On the Homotopy Classification of Elliptic Operators on Manifolds with Edges

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Abstract

We obtain a stable homotopy classification of elliptic operators on manifolds with edges.

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Introduction

The paper deals with the classification of elliptic operators on manifolds with edges, i.e., a description of the set of elliptic operators up to stable homotopy.

For the first time, such a classification for the case of smooth manifolds was given by Atiyah and Singer [1] in terms of the topological K-functor. Later, an approach to the homotopy classification on manifolds with singularities in terms of the *analytic K-homology* of the manifold was suggested in [2]. Namely, an elliptic operator on a manifold $\mathcal{M}$ with singularities is represented as an abstract (elliptic) operator in the sense of Atiyah

[3] and hence defines a cycle in the analytic K-homology of the manifold $\mathcal{M}$ viewed as a compact topological space. In other words, there is a well-defined homomorphism

$$\operatorname{Ell}_0(\mathcal{M}) \longrightarrow K_0(\mathcal{M}), \tag{0.1}$$

where $\operatorname{Ell}_0(\mathcal{M})$ is the group of stable homotopy classes of elliptic operators. For the case of isolated singularities, the fact that (0.1) is an isomorphism was proved in [2]. (See also [4], where the case of one singular point was considered.)

The classification (0.1) has many corollaries and applications, including a formula for the obstruction, similar to the Atiyah–Bott obstruction [5], to the existence of Fredholm problems for elliptic equations, the fact that the group $\operatorname{Ell}_0(\mathcal{M})$ is equal modulo torsion to the homology $H_{ev}(\mathcal{M})$, a generalization of Poincaré duality in K-theory to manifolds with singularities, etc.

In the present paper, we prove that the mapping (0.1) is an isomorphism for elliptic operators on manifolds with edges in the sense of [6].

The idea underlying the proof is simple. A manifold $\mathcal{M}$ with edges is a stratified manifold with two strata, the singularity stratum X and the open stratum $\mathcal{M} \setminus X$; both strata are smooth. The isomorphism (0.1) on smooth manifolds is known [7, 8], and hence it is natural to extend the assertion about the isomorphism to the union of these two strata using the sequences

$$\begin{array}{ccccccccc} \operatorname{Ell}_1(\mathcal{M}\setminus X) & \to & \operatorname{Ell}_0(X) & \to & \operatorname{Ell}_0(\mathcal{M}) & \to & \operatorname{Ell}_0(\mathcal{M}\setminus X) & \to & \operatorname{Ell}_1(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_1(\mathcal{M}\setminus X) & \to & K_0(X) & \to & K_0(\mathcal{M}) & \to & K_0(\mathcal{M}\setminus X) & \to & K_1(X) \end{array} \tag{0.2}$$

of the pair $X \subset \mathcal{M}$ in K-homology and Ell-theory. Once this commutative diagram is defined, the desired isomorphism follows from the five lemma.

Actually, we construct Ell-theory (Ell-functor) for the manifold $\mathcal{M}$ with edges and then establish the isomorphism (0.1) by mimicking the proof of the uniqueness theorem for extraordinary cohomology theory. We however point out the following important facts.

1. Exact sequences in elliptic theory are not known in general. To construct these exact sequences, we represent the group $\operatorname{Ell}_0(\mathcal{M})$ as the K-group of some C^*-algebra and then use the exact sequence of K-theory of algebras.

2. The main difficulty is the boundary map in the upper row (i.e., the boundary map in K-theory of C^*-algebras). We use *semiclassical quantization* (e.g., see [9]), which permits us to replace the algebra of edge symbols by a simpler algebra of families of parameter-dependent operators in the computation of the boundary map. For the latter algebra, the boundary map is given by the index theorem in [10].

Let us briefly outline the contents of the paper. We recall the main notions of elliptic theory on manifolds with edges in Sec. 1. Section 2 describes a construction that assigns a cycle in K-homology to each elliptic operator. Then (Sec. 3) we state a homotopy classification theorem and present its proof except for two especially lengthy computations, which are given separately in Secs. 4 and 5. The last section contains some additional remarks (the classification of edge morphisms and the topological obstruction to the existence of elliptic edge problems). The desired properties of semiclassical quantization are established in the Appendix.

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1 Operators on Manifolds with Edges

First, we recall some facts of the theory of elliptic operators on manifolds with edges. We systematically use the results of the paper [6].[1]

1. Manifolds with edges. Let M be a smooth compact manifold with boundary ∂M equipped with the structure of a smooth locally trivial bundle $\pi : \partial M \to X$ with base X and fiber Ω. A *manifold* $\mathcal{M}$ *with edge* $X \subset \mathcal{M}$ is the space obtained from M by identifying the points lying in the same fiber:

$$\mathcal{M} = M/\sim, \qquad x \sim y \Longleftrightarrow x = y, \text{ or } (x, y \in \partial M \text{ and } \pi(x) = \pi(y)).$$

The complement $M^\circ = \mathcal{M} \setminus X$ is an open smooth manifold, and an arbitrary point of the edge X has a neighborhood homeomorphic to the *model wedge*

$$\mathbb{W} = \mathbb{R}^n \times K_\Omega, \tag{1.1}$$

where n is the dimension of X and $K_\Omega = \Omega \times \overline{\mathbb{R}}_+ / \Omega \times \{0\}$ is the cone with base Ω. In this local model, the points of the edge form the subset $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times K_\Omega$.

We often use local coordinates (x, ω, r). For the model wedge (1.1), these coordinates have the following form: $x \in \mathbb{R}^n$, $r \in \mathbb{R}_+$, and ω is a coordinate on Ω.

2. Differential operators and function spaces. Consider a differential expression with smooth coefficients on M° having the form

$$D = \sum_{|\alpha|+|\beta|+j+l \le m} a_{\alpha\beta jl}(r, \omega, x) \left(-i\frac{\partial}{\partial x}\right)^\alpha \left(-\frac{i}{r}\frac{\partial}{\partial \omega}\right)^\beta \left(-i\frac{\partial}{\partial r}\right)^j \left(\frac{1}{r}\right)^l \tag{1.2}$$

in a neighborhood of the edge, where m is the *order* of the expression and the coefficients $a_{\alpha\beta jl}(r, \omega, x)$ are smooth functions up to $r = 0$. Such differential expressions can be realized as bounded operators

$$D : \mathcal{W}^{s,\gamma}(\mathcal{M}) \longrightarrow \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}) \tag{1.3}$$

[1]The paper [6] contains a version of the theory of elliptic operators on manifolds with edges [11, 12] especially suited for the study of topological aspects of the theory. In particular, one deals only with smooth (or continuous) symbols and does not impose any analyticity requirements.

in the *edge Sobolev spaces* $\mathcal{W}^{s,\gamma}$ obtained by gluing of the standard Sobolev space H^s on the smooth part of the manifold and the function space on the infinite wedge (1.1) with the norm

$$\|u\|_{s,\gamma} = \left(\int [\xi]^{2s} \|\varkappa_{[\xi]}^{-1} \widetilde{u}\|_{\mathcal{K}^{s,\gamma}} d\xi \right)^{1/2}, \quad s, \gamma \in \mathbb{R},$$

where $[\xi] = \sqrt{1+\xi^2}$, $\widetilde{u}$ is the Fourier transform of u with respect to the variable x,

$$\varkappa_\lambda u(r) = \lambda^{(k+1)/2} u(\lambda r)$$

in a unitary action of the group $\mathbb{R}_+$ in the space $L^2(K_\Omega, d\mathrm{vol})$ with the volume form $d\mathrm{vol} = r^k dr d\omega$ corresponding to the cone metric $dr^2 + r^2 d\omega^2$ (here $k = \dim \Omega$), and the norm $\|\cdot\|_{\mathcal{K}^{s,\gamma}}$ for functions on the cone K_Ω is determined by the formula

$$\|u\|_{\mathcal{K}^{s,\gamma}} = \|(1 + r^{-2} + \Delta_{K_\Omega})^{s/2} \rho^{s-\gamma} u\|_{L^2(K_\Omega, d\mathrm{vol})},$$

in which Δ_{K_Ω} is the Beltrami–Laplace operator with respect to the conical metric and ρ is a smooth weight function equal to r in a neighborhood of $r = 0$ and equal to unity for large r.

3. Pseudodifferential operators and symbols. The paper [6] describes a calculus of pseudodifferential operators on manifolds with edges extending the calculus of edge-degenerate differential operators. A pseudodifferential operator D of order m in the spaces (1.3) has a well-defined *interior symbol*, which is a function $\sigma(D)$ homogeneous of degree m on the cotangent bundle $T_0^*\mathcal{M}$ of the manifold with edges without the zero section (the definition of $T^*\mathcal{M} \in \mathrm{Vect}(M)$ can be found in the cited paper) and a well-defined *edge symbol*, which is an operator-valued function

$$\sigma_\wedge(D)(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega), \quad (x,\xi) \in T_0^* X, \tag{1.4}$$

in the function space on the infinite wedge. The edge symbol possesses the *twisted homogeneity* property

$$\sigma_\wedge(D)(x, \lambda\xi) = \lambda^m \varkappa_\lambda \sigma_\wedge(D)(x,\xi) \varkappa_\lambda^{-1}, \quad \lambda \in \mathbb{R}_+. \tag{1.5}$$

In particular, the edge symbol of the operator (1.2) is equal to

$$D = \sum_{|\alpha|+|\beta|+j+l=m} a_{\alpha\beta jl}(0,\omega,x) \xi^\alpha \left(-\frac{i}{r} \frac{\partial}{\partial \omega} \right)^\beta \left(-i \frac{\partial}{\partial r} \right)^j \left(\frac{1}{r} \right)^l.$$

The notion of interior and edge symbols is important in that the following assertion holds.

Proposition 1.1. *A pseudodifferential operator D of order m in the spaces* (1.3) *is compact if and only if $\sigma(D) = 0$ and $\sigma_\wedge(D) = 0$.*

4. The composition theorem and ellipticity. The main property of the calculus of pseudodifferential operators on manifolds with edges is expressed by the composition theorem.

Theorem 1.2. *The composition of edge-degenerate pseudodifferential operators corresponds to the composition of their symbols* (*interior and edge*):

$$\sigma(D_1D_2) = \sigma(D_1)\sigma(D_2), \quad \sigma_\wedge(D_1D_2) = \sigma_\wedge(D_1)\sigma_\wedge(D_2).$$

In conjunction with the compactness criterion given by proposition 1.1, the composition formula results in the following finiteness theorem.

Theorem 1.3. *If a pseudodifferential operator D of order m acting in the space* (1.3) *is elliptic* (*i.e., its interior symbol is invertible everywhere outside the zero section on $T^*\mathcal{M}$ and the edge symbol is invertible in the spaces* (1.4) *everywhere outside the zero section on T^*X*), *then it is Fredholm.*

Naturally, all the preceding is also valid for operators acting in spaces of sections of vector bundles on M.

5. Order reduction. In this paper, we are interested in the classification of elliptic pseudodifferential operators (1.3) modulo stable homotopies. (The precise definition of stable homotopy will be given below.) Let us show that this problem can actually be reduced to the special case of zero-order operators in the spaces $\mathcal{W}^{s,\gamma}$ for $s = \gamma = 0$. Indeed, an operator D is elliptic for given s (and fixed γ) if and only if it is elliptic for any s (with the same γ); see, e.g., [6]. Consequently, a homotopy in the class of elliptic operators for some s is valid for all s. Hence without loss of generality we can assume that $s = \gamma$, i.e., consider the operators

$$D : \mathcal{W}^{\gamma,\gamma}(\mathcal{M}) \longrightarrow \mathcal{W}^{\gamma-m,\gamma-m}(\mathcal{M}). \tag{1.6}$$

Next, there exist elliptic operators

$$V : \mathcal{W}^{0,0}(\mathcal{M}) \longrightarrow \mathcal{W}^{\gamma,\gamma}(\mathcal{M}), \quad \widetilde{V} : \mathcal{W}^{\gamma-m,\gamma-m}(\mathcal{M}) \longrightarrow \mathcal{W}^{0,0}(\mathcal{M}) \tag{1.7}$$

of index zero. Then the mapping

$$D \longmapsto \widetilde{V}DV$$

reduces elliptic operators (and homotopies) in the spaces (1.6) to those in the space $\mathcal{W}^{0,0}(\mathcal{M})$. The inverse mapping (modulo compact operators, which plays no role if homotopies in the class of elliptic operators are considered) naturally has the form

$$D \longmapsto \widetilde{V}^{-1}DV^{-1},$$

where $\widetilde{V}^{-1}$ and V^{-1} are almost inverses of $\widetilde{V}$ and V, respectively.

6. Pseudodifferential operators of order zero. By virtue of order reduction, in what follows we are mainly interested in pseudodifferential operators of order zero in the space $\mathcal{W}^{0,0}(\mathcal{M})$. Hence we give a more detailed description of their construction and properties, mainly following [6] with some simplifications (related to the fact that the paper [6] deals with operators of arbitrary order and not only compactness, but also smoothing properties of remainders in composition formulas are taken into account). For simplicity, we present all facts for operators acting in function spaces on $\mathcal{M}$. The generalization to operators acting in spaces of sections of vector bundles on $\mathcal{M}$ is trivial.

Edge symbols. First, we describe the class of edge symbols used here.

Definition 1.4. An *edge symbol* is a family $D(x,\xi)$, $(x,\xi)\in T_0^*X$, of operators in function spaces on the cones K_Ω with the following properties.

1. For any multi-indices α,β, $|\alpha|+|\beta|=0,1,2,\dots$, the derivatives

$$D^{(\alpha,\beta)}(x,\xi):\mathcal{K}^{0,0}(K_\Omega)\longrightarrow\mathcal{K}^{0,0}(K_\Omega)$$

   are continuous operators.

2. The *twisted homogeneity condition*

$$D(x,\lambda\xi)=\varkappa_\lambda D(x,\xi)\varkappa_\lambda^{-1},\quad \lambda\in\mathbb{R}_+,$$

   holds.

3. Modulo compact operators, one has the representation

$$D(x,\xi)=d\left(x,\overset{2}{\xi r},ir\overset{1}{\frac{\partial}{\partial r}}\right),\tag{1.8}$$

   where $d(x,\eta,p)$ is a classical pseudodifferential operator with parameters $(\eta,p)\in T_x^*X\times\mathcal{L}_{-(k+1)/2}$ in the sense of Agranovich–Vishik [13] of order zero on Ω smoothly depending on the additional parameter $x\in X$.

Here $\mathcal{L}_{-(k+1)/2}=\{\operatorname{Im}p=-(k+1)/2\}$ is the weight line, and the function of the operator $ir\partial/\partial r$ in (1.8) is defined with the help of the Mellin transform on this weight line.

Definition 1.5. The *interior symbol* of the edge symbol $D(x,\xi)$ is the principal symbol $\sigma(D)=\sigma(d)$ in the sense of Agranovich–Vishik of the corresponding pseudodifferential operator $d(x,\eta,p)$. The *conormal symbol* of $D(x,\xi)$ is the operator family $\sigma_c(D)=d(x,0,p)$ in the space $L^2(\Omega)$.

The main properties of edge symbols are expressed by the following theorem.

Theorem 1.6. *The following assertions hold.*

1. *Definition* 1.4 *is consistent*; *i.e., the operator* (1.8) *is always bounded in* $\mathcal{K}^{0,0}(K_\Omega)$.

2. *The operator* (1.8) *is compact if and only if its interior and conormal symbol are zero.* (*In particular, it follows that the interior and conormal symbol of an edge symbol are well defined.*)

3. *Edge symbols form a local* C^**-algebra, and, modulo compact edge symbols, the product of pseudodifferential operators* $d(x,\eta,p)$ *corresponds to the product of the respective edge symbols* $D(x,\xi)$ *and the adjoint operator corresponds to the adjoint edge symbol.*

4. (*Corollary.*) *The mapping that takes each edge symbol to its interior and conormal symbols is linear and multiplicative and commutes with the passage to the adjoint operator.*

5. (*Norm estimates modulo compact operators.*) *For an edge symbol* $D(x,\xi)$ *of order zero, one has*[2]

$$\inf_{K\in C(S^*X,\mathcal{K})}\max_{S^*X}\|D+K\|_{\mathcal{B}(\mathcal{K}^{0,0}(K_\Omega))}=\max\left(\max_{\partial S^*\mathcal{M}}|\sigma(D)|,\sup_{X\times\mathbb{R}}\|\sigma_c(D)\|_{\mathcal{B}(L^2(\Omega))}\right). \tag{1.9}$$

6. *The commutator* $[D(x,\xi),\varphi]$ *is compact for any continuous function* $\varphi(r)$ *on* $\mathbb{R}_+$ *equal to zero for sufficiently large* r.

7. *The product* $\varphi(r)D(x,\xi)$ *is a symbol of order zero and has a compact fiber variation* [14] *on* T_0^*X *if* $\varphi(r)$ *is the same as in item* 6.

8. *For any pair* (*interior symbol, conormal symbol*) *satisfying the compatibility condition* ($\sigma(d)$ *for* $\eta=0$ *is equal to* $\sigma(\sigma_c)$) *one can construct an edge symbol.*

The algebra of zero-order edge symbols will be denoted by $\Psi_\wedge(X)$.

Pseudodifferential operators. Now we can describe the class of zero-order pseudodifferential operators.

Let $\mathcal{A}$ be the algebra of classical zero-order pseudodifferential operator A on the open manifold M° with the following properties:

1) the principal symbol $\sigma(A)$ is a smooth function on $T_0^*\mathcal{M}$ up to the boundary;

2) the operator A is continuous in the space $\mathcal{W}^{0,0}(\mathcal{M})$;

3) A compactly commutes with $C(\mathcal{M})$.

[2]Here and in the following, $\mathcal{K}$ is either the ideal of compact operators or (where there is a bundle) a bundle of algebras of compact operators acting in function spaces on the fibers. Likewise, $\mathcal{B}$ corresponds to the algebra of bounded operators.

Definition 1.7. A *zero-order pseudodifferential operator* on $\mathcal{M}$ is a continuous operator

$$B : \mathcal{W}^{0,0}(\mathcal{M}) \longrightarrow \mathcal{W}^{0,0}(\mathcal{M}) \tag{1.10}$$

representable modulo compact operators in the form

$$B = (\varphi(r)D)\left(\overset{2}{x}, -i\frac{\overset{1}{\partial}}{\partial x}\right) + rA, \tag{1.11}$$

where $D(x,\xi) \in \Psi_\wedge(X)$ is an edge symbol, $\varphi(r)$ is a smooth function on M equal to 1 on ∂M and zero outside a sufficiently small neighborhood of the boundary, and $A \in \mathcal{A}$ is a classical pseudodifferential operator on M°. (Here r is the distance to ∂M.)

Definition 1.8. The *edge symbol* of the operator (1.10) is the operator family

$$\sigma_\wedge(B) = D(x,\xi). \tag{1.12}$$

The *interior symbol* of the operator (1.10) is the function

$$\sigma(B) = \varphi(r)\sigma(D) + r\sigma(A). \tag{1.13}$$

The set of zero-order pseudodifferential operator on $\mathcal{M}$ will be denoted by $\Psi(\mathcal{M})$. The main properties of pseudodifferential operators are expressed by the following theorem.

Theorem 1.9. *The following assertions hold.*

1. *The definition is consistent*; *i.e., the operator* (1.11) *is always bounded in* $\mathcal{W}^{0,0}(\mathcal{M})$.

2. *The interior and edge symbols of the operator* (1.11) *are well defined. It is compact if and only if both symbols are zero.*

3. *The interior and edge symbols satisfy the compatibility condition*

$$\sigma(B)|_{\partial T_0^*\mathcal{M}} = \sigma(\sigma_\wedge(B)). \tag{1.14}$$

   For any pair (*interior symbol, edge symbol*) *satisfying the compatibility condition, one can construct the corresponding pseudodifferential operator.*

4. *Pseudodifferential operators form a local* C^**-algebra, and the mapping taking each pseudodifferential operator to its interior and edge symbols is a* $*$*-homomorphism.*

5. (*Norm estimates modulo compact operators.*) *For a zero-order edge-degenerate pseudodifferential operator* D *one has*

$$\inf_{K\in\mathcal{K}} \|D+K\|_{\mathcal{B}(\mathcal{W}^{0,0}(\mathcal{M}))} = \max\left(\max_{S^*\mathcal{M}} |\sigma(D)|, \max_{S^*X} \|\sigma_\wedge(D)\|_{\mathcal{B}(\mathcal{K}^{0,0}(K_\Omega))}\right) \tag{1.15}$$

6. *If* $B \in \Psi(\mathcal{M})$, *then the commutator* $[B,\varphi]$ *is compact for any function* $\varphi \in C(\mathcal{M})$.

The norm closure. The norm estimates modulo compact operators imply the following description of the Calkin algebras of the closures $\overline{\Psi(\mathcal{M})}$ and $\overline{\Psi_\Lambda(X)}$.

Corollary 1.10. *The interior and edge symbol homomorphisms (for operators on* $\mathcal{M}$*)*

$$\Psi(\mathcal{M}) \overset{\sigma}{\to} C^\infty(S^*\mathcal{M}), \quad \Psi(\mathcal{M}) \overset{\sigma_\Lambda}{\to} \Psi_\Lambda(X),$$

induce the isomorphism

$$\overline{\Psi(\mathcal{M})}/\mathcal{K} \simeq \left\{ (a, a_\Lambda) \middle| \begin{array}{c} a \in C(S^*\mathcal{M}), \quad a_\Lambda \in \overline{\Psi_\Lambda(X)} : \\ a|_{\partial S^*\mathcal{M}} = \sigma(a_\Lambda) \end{array} \right\}.$$

The interior and conormal symbol homomorphisms (for edge symbols)

$$\Psi_\Lambda(X) \overset{\sigma}{\to} C^\infty(\partial S^*\mathcal{M}), \quad \Psi_\Lambda(X) \overset{\sigma_c}{\to} \Psi_c(X),$$

where Ψ_c *is the algebra of conormal symbols, induce the isomorphism*

$$\overline{\Psi_\Lambda(X)}/C(S^*X, \mathcal{K}) \simeq \left\{ (a, a_c) \middle| \begin{array}{c} a \in C(\partial S^*\mathcal{M}), \quad a_c \in \overline{\Psi_c(X)} : \\ a|_{S(T^*X \oplus \mathbf{1})} = \sigma(a_c) \end{array} \right\}.$$

2 An Element in the K-Homology of the Singular Space

In this section, we show how an elliptic operator on a manifold with edges gives rise to an element in the analytic K-homology of the space $\mathcal{M}$. (A detailed exposition of the theory of analytic K-homology can be found in [15], [16], and [8]. An introduction to the theory can be found in [17].) To the best of the authors' knowledge, this correspondence was used for the first time in [18]. In accordance with the preceding, we consider only zero-order operators.

Let

$$D : \mathcal{W}^{0,0}(\mathcal{M}, E) \longrightarrow \mathcal{W}^{0,0}(\mathcal{M}, F)$$

be an elliptic operator of order zero in sections of vector bundles $E, F \in \mathrm{Vect}(M)$. The commutator $[D, f]$ with a function $f \in C^\infty(M)$ is compact if the restrictions of the function to the fibers of π are constant functions. (This follows from the composition formula.) Thus D is a generalized elliptic operator on $\mathcal{M}$ in the sense of Atiyah [3] and hence defines a class in the analytic K-homology $K_0(\mathcal{M})$ of the singular space $\mathcal{M}$. Let us give a precise construction of the corresponding cycle.

If D is self-adjoint (and $E = F$), then we consider the *normalization*

$$\mathcal{D} = (P_{\ker D} + D^2)^{-1/2} D : \mathcal{W}^{0,0}(\mathcal{M}, E) \longrightarrow \mathcal{W}^{0,0}(\mathcal{M}, E), \tag{2.1}$$

where $P_{\ker D}$ is the orthogonal projection on the kernel.

In the general case, we consider the self-adjoint operator

$$\mathcal{D} = \begin{pmatrix} 0 & D(P_{\ker D} + D^*D)^{-1/2} \\ (P_{\ker D} + D^*D)^{-1/2}D^* & 0 \end{pmatrix} : \mathcal{W}^{0,0}(\mathcal{M}, E\oplus F) \to \mathcal{W}^{0,0}(\mathcal{M}, E\oplus F), \tag{2.2}$$

which is odd with respect to the $\mathbb{Z}_2$-grading of the space $\mathcal{W}^{0,0}(\mathcal{M}, E) \oplus \mathcal{W}^{0,0}(\mathcal{M}, F)$. By $C(\mathcal{M})$ we denote the algebra of continuous functions on $\mathcal{M}$.

Proposition 2.1. *The operators* (2.1) *and* (2.2) *are zero-order elliptic pseudodifferential operators and define elements in* K*-homology*; *these elements will be denoted by*

$$[D] \in K_*(\mathcal{M}),$$

where $* = 1$ *for self-adjoint operators and* $* = 0$ *in the general case.*

Proof. 1. The operator D^*D is pseudodifferential, and the same is true for $P_{\ker D}$, since the latter operator is finite rank and hence compact. Thus $D^*D+P_{\ker D}$ is a pseudodifferential operator, and since it is invertible, it follows that the inverse is also a pseudodifferential operator (recall that $\overline{\Psi(\mathcal{M})}$ is a C^*-algebra). To prove that $(D^*D + P_{\ker D})^{-1/2}$ is a pseudodifferential operator, it remains to use the formula

$$A^{-1/2} = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(A+\lambda)^{-1}d\lambda$$

for a self-adjoint strongly positive operator A (e.g., see [16, p. 165]).

2. The operators $\mathcal{D}$ in (2.1) and (2.2) are self-adjoint operators acting in a $*$-module over the C^*-algebra $C(\mathcal{M})$ and have the properties

$$[\mathcal{D}, f] \in \mathcal{K}, \qquad (\mathcal{D}^2 - 1)f \in \mathcal{K}.$$

Thus we have the Fredholm modules

$$[D] = \begin{cases} [\mathcal{D}, \mathcal{W}^{0,0}(\mathcal{M}, E)] \in K_1(\mathcal{M}) & \text{if } D = D^* \\ [\mathcal{D}, \mathcal{W}^{0,0}(\mathcal{M}, E\oplus F)] \in K_0(\mathcal{M}) & \text{if } D \neq D^*. \end{cases} \tag{2.3}$$

□

Remark 2.2. If only the interior symbol of D is elliptic, then there is a well-defined element in the K-homology of the open smooth part of $\mathcal{M}$:

$$[D] \in K_*(\mathcal{M} \setminus X). \tag{2.4}$$

Here in the definition of the elements (2.1) and (2.2) one should replace the expression $(P_{\ker D}+D^*D)^{-1/2}$ by an arbitrary self-adjoint edge-degenerate pseudodifferential operator with interior symbol $(\sigma(D)^*\sigma(D))^{-1/2}$.

3 The Homotopy Classification

We recall the standard equivalence relation on the set of pseudodifferential operator acting in sections of vector bundles, namely, *stable homotopy.*

Definition 3.1. Two operators

$$D : \mathcal{W}^{0,0}(\mathcal{M}, E) \to \mathcal{W}^{0,0}(\mathcal{M}, F), \quad D' : \mathcal{W}^{0,0}(\mathcal{M}, E') \to \mathcal{W}^{0,0}(\mathcal{M}, F')$$

are said to be *stably homotopic* if they are homotopic modulo stabilization by vector bundle isomorphism, i.e., if there exists an continuous homotopy of elliptic operators

$$D \oplus 1_{E_0} \sim f^*\big(D' \oplus 1_{F_0}\big)e^*,$$

where $E_0, F_0 \in \mathrm{Vect}(M)$ are vector bundles and

$$e : E \oplus E_0 \longrightarrow E' \oplus F_0, \qquad f : F' \oplus F_0 \longrightarrow F \oplus E_0$$

are vector bundle isomorphisms.

Stable homotopy is an equivalence relation on the set of all elliptic edge-degenerate pseudodifferential operators acting in sections of vector bundles. By $\mathrm{Ell}_0(\mathcal{M})$ we denote the set of elliptic operators modulo stable homotopies. This set is a group with respect to the direct sum of elliptic operators. The inverse element corresponds to an almost inverse (i.e., an inverse modulo compact operators), and the unit is the equivalence class of trivial operators.

In a similar way, one defines odd elliptic theory $\mathrm{Ell}_1(\mathcal{M})$ as the group of stable homotopy equivalence classes of elliptic self-adjoint operators. Here the class of trivial operators consists of Hermitian isomorphisms of vector bundles.

The *homotopy classification problem for elliptic operators* on the manifold $\mathcal{M}$ is the problem of computing the groups $\mathrm{Ell}_*(\mathcal{M})$.

The following theorem solves the classification problem for manifolds with edges and is the main result of the paper.

Theorem 3.2. *There is an isomorphism*

$$\mathrm{Ell}_*(\mathcal{M}) \overset{\wp}{\simeq} K_*(\mathcal{M}),$$

which takes each elliptic operator D to the element defined in Proposition 2.1.

Corollary 3.3. *Two elliptic operators D_1 and D_2 are stably rationally homotopic if and only if they have the same indices with coefficients in an arbitrary bundle on $\mathcal{M}$:*

$$\mathrm{ind}(1 \otimes p)(D_1 \otimes 1_N)(1 \otimes p) = \mathrm{ind}(1 \otimes p)(D_2 \otimes 1_N)(1 \otimes p), \tag{3.1}$$

where $p \in \mathrm{Mat}(N \times N, C(\mathcal{M}))$ is a matrix projection.

This follows from the nondegeneracy (on the free parts of the groups) of the natural pairing $K_0(\mathcal{M}) \times K^0(\mathcal{M}) \longrightarrow \mathbb{Z}$, which is just defined by the formula (3.1).

Proof of the theorem. The mapping is well defined, since homotopies of elliptic operators give rise to homotopies of the corresponding Fredholm modules, i.e., result in the same element in K-homology. Bundle isomorphisms give degenerate modules.

Let us now prove that the mapping is an isomorphism. We split the proof into three stages.

1. Reduction of Ell**-groups to** K**-groups of** C^***-algebras (see [2]).** We interpret edge-degenerate operators in sections of vector bundles as operator generated by the pair of algebras

$$C^\infty(M) \subset \Psi(\mathcal{M})$$

of scalar operators. The embedding corresponds to the conventional action of functions as multiplication operators. Namely, an arbitrary edge-degenerate pseudodifferential operator of order zero can be represented in the form

$$D' : \operatorname{Im} P \longrightarrow \operatorname{Im} Q,$$

where $P = P^2$ and $Q = Q^2$ are matrix projections with coefficients in the function algebra $C^\infty(M)$ and D' is a matrix operator whose entries belong to the operator algebra $\Psi(\mathcal{M})$.

We obtain a group isomorphic to $\mathrm{Ell}(\mathcal{M})$ if, instead of operators with smooth symbols, we consider operators whose symbols are only *continuous*, i.e., pass to the closure $\overline{\Psi(\mathcal{M})}$ of the algebra of pseudodifferential operators with respect to the operator norm. (The fact that these groups are isomorphic follows from Theorem 1.9). By $\Sigma \stackrel{\text{def}}{=} \overline{\Psi(\mathcal{M})}/\mathcal{K}$ we denote the algebra of continuous symbols.

The results of [2] give the isomorphisms[3]

$$\mathrm{Ell}_*(\mathcal{M}) \stackrel{\chi}{\simeq} K_*(\mathrm{Con}(C(M) \to \Sigma)).$$

Here

$$\mathrm{Con}(A \stackrel{f}{\longrightarrow} B) = \Big\{(a, b(t)) \in A \oplus C_0([0,1), B) \mid f(a) = b(0)\Big\}$$

is the cone of the algebra homomorphism $f : A \to B$. In the odd case, one can rewrite the K-group in the form

$$K_1(\mathrm{Con}(C(M) \to \Sigma)) \simeq K_0(\Sigma)/K^0(M).$$

The composition of the last isomorphism with χ is a generalization of the Atiyah–Patodi–Singer isomorphism [19]; i.e., self-adjoint elliptic operators modulo stable homotopy are isomorphic to symbols-projections modulo projections determining sections of bundles.

[3]Note that if the edges are absent, then the isomorphism is just the Atiyah–Singer difference construction [1].

2. A diagram relating K-theory of algebras and K-homology. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0\to & \Sigma_0 & \longrightarrow & \Sigma & \stackrel{\sigma}{\longrightarrow} & C(S^*\mathcal{M}) & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0\to & 0 & \longrightarrow & C(M) & = & C(M) & \to 0 \end{array} \qquad (3.2)$$

with exact rows. Here σ is the interior symbol and $\Sigma_0 = \ker\sigma$ is the corresponding ideal. The diagram induces the exact sequence

$$0 \to S\Sigma_0 \longrightarrow \operatorname{Con}(C(M)\to\Sigma) \longrightarrow C_0(T^*\mathcal{M}) \to 0 \qquad (3.3)$$

of the cones of the vertical homomorphisms. Here S stands for the suspension: $S\Sigma_0 = C_0((0,1),\Sigma_0)$.

For brevity, we set $A = \operatorname{Con}(C(M)\to\Sigma)$.

The key point of the proof is the construction of the commutative diagram

$$\begin{array}{ccccccccc} K_c^{*+1}(T^*\mathcal{M}) & \stackrel{\partial}{\to} & K_*(S\Sigma_0) & \to & K_*(A) & \to & K_c^*(T^*\mathcal{M}) & \stackrel{\partial}{\to} & K_{*+1}(S\Sigma_0) \\ \downarrow & \boxed{A} & \downarrow & \boxed{B} & \downarrow & \boxed{C} & \downarrow & \boxed{D} & \downarrow \\ K_{*+1}(M\setminus\partial M) & \stackrel{\partial}{\to} & K_*(X) & \to & K_*(\mathcal{M}) & \to & K_*(M\setminus\partial M) & \to & K_{*+1}(X) \end{array} \qquad (3.4)$$

relating the exact sequence induced by (3.3) in K-theory of algebras to the exact sequence of the pair $X\subset\mathcal{M}$ in K-homology.

The vertical arrows in (3.4) are induced by quantizations. Namely, the elements of K-groups in the upper row correspond to some symbols, and the vertical mappings take these symbols to the corresponding operators. More precisely,

- the mappings $K_c^*(T^*\mathcal{M})\to K_*(M\setminus\partial M)$ and $K_*(A)\to K_*(\mathcal{M})$ are determined by quantizations (see the preceding section);
- the mappings $K_*(S\Sigma_0)\longrightarrow K_*(X)$ are induced by quantization of edge symbols. In more detail, the mapping $K_1(\Sigma_0)\longrightarrow K_0(X)$ takes an edge symbol $1+u(x,\xi)$, $u\in\Sigma_0$, invertible on S^*X to the operator

$$1+u\left(x,-i\frac{\partial}{\partial x}\right) \qquad (3.5)$$

in the space $L^2(X,\mathcal{K}^{0,0}(K_\Omega))$ on the infinite wedge. The quantization is well defined, since $u(x,\xi)$ has a compact fiber variation (the interior symbol of the edge symbol is zero). The mapping $K_0(\Sigma_0)\longrightarrow K_1(X)$ takes a self-adjoint edge symbol-projection $p(x,\xi)$ to the operator

$$2p\left(x,-i\frac{\partial}{\partial x}\right)-1. \qquad (3.6)$$

3. An application of the five lemma. Once we prove that the diagram (3.4) commutes, it follows from the five lemma that the middle vertical arrow is an isomorphism, since the quantizations $K_c^*(T^*\mathcal{M}) \to K_*(M\backslash\partial M)$ are defined on the interior of the smooth manifold M with boundary and are isomorphisms (e.g., see [8]). The isomorphism of edge quantizations $K_*(S\Sigma_0) \to K_*(X)$ will be established below in Sec. 4.

The fact that the diagram commutes can be established as follows:

- The square $\boxed{C}$ in (3.4) commutes automatically, since the horizontal arrows are forgetful mappings;
- The fact that the square $\boxed{B}$ commutes will be proved in Sec. 4.
- The fact that the square $\boxed{A}$, including the boundary maps, commutes will be verified in Sec. 5.

This completes the proof of the theorem up to the above-mentioned computations. $\square$

4 Computations on the Edge

1. The isomorphism $K_*(\Sigma_0) \simeq K_{*+1}(X)$. Consider the diagram

$$\begin{array}{ccccccc} \to & K^*(S^*X) & \to & K_*(\Sigma_0) & \to & K_c^*(X\times\mathbb{R}) & \to \\ & \| & & \downarrow L & & \| & \\ \to & K^*(S^*X) & \to & K_c^{*+1}(T^*X) & \to & K_c^*(X\times\mathbb{R}) & \to, \end{array} \tag{4.1}$$

which compares the K-theory sequence corresponding to the short exact sequence

$$0 \to C(S^*X,\mathcal{K}) \longrightarrow \Sigma_0 \xrightarrow{\sigma_c} C_0(X\times\mathbb{R},\mathcal{K}) \to 0 \tag{4.2}$$

(where σ_c is the conormal symbol) with the sequence of K-groups of the pair $S^*X \subset B^*X$ formed by the unit sphere and ball bundles in T^*X. The mapping L is the difference constructions for pseudodifferential operators (3.5), (3.6) with operator-valued symbols in the sense of Luke (see [14] and [20]).

Lemma 4.1. *The diagram* (4.1) *commutes.*

Proof. 1. The commutativity of the squares

$$\begin{array}{ccc} K^*(S^*X) & \longrightarrow & K_*(\Sigma_0) \\ \| & & \downarrow L \\ K^*(S^*X) & \longrightarrow & K^{*+1}(T^*X) \end{array}$$

follows from the fact that for finite-dimensional symbols the difference constriction coincides with the Atiyah–Singer difference constriction.

2. The commutativity of the squares

$$\begin{array}{ccc} K_*(\Sigma_0) & \stackrel{\sigma_c}{\longrightarrow} & K^*_c(X\times\mathbb{R}) \\ \downarrow L & & \| \\ K^{*+1}_c(T^*X) & \stackrel{j^*}{\longrightarrow} & K^{*+1}(X), \end{array} \qquad j: X\to T^*X,$$

follows from the index formula [21]

$$\beta\,\mathrm{ind}\, D_y = \mathrm{ind}\,\sigma_c(D_y) \in K^1_c(Y\times\mathbb{R}) \tag{4.3}$$

for a family of elliptic operators D_y, $y\in Y$, with unit interior symbol on the infinite cone. Here Y is a compact parameter space and β is the periodicity isomorphism $K(Y)\simeq K^1_c(Y\times\mathbb{R})$.

The formula (4.3) applies directly to the group $K_1(\Sigma_0)$, and for the K_0-group one uses the suspension (cf. (3.6)).

3. The commutativity of the squares

$$\begin{array}{ccc} K^{*+1}_c(X\times\mathbb{R}) & \stackrel{\partial}{\longrightarrow} & K^*(S^*X) \\ \| & & \| \\ K^*(X) & \stackrel{p^*}{\longrightarrow} & K^*(S^*X) \end{array}, \qquad p: S^*X\to X,$$

also follows from the above-mentioned index formula, since the boundary mapping in K-theory of algebras is an index mapping. We leave details to the reader. □

By applying the five lemma, we arrive at the desired corollary.

Corollary 4.2. *The quantization* $K_*(\Sigma_0)\to K_{*+1}(X)$ *is an isomorphism*: $K_*(\Sigma_0)\simeq K^{*+1}_c(T^*X)\simeq K_{*+1}(X)$.

2. Commutativity of the square $\boxed{B}$. To be definite, we consider the even case. The odd case can be treated in a similar way.

The image of the composite mapping $K_0(S\Sigma_0)\to K_0(A)\to K_0(\mathcal{M})$ corresponds to elliptic operators on $\mathcal{M}$ of the form $1+\mathbf{G}$, where $\mathbf{G}$ is an operator with zero interior symbol. We must show that the element

$$[1+\mathbf{G}:\mathcal{W}^{0,0}(\mathcal{M})\to\mathcal{W}^{0,0}(\mathcal{M})]\in K_0(\mathcal{M})$$

coincides with the element

$$\left[1+g\left(x,-i\frac{\partial}{\partial x}\right):\mathcal{W}^{0,0}(W)\longrightarrow\mathcal{W}^{0,0}(W)\right]\in K_0(\mathcal{M})$$

determined by the operator on the infinite wedge W with symbol $g(x,\xi)=\sigma_\Lambda(\mathbf{G})$. For the latter operator, the module structure on the spaces is defined as follows: a function $f\in C(\mathcal{M})$ acts as the multiplication by its restriction to the edge.

The equality of these two elements can be established in two steps:

1. (cutting away the smooth part of the manifold) without changing the element of the K-homology group, one can proceed to the restriction of the operator $1+\mathbf{G}$ to a neighborhood of the edge (the original operator and its restriction define stably equivalent Fredholm modules and hence the same elements in $K_0(\mathcal{M})$);

2. (homotopy of the module structure) in the neighborhood of the edge, the module structure $f, u \mapsto fu$ cam be homotopied to $f, u \mapsto f(x,0)u(x,\omega,r)$ by scaling represented by the formula $f(x,\omega,\varepsilon r)u(x,\omega,r)$ in local coordinates (x,ω,r).

We have established that the square $\boxed{\mathrm{B}}$ commutes.

5 Comparison of the Boundary Mappings

Let us verify that the boundary mappings in K-theory of algebras and K-homology of spaces are compatible.

1. First, we show that the boundary mappings can be defined in terms of the restriction of structures to the boundary. Consider the commutative diagram of C^*-algebras

$$\begin{array}{ccccccc} 0\to & S\Sigma_0 & \to & A & \to & C_0(T^*\mathcal{M}) & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0\to & S\Sigma_0 & \to & \mathrm{Con}(C(\partial M)\to \overline{\Psi_\Lambda(X)}) & \to & C_0(\partial T^*\mathcal{M}) & \to 0 \end{array}$$

relating two short exact sequences. Next, we consider the diagram of spaces and continuous mappings

$$\begin{array}{ccc} \partial M & \subset & M \\ \pi\downarrow & & \downarrow \\ X & \subset & \mathcal{M}. \end{array}$$

Since the boundary mapping is natural, we have the following lemma.

Lemma 5.1. *The diagrams*

$$\begin{array}{ccc} K_c^*(T^*\mathcal{M}) & \stackrel{\partial}{\longrightarrow} & K_{*+1}(S\Sigma_0) \\ \downarrow & & \| \\ K_c^*(\partial T^*\mathcal{M}) & \stackrel{\partial'}{\longrightarrow} & K_{*+1}(S\Sigma_0), \end{array} \tag{5.1}$$

$$\begin{array}{ccc} K_*(M\setminus\partial M) & \stackrel{\partial}{\longrightarrow} & K_{*+1}(X) \\ \partial'\downarrow & & \| \\ K_{*+1}(\partial M) & \stackrel{\pi_*}{\longrightarrow} & K_{*+1}(X) \end{array}, \quad \pi:\partial M\longrightarrow X, \tag{5.2}$$

commute.

2. To compute the boundary mappings $K_c^*(\partial T^*\mathcal{M}) \xrightarrow{\partial'} K_{*+1}(S\Sigma_0)$, we reduce the algebra of edge symbols to the simpler algebra of operator families on the fibers Ω with parameters in the sense of Agranovich–Vishik [13]). We relate these two algebras by the semiclassical quantization method (e.g., see [9]).

In our case, semiclassical quantization is a family of linear mappings (see the Appendix)

$$T_h : \Psi_{T^*X\times\mathbb{R}}(\Omega) \longrightarrow \Sigma|_{\partial M}, \quad h \in (0,1],$$

$$(T_h u)(\xi) = u\left(\overset{2}{r}\,\xi, ih\,\overset{1}{r\frac{\partial}{\partial r}} + ih\frac{n+1}{2}\right), \quad \xi \in T^*X, \tag{5.3}$$

where by $\Psi_{T^*X\times\mathbb{R}}(\Omega)$ we denote the algebra of smooth families of pseudodifferential operators on the fibers Ω with parameters in $T^*X\times\mathbb{R}$; these mappings satisfy the following relations as $h \to 0$:

$$T_h(ab) = T_h(a)T_h(b) + o(1), \quad (T_h(a))^* = T_h(a^*) + o(1),$$

where $a, b \in \Psi_{T^*X\times\mathbb{R}}(\Omega)$ are arbitrary elements and the estimate $o(1)$ holds in the operator norm. This semiclassical quantization is a special case of so-called *asymptotic homomorphisms*, which play an important role in the theory of C^*-algebras [22], [23], [24]. In particular, it follows that the quantization T_h induces the K-group homomorphism

$$T : K_*(\overline{\Psi_{T^*X\times\mathbb{R}}(\Omega)}) \to K_*(\overline{\Sigma|_{\partial M}}).$$

Here we have used the fact that the algebras of operators with smooth symbols in question are subalgebras in their closures and are closed with respect to holomorphic functional calculus.

Consider the commutative diagram

$$\begin{array}{ccccccc}
0 \to & \Psi^{-1}_{T^*X\times\mathbb{R}}(\Omega) & \to & \Psi_{T^*X\times\mathbb{R}}(\Omega) & \to & C^\infty(\partial S^*\mathcal{M}) & \to 0 \\
& \downarrow T_h & & \downarrow T_h & & \downarrow t_h & \\
0 \to & \Sigma_0 & \to & \Sigma|_{\partial M} & \to & C(\partial S^*\mathcal{M}) & \to 0,
\end{array} \tag{5.4}$$

where on the ideal $\Psi^{-1}_{T^*X\times\mathbb{R}}(\Omega)$ consisting of families of order ≤ -1 we consider the restriction of the mapping T_h and by t_h we denote the homomorphism on principal symbols.

The algebra $C^\infty(\partial M)$ is embedded in each of the algebras $\Psi_{T^*X\times\mathbb{R}}(\Omega)$, $C^\infty(\partial S^*\mathcal{M})$, and $\Sigma|_{\partial M}$. The diagram formed by the cones of these embeddings gives the square

$$\begin{array}{ccc}
K_c^*(\partial T^*\mathcal{M}) & \xrightarrow{\partial''} & K_c^*(T^*X\times\mathbb{R}) \\
\| & & \downarrow T \\
K_c^*(\partial T^*\mathcal{M}) & \xrightarrow{\partial'} & K_*(\Sigma_0)
\end{array} \tag{5.5}$$

of K-groups. (The mappings in this diagram are ordinary homomorphisms.) Here we have used the isomorphism $K_*(\Psi^{-1}_{T^*X\times\mathbb{R}}(\Omega)) \simeq K_*(C_0(T^*X\times\mathbb{R}))$. The commutativity in (5.5) follows from the fact that the boundary mapping is natural with respect to asymptotic homomorphisms. (We leave the easy verification of this fact to the reader; cf. [23].)

We embed the square (5.5) in the diagram

$$
\begin{array}{ccc}
K_{*+1}(\partial M) & \xrightarrow{\pi_*} & K_{*+1}(X) \\
\uparrow & & \uparrow \\
K_c^*(\partial T^*\mathcal{M}) & \xrightarrow{\partial''} & K_c^*(T^*X\times\mathbb{R}) \\
\| & & \downarrow T \\
K_c^*(\partial T^*\mathcal{M}) & \xrightarrow{\partial'} & K_*(\Sigma_0)
\end{array}
$$

involving K-homology groups. The upper square in the diagram commutes. Indeed, the boundary mapping ∂'' is given by the index of families with parameters ranging in $T^*X\times\mathbb{R}$. This index can be computed by the index theorem in [10]: $\partial''=\pi_!$, where

$$\pi_!: K_c^*(T^*\partial M)\longrightarrow K_c^*(T^*X)$$

is the direct image mapping in topological K-theory. It is also known [7] that the direct image $\pi_!$ gives the induced mapping π_* after the passage to K-homology.

The commutativity of the square $\boxed{\mathrm{A}}$ in (3.4) will be proved once we establish that the homomorphism T is a K-group isomorphism inverse to the isomorphism L described in the preceding section.

Lemma 5.2. *The mapping*

$$T: K_*(C_0(T^*X\times\mathbb{R}))\longrightarrow K_*(\Sigma_0)$$

is the isomorphism and the inverse of L*, see* (3.5).

Proof. To be definite, consider the mapping

$$T: K_1(C_0(T^*X\times\mathbb{R},\mathcal{K}))\longrightarrow K_1(\Sigma_0).$$

Let us prove that the composition LT gives the identity mapping of the K-group.

1. Indeed, on a symbol $u(\xi,p)$ invertible for $(\xi,p)\in T^*X\times\mathbb{R}$ and identically equal to unity outside a compact set, this mapping is defined by the formula

$$LT[u]=\operatorname{ind} T_h u\in K_c^0(T^*X),\quad [u]\in K_c^1(T^*X\times\mathbb{R}),$$

for sufficiently small h. The index element is well defined, since the operator-valued function$(T_hu)(\xi)$ has a compact fiber variation on $T^*X\setminus\mathbf{0}$ and is invertible for $\xi\neq 0$.

2. By $\overline{T_h u}$ we denote the family of Fredholm operators on $T^*X\setminus\mathbf{0}$ coinciding with the family T_hu for $|\xi|<1$ and equal to the family

$$u\left((\overset{2}{r}+|\xi|-1)\xi, ih\,\overset{1}{r\frac{\partial}{\partial r}}+ih\frac{n+1}{2}\right)$$

for $|\xi| \geq 1$. For sufficiently small h, this family will be invertible for all ξ (this follows from the boundedness of the support of $1-u$ and also from the fact that the estimates (7.7) hold uniformly with respect to the parameter λ, const $> \lambda \geq 0$, for the symbol $u(\xi(e^{-t}+\lambda), p))$. By construction, we have

$$\operatorname{ind} T_h u = \operatorname{ind} \overline{T_h u}.$$

But the family $\overline{T_h u}$ consists of identity operators for ξ outside a compact set. Hence its index can be computed by formula (4.3) and is equal to the index of the family of conormal symbols $u(\xi, p)$ (modulo periodicity), i.e., indeed gives the original element

$$LT[u] = [u]$$

of the K-group. □

Remark 5.3. The introduction of the semiclassical parameter $h \to 0$ can also be viewed as an adiabatic limit (e.g., see [25]) reducing the study of edge symbols as operators on the infinite cone K_Ω to families with parameters of operators on sections of the cone. Note that families with parameters play a role similar to that of the edge symbol in the theory of elliptic operators on manifolds with fibered cusps [26]. It would be of interest to clarify the relationship not only between symbols, but also between operators in these two theories.

6 Some Remarks

1. The classification of edge morphisms. Let us show that the classification of elliptic edge morphisms with conditions and co-conditions on the edge (e.g., see [6]) follows from the classification of edge operators. By $\widetilde{\Sigma} \supset \Sigma$ we denote the symbol algebra corresponding to the algebra of zero-order pseudodifferential operators with edge and co-edge conditions and with pseudodifferential operators on the edge.

Lemma 6.1. *The embedding* $\operatorname{Con}(C(M) \to \Sigma) \subset \operatorname{Con}(C(M) \to \widetilde{\Sigma})$ *induces an isomorphism in K-theory. The embedding* $\Sigma_0 \subset \widetilde{\Sigma}_0$ *has a similar property.*

Proof. The first fact follows from the commutative diagram

$$\begin{array}{ccccc} SC(S^*X, \mathcal{K}) & \to & \operatorname{Con}(C(M) \to \Sigma) & \stackrel{(\sigma,\sigma_c)}{\longrightarrow} & \operatorname{Im}(\sigma, \sigma_c) \\ \cap & & \cap & & \| \\ SC(S^*X, \widetilde{\mathcal{K}}) & \to & \operatorname{Con}(C(M) \to \widetilde{\Sigma}) & \stackrel{(\sigma,\sigma_c)}{\longrightarrow} & \operatorname{Im}(\sigma, \sigma_c) \end{array}$$

(the projections correspond to taking the interior and conormal symbols), since the left vertical embedding induces an isomorphism of K-groups. Here by $\widetilde{\mathcal{K}}$ we denote the algebra of compact operators in the direct sum $\mathcal{K}^{0,0}(K_\Omega) \oplus \mathbb{C}$, and the embedding is induced by the representation of operators in $\mathcal{K}$ as the upper left corner of the matrix.

The second isomorphism $K_*(\Sigma_0) \simeq K_*(\widetilde{\Sigma}_0)$ can be obtained in a similar way. □

We set $\widetilde{A} = \text{Con}(C(M)\oplus C(X) \to \widetilde{\Sigma})$. The group $K_0(\widetilde{A})$ classifies the stable homotopy classes of elliptic *edge problems* (see [6])

$$\mathcal{D} = \begin{pmatrix} D & C \\ B & P \end{pmatrix} : \mathcal{W}^{0,0}(\mathcal{M}, E) \oplus L^2(X, E_0) \longrightarrow \mathcal{W}^{0,0}(\mathcal{M}, F) \oplus L^2(X, F_0),$$

$E, F \in \text{Vect}(M)$, $E_0, F_0 \in \text{Vect}(X)$.

Proposition 6.2. *One has the isomorphism* $K_0(\widetilde{A}) \simeq K_0(A) \oplus K^0(X)$.

Proof. Consider the sequence

$$0 \to \text{Con}(C(M) \to \widetilde{\Sigma}) \subset \text{Con}(C(M) \oplus C(X) \to \widetilde{\Sigma}) \to C(X) \to 0.$$

The corresponding sequence

$$\ldots \to K_*(A) \longrightarrow K_*(\widetilde{A}) \longrightarrow K^*(X) \to \ldots \tag{6.1}$$

of K-groups (here we have used Lemma 6.1) splits.

First, let us indicate a splitting $j : K^0(X) \to K_0(\widetilde{A})$. To this end, we choose an edge symbol

$$\sigma_\Lambda^0(x) : \mathcal{K}^{0,0}(K_{\Omega_x}) \longrightarrow \mathcal{K}^{0,0}(K_{\Omega_x})$$

such that for all $(x, \xi) \in T^*X \setminus \mathbf{0}$ it is Fredholm, has a one-dimensional kernel, a trivial cokernel, and unit principal symbol (e.g., see [21]).

Now we define an element $j[E]$ for each vector bundle $E \in \text{Vect}(X)$ as the following symbol:

$$a = 1, a_\Lambda = \begin{pmatrix} \sigma_\Lambda^0 \otimes 1_E \oplus 1 \\ i^* \end{pmatrix} : \mathcal{K}^{0,0}(K_\Omega, E \oplus E^\perp) \longrightarrow \begin{matrix} \mathcal{K}^{0,0}(K_\Omega, E \oplus E^\perp) \\ \oplus \\ E \end{matrix},$$

where by $i : E \to \mathcal{K}^{0,0}(K_\Omega)$ we denote some embedding of the finite-dimensional bundle into the infinite-dimensional bundle, i^* is the adjoint mapping, and $E^\perp$ is the complementary bundle.

A similar splitting in the term $K_1(\widetilde{A})$ can be obtained with the use of suspension $(K^1(X) \simeq K(X \times \mathbb{S}^1)/K(X))$. □

2. Topological obstructions in edge theory. The commutativity of the square $\boxed{\text{A}}$ in diagram (3.4) can be interpreted as a topological obstruction.

Corollary 6.3 ([27]). *The stable homotopy class of an elliptic interior symbol* σ *on* $T^*\mathcal{M}$ *contains a representative possessing a compatible elliptic edge symbol (i.e., there exists a Fredholm operator) if and only if*

$$\pi_![\sigma|_{\partial T^*\mathcal{M}}] = 0 \in K_c^1(T^*X), \qquad \pi : \partial M \longrightarrow X,$$

where $\pi_! : K_c^0(\partial T^*\mathcal{M}) \to K_c^1(T^*X)$ *is the direct image mapping.*

A similar obstruction for self-adjoint elliptic operators is given by the same formula. We point out that there is an essential difference between the even and odd cases. Namely, the value of the boundary mapping $K_c^0(T^*\mathcal{M}) \xrightarrow{\partial} K_1(S\Sigma_0) \simeq K_c^1(T^*X)$ on the element determined by an elliptic symbol σ is expressed in terms of the index

$$\partial[\sigma] = p\,\mathrm{ind}\,\sigma_\Lambda \in K^0(S^*X)/K^0(X) \simeq K_c^1(T^*X),$$

where $p : K(S^*X) \to K(S^*X)/K(X)$, of a compatible edge symbol σ. (The edge symbol is Fredholm, since by [28] for an elliptic interior symbol σ there always exists a compatible invertible conormal symbol.)

However, for self-adjoint elliptic operators there does not necessarily exist a compatible elliptic self-adjoint conormal symbol. This is shown by the following example even in the case of conical singularities.

Example 6.4. Consider an interior elliptic symmetric operator of Dirac type on a manifold with N conical points of the form

$$D = \Gamma\left(\frac{\partial}{\partial t} + A\right), \qquad r = e^{-t}$$

in a neighborhood of the singularities, where $\Gamma^2 = -1$, $\Gamma A + A\Gamma = 0$, and $\Gamma^* = -\Gamma$, $A = A^*$. Then the boundary mapping $K_c^1(T^*\mathcal{M}) \to K_c^0(T^*X) = \mathbb{Z}^N$, whose triviality on the element $[\sigma(D)]$ is the obstruction to the existence of a self-adjoint elliptic operator, takes $[\sigma(D)]$ to the sequence of indices of operators induced on the bases of the cones:

$$\mathrm{ind}\big(A : \mathrm{Im}(\Gamma + i) \longrightarrow \mathrm{Im}(\Gamma - i)\big).$$

Note that the sum of these indices is zero, since the index is cobordism invariant [29], but separate terms can be nonzero. For example, one can take $M = [0,1] \times \mathbb{CP}^{2n}$ with the signature operator.

7 Appendix

1. Semiclassical quantization. Let Ω be a smooth closed manifold. By $\Psi^0 \equiv \Psi^0(\Omega, \mathbb{R}^{l+1}_{\eta,p})$ we denote the algebra of classical pseudodifferential operator of order zero with parameters $(\eta, p) \in \mathbb{R}^{l+1}$ in the sense of Agranovich–Vishik on Ω, and by $\Psi^0_\wedge \equiv \Psi^0_\wedge(K_\Omega)$ we denote the algebra of edge symbols[4] of order zero depending on the parameter $\xi \in \mathbb{S}^{l-1} \subset \mathbb{R}^l_\xi$ in the space $\mathcal{K}^{0,0}(K_\Omega)$ on the cone K_Ω.

The problem is to construct an asymptotic homomorphism (quantization)

$$T_h : \Psi^0 \longrightarrow \Psi^0_\wedge, \quad h \in (0,1],$$

[4]Edge symbols depend also on x, but we omit this dependence, since it does not affect the construction in any way.

such that the interior symbol and the conormal symbol of the edge symbol given by the quantization satisfy the relations

$$\sigma(T_h a)(\eta, p) = \sigma(a)(q, \eta, hp), \tag{7.1}$$

$$\sigma_c(T_h a)(\eta, p) = a\left(0, h\left(p + i(n+1)/2\right)\right). \tag{7.2}$$

Theorem 7.1. *The mapping T_h can be defined by the formula*

$$(T_h a)(\xi) = a\left(\xi \overset{2}{r}, ihr\overset{1}{\frac{\partial}{\partial r}} + ih\frac{n+1}{2}\right), \tag{7.3}$$

where numbers over operators are treated in the sense of noncommutative analysis [30] *and functions of $ir\partial/\partial r$ are defined with the help of the Mellin transform on the weight line* $\operatorname{Im} p = -(n+1)/2$*, where* $n = \dim \Omega$.

Proof. First, let us show that the mapping (7.3) is well defined and continuous in the spaces

$$T_h : \Psi^0 \longrightarrow \mathcal{B}(\mathcal{K}^{0,0}(K_\Omega))$$

and indeed specifies an asymptotic quantization, i.e., satisfies

$$T_h(a)T_h(b) = T_h(ab) + o(1), \quad h \to 0,$$

where the estimate $o(1)$ on the right-hand side is in the operator norm. To prove this, we make the change of variables $r = e^{-t}$. It takes the space $\mathcal{K}^{0,0}(K_\Omega)$ to the space $L^2(\mathbb{R}_t, L^2(\Omega), e^{(n+1)t}dt)$ of functions ranging in $L^2(\Omega)$ and square integrable with respect to the measure $e^{(n+1)t}dt$ on the real line, and the quantization formula (7.3) acquires the standard h-pseudodifferential form

$$T_h(a(\eta, p)) = e^{(n+1)t/2} A\left(\overset{2}{t}, -ih\overset{1}{\frac{\partial}{\partial t}}\right) e^{-(n+1)t/2}, \tag{7.4}$$

where

$$A(t, p) = a(\xi e^{-t}, p). \tag{7.5}$$

Note that hence it suffices to study the operator

$$A\left(\overset{2}{t}, -ih\overset{1}{\frac{\partial}{\partial t}}\right) : L^2(\mathbb{R}_t \times \Omega, dtd\omega) \longrightarrow L^2(\mathbb{R}_t \times \Omega, dtd\omega)$$

in the standard Lebesgue space $L^2(\mathbb{R} \times \Omega) = L^2(\mathbb{R}_t \times \Omega, dtd\omega)$. Since the operator-valued function $a(\eta, p)$ is a zero-order pseudodifferential operator with parameters in the sense of Agranovich–Vishik, it follows that its norm in $L^2(\Omega)$ satisfies the estimates

$$\left\| \frac{\partial^{\alpha+k} a(\eta, p)}{\partial \eta^\alpha \partial p^k} \right\| \leq C_{\alpha k}(1 + |\eta| + |p|)^{-|\alpha|-k} \tag{7.6}$$

for $|\alpha| + k = 0, 1, 2, \ldots$ and hence the norm of the function (7.5) satisfies the estimates

$$\left\| \frac{\partial^{l+k} A(t,p)}{\partial t^l \partial p^k} \right\| \le \widetilde{C}_{lk} (1 + |p|)^{-k}. \tag{7.7}$$

Indeed,

$$\frac{\partial^{l+k} A(t,p)}{\partial t^l \partial p^k} = \sum_{|\alpha| \le l} M_{l\alpha} e^{-|\alpha| t} \frac{\partial^{\alpha+k} a}{\partial \eta^\alpha \partial p^k} (\xi e^{-t}, p),$$

where the coefficients $M_{l\alpha}$ depend only on ξ, and hence (we assume that ξ lies on the sphere)

$$\left\| \frac{\partial^{l+k} A(t,p)}{\partial t^l \partial p^k} \right\| \le \sum_{|\alpha| \le l} |M_{l\alpha}| C_{\alpha k} e^{-|\alpha| t} (1 + e^{-t} + |p|)^{-|\alpha| - k}.$$

It remains to note that

$$e^{-t}(1 + e^{-t} + |p|)^{-1} \le 1, \quad (1 + e^{-t} + |p|)^{-1} \le (1 + |p|)^{-1}.$$

Hence we have shown that the operator-value symbol $A(t,p)$ belongs to the Hörmander class $S^0 \equiv S^0_{1,0}$.

Lemma 7.2. *The following assertions hold.*

(i) *If $A \in S^0$, then the operator*

$$\widehat{A} \equiv A\left(\overset{2}{t}, -ih\overset{1}{\frac{\partial}{\partial t}}\right) : L^2(\mathbb{R} \times \Omega) \longrightarrow L^2(\mathbb{R} \times \Omega)$$

is bounded uniformly with respect to $h \in (0, 1]$.

(ii) *If $A, B \in S^0$, then*

$$\widehat{AB} - \widehat{A}\widehat{B} = O(h).$$

Proof. The proof of this lemma is standard. It follows the proof of Theorem IV.6 in [30, p. 282] almost word for word, and we omit it. □

Thus we have shown that T is an asymptotic homomorphism. It remains to verify that the interior and conormal symbols of the operator $T(a)$ satisfy relations (7.1) and (7.2). But this follows directly from Definition 1.5. □

2. Estimates modulo compact operators.

Theorem 7.3. (i) *The norm modulo compact operators of an edge-degenerate zero-order pseudodifferential operator in the space$\mathcal{W}^{0,0}(\mathcal{M})$ is equal to*

$$\inf_{K \in \mathcal{K}} \|D + K\|_{\mathcal{B}(\mathcal{W}^{0,0}(\mathcal{M}))} = \max\left(\max_{S^*\mathcal{M}} |\sigma(D)|, \max_{S^*X} \|\sigma_\Lambda(D)\|_{\mathcal{B}(\mathcal{K}^{0,0}(K_\Omega))} \right) \tag{7.8}$$

(ii) *The norm modulo compact operators of an edge symbol is equal to*

$$\inf_{K\in C(S^*X,\mathcal{K})} \max_{S^*X} \|a_\Lambda + K\|_{\mathcal{B}(\mathcal{K}^{0,0}(K_\Omega))}$$
$$= \max\left(\max_{\partial S^*\mathcal{M}} |\sigma(a_\Lambda)|, \sup_{X\times\mathbb{R}} \|\sigma_c(a_\Lambda)\|_{\mathcal{B}(L^2(\Omega))}\right) \quad (7.9)$$

Proof. The proof of both assertions follows the same scheme and is very simple. The symbol (interior or edge) of the operator D can be computed (up to Fourier transform with respect to part of variables, which does not change the norm) as the strong limit of the expression $U_\lambda^{-1}AU_\lambda$, where U_λ is a unitary local dilation group in a neighborhood of the corresponding point of $\mathcal{M}$. It follows that the norm of the symbol does not exceed the norm of the operator. Next, the product of operators corresponds to the product of symbols, and the adjoint operator corresponds to the adjoint symbol. It follows that the mapping "operator $\mapsto$ symbols" extends to a $*$-homomorphism of the C^*-algebra obtained by the closure of the algebra of pseudodifferential operator into the corresponding C^*-algebra of symbols. The kernel of this homomorphism coincides with the set of compact operators, since the equality of all symbols to zero is a necessary and sufficient condition for compactness. By the general properties of $*$-homomorphisms, we see that the mapping "operator $\mapsto$ symbols" is an isometric isomorphism of the quotient algebra of pseudodifferential operators modulo compact operators onto the symbol algebra, which proves the desired assertion.

To prove the corresponding assertion for edge symbols, we should write out a local one-parameter group computing the interior symbol of the edge symbol. This (additive) group acts in a neighborhood of the point (r,ω_0) of the cone K_Ω in the coordinates $(r,\tau) = (r, r(\omega-\omega_0))$ by the formula

$$U_\lambda\varphi(r,\tau) = \varphi(r+\lambda,\tau). \quad (7.10)$$

The conormal symbol $a(0,p)$ (more precisely, its Mellin transform) can be computed as the strong limit

$$\sigma_c(T(a)) = s\text{-}\lim_{\lambda\to\infty} U_\lambda^{-1}T(a)U_\lambda,$$

where U_λ is the group of dilations with respect to the variable r. □

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