cap \mathcal{D}) = O \cap (\mathbb{R}^q \times \mathcal{C}(X))$ where

$\mathcal{C}(X) = \{(z_{n-q}z', z_{n-q}) : z' \in X, \, z_{n-q} > 0\}$ is a cone over a bounded domain $X \subset \mathbb{R}^{n-q-1}$.

In $U \cap \mathcal{D}$ we can thus use local coordinates $(y, r, x) \in \mathbb{R}^q \times \mathbb{R}_+ \times X$, where $r = z_{n-q}$ and $x = z_{n-q}^{-1} z'$. Then

$$\begin{aligned} D_{z_j} &= r^{-1} D_{x_j}, \qquad \text{for } j = 1, \ldots, n-q-1, \\ D_{z_{n-q}} &= D_r - r^{-1} \sum_{j=1}^{n-q-1} x_j D_{x_j}, \end{aligned}$$

hence any differential operator L of order $2m$ in $\mathcal{D}$ takes in the local coordinates of $U \cap \mathcal{D}$ the form

$$\pi_* L = \frac{1}{r^{2m}} \sum_{|\beta|+k+|\alpha| \leq 2m} L_{\beta,k,\alpha}(y, r, x) \Big(r D_y \Big)^{\beta} \Big(r D_r \Big)^{k} D_x^{\alpha}, \tag{3.1}$$

the coefficients $L_{\beta,k,\alpha}$ being $(k \times k)$-matrices of smooth functions in $U \cap \mathcal{D}$. When posing boundary value problems for solutions of $Lu = f$ in $\mathcal{D}$, one usually assumes that the coefficients of L are C^∞ up to the smooth part of the boundary, i.e., $\partial \mathcal{D} \setminus Y$. We restrict our discussion to those differential operators L, for which the coefficients $L_{\beta,k,\alpha}$ are actually smooth up to the edge, the latter corresponding to $r = 0$. This condition can be relaxed by requiring the continuity of $L_{\beta,k,\alpha}$ up to $r = 0$ along with certain asymptotic expansions for $r \to 0$.

The vector field $\mathbf{D}_r = r D_r$ is called the Fuchs derivative. The analysis of Fuchs type operators is typical for conical singularities.

As is easy to check, the differential operators (3.1) are invariant under those local diffeomorphisms of $\mathcal{D}$, which preserve the cone bundle structure of $\mathcal{D}$ close to the edge Y. In this way we obtain what will be referred to as edge degenerate operators.

The natural domains of edge degenerate operators are weighted Sobolev spaces $H^{s,\gamma}(\mathcal{D}, \mathbb{C}^k)$ depending on two real parameters s and γ. The index s specifies the smoothness over $\mathcal{D}$, while the index γ specifies the growth near Y. For $s = 0, 1, \ldots$, the space $H^{s,\gamma}(\mathcal{D}, \mathbb{C}^k)$ is defined to be the completion of $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y, \mathbb{C}^k)$ with respect to the norm

$$\|u\|^2_{H^{s,\gamma}(\mathcal{D},\mathbb{C}^k)} = \int_{\mathcal{D}} \rho^{-2\gamma} \Big(\sum_{|\vartheta| \leq s} \rho^{2|\vartheta|} \, |D^{\vartheta} u(w)|^2 \Big) \, dw, \tag{3.2}$$

where ρ is a smooth positive function on $\overline{\mathcal{D}} \setminus Y$ which is equal to the distance from Y near the edge Y. For non-integral $s > 0$ the space $H^{s,\gamma}(\mathcal{D}, \mathbb{C}^k)$ is defined by complex interpolation, and for $s < 0$ it is introduced by a familiar duality argument.

Our next objective is to microlocalise the norm (3.2). Since Y is compact, there is a finite covering $\{U_i\}$ of Y by open sets in $\mathbb{R}^n$ and a family of diffeomorphisms $h_i : U_i \to O_i$, such that $h_i(U_i \cap \mathcal{D}) = O_i \cap (\mathbb{R}^q \times \mathcal{C}(X_i))$. Pick a partition of unity $\{\varphi_i\}$ in some neighbourhood of Y subordinate to the covering $\{U_i\}$, i.e., $\varphi_i \in C^\infty_{\mathrm{comp}}(U_i)$ and $\sum \varphi_i = 1$ near Y. Set $\varphi_0 = 1 - \sum \varphi_i$, then φ_0 is a C^∞ function on the closure of $\mathcal{D}$, vanishing close to Y. Hence it follows that

$$\|u\|^2_{H^{s,\gamma}(\mathcal{D},\mathbb{C}^k)} \sim \|\varphi_0 u\|^2_{H^s(\mathcal{D},\mathbb{C}^k)} + \sum \|\varphi_i u\|^2_{H^{s,\gamma}(\mathcal{D},\mathbb{C}^k)}.$$

For each i, the product $\varphi_i u$ has compact support in $U_i \cap \overline{\mathcal{D}}$ with local coordinates (y, r, x). An easy verification shows that the norm of $\varphi_i u$ in $H^{s,\gamma}(\mathcal{D}, \mathbb{C}^k)$ is actually equivalent to

$$\Big(\int\limits_{\mathbb{R}^q} \int\limits_0^\infty r^{-2\gamma+(n-q-1)} \sum_{|\beta|+j+A\le s} (1+r^2)^{s-|\beta|-j-A} \big\|(rD_y)^\beta \mathbf{D}_r^j h_{i,*}(\varphi_i u)\big\|^2_{H^A(X_i)}\, dydr \Big)^{\frac{1}{2}} \tag{3.3}$$

for all $s = 0, 1, \ldots$.

Note that the factors $(1+r^2)^{s-|\beta|-j-A}$ do not affect the norm at $r = 0$. However, they are of great importance in the analysis on infinite model wedges $\mathbb{R}^q \times \mathcal{C}(X_i)$.

Lemma 3.1 *As defined by (3.3), the norm survives under 'edgification', i.e.,*

$$\|u\|_{H^{s,\gamma}(\mathbb{R}^q\times\mathcal{C}(X))} \sim \Big(\int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\to\eta} u\|^2_{H^{s,\gamma}(\mathcal{C}(X))} d\eta \Big)^{\frac{1}{2}}$$

where $\kappa_\lambda \in \mathcal{L}(H^{s,\gamma}(\mathcal{C}(X)))$ *is given by* $(\kappa_\lambda u)(r,x) = \lambda^{s-\gamma+\frac{n-q}{2}} u(\lambda r, x)$ *for* $\lambda > 0$.

Proof. Let u be a C^∞ function of compact support on $\mathbb{R}^q$ taking its values in $H^{s,\gamma}(\mathcal{C}(X))$. By abuse of notation, we write it simply $u(y, r, x)$ and omit r and x unless it may cause a confusion. Using explicit formula (3.3) for the norm in $H^{s,\gamma}(\mathcal{C}(X))$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\to\eta} u\|^2_{H^{s,\gamma}(\mathcal{C}(X))} d\eta \\
&= \int_{\mathbb{R}^q} \langle\eta\rangle^{2\gamma-(n-q)} \|\mathcal{F}_{y\to\eta} u\Big(y, \frac{r}{\langle\eta\rangle}, x\Big)\|^2_{H^{s,\gamma}(\mathcal{C}(X))} d\eta \\
&= \int_{\mathbb{R}^q} \int_0^\infty r^{-2\gamma+(n-q-1)} \sum_{j+A\le s} \Big(1 + (\langle\eta\rangle r)^2\Big)^{s-j-A} \|\mathbf{D}_r^j \mathcal{F}_{y\to\eta} u\|^2_{H^A(X)} dr d\eta,
\end{aligned}$$

the latter equality being a consequence of the obvious commutativity relation $\mathbf{D}\,(U(\lambda r)) = (\mathbf{D}U)(\lambda r)$.

Since $\langle \eta \rangle^2 \sim 1 + |\eta|^2$ holds on $\mathbb{R}^q$, the equivalence meaning that the ratio of the functions is bounded both below und above by positive constants, we conclude that

$$\begin{aligned}\Big(1+(\langle \eta \rangle r)^2\Big)^{s-j-A} &\sim \Big(1+r^2+r^2|\eta|^2\Big)^{s-j-A} \\ &= \sum_{k=0}^{s-j-A} \binom{s-j-A}{k} \Big(1+r^2\Big)^{s-j-A-k} \Big(r|\eta|\Big)^{2k}\end{aligned}$$

whence

$$\begin{aligned}&\int_{\mathbb{R}^q} \langle \eta \rangle^{2s} \|\kappa_{\langle \eta \rangle}^{-1} \mathcal{F}_{y\to\eta} u\|^2_{H^{s,\gamma}(\mathcal{C}(X))} d\eta \\ \sim\ &\int_0^\infty r^{-2\gamma+(n-q-1)} \sum_{k+j+A\le s} (1+r^2)^{s-k-j-A} \Big(\int_{\mathbb{R}^q} (r|\eta|)^{2k}\, \|\mathbf{D}_r^j \mathcal{F}_{y\to\eta} u\|^2_{H^A(X)} d\eta \Big) dr \\ \sim\ &\int_{\mathbb{R}^q} \int_0^\infty r^{-2\gamma+(n-q-1)} \sum_{|\beta|+j+A\le s} (1+r^2)^{s-|\beta|-j-A} \left\| (rD_y)^\beta \mathbf{D}_r^j u \right\|^2_{H^A(X)} dy dr,\end{aligned}$$

which is due to Parseval's formula.

□

Denote by $H^{s-1/2,\gamma-1/2}(\partial\mathcal{D},\mathbb{C}^k)$ the space of traces on $\partial\mathcal{D}\setminus Y$ of functions in $H^{s,\gamma}(\mathcal{D},\mathbb{C}^k)$. Any $(k\times k)$-matrix L of scalar edge degenerate differential operators of order $2m$ in $\mathcal{D}$, and $(k_j\times k)$-matrices B_j of scalar edge degenerate differential operators of order m_j near the boundary of $\mathcal{D}$ give rise to a continuous operator

$$\begin{array}{c} L \\ \oplus B_j \end{array} : \ H^{s,\gamma}(\mathcal{D},\mathbb{C}^k) \to \begin{array}{c} H^{s-2m,\gamma-2m}(\mathcal{D},\mathbb{C}^k) \\ \oplus H^{s-m_j-1/2,\gamma-m_j-1/2}(\partial\mathcal{D},\mathbb{C}^{k_j}) \end{array} \tag{3.4}$$

for $s \ge \max\{2m, m_j+1\}$.

The principal parts of the boundary value problem defined by (3.4) are evaluated by the local principle of Simonenko, cf. [Sim65a, Sim65b]. To this end, one uses the local geometry of $\mathcal{D}$. At each interior point of the domain the translation group $\mathbb{R}^n$ acts which leads to the principal homogeneous symbol of L at this point. At any point of $\partial\mathcal{D}\setminus Y$ the local translation group of the tangent boundary hyperplane acts as well as the dilatation group along the inward normal direction. This results in the usual boundary symbol which is a family of boundary value problems for a system of ordinary differential equations on the semiaxis, parametrised by the boundary covariable. The invertibility of the boundary symbol away from the zero section of the cotangent bundle of the boundary just amounts to Lopatinskii's condition. It remains to compute the principal part of (3.4) at the points of the edge Y. At each $y \in Y$ the local translation group of the tangent plane of Y acts as well as the dilatation group along the tangent cone of $\mathcal{D}$ in the normal plane of Y at y. This leads to a family of boundary value problems in the infinite cone which is parametrised by the covariable of the edge. In this way we arrive at what is known as the edge symbol of (3.4), cf. [Sch98]. If L is given by (3.1) near $y \in Y$ then an easy computation yields

$$\begin{aligned} L_0(y;\eta,\mathbf{D}_r,D_x) &= \lim_{\lambda\to\infty} \lambda^{-2m}\, \tilde{\kappa}_\lambda^{-1} \frac{1}{r^{2m}} \sum_{|\beta|+k+|\alpha|\le 2m} L_{\beta,k,\alpha}(y,r,x)\Big(r\lambda\eta\Big)^\beta \mathbf{D}_r^k D_x^\alpha\, \kappa_\lambda \\ &= \frac{1}{r^{2m}} \sum_{|\beta|+k+|\alpha|\le 2m} L_{\beta,k,\alpha}(y,0,x)\Big(r\eta\Big)^\beta \mathbf{D}_r^k D_x^\alpha \end{aligned} \qquad (3.5)$$

for $\eta \in \mathbb{R}^q$. Similarly we localise the boundary operators B_j at $y \in Y$, thus obtaining an edge symbol mapping

$$\begin{matrix} L_0(y;\eta,\mathbf{D}_r,D_x) \\ \oplus B_{j,0}(y;\eta,\mathbf{D}_r,D_x) \end{matrix} : \ H^{s,\gamma}(\mathcal{C}(X_y),\mathbb{C}^k) \to \begin{matrix} H^{s-2m,\gamma-2m}(\mathcal{C}(X_y),\mathbb{C}^k) \\ \oplus H^{s-m_j-1/2,\gamma-m_j-1/2}(\partial\mathcal{C}(X_y),\mathbb{C}^{k_j}) \end{matrix} \qquad (3.6)$$

for all $(y,\eta) \in T^*Y$.

The family of boundary value problems (3.6) on $\mathcal{C}(X_y)$ is a homogeneous function of η, the homogeneity being appropriately defined, cf. [Sch98]. Hence it suffices to study mapping properties of (3.6) for η varying over the unit sphere in $\mathbb{R}^q$.

The calculus of boundary value problems (3.6) over infinite cones $\mathcal{C}(X_y)$ is well understood, cf. [Tro77], [NP91], et al. The solvability is controlled by the so-called conormal symbol of the boundary value problem which is a family of boundary value problems on the link X_y, parametrised by the covariable of the Fuchs derivative which varies over the line $\Im\varrho = -\gamma + (n-q)/2$ in the complex plane. For the edge symbol of L given by (3.5), the conormal symbol is

$$\sigma_M(L_0)(y;\varrho) = \sum_{k+|\alpha|\le 2m} L_{\beta,k,\alpha}(y,0,x)\varrho^k D_x^\alpha,$$

and similarly for boundary operators B_j. Mention that the conormal symbol originates from the Mellin representation of differential operators, which justifies the use of the subscript 'M'. In this way we get the conormal symbol mapping

$$\begin{matrix} \sigma_M(L_0)(y;\varrho) \\ \oplus\sigma_M(B_{j,0})(y;\varrho) \end{matrix} : \ H^s(X_y,\mathbb{C}^k) \to \begin{matrix} H^{s-2m}(X_y,\mathbb{C}^k) \\ \oplus H^{s-m_j-1/2}(\partial X_y,\mathbb{C}^{k_j}) \end{matrix} \qquad (3.7)$$

where $\Im\varrho = -\gamma + (n-q)/2$.

Theorem 3.2 *Let $y \in Y$. The operator (3.6) is Fredholm if and only if the line $\Im\varrho = -\gamma+(n-q)/2$ is free from the spectrum of the conormal symbol (3.7).*

Recall that a number $\varrho \in \mathbb{C}$ is said to belong to the spectrum of (3.7) if the mapping (3.7) fails to be bijective. We mention a few properties of the model problems.

Theorem 3.3 *Let $u \in H^{s,\gamma}(\mathcal{C}(X_y), \mathbb{C}^k)$ be a solution of the homogeneous problem corresponding to (3.6). If χ is a C^∞ function on the closure of $\mathcal{C}(X_y)$ vanishing near the vertex y, then $\chi u \in H^{\infty,\infty}(\mathcal{C}(X_y), \mathbb{C}^k)$.*

In other words, all solutions of the homogeneous problem corresponding to (3.6) are smooth away from the vertex on the closure of $\mathcal{C}(X_y)$ and vanish more rapidly than any power of r, as $r \to \infty$.

Theorem 3.4 *Let $\gamma_1, \gamma_2 \in \mathbb{R}$ satisfy $\gamma_1 < \gamma_2$. Assume that the closed strip between the lines $\Im \varrho = -\gamma_2 + (n-q)/2$ and $\Im \varrho = -\gamma_1 + (n-q)/2$ does not contain any eigenvalue of the conormal symbol (3.7). If $u \in H^{s,\gamma_1}(\mathcal{C}(X_y), \mathbb{C}^k)$ satisfies*

$$\begin{array}{rcl} L_0 u & \in & H^{s-m,\gamma_2-m}(\mathcal{C}(X_y), \mathbb{C}^k), \\ B_{j,0} u & \in & H^{s-m_j-1/2,\gamma_2-m_j-1/2}(\partial\mathcal{C}(X_y), \mathbb{C}^{k_j}), \end{array}$$

then actually $u \in H^{s,\gamma_2}(\mathcal{C}(X_y), \mathbb{C}^k)$.

Theorem 3.4 shows that solutions of the boundary value problem in the cone $\mathcal{C}(X_y)$ that is associated with (3.6) may change their genuine weight exponents γ only over the projection of the spectrum of (3.7) onto the real axis $\mathbb{R}$.

Theorem 3.5 *Suppose that each of the lines $\Im \varrho = -\gamma_2 + (n-q)/2$ and $\Im \varrho = -\gamma_1 + (n-q)/2$ contains eigenvalues of (3.7) while the strip between these lines is free from the spectrum of (3.7). If (3.6) is an isomorphism for some $\gamma \in (\gamma_1, \gamma_2)$, then this operator is an isomorphism for all $\gamma \in (\gamma_1, \gamma_2)$, but not for $\gamma \notin (\gamma_1, \gamma_2)$.*

When combined with Theorem 3.2, the latter result shows that there exists at most one interval (γ_1, γ_2), such that the mapping (3.6) is bijective for all $\gamma \in (\gamma_1, \gamma_2)$. However, it may happen that no interval (γ_1, γ_2) with this property exists. This is in particular the case for the Neumann problem. The dependence of (γ_1, γ_2) on y will not cause additional troubles if we allow γ to depend on y over the edge Y.

We now turn to the original boundary value problem (3.4). We require that L be elliptic and (3.4) satisfy Lopatinskii's condition on $\partial\mathcal{D} \setminus Y$. To each point $y \in Y$ we assign the edge symbol mapping (3.6) depending on y. Suppose that for any point $y \in Y$ there is a number $\gamma(y)$, such that (3.6) is an isomorphism for all $\eta \in T^*_y(Y) \setminus \{0\}$. Then, by Theorem 3.5, this property of the mapping (3.6) remains valid for all $\gamma(y) \in (\gamma_l(y), \gamma_u(y))$. Without loss of generality we can assume that this interval can not be further expanded. The functions $y \mapsto \gamma_{l,u}(y)$ are continuous.

Theorem 3.6 *Under the above assumptions, if moreover γ is a C^∞ function on $\overline{\mathcal{D}}$ with $\gamma(y) \in (\gamma_l(y), \gamma_u(y))$ for all $y \in Y$, then the operator (3.4) is Fredholm.*

This theorem is proved by constructing a regulariser patched by means of a partition of unity on $\overline{\mathcal{D}}$ from 'local' regularisers for model problems in the wedge $\mathbb{R}^q \times \mathcal{C}(X_y)$ and on all of $\mathbb{R}^n$. If the conditions of Theorem 3.6 are not satisfied then (3.4) fails to be a Fredholm operator. Thus, to get the Fredholm property of problem (3.4) in the scale $H^{s,\gamma}(\mathcal{D}, \mathbb{C}^k)$ one has to ensure that the edge symbol (3.6) be an isomorphism outside of the zero section on the cotangent bundle T^*Y. In general, it is a difficult task. As was mentioned, for some problems it is impossible to find a number γ such that the operator (3.6) is an isomorphism. This suggests us to look for supplementary conditions for solutions along the edge, a new scale of spaces, and so on, cf. for instance [Sch98], [Sch01]. At present the situation is well understood in the case of Dirichlet and Neumann problems for formally self-adjoint elliptic systems L, cf. [NP91].

4 Asymptotics of solutions

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with an edge Y on the boundary, as in Section 3.

Consider an elliptic boundary value problem

$$\left\{ \begin{array}{rcll} Lu & = & f & \text{in} \ \ \mathcal{D}, \\ B_j u & = & u_j & \text{on} \ \ \partial\mathcal{D} \setminus Y \end{array} \right.$$

for $j = 0, 1, \ldots, m-1$. Let y^0 be an arbitrary point of the edge Y. Pick a neighbourhood U of y^0 in $\mathbb{R}^n$ and a diffeomorphism h of U onto an open set $O \subset \mathbb{R}^n$, such that $h(U \cap \mathcal{D}) = O \cap (\mathbb{R}^q \times \mathcal{C}(X))$. Here $\mathbb{R}^q$ substitutes the tangential plane of Y at y^0, and $\mathcal{C}(X)$ is a cone in the normal plane of Y at y^0 whose link is a compact $(n-q-1)$-dimensional manifold X with smooth boundary. We identify the normal plane with $\mathbb{R}^{n-q}$ and choose any coordinates z on it.

Fix a cut-off function $\omega \in C^\infty_{\mathrm{comp}}(U)$ which is equal to 1 in a neighbourhood of y^0. Write $u = \omega u + (1-\omega)u$, then the product ωu is supported in $U \cap \mathcal{D}$ and satisfies

$$\left\{ \begin{array}{rcll} L\,(\omega u) & = & \omega f + [L, \omega] u & \text{in} \ \ U \cap \mathcal{D}, \\ B_j\,(\omega u) & = & \omega u_j + [B_j, \omega] u & \text{on} \ \ U \cap \partial\mathcal{D} \end{array} \right. \tag{4.1}$$

for $j = 0, 1, \ldots, m-1$, while $(1-\omega)u$ vanishes near y^0. Using the coordinates (y, z) thus reduces the problem of asymptotic expansion of u near y^0 to that in the model wedge $W := \mathbb{R}^q \times \mathcal{C}(X)$.

As in Section 3, to each point of the edge $y \in \mathbb{R}^q$ we assign the principal edge symbol

$$\begin{array}{c} L_0(y;\eta,\mathbf{D}_r,D_x) \\ \oplus B_{j,0}(y;\eta,\mathbf{D}_r,D_x) \end{array}$$

whose domain is a weighted Sobolev space $H^{s,\gamma}(\mathcal{C}(X),\mathbb{C}^k)$, cf. (3.6). Being in an algebra of boundary value problems on the infinite cone $\mathcal{C}(X)$, it in turn bears a conormal symbol

$$\begin{array}{c} \sigma_M(L_0)(y;\varrho) \\ \oplus \sigma_M(B_{j,0})(y;\varrho) \end{array}$$

taking its values in boundary value problems on the link X, cf. (3.7). Denote by $(\gamma_l(y),\gamma_u(y))$ the largest interval where the principal edge symbol is an isomorphism for each $\eta \in \mathbb{S}^{q-1}$. The existence of such an interval should be postulated, and $\gamma_{l,u}$ are continuous functions on $\mathbb{R}^q$ which are constant for $|y|$ large enough.

We consider a solution $u \in H^{s,\gamma}(W,\mathbb{C}^k)$ of problem (4.1) with right-hand sides $f \in H^{s-m,\delta-m}(W,\mathbb{C}^k)$ and $u_j \in H^{s-m_j-1/2,\delta-m_j-1/2}(\partial W,\mathbb{C}^{k_j})$. Here γ and δ are C^∞ functions on $\mathbb{R}^q$ which are constant for $|y| > R$ and satisfy $\gamma(y) < \delta(y) < \gamma_u(y)$ for all $y \in \mathbb{R}^q$.

Denote by $\lambda_1(y),\ldots,\lambda_N(y)$ the eigenvalues of the conormal symbol which are situated between the lines $\Im\varrho = -\delta + (n-q)/2$ and $\Im\varrho = -\gamma + (n-q)/2$. Let moreover

$$\left(\varphi^{(i)}_{\nu,j}\right)_{\substack{i=1,\ldots,I_\nu \\ j=0,1,\ldots,r_{\nu,i}-1}}$$

be a canonical system of Jordan chains corresponding to the eigenvalue $\lambda_\nu(y)$. We suppose that the following conditions are fulfilled: 1) for all $y \in \mathbb{R}^q$, the above lines are free from the spectrum of the conormal symbol; 2) the numbers N, I_ν and $r_{\nu,i}$ are independent of $y \in \mathbb{R}^q$; and 3) the Jordan chains may be chosen in such a way that the functions $y \mapsto \varphi^{(i)}_{\nu,j}(y,x)$ are smooth on $\mathbb{R}^q$ for all $x \in \overline{\mathcal{D}}$.

Theorem 4.1 *Under the hypotheses 1)-3), let $\delta-\gamma \in (0,1)$ for all $y \in \mathbb{R}^q$ and $D^\beta_y f$, $D^\beta_y u_i$ belong to $H^{s-m,\delta-m}(W,\mathbb{C}^k)$ and $H^{s-m_j-1/2,\delta-m_j-1/2}(\partial W,\mathbb{C}^{k_j})$, respectively, for all β. Then every solution $u \in H^{s,\gamma}(W,\mathbb{C}^k)$ of (4.1) has a representation*

$$u(y,z) = \sum_{\nu=1}^{N}\sum_{i=1}^{I_\nu}\sum_{j=0}^{r_{\nu,i}-1} c^{(i)}_{\nu,j}(y)\, u^{(i)}_{\nu,j}(y,z) + R(y,z), \tag{4.2}$$

the remainder R being in $H^{s,\delta}(W,\mathbb{C}^k)$, the coefficients $c^{(i)}_{\nu,j}(y)$ being smooth functions on $\mathbb{R}^q$, and

$$u^{(i)}_{\nu,j}(y,z) = r^{\imath\lambda_\nu} \sum_{k=1}^{r_{\nu,i}-j} \frac{1}{(k-1)!}\,(\imath\,\ln r)^{k-1}\varphi^{(i)}_{\nu,r_{\nu,i}-j-k}(x).$$

If we drop the restriction $\delta - \gamma < 1$, the asymptotic formula (4.2) becomes more complicated because of lower order terms. In general, the form of these terms does not coincide with that of ingredients of asymptotic expansions in a cone. The lower order terms may contain differentiations in y. If we impose no additional assumptions on the smoothness in y of the right-hand side of (4.1) then the coefficients fail to be smooth. Explicit formulas for the coefficients $c^{(i)}_{\nu,j}(y)$ by means of the Green formula were first obtained in [MP76], cf. also [NP91].

5 Cuspidal singularities

Let $\mathcal{D} \subset \mathbb{R}^n$ be a bounded domain whose boundary is smooth except a smooth closed manifold $Y \subset \partial\mathcal{D}$ of dimension q, where $0 \le q \le n-2$. In $\mathcal{D}$ we consider a boundary value problem

$$\left\{ \begin{array}{rcll} Lu & = & f & \text{in } \mathcal{D}, \\ B_j u & = & u_j & \text{on } \partial\mathcal{D} \end{array} \right. \tag{5.1}$$

for $j = 0, 1, \ldots, m-1$, where L is a $(k \times k)$-matrix of scalar differential operators of order $2m$ in $\mathcal{D}$, and B_j is a $(k_j \times k)$-matrix of scalar differential operators of order m_j close to boundary in $\mathcal{D}$.

Pick a sufficiently small tubular neighbourhood U of Y in $\mathbb{R}^n$. For points $p \in U$ we introduce new coordinates (y, z), where $y = y(p)$ is the intersection point of Y and the $(n-q)$-dimensional plane F_y through p which is orthogonal to Y, and $z = z(p)$ are coordinates of p in F_y. Suppose there is a coordinate system $z(p)$ in F_y with origin at y, which depends on p continuously and such that $\mathcal{D} \cap U$ is given by the inequality $z^d_{n-q} > \varphi(y, z') + r(y, z')$, where $d > 0$, $z' = (z_1, \ldots, z_{n-q-1})$,

$$\begin{array}{rcl} \varphi(y, z') & = & \displaystyle\sum_{|\alpha|=k} \varphi_\alpha(y)\, z'^\alpha, \\ r(y, z') & = & O\left(|z'|^{k+1}\right), \end{array}$$

the functions $\varphi_\alpha(y)$ and $r(y, z')$ being C^∞, and $\varphi(y, z') > 0$ for $z' \neq 0$. We thus represent $\mathcal{D} \cap U$ as a fibre bundle over Y, the fibre being an $(n-q)$-dimensional quasi-homogeneous cone.

Set $o = k/d$. For $o = 1$ the singularity along Y is conical, for $o < 1$ the singularity is cuspidal, and the case $o > 1$ actually corresponds to finite smoothness.

Assume that the coefficients of L are infinitely differentiable in a neighbourhood of $\overline{\mathcal{D}}$, and the coefficients $B_{j,\vartheta}$ of B_j possess near Y asymptotic expansions

$$B_{j,\vartheta} \sim \sum_{k,l=0}^{\infty} B^{(k,l)}_{j,\vartheta} \left(y, z_{n-q}^{-\frac{1}{o}} z' \right) z_{n-q}^{\frac{|\vartheta| - m_j + k}{o} + l}$$

where $B_{j,\vartheta}^{(k,l)}$ are C^∞ and $B_{j,\vartheta}^{(0,0)} = 0$ for $|\vartheta| < m_j$. When changing the coordinates by

$$\begin{aligned} r &= z_{n-q}, \\ x &= z_{n-q}^{-\frac{1}{o}} z' \end{aligned}$$

we get

$$\begin{aligned} D_{z_j} &= r^{-\frac{1}{o}} D_{x_j}, \qquad \text{for } j = 1, \ldots, n-q-1, \\ D_{z_{n-q}} &= D_r - \frac{1}{o}\frac{1}{r} \sum_{j=1}^{n-q-1} x_j D_{x_j}. \end{aligned}$$

The inequality $z_{n-q}^d > \varphi(y, z') + r(y, z')$ transforms to $1 > \varphi(y, x) + O(r^{\frac{k+1}{k}})$, which describes the link of Y, namely $V_y = \{x \in \mathbb{R}^{n-q-1} : \varphi(y, x) < 1\}$. The push-forward of L is

$$\pi_* L = r^{-\frac{2m}{o}} \sum_{|\beta|+k+|\alpha|=2m} L_{\beta,k,\alpha}(y, r^{\frac{1}{o}} x, r) \left(r^{\frac{1}{o}} D_y\right)^\beta \left(r^{\frac{1}{o}} D_r\right)^k D_x^\alpha$$

up to terms of the form $r^{-\frac{2m}{o}} O(1)$ for $r \to 0$. Such terms are negligible in the calculus of boundary value problems of [RST04], hence we are able to localise (5.1) to

$$\left\{ \begin{array}{rcll} L_0(y; \eta, \varrho, D_x) u &=& f & \text{in } V_y, \\ B_{j,0}(y, x, \eta, \varrho, D_x) u &=& u_j & \text{on } \partial V_y \end{array} \right. \tag{5.2}$$

for $j = 0, 1, \ldots, m-1$, where L_0 stands for the principal homogeneous part of L in the coordinates (y, z) with coefficients evaluated at $z_{n-q} = 0$, and $B_{j,0}$ is the principal homogeneous part of B_j in the coordinates (y, z), with $B_{j,\vartheta}$ replaced by $B_{j,\vartheta}^{(0,0)}$.

The mapping (5.2), which assigns a boundary value problem on the link V_y to any $(y, \eta, \varrho) \in Y \times \mathbb{R}^{q+1}$, is called the symbol of (5.1) along the edge Y. It controls the Fredholm property of (5.1) in the weighted Sobolev spaces $H^{s,\mu}(\mathcal{D}, \mathbb{C}^k)$ that are defined as completions of $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y, \mathbb{C}^k)$ with respect to the norm

$$\|u\|^2 = \|u\|^2_{H^s(\mathcal{D},\mathbb{C}^k)} + \int_{\mathcal{D}\cap U} z_{n-q}^{-2\mu} \sum_{|\vartheta| \le s} z_{n-q}^{2\frac{|\vartheta|}{o}} |D^\vartheta u(y, z)|^2 \, dy dz. \tag{5.3}$$

As usual, the space of restrictions to the boundary of various functions in $H^{s,\mu}(\mathcal{D}, \mathbb{C}^k)$ is denoted by $H^{s-1/2,\mu-1/2o}(\partial\mathcal{D}, \mathbb{C}^k)$. The norm of u in this latter space is defined to be the infimum of the norms in $H^{s,\mu}(\mathcal{D}, \mathbb{C}^k)$ over all continuations of u to $\mathcal{D}$.

The boundary value problem (5.1) is said to be elliptic if: 1) the operator L is elliptic; 2) the Lopatinskii condition is fulfilled at all points of $\partial\mathcal{D}\setminus Y$; and 3) for each $y \in Y$, the problem (5.2) is elliptic with parameter $(\eta,\varrho) \in \mathbb{R}^{q+1}$ and the resolvent $R(\eta,\varrho)$ has no poles on all of $\mathbb{R}^{q+1}$. The following theorem was first completely proved in [RST04] while it has been announced in the paper [Fei72].

Theorem 5.1 *Let $o < 1$. Suppose $s \geq \max\{2m, m_j + 1\}$ and $\mu \in \mathbb{R}$. In order that the operator*

$$\begin{array}{c} L \\ \oplus B_j \end{array} : H^{s,\mu}(\mathcal{D},\mathbb{C}^k) \to \begin{array}{c} H^{s-2m,\mu-2m/o}(\mathcal{D},\mathbb{C}^k) \\ \oplus H^{s-m_j-1/2,\mu-m_j/o-1/2o}(\partial\mathcal{D},\mathbb{C}^{k_j}) \end{array}$$

would be Fredholm, it is necessary and sufficient that the problem (5.1) be elliptic.

For smooth $\{f, u_j\}$, the solution u of the problem (5.1) need not be smooth in general. One can show an asymptotic of the solution in a neighbourhood of Y.

Theorem 5.2 *Let $o < 1$ and (5.1) be elliptic. Assume that $f \in C^\infty(\overline{\mathcal{D}},\mathbb{C}^k)$ and $u_j \in C^\infty(\partial\mathcal{D},\mathbb{C}^{k_j})$. Then there exist C^∞ functions $u_{k,l}$, such that for any N there is b_N with*

$$u = \sum_{k+ol\leq b_N} u_{k,l}\Big(y, z_{n-q}^{-\frac{1}{o}} z'\Big) z_{n-q}^{\frac{k}{o}+l} + R_N,$$

where $R_N \in H^{\infty,\mu}(\mathcal{D},\mathbb{C}^k)$ provided $o\mu < N$.

As but one application of Theorem 5.2 we mention that if $q = n-2$ and $1/o$ is integer then, under the hypotheses of Theorem 5.2, the solution u of (5.1) belongs to $C^\infty(\overline{\mathcal{D}},\mathbb{C}^k)$.

As is shown in [RST04], the Dirichlet problem in $\mathcal{D}$ is elliptic in the above sense.

6 Boundary integral equations

In the early 1950s Lopatinskii [Lop53] showed a way of reducing a general boundary value problem for an elliptic system in a bounded domain to a regular integral equation on the boundary. This approach to boundary value problems for elliptic equations in a smooth domain D was developed by Calderón [Cal63] who reduced them to pseudodifferential equations on ∂D via what is now called the Calderón projector. When the surface ∂D is merely C^1 or Lipschitzian, the corresponding boundary operator fails to be pseudodifferential and, typically, can only be described in terms of singular integrals. Calderón's intuition that such problems are ultimately tractable via harmonic analysis techniques stimulated a long-term program with profound implications in the field.

As is mentioned in [Maz88], the signification 'boundary integral equation' appeared relatively recently. Earlier one spoke on integral equations of potential theory.

Both applications and the inner logic of the theory itself require investigation of non-regular curves and surfaces. First steps in this direction were made by Korn [Kor02], Zaremba [Zar04] and Carleman [Car16] who studied boundary value problems for the Laplace equation in a plane domain with corners on the boundary by potential theory methods. Radon [Rad19] generalised the results of [Kor02] and [Zar04] to plane contours of bounded rotation variation without cuspidal points. In [Rad19] the integral operators are treated in the space of continuous functions. A new feature of analysis of boundary integral equations on non-regular curves and surfaces is that the relevant operators lose compactness while being bounded. To coup with this difficulty Radon made use of the operator theory in function spaces based on the concept of Fredholm radius due to him.

Let T be a bounded operator in a Banach space B. After Radon, the Fredholm radius $R(T)$ of T is the radius of the largest disk with centre $\lambda = 0$ in the complex plane, in whose interior the equation $(1 - \lambda T)u = f$ obeys Fredholm theory. It coincides with the supremum of the radii of uniform convergence of series $I + \lambda(T-K) + \lambda^2(T-K)^2 + \ldots$ over all compact operators K on B. By the essential norm of T in B is meant $|T| := \inf \|T - K\|_{\mathcal{L}(B)}$, the infimum being over all compact operators K on B. This concept stems from [Rad19] where it is used to estimate $R(T)$ from below. Namely, since the Neumann series converges for $|\lambda| < \|T - K\|^{-1}_{\mathcal{L}(B)}$ it follows that $R(T) \geq |T|^{-1}$. The results of Radon actually originated an important branch of functional analysis, cf. [GK57].

If $T = 2\mathcal{P}_{\mathrm{dl},0}$ is the (doubled) principal value of the double layer potential over a closed curve of bounded rotation variation then the Fredholm radius of T in the space of continuous functions just amounts to $|T|^{-1} = \pi/\alpha$, where α is the largest jump of the tangential angle at the points of the curve, cf. [Rad19]. In case the curve does not contain cuspidal points one obtains $\alpha < \pi$, and so $R(T) > 1$. Hence the Dirichlet and Neumann problems are Fredholm in such a plane domain.

Long time after the paper of Radon the opinion existed that he developed the potential theory in the space of continuous functions up to its natural extents. First in the 1960s Burago and Maz'ya [BM67], Král [Krá80] et al. carried over the theory of Radon to more general curves described in terms of variation of the angle at which the boundary subsets are observable from any boundary point. Yet another approach to boundary integral equations $(1-T)u = f$ on piecewise smooth curves without cuspidal points originates from Lopatinskii [Lop63] who suggested to treat them in weighted Lebesque spaces, the weight being a power of the distance to corner points. The operator T is shown to coincide up to a compact term with an integral operator on the angle whose kernel is homogeneous of degree -1. To explicitly invert this latter operator called the model operator, Lopatinskii made use of the Mellin transform.

In [BM67] a proper substitute of curves of bounded rotation variation was found for surfaces of higher dimension in $\mathbb{R}^n$. It corresponds to the intuitive idea of spatial angle $\omega(x,\sigma)$, at which a boundary subset σ is observable from a boundary point x. The results of Radon extend to those domains D for which the supremum of $\operatorname{var}\omega(x,\cdot)(\partial D \cap B(x,r))$ over all $x \in \partial D$ is less than $\sigma_n/2$, provided that r is small enough. This condition proves to be equivalent to $|T| < 1$ where $T = 2\mathcal{P}_{\mathrm{dl},0}$ is the (doubled) principal value of the double layer potential over ∂D. The results of [BM67] thus apply to integral equations of harmonic potential theory only in the case $|T| < 1$, the essential norm being evaluated in the space of continuous functions. However, this condition fails to hold even for fairly simple polyhedrons in $\mathbb{R}^n$. Sometimes the inequality $|T| < 1$ can be attained by using a weight norm in $C(\partial D)$ which depends on the geometry of ∂D, cf. [KW86].

An essential progress in the study of boundary value problems for second order equations in domains with Lipschitz boundary was achieved in the 1980s. It was initiated by the paper of Calderón [Cal77] who proved the L^2 boundedness of the Cauchy integral on C^1 and Lipschitz curves, cf. also [CMM82]. The result of Calderón was used by [FJR78] to construct an L^p potential theory for C^1 surfaces in $\mathbb{R}^n$. Yet another striking advance was the treatment of the Laplacian under Dirichlet and Neumann boundary conditions on C^1 and Lipschitz domains, cf. [FJR78], [DK87], [Ver84], et al. Since the end of the 1980s very few new elliptic boundary problems have been attacked from this point of view. One notable related development was the treatment of the Neumann problem for the Hodge Laplacian $\Delta = \delta d + d\delta$ on arbitrary Lipschitz domains in [MM96], [MT99], [MMT00].

Although Lipschitz surfaces make a considerable class of general surfaces, they are still not sufficient for many applications. Such simple surfaces as polyhedrons or cones with smooth generatrix are not exhausted by Lipschitz surfaces. Moreover, the Lipschitz surfaces do not well suit to analysis of smoothness and local singularities of solutions.

The difficulties in the study of such questions for arbitrary piecewise smooth surfaces are evoked to considerable extent by imperfection of the existing theory of integral and pseudodifferential operators on non-smooth manifolds, cf. [Pla86, Pla97], [Sch91, Sch98] et al.

It was the paper [FJL77] that first elucidated anomalies of the potential theory in domains with corners and edges. It studies the solvability in L^p of boundary integral equations of the Dirichlet problem for the Laplace equation in several special domains, namely in a quarter-plane, a circular cone, a quarter-space and a bounded plane domain with angle $\pi/2$ on the contour. In the case of quarter-plane the integral equation is solvable in L^p for $p \neq 3/2$, while the value $p = 3/2$ is not exceptional for the explicit solution of the Dirichlet problem by Poisson integral. The integral equation of the inner Dirichlet problem in the cone $\{(x,y,z): \; x^2+y^2 \leq z^2, \, z > 0\}$ is uniquely solvable in L^p for all $p > 1$ but a sequence $\{p_k\}_{k=1}^{\infty}$ lying in the interval $(1,2)$. In the case of quarter-space the integral equation is uniquely solvable in L^p for all $p > 3/2$. If $1 < p < 3/2$ the space of solutions of the homogeneous equation is infinite dimensional. Moreover, the norm of the operator is shown to be less than 1 provided $p > 3/2$.

An important aspect of the theory of boundary integral equations is techniques of numerical solution thereof. It is often referred to as the boundary element method. This direction makes an independent area of investigations, cf. [KK62], [Wen82], [BB81]. Ryaben'kii [Rya69] developed a potential method for systems of difference equations with constant coefficients which leads to discrete analogues of boundary integral equations.

7 Hodge theory

In this paper we deal with the Neumann problem for the Hodge Laplacian on domains with non-regular boundary. Another perspective from which the results can be understood has to do with the direction initiated by Hodge's work on harmonic integrals in the 1930s. In [Hod41] he generalised to arbitrary compact Riemannian manifolds the potential theory used by Riemann in his study of Riemannian surfaces. The main question that we address is that of the efficiency of the layer potential method in the global analysis, arbitrary topology and in the presence of non-smooth structures.

The topology of the underlying domain plays a major role in the problem under discussion. For instance, there are natural topological obstructions to the unique solvability of the Neumann problem as well as to the invertibility of the boundary operators under consideration. Among the main ingredients used here we mention a regularity result allowing for the transition between variational methods in D and boundary integral equation techniques, and de Rham theory dealing with the topological information encoded in the problem. We contribute to the case of non-Lipschitz singularities.

In the context of arbitrary domains in manifolds and for arbitrary degrees, the only reasonably well understood case is that when all structures involved are smooth. In this context, the Neumann problem have been treated by

Conner [Con53], Duff [Duf54], Garabedian and Spencer [GS53].

A notable exception is the work by Morrey and Eells [ME56, Mor56] who use variational methods and a priori regularity estimates. However, in their formulation boundary traces are taken in a weak distributional sense which actually affects the character of the problem.

The methods for proving the Hodge theory on smooth domains of Conner [Con53] and Morrey and Eells [ME56, Mor56] fails if the boundary bears singularities. On smooth domains the Neumann problem for the Hodge Laplacian is coercive and the Gårding inequality holds. On domains with singularities the Gårding inequality is still true near the smooth part of the boundary. However, the boundary conditions are meaningless at the singularities, hence the analysis of differentiability is much more difficult.

The L^2 Hodge theory on domains with boundary containing conical points was constructed by Shaw [Sha83], who used the results of Kondrat'ev [Kon67] on elliptic equations with general boundary conditions to prove the existence and compactness of the Neumann operator that solves the Neumann problem. Shaw measured the regularity of solutions at the singularities by carefully analysing the first eigenvalue of the Laplace-Beltrami operator on subdomains of $\mathbb{S}^{n-1}$ under certain boundary conditions.

In 1970, Singer [Sin71] presents a comprehensive program aimed at extending the theory of elliptic operators and their index to more general situations, namely to "non-smooth manifolds, non-manifolds of special type and to a context where it is natural that integer (index) be replaced by real number". As but one part of this program Teleman [Tel80] produced a Hodge theory and signature operators on PL manifolds.

A couple of years later, Sullivan [Sul79] formulated the problem of constructing an index theory on Lipschitz manifolds. The interest in studying Lipschitz manifolds derives from the following two desirable but conflicting features of the Lipschitz homeomorphisms in $\mathbb{R}^n$, cf. [Sul79]. They preserve a rich analytic structure, whereas, from the topological point of view, they are very manageable. By [Sul79], any topological manifold of dimension $\neq 4$ admits a Lipschitz structure which is unique up to a Lipschitz homeomorphism close to the identity. On any Lipschitz manifold L^2-forms, exterior derivatives and currents may be defined; all these objects are basic for the Hodge theory. The Lipschitz Hodge theory presented in [Tel83] is a slight modification of the combinatorial Hodge theory [Tel80].

Cheeger [Che80] studied Hodge theory for manifolds with interior conical points using a geometric approach. Similarly to [Tel83], this set up gives however no information about the regularity of the harmonic forms near singularities.

8 The Neumann problem

In this section we formulate the Neumann problem on domains with edges on the boundary and solve it. Our set up gives us information about the regularity of the harmonic forms near singularities. Although our method can be easily generalized to manifolds with edges, we only consider domains in $\mathbb{R}^n$ in this paper.

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\mathcal{D}$, such that $\partial\mathcal{D}$ is smooth everywhere except of a closed manifold Y of dimension $0 \le q \le n-1$. We assume that $\partial\mathcal{D}$ has along Y edge type singularities, cf. Section 3. This means that the domain $\mathcal{D}$ is a cone bundle close to Y, the base being Y and a fibre being the cone $\mathcal{C}(X)$ over a bounded domain with smooth boundary in $\mathbb{R}^{n-q-1}$.

We shall formulate the Neumann problem on $\mathcal{D}$ to study the L^2 Hodge theory, cf. [ST03]. The formulation is exactly the same for smooth domains, cf. [Con53]. We shall only give a short description here.

Let $C^\infty(\mathcal{D}, \bigwedge^i)$ and $C^\infty_{\mathrm{comp}}(\mathcal{D}, \bigwedge^i)$ denote the spaces of C^∞ differential forms of degree i and C^∞ differential forms of degree i of compact support, respectively, and $C^\infty(\overline{\mathcal{D}}, \bigwedge^i)$ denotes the C^∞ forms which can be extended smoothly to all of $\mathbb{R}^n$. Furthermore, we write $L^2(\mathcal{D}, \bigwedge^i)$ for all i-forms with square integrable coefficients. The Laplace operator on $\bigwedge^i(\mathcal{D})$ is defined to be $\Delta = \delta d + d\delta$ where d is the exterior derivative and δ is the formal adjoint of d. It is easy to check that Δ is formally self-adjoint.

In order to form an L^2 self-adjoint extension of Δ, we define the L^2 strong closure of d, denoted T. Given $u \in L^2(\mathcal{D}, \bigwedge^i)$, we set $u \in \mathcal{D}^i_T$ if there is a sequence $u_\nu \in L^2(\mathcal{D}, \bigwedge^i) \cap C^\infty(\mathcal{D}, \bigwedge^i)$ such that $u_\nu \to u$ in $L^2(\mathcal{D}, \bigwedge^i)$ and $\{du_\nu\}$ is a Cauchy sequence in $L^2(\mathcal{D}, \bigwedge^{i+1})$. We define $Tu = f$ where $f \in L^2(\mathcal{D}, \bigwedge^{i+1})$ is the limit of $\{du_\nu\}$.

The L^2 adjoint T^* of T is defined in a familiar way. Namely, $g \in \mathcal{D}^{i+1}_{T^*}$ and $T^* g = v$ means that $g \in L^2(\mathcal{D}, \bigwedge^{i+1})$ and $v \in L^2(\mathcal{D}, \bigwedge^i)$ satisfy $(Tu, g) = (u, v)$ for all $u \in \mathcal{D}^i_T$. It is easy to check that both T and T^* are closed densely defined operators satisfying $T^2 = T^{*2} = 0$.

We are now in position to introduce an L^2 self-adjoint extension of Δ. Namely, set

$$\mathcal{D}^i_L := \{u \in \mathcal{D}^i_T \cap \mathcal{D}^i_{T^*} : Tu \in \mathcal{D}^{i+1}_{T^*} \text{ and } T^* u \in \mathcal{D}^{i-1}_T\}$$

and $Lu = T^* T u + T T^* u$ for $u \in \mathcal{D}^i_L$.

The L^2 self-adjointness of L was proved by [Gaf55]. We simply summarise the results here.

Lemma 8.1 *The operator L is self-adjoint. Furthermore, $(L+1)^{-1}$ exists, is bounded and everywhere defined.*

The space of harmonic i-forms $\mathcal{H}^i(\mathcal{D})$ is defined to consist of all forms $u \in \mathcal{D}^i_T \cap \mathcal{D}^i_{T^*}$ with the property that $Tu = T^*u = 0$. It is easy to check that $\mathcal{H}^i(\mathcal{D}) = \ker L$. From this and Lemma 8.1 we get immediately the so-called Weak Hodge Decomposition Theorem.

Lemma 8.2 *The range of L is orthogonal to $\mathcal{H}^i(\mathcal{D})$ and we have an orthogonal decomposition*

$$L^2(\mathcal{D}, \textstyle\bigwedge^i) = \mathcal{H}^i(\mathcal{D}) \oplus \overline{T^*T\,\mathcal{D}^i_L} \oplus \overline{TT^*\,\mathcal{D}^i_L},$$

overline denoting the closure in $L^2(\mathcal{D}, \bigwedge^i)$.

The Neumann problem for the de Rham complex actually consists of proving the existence of an operator N on $L^2(\mathcal{D}, \bigwedge^i)$ which possesses the following properties:

- 1^o N is bounded and self-adjoint.
- 2^o $N\,\mathcal{H}^i(\mathcal{D}) = 0$, $\mathrm{Ran}\,N \subset \mathcal{D}^i_L$ and $\mathrm{Ran}\,N$ is orthogonal to $\mathcal{H}^i(\mathcal{D})$.
- 3^o $LN = I - H$ on $L^2(\mathcal{D}, \bigwedge^i)$ and $NL = I - H$ on $\mathcal{D}^i_L$, where H is the orthogonal projection onto $\mathcal{H}^i(\mathcal{D})$.
- 4^o $NT = TN$ on $\mathcal{D}^i_T$.
- 5^o $NC^\infty(\mathcal{D}, \bigwedge^i) \subset C^\infty(\mathcal{D}, \bigwedge^i)$.

If $\mathrm{Ran}\,L$ is closed, by Lemma 8.2, we can decompose every $u \in L^2(\mathcal{D}, \bigwedge^i)$ into $u = Hu + Lv$, where $v \in \mathcal{D}^i_L$. Define $N : L^2(\mathcal{D}, \bigwedge^i) \to L^2(\mathcal{D}, \bigwedge^i)$ by $Nu = v - Hv$. It is easily verified that the operator N satisfies properties 1^o to 5^o. Hence, the Neumann problem can be solved whenever the range of L is closed.

Lemma 8.3 *If $(L + 1)^{-1}$ is compact then* $\mathrm{Ran}\,L$ *is closed. Furthermore, $\mathcal{H}^i(\mathcal{D})$ is finite dimensional and the Neumann operator N is compact.*

Proof. Since $(L+1)^{-1}$ is self-adjoint and compact, the lemma follows from the Riesz theory for compact self-adjoint operators on Hilbert spaces.

□

Our next goal is to show that $(L + 1)^{-1}$ is compact, thus solving the Neumann problem.

Let $C^\infty(\bar{\mathcal{D}} \setminus Y, \bigwedge^i)$ denote the i-forms which are smooth up to the boundary except at Y. There exists a real-valued function $\varrho \in C^\infty(\mathbb{R}^n \setminus Y)$ such that $\partial\mathcal{D} \setminus Y = \{x \in \mathbb{R}^n \setminus Y : \varrho(x) = 0\}$. We normalise ϱ so that $|d\varrho| = 1$ on $\partial\mathcal{D} \setminus Y$. The sign of ϱ is chosen in such a way that $\varrho < 0$ in $\mathcal{D}$ and $\varrho > 0$ outside of $\bar{\mathcal{D}}$. As usual, we write $\sigma(\delta)(x, d\varrho)$ for the symbol of δ in the direction $d\varrho$ at a point $x \in \partial\mathcal{D} \setminus Y$. We shall give an explicit description of the forms in $\mathcal{D}^i_L \cap C^\infty(\bar{\mathcal{D}} \setminus Y, \bigwedge^i)$.

Lemma 8.4 *If $g \in \mathcal{D}^i_{T^*} \cap C^\infty(\bar{\mathcal{D}} \setminus Y, \bigwedge^i)$, then $\sigma(\delta)(x, d\varrho)g = 0$ holds on $\partial\mathcal{D} \setminus Y$.*

Proof. For every $u \in C^\infty(\bar{\mathcal{D}}, \bigwedge^{i-1})$ having support away from the edge Y, we have

$$\begin{aligned}(g, du) &= (\delta g, u) - \int_{\partial\mathcal{D}} (\sigma(\delta)(x, d\varrho)g, u)\, ds \\ &= (T^* g, u),\end{aligned}$$

by Green's formula, where ds is the surface element on $\partial\mathcal{D}$ which is well defined away from the edge. Since $C^\infty_{\mathrm{comp}}(\mathcal{D}, \bigwedge^{i-1})$ is dense in $L^2(\mathcal{D}, \bigwedge^{i-1})$, we have $\delta g = T^* g$ and

$$\int_{\partial\mathcal{D}} (\sigma(\delta)(x, d\varrho)g, u)\, ds = 0,$$

which implies $\sigma(\delta)(x, d\varrho)g = 0$ on $\partial\mathcal{D} \setminus Y$.

□

Lemma 8.5 *If $u \in \mathcal{D}^i_L \cap C^\infty(\bar{\mathcal{D}} \setminus Y, \bigwedge^i)$, then*

$$\begin{aligned}\sigma(\delta)(x, d\varrho)u &= 0, \\ \sigma(\delta)(x, d\varrho)du &= 0\end{aligned}$$

on $\partial\mathcal{D} \setminus Y$.

Proof. Since $u \in \mathcal{D}^i_{T^*}$ and $du \in \mathcal{D}^{i+1}_{T^*}$, we can apply Lemma 8.4.

□

From Lemmas 8.4 and 8.5 we see that every element in $\mathcal{D}^i_L \cap C^\infty(\overline{\mathcal{D}} \setminus Y, \bigwedge^i)$ satisfies the same boundary conditions away from the edge as in the usual Neumann problem on smooth domains. Therefore, if $Lu = v$ and moreover $v \in C^\infty(\overline{\mathcal{D}} \setminus Y, w^i)$, then $u \in C^\infty(\overline{\mathcal{D}} \setminus Y, \bigwedge^i)$, which is due to the local elliptic reqularity.

Writing in components, any sufficiently smooth solution u of the Neumann problem in $\mathcal{D}$ satisfies the system

$$\begin{aligned}\Big(\sum_{j=1}^n D_j^2\Big) u_J &= f_J \quad \text{in} \quad \mathcal{D} \quad \text{for every } J, \\ \sigma(\delta)(x, d\varrho)u &= 0 \quad \text{on} \quad \partial\mathcal{D} \setminus Y, \\ \sigma(\delta)(x, d\varrho)du &= 0 \quad \text{on} \quad \partial\mathcal{D} \setminus Y,\end{aligned} \tag{8.1}$$

where

$$\begin{aligned}u &= \textstyle\sum'_{\#J=i} u_J dx_J, \\ f &= \textstyle\sum'_{\#J=i} f_J dx_J\end{aligned}$$

and $\sum'$ means the sum over all increasing ordered i-tuples $J = (j_1, \ldots, j_i)$, where $1 \le j_1 < \ldots < j_i \le n$.

Our main concern is to study the regularity of system (8.1) near the edge. To this end, we invoke the weighted Sobolev spaces $H^{s,\gamma}(\mathcal{D}, \bigwedge^i)$ introduced in Section (3.2). An i-form is in $H^{s,\gamma}(\mathcal{D}, \bigwedge^i)$ if and only if every component is in $H^{s,\gamma}(\mathcal{D})$.

First we consider the boundary value problem (8.1) on an infinite model wedge $W = \mathbb{R}^q \times \mathcal{C}(X)$. Fix $y \in Y$. Since Y is a smooth q-dimensional manifold, there is a plane F_y of dimension $n-q$ through y which is orthogonal to Y. Near y it meets $\mathcal{D}$ in a smooth domain with the only singilar point y on the boundary. We denote by $\mathcal{C}(X)$ the tangent cone of this domain at the point y. When introducing polar coordinates in F_y with centre y, we specify the link X as a smooth domain on the $(n-q-1)$-dimensional unit sphere with centre y. This just amounts to the intersection of the unit sphere with centre y in F_y with $\mathcal{D}$.

Choose a sufficiently small tubular neighbourhood U if Y in $\mathbb{R}^n$. For points $p \in U$ we introduce new coordinates (y, r, x), where $y = y(p)$ is the intersection point of Y and the $(n-q)$-dimensional plane F_y through p which is orthogonal to Y, and $r = r(p)$ and $x = x(p)$ are polar coordinates of p in F_y. In these coordinates the tangent cone at y is $\mathcal{C}(X) = \{rx \in F_y : r > 0,\ x \in X\}$, rx being the dilation of x.

For the Fourier transform $\hat{u}_J(\eta, r, x)$ of $u_J(y, r, x)$ in $y \in \mathbb{R}^q$ we obtain a family of boundary value problems on the cone $\mathcal{C}(X)$ depending on a parameter $\eta \in \mathbb{R}^q$,

$$\begin{array}{rcll} \dfrac{1}{r^2}\Big(r^2|\eta|^2 + (rD_r)^2 - \imath(n-q-2)(rD_r) + \Delta_X\Big)\hat{u}_J & = & \hat{f}_J & \text{in} \quad \mathbb{R}_+ \times X, \\ B_0(x)\,\hat{u} & = & 0 & \text{on} \quad \mathbb{R}_+ \times \partial X, \\ B_1(x)\,d_x\hat{u} & = & 0 & \text{on} \quad \mathbb{R}_+ \times \partial X, \end{array} \tag{8.2}$$

for every J, where Δ_X is the Laplace-Beltrami operator on the unit sphere in F_y, and both $B_0(x)$ and $B_1(x)$ are matrices of smooth functions on the boundary of X.

When applying the Fourier separation of variables to (8.2), one encounters Bessel's differential equations which are generally typical for the analysis on a cone. It is a happy case if such an equation may be solved explicitly. A standard argument leads to power series solutions and Bessel's functions. Hence the verification of the bijectivity of (8.2) in weigthed Sobolev spaces is an immense problem in general.

To see that the problem (8.2) is Fredholm we evaluate its conormal symbol.

This is

$$\begin{array}{rcll} (\varrho^2 - \imath(n{-}q{-}2)\varrho + \Delta_X)\, M_{r\mapsto\varrho}\hat{u}_J & = & M_{r\mapsto\varrho+2\imath}\hat{f}_J & \text{in} \quad X, \\ B_0(x)\, M_{r\mapsto\varrho}\hat{u} & = & 0 & \text{on} \quad \partial X, \\ B_1(x)\, d_x M_{r\mapsto\varrho}\hat{u} & = & 0 & \text{on} \quad \partial X \end{array} \qquad (8.3)$$

for all J, where $\Im\varrho = -\gamma + (n-q)/2$.

Since the boundary value problem $\{\Delta_X, B_0, B_1 \circ d_x\}$ satisfies the Lopatinskii condition and X is a bounded domain with smooth boundary, equations (8.3) are well understood. This problem has a discrete spectrum σ of finite multiplicity. Furthermore, there exists a resolvent $R(\varrho)$ for the problem (8.3) and $R(\varrho)$ is actually a meromorphic function of ϱ with poles at those points, such that $-\varrho^2 + \imath(n{-}q{-}2)\varrho$ is in the spectrum σ. Using the estimates of [AV64] for elliptic boundary value problems depending on a parameter, we are able to prove the following.

We restrict our discussion to the natural case where Y is of dimension $n-2$. This actually corresponds to domains with piecewise boundary because the intersection of two boundary pieces at a non-zero angle leads to a smooth edge on the boundary, whose codimension in $\mathbb{R}^n$ is 2. The condition $q = n-2$ is essential to the proof.

Lemma 8.6 *Suppose that $\eta \in \mathbb{R}^q \setminus \{0\}$. If the resolvent $R(\varrho)$ has no pole on the line $\Im\varrho = -\gamma + 1$, then for any i-form $\hat{f} \in H^{s-2,\gamma-2}(\mathcal{C}(X), \bigwedge^i)$ there exists a unique i-form $\hat{u}$ in $H^{s,\gamma}(\mathcal{C}(X), \bigwedge^i)$, such that $\hat{u}$ fulfills the boundary value problem (8.2).*

The main result of the edge theory now tells us that under the hypotheses of Lemma 8.6 the Neumann problem is Fredholm.

Lemma 8.7 *$\mathcal{D}_L^i \subset H^{2,\epsilon}(\mathcal{D}, \bigwedge^i)$ for some positive number ϵ and $(L+1)^{-1}$ is compact on $L^2(\mathcal{D}, \bigwedge^i)$.*

Proof. As is easy to check, the domain of L lies in $H^{2,0}(\mathcal{D}, \bigwedge^i)$. Since the poles of $R(\varrho)$ are discrete and pure imaginary (see the next section), there exists a small number $\epsilon > 0$ such that there is no pole between the lines $\Im\varrho = -2\epsilon + 1$ and $\Im\varrho = 1$. By Theorem 3.5, we conclude that $\mathcal{D}_L^i \subset H^{2,\epsilon}(\mathcal{D}, \bigwedge^i)$ and the following estimate holds

$$\begin{array}{rcl} \|u\|^2_{H^{2,\epsilon}(\mathcal{D},\bigwedge^i)} & \leq & C\Big(\|Lu\|^2 + \|u\|^2\Big) \\ & \leq & 2C\left(\|(L+1)u\|^2 + \|u\|^2\right) \end{array}$$

for each $u \in \mathcal{D}_L$, with some constant C independent of u. It is a standard fact that the inclusion map $H^{2,\epsilon}(\mathcal{D}, \bigwedge^i) \hookrightarrow L^2(\mathcal{D}, \bigwedge^i)$ is compact. Hence we deduce that $(L+1)^{-1}$ is compact, showing the lemma. □

Combining Lemma 8.3 and Lemma 8.7, we conclude that the Neumann problem is solvable.

Theorem 8.8 (Strong Hodge Decomposition) *Let $\mathcal{D}$ be a domain in $\mathbb{R}^n$ with smooth edges of codimension 2 on the boundary. Then $\mathcal{H}^i(\mathcal{D})$ is of finite dimension for each $i = 0, 1, \ldots, n$ and*

$$L^2(\mathcal{D}, \textstyle\bigwedge^i) = \mathcal{H}^i(\mathcal{D}) \oplus T^*T\,\mathcal{D}^i_L \oplus TT^*\,\mathcal{D}^i_L.$$

Note that the condition of Theorem 5.1 is much easier verified than the bijectivity of (8.2). Hence Theorem 5.1 effectively applies to derive the solvability of the Neumann problem in domains with cuspidal edges, at least for differential forms of higher degree. Cuspidal singularities fall out the class of Lipschitz singularities.

9 Regularity up to the boundary

From Section 8 we see that the regularity of the Neumann problem, or the operator L, depends on the distribution of the poles of the corresponding resolvent $R(\varrho)$. The resolvent is related to any point y of the edge Y and it amounts to the inverse operator of a boundary value problem on the link X_y of singularitity at y. More precisely, we consider the section of $\mathcal{D}$ by an $(n-q)$-dimensional plane through y orthogonal to Y. This section has a singular point at y, and we consider the tangent cone to the section at y. The link X_y is a bounded domain with smooth boundary on the sphere of sufficiently small radius with centre y in the normal plane at y, that is cut out by the tangent cone. The Neumann problem by freezing coefficients along the edge Y and passing to the Mellin transform in the cone axis variable yields a family of boundary value problems on the link X_y, cf. (8.3), whose resolvent specifies the regularity of solutions to the Neumann problem up to the boundary. In the sequel we omit the sub y of X_y assuming without restriction of generality that the opening angle of the wedge along Y varies inessentially. In this section we give some estimates of the poles of $R(\lambda)$. Our result is by no means complete, but it gives us a sufficient condition of continuity up to the boundary for the operator L.

From now on we identify an i-form u on $\mathcal{D}$ with the collection of its coefficients u_J. Let

$$n_i = \binom{n}{i}$$

and $N^i_{\partial X} := \{U \in C^\infty(\overline{X})^{n_i} :\ B_0 U = B_1 dU = 0 \text{ on } \partial X\}$.

Lemma 9.1

1° Δ_X *is a symmetric operator on* $N^i_{\partial X}$.

2° *The eigenvalues of* $\{\Delta_X, B_0, B_1 \circ d\}$ *are real.*

Proof. If $U, V \in N^i_{\partial X}$, we have

$$\begin{array}{rcccl} B_0 U & = & B_1 dU & = & 0, \\ B_0 V & = & B_1 dV & = & 0 \end{array}$$

on ∂X.

Let φ be a C^∞ function with compact support on $\mathbb{R}_+$, such that $\varphi(r) \geq 0$ for all r and

$$\varphi(r) = \begin{cases} 1 & \text{if} \quad r \in [2,3], \\ 0 & \text{if} \quad r \in (0,1] \cup [4,\infty). \end{cases}$$

Then the vector φU can be identified as an i-form u with components φU_j, $j = 1, \ldots, n_0$, on $(0,\infty) \times X$. We think of φU as being constant in y along $Y \cong \mathbb{R}^q$. Similarly $v = \varphi V$ can be identified with an i-form on $(0,\infty) \times X$ which is constant in y.

Write $\Delta_{\mathcal{C}(X)}$ for the Laplace-Beltrami operator on $\mathcal{C}(X)$ with respect to the cone metric $dr^2 + r^2 d\omega^2$, cf. (8.2) with $\eta = 0$. The integration by parts readily yields

$$(\Delta_{\mathcal{C}(X)} u, v) = (u, \Delta_{\mathcal{C}(X)} v), \tag{9.1}$$

the left-hand side being equal

$$\begin{array}{cl} & \displaystyle\sum_{j=1}^{n_0} \Big(\Big(-\partial_r^2 - \frac{n-q-1}{r}\, \partial_r + \frac{1}{r^2} \Delta_X \Big) u_j, v_j \Big) \\ = & \displaystyle\Big(\int_0^\infty \Big(-\varphi''(r) - \frac{n-q-1}{r}\, \varphi'(r) \Big) \varphi(r) r^{n-q-1} dr \Big) \sum_{j=1}^{n_0} \int_X u_j \bar{v}_j d\omega \\ + & \displaystyle\Big(\int_0^\infty (\varphi(r))^2 r^{n-q-3} dr \Big) \sum_{j=1}^{n_0} \int_X (\Delta_X u_j)\, \bar{v}_j d\omega \end{array}$$

where $d\omega$ is the volume element on X. Similarly, the right-hand side of (9.1) is equal to

$$\begin{array}{cl} & \displaystyle\Big(\int_0^\infty \Big(-\varphi''(r) - \frac{n-q-1}{r}\, \varphi'(r) \Big) \varphi(r) r^{n-q-1} dr \Big) \sum_{j=1}^{n_0} \int_X u_j \bar{v}_j d\omega \\ + & \displaystyle\Big(\int_0^\infty (\varphi(r))^2 r^{n-q-3} dr \Big) \sum_{j=1}^{n_0} \int_X u_j\, \overline{\Delta_X v_j} d\omega. \end{array}$$

Since $\displaystyle\int_0^\infty (\varphi(r))^2 r^{n-q-3} dr \neq 0$, we get

$$(\Delta_X u, v)_{L^2(X, \bigwedge^i)} = (u, \Delta_X v)_{L^2(X, \bigwedge^i)}$$

for all $u, v \in N^i_{\partial X}$. It follows that $\{\Delta_X, B_0, B_1 \circ d\}$ is symmetric on $N^i_{\partial X}$ and its eigenvalues are real.

□

In the case of edges of codimension 2 all the eigenvalues of the boundary value problem $\{\Delta_X, B_0, B_1 \circ d\}$ are non-negative.

Lemma 9.2 *The eigenvalues of* $\{\Delta_X, B_0, B_1 \circ d\}$ *are bounded below by the quantity*

$$-\frac{(n-q-2)^2}{4}.$$

Proof. For all $U \in N^i_{\partial X}$ we use the same notation as in Lemma 9.1. Then we have $(\Delta_{\mathcal{C}(X)} u, u) \geq 0$, or equivalently

$$\Big(\int_0^\infty (\varphi'(r))^2 r^{n-q-1} dr\Big) \sum_{j=1}^{n_0} \int_X |u_j|^2 d\omega + \Big(\int_0^\infty (\varphi(r))^2 r^{n-q-3} dr\Big) \sum_{j=1}^{n_0} \int_X (\Delta_X u_j)\, \bar{u}_j d\omega \geq 0.$$

If $\lambda_0 < 0$ is an eigenvalue of $\{\Delta_X, B_0, B_1 \circ d\}$ and u is the corresponding eigenfunction, then we get

$$\int_0^\infty (\varphi(r))^2 r^{n-q-3} dr \leq \frac{1}{|\lambda_0|} \int_0^\infty (\varphi'(r))^2 r^{n-q-1} dr. \tag{9.2}$$

Since (9.2) holds for any function $\varphi \in C_0^\infty(\mathbb{R}_+)$, we deduce by Hardy's inequality that

$$\int_0^\infty (\varphi(r))^2 r^{n-q-3} dr \leq \frac{4}{(n-q-2)^2} \int_0^\infty (\varphi'(r))^2 r^{n-q-1} dr$$

and $\dfrac{4}{(n-q-2)^2}$ is the best possible constant, hence from (9.2) we obtain

$$\frac{1}{|\lambda_0|} \geq \frac{4}{(n-q-2)^2},$$

showing

$$\lambda_0 \geq -\frac{(n-q-2)^2}{4}.$$

□

In fact, since $\mathcal{C}(X)$ is simply connected, we conclude by the positive definiteness of $(\Delta u, v)_{L^2(\mathcal{C}(X), \Lambda^i)}$ that

$$\lambda_0 > -\frac{(n-q-2)^2}{4}.$$

Lemma 9.3 *The poles of $R(\varrho)$ are purely imaginary.*

Proof. The poles of $R(\varrho)$ are those such that $-\varrho^2 + \imath(n-q-2)\varrho$ is an eigenvalue of $\{\Delta_X, B_0, B_1 \circ d\}$. Lemma 9.1 yields $\Im\,(-\varrho^2 + \imath(n-q-2)\varrho) = 0$, so we have either $\varrho = \imath b$ or

$$\varrho = a + \imath\,\frac{n-q-2}{2}.$$

In this latter case we get

$$\begin{array}{rcl} -\varrho^2 + \imath(n-q-2)\varrho & = & -a^2 - \dfrac{(n-q-2)^2}{4} \\ & \leq & -\dfrac{(n-q-2)^2}{4}, \end{array}$$

which contradicts Lemma 9.2. This proves the lemma.

□

Lemmas 9.1, 9.2 and 9.3 are true for any domain X. In order to improve the results on the poles of $R(\varrho)$, Shaw imposes familiar conditions on the number of negative eigenvalues of the Hesse form of $\mathcal{C}(X)$, cf. [Sha83].

10 The Green formula

In order to reduce the Neumann problem for the Hodge Laplacian to the boundary we invoke as usual a suitable Green formula. Green formulas are actually well understood in the context of general elliptic complexes, cf. [Tar95, 2.5.4].

Let $g(x)$ stands for the standard fundamental solution of convolution type for the Laplace equation on $\mathbb{R}^n$, i.e.,

$$g(x) = \begin{cases} \dfrac{1}{2\pi} \ln|x|, & \text{if } \; n = 2, \\ \dfrac{1}{\sigma_n} \dfrac{1}{2-n} \dfrac{1}{|x|^{n-2}}, & \text{if } \; n \geq 3, \end{cases}$$

σ_n being the area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$.

Suppose $\mathcal{D}$ is an arbitrary bounded Lipschitz domain in $\mathbb{R}^n$, cf. [Mor66] and elsewhere. The boundary $\partial\mathcal{D}$ is regarded as a Lipschitz submanifold of codimension one in $\mathbb{R}^n$. At almost every point $y \in \partial\mathcal{D}$ it has a well-defined outward unit normal vector $n(y)$. Using the Riemannian metric on $T\mathbb{R}^n$ one specifies $n(y)$ within the cotangent space $T^*_y\mathbb{R}^n$. In this way we obtain what is usually referred to as the outward unit conormal vector for $\partial\mathcal{D}$ at y, denoted $\nu(y)$. Also, $d\sigma$ stands for the surface measure induced by the the Riemannian metric of $T\mathbb{R}^n$ on $\partial\mathcal{D}$.

As usual we define the interior product of a 1-form ξ and a q-form u by setting

$$\xi \rfloor u := (-1)^{(\deg u - 1)n} * (\xi \wedge *u)\,.$$

Set $\bigwedge^q = \bigwedge^q T^*\mathbb{R}^n$. A measurable section $u : \partial\mathcal{D} \to \bigwedge^q$ is called tangential if $\nu \rfloor u = 0$ a.e. on $\partial\mathcal{D}$, and normal if $\nu \wedge u = 0$ a.e. on $\partial\mathcal{D}$. Every $u : \partial\mathcal{D} \to \bigwedge^q$ can be written as

$$u = t(u) + \nu \wedge n(u)$$

where

$$\begin{array}{rcl} t(u) & = & \nu \rfloor (\nu \wedge u\,|_{\partial\mathcal{D}})\,, \\ n(u) & = & \nu \rfloor u\,|_{\partial\mathcal{D}} \end{array}$$

cf. [Tar95, 3.2.2]. Since the operator $i_n = \nu\rfloor$ is nilpotent of index 2 both $t(u)$ and $n(u)$ are tangential forms of degree q and $q-1$, respectively. In fact, the tangential forms of degree q give rise to a vector bundle $\bigwedge_t^q$ over $\partial\mathcal{D}$, cf. equality (3.2.1) *ibid.*

It is well known that

$$\begin{array}{rcl} \sigma(d)(x,\xi)u & = & \imath\xi \wedge u, \\ \sigma(\delta)(x,\xi)g & = & -\imath\xi \rfloor g \end{array}$$

for all $x \in \mathbb{R}^n$ and $\xi \in T_x^*\mathbb{R}^n \setminus \{0\}$. We are now in a position to introduce the layer potentials that are relevant in the Neumann problem. More precisely, given any

$$\begin{array}{rclcrcl} u_1 & \in & L(\partial\mathcal{D}, \bigwedge_t^q), & & u_2 & \in & L(\partial\mathcal{D}, \bigwedge_t^{q-1}), \\ u_3 & \in & L(\partial\mathcal{D}, \bigwedge_t^{q-1}), & & u_4 & \in & L(\partial\mathcal{D}, \bigwedge_t^q), \end{array}$$

these are the double layer potentials

$$\begin{array}{rcl} \mathcal{P}_t(u_1)(x) & = & \delta \displaystyle\int_{\partial\mathcal{D}} g(x-y)\,\nu(y) \wedge u_1(y)\,d\sigma(y), \\ \mathcal{P}_n(u_2)(x) & = & -\,d \displaystyle\int_{\partial\mathcal{D}} g(x-y)\,u_2(y)\,d\sigma(y), \end{array}$$

and single layer potentials

$$\begin{array}{rcl} \mathcal{P}_{t\circ\delta}(u_3)(x) & = & \displaystyle\int_{\partial\mathcal{D}} g(x-y)\,\nu(y) \wedge u_3(y)\,d\sigma(y), \\ \mathcal{P}_{n\circ d}(u_4)(x) & = & -\displaystyle\int_{\partial\mathcal{D}} g(x-y)\,u_4(y)\,d\sigma(y), \end{array}$$

cf. [Tar95, 3.3.2]. For any $f \in L(\mathcal{D}, \bigwedge^q)$, we also define the so-called volume potential

$$\mathcal{P}_\Delta(f)(x) = -\int_{\mathcal{D}} g(x-y)\,f(y)\,dy.$$

The coexact current $\mathcal{P}_t(u_1)$ is also known as the Biot-Savart potential of the form u_1. The exact current $\mathcal{P}_n(u_2)$ is said to be the Coulomb potential of the form u_2.

Lemma 10.1 *For any differential form u of degree $0 \le q \le n$ and of class $C^2(\overline{\mathcal{D}})$,*

$$\mathcal{P}_t(t(u))(x) + \mathcal{P}_n(n(u))(x) + \mathcal{P}_{t\circ\delta}(t(\delta u))(x) + \mathcal{P}_{n\circ d}(n(du))(x) + \mathcal{P}_\Delta(\Delta u)(x)$$
$$= \begin{cases} u(x), & \text{if } x \in \mathcal{D}, \\ 0, & \text{if } x \notin \overline{\mathcal{D}}. \end{cases} \tag{10.1}$$

Proof. This is a very particular case of the Green formula (2.5.11) of [Tar95] related to the Laplacian of an arbitrary elliptic complex on a compact smooth manifold with boundary.

□

11 Boundary reduction of Neumann problem

We shall find it necessary to work with classes of symbols $\mathcal{S}^m_{1,0}$ which only exhibit a limited amount of regularity in the spatial variable while being still C^∞ in the Fourier variable. The generic case in the scalar-valued Euclidean setting is as follows. For a normed function space $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ with the property that $C^\infty_{\mathrm{comp}} \hookrightarrow \mathcal{F} \hookrightarrow C^0$ (most typically $\mathcal{F} = C^s$ or a similar space) we say that $a(x,\xi) \in \mathcal{F}\mathcal{S}^m_{1,0}$ if

$$\|D^\beta_\xi a(\cdot,\xi)\|_{\mathcal{F}} \le C_\beta \, \langle\xi\rangle^{m-|\beta|}$$

for all $\alpha \in \mathbb{Z}^n_+$. The class of pseudodifferential operators $\mathrm{op}(a)$ with symbols in $\mathcal{F}\mathcal{S}^m_{1,0}$ is denoted by $\mathcal{F}\Psi^m_{1,0}$. As usual, we write $\mathcal{F}\mathcal{S}^m_{\mathrm{cl}}$ for the subclass of classical symbols, i.e., all $a(x,\xi) \in \mathcal{F}\mathcal{S}^m_{1,0}$ with $a \sim a_m + a_{m-1} + \ldots$ where $a_j(x,\xi)$ is of class $\mathcal{F}$ in x and is smooth and homogeneous of degree j in ξ for $|\xi| \ge 1$.

Given any two smooth vector bundles V and $\tilde{V}$ over the same manifold, let $\mathcal{F}\Psi^m_{1,0}(V,\tilde{V})$, etc., stand for for the space of pseudodifferential operators mapping sections of V to sections of $\tilde{V}$ which, in local coordinates and over local trivialisations of V and $\tilde{V}$, can be represented as matrices with entries from $\mathcal{F}\Psi^m_{1,0}$.

For a pseudodifferential operator $A \in C^0\Psi^{-1}_{\mathrm{cl}}(V,\tilde{V})$, we denote by $\sigma(A)$ its principal symbol belonging to $C^0\mathcal{S}^{-1}_{\mathrm{cl}}(T^*\mathbb{R}^n, \mathrm{Hom}(V,\tilde{V}))$, and by K_A the Schwartz kernel of A. This latter is a distribution over $\mathbb{R}^n \times \mathbb{R}^n$ with values in $\tilde{V} \otimes V^*$.

Next, we introduce a layer potential operator $P_{A,0}$ by formally writing, given a section $u : \partial\mathcal{D} \to V$,

$$P_{A,0}u\,(x) = \mathrm{p.v.} \int_{\partial\mathcal{D}} \langle K_A(x,y), u(y)\rangle_y \, d\sigma(y) \tag{11.1}$$

for $x \in \partial\mathcal{D}$.

Lemma 11.1 *Let $A \in C^0\Psi^{-1}_{\text{cl}}(V,\tilde{V})$ be an operator whose principal symbol $\sigma(A)(x,\xi)$ is odd in $\xi \in T^*_x\mathbb{R}^n$. Then, for each $u \in L^p(\partial\mathcal{D}, V)$ with $1 < p < \infty$, the principal value integral $P_{A,0}u(x)$ exists at almost every point $x \in \partial\mathcal{D}$. Moreover,*

$$P_{A,0} : L^p(\partial\mathcal{D}, V) \to L^p(\partial\mathcal{D}, \tilde{V})$$

is a bounded operator.

Proof. The problem localizes and, given the invariance of the class of symbols and pseudodifferential operators under discussion, it can be transported in $\mathbb{R}^{n-1}$ via coordinate mappings. For detail we refer the reader to [MT99] and elsewhere.

□

To state the main result of this section we need some more notation. Set $\mathcal{D}^+ := \mathcal{D}$ and $\mathcal{D}^- := \mathbb{R}^n \setminus \overline{\mathcal{D}}$. For a section $u \in \mathcal{D}'(\mathbb{R}^n, V)$, we denote by $u^\pm$ its restriction to $\mathcal{D}^\pm$.

Since the kernel K_A is continuous away from the diagonal, the layer potential (11.1) is always well defined for $x \notin \partial\mathcal{D}$. It just amounts to $A(u\delta_{\partial\mathcal{D}})$ where $\delta_{\partial\mathcal{D}}$ is the surface layer on $\partial\mathcal{D}$. We write $P_{A,\pm}u$ for $A(u\delta_{\partial\mathcal{D}})$ restricted to $\mathcal{D}^\pm$, respectively.

If $u \in \mathcal{D}'(\mathbb{R}^n, V)$ is continuous away from $\partial\mathcal{D}$ and $x^0 \in \partial\mathcal{D}$, we denote by $u^\pm(x^0)$ the non-tangential limit values of $u^\pm$ at x^0, that is

$$u^\pm(x^0) = \lim_{x \in \mathcal{C}^\pm(x^0)} u(x)$$

where $\gamma^\pm(x^0) \subset \mathcal{D}^\pm$ are appropriate non-tangential approach regions. As usual, they are controlled by the non-tangential maximal operator defined for $u : \mathbb{R}^n \setminus \partial\mathcal{D} \to V$ by

$$\mathcal{N}u^\pm\,(x^0) = \sup_{x \in \mathcal{C}^\pm(x^0)} |u(x)|$$

if $x^0 \in \partial\mathcal{D}$.

Lemma 11.2 *Under hypotheses of Lemma 11.1, there is a constant $C > 0$ such that $\|\mathcal{N}(P_{A,\pm}u)\|_{L^p(\partial\mathcal{D})} \le C\,\|u\|_{L^p(\partial\mathcal{D},V)}$ for any $u \in L^p(\partial\mathcal{D}, V)$. Moreover, $P_{A,\pm}u$ have non-tangential limit values at almost every point $x \in \partial\mathcal{D}$, namely*

$$P_{A,\pm}u\,(x) = \pm\frac{1}{2\imath}\,\sigma(A)(x,\nu(x))\,u(x) + P_{A,0}u\,(x). \tag{11.2}$$

Proof. Cf. Theorem 1.1 in [MMT00].

□

In [MMT00] it is also proved that if actually $A \in C^{0,\lambda}\Psi^{-1}_{\text{cl}}(V,\tilde{V})$ for some $\lambda > 1/2$, then $P_{A,+}$ maps $L^2(\partial\mathcal{D}, V)$ continuously to $H^{1/2}(\mathcal{D}, \tilde{V})$. Similar results are also valid for all formal adjoints of pseudodifferential operators in $C^0\Psi^{-1}_{\mathrm{cl}}(\tilde{V}, V)$.

A typical context to which Lemma 11.2 applies is when A has the form $D\Pi$ or its formal adjoint. Here D is a first order differential operator of type $V \to \tilde{V}$ and Π is a parametrix for an elliptic operator of $\Psi^2_{\mathrm{cl}}(V)$ with even principal symbol. In fact, similar arguments are still valid under considerably more relaxed smoothness assumptions on the coefficients of the elliptic operator in question.

Corollary 11.3 *Assume that $\mathcal{D}$ is a relatively compact domain with Lipschitz boundary and $u \in L^p(\partial\mathcal{D}, \bigwedge^q_t)$, where $1 < p < \infty$. Then the principal value integral $\mathcal{P}_{t,0}u$ exists a.e. on $\partial\mathcal{D}$ and defines a form in $L^p(\partial\mathcal{D}, \bigwedge^q)$. Moreover, $\mathcal{P}_{t,+}u$ has non-tangential limit values a.e. on the boundary of $\mathcal{D}$ which satisfy*

$$\mathcal{P}_{t,+}u = \frac{1}{2}\,u + \mathcal{P}_{t,0}u. \tag{11.3}$$

Proof. In the case of smooth domains $\mathcal{D}$ and data u this result is well known, cf. for instance Theorem 3.2.6 in [Tar95]. The additional difficulties which are caused by boundary singularities are settled by Lemmas 11.1 and 11.2.

□

Jump formulas (11.2) are known as the Sokhotskii-Plemelj formulas for the double layer potential.

We now return to the Neumann problem for the de Rham complex. Given any $f \in L^2(\mathcal{D}, \bigwedge^q)$, it consists of finding a differential form $u \in L^2(\mathcal{D}, \bigwedge^q)$ satisfying

$$\begin{array}{rcll} (\delta d + d\delta)u & = & f & \text{in} \quad \mathcal{D}, \\ n(u) & = & 0 & \text{on} \quad \partial\mathcal{D}, \\ n(du) & = & 0 & \text{on} \quad \partial\mathcal{D}, \end{array} \tag{11.4}$$

the boundary conditions being understood in a suitable weak sense. Since $\{t, n, t \circ \delta, n \circ d\}$ is in fact a Dirichlet system of order 1 on $\partial\mathcal{D}$, the case of non-zero boundary data reduces to (11.4) in a familiar way, and we omit the details.

If $u \in L^2(\mathcal{D}, \bigwedge^q)$ is a solution of (11.4) then the Green formula (10.1) shows that

$$u = \mathcal{P}_t(t(u)) + \mathcal{P}_{t\circ\delta}(t(\delta u)) + \mathcal{P}_{\Delta}(f) \tag{11.5}$$

in $\mathcal{D}$. In order to find u it remains to determine $u_1 := t(u)$ and $u_3 := t(\delta u)$ on $\partial\mathcal{D}$. For this purpose, we invoke Corollary 11.3 to derive a system of two boundary integral equations for u_1 and u_3. More specifically, we evaluate both $t(u)$ and $t(\delta u)$ from (11.5), obtaining

$$\begin{array}{rcl} t(u) & = & t\left(\mathcal{P}_{t,+}(t(u))\right) + t\left(\mathcal{P}_{t\circ\delta,+}(t(\delta u))\right) + t\left(\mathcal{P}_{\Delta,+}(f)\right), \\ t(\delta u) & = & t\left(\delta\mathcal{P}_{t,+}(t(u))\right) + t\left(\delta\mathcal{P}_{t\circ\delta,+}(t(\delta u))\right) + t\left(\delta\mathcal{P}_{\Delta,+}(f)\right) \end{array}$$

a.e. on $\partial\mathcal{D}$. As is easy to check,

$$\begin{array}{rcl} \delta\mathcal{P}_{t,+}(t(u)) & = & 0, \\ \delta\mathcal{P}_{t\circ\delta,+}(t(\delta u)) & = & \mathcal{P}_{t,+}(t(\delta u)) \end{array}$$

whence

$$\begin{array}{rcl} t(u) - t\left(\mathcal{P}_{t,+}(t(u))\right) - t\left(\mathcal{P}_{t\circ\delta,+}(t(\delta u))\right) & = & t\left(\mathcal{P}_{\Delta,+}(f)\right), \\ t(\delta u) - t\left(\mathcal{P}_{t,+}(t(\delta u))\right) & = & t\left(\delta\mathcal{P}_{\Delta,+}(f)\right) \end{array}$$

a.e. on $\partial\mathcal{D}$.

Theorem 11.4 *Let $u_1 \in L^2(\partial\mathcal{D}, \bigwedge_t^q)$ and $u_3 \in L^2(\partial\mathcal{D}, \bigwedge_t^{q-1})$ satisfy the system of integral equations*

$$\begin{array}{rcl} \frac{1}{2}u_1 - t\left(\mathcal{P}_{t,0}(u_1)\right) - t\left(\mathcal{P}_{t\circ\delta,+}(u_3)\right) & = & t\left(\mathcal{P}_{\Delta,+}(f)\right), \\ \frac{1}{2}u_3 - t\left(\mathcal{P}_{t,0}(u_3)\right) & = & t\left(\delta\mathcal{P}_{\Delta,+}(f)\right) \end{array} \tag{11.6}$$

a.e. on $\partial\mathcal{D}$. Then $u = \mathcal{P}_{t,+}(u_1) + \mathcal{P}_{t\circ\delta,+}(u_3) + \mathcal{P}_{\Delta,+}(f)$ is a solution of the Neumann problem (11.4) up to a finite-dimensional subspace of $L^2(\partial\mathcal{D}, \bigwedge^{q-1})$.

Proof. From the remark after Lemma 11.2 it follows that $u \in H^{1/2}(\mathcal{D}, \bigwedge^q)$. Since the volume potential solves the equation $\Delta\mathcal{P}_{\Delta,+}(f) = f$ in $\mathcal{D}$ we conclude that $\Delta u = f$ in $\mathcal{D}$.

When combined with the jump formula (11.3), equations (11.6) easily yield $t(u) = u_1$ and $t(\delta u) = u_3$ a.e. on ∂D. Hence u represents by the formula (11.5) in $\mathcal{D}$.

Comparing (10.1) with (11.5) we see that $\mathcal{P}_n(n(u)) + \mathcal{P}_{n\circ d}(n(du)) \equiv 0$ in $\mathcal{D}$. Denote the current on the left-hand side by T. This is a harmonic form away from the boundary of $\mathcal{D}$ vanishing at $x = \infty$ as $|x|^{2-n}$. By Corollary 11.3, we get

$$\begin{array}{rclcrcl} t(T^+) - t(T^-) & = & 0, & \quad & t(\delta T^+) - t(\delta T^-) & = & 0, \\ n(T^+) - n(T^-) & = & n(u), & & n(dT^+) - n(dT^-) & = & n(du) \end{array}$$

on ∂D, cf. also Theorem 3.3.9 of [Tar95]. Since $T^+ \equiv 0$ we conclude that both $t(T^-)$ and $t(\delta T^-)$ vanish, and so

$$\begin{array}{rcl} \|dT^-\|^2_{L^2(\mathbb{R}^n\setminus\mathcal{D},\bigwedge^{q+1})} + \|\delta T^-\|^2_{L^2(\mathbb{R}^n\setminus\mathcal{D},\bigwedge^{q-1})} & = & \left(\Delta T^-, T^-\right)_{L^2(\mathbb{R}^n\setminus\mathcal{D},\bigwedge^q)} \\ & = & 0 \end{array}$$

by Stokes' formula. It follows that $dT^- = 0$ and $\delta T^- = 0$, which means that $*T^-$ is a solution of the Neumann problem in $\mathbb{R}^n \setminus \overline{\mathcal{D}}$ with zero data. Since the space of such $*T^-$ is isomorphic to the de Rham cohomology of $\mathbb{R}^n \setminus \mathcal{D}$ at step $n-q$, it is finite dimensional. To complete the proof it suffices to observe that $n(u) = -n(T^-)$.

□

Were the cohomology of $\mathbb{R}^n \setminus \mathcal{D}$ trivial at step $n-q$, we would be able to conclude that $T^- \equiv 0$, showing $n(u) = 0$.

Note that $u = \mathcal{P}_{t,+}(u_1) + \mathcal{P}_{t\circ\delta,+}(u_3) + \mathcal{P}_{\Delta,+}(f)$ need not satisfy (11.4) in general. To see this, pick a non-trivial form $T^- \in C^\infty(\mathbb{R}^n \setminus \mathcal{D}, \bigwedge^q)$ vanishing at infinity and satisfying $dT^- = 0$, $\delta T^- = 0$ and $t(T^-) = 0$. Choose a differential form $u \in C^\infty(\overline{\mathcal{D}}, \bigwedge^q)$ with the property that $n(u) = -n(T^-)$ and $n(du) = 0$. Then $\mathcal{P}_n(n(u)) = -\mathcal{P}_n(n(T^-)) = 0$ in $\mathcal{D}$, as follows from the Green formula (10.1). Hence u represents by the formula $u = \mathcal{P}_{t,+}(u_1) + \mathcal{P}_{t\circ\delta,+}(u_3) + \mathcal{P}_{\Delta,+}(f)$ where $u_1 = t(u)$, $u_3 = t(\delta u)$ and $f = \Delta u$. Clearly, u_1 and u_3 satisfy (11.6) while $n(u) \neq 0$.

Corollary 11.5 *If the operator $I - 2t\mathcal{P}_{t,0}$ in $L^2(\partial\mathcal{D}, \bigwedge^q_t)$ is Fredholm, then so is the Neumann problem (11.4).*

12 Topological ingredients

We first recall the Abstract de Rham Theorem, cf. for instance [Wel73]. Let X be a Hausdorff paracompact topological space, and let $\mathcal{S}$ be a sheaf over X. If

$$\mathcal{F}^\cdot : \; 0 \longrightarrow \mathcal{S} \stackrel{\iota}{\longrightarrow} \mathcal{F}^0 \stackrel{d^0}{\longrightarrow} \mathcal{F}^1 \stackrel{d^1}{\longrightarrow} \ldots$$

is a resolution of $\mathcal{S}$ by fine sheaves over X, then $H^i(X, \mathcal{S}) \cong H^i(\mathcal{F}^\cdot(X))$ holds for every $i = 0, 1, \ldots$. Here $H^i(X, \mathcal{S})$ stands for the i-th cohomology of X with coefficients in $\mathcal{S}$.

Theorem 12.1 *Suppose that $\mathcal{D}$ is a relatively compact domain with Lipschitz boundary. Then, for every $j = 0, 1, \ldots$, there is an isomorphism of vector spaces*

$$H^i(\mathcal{D}, \mathbb{R}) \cong \frac{\{u \in L^2(\mathcal{D}, \bigwedge^i) : du = 0 \text{ in } \mathcal{D}\}}{d\{u \in L^2(\mathcal{D}, \bigwedge^{i-1}) : du \in L^2(\mathcal{D}, \bigwedge^i)\}}. \tag{12.1}$$

Proof. For any fixed $i = 0, 1, \ldots$, we consider the sheaf $\mathcal{L}^i$ over the closure of $\mathcal{D}$, defined by

$$\mathcal{L}^i(U) = \{u \in L^2_{\mathrm{loc}}(U, \bigwedge^i) : du \in L^2_{\mathrm{loc}}(U, \bigwedge^{i+1})\}$$

for any open subset U of $\overline{\mathcal{D}}$, and the sheaf morphism $d : \mathcal{L}^i \to \mathcal{L}^{i+1}$ given by the restriction of d to the interior of U in $\mathcal{D}$. We claim that the sequence of sheaves $\{\mathcal{L}^i\}_{i=0,1,\ldots}$ forms a fine resolution of the sheaf $\mathbb{R}$. Indeed, the only thing to be checked is the exactness on stalks. However, the L^2 Poincaré lemma to be verified in this context is clearly invariant under pull-back by bi-Lipschitz homeomorphisms and, hence, can be transported to a ball in $\mathbb{R}^n$. In the latter case such a result is essentially well known, for the standard proof of the Poincaré lemma can be adapted to this setting by using a mollifying argument. Note that this proof utilizes only the Lipschitz structure of the underlying Riemannian manifold. Hence the Abstract de Rham Theorem yields (12.1), as desired.

□

If the cohomology $H^i(\mathcal{D}, \mathbb{R})$ is finite dimensional then it just amounts to the i-th singular homology group of $\mathcal{D}$ over real numbers, i.e., $H_i^{\mathrm{sing}}(\mathcal{D}, \mathbb{R})$. The dimension of this latter is said to be the i-th Betti number of $\mathcal{D}$. The cohomology on the right-hand side of (12.1) is actually $H^i(\mathcal{L}^\cdot(\mathcal{D}))$. It fails to coincide with the L^2 cohomology of $\mathcal{D}$ in general. By this latter is usually meant

$$\frac{\{u \in \mathcal{D}_T^i : Tu = 0\}}{d\mathcal{D}_T^{i-1}}, \tag{12.2}$$

cf. § 8. However, if the Neumann problem is normally solvable then $H^i(\mathcal{L}^\cdot(\mathcal{D}))$ just amounts to (12.2). To prove this, it suffices to use the strong Hodge decomposition.

Corollary 12.2 *If the operator $(L+1)^{-1}$ is compact then the L^2 cohomology of $\mathcal{D}$ at step i is isomorphic to $H_i^{\mathrm{sing}}(\mathcal{D}, \mathbb{R})$.*

Note that in a recent paper [KS03] sufficient conditions of compact solvability for the de Rham complex are derived.

References

[Agr96] M. S. Agranovich, *Estimates of s-numbers and spectral asymptotics for layer potential type integral operators on nonsmooth surfaces*, Funkts. Analiz **30** (1996), no. 2, 1–18.

[AV64] M. S. Agranovich and M. I. Vishik, *Elliptic problems with parameter and parabolic problems of general type*, Uspekhi Mat. Nauk **19** (1964), no. 3, 53–160.

[BB81] P. K. Banerjee and R. Butterfield, *Boundary element methods in engineering science*, McGraw-Hill Book Co., London et al., 1981.

[BM67] Yu. D. Burago and V. G. Maz'ya, *Multidimensional potential theory and solution of boundary value problems for domains with nonregular boundary*, Zap. Nauchn. Semin. LOMI, vol. 3, LOMI Akad. Nauk SSSR, Leningrad, 1967, pp. 5–86.

[Cal63] A. P. Calderón, *Boundary value problems for elliptic equations*, Outlines of the Joint Soviet-American Symp. on Part. Diff. Eq., Nauka, Novosibirsk, 1963, pp. 303–304.

[Cal77] A. Calderón, *Cauchy integrals on lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 1324–1327.

[Car16] T. Carleman, *Über das Neumann-Poincarésche Problem für ein Gebiet mit Ecken*, Almquist and Wiksell, Uppsala, 1916, 195 pp.

[Che80] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Proc. Sympos. Pure Math. **36** (1980), 21–45.

[CMM82] R. R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes Lipschitziennes*, Ann. Math. **116** (1982), no. 2, 361–387.

[Con53] P. E. Conner, *The Neumann's problem for differential forms on Riemannian manifolds*, Mem. Amer. Math. Soc. **20** (1953), 1–58.

[DK87] B. Dahlberg and C. Kenig, *Hardy spaces and the L^p Neumann problem for Laplace's equation in a Lipschitz domain*, Ann. Math. **125** (1987), 437–465.

[Duf54] G. F. Duff, *A tensor boundary value problem of mixed type*, Can. J. Math. **6** (1954), 427–440.

[FJL77] J. E. Fabes, M. Jodeit, and J. E. Lewis, *Double layer potentials for domains with corners and edges*, Indiana Univ. Math. J. **26** (1977), no. 1, 95–114.

[FJR78] J. E. Fabes, M. Jodeit, and N. M. Riviére, *Potential techniques for boundary value problems in C^1 domains*, Acta Math. **141** (1978), no. 3–4, 165–186.

[Fei72] V. I. Feigin, *Elliptic equations in domains with multidimensional singularities on the boundary*, Uspekhi Mat. Nauk **27** (1972), no. 2, 183–184.

[Gaf55] M. P. Gaffney, *Hilbert space methods in the theory of harmonic integrals*, Trans. Amer. Math. Soc. **78** (1955), 551–590.

[GS53] P. R. Garabedian and D. C. Spencer, *A complex tensor calculus for Kähler manifolds*, Acta Math. **89** (1953), 279–331.

[GK57] I. Ts. Gokhberg and M. G. Krein, *Basic propositions on defect numbers, root numbers and indices of linear operators*, Uspekhi Mat. Nauk **12** (1957), no. 2 (74), 43–118.

[Gru71] V. V. Grushin, *On a class of elliptic pseudodifferential operators degenerate on a submanifold*, Math. USSR Sbornik **13** (1971), no. 2, 155–183.

[Hod41] W. V. D. Hodge, *The Theory and Application of Harmonic Integrals*, Cambridge Univ. Press, New York, 1941.

[KK62] L. V. Kantorovich and V. I. Krylov, *Approximate Methods of Higher Analysis*, Fizmatgiz, Moscow, 1962, 695 pp.

[KO83] V. A. Kondrat'ev and O. A. Oleinik, *Boundary value problems for partial differential equations in nonsmooth domains*, Uspekhi Mat. Nauk **38** (1983), no. 2, 3–76.

[Kon67] V. A. Kondrat'ev, *Boundary value problems for elliptic equations in domains with conical points*, Trudy Mosk. Mat. Obshch. **16** (1967), 209–292.

[Kor02] A. Korn, *Abhandlungen zur Potentialtheorie in 5 Hefte*, Dümmler, Berlin, 1901–1902.

[Krá80] J. Král, *Integral operators in potential theory*, Lect. Notes Math., vol. 823, Springer-Verlag, Berlin, 1980.

[KW86] J. Král and W. Wendland, *Some examples concerning applicability of the Fredholm-Radon method in potential theory*, Apl. Mat. **31** (1986), 293–318.

[KS03] V. I. Kuz'minov and I. A. Shvedov, *On compact solvability of differentials of an elliptic differential complex*, Sibirsk. Mat. Zh. **44** (2003), no. 6, 1280–1294.

[Lop53] Ya. B. Lopatinskii, *On a reduction of boundary value problems for elliptic systems to regular integral equations*, Ukrainsk. Mat. Zh. **5** (1953), no. 2, 123–151.

[Lop63] Ya. B. Lopatinskii, *Fundamental solutions to a system of differential equations of elliptic type*, Theor. and Applied Math. **2** (1963), 53–57.

[Maz88] V. G. Maz'ya, *Boundary integral equations*, Current Problems of Mathematics. Fundamental Directions. Vol. 27, VINITI, Moscow, 1988, pp. 131–228.

[MP76] V. G. Maz'ya and B. A. Plamenevskii, *On the coefficients in the asymptotics of solutions of solutions of elliptic boundary value problems near the edge*, Sov. Math. Dokl. **17** (1976), 970–974.

[MP77] V. G. Maz'ya and B. A. Plamenevskii, *Elliptic boundary value problems on manifolds with singularities*, Problems of Mathematical Analysis, vol. 6, Univ. of Leningrad, 1977, pp. 85–142.

[MM96] D. Mitrea and M. Mitrea, *Boundary integral methods for harmonic differential forms in Lipschitz domains*, ERA Amer. Math. Soc. **2** (1996), 92–97.

[MMT00] D. Mitrea, M. Mitrea, and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Mem. Amer. Math. Soc. **???** (2000), ???–???

[MT99] M. Mitrea and M. Taylor, *Boundary layer methods for Lipschitz domains in Riemannian manifolds*, J. Funct. Anal. **163** (1999), no. 3, 181–251.

[ME56] C. B. Morrey and J. Eells, *A variational method in the theory of harmonic integrals. I*, Amer. J. Math. **63** (1956), no. 2, 91–128.

[Mor56] C. B. Morrey, *A variational method in the theory of harmonic integrals. II*, Amer. J. Math. **78** (1956), no. 1, 137–170.

[Mor66] C. B. Morrey, *Multiple Iintegrals in the Calculus of Variations*, Springer Verlag, New York, 1966.

[NP91] S. A. Nazarov and B. A. Plamenevskii, *Elliptic Boundary Value Problems in Domains with Piecewise Smooth Boundary*, Nauka, Moscow, 1991.

[Pla86] B. A. Plamenevskii, *Algebras of Pseudodifferential Operators*, Nauka, Moscow, 1986.

[Pla97] B. A. Plamenevskii, *Elliptic Boundary Problems in Domain with Piecewise Smooth Boundary*, Partial Differential Equations IX, Encyclopaedia of Mathematical Sciences, vol. 79, Springer, Berlin et al., 1997, pp. 217–273.

[Rad19] J. Radon, *Über die Randwertaufgaben beim logarithmischen Potential*, Sitzungsber. Akad. Wiss. Wien **128** (1919), no. 7, 1123–1167.

[RST04] V. Rabinovich, B.-W. Schulze, and N. Tarkhanov, *Boundary value problems in oscillating cuspidal wedges*, Rocky Mountain Journal of Mathematics **34** (Fall 2004), no. 3, 1397–1469.

[Rya69] V. S. Ryaben'kii, *Green formula for systems of difference equations with constant coefficients*, Mat. Zam. **5** (1969), no. 6, 615–622.

[Sch91] B.-W. Schulze, *Pseudo-Differential Operators on Manifolds with Singularities*, North-Holland, Amsterdam, 1991.

[Sch98] B.-W. Schulze, *Boundary Value Problems and Singular Pseudo-Differential Operators*, J. Wiley, Chichester, 1998.

[Sch01] B.-W. Schulze, *An Algebra of Boundary Value Problems Not Requiring Shapiro-Lopatinskij Conditions*, J. Funct. Anal. **179** (2001), 374–408.

[ST99] B.-W. Schulze and N. N. Tarkhanov, *Elliptic complexes of pseudodifferential operators on manifolds with edges*, Evolution Equations, Feshbach Resonances, Singular Hodge Theory. Advances in Partial Differential Equations 16, Wiley-VCH, Berlin et al., 1999, pp. 287–431.

[Sha83] M.-C. Shaw, *Hodge theory on domains with conic singularities*, Comm. Part. Diff. Equ. **8** (1983), no. 1, 65–88.

[Sim65a] I. B. Simonenko, *A new general method for investigating linear operator equations of the singular integral operator type. I*, Izv. Akad. Nauk SSSR. Ser. Mat. **29** (1965), 567–586.

[Sim65b] I. B. Simonenko, *A new general method for investigating linear operator equations of the singular integral operator type. II*, Izv. Akad. Nauk SSSR. Ser. Mat. **29** (1965), 757–782.

[Sin71] I. M. Singer, *Future extensions of index theory and elliptic operators*, Prospects in Mathematics. Annals Math. Studies 70, Princeton, 1971, pp. 171–185.

[ST03] A. Shlapunov and N. Tarkhanov, *Duality by reproducing kernels*, IJMMS **6** (2003), January, 327–395.

[Sul79] D. Sullivan, *Hyperbolic geometry and homeomorphisms*, Geometric Topology (Proc. Georgia Topology Conference, Athens, Georgia, 1977) (J. C. Cantrell, ed.), Academic Press, 1979, pp. 543–555.

[SZ83] B. A. Solonnikov and B. Zayonchkovskii, *On the Neumann problem for second order elliptic equations on domains with edges*, Zap. Nauchn. Semin. LOMI, vol. 126, LOMI Akad. Nauk SSSR, Leningrad, 1983, pp. 7–48.

[Tar95] N. N. Tarkhanov, *Complexes of Differential Operators*, Kluwer Academic Publishers, Dordrecht, NL, 1995.

[Tel80] N. Teleman, *Combinatorial Hodge theory and signature operator*, Inventiones Math. **61** (1980), 227–249.

[Tel83] N. Teleman, *The index of signature operators on Lipschitz manifolds*, Publ. Math. I.H.E.S. **58** (1983), 39–78.

[Tro77] Guido Trombetti, *Problemi ellittici in un cono*, Ricerche Mat. **26** (1977), no. 1, 103–134.

[VE67] M. I. Vishik and G. I. Eskin, *Elliptic equations in convolution in a bounded domain and their applications*, Russian Math. Surveys **22** (1967), no. 1, 13–75.

[Ver84] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), no. 3, 572–611.

[Wel73] R. Wells, *Differential Analysis on Complex Manifolds*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

[Wen82] W. L. Wendland, *Boundary elements methods and their asymptotic convergence*, Preprint 690, Technische Hochschule, Darmstadt, 1982.

[Whi46] H. Whitney, *Complexes of manifolds*, Proc. Nat. Acad. Sci. U.S.A. **33** (1946), 10–11.

[Zar04] S. Zaremba, *Les fonctions fondamentales de H. Poincaré et méthode de Neumann pour une frontiére composée de polygones curvilignes*, J. math. pures et appl. **4** (1904), 395–444.

E-mail addresses :

massing@rz.uni-potsdam.de

tarkhanov@math.uni-potsdam.de