Elliptic Theory on Manifolds with Edges[1]

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Analysis and Topology

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Chapter 6

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Introduction

In this paper, we acquaint the reader with analytical aspects of elliptic theory on manifolds with singularities of the simplest type, that is, edges. This theory was developed by Schulze (e.g., see (Schulze 1991), where one can find further references, and also (Egorov and Schulze 1997)). Although the geometry of such manifolds is hardly more complicated than that of manifolds with isolated singularities, the construction of elliptic theory proves to be much more involved, and the choice of "right" spaces where the operators should act is by no means obvious. Hence in the first section we try to give a clear presentation of related definitions and constructions, emphasizing the motivations and omitting less important details. In particular, the reader will learn

- how edge-degenerate operators are defined and in which spaces they should be considered;
- what the notion of symbol looks like for such operators;
- why these operators, even under the ellipticity condition for the principal symbol, are Fredholm only in exceptional cases, and how to obtain well-posed problems for them;
- what notion of ellipticity is suitable for edge problems;
- how one states the finiteness theorem in this case.

Issued related to *pseudodifferential* operators are touched in the first section only as long as they are necessary for the understanding of statements and motivations. The theory of edge-degenerate pseudodifferential operators, which is not only the main tool in the proof of the finiteness theorem but also an important basis for homotopies used in index theory, is developed in Section 2. In the third section, by means of the theory of pseudodifferential operators, we construct regularizers and thus prove the finiteness theorem for elliptic edge *problems* (including the case of *pseudo*differential operators in the main equation of the problem).

The techniques used in this paper involve some facts concerning pseudodifferential operators in spaces of sections of infinite-dimensional bundles. These facts are included in the Appendix.

6.1. Motivation and Main Constructions

6.1.1. Manifolds with edges

First, we introduce the main geometric object on which elliptic theory will be developed in this paper, namely, manifolds with edges.

Definition. Let M be a smooth compact manifold with boundary ∂M. Suppose that the boundary is equipped with the structure of a locally trivial bundle with base X and fiber Ω, where Ω and X are also smooth compact manifolds:

$$\pi : \partial M \xrightarrow{\Omega} X. \tag{6.1}$$

We identify all points in each fiber $\Omega_x = \pi^{-1}(x)$ of π with one another, i.e., introduce the following equivalence relation on M:

two points $a_1, a_2 \in M$ are equivalent ($a_1 \sim a_2$) if and only if either $a_1 = a_2$ or both points belong to ∂M and $\pi(a_1) = \pi(a_2)$. (6.2)

DEFINITION 6.1. The quotient space $\mathcal{M} = M/\sim$ with respect to the equivalence relation (6.2) is called a *manifold with edges* or, more precisely, a *manifold with edge* X.

The structure of a neighborhood of the edge. Let U be a collar neighborhood of the boundary in M. Once and for all, we choose a direct product structure

$$U \simeq \partial M \times [0, 1); \tag{6.3}$$

then the bundle (6.1) lifts to the bundle (denoted by the same letter)

$$\pi : U \longrightarrow X \times [0, 1). \tag{6.4}$$

(The projection acts as the identity operator with respect to the second argument.) Passing to the quotient space, we see that the neighborhood $\mathcal{U} = U/\sim$ of the edge in $\mathcal{M}$ is also a bundle over X. The fiber of this bundle is a manifold with isolated singularity, namely, the neighborhood $\{r < 1\}$ of the vertex in the infinite cone

$$K_\Omega = (\Omega \times \mathbb{R}_+) \bigm/ \bigl(\Omega \times \{0\}\bigr) \tag{6.5}$$

with base Ω.

Thus geometrically a manifold with edges is a space with singularities which looks like the direct product of a domain V in Euclidean space $\mathbb{R}^n$ by the cone K_Ω with smooth compact base Ω in a neighborhood of any singular point. The *edge* X (the set of singular points) itself is a smooth manifold and is locally represented by the product of V by the cone vertex.

Next, there is a natural diffeomorphism $\mathcal{M} \setminus X \equiv \overset{\circ}{\mathcal{M}} \simeq \overset{\circ}{M} \equiv M \setminus \partial M$ of the interiors.

Remark 6.2. Note that even though the manifold $\mathcal{M}$ with edges is the main geometric object of the theory and, in particular, it is this manifold that arises in examples and applications of the theory, it will be more convenient to carry out most of the main analytical constructions on the original manifold M with boundary. We usually do so in what follows.

The infinite wedge associated with $\mathcal{M}$**.** If $\mathcal{M} = M/\sim$ is a compact manifold with edge X, then it has a naturally associated manifold W with edge X called an *infinite wedge* with edge X. It is defined as the quotient space of the Cartesian product $\partial M \times \overline{\mathbb{R}}_+$ by the equivalence relation

points $a_1, a_2 \in \partial M \times \overline{\mathbb{R}}_+$ are equivalent ($a_1 \sim a_2$) if and only if either $a_1 = a_2$ or both points belong to $\partial M \times \{0\}$ and satisfy $\pi(a_1) = \pi(a_2)$, (6.6)

similar to (6.2):

$$W = \left(\partial M \times \overline{\mathbb{R}}_+\right)/\sim \ .$$

Thus W is a locally trivial bundle over X with fiber the infinite cone K_Ω. The projection of this bundle will also be denoted by π.

In W as well as in $\mathcal{M}$, the neighborhood $\mathcal{U}$ of the edge is well defined. We sometimes identify it with the corresponding neighborhood in $\mathcal{M}$, which permits us to treat functions on $\mathcal{M}$ supported in $\mathcal{U}$ as functions on W and, conversely, functions on W supported in $\mathcal{U}$ with functions on $\mathcal{M}$.

Coordinates on $\mathcal{M}$**.** We understand coordinates on $\mathcal{M}$ as *admissible* coordinates on M. Specifically, we consider coordinate neighborhoods of two types on M. First, these are coordinate neighborhoods of interior points, where arbitrary smooth coordinates systems will be called admissible. Second, these are neighborhoods of boundary points. The boundary ∂M has the structure of a vector bundle, and admissible coordinates on M in these neighborhoods are coordinates compatible with this structure, namely, coordinates of the form (x, ω, r), where r is a defining function of the boundary ($r \geq 0$ and the equation of the boundary has the form $r = 0$) and the variables x are coordinates on the base X of the bundle π (i.e., they are constant in the fibers of π). Then for each given x and $r = 0$ the variables ω are automatically local coordinates in the fiber Ω_x. The coordinate transition maps

$$(x, \omega, r) = h(\widetilde{x}, \widetilde{\omega}, \widetilde{r})$$

near the boundary have the form

$$x = f(\widetilde{x}) + \widetilde{r} f_1(\widetilde{x}, \widetilde{\omega}, \widetilde{r}), \quad \omega = g(\widetilde{x}, \widetilde{\omega}, \widetilde{r}), \quad r = F(\widetilde{x}, \widetilde{\omega}, \widetilde{r})\widetilde{r}, \tag{6.7}$$

where F, f, f_1, g are smooth functions and

$$F(x, \omega, 0) > 0, \quad \det \frac{\partial f(\widetilde{x})}{\partial \widetilde{x}} \neq 0, \quad \det \frac{\partial g(\widetilde{x}, \widetilde{\omega}, 0)}{\partial \widetilde{\omega}} \neq 0. \tag{6.8}$$

Similar admissible coordinate systems (x, ω, r) are defined on the infinite wedge W. In this case, we always assume that the variable r ranges over the entire $\overline{\mathbb{R}}_+$ (i.e., the coordinate neighborhood is an infinite cone). With the change of variables (6.7) we associate the *derived change of variables*

$$(x, \omega, r) = h_*(\widetilde{x}, \widetilde{\omega}, \widetilde{r})$$

on the infinite wedge given by the formulas

$$x = f(\widetilde{x}), \quad \omega = g(\widetilde{x}, \widetilde{\omega}, 0), \quad r = F(\widetilde{x}, \widetilde{\omega}, 0)\widetilde{r}. \tag{6.9}$$

(We have essentially passed to differentials only with respect to the variable r at $r = 0$.) The derived change of variables is obviously compatible with the bundle structure $\pi : W \longrightarrow X$.

The cotangent bundle of $\mathcal{M}$**.** The *cotangent bundle* of $\mathcal{M}$ plays an important role in the subsequent constructions. By definition, it is some special vector bundle over the corresponding manifold M with boundary. Let us describe it.[1] In the space $\Lambda^1(M)$ of smooth differential 1-forms on M, consider the subspace $\Lambda^1_\wedge(M) \subset \Lambda^1(M)$ of forms that vanish on the tangent vectors to the fibers $\Omega_x = \pi^{-1}(x)$ of π:

$$\Lambda^1_\wedge(M) = \left\{ \alpha \in \Lambda^1(M) \;\middle|\; \alpha\big|_{T\Omega_x} = 0 \quad \text{for all } x \in X. \right\}$$

In admissible local coordinates in a neighborhood of the boundary, such forms can be represented as

$$\alpha = \xi\, dx + qr\, d\omega + p\, dr, \tag{6.10}$$

where the functions ξ, q, and p are smooth up to the boundary.[2] It follows that $\Lambda^1_\wedge(M)$ is a locally free $C^\infty(M)$-module and hence, by the Serre–Swan theorem, the module of sections of a vector bundle on M, which is called the *cotangent bundle of the manifold* $\mathcal{M}$ *with edges* and denoted by $T^*\mathcal{M}$. The embedding $\Lambda^1_\wedge(M) \subset \Lambda^1(M)$ induces the natural embedding

$$\jmath : T^*\mathcal{M} \hookrightarrow T^*M, \tag{6.11}$$

which is an isomorphism over the interior $\overset{\circ}{M}$ of M. The manifold $T^*\mathcal{M}$ is a manifold with boundary. There are two types of canonical coordinates on $T^*\mathcal{M}$: in neighborhoods of interior points, these are standard canonical coordinates (y, θ) induced from the cotangent bundle T^*M, and in neighborhoods of boundary points, these are coordinates of the form $(x, \omega, r; \xi, q, p)$, where (x, ω, r) are admissible coordinates near the boundary on M and (ξ, q, p) are the coordinates corresponding to the representation (6.10) of differential forms $\alpha \in \Lambda^1_\wedge(M)$.

Note that there is a natural diffeomorphism

$$\partial T^*\mathcal{M} \simeq \partial T^*W, \tag{6.12}$$

where the cotangent bundle of the infinite wedge W is obtained by a similar construction over $\partial M \times \overline{\mathbb{R}}_+$. To obtain this diffeomorphism, it suffices to use the identification of the neighborhood $\mathcal{U}$ of the edge in $\mathcal{M}$ with the corresponding neighborhood of the edge in W. (This identification depends on the choice of the trivialization of a collar neighborhood of the boundary in M, but a straightforward verification shows that the diffeomorphism (6.12) is independent of this choice.)

In what follows, we need a special direct sum decomposition of the bundle $\partial T^*\mathcal{M}$ over ∂M. This decomposition is constructed as follows. We have already noted that the wedge W is a bundle over X with fiber K_Ω. Let $T^*K_{\Omega_x}$ be the cotangent bundle of the cone K_{Ω_x} over $x \in X$ constructed by the same recipe as $T^*\mathcal{M}$. (The boundary bundle π_x for K_{Ω_x} is just a mapping into the point x: $\pi_x : K_{\Omega_x} \longrightarrow \{x\}$.) This cotangent bundle is a vector bundle over the manifold $\Omega_x \times \mathbb{R}_+$. Taking the disjoint union of $T^*K_{\Omega_x}$ over all $x \in X$ and equipping this union with a natural topology, we obtain a bundle over $\partial M \times \overline{\mathbb{R}}_+$, which will be denoted by $T^*K \longrightarrow \partial M \times \overline{\mathbb{R}}_+$.

PROPOSITION 6.3. *There exists a natural decomposition*

$$\partial T^*\mathcal{M} = \pi^*(T^*X) \oplus \partial T^*K \tag{6.13}$$

[1] This construction is similar to, but different from, Melrose's construction of the "compressed cotangent bundle" (Melrose 1981).

[2] One can readily see that the form (6.10) is invariant with respect to admissible changes of variables (6.7).

into a direct sum of vector bundles over ∂M. Here $\pi^*(T^*X)$ is the lift of the bundle T^*X from X to ∂M via the projection π and $\partial T^*K = T^*K\big|_{\partial M\times\{0\}}$.

Proof. It suffices to construct natural projections of the left-hand side *onto* each of the terms on the right-hand side. (Then the fact that the mapping (6.13) is an isomorphism follows by the computation of dimensions of the bundles occurring in the formula.) The projection

$$p_1 : \partial T^*\mathcal{M} \longrightarrow \pi^*(T^*X)$$

is defined as follows. Consider the restriction

$$\jmath : \partial T^*\mathcal{M} \longrightarrow \partial T^*M$$

of the mapping (6.11) to the boundary. The range of this mapping at each point consists of all (algebraic) forms vanishing on the tangent space to the fiber of π through this point, i.e., exactly coincides with the corresponding fiber of $\pi^*(T^*X)$, and we can take $p_1 = \jmath$. To construct the projection

$$p_2 : \partial T^*\mathcal{M} \longrightarrow \partial T^*K,$$

we use the isomorphism (6.12). Let β be an element of the fiber of ∂T^*W over a point $v \in \partial M$. This means that $\beta = \alpha(v)$ for some section

$$\alpha \in \Gamma(T^*W) = \Lambda^1_\wedge(\partial M \times \overline{\mathbb{R}}_+) \subset \Lambda^1(\partial M \times \overline{\mathbb{R}}_+).$$

We interpret this section as a differential form on $\partial M \times \overline{\mathbb{R}}_+$ and restrict it to $\Omega_x \times \overline{\mathbb{R}}_+$, where $x = \pi(v)$ (i.e., Ω_x is the fiber of π through v). The resulting restriction, which we denote by $\widetilde{\alpha}$, still vanishes on tangent vectors to $\Omega_x \times \{0\}$ and hence lies in $\Lambda^1_\wedge(\Omega_x \times \overline{\mathbb{R}}_+)$ and can be interpreted as a section of $\widetilde{\alpha} \in \Gamma(T^*K_{\Omega_x})$. The restriction of this section to v specifies an element of the bundle ∂T^*K, and we set

$$p_2(\beta) = \widetilde{\alpha}\big|_v.$$

Easy computations with the use of the coordinate representation (6.10) show that the mapping p_2 is well defined (i.e., is independent of the choice of the section α) and is an epimorphism. The proof of the proposition is complete. ☐

A description of $\mathcal{M}$ with the help of a metric. One can also describe a manifold with edges by differential-geometric means. To this end, one equips M with a Riemannian metric degenerating at the boundary in such a way that the distance between points lying in the same fiber is zero. One can readily obtain such metrics using the cotangent bundle of $\mathcal{M}$. Specifically, the mapping (6.11) induces a mapping

$$\jmath : S^2(T^*\mathcal{M}) \longrightarrow S^2(T^*M) \tag{6.14}$$

(denoted by the same letter) of symmetric powers of these bundles. By applying the latter mapping to an arbitrary positive definite section of $S^2(T^*\mathcal{M})$, we obtain a Riemannian metric on $\overset{\circ}{M}$ degenerating on ∂M and smooth up to the boundary. The simplest metric of this sort has the form

$$d\rho^2 = dr^2 + dx^2 + r^2 d\omega^2 \tag{6.15}$$

in U, where $r \in [0,1)$ is the second coordinate in the decomposition (6.3) (one can readily see that this coordinate is just the distance to the boundary in the metric (6.15)) and dx^2 and $d\omega^2$ are some smooth nondegenerate metrics on X and Ω_x, respectively. (The latter metric is naturally assumed to depend smoothly on $x \in X$). In terms of the metric (6.15), the equivalence relation (6.2) can be described as follows:

$$a_1 \sim a_2 \text{ if and only if } \rho(a_1, a_2) = 0. \tag{6.16}$$

Remark 6.4. Needless to say, there are many other metrics satisfying (6.16). For example, if we replace the factor r^2 multiplying $d\omega^2$ in (6.15) by r^{2k}, then we obtain "cuspidal edges of order k." We restrict ourselves to the case of the metric (6.15); the corresponding edges are said to be *conical*.

6.1.2. Edge-degenerate differential operators

Definition. What differential operators will be studied on manifolds with edges? In the interior $\overset{\circ}{\mathcal{M}} = \mathcal{M} \setminus X$ of the manifold, they will be just arbitrary differential operators with smooth coefficients, but how do they behave near the edge? To define a natural class of such operators, we use the space $\Lambda^1_\wedge(M)$ of differential 1-forms corresponding to the edge structure.

DEFINITION 6.5. By $\mathrm{Vect}_\wedge(\mathcal{M})$ we denote the space of vector fields V on $\overset{\circ}{M}$ such that $\alpha(V) \in C^\infty(M)$ for every $\alpha \in \Lambda^1_\wedge(M)$. (More precisely, the function $\alpha(V)$ defined on $\overset{\circ}{M}$ extends by continuity to a smooth function on the entire M.)

Next, by $\mathcal{D} = \mathcal{D}(\mathcal{M})$ we denote the set of linear differential operators on $\overset{\circ}{M}$ that are finite linear combinations of terms of the form $V_1 \cdots V_j a V_{j+1} \cdots V_s$ with various s (the case $s = 0$ is not excluded), where a is a smooth function on M and all V_j are vector fields belonging to $\mathrm{Vect}_\wedge(M)$. The subspace of operators that can be represented by linear combinations in which $s \le m$ for each of the terms will be denoted by $\mathcal{D}_m = \mathcal{D}_m(\mathcal{M})$. The elements of $\mathcal{D}$ ($\mathcal{D}_m$) are called *edge-degenerate differential operators* on $\mathcal{M}$ (*of order* $\le m$).

A coordinate description. Let (x, ω, r) be an admissible coordinate system on M in a neighborhood of a boundary point. Then the vector fields

$$A_j = -i\frac{\partial}{\partial x_j},\ j = 1, \dots, n, \quad B = -i\frac{\partial}{\partial r}, \quad C_j = -\frac{i}{r}\frac{\partial}{\partial \omega_j},\ j = 1, \dots, k, \tag{6.17}$$

belong to $\mathrm{Vect}_\wedge(\mathcal{M})$. As one might expect, they all have lengths of the order of unity (uniformly bounded above and below by positive constants) in the metric (6.15). We would like to represent our differential operators as polynomials of such vector fields with coefficients being smooth functions on M. Note that the operator tuple (6.17) is not closed with respect to commutators: for example, the commutator

$$[B, C_j] = -i\frac{1}{r}C_j, \tag{6.18}$$

as well as the commutator

$$[C_j, \omega_j] = -\frac{i}{r}, \tag{6.19}$$

cannot be expressed as an ordered polynomial[3] with smooth coefficients of the operators (6.17). Hence the set of *ordered* polynomials with smooth coefficients of the operators (6.17) is not closed with respect to composition: when moving the operator arguments in the product to the corresponding positions, we gain the singular factor $1/r$. To rectify the situation, we supplement our operator tuple with yet another operator

$$D = \frac{1}{r}. \tag{6.20}$$

PROPOSITION 6.6. *An edge-degenerate differential operator P of order $\leq m$ on a manifold $\mathcal{M}$ with edge X can always be represented in local coordinates (x, r, ω) in a neighborhood of the boundary as an ordered polynomial of degree $\leq m$ of the operators* (6.17) *and* (6.20) *with coefficients smooth up to $r = 0$:*

$$P = \sum_{|\alpha|+|\beta|+j+l\leq m} a_{\alpha\beta jl}(x, \omega, r) A^{\alpha} B^{j} C^{\beta} D^{l}. \tag{6.21}$$

We point out that in this representation the order of an operator near the edge counts *not only differentiations but also the factors* $1/r$.

Remark 6.7. Needless to say, the representability in the form (6.21) is independent of the choice of local coordinates near the boundary. Away from the boundary, edge-degenerate differential operators can be arbitrary differential operators with smooth coefficients.

PROPOSITION 6.8. *The set $\mathcal{D}$ of edge-degenerate differential operators is an algebra, and*

$$\mathcal{D}_m \mathcal{D}_{m'} \subseteq \mathcal{D}_{m+m'}. \tag{6.22}$$

(*In other words, the algebra $\mathcal{D}$ is filtered by the subspaces $\mathcal{D}_m$.*)

The proof readily follows from the definitions. □

6.1.3. Symbols

Definition. Let us find out what the notion of *principal symbol* looks like for edge-degenerate operators. Recall that for the case of a smooth manifold the space of mth-order principal symbols, i.e., homogeneous polynomials of degree m with respect to the momentum variables is naturally identified with the quotient space of operators of order m by operators of order $m-1$: the symbol is uniquely determined by the principal part of the operator and in turn uniquely determines the operator modulo lower-order terms. By analogy with this, we give the definition of the principal symbol in our case. We know that the algebra $\mathcal{D}$ of edge-degenerate differential operators is filtered by the order of the operator, and we define symbols as elements of the associated graded algebra

$$\operatorname{gr} \mathcal{D} = \bigoplus_{m=0}^{\infty} \mathcal{D}_m/\mathcal{D}_{m-1} \qquad (\mathcal{D}_{-1} \overset{\text{def}}{=} \mathbf{0}).$$

DEFINITION 6.9. The *principal symbol* (of order m) of an edge-degenerate differential operator $P \in \mathcal{D}_m$ of order $\leq m$ is the image of P in the quotient space $\mathcal{D}_m/\mathcal{D}_{m-1}$ under the natural projection:

$$\mathcal{D}_m \ni P \longmapsto \Sigma(P) \in \mathcal{D}_m/\mathcal{D}_{m-1}. \tag{6.23}$$

[3]That is, as a polynomial in which the order of operator arguments is fixed; say, the A_j act first, B acts second, and the C_j act third; see (Maslov 1973) and also (Nazaikinskii, Sternin and Shatalov 1995) for a systematic exposition of the noncommutative operator calculus.

The interior principal symbol and the edge symbol. To use this definition in practice, we however need to describe the structure of the principal symbol in more detail. It turns out that it can be represented by a pair consisting of the *interior principal symbol* and the *edge symbol*. Let us describe both components of the pair. Let $D \in \mathcal{D}_m$ be an edge-degenerate differential operator of order m. Its principal symbol $\sigma_{clas}(D)$ in the traditional sense is a function on $T^*\overset{\circ}{M}$ polynomial in the fibers and having singularities (growing unboundedly) near ∂T^*M. We set

$$\sigma(D) = \jmath^* \sigma_{clas}(D), \tag{6.24}$$

where $\jmath : T^*\mathcal{M} \longrightarrow T^*M$ is the mapping introduced above. This is a function on the cotangent bundle $T^*\mathcal{M}$. Let us describe the behavior of this function near the boundary. Let

$$D = \sum_{\alpha+\beta+j+l\leq m} a_{\alpha\beta jl}(x,\omega,r)\left(-i\frac{\partial}{\partial x}\right)^{\alpha}\left(-\frac{i}{r}\frac{\partial}{\partial\omega}\right)^{\beta}\left(-i\frac{\partial}{\partial r}\right)^{j}\left(\frac{1}{r}\right)^{l} \tag{6.25}$$

in admissible coordinates. Then one can readily compute that the function $\sigma(D)$ in the canonical local coordinates $(x,\omega,r;\xi,q,p)$ on $T^*\mathcal{M}$ has the form

$$\sigma(D) = \sum_{\alpha+\beta+j=m} a_{\alpha\beta j0}(x,\omega,r)\xi^{\alpha}q^{\beta}p^{j}. \tag{6.26}$$

In particular, it is smooth up to the boundary.

DEFINITION 6.10. The function $\sigma(D)$ on $T^*\mathcal{M}$ given by the formula (6.24) is called the *interior principal symbol* of the edge-degenerate differential operator $D \in \mathcal{D}_m$.

Let us now proceed to the definition of the edge symbol. Let D again have the form (6.25).

DEFINITION 6.11. The operator family

$$\sigma_\wedge(D)(x,\xi) = \sum_{\alpha+\beta+j+l=m} a_{\alpha\beta jl}(x,\omega,0)\xi^{\alpha}\left(-\frac{i}{r}\frac{\partial}{\partial\omega}\right)^{\beta}\left(-i\frac{\partial}{\partial r}\right)^{j}\left(\frac{1}{r}\right)^{l}, \tag{6.27}$$

depending on the parameters (x,ξ), is called the *edge symbol* of D.

We have defined the edge symbol in local coordinates. Needless to say, now we need to globalize our definition, i.e. find how this family is transformed under changes of coordinates. Consider the bundle $\widetilde{W}$ with fiber the cone K_Ω over T^*X obtained by lifting the bundle

$$\pi : W \overset{K_\Omega}{\longrightarrow} X$$

to T^*X via the natural projection $p : T^*X \longrightarrow X$. Thus the local coordinates on $\widetilde{W}$ are (x,ξ,ω,r), where (x,ξ) are canonical coordinates on T^*X and (x,ω,r) are admissible coordinates on W.

PROPOSITION 6.12. *The local expression* (6.27) *specifies a well-defined operator family on the fibers K_{Ω_x} of the bundle $\widetilde{W} \longrightarrow T^*X$. This family is parametrized by points of the cotangent bundle T^*X. More precisely, under a change of coordinates h on M given by the formula* (6.7)*, the operator* (6.27) *is transformed by the derived change of coordinates h_* given by* (6.9)*, and the parameter ξ is transformed as a momentum variable, i.e., is multiplied by the matrix $({}^t\partial\widetilde{f}/\partial\widetilde{x})^{-1}$.*

The proof is by a straightforward computation. □

The following proposition describes the relation of the interior principal symbol and the edge symbol to the above-introduced principal symbol as an element of the associated gradation.

PROPOSITION 6.13. (1) *The equations* $\sigma(D) = 0$ *and* $\sigma_\wedge(D) = 0$ *for the operator* $D \in \mathcal{D}_m$ *hold if and only if* $D \in \mathcal{D}_{m-1}$. (*Thus the pair* $(\sigma(D), \sigma_\wedge(D))$ *isomorphically represents the class* $\Sigma(D) \in \mathcal{D}_m/\mathcal{D}_{m-1}$.)

(2) *The correspondence* $D \longmapsto (\sigma(D), \sigma_\wedge(D))$ *is linear and multiplicative. This means that*

$$\sigma(D_1D_2) = \sigma(D_1)\sigma(D_2), \quad \sigma_\wedge(D_1D_2) = \sigma_\wedge(D_1)\sigma_\wedge(D_2).$$

The compatibility condition. The interior principal symbol and the edge symbol of an operator $D \in \mathcal{D}_m$ are not independent but satisfy some compatibility condition. To state this condition, let us view the family $\sigma_\wedge(D)(x,\xi)$ as a cone-degenerate *differential operator* on the cone K_Ω *with parameter* $\xi \in T^*X$ *in the sense of Agranovich–Vishik* (Agranovich and Vishik 1964). (Needless to say, this operator also depends on the additional parameter $x \in X$.) Let D have the form (6.25). Then the principal symbol of the family $\sigma_\wedge(D)(x,\xi)$ viewed as an operator of order m with parameter ξ has the form

$$\sigma(\sigma_\wedge(D))(x,\omega,\xi,q,p) = \sum_{\alpha+\beta+j=m} a_{\alpha\beta j0}(x,\omega,0)\xi^\alpha q^\beta p^j \tag{6.28}$$

(and is independent of r). By comparing this with the expression (6.26) for the principal symbol, we obtain the compatibility condition

$$\sigma(\sigma_\wedge(D)) = \sigma_\partial(D) \equiv \sigma(D) \big|_{\partial T^*\mathcal{M}} . \tag{6.29}$$

Condition (6.29) is invariant, which readily follows with regard to the decomposition (6.13).

This condition is obviously necessary and sufficient for the existence of a differential operator with given edge and principal symbols.

We can summarize this as follows:

The homogeneous component of degree m *of the graded algebra* $\operatorname{gr}\mathcal{D}$ *associated with the algebra* $\mathcal{D}$ *of edge-degenerate differential operators on a manifold* M *with edges consists of pairs* (*principal symbol, edge symbol*) *satisfying the compatibility condition* (6.29). *These pairs can naturally be viewed as symbols of such differential operators.*

Homogeneity properties of the symbols. In the case of differential operators on smooth manifolds, the principal symbol $\sigma(D)$ of an operator of order m is a homogeneous polynomial of degree m and hence satisfies

$$\sigma(D)(y,\lambda\eta) = \lambda^m\sigma(D)(y,\eta), \quad \lambda > 0.$$

The same is true for the principal symbol of an edge-degenerate differential operator. Does the edge symbol also have some homogeneity property? The answer is "yes," but the property is more complicated. Let $\varkappa_\lambda$ be the operator multiplying the argument r of a function by λ:

$$\varkappa_\lambda u(r) = u(\lambda r). \tag{6.30}$$

(The function u may well have other arguments, which remain unaffected.) The homogeneity property of the edge symbol can be written in the form

$$\sigma_\wedge(D)(x,\lambda\xi) = \lambda^m \varkappa_\lambda \sigma(D)(x,\xi)\varkappa_\lambda^{-1}, \quad \lambda \in \mathbb{R}_+. \tag{6.31}$$

This property is referred to as *twisted homogeneity*.

Remark 6.14. The family $\varkappa_\lambda$ of scaling operators is a multiplicative one-parameter group of linear operators: $\varkappa_{\lambda\mu} = \varkappa_\lambda \varkappa_\mu$, $\lambda, \mu \in \mathbb{R}_+$.

6.1.4. Elliptic problems

Statement of the problem. We say that an edge-degenerate operator D on a manifold $\mathcal{M}$ with edges is *formally elliptic* if its principal symbol is everywhere invertible on the cotangent bundle $T^*\mathcal{M}$ outside the zero section. We wish to study an equation of the form

$$Du = f \tag{6.32}$$

on the manifold $\mathcal{M}$. The main question of interest to us is as follows:

- In what function spaces is it natural to consider this equation, and should one subject the solution u and the right-hand side f to additional conditions to make the equation uniquely solvable (or at least Fredholm)? What is the form of these conditions?

An example of an edge problem. Let $\mathcal{M} = M$ be a manifold with boundary, so that $\Omega = \{pt\}$ and $\pi = \mathrm{id} : \partial M \longrightarrow X = \partial M$. Consider the equation

$$Du \equiv \triangle u - \frac{a}{r^2}u = f,$$

where $\triangle$ is the Beltrami–Laplace operator and a is a nonnegative constant. In the simplest case, this equation in local coordinates has the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial x^2} - \frac{a}{r^2}u = f.$$

Let us study the equation near the edge. To this end, we freeze the coefficients at a point of the edge and pass to the Fourier transform with respect to the variables x. We obtain the family of equations

$$\widehat{D}(\xi)\widetilde{u} = \widetilde{f}$$

on the half-line $\mathbb{R}_+$, where

$$\widehat{D}(\xi) \equiv \sigma_\wedge(D)(\xi) = \frac{\partial^2}{\partial r^2} - |\xi|^2 - \frac{a}{r^2}$$

is the edge symbol of the operator D. In what spaces is it natural to study this family of equations?

For large r, the operator $\widehat{D}(\xi)$ is essentially an operator with constant coefficients, and it is natural to study it in the ordinary Sobolev spaces H^s. For small r, we can rewrite the operator in the form

$$\widehat{D}(\xi) = -r^{-2}\left[\left(ir\frac{\partial}{\partial r}\right)^2 - i\left(ir\frac{\partial}{\partial r}\right) + |r\xi|^2 + a\right].$$

We see that the operator is cone-degenerate and can naturally be studied for these r in the weighted Sobolev spaces $H^{s,\gamma}$ (Schulze 1991).

DEFINITION 6.15. The space $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ is glued from $H^{s,\gamma}(\mathbb{R}_+)$ for small r and $H^s(\mathbb{R}_+)$ for large r.

Thus we consider the operator family

$$\widehat{D}(\xi) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+).$$

We are interested in its kernel and cokernel, so we shall consider also the adjoint family

$$\widehat{D}(\xi)^* : \mathcal{K}^{2-s,2-\gamma}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{-s,-\gamma}(\mathbb{R}_+),$$

which is given on $C_0^\infty(\mathbb{R}_+)$ by the same differential expression as $\widehat{D}(\xi)$. We wish to keep the exposition as elementary as possible, and so in what follows we set $a = 0$.

To obtain a Fredholm problem for the original operator, we must ensure the invertibility of the operator $\widehat{D}(\xi)$ for large ξ.

Since the edge symbol is twisted-homogeneous, it suffices to do this for $|\xi| = 1$ and then extend the result to all $\xi \neq 0$ by homogeneity.

The conormal symbol

$$\sigma_c(\widehat{D}(\xi)) = -p^2 + ip$$

of the edge symbol in question is invertible for $p \neq 0, i$, so that the operator $\widehat{D}(\xi)$ is Fredholm for $\gamma \neq \frac{1}{2}, \frac{3}{2}$ (Schulze 1991).

Let us compute the kernel and cokernel of the edge symbol for these γ.

Formally, the kernel of $\widehat{D}(\xi)$ and $\widehat{D}(\xi)^*$ is given by the expression

$$u = C_1 e^{-r} + C_2 e^r.$$

However, the constant C_2 is always zero, which follows from the integrability at infinity.

Next, the presence of the weight factor $r^{-2\gamma}$ in the definition of the norm in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ results in the assertion that $e^{-r} \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ if and only if $\gamma < \frac{1}{2}$.

Now let us exhaust all possible cases.

1. If $\gamma < \frac{1}{2}$, then $\widehat{D}(\xi)$ has the one-dimensional *kernel* $\{Ce^{-r}\}$ and a trivial cokernel. To make the edge symbol invertible, one can equip it, say, with a condition of the form

$$\int_0^\infty \phi(r)\widetilde{u}(r)\, dr = g,$$

where $\phi \in C_0^\infty(\mathbb{R}_+)$ is a given function nonorthogonal to e^{-r}.

2. For $\frac{1}{2} < \gamma < \frac{3}{2}$, the edge symbol $\widehat{D}(\xi)$ is invertible.

3. If $\gamma > \frac{3}{2}$, then the edge symbol $\widehat{D}(\xi)$ has a trivial kernel and the one-dimensional *cokernel* $\{Ce^{-r}\}$. To make the edge symbol invertible, we equip it with a *co-condition* including a numerical unknown $\mu \in \mathbb{C}$:

$$\widehat{D}(\xi)\widetilde{u} + \mu\phi = \widetilde{f}.$$

Here ϕ is a given function satisfying the same nonorthogonality condition as above.

In all three cases, adding a finite-dimensional (in this example, one-dimensional) condition or co-condition where necessary, we have made the edge symbol for $|\xi| = 1$ an *invertible operator* $A(x,\xi)$. For $\gamma < 1/2$, this operator has the form

$$A(x,\xi) = \begin{pmatrix} \widehat{D}(\xi) \\ \widehat{B} \end{pmatrix} : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \begin{matrix} \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix}, \tag{6.33}$$

where

$$\widehat{B}v = \int_0^\infty \phi(r)v(r)\,dr.$$

For $\frac{1}{2} < \gamma < \frac{3}{2}$, we have

$$A(x,\xi) = \widehat{D}(\xi) : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+). \tag{6.34}$$

Finally, for $\gamma > \frac{3}{2}$ the operator $A(x,\xi)$ is given by the formula

$$A(x,\xi) = \begin{pmatrix}\widehat{D}(\xi), & \widehat{C}\end{pmatrix} : \begin{matrix}\mathcal{K}^{s,\gamma}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}\end{matrix} \longrightarrow \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}_+), \tag{6.35}$$

where

$$\widehat{C}\mu = \mu\phi(r).$$

Now we extend the operator-valued symbol $A(x,\xi)$ to all values $\xi \neq 0$ by twisted homogeneity. Here we assume that the group $\varkappa_\lambda$ acts trivially (as the identity operator) on the one-dimensional complex space $\mathbb{C}$. The component B (or C) of the symbol will be taken homogeneous of degree l, which does not necessarily coincide with m. (In fact, we deal with homogeneity in the sense of Douglis–Nirenberg.) By extending the symbol $\widehat{D}$ of the main operator to all $\xi \neq 0$, we obtain the same symbol, since it is twisted-homogeneous. The continuation of the symbols $\widehat{B}$ and $\widehat{C}$ gives the symbols

$$\begin{aligned}\widehat{B}(x,\xi)v &= |\xi|^{l+1/2}\int_0^\infty \phi(r|\xi|)v(r)\,dr,\\ \widehat{C}(x,\xi)\mu &= \mu|\xi|^{l+1/2}\phi(r|\xi|).\end{aligned}$$

By smoothing these symbols near $\xi = 0$ and then by applying the inverse Fourier transform, we obtain the following problems for the operator D:

$$\begin{aligned}
&Du = f, \quad \int_0^\infty \phi\left(r\left[-i\frac{\partial}{\partial x}\right]\right)\left[-i\frac{\partial}{\partial x}\right]^{l+1/2} u(x,r)\,dr = g(x), && \gamma < 1/2;\\
&Du = f, && \frac{1}{2} < \gamma < \frac{3}{2};\\
&Du + \phi\left(r\left[-i\frac{\partial}{\partial x}\right]\right)\left[-i\frac{\partial}{\partial x}\right]^{l+1/2}\mu(x) = f, && \gamma > \frac{3}{2}.
\end{aligned}$$

Here $[\xi]$ is a smooth positive function such that

$$[\xi] = |\xi| \quad \text{for large } |\xi|.$$

In what spaces will these problems be Fredholm? Obviously, these spaces are obtained from the spaces $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ with norm depending on the parameter ξ by the inverse Fourier transform $\mathcal{F}^{-1}_{\xi\to x}$. They will be described in the general case at the end of this section.

The general case. The following scheme for studying elliptic equations on manifolds with multidimensional singularities was apparently suggested for the first time in the case of Sobolev problems in (Sternin 1967*a*).

We say that an operator P is *formally elliptic* if its principal symbol is invertible everywhere outside the zero section. We shall study the formally elliptic equation $Pu = f$ in appropriate spaces. Our main task is to find out whether the equation is Fredholm and, if this is not the case, whether it is possible to impose some conditions on the solution u and the right-hand side f to make the equation Fredholm. A traditional way to prove the Fredholm property is to construct a regularizer. It is usually constructed locally, and then the local regularizers are glued together with the help of a partition of unity. Consider a formally elliptic edge-degenerate differential operator P of order m on a manifold $\mathcal{M}$ with edge X. In a neighborhood of any interior point, formal ellipticity coincides with the ordinary ellipticity, and one can construct the regularizer by usual methods of elliptic theory in the form of a pseudodifferential operator whose symbol is the inverse of the principal symbol $\sigma(P)$. Now consider a neighborhood of some point of the edge. Locally, we can assume that $X = \mathbb{R}^n$. Let us freeze the coefficients $a_{\alpha\beta l}$ of the operator P at some point of the edge and proceed in the equation $Pu = f$ to the Fourier transform with respect to the variables x (i.e., the variables along the edge). Then we obtain the family of equations

$$\sigma_\wedge(P)(x,\xi)\widetilde{u} = \widetilde{f}, \tag{6.36}$$

where the tilde stands for the Fourier transform with respect to the edge variables. To construct a regularizer of the operator P in a neighborhood of the edge, we should exactly solve Eq. (6.36) in the domain $r < \varepsilon$, where ε is a given (independent of r) positive number, for any $(x,\xi) \in T_0^*X$ (or at least for large $|\xi|$). The edge symbol possesses the easy-to-verify homogeneity property (6.31) and hence it suffices to solve (6.36), say, only for $|\xi| = 1$ on the *entire infinite cone* K_Ω. (Indeed, any solution of (6.36) has the form

$$\widetilde{u}(r,\omega) = |\xi|^{-m} v(|\xi| r, \omega), \tag{6.37}$$

where $v(r,\omega)$ is a solution of the equation

$$\sigma_\wedge(P)(x,\xi/|\xi|)v(r,\omega) = \widetilde{f}(r/|\xi|,\omega), \tag{6.38}$$

so that for arbitrarily large $|\xi|$, to know the solution of Eq. (6.36) in the domain $r \le \varepsilon$, one should know the solution of (6.38) for arbitrarily large r.)

Thus let us study Eq. (6.36) on the cone K_Ω. This is an elliptic equation that is cone-degenerate at the vertex of the cone K_Ω and behaves for large r as a usual elliptic equation with slowly varying coefficients at infinity in $\mathbb{R}^{k+1}$ written in polar coordinates.[4] What can one say about its solutions? Since the equation has variable coefficients, one cannot hope in general that it is uniquely solvable (invertible); we can only hope that it is Fredholm in suitable spaces, and then one can make it uniquely solvable by

[4]An operator with coefficients slowly varying at infinity in $\mathbb{R}^{k+1}$ is an operator of the form

$$D = \sum a_\alpha(y)(-i\partial/\partial y)^\alpha,$$

where the coefficients $a_\alpha(y)$, $y \in \mathbb{R}^{k+1}$, satisfy the estimates

$$|a_\alpha^{(\beta)}(y)| \le \operatorname{const}(1+|y|)^{-|\beta|}.$$

adding finitely many conditions. As we return from the model equation to the original equation with the operator P with the help of the inverse Fourier transform, these conditions become some conditions on the solution and the right-hand side of the original equation providing its Fredholm property. Thus first of all we should study the Fredholm property of the symbol $\sigma_\wedge(P)(x,\xi)$ on the infinite cone K_Ω by constructing a regularizer. We cover the cone K_Ω by two overlapping neighborhoods, a neighborhood U_1 of the vertex $r=0$ and a neighborhood U_2 of infinity. One can construct the regularizer locally, in each of these neighborhoods. In U_2, our operator is a usual elliptic operator with coefficients slowly varying at infinity, and hence the regularizer can be constructed in the usual Sobolev spaces $H^s(U_2)$. In U_1, our operator is cone-degenerate, and hence the regularizer can be constructed in weighted Sobolev spaces $H^{s,\gamma}(U_1)$. On the intersection $U_1\cap U_2$ these spaces coincide,

$$H^{s,\gamma}(U_1\cap U_2)=H^s(U_1\cap U_2), \tag{6.39}$$

and so we can glue them together by using a partition of unity subordinate to the cover (U_1,U_2). The resulting space will be denoted by $\mathcal{K}^{s,\gamma}(K_\Omega)$.

For the norm in $\mathcal{K}^{s,\gamma}(K_\Omega)$ one can readily write out a simple equivalent global expression that does not use partitions of unity. Namely, consider the space $L^2(K_\Omega)\equiv L^2(K_\Omega, r^k\,dr\,d\omega)$ on the cone K_Ω, where

$$r^k\,dr\,d\omega,\quad k=\dim\Omega, \tag{6.40}$$

is the Riemannian volume form corresponding to the metric

$$dr^2+r^2d\omega^2 \tag{6.41}$$

on K_Ω. In this space, we consider the operator

$$T=1+r^{-2}+\triangle_{K_\Omega}, \tag{6.42}$$

where $\triangle_{K_\Omega}$ is the Beltrami–Laplace operator on K_Ω associated with the cone-degenerate metric (6.41).

PROPOSITION 6.16. *The operator T with domain $C_0^\infty(K_\Omega^\circ)$ is essentially self-adjoint in $L^2(K_\Omega)$.*

Proof. The proof of this fact follows the proof of Theorem X.11 in (Reed and Simon 1975, p. 161) almost word for word. □

Consider also a weight function $\rho(r)$ equal to r in a neighborhood of $r=0$ and tending to 1 as $r\to\infty$. The following assertion holds.

PROPOSITION 6.17. *An equivalent norm in $\mathcal{K}^{s,\gamma}(K_\Omega)$ can be given by the formula*

$$\|u\|_{s,\gamma}=\left\|T^{s/2}\rho^{s-\gamma}u\right\|_{L^2(K_\Omega)}, \tag{6.43}$$

The proof is by a straightforward computation. □

Thus we shall study the edge symbol in the spaces $\mathcal{K}^{s,\gamma}(K_\Omega)$. The following theorem holds.

THEOREM 6.18. (1) *The edge symbol of an operator P of order m is continuous in the spaces*

$$\sigma_\wedge(P)(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega). \tag{6.44}$$

(2) *If the operator P is formally elliptic, then its edge symbol is Fredholm in the spaces* (6.44) *if and only if the conormal symbol*[5]

$$\sigma_c(\sigma_\wedge(P))(p) : H^s(\Omega) \longrightarrow H^{s-m}(\Omega) \tag{6.45}$$

of the edge symbol is invertible on the weight line $\operatorname{Im} p = \gamma$. Moreover, the kernel and cokernel of the operator (6.44) *are independent of the smoothness exponent s.*

The first assertion of the theorem is obvious, and the second assertion can be proved by the construction of a regularizer in accordance with the scheme given above.

Boundary and coboundary conditions on the edge. Now assume that we have managed to find a weight exponent γ in such a way that the edge symbol is Fredholm for all $(x,\xi) \in T_0^*X$. For each (x,ξ) we can make it an invertible operator by subjecting the solution and the right-hand sides of Eq. (6.36) to finitely many orthogonality conditions

$$(\widetilde{u}, b_j) = 0, \quad j = 1, ..., M; \qquad (\widetilde{f}, c_j^*) = 0, \quad j = 1, ..., N. \tag{6.46}$$

(the inner product is taken in the space $L^2(K_\Omega)$), and the functions b_j and c_j^* can be chosen to be infinitely smooth:

$$b_j \in \bigcap_s \mathcal{K}^{s,-\gamma}(K_\Omega), \quad c_j^* \in \bigcap_s \mathcal{K}^{s,m-\gamma}(K_\Omega). \tag{6.47}$$

We can pass from problem (6.36), (6.46) to the nonhomogeneous problem

$$\sigma_\wedge(P)(x,\xi)\widetilde{u} + C\widetilde{v} = \widetilde{f}, \tag{6.48}$$

$$B\widetilde{u} = \widetilde{g}, \tag{6.49}$$

where the operators

$$B : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathbb{C}^M, \qquad C : \mathbb{C}^N \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega) \tag{6.50}$$

are given by the formulas

$$B\widetilde{u} = \big((\widetilde{u}, b_1), \dots, (\widetilde{u}, b_M)\big), \quad C\widetilde{v} = \widetilde{v}_1 c_1 + \dots + \widetilde{v}_N c_N, \tag{6.51}$$

in which

$$c_1, \dots, c_N \in \bigcap_s \mathcal{K}^{s,\gamma-m}(K_\Omega) \tag{6.52}$$

are functions such that

$$\det \begin{pmatrix} (c_1, c_1^*) & \dots & (c_1, c_N^*) \\ \dots & \dots & \dots \\ (c_N, c_1^*) & \dots & (c_N, c_N^*) \end{pmatrix} \neq 0. \tag{6.53}$$

[5] One readily sees that it is independent of ξ.

Now suppose that these conditions and co-conditions can be chosen in such a way that they continuously depend on (x,ξ) and, together with the edge symbol, form a family of invertible operators

$$\begin{pmatrix} \sigma_\wedge(P)(x,\xi) & C(x,\xi) \\ B(x,\xi) & 0 \end{pmatrix} : \mathcal{K}^{s,\gamma}(K_\Omega) \oplus E_x \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega) \oplus F_x, \tag{6.54}$$

where E and F are finite-dimensional vector bundles over X. (This is not always possible; the obstruction, which has a topological nature, was computed in (Nazaikinskii, Savin, Sternin and Schulze 2004).) Since the edge symbol is twisted homogeneous, we can assume without loss of generality that the operators (6.50) are also twisted-homogeneous in ξ (the group $\varkappa_\lambda$ acts in the fibers of E and F as the identity operator by definition):

$$B(x,\lambda\xi) = \lambda^l B(x,\xi)\varkappa_\lambda^{-1}, \quad C(x,\lambda\xi) = \lambda^{l'}\varkappa_\lambda C(x,\xi). \tag{6.55}$$

(It suffices to define these operators for $|\xi| = 1$ and then extend them by continuity.)

Passing to the inverse Fourier transform and unfreezing the coefficients, from the family (6.54) we obtain the *edge problem*

$$Pu + C(x,-i\partial/\partial x)v = f, \tag{6.56}$$

$$B(x,-i\partial/\partial x)u = g, \tag{6.57}$$

where v and g are functions (more precisely, sections of the bundles E and F) on X.

Here $\widehat{B} = B(x,-i\partial/\partial x)$ is an operator of *edge boundary conditions* (or a *boundary operator* for short[6]), which takes each function $u(x,r,\omega)$ on $\mathcal{M}$ to the function

$$[\widehat{B}u](x) = \int e^{ix\xi} \int_{K_\omega} \chi(|\xi|)\vec{b}(x,\xi,r\xi,\omega)\widetilde{u}(\xi,r,\omega)r^{k+1}dr\,d\omega\,d\xi \tag{6.58}$$

on the boundary (here $\chi(|\xi|)$ is a cutoff function equal to unity in a neighborhood of infinity), and $\widehat{C} = C(x,-i\partial/\partial x)$ is an operator of *edge coboundary conditions*, or a *coboundary operator*. (The adjoint of $\widehat{C}$ has a form similar to (6.58).)

Weighted Sobolev spaces. In what spaces is problem (6.56), (6.57) Fredholm? These spaces are obtained from $K^{s,\gamma}(K_\Omega)$ by direct integration with respect to ξ (see the Appendix concerning the general theory).

DEFINITION 6.19 ((Schulze 1991)). By $\mathcal{W}^{s,\gamma}(W)$ we denote the space of functions on W obtained by the completion of $C_0^\infty(W^\circ)$ with respect to the norm

$$\|u\|_{s,\gamma} = \left(\int [\xi]^{2s} \left\|\varkappa_{[\xi]}^{-1}\widetilde{u}\right\|_{\mathcal{K}^{s,\gamma}} d\xi\right)^{1/2}, \tag{6.59}$$

where

$$\varkappa_\lambda v(r,\omega,\xi) = \lambda^{(k+1)/2} v(\lambda r,\omega,\xi), \quad k = \dim\Omega.$$

(From now on, in contrast to (6.30), we include the scalar factor $\lambda^{(k+1)/2}$ in the definition of the group $\varkappa_\lambda$; this normalization agrees with the one adopted in (Schulze 1991).)

The space $\mathcal{W}^{s,\gamma}(\mathcal{M})$ is obtained from $\mathcal{W}^{s,\gamma}(W)$ by a standard gluing with the Sobolev space $H^s(M)$ in the interior of M.

[6] We use terminology introduced for the case of Sobolev problems in (Sternin 1967*a*).

The spaces $\mathcal{W}^{s,\gamma}(W)$ have the simplest form for $s = \gamma$. Consider the space $L^2(\mathcal{M}, d\mathrm{vol})$, where $d\mathrm{vol}$ is the Riemannian volume form corresponding to the metric $d\rho^2$. Near the edge, this form is in the simplest case given by

$$d\mathrm{vol} = r^k dr d\omega dx, \quad k = \dim \Omega, \tag{6.60}$$

where $d\omega$ and dx are Riemannian volume forms on Ω_x and X, respectively. Consider also the operator

$$\widetilde{T} = 1 - \triangle_M + \frac{1}{r^2}, \tag{6.61}$$

where $\triangle_M$ is the Beltrami–Laplace operator on M corresponding to the metric $d\rho^2$ and r is the globally defined function on M equal to the distance from the boundary.

PROPOSITION 6.20. *For integer $s \geq 0$, an equivalent norm on the space $\mathcal{W}^{s,s}(\mathcal{M})$ is given by the formula*

$$\|u\|_s = \sqrt{(\widetilde{T}^s u, u)_0}, \quad u \in C_0^\infty(\mathcal{M}^\circ), \tag{6.62}$$

where $(\,\cdot\,,\,\cdot\,)_0$ is the inner product in $L^2(\mathcal{M}, d\mathrm{vol})$.

The proof goes by a standard functional-analytic argument. □

The following theorem can be proved by a straightforward computation.

THEOREM 6.21. *Outside an arbitrarily small neighborhood of the edge, the space $\mathcal{W}^{s,\gamma}(\mathcal{M})$ coincides with the corresponding Sobolev space $H^s(\mathcal{M})$. Every edge-degenerate operator $P \in \mathcal{D}_m$ of order m is continuous in the spaces*

$$P : \mathcal{W}^{s,\gamma}(\mathcal{M}) \longrightarrow \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}) \tag{6.63}$$

for every s and γ.

The finiteness theorem. Now we are in a position to state the finiteness theorem.

THEOREM 6.22. *Suppose that the operator P is formally elliptic and the edge symbol of problem* (6.56), (6.57) *is invertible in the spaces*

$$\begin{pmatrix} \sigma_\wedge(P)(x,\xi) & C(x,\xi) \\ B(x,\xi) & 0 \end{pmatrix} : \mathcal{K}^{s,\gamma}(K_\Omega) \oplus E_x \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega) \oplus F_x.$$

Then this problem is Fredholm in the spaces

$$\begin{pmatrix} P & C(x,-i\partial/\partial x) \\ B(x,-i\partial/\partial x) & 0 \end{pmatrix} : \mathcal{W}^{s,\gamma}(\mathcal{M}) \oplus H^{s-m+l'+(k+1)/2}(X,E)$$
$$\longrightarrow \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}) \oplus H^{s-l-(k+1)/2}(X,F).$$

The kernel and cokernel of the problem are independent of the smoothness parameter s.

The proof of the finiteness theorem is based on the calculus of edge pseudodifferential operators developed in forthcoming sections.

6.2. Pseudodifferential Operators

6.2.1. Edge symbols

The edge symbols of edge-degenerate differential operators are degenerate differential operators of special form with parameter ξ on the cone K_Ω. Accordingly, the edge symbols of edge-degenerate *pseudo*-differential operators should obviously be degenerate pseudodifferential operators with parameter. Our aim in this subsection is to construct some algebra of pseudodifferential operators with parameter on K_Ω including, in particular, the edge symbols of differential operators and (for elliptic symbols) their (almost) inverses.

These operators will depend on the parameters $(x,\xi) \in T_0^*X$ and act in the spaces $\mathcal{K}^{s,\gamma}(K_\Omega)$ on the cone K_Ω. Our considerations are mainly local with respect to x, and so without loss of generality we can work in local coordinates and assume that the parameter ξ ranges over $\mathbb{R}^n \setminus \{0\}$, where $n = \dim X$. Next, our operator classes are "bound" to a specific value of the weight exponent γ, whereas the smoothness parameter s runs over the entire real line. To describe continuous operators in families of Hilbert spaces, one could use the adequate technique of polynormed spaces or at least Fréchet spaces, but we refrain from using this complicated machinery and speak each time of individual pairs of Hilbert spaces. By K_Ω° we denote the open half-cylinder $(0,+\infty) \times \Omega$.

In what follows we especially often use a specific class of cutoff functions depending on the variable r. Thus it will be reasonable to give such functions a name.

DEFINITION 6.23. A smooth function $\psi(r)$ defined for $r \geq 0$, equal to unity for sufficiently small r, and vanishing for sufficiently large r is called an *R-function*.

Continuity properties. Prior to defining the set of edge symbols constructively, we shall describe the mapping properties that we wish to hold for these symbols. In a sense, we require as little as possible so as still to be able to prove the finiteness theorem and the smoothness of solutions.

Our edge symbols will be families of order m and weight γ in the sense of the following definition.

DEFINITION 6.24. A *family of order m and weight γ* is a smooth operator family

$$D(x,\xi) : C_0^\infty(K_\Omega^\circ) \longrightarrow \mathcal{D}'(K_\Omega^\circ)$$

parametrized by points $(x,\xi) \in T_0^*X$ and possessing the following properties:

1) (**twisted homogeneity**)

$$D(x,\lambda\xi) = \lambda^m \varkappa_\lambda D(x,\xi) \varkappa_\lambda^{-1}, \quad \lambda > 0$$

(recall that $\varkappa_\lambda u(r,\omega) = \lambda^{(k+1)/2} u(\lambda r,\omega)$);

2) (**continuity**) the family $D(x,\xi)$ and all of its derivatives extend by closure to smooth families of continuous operators in the spaces

$$D^{(\alpha,0)}(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega), \qquad |\alpha| = 0,1,2,\dots, \tag{6.64}$$

$$D^{(\alpha,\beta)}(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m+1,\gamma-m}(K_\Omega), \qquad |\alpha| = 0,1,2,\dots, \; |\beta| = 1,2,\dots, \tag{6.65}$$

for a given weight $\gamma \in \mathbb{R}$ and for each $s \in \mathbb{R}$;

3) (**almost compact fiber variation**) for an arbitrary R-function $\varphi(r)$, the operator families

$$\varphi(r)D(x,\xi),\ D(x,\xi)\varphi(r) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega) \tag{6.66}$$

have compact fiber variation (i.e., the operators $\varphi(r)\partial D/\partial\xi(x,\xi)$ and $\partial D/\partial\xi(x,\xi)\varphi(r)$ are compact in this pair of spaces);

4) conditions 1)–3) remain valid for the family $(1+r|\xi|)^l D(x,\xi)(1+r|\xi|)^{-l}$ for arbitrary $l \in \mathbb{R}$.

Remark 6.25. 1. The smoothness of our families is assumed in the strong operator topology and hence[7] in the uniform operator topology.

2. In what follows, we do not usually mention extension by closure explicitly and speak merely of the continuity of the operator in the corresponding spaces; the closure is denoted by the same letter as the original operator.

PROPOSITION 6.26. *If $D \in \mathcal{D}_m(\mathcal{M})$ is an edge-degenerate differential operator of order m, then its edge symbol is a family of order m for each γ.*

Proof. The proof is by straightforward verification. □

Remark 6.27. Note that the estimate (6.65) remains valid for edge symbols of differential operators if we replace the right-hand side by $\mathcal{K}^{s-m+|\beta|,\gamma-m+|\beta|}(K_\Omega)$.

Compact smoothing edge symbols. Instead of defining the entire set of edge symbols in one step, it is more convenient to describe the subset of compact smoothing edge symbols first.

DEFINITION 6.28. A family $D(x,\xi)$ of order m and weight γ is said to be a *compact smoothing edge symbol* if it additionally satisfies the following properties:

5) the operators $D(x,\xi)$ are compact in the spaces

$$D(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega);$$

6) the operators (6.65) are continuous also for $\beta = 0$;

7) properties 5) and 6) remain valid for the operators for the families $rD(x,\xi)$ and $D(x,\xi)r$.

The set of compact smoothing edge symbols of order m and weight γ will be denoted by I^m_γ.

PROPOSITION 6.29. 1. *The multiplication of operators induces a bilinear mapping*

$$I^l_{\gamma-m} \times I^m_\gamma \longrightarrow I^{m+l}_\gamma$$

for any $\gamma, m, l \in \mathbb{R}$.

2. *The passage to the adjoint operator (with respect to the inner product in $\mathcal{K}^{0,0}(K_\Omega)$) induces an antilinear mapping*

$$I^m_\gamma \longrightarrow I^m_{m-\gamma}$$

for any $\gamma, m \in \mathbb{R}$.

[7] The resonance theorem (Yosida 1968) guarantees the uniform boundedness of all derivatives.

Proof. Properties 1), 2), and 6) are respected by multiplication as well as by the passage to the adjoint operator. Property 3) is not respected in general, but for compact smoothing edge symbols it is covered by the stronger property 5), which is already preserved under these operations. The conjugation with the operator $(1+r|\xi|)^l$, which is self-adjoint in $\mathcal{K}^{0,0}(K_\Omega)$, is a homomorphism and hence preserved under multiplication; when passing to the adjoint operator, we should just replace l by $-l$. This proves that property 4) is preserved. Property 7) obviously remains valid for the adjoint operators; as to the products, one should only note that property 7) is equivalent to the same property with the operators $(1+r|\xi|)D(x,\xi)$ and $D(x,\xi)(1+r|\xi|)$ instead of $rD(x,\xi)$ and $D(x,\xi)r$ and then use property 4). □

The definition of general edge symbols. Now we are in a position to describe general edge symbols. These symbols are families of pseudodifferential operators on the infinite cones K_Ω and, as such, they have principal symbols and conormal symbols.

Let $H = H(x,\omega,\xi,p,q) \in \mathcal{O}^m(\partial T_0^*\mathcal{M})$ be a smooth homogeneous function of order m on the boundary $\partial T_0^*\mathcal{M}$ of the cotangent bundle of $\mathcal{M}$. Next, let $h(x,p)$ be an mth-order conormal symbol on the weight line

$$p \in \mathcal{L}_\gamma = \left\{\operatorname{Im} p = \gamma + \frac{k+1}{2}\right\}. \tag{6.67}$$

We assume that $h(x,p)$ depends on the parameter $x \in X$ but is *independent of the variable ξ in the fibers of T^*X*. Next, we assume that the principal symbol H and the conormal symbol h satisfy the *compatibility condition*

$$\sigma(h) = H(x,\omega,0,-p,q), \tag{6.68}$$

where the symbol on the left-hand side is the principal symbol of the operator h viewed as a pseudodifferential operator on Ω with parameter $p \in \mathcal{L}_\gamma$ in the sense of Agranovich–Vishik (Agranovich and Vishik 1964).

Our aim is to define an edge symbol

$$D(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega)$$

with the symbol pair (H,h). The construction is carried out separately in a neighborhood of the cone vertex and outside the neighborhood; the results are then glued together with the use of a partition of unity constructed from R-functions.

We proceed with the definition for $|\xi| = 1$. Later on we extend the definition to all ξ by twisted homogeneity.

Near the vertex, the spaces $\mathcal{K}^{s,\gamma}(K_\Omega)$ coincide with their counterparts $H^{s,\gamma}(K_\Omega)$, and we define $D(x,\xi)$ there as the family of cone-degenerate operators

$$\widehat{h} = r^{-m} h\left(x, ir\frac{\partial}{\partial r}\right) : H^{s,\gamma}(K_\Omega) \longrightarrow H^{s-m,\gamma-m}(K_\Omega), \quad x \in X. \tag{6.69}$$

Note that this operator is actually independent of ξ, but this is no surprise, because it is near infinity on the cone that the dependence on ξ becomes essential.

Away from a neighborhood of the vertex, the conormal symbol plays no role at all, and all we need to define our operator family there is the principal symbol $H(x,\omega,\xi,p,q)$. Informally speaking, the operator, $\widehat{H}$, should be obtained from the symbol H by the substitution

$$p \mapsto -i\frac{\partial}{\partial r}, \quad q \mapsto -\frac{i}{r}\frac{\partial}{\partial \omega}.$$

To make this precise, we first note that since we work away from the vertex, we can multiply the symbol by a smooth cutoff function $\rho(r)$ equal to zero for small r and unity for large r. Next, consider a partition $\sum e_j = 1$ of unity subordinate to some finite coordinate cover on Ω and excision functions f_j supported in the same coordinate neighborhoods as the respective e_j and satisfying the condition $e_j f_j = e_j$. We set

$$F_j(x, r, \omega, \xi, p, q) = \rho(r) e_j(\omega) H(x, \omega, \xi, p, q).$$

As is customary in the theory of pseudodifferential operators, we shall define $\widehat{H}$ by the formula

$$\widehat{H} = \sum_j \widehat{F}_j \circ f_j, \tag{6.70}$$

and so the problem is to define the local representatives $\widehat{F}_j$ corresponding to coordinate neighborhoods on Ω. Consider the change of variables

$$\alpha : [\varepsilon, \infty) \times \mathbb{R}^k_\omega \longrightarrow \mathbb{R}^{k+1}_y$$

given by the formulas

$$y_0 = r, \quad y' \equiv (y_1, \dots, y_k) = r\omega.$$

In the new variables y, we define $\widehat{F}$ to be the pseudodifferential operator

$$\widehat{F}_j = P\left(x, \frac{\overset{2}{y'}}{y_0}, \xi, -i\frac{\overset{1}{\partial}}{\partial y}\right) \tag{6.71}$$

in $\mathbb{R}^{k+1}_y$ with symbol

$$P(y, \eta) = F_j(x, y/y_0, \xi, \eta).$$

(Here η is the variable dual to y.)

The symbol $P(y, \eta)$ satisfies the estimates (recall that $|\xi| = 1$)

$$|P^{(\alpha,\beta)}(y, \eta)| \le C_{\alpha\beta}(1 + |\eta|)^{m-|\beta|}(1 + |y|)^{-\alpha} \tag{6.72}$$

and hence the operator $\widehat{F}_j$ in the new variables y is continuous in the Sobolev spaces

$$\widehat{F}_j : H^s(\mathbb{R}^{k+1}_y) \longrightarrow H^{s-m}(\mathbb{R}^{k+1}_y).$$

Returning to the original variables and recalling the definition of $\mathcal{K}^{s,\gamma}$, we see that the operator $\widehat{F}_j$ is continuous in the spaces

$$\widehat{F}_j : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega).$$

Now we patch together our constructions and extend to arbitrary $|\xi|$ by twisted homogeneity, thus arriving at the following definition.

DEFINITION 6.30. An *edge symbol of order* m *and weight* γ *with principal symbol* H *and conormal symbol* h is a family of order m representable modulo compact smoothing edge symbols (elements of the space I^γ_m) in the form

$$D(x, \xi) = \chi_1(r|\xi|)\widehat{h}\psi(r|\xi|) + (1 - \chi_2(r|\xi|))\widehat{H}(1 - \psi(r|\xi|)), \tag{6.73}$$

where χ_1, χ_2, and ψ are R-functions such that

$$\chi_1\psi = \psi, \quad (1-\chi_2)(1-\psi) = 1-\psi, \tag{6.74}$$

the operator $\widehat{h}$ is defined by the formula (6.69), and the operator $\widehat{h}$ is defined by the formula (6.70). The set of edge symbols of order m and weight γ will be denoted by $\mathrm{Edge}^m_\gamma \equiv \mathrm{Edge}^m_\gamma(T^*_0X)$. We write $H = \sigma(D(x,\xi))$ and $h = \sigma_c(D(x,\xi))$.

THEOREM 6.31. 1) *The family* (6.73) *is indeed a family of order m and is independent of the ambiguity in the construction modulo compact smoothing operators. In other words, if $D \in \mathrm{Edge}^m_\gamma$, then the principal and conormal symbols of D are well defined, and $D \in I^m_\gamma$ if and only if $\sigma(D) = 0$ and $\sigma_c(D)$=0.*

2) *The product of operators induces a bilinear mapping*

$$\mathrm{Edge}^l_{\gamma-m} \times \mathrm{Edge}^m_\gamma \longrightarrow \mathrm{Edge}^{m+l}_\gamma,$$

and the composition law

$$\sigma(D_1D_2) = \sigma(D_1)\sigma(D_2), \quad \sigma_c(D_1D_2) = \sigma_c(D_1)\sigma_c(D_2) \tag{6.75}$$

holds.

3) *The passage to the adjoint operator* (*with respect to the inner product in $\mathcal{K}^{0,0}$*) *induces an antilinear mapping*

$$\mathrm{Edge}^m_\gamma \longrightarrow \mathrm{Edge}^m_{m-\gamma},$$

and

$$\sigma(D^*) = \sigma(D)^*, \sigma_c(D^*) = \sigma_c(D)^*. \tag{6.76}$$

4) *If $D \in \mathrm{Edge}^m_\gamma$ and $\varphi(r)$ is a smooth function bounded together with all derivatives and equal to zero for large r, then the operator families $\varphi(r)D$ and $D\varphi(r)$ have compact fiber variation in the spaces*

$$\mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega).$$

Proof. 1) Let us verify conditions 1) and 2) in the definition of a family of order m. Condition 1) (twisted homogeneity) follows by a straightforward computation from (6.73). Let us verify condition 2), i.e., the continuity of the family $D(x,\xi)$ and its derivatives in the spaces (6.64) and. The continuity of the operators (6.64) is clear from the construction for both terms in (6.73) separately. To prove the continuity of the operators (6.65), which involve ξ-derivatives, note that the only troublesome terms are those arising from the differentiation of ψ and $1-\psi$ in (6.73). (Indeed, the differentiation of $\widehat{H}$ gives an operator with the desired properties, and so does the differentiation of χ_1 and χ_2, since the derivatives of these functions are zero on the supports of ψ and $1-\psi$, respectively.) Now note that $\chi_1 = \chi_2 = 1$ on the support and that, owing to the compatibility condition (6.68), $\widehat{h}$ and $\widehat{H}$ are operators with the same principal symbol on the (compact) support of ψ'; it follows that the troublesome derivatives cancel each other out modulo terms satisfying the desired estimates.

The family $D(x,\xi)$ is independent modulo smoothing operators of the ambiguity in the construction (the choice of the R-functions χ_1, χ_2, and φ as well as partitions of unity and excision functions on Ω).

This follows from the composition formulas and formulas for the change of variables in pseudodifferential operators in conjunction with the estimate (6.72). The key point is as follows: this estimate implies that the remainder terms in these formulas contain the factor $(1+r)^{-1}$, which guarantees the validity of condition 7) in Definition 6.28. Finally, the validity of condition 4) in the definition of a family of order m follows from the fact that the set of pseudodifferential operators with symbols satisfying the estimate (6.72) is invariant with respect to conjugation by $(1+y^2)^{l/2}$ for any l.

2) This property follows from the composition formula for pseudodifferential operators; the desired estimates guaranteeing that the remainders belong to I_γ^m again follow from (6.72).

3) To prove this property, one passes to the adjoint operator in the formula (6.73) and uses the fact that the remainder arising from the change in the order of action of operator arguments in the pseudodifferential operator belongs to the space I_γ^m by virtue of the estimates (6.72).

4) It suffices to prove that the fiber variation of the product of the symbol (6.73) by $\varphi(r)$ on the right or on the left is compact. (Compact smoothing symbols have compact fiber variation even without this factor.) We represent the symbol (6.73) in the form

$$D(x,\xi)=B(x,\xi,\xi),$$

where the first argument ξ of B corresponds to the argument ξ in the cutoff functions χ_1, χ_2, and ψ, and the second to the argument ξ in the symbol $\widetilde{H}$ itself. Then we have

$$D(x,\xi)-D(x,\xi')=\bigl[B(x,\xi,\xi)-B(x,\xi',\xi)\bigr]+\bigl[B(x,\xi',\xi)-B(x,\xi',\xi')\bigr].$$

The expression in the first brackets is a pseudodifferential operator of order $m-1$ with compactly supported Schwartz kernel and hence is compact. Hence it suffices to prove that the second expression multiplied by $\varphi(r)$ on the right or on the left is compact. But this assertion is obvious, since the symbol $\partial H/\partial\xi$ is homogeneous of degree $m-1$ in the momentum variables and hence the operator $\partial\widehat{H}/pa\xi$ (without the factor $\varphi(r)$) is continuous in the spaces $\mathcal{K}^{s,\gamma}(K_\Omega)\longrightarrow\mathcal{K}^{s-m+1,\gamma-m}(K_\Omega)$. The multiplication by $\varphi(r)$ with regard to the fact that the support of the kernel is separated from $r=0$ gives the desired compactness result.

The proof of the theorem is complete. □

PROPOSITION 6.32. *If $D\in\mathcal{D}_m(\mathcal{M})$ is an edge-degenerate operator of order m, then $\sigma_\wedge(D)\in\mathrm{Edge}_\gamma^m$ for each γ.*

The proof is by a straightforward verification. □

Fredholm property and smoothness. To construct almost inverses of edge-degenerate operators in what follows, we need the inverses of their edge symbols. As a rule, the edge symbol of an interior-elliptic operator is not invertible but (in the best possible case) only Fredholm, and to make it invertible one has to supplement it with conditions and co-conditions (which results in the appearance of edge boundary and coboundary conditions for the original operator). Let us give some facts concerning Fredholm edge symbols.

THEOREM 6.33. *Let $D(x,\xi)\in\mathrm{Edge}_\gamma^m$ be a given edge symbol. If it is Fredholm* (*respectively, invertible*) *for some $s\in\mathbb{R}$, then its principal symbol is everywhere invertible outside the zero section, the conormal symbol is invertible on the weight line, and the almost inverse* (*respectively, inverse*) *of D lies in $\mathrm{Edge}_{\gamma-m}^{-m}$. Next, the following conditions are equivalent*:

(i) *The family*

$$D(x,\xi):\mathcal{K}^{s,\gamma}(K_\Omega)\longrightarrow\mathcal{K}^{s-m,\gamma-m}(K_\Omega) \tag{6.77}$$

is Fredholm for some $s\in\mathbb{R}$.

(ii) *The family* (6.77) *is Fredholm for all* $s\in\mathbb{R}$.

(iii) *The principal symbol* $\sigma(D)(\omega,\xi,p,q)$ *is invertible for* $(\xi,p,q)\neq 0$, *and the conormal symbol* $\sigma_c(D)(p)$ *is invertible for* $p\in\mathcal{L}_\gamma$.

Moreover, if the edge symbol is Fredholm, then its kernel, cokernel, and index are independent of s.

Proof. Let $D(x,\xi)\in\mathrm{Edge}^m_\gamma$ be a Fredholm family in the spaces

$$D(x,\xi):\mathcal{K}^{s,\gamma}(K_\Omega)\longrightarrow\mathcal{K}^{s-m,\gamma-m}(K_\Omega)$$

Let us prove that the principal symbol $\sigma(D(x,\xi))$ vanishes nowhere outside the zero section on $\partial T^*\mathcal{M}$. Suppose the contrary: $\sigma(D(x,\xi))=0$ at some point $(x_*,\omega_*,\xi_*,q_*,p_*)\in\partial T^*_0\mathcal{M}$. Then one can readily construct a sequence of functions destroying the standard *a priori* estimate that follows from the Fredholm property. Here we distinguish between two cases.

i) $\xi_*=0$. Then the *a priori* estimate is violated for $D(x_*,\xi)$ for any $\xi\neq 0$ on the sequence

$$\psi_\lambda(\omega,r)=e^{i\lambda(\omega q_*+rp_*)}\varphi(\sqrt{\lambda}(r-1),\sqrt{\lambda}(\omega-\omega_*)),$$

weakly convergent to zero, where φ is a smooth function supported in a neighborhood of the origin.

ii) $\xi_*\neq 0$. Then the *a priori* estimate is violated for $D(x_*,\xi_*)$ on the sequence

$$\psi_\lambda(\omega,r)=e^{i\lambda(\omega q_*+rp_*)}\varkappa_\lambda\varphi(\sqrt{\lambda}(r-1),\sqrt{\lambda}(\omega-\omega_*)),$$

weakly convergent to zero.

Thus $\sigma(D(x,\xi))$ vanishes nowhere outside the zero section, and hence there exists an edge symbol $B(x,\xi)$ such that

$$\sigma(B(x,\xi))=\sigma(D(x,\xi))^{-1}.$$

By the composition formulas, we then have

$$D(x,\xi)B(x,\xi)=1+R(x,\xi),\quad B(x,\xi)D(x,\xi)=1+\widetilde{R}(x,\xi), \tag{6.78}$$

where $R\in J^0_\gamma(K_\Omega)$, $\widetilde{R}\in J^0_{\gamma-m}(K_\Omega)$.

Now if

$$Q(x,\xi):\mathcal{K}^{s-m,\gamma-m}(K_\Omega)\longrightarrow\mathcal{K}^{s,\gamma}(K_\Omega)$$

is the inverse of $D(x,\xi)$, then we obtain

$$B-Q=RQ=Q\widetilde{R},$$

whence the desired assertion follows readily. The computations for the case in which Q is an almost inverse (modulo the projections onto the kernel and cokernel of D) are similar.

Now let us prove that the kernel, cokernel, and index of $D(x,\xi)$ are independent of s. Indeed, if $u\in\ker D(x,\xi)$, then

$$0=BDu=u+Ru,$$

and the operator R increases the smoothness by one. It follows that the elements of the kernel are infinitely smooth and the kernel is independent of s. The corresponding assertion for the cokernel is obtained from the second equation in (6.78). The assertion about the index follows from the assertions about the kernel and the cokernel. ☐

Smoothing edge symbols of order m. In what follows we also use the ideal of smoothing edge symbols.

DEFINITION 6.34. An edge symbol $D(x,\xi)$ of order m and weight γ is said to be *smoothing* if its principal symbol is zero. The set of smoothing edge symbols of order m and weight γ will be denoted by J^m_γ.

Smoothing edge symbols possess the following properties.

PROPOSITION 6.35. *Let* $D \in J^m_\gamma$.

1) (*gain in smoothness*) *the operator* $D(x,\xi)$ *and all of its derivatives are continuous in the spaces*

$$D^{(\alpha,\beta)}(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m+1,\gamma-m}(K_\Omega), \quad s \in \mathbb{R};$$

2) *the operators* $rD(x,\xi)$ *and* $D(x,\xi)r$ *possess the same property;*

3) *the family*

$$D(x,\xi) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega)$$

has a compact fiber variation (*i.e., the differences* $D(x,\xi)-D(x,\xi')$ *are compact in these spaces*)*;*

4) *conditions* 1)–3) *remain valid for the family* $(1+r^2)^l D(x,\xi)(1+r^2)^{-l}$ *for arbitrary* $l \in \mathbb{R}$*;*

5) *if* $\varphi(r)$ *is a smooth function such that* $\varphi(r)=0$ *in a neighborhood of zero and* $\varphi(r)=1$ *for large* r*, then the operators*

$$\varphi(r)D(x,\xi),\ D(x,\xi)\varphi(r) : \mathcal{K}^{s,\gamma}(K_\Omega) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(K_\Omega)$$

are compact.

Proof. Properties 1)–4) are an immediate consequence of the definitions. Let us prove 5). The operators $\varphi(r)D(x,\xi)$ and $D(x,\xi)\varphi(r)$ in these spaces can be represented as the compositions

$$\mathcal{K}^{s,\gamma}(K_\Omega) \xrightarrow{rD(x,\xi)} \mathcal{K}^{s-m+1,\gamma-m}(K_\Omega) \xrightarrow{\varphi(r)/r} \mathcal{K}^{s-m,\gamma-m}(K_\Omega)$$

and

$$\mathcal{K}^{s,\gamma}(K_\Omega) \xrightarrow{\varphi(r)/r} \mathcal{K}^{s-1,\gamma}(K_\Omega) \xrightarrow{D(x,\xi)r} \mathcal{K}^{s-m,\gamma-m}(K_\Omega),$$

respectively, of continuous operators. Moreover, the operator $\varphi(r)/r$ is compact in both cases. (Indeed, the smoothness exponent in both cases is diminished by one, the weight exponent remains unchanged, and the function $\varphi(r)/r$ is equal to zero in a neighborhood of zero and tends to zero at infinity.) □

6.2.2. Pseudodifferential operators

Now we shall define pseudodifferential operators. This shall be done in several steps. First, we describe their general continuity properties. Then from the set of all operators satisfying these continuity properties we single out pseudodifferential operators as operators that possess edge and interior principal symbols. Finally, we define quantization, i.e., a mapping that assigns pseudodifferential operators to compatible pairs (interior principal symbol, edge symbol) and show that this mapping possesses all necessary properties.

Operators of order m***.*** We choose and fix some value of the weight exponent $\gamma \in \mathbb{R}$.

DEFINITION 6.36. An operator

$$D : C_0^\infty(\mathcal{M}^\circ) \longrightarrow \mathcal{D}'(\mathcal{M}^\circ)$$

is called an *operator of order* m (*with weight exponent* γ) if it can be extended to a continuous operator in the spaces

$$D : \mathcal{W}^{s,\gamma}(\mathcal{M}) \longrightarrow \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}) \tag{6.79}$$

for every $s \in \mathbb{R}$. An operator of order m is said to be *negligible* if it is compact in the spaces (6.79) and continuous in the spaces

$$D : \mathcal{W}^{s,\gamma}(\mathcal{M}) \longrightarrow \mathcal{W}^{s-m+1,\gamma-m}(\mathcal{M}) \tag{6.80}$$

for every $s \in \mathbb{R}$.

The space of operators of order m will be denoted by $\mathrm{Op}_\gamma^m = \mathrm{Op}_\gamma^m(\mathcal{M})$, and the subspace of negligible operators by $J\,\mathrm{Op}_\gamma^m(\mathcal{M}) \subset \mathrm{Op}_\gamma^m(\mathcal{M})$.

The following assertion is obvious.

PROPOSITION 6.37. *The operator product induces bilinear mappings*

$$\mathrm{Op}_{\gamma-m}^l \times \mathrm{Op}_\gamma^m \longrightarrow \mathrm{Op}_\gamma^{l+m},$$
$$J\,\mathrm{Op}_{\gamma-m}^l \times \mathrm{Op}_\gamma^m \longrightarrow J\,\mathrm{Op}_\gamma^{l+m}, \quad \mathrm{Op}_{\gamma-m}^l \times J\,\mathrm{Op}_\gamma^m \longrightarrow J\,\mathrm{Op}_\gamma^{l+m}$$

for any l *and* m*.*

The compatibility condition. Our pseudodifferential operators will have interior principal symbol σ and edge symbols $\sigma_\wedge$.

What pairs $(\sigma, \sigma_\wedge)$ should be quantized? Note that the interior principal symbol and the edge symbol of a *differential* operator are related by the compatibility condition (6.29).

We impose the same compatibility condition on the symbols of pseudodifferential operators.

The definition of pseudodifferential operators. We introduce the notion of smooth functions on $\mathcal{M}$.

DEFINITION 6.38. A function φ on a manifold $\mathcal{M}$ with edges is said to be *smooth* if the lift of φ to the corresponding manifold M with boundary via the natural projection $M \longrightarrow \mathcal{M}$ is a smooth function. In other words, $\varphi \in C^\infty(\mathcal{M})$ if and only if $\varphi \in C^\infty(M)$ and the restriction $\varphi|_{\partial M}$ is constant along the fibers of π.

Now we are in a position to give the definition of pseudodifferential operators.

DEFINITION 6.39. An operator $P \in \mathrm{Op}_\gamma^m$ is called a *pseudodifferential operator of order* m *and weight* γ on the manifold $\mathcal{M}$ if the following conditions are satisfied.

1) The inclusion $[\varphi, P] \in J\,\mathrm{Op}_\gamma^m$ holds for each function $\varphi \in C^\infty(M)$.

Next, there exists a smooth interior principal symbol σ of order m on T_0^*M and an edge symbol $\sigma_\wedge \in \mathrm{Edge}_\gamma^m$ satisfying the compatibility condition (6.29) and such that

2) the operator P is pseudodifferential with principal symbol σ on the open manifold M°;

3) the operator P can be represented modulo $J\operatorname{Op}_\gamma^m$ in the form

$$P = (\varphi(r)\sigma_\wedge)\left(x, -i\frac{\partial}{\partial x}\right) + rQ, \tag{6.81}$$

where $\varphi(r)$ is an R-function and $Q \in \operatorname{Op}_\gamma^m$ is a pseudodifferential operator on the open manifold M° such that Q can be represented in a neighborhood of X (that is, after the multiplication by R-functions on the right and on the left) as a pseudodifferential operator on X with operator-valued symbol

$$q(x,\xi) \in S^m_{CV}(X; K^{s,\gamma}(K_\Omega), K^{s-m,\gamma-m}(K_\Omega))$$

of compact variation (see (Luke 1972)) for each $s \in \mathbb{R}$.

The set of pseudodifferential operators of order m and weight γ will be denoted by $\operatorname{PSD}_\gamma^m \equiv \operatorname{PSD}_\gamma^m(\mathcal{M})$.

Remark 6.40. The principal symbol of the operator Q in (6.81) is of course

$$\sigma(Q) = r^{-1}\big(\sigma - \sigma|_{\partial T^*M}\varphi(r)\big).$$

PROPOSITION 6.41. *The interior principal symbol and the edge symbol of a pseudodifferential operator are uniquely determined.*

Proof. The assertion concerning the interior principal symbol is known from the usual theory of pseudodifferential operators. To prove that the edge symbol is also uniquely determined, let us represent P in a collar neighborhood of the edge in the form of a pseudodifferential operator on X with operator-valued symbol of compact variation. This is clearly possible for both terms in the representation (6.81). Thus we have

$$P = F\left(x, -i\frac{\partial}{\partial x}\right),$$

where

$$F(x,\xi) = \varphi(r)\sigma_\wedge(x,\xi) + rq(x,\xi).$$

(Here q is the operator-valued symbol of Q.) Now we apply $\varkappa_\lambda^{-1}F(x,\xi)\varkappa_\lambda$ to an arbitrary function $u \in C_0^\infty(K_\Omega^\circ)$ and let $\lambda \to \infty$, thus obtaining

$$\begin{aligned}\varkappa_\lambda^{-1}F(x,\xi)\varkappa_\lambda u &= \varphi(r/\lambda)\lambda^m\sigma_\wedge(x,\xi)u + \lambda^{-1}rq(x,\lambda\xi)u\\ &= \varphi(r/\lambda)\lambda^m\sigma_\wedge(x,\xi)u + O(\lambda^{m-1}).\end{aligned}$$

This permits us to reconstruct the edge symbol from the operator and completes the proof of the proposition. □

6.2.3. Quantization

The computation of symbols is embedded in the very definition of pseudodifferential operators. Now our task is to construct an inverse mapping, *quantization*. Thus pseudodifferential operators will be obtained by quantization of *pairs* $(\sigma, \sigma_\wedge)$, where σ is the interior principal symbol (a homogeneous function on $T_0^*\mathcal{M}$) and $\sigma_\wedge = \sigma_\wedge(x,\xi)$ is the edge symbol.

Let $\mathcal{A}^m_\gamma$ be the set of pairs $(\sigma,\sigma_\wedge)$ satisfying the compatibility condition (6.29), where σ is an interior principal symbol of order m and $\sigma_\wedge\in\mathrm{Edge}^m_\gamma$. This is obviously a linear space, and we have the natural embedding

$$\begin{aligned} J^m_\gamma &\hookrightarrow \mathcal{A}^m_\gamma\\ \sigma_\wedge &\longmapsto (0,\sigma_\wedge). \end{aligned}$$

Having this embedding in mind, we sometimes denote the corresponding element of $\mathcal{A}^m_\gamma$ merely by $\sigma_\wedge$ instead of $(0,\sigma_\wedge)$.

PROPOSITION 6.42. *The componentwise multiplication induces a bilinear mapping*

$$\mathcal{A}^l_{\gamma-m}\times\mathcal{A}^m_\gamma\longrightarrow\mathcal{A}^{m+l}_\gamma,$$

and the subspace of smoothing edge symbols is an "ideal" in the sense that

$$J^l_{\gamma-m}\times\mathcal{A}^m_\gamma\longrightarrow J^{m+l}_\gamma,\quad \mathcal{A}^l_{\gamma-m}\times J^m_\gamma\longrightarrow J^{m+l}_\gamma.$$

Proof. Both assertions follow from item 2) of Theorem 6.31, since the compatibility condition is linear and multiplicative. ☐

Quantization of the ideal J^m_γ**.** Note that the above-mentioned embedding $J^m_\gamma\hookrightarrow\mathcal{A}^m_\gamma$ gives rise to the short exact sequence

$$0\longrightarrow J^m_\gamma\longrightarrow\mathcal{A}^m_\gamma\longrightarrow\mathcal{O}^m\longrightarrow 0,$$

where $\mathcal{O}^m\equiv\mathcal{O}^m(T^*_0\mathcal{M})$ is the space of interior principal symbols of order m. We use this exact sequence to construct quantization modulo negligible operators. Namely, first we quantize the elements of the ideal J^m_γ modulo negligible operators and the elements of the quotient

$$\mathcal{A}^m_\gamma\;/\;J^m_\gamma\equiv\mathcal{O}^m \tag{6.82}$$

modulo operators with zero interior principal symbol. The resulting pair of quantizations of the extreme terms of the sequence is then lifted in a standard way to a quantization of the middle term.

Thus let us quantize the ideal J^m_γ. Let $a\in J^m_\gamma$, i.e., $a=(0,\sigma_\wedge)$, where $\sigma_\wedge\in J^m_\gamma$. We take a smooth function

$$\varphi(r)=\begin{cases}1 & \text{for small } r,\\ 0 & \text{for } r>1/2\end{cases}$$

and set

$$\widehat{a}=\varphi(r)\sigma_\wedge\left(x,-i\frac{\partial}{\partial x}\right)\varphi(r), \tag{6.83}$$

where the pseudodifferential operator $\sigma_\wedge(x,\xi)$ with operator-valued symbol is defined in the usual way with the help of a partition of unity subordinate to a cover of X by local charts (see the Appendix). Owing to the presence of the factors $\varphi(r)$ on the right and on the left, we can interpret the operator (6.83) not only as an operator on the infinite wedge $\mathcal{W}$ but also as an operator on $\mathcal{M}$. (The support of its kernel is contained in the Cartesian product $U\times U$.)

THEOREM 6.43. *The following assertions hold:*

1) *The interior principal symbol of the operator* $\widehat{a}$ *is zero.*

2) *The operator* $\widehat{a}$ *is modulo the subspace* $J\operatorname{Op}_\gamma^m$ *independent of the choice of a cutoff function* φ *and other ambiguous elements of the construction.*

Proof. If $a \in \mathcal{J}_\gamma^m$, then it follows from property 1 in Proposition 6.35 that the products $(1-\psi(r))\widehat{a}$ and $\widehat{a}(1-\psi(r))$ raise the smoothness by one and are compact in the corresponding spaces. From this we readily obtain both assertions of the theorem. □

Quantization of the entire symbol algebra $\mathcal{A}_\gamma^m$**.** Let $a = (\sigma, \sigma_\wedge) \in \mathcal{A}_\gamma^m$ be a given symbol. We wish to assign a pseudodifferential operator to it. This can be done as follows.

1) First, we construct some pseudodifferential operator $P_0 \in \mathrm{PSD}_\gamma^m$ with interior principal symbol σ.

2) Then the edge symbols $\sigma_\wedge$ and $\sigma_\wedge(P_0)$ are compatible with the same interior principal symbol σ, so that their difference is compatible with the zero interior principal symbol. This means that

$$\sigma_\wedge - \sigma_\wedge(P_0) \in J_\gamma^m.$$

3) Thus, using the construction from the preceding item, we can construct a pseudodifferential operator P_1 with interior principal symbol $\sigma(P_1) = 0$ and edge symbol

$$\sigma_\wedge(P_1) = \sigma_\wedge - \sigma_\wedge(P_0).$$

4) It remains to set

$$\widehat{a} = P_0 + P_1;$$

now we have

$$\sigma(\widehat{a}) = \sigma, \quad \sigma_\wedge(\widehat{a}) = \sigma_\wedge$$

by construction.

All steps except for the first are obvious, and it remains to explain the first step. It suffices to consider the case in which the interior principal symbol σ is supported in a neighborhood of the edge. (The general case then follows with the use of a partition of unity.) We set

$$P_0 = e(r) D\left(x, -i\frac{\partial}{\partial x}\right) e(r), \tag{6.84}$$

where $e(r)$ is an R-function equal to unity in a wider neighborhood of the edge and the symbol $D(x,\xi)$ is given by the formula

$$D(x,\xi) = \chi_1(r|\xi|)\widehat{h}\psi(r|\xi|) + (1-\chi_2(r|\xi|))\widehat{\sigma}(1-\psi(r|\xi|)). \tag{6.85}$$

Here χ_1, χ_2, and ψ are R-functions such that

$$\chi_1\psi = \psi, \quad (1-\chi_2)(1-\psi) = 1-\psi, \tag{6.86}$$

h is an arbitrary conormal symbol compatible with σ, and the operator $\widehat{\sigma}$ is constructed in the same way as in the definition of edge symbols. (The only difference is that the symbol σ has an additional dependence on the variable r. This does not cause any complications.)

PROPOSITION 6.44. *The operator* (6.84) *satisfies* $P_0 \in \mathrm{PSD}^m_\gamma$ *and*

$$\sigma(P_0) = \sigma.$$

Proof. The fact that the operator (6.84) is a pseudodifferential operator on the open manifold M° and the relation $\sigma(P_0) = \sigma$ are trivial. To prove that $P_0 \in \mathrm{PSD}^m_\gamma$, it suffices to show that P_0 has an edge symbol. This is clear directly from (6.84). Namely, applying the definition of the edge symbol, we see that the edge symbol of P_0 can be obtained by setting $r = 0$ in the coefficients of the operator $\widehat{\sigma_c}$ and in the symbol σ. □

Calculus. The quantization mapping constructed above has the following properties.

THEOREM 6.45. *The quantization mapping is well defined and unique modulo negligible operators. This mapping is the right inverse and the left almost inverse of the symbol mapping:*

$$\textit{if}\quad a = (\sigma, \sigma_\wedge) \in \mathcal{A}^m_\gamma, \quad \textit{then}\quad \sigma(\widehat{a}) = \sigma,\ \ \sigma_\wedge(\widehat{a}) = \sigma_\wedge;$$
$$\textit{if}\quad P \in \mathrm{PSD}^m_\gamma, \quad \textit{then}\quad (\sigma(P), \sigma_\wedge(P))^\wedge - P \in J\,\mathrm{Op}^m_\gamma\,.$$

The proof readily follows from the construction. □

COROLLARY 6.46. *An operator* $P \in \mathrm{PSD}^m_\gamma$ *is compact in the spaces*

$$P : \mathcal{W}^{s,\gamma}(\mathcal{M}) \longrightarrow \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}), \quad s \in \mathbb{R},$$

if and only if

$$\sigma(P) = 0, \quad \sigma_\wedge(P) = 0.$$

Let us now state the main theorems of the calculus of edge-degenerate pseudodifferential operators.

THEOREM 6.47 (on the composition of edge-degenerate operators). *The product of operators induces a bilinear mapping*

$$\mathrm{PSD}^l_{\gamma-m}(\mathcal{M}) \times \mathrm{PSD}^m_\gamma(\mathcal{M}) \longrightarrow \mathrm{PSD}^{l+m}_\gamma(\mathcal{M})$$

for any l *and* m*. The symbols of the product are given by the formulas*

$$\sigma(D_2D_1) = \sigma(D_2)\sigma(D_1), \quad \sigma_\wedge(D_2D_1) = \sigma_\wedge(D_2)\sigma_\wedge(D_1). \tag{6.87}$$

The last assertion can also be represented in the following form. If $a \in \mathcal{A}^l_{\gamma-m}$ and $b \in \mathcal{A}^m_\gamma$, then

$$\widehat{(ab)} = \widehat{a}\widehat{b} \mod J\,\mathrm{Op}^{m+l}_\gamma\,.$$

THEOREM 6.48 (on the adjoint operator). *The passage to the adjoint operator* (*with respect to the inner product in* $\mathcal{W}^{0,0}(\mathcal{M})$) *induces an antilinear mapping*

$$\mathrm{PSD}^m_\gamma(\mathcal{M}) \longrightarrow \mathrm{PSD}^m_{m-\gamma}(\mathcal{M}).$$

The symbols of the adjoint operator are given by the formulas

$$\sigma(D^*) = \sigma(D)^*, \quad \sigma_\wedge(D^*) = \sigma_\wedge(D)^*. \tag{6.88}$$

Proof. The proof of both theorems is by a straightforward computation. It is based on composition theorems for usual pseudodifferential operators and for pseudodifferential operators with operator-valued symbols. □

Ellipticity, Fredholm property, and smoothness. Let us now state the main assertions of elliptic theory for edge-degenerate pseudodifferential operators without additional (co-)conditions. These assertions are primarily of illustrative nature, since the supply of operators that are elliptic without additional conditions is very restricted.

DEFINITION 6.49. An operator $D \in \mathrm{PSD}^m_\gamma(\mathcal{M})$ is said to be *interior elliptic* if its interior principal symbol $\sigma(D)$ is invertible everywhere (up to the boundary) on the cotangent bundle $T^*_0\mathcal{M}$ without the zero section. The operator $\mathcal{D}$ is said to be *elliptic* if it is interior elliptic and its edge symbol is invertible everywhere on T^*_0X.

THEOREM 6.50. *Let $D \in \mathrm{PSD}^m_\gamma(\mathcal{M})$. The following conditions are equivalent*:

(i) *The operator*

$$D : \mathcal{W}^{s,\gamma}(\mathcal{M}) \longrightarrow \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}) \tag{6.89}$$

is Fredholm for some $s \in \mathbb{R}$.

(ii) *The operator* (6.89) *is Fredholm for all $s \in \mathbb{R}$.*

(iii) *The operator* (6.89) *is elliptic.*

Under any of these conditions, the kernel, cokernel and index of the operator (6.89) *are independent of s.*

This theorem is a special case of the more general Theorem 6.72 below .

6.3. Elliptic Morphisms and the Finiteness Theorem

Now we proceed to edge boundary value problems. Let $A \in \mathrm{PSD}_\gamma^m(\mathcal{M})$ be an edge-degenerate operator. Suppose that it is interior elliptic. Then the principal symbol of its edge symbol is everywhere invertible on $T_0^*\mathcal{M}\big|_{\partial M}$, and (under the assumption that the conormal symbol exists and is also invertible for all $x \in X$ on the weight line $\mathcal{L}_\gamma$) the edge symbol $\sigma_\wedge(D)(x,\xi)$ proves to be a Fredholm family on T_0^*X. As was already explained in Section 1, one can try to make the edge symbol invertible by supplementing it with some conditions and co-conditions. Next, these conditions are quantized, and we arrive at matrix operators of the form[8]

$$\mathbf{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(\mathcal{M}) \\ \oplus \\ H^s(X) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}) \\ \oplus \\ H^{s-m}(X) \end{matrix}, \tag{6.90}$$

where B and C are edge boundary and coboundary operators and D is a pseudodifferential operator on X. (The last component may be lacking in natural statements of edge problems, but it inevitably arises in products of matrix operators of this form.)

DEFINITION 6.51. A matrix operator $\mathbf{A}$ acting in the spaces (6.90) for all $s \in \mathbb{R}$ is called an *operator of order* m (with *weight exponent* γ). The operator $\mathbf{A}$ is said to be *negligible* if it is compact in the spaces (6.90) and continuous in the spaces

$$\mathbf{A} : \mathcal{W}^{s,\gamma}(\mathcal{M}) \oplus H^s(X) \longrightarrow \mathcal{W}^{s-m+1,\gamma-m}(\mathcal{M}) \oplus H^{s-m+1}(X)$$

for every $s \in \mathbb{R}$.

The space of matrix operators of order m and weight γ will be denoted by $\mathbf{Op}_\gamma^m = \mathbf{Op}_\gamma^m(\mathcal{M})$, and the subspace of negligible operators will be denoted by $J\,\mathbf{Op}_\gamma^m(\mathcal{M})$.

Following (Sternin 1967*b*), we refer to matrix operators of the form (6.90) as *morphisms*. The study of morphisms of this form and conditions ensuring that they are Fredholm gives the main analytical results pertaining to edge boundary value problems. First, we study a subalgebra of matrix operators containing boundary and coboundary operators as well as pseudodifferential operators on X. The full class of matrix operators is the (nondirect) sum of this subclass and the class of pseudodifferential operators on $\mathcal{M}$.

6.3.1. Matrix Green operators

Matrix Green operators are obtained by quantization of matrix Green symbols. Since these operators are concentrated (modulo negligible operators) in an arbitrarily small neighborhood of the edge X, one can consider them either on $\mathcal{M}$ or on the infinite wedge W associated with $\mathcal{M}$. The latter is often more convenient. Recall (see Sec. 1) that this wedge is a locally trivial bundle over X whose fiber is the infinite cone K_Ω. The fiber over a point x will be denoted by K_x. The neighborhood U of the edge in $\mathcal{M}$ is naturally identified with a similar neighborhood in W. If E is a vector bundle over $\mathcal{M}$, then in U it can be identified with the lift to W of its restriction to the boundary ∂M of the stretched manifold $M = \mathcal{M}^\wedge$. This lift, as well as its restrictions to the cones K_x, will be denoted by the same letter E. Operators of order m and negligible operators are introduced as in Definition 6.51 with $\mathcal{M}$ replaced by W.

[8]For simplicity, we avoid considering operators of vector order in the sense of Douglis–Nirenberg.

Green symbols.

DEFINITION 6.52. A *matrix Green symbol of order* m *and weight* γ is a smooth family of linear operators

$$\mathbf{g}_\wedge(x,\xi) = \begin{pmatrix} g_\wedge(x,\xi) & c_\wedge(x,\xi) \\ b_\wedge(x,\xi) & d_\wedge(x,\xi) \end{pmatrix} : \begin{matrix} C_0^\infty(K_x,E_1) \\ \oplus \\ J_{1x} \end{matrix} \longrightarrow \begin{matrix} \mathcal{D}'(K_x,E_2) \\ \oplus \\ J_{2x} \end{matrix},$$

where $E_{1,2}$ are finite-dimensional vector bundles over the infinite wedge W and $J_{1,2}$ are finite-dimensional vector bundles over X such that the following conditions hold.

1. The family $\mathbf{g}_\wedge(x,\xi)$ is twisted homogeneous of degree m:

$$\mathbf{g}_\wedge(x,\lambda\xi) = \lambda^m \varkappa_\lambda \mathbf{g}_\wedge(x,\xi)\varkappa_\lambda^{-1}, \quad \lambda > 0, \tag{6.91}$$

   where the group $\varkappa_\lambda$ acts in the fibers $J_{1,2x}$ as the identity mapping for all λ.

2. The family $\mathbf{g}_\wedge(x,\xi)$ extends by closure to a family of compact linear operators in the spaces

$$\mathbf{g}_\wedge(x,\xi) : \begin{matrix} \mathcal{K}^{s,\gamma}(K_x,E_1) \\ \oplus \\ J_{1x} \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{l,\gamma}(K_x,E_2) \\ \oplus \\ J_{2x} \end{matrix}$$

   for any $s,l \in \mathbb{R}$.

3. The same is true of the families $r\mathbf{g}_\wedge(x,\xi)$, $\mathbf{g}_\wedge(x,\xi)r$, and

$$(1+r)^l \mathbf{g}_\wedge(x,\xi)(1+r)^{-l}, \quad l \in \mathbb{R},$$

   where the operator of multiplication by r acts as the zero operator in the fibers of the bundles $J_{1,2x}$.

The space of matrix Green symbol of order m and weight γ will be denoted by[9] $S^m_{\gamma,G}(T_0^*X)$.

Remark 6.53. In particular, it follows from the definition that the upper left entry of a matrix Green symbol satisfies $g_\wedge(x,\xi) \in J^m_\gamma$, i.e., is an edge symbol. However, the smoothing properties of such a symbol with respect to the variables (r,ω) for $r>0$ are much stronger that those of an arbitrary element of J^m_γ.

Green operators. The following proposition shows that matrix Green symbols have the compact fiber variation property. This property ensures (see the Appendix) that the quantization of such symbols gives operators with a "good" composition rule.

PROPOSITION 6.54. *For each* s, *one has the embedding*

$$S^m_{\gamma,G}(T_0^*X) \subset S^0_{CV}(T_0^*X, \mathcal{K}^{s,\gamma}(K_x)\oplus\mathbb{C}, \mathcal{K}^{s-m,\gamma}(K_x)\oplus\mathbb{C}),$$

where the space of symbols with compact fiber variation on the right-hand side is introduced in Definition A.9 *in the Appendix and the spaces* $\mathcal{K}^{s,\gamma}\oplus\mathbb{C}$ *are equipped with families of norms*

$$\|u\oplus z\|_\xi = [\xi]^s \left(\left\| \varkappa_{[\xi]}^{-1} u \right\|_{\mathcal{K}^{s,\gamma}} + |z| \right) \tag{6.92}$$

of tempered growth with respect to the parameter ξ.

[9] To simplify the notation, in what follows we assume that $E_{1,2}$ and $J_{1,2}$ are trivial one-dimensional complex vector bundles and omit them from the notation.

The proof follows directly from the definition. □

COROLLARY 6.55. *The quantization of a Green symbol results in a well-defined bounded operator*

$$\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right):\begin{matrix}\mathcal{W}^{s,\gamma}(W)\\ \oplus\\ H^s(X)\end{matrix}\longrightarrow\begin{matrix}\mathcal{W}^{s-m,\gamma-m}(W)\\ \oplus\\ H^{s-m}(X)\end{matrix},\tag{6.93}$$

and the product of such operators (*modulo negligible operators*) *corresponds to the product of their Green symbols.*

Proof. Indeed, the space $\mathcal{W}^{s,\gamma}(W)\oplus H^s(X)$ is obtained from the spaces with the family of norms (6.92) by the construction in the Appendix so that the desired result follows from Theorem A.81. □

Unfortunately, the operator (6.93) acts (in the first component) on the infinite wedge W rather than on the manifold $\mathcal{M}$. However, one can readily rectify this by using the following assertion. To unify the notation, we adopt the convention that if $\psi(r)$ is an arbitrary R-function and $v(x)$ is a function on the edge X, then

$$\psi(r)v(x)\overset{\text{def}}{\equiv}v(x)$$

(in agreement with the fact that $r=0$ on the edge).

PROPOSITION 6.56. *Let $\psi(r)$ be an arbitrary R-function. Then the operators*

$$(1-\psi(r))\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right),\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right)(1-\psi(r)):\begin{matrix}\mathcal{W}^{s,\gamma}(W)\\ \oplus\\ H^s(X)\end{matrix}\to\begin{matrix}\mathcal{W}^{s-m,\gamma-m}(W)\\ \oplus\\ H^{s-m}(X)\end{matrix}$$

are compact for every $s\in\mathbb{R}$. Moreover, these operators are continuous in the spaces

$$(1-\psi(r))\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right),\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right)(1-\psi(r)):\begin{matrix}\mathcal{W}^{s,\gamma}(W)\\ \oplus\\ H^s(X)\end{matrix}\to\begin{matrix}\mathcal{W}^{s-m+1,\gamma-m}(W)\\ \oplus\\ H^{s-m+1}(X)\end{matrix}$$

for every $s\in\mathbb{R}$, i.e., belong to the space $J\,\mathbf{Op}^m_\gamma$.

Proof. This proof of the second assertion is completely similar to that of Theorem 6.43. The first assertion follows from the fact that the symbol $\mathbf{g}_\wedge$ is compact. □

DEFINITION 6.57. The operator

$$\operatorname{op}_G(\mathbf{g}_\wedge)=\psi(r)\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right)\psi(r),\tag{6.94}$$

where $\psi(r)$ is an arbitrary R-function with support in U, is called a *Green operator with* (*edge*) *symbol*[10] $\mathbf{g}_\wedge(x,\xi)$.

By Proposition 6.56, the operator (6.94) differs from the operator (6.93) by a negligible operator. The new operator (6.94) can obviously be treated as an operator on $\mathcal{M}$, since its Schwartz kernel is supported in $U\times U$.

[10] Note that the principal symbol of a Green operator is always zero.

Remark 6.58. Note that the upper left entry of the operator (6.94) is a pseudodifferential operator on $\mathcal{M}$ whose principal symbol (of order m) is zero and whose edge symbol coincides with the upper left entry of the Green symbol.

Corollary 6.55 implies the following assertion.

PROPOSITION 6.59. *Modulo negligible operators, the product of two Green operators is again a Green operator, and moreover,*

$$\sigma_\wedge\big(\operatorname{op}_G(\mathbf{g}_\wedge^{(1)})\operatorname{op}_G(\mathbf{g}_\wedge^{(2)})\big) = \mathbf{g}_\wedge^{(1)}\mathbf{g}_\wedge^{(2)},$$

i.e., the correspondence between Green symbols and Green operators is multiplicative.

6.3.2. General morphisms

Now we can describe general pseudodifferential morphisms on a manifold with edges.

DEFINITION 6.60. A *morphism of order* m *and weight* γ is an operator that, modulo elements of the space $J\,\mathbf{Op}_\gamma^m(\mathcal{M})$, can be represented as the sum

$$\mathbf{A} = A + \operatorname{op}_G(\mathbf{g}_\wedge), \tag{6.95}$$

where the first term is a pseudodifferential operator[11] on $\mathcal{M}$ of order m and weight γ and the second term is a matrix Green operator of order m and weight γ.

The set of morphisms of order m and weight γ is denoted by $\operatorname{Mor}_\gamma^m(\mathcal{M})$.

Remark 6.61. The representation (6.95) is obviously nonunique by virtue of Remark 6.58. However, Proposition 6.63 below shows that this does not result in any difficulties in the theory.

Symbols.

DEFINITION 6.62. Let $\mathbf{A} \in \operatorname{Mor}_\gamma^m(\mathcal{M})$ be the morphism (6.95). We define the interior principal and edge symbols of $\mathbf{A}$ by the formulas[12]

$$\sigma(\mathbf{A}) = \sigma(A), \tag{6.96}$$

$$\sigma_\wedge(\mathbf{A})(x,\xi) = \sigma_\wedge(A)(x,\xi) + \mathbf{g}_\wedge(x,\xi). \tag{6.97}$$

PROPOSITION 6.63. *Definition* 6.62 *is well-posed*: *the interior principal and edge symbols of* $\mathbf{A}$ *are independent of the specific choice of the representation* (6.95).

Proof. The independence of the interior principal symbol of the choice of the representation (6.95) is obvious, since the interior principal symbol of the upper left entry of a Green operator is zero. Relation (6.97) is also obvious, since the edge symbol of the upper left entry can be computed according to Definition 6.39. □

[11]For brevity, we write A instead of the 2×2 matrix $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and use similar notation for symbols.

[12]Where the first term is the shorthand notation for the matrix $\begin{pmatrix} \sigma_\wedge(A)(x,\xi) & 0 \\ 0 & 0 \end{pmatrix}$.

Quantization. Now our task is to construct *quantization*, i.e., a mapping that takes each compatible pair (interior principal symbol, edge symbol) to the corresponding morphism. First, we should describe the set of edge symbols in question and state the compatibility condition. By analogy with the set of operators (6.95), we define the set of matrix 2×2 *edge symbols* of order m and weight γ as

$$\mathbf{Edge}^m_\gamma(T^*_0X) = \begin{pmatrix} \mathrm{Edge}^m_\gamma(T^*_0X) & 0 \\ 0 & 0 \end{pmatrix} + S^m_{\gamma,G}(T^*_0X).$$

DEFINITION 6.64. The *principal symbol* of a matrix edge symbol

$$\boldsymbol{\sigma}_\wedge = \begin{pmatrix} \sigma_{\wedge 11} & \sigma_{\wedge 12} \\ \sigma_{\wedge 21} & \sigma_{\wedge 22} \end{pmatrix} \in \mathbf{Edge}^m_\gamma(T^*_0X) \tag{6.98}$$

of order m and weight γ is the principal symbol of its upper left entry:

$$\sigma(\boldsymbol{\sigma}_\wedge) \stackrel{\text{def}}{=} \sigma(\sigma_{\wedge 11}). \tag{6.99}$$

One says that the edge symbol (6.98) is *compatible* with an interior principal symbol $\sigma \in \mathcal{O}^n(T^*_0\mathcal{M})$ of order m on $T^*_0\mathcal{M}$ if the restriction of the latter to the boundary $\partial T^*_0\mathcal{M}$ coincides with the principal symbol of the edge symbol (6.98):

$$\sigma(\boldsymbol{\sigma}_\wedge) = \sigma\big|_{\partial T^*_0\mathcal{M}}. \tag{6.100}$$

PROPOSITION 6.65. *If a matrix edge symbol $\boldsymbol{\sigma}_\wedge \in \mathbf{Edge}^m_\gamma(T^*_0X)$ is invertible for all $(x,\xi) \in T^*_0X$, then*

$$\boldsymbol{\sigma}_\wedge^{-1} \in \mathbf{Edge}^{-m}_{\gamma-m}(T^*_0X) \quad \textit{and} \quad \sigma\big(\boldsymbol{\sigma}_\wedge^{-1}\big) = \sigma(\boldsymbol{\sigma}_\wedge)^{-1}.$$

Now let a compatible pair

$$(\sigma, \boldsymbol{\sigma}_\wedge) \in \mathcal{O}^m(T^*_0\mathcal{M}) \times \mathbf{Edge}^m_\gamma(T^*_0X), \quad \boldsymbol{\sigma}_\wedge = \begin{pmatrix} \sigma_{\wedge 11} & \sigma_{\wedge 12} \\ \sigma_{\wedge 21} & \sigma_{\wedge 22} \end{pmatrix} \tag{6.101}$$

be given.

DEFINITION 6.66. The *quantization of the pair* (6.101) is the morphism of order m and weight γ given by the formula

$$(\sigma, \boldsymbol{\sigma}_\wedge)^\wedge = (\sigma, \sigma_{\wedge 11})^\wedge + \mathrm{op}_G\left[\begin{pmatrix} 0 & \sigma_{\wedge 12} \\ \sigma_{\wedge 21} & \sigma_{\wedge 22} \end{pmatrix}\right]. \tag{6.102}$$

One can readily see that the principal and edge symbols of this morphisms are, respectively,

$$\sigma((\sigma, \boldsymbol{\sigma}_\wedge)^\wedge) = \sigma, \quad \sigma_\wedge((\sigma, \boldsymbol{\sigma}_\wedge)^\wedge) = \boldsymbol{\sigma}_\wedge.$$

Just as with pseudodifferential operators, the symbols of a pseudodifferential morphism determine it modulo negligible operators. Thus quantization modulo negligible operators is the two-sided inverse for the symbol mapping taking the operator to the pair (interior principal symbol, edge symbol). Recall that negligible operators are compact, so that a given pair of symbols completely determines whether the corresponding operator is Fredholm and, if this is the case, determines its index.

PROPOSITION 6.67. *One has the implications*

$$(\sigma, \boldsymbol{\sigma}_\wedge)^\wedge \textit{ is compact in the spaces } (6.90) \Longrightarrow \{\sigma = 0 \textit{ and } \boldsymbol{\sigma}_\wedge = 0\} \Longrightarrow (\sigma, \boldsymbol{\sigma}_\wedge)^\wedge \in J\,\mathbf{Op}^m_\gamma\,.$$

Remark 6.68. Thus compact operators in our operator algebra are not only compact but also smoothing.

Calculus. Now we are in a position to state the main theorems of the calculus of pseudodifferential morphisms on manifolds with edges.

THEOREM 6.69. *The product of operators induces a bilinear mapping*

$$\mathrm{Mor}^{l}_{\gamma-m}(\mathcal{M}) \times \mathrm{Mor}^{m}_{\gamma}(\mathcal{M}) \longrightarrow \mathrm{Mor}^{l+m}_{\gamma}(\mathcal{M}). \quad (6.103)$$

The principal and edge symbols of the product are given by the formulas

$$\sigma(\mathbf{AB}) = \sigma(\mathbf{A})\sigma(\mathbf{B}), \quad \sigma_\wedge(\mathbf{AB}) = \sigma_\wedge(\mathbf{A})\sigma_\wedge(\mathbf{B}). \quad (6.104)$$

THEOREM 6.70. *The passage to the adjoint operator induces the antilinear mapping*

$$\mathrm{Mor}^{m}_{\gamma}(\mathcal{M}) \longrightarrow \mathrm{Mor}^{m}_{m-\gamma}(\mathcal{M}); \quad (6.105)$$

moreover,

$$\sigma(\mathbf{A}^*) = \sigma(\mathbf{A})^*, \quad \sigma_\wedge(\mathbf{A}^*) = \sigma_\wedge(\mathbf{A})^*. \quad (6.106)$$

Proof. The proof of both theorems is by a straightforward computation based on the composition theorems for pseudodifferential operators and pseudodifferential operators with operator-valued symbols. □

6.3.3. Ellipticity, Fredholm property, and smoothness

The calculus constructed above implies, in the standard way, the main assertions of elliptic theory for pseudodifferential morphisms (or, which is the same, for elliptic edge problems, i.e., edge-degenerate pseudodifferential operators equipped with edge boundary and coboundary conditions). The finiteness and smoothness theorems for elliptic edge problems for *differential* operators stated in Sec. 1 are a special case of these general theorems.

DEFINITION 6.71. A morphism $\mathbf{A} \in \mathrm{Mor}^{m}_{\gamma}(\mathcal{M})$ is said to be *interior elliptic* if its interior principal symbol $\sigma(\mathbf{A})$ is invertible up to the boundary of the cotangent bundle $T^*_0\mathcal{M}$ without the zero section. The morphism $\mathbf{A}$ is said to be *elliptic* if it is interior elliptic and its edge symbol is invertible on $T^*_0 X$.

THEOREM 6.72. *Let* $\mathbf{A} \in \mathrm{Mor}^{m}_{\gamma}(\mathcal{M})$. *The following conditions are equivalent*:

(i) *The morphism* $\mathbf{A}$ *is Fredholm in the spaces* (6.90) *for some* $s \in \mathbb{R}$.

(ii) *The morphism* $\mathbf{A}$ *is Fredholm in the spaces* (6.90) *for all* $s \in \mathbb{R}$.

(iii) *The morphism* $\mathbf{A}$ *is elliptic.*

Under any of these conditions, the kernel and cokernel of $\mathbf{A}$ *are independent of* s.

Proof. We shall only prove that the ellipticity of a morphism implies its Fredholm property and the smoothness (independence of s) of the kernel and cokernel.

Thus let $\mathbf{A}$ be elliptic. Then $\sigma(\mathbf{A})^{-1}$ is a well-defined principal symbol on $T^*_0\mathcal{M}$, and $\sigma_\wedge(\mathbf{A})^{-1}$ is a well-defined edge symbol on $T^*_0 X$ by Proposition 6.65. By the same proposition,

$$\sigma(\sigma_\wedge(\mathbf{A})^{-1}) = \sigma(\sigma_\wedge(\mathbf{A}))^{-1},$$

whence it readily follows that the pair $\left(\sigma(\mathbf{A})^{-1}, \sigma_\wedge(\mathbf{A})^{-1}\right)$ satisfies the compatibility condition (6.100), since so does the pair $(\sigma(\mathbf{A}), \sigma_\wedge(\mathbf{A}))$.

According to Definition 6.66, we construct a morphism $\mathbf{R} \in \operatorname{Mor}^{-m}_{\gamma-m}(\mathcal{M})$ such that

$$\sigma(\mathbf{R}) = \sigma(\mathbf{A})^{-1}, \quad \sigma_\wedge(\mathbf{R}) = \sigma_\wedge(\mathbf{A})^{-1}.$$

Using the composition theorem 6.69 and Proposition 6.67, we obtain

$$\mathbf{AR} = 1 + \mathbf{Q}_1, \quad \mathbf{RA} = 1 + \mathbf{Q}_2, \tag{6.107}$$

where

$$\mathbf{Q}_1 \in J\,\mathbf{Op}^0_{\gamma-m}, \quad \mathbf{Q}_2 \in J\,\mathbf{Op}^0_\gamma$$

are negligible operators. Since negligible operators are compact, from (6.107) we readily find that the morphism $\mathbf{A}$ is Fredholm in the spaces (6.90) for all $s \in \mathbb{R}$.

Now let us prove that the kernel of $\mathbf{A}$ is independent of s. Let $u \in \ker \mathbf{A}$. By applying the morphism $\mathbf{RA}$ to u, we obtain, by virtue of the second equation in (6.107),

$$u = -\mathbf{Q}_2 u. \tag{6.108}$$

The negligible operator $\mathbf{Q}_2$ by Definition 6.51 is continuous in the spaces

$$\mathbf{Q}_2 : \mathcal{W}^{s,\gamma}(\mathcal{M}) \oplus H^s(X) \longrightarrow \mathcal{W}^{s+1,\gamma}(\mathcal{M}) \oplus H^{s+1}(X),$$

so that it follows from (6.108) that

$$u \in \mathcal{W}^{s,\gamma}(\mathcal{M}) \oplus H^s(X) \implies u \in \mathcal{W}^{s+1,\gamma}(\mathcal{M}) \oplus H^{s+1}(X).$$

In turn, it follows that u is contained in all spaces $\mathcal{W}^{s,\gamma}(\mathcal{M}) \oplus H^s(X)$, $s \in \mathbb{R}$, simultaneously. (The opposite implication is trivial in view of the embedding

$$\mathcal{W}^{s,\gamma}(\mathcal{M}) \oplus H^s(X) \subset \mathcal{W}^{s',\gamma}(\mathcal{M}) \oplus H^{s'}(X)$$

for $s' < s$.) In a similar way (by passing to adjoint operators) one can prove the independence of s of the cokernel of the morphism $\mathbf{A}$. The proof is complete. □

Appendix A. Fiber Bundles and Direct Integrals

In this appendix, we outline a simple general theory of pseudodifferential operators in Hilbert spaces of Hilbert-valued functions. The main examples arise in applications if one describes Hilbert function spaces on the total space of a locally trivial bundle

$$\pi : Y \longrightarrow X$$

with compact base as spaces of Hilbert-valued functions (or distributions) on X. These spaces behave "along X" very similarly to usual Sobolev spaces. (There are no distinguished or degenerate directions.) It is convenient to define such spaces in the Fourier transform as direct integrals of Hilbert spaces. We define pseudodifferential operators in such spaces and establish a boundedness theorem for such operators.

Throughout the following, we use the notation $[\xi] = (1 + |\xi|^2)^{1/2}$.

A.1. Local theory

Let H be a Hilbert space with norm $\| \cdot \|$ and inner product $(\cdot , \cdot)$. Suppose that it is equipped with a family $\{\| \cdot \|_\xi\}$ of Hilbert norms depending on the parameter $\xi \in \mathbb{R}^n$, equivalent to the original norm $\| \cdot \|$, and satisfying the following conditions:

- $\|u\|_\xi$ is a measurable function of ξ for any $u \in H$;
- there exist constants C and N such that

$$\|u\|_\eta \leq C\left(\frac{[\eta]}{[\xi]} + \frac{[\xi]}{[\eta]}\right)^N \|u\|_\xi \quad \text{for any } \xi, \eta \in \mathbb{R}^n \text{ and } u \in H,\ u \neq 0. \tag{A.109}$$

DEFINITION A.73. A family of norms satisfying the above-mentioned conditions will be called a *tempered family of norms*.

We denote the space H equipped with the norm $\| \cdot \|_\xi$ by H_ξ and define the *direct integral* of H_ξ by the formula

$$\int H_\xi \, d\xi \stackrel{\text{def}}{=} \left\{ u : \mathbb{R}^n \longrightarrow H \;\middle|\; u(\xi) \text{ is measurable and } \int \|u(\xi)\|_\xi^2 \, d\xi < \infty \right\}. \tag{A.110}$$

Remark A.74. (1) Strong and weak measurability are the same in a separable Hilbert space, so we do not specify the kind of measurability in the definition.

(2) If $u(\xi)$ is measurable, then so is $\|u(\xi)\|_\xi$, and hence (A.110) is well defined.

The space $\mathcal{H} = \int H_\xi \, d\xi$ is equipped with the natural norm

$$\|u\| = \left\{ \int \|u(\xi)\|_\xi^2 \, d\xi \right\}^{1/2}, \tag{A.111}$$

which makes it a Hilbert space.

LEMMA A.75. *The space $\mathcal{H}$ is the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^n, H)$ in the norm* (A.111).

The proof is standard.

We are actually interested in the space $\mathcal{F}^{-1}\mathcal{H}$ obtained from $\mathcal{H}$ by the inverse Fourier transform. It follows from the preceding lemma that $\mathcal{F}^{-1}\mathcal{H}$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^n, H)$ with respect to the norm

$$\|u\|_{(0)} = \|\widetilde{u}\| .$$

EXAMPLE A.76. If $H = \mathbb{C}$ and $|\!|u|\!|_\xi = [\xi]|u|$, then $\mathcal{F}^{-1}\mathcal{H} = H^s(\mathbb{R}^n)$ is the ordinary Sobolev space in $\mathbb{R}^n$.

If $H = H^s(M)$, where M is a smooth compact manifold, and

$$|\!|u|\!|_\xi = \left\| \left(\frac{1+|\xi|^2-\triangle}{1-\triangle} \right)^{s/2} u \right\|_{H^s(M)},$$

where $\triangle$ is the Beltrami–Laplace operator on M, then $\mathcal{F}^{-1}\mathcal{H} = H^s(\mathbb{R}^n \times M)$ is the Sobolev space on $\mathbb{R}^n \times M$.

Let

$$\mathcal{G} = \int G_\xi \, d\xi$$

be another direct integral of Hilbert spaces.

Let us study the continuity of pseudodifferential operators from $\mathcal{F}^{-1}\mathcal{H}$ to $\mathcal{F}^{-1}\mathcal{G}$. We consider operator-valued symbols $D(x,\xi)$ ranging in $\mathcal{L}(H,G)$ and satisfying the estimates

$$\left\| \frac{\partial^\alpha D}{\partial x^\alpha}(x,\xi) : H_\xi \longrightarrow G_\xi \right\| \le C_{\alpha l} \, [x]^{-l} , \quad l, |\alpha| = 0,1,2,\dots . \tag{A.112}$$

THEOREM A.77. *If the estimates* (A.112) *hold, then the operator*

$$D\left(\overset{2}{x}, -i\frac{\overset{1}{\partial}}{\partial x}\right) : \mathcal{F}^{-1}\mathcal{H} \longrightarrow \mathcal{F}^{-1}\mathcal{G}$$

is continuous.

Proof. We shall prove an equivalent assertion, namely, the continuity of the operator

$$\widehat{D} = D\left(i\frac{\overset{2}{\partial}}{\partial \xi}, \overset{1}{\xi}\right) : \mathcal{H} \longrightarrow \mathcal{G}.$$

The operator acts by the formula

$$\left[\widehat{D}u\right](\xi) = \int \widetilde{D}(\xi-\eta,\eta)u(\eta)\, d\eta = \int \widetilde{D}(z, \xi - z)u(\xi - z)\, dz, \tag{A.113}$$

where $\widetilde{D}$ is the Fourier transform of the symbol D with respect to the first argument. By virtue of the estimates (A.112), the Fourier transform is continuous and satisfies the estimates

$$\left\| \widetilde{D}(z,\eta) : H_\eta \longrightarrow G_\eta \right\| \le C_l \, [z]^{-l} , \quad l = 0,1,2,\dots .$$

We can rewrite (A.113) in the form

$$\widehat{D}u = \int U(\,\cdot\,, z)\, dz, \tag{A.114}$$

where

$$U(\xi, z) = \widetilde{D}(z, \xi - z)u(\xi - z)$$

By virtue of the properties of the norm and the estimates imposed on $\widetilde{D}$, we have

$$|U(\xi, z)|_\xi \le \text{const}\, [z]^N\, |U(\xi, z)|_{\xi - z} \le \text{const}\, [z]^{-M}\, |u(\xi - z)|_{\xi - z},$$

where M is arbitrarily large. Hence

$$\|U(\,\cdot\,, z)\| = \left\{ \int |U(\xi, z)|^2 d\xi \right\}^{1/2} \le \text{const}\, [z]^{-M/2} \left\{ \int |u(\xi - z)|^2_{\xi - z} dz \right\}^{1/2} = \text{const}\, [z]^{-M/2}\, \|u\|\,;$$

by substituting this estimate into (A.114) and by integrating over z, we arrive at the desired result. □

COROLLARY A.78. *The operator of multiplication by a smooth compactly supported function is continuous in the space $\mathcal{F}^{-1}\mathcal{H}$.*

A.2. Globalization

So far we have defined spaces of Hilbert-valued functions on $\mathbb{R}^n$. To proceed to a manifold, we should study how the norms are affected by changes of variables. Let $|\cdot|_\xi$ be a given tempered family of norms in a Hilbert space H, and let $\|\cdot\|$ be the corresponding norm in the direct integral $\mathcal{H}$ of the spaces H_ξ.

PROPOSITION A.79. *Let $U \subset \mathbb{R}^n$ be a bounded domain, and let $f : U \longrightarrow V \subset \mathbb{R}^n$ be a diffeomorphism extendible into some neighborhood of the closure $\overline{U}$ of the domain U. Then there exist positive constants C_1 and C_2 such that the inequalities*

$$c\, \|\widetilde{u}\| \le \left\| \widetilde{f^* u} \right\| \le \|\widetilde{u}\|$$

(*where the tilde stands for the Fourier transform*) *hold for each smooth H-valued function $u(x)$ supported in V.*

Proof. Let u be an H-valued function supported in V, let $v(\xi)$ be its Fourier transform, and let $w(\eta)$ be the Fourier transform of f^*u. Next, let $\rho(y)$ be a smooth compactly supported function whose support is contained in the domain of the extended diffeomorphism f and which is equal to unity in U. Then

$$w(\eta) = \left(\frac{1}{2\pi}\right)^n \iint e^{i(f(y)\xi - y\eta)} \rho(y) v(\xi)\, d\xi dy. \tag{A.115}$$

Using the operator

$$L = \frac{1 - i\langle f'(y)\xi - \eta, \partial/\partial y\rangle}{1 + |f'(y)\xi - \eta|^2},$$

which satisfies the relation

$$Le^{i(f(y)\xi-y\eta)}=e^{i(f(y)\xi-y\eta)},$$

and integrating by parts M times in (A.115), we obtain

$$w(\eta)=\left(\frac{1}{2\pi}\right)^n\iint e^{i(f(y)\xi-y\eta)}[({}^tL)^M\rho(y)]v(\xi)\,d\xi dy, \tag{A.116}$$

where ${}^tL$ is the transpose of L. Moreover, the estimate

$$|({}^tL)^M\rho(y)|\le \operatorname{const}\big[1+|f'(y)\xi-\eta|^2\big]^{-M/2}$$

is obviously valid. In the integral (A.116), we make the change of variables

$$\xi=\xi(\eta,t)\equiv f'(y)^{-1}(t+\eta);$$

then it becomes

$$w(\eta)=\left(\frac{1}{2\pi}\right)^n\iint e^{i(f(y)\xi(\eta,t)-y\eta)}\frac{({}^tL)^M\rho(y)}{\det f'(y)}v(f'(y)^{-1}(t+\eta))\,dtdy, \tag{A.117}$$

or

$$w(\eta)=\int U(\eta,t,y)\,dtdy,$$

where the integrand U in (A.117) satisfies the estimate

$$\|U(\eta,t,y)\|_\eta\le \operatorname{const}\,[t]^{(N-M)/2}\,\|v(f'(y)^{-1}(t+\eta))\|_{f'(y)^{-1}(t+\eta)}.$$

Now the end of the proof is the same as in Lemma A.77 from the preceding subsection. ☐

This proposition, in conjunction with the corollary to Lemma A.77 in the preceding subsection, shows that by using partitions of unity we can introduce a well-defined space $\mathcal{F}^{-1}\mathcal{H}(X)$ of H-valued functions on X whose local model is the space $\mathcal{F}^{-1}\mathcal{H}$ constructed in the preceding section. Pseudodifferential operators whose complete symbols in local coordinates satisfy the estimates (A.112) are continuous in such spaces.

We now give a modified definition of symbols with compact fiber variation. (The original definition can be found in (Luke 1972).)

DEFINITION A.80. Let $\|\cdot\|^1_\xi$ and $\|\cdot\|^2_\xi$ be two tempered families of norms in Hilbert spaces H_1 and H_2, respectively. By $S^0_{CV}\equiv S^0_{CV}(\mathbb{R}^{2n},H_1,H_2)$ we denote the space of operator-valued symbols $a(x,\xi)$ ranging in $\mathcal{L}(H_1,H_2)$, satisfying the estimates

$$\left\|\frac{\partial^{\alpha+\beta}a(x,\xi)}{\partial x^\alpha\partial\xi^\beta}:H_{1\xi}\longrightarrow H_{2\xi}\right\|\le C_{\alpha\beta}\,[\xi]^{-|\beta|}, \tag{A.118}$$

and having compact fiber variation. The space $S^0_{CV}(T^*X,H_1,H_2)$ is defined as the space of operator-valued symbols whose coordinate representatives satisfy the estimates (A.118).

Let $a \in S^0_{CV}(T^*X, H_1, H_2)$, and let $1 = \sum_j e_j^2$ be a finite smooth partition of unity subordinate to a cover of X by coordinate neighborhoods. We set

$$\widehat{a} = \sum_j \widehat{(e_j a)} e_j,$$

where the operator $\widehat{(e_j a)}$ is defined in local coordinates as the pseudodifferential operator

$$\widehat{(e_j a)} = (e_j a)\left(\overset{2}{x}, -i\frac{\overset{1}{\partial}}{\partial x}\right).$$

The preceding arguments, in conjunction with the results and arguments in (Luke 1972), imply the following theorem.

THEOREM A.81. *The operator $\widehat{a}$ is bounded in the spaces*

$$\widehat{a} : \mathcal{F}^{-1}\mathcal{H}_1 \longrightarrow \mathcal{F}^{-1}\mathcal{H}_2$$

and independent, modulo compact operators in these spaces, of the choice of local coordinates and the partition of unity. The product of such operators corresponds, modulo compact operators, to the product of symbols.

A.3. Versions of the Definition of the Norm

A tempered family of norms can always be given by the formula

$$\|u\|_\xi = \|A(\xi)u\|, \tag{A.119}$$

where $A(\xi)$ is a strongly measurable family of bounded operators in H and $\|\cdot\|$ is the norm in H.

The operator $A(\xi)$ is uniquely determined if we require that it be positive and self-adjoint. However, these requirements are not necessary in applications; moreover, one can proceed to equivalent norms.

LEMMA A.82. *Two tempered families of norms determined by operator families $A(\xi)$ and $B(\xi)$ in accordance with* (A.119) *determine the same* (*up to norm equivalence*) *space $\mathcal{H}$ if and only if the operator families*

$$A(\xi)B^{-1}(\xi), \quad B(\xi)A^{-1}(\xi)$$

are bounded uniformly with respect to ξ.

The proof is obvious.

Although the assertion is trivial, it permits one to give quite remarkable equivalent expressions for the norm. For example, let $\varkappa_\lambda$ be the group of dilations in $H^s(\mathbb{R}^k)$ acting by the formula

$$\varkappa_\lambda u(x) = \lambda^{k/2} u(\lambda x), \quad \lambda \in \mathbb{R}_+.$$

The operator families

$$A(\xi) = \left(\frac{1 + |\xi|^2 - \triangle}{1 - \triangle}\right)^{s/2}, \quad B(\xi) = [\xi]^s \, \varkappa_{[\xi]}^{-1}$$

satisfy the assumptions of the lemma, and hence the norms associated with these families are equivalent. Moreover, the family $B(\xi)$ defines the space $\mathcal{W}^s(\mathbb{R}^n, H^s(\mathbb{R}^k))$. We obtain the well-known identity

$$H^s(\mathbb{R}^{n+k}) = \mathcal{W}^s(\mathbb{R}^n, H^s(\mathbb{R}^k)).$$

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