Boundary Value Problems in Edge Representation

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Abstract

Edge representations of operators on closed manifolds are known to induce large classes of operators that are elliptic on specific manifolds with edges, cf. [9]. We apply this idea to the case of boundary value problems. We establish a correspondence between standard ellipticity and ellipticity with respect to the principal symbolic hierarchy of the edge algebra of boundary value problems, where an embedded submanifold on the boundary plays the role of an edge. We first consider the case that the weight is equal to the smoothness and calculate the dimensions of kernels and cokernels of the associated principal edge symbols. Then we pass to elliptic edge operators for arbitrary weights and construct the additional edge conditions by applying relative index results for conormal symbols.

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Contents

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Introduction

Mixed elliptic problems (such as the Zaremba problem for the Laplace operator) or crack problems (e.g., the Lamé system with two-sided elliptic conditions on a crack in a medium) belong to the category of boundary value problems on a manifold with edge and boundary (where the interface in the case of mixed problems or the boundary of the crack play the role of the edge). Such problems are embedded in a corresponding general calculus of edge-boundary value problems with the transmission property at the smooth part of the boundary, cf. [14]. They are controlled by a principal symbolic hierarchy $(\sigma_\psi, \sigma_\partial, \sigma_\wedge)$ consisting of the interior, the boundary and the edge component σ_ψ, σ_∂ and $\sigma_\wedge$, respectively. The solvability in weighted edge Sobolev spaces is connected with additional conditions of trace and potential type along the edge (analogously as boundary conditions in the theory of boundary value problems), provided that some topological obstruction for their existence vanishes.

In general, certain weights have to be excluded, and the chosen admissible weights (if they exist at all) affect the number of edge conditions (the difference of the number of potential and trace conditions is equal to the Fredholm index of an elliptic boundary value problem on an infinite cone transversal to the interface). It also happens that such conditions are unnecessary. These data (which depend on the geometry of the configuration and on the coefficients of the operators) are often unknown or very difficult to calculate, also for the concrete examples mentioned in the beginning (although there are 'abstract' characterisations of the qualitative effects in general terms). It is therefore an interesting extra chapter of the general edge calculus to explicitly construct sufficiently large classes of operators which are elliptic with respect to all components of $(\sigma_\psi, \sigma_\partial, \sigma_\wedge)$. Even for the case without boundary (i.e., when σ_∂ disappears) it is a non-trivial task to establish new classes of elliptic elements.

For closed manifolds with edges there are different strategies, e.g., external multiplications between elliptic operators on a cone and on a C^∞ manifold, cf. [19], or edge representations of elliptic operators on a smooth manifold with respect to an embedded submanifold that plays the role of an edge, cf. [9]. The constructions of [9] are crucial for the characterisation of the number of elliptic interface conditions (together with the admissible weights) for the case of the Zaremba problem and for other mixed problems in the framework of [8].

In the present paper we construct edge representations of boundary value problems with respect to an embedded submanifold of the boundary. We obtain in this way new families of operators which are elliptic with respect to $(\sigma_\psi, \sigma_\partial, \sigma_\wedge)$ on an associated manifold with edge and boundary.

In Chapter 1 we outline the basic material on boundary value problems for differential operators on a manifold W with edge $Z \subset W$ and boundary (further details may be found in [16]). The operators are realised in weighted edge Sobolev spaces. Ellipticity refers to additional conditions of trace and potential type along the edge which exist when a topological obstruction on the operator vanishes; this is an analogue of [2] for edge problems, see also [32, 34]. We show (Theorem 1.3.2) that vanishing of the topological obstruction for the existence of elliptic edge conditions is independent of the weight.

Chapter 2 gives the edge representation of boundary value problems $\mathcal{A}$ in the half-space with respect to an embedded hypersurface on the boundary. We show that the resulting operators belong to the edge calculus of Chapter 1 when the weight is equal to the smoothness s, up to certain discrete exceptional values. In particular, the ellipticity of $\mathcal{A}$ in the standard sense (i.e., with respect to the interior and the boundary symbol) entails the ellipticity with respect to the edge symbolic hierarchy (Theorem 2.3.2). At the same time we explicitly calculate extra edge conditions, and we see that the above mentioned topological obstruction vanishes in our situation.

In Chapter 3 we pass to boundary value problems on a C^∞ manifold with boundary and obtain the global versions of the results of Chapter 2 (Theorem 3.1.3).

In Chapter 4 we study boundary value problems in edge spaces with arbitrary weights. With the help of relative index results for the principal conormal symbols (Proposition 4.1.2 and Corollary 4.1.3) we calculate the elliptic edge conditions also in this case, up to discrete exceptional weights and using the fact that the topological obstruction remains trivial for arbitrary weights (Theorem 4.2.2). We finally consider the parametrices of our elliptic edge boundary value problems (4.4.1).

Our paper belongs to the program of studying ellipticity of operators on manifolds with geometric singularities, here with edge and boundary.

To give a few references with relations to our results let us first mention the classical works of Agranovich and Vishik [1] on parameter-dependent elliptic problems, and Kondrat'ev [15] on elliptic boundary value problems for conical singularities; another background is the analysis on manifolds with conical exits to infinity in the sense of Parenti [25] and Cordes [7] and the (pseudo-differential) calculus of boundary value problems of Boutet de Monvel [5]. Our approach also develops new aspects of the Sobolev problems, see the paper [36] of Sternin. As is well known the study of operators for conical singularities is also motivated by the index theory and the geometric analysis, see Atiyah, Patodi and Singer [4], Cheeger [6], or Melrose and Mendoza [18] (the latter paper also considers relative index expressions in terms of the poles and zeros of principal conormal symbols). Moreover, Nistor [23, 24], Gil and Mendoza [11], Loya [17] and many other authors contributed to the field. A calculus of operators on manifolds with edges with extra trace and potential contributions was formulated first in [29]; concerning index theory in this kind of edge and corner algebras, cf. Schrohe and Seiler [28], Fedosov, Tarkhanov, and Schulze [10], or Nazaikinskij, Savin, Schulze and Sternin [20]. The paper [21] gives a detailed description of connections between different branches of the analysis on manifolds with edges, see also the references there.

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1 Operators on manifolds with edges and boundary

1.1 Manifolds with edges and weighted spaces

A manifold W with boundary and edge Z is a topological space that is modelled on a wedge $X^\Delta \times \Omega$ near any point $z \in Z$ with the model cone $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$, where X is a (in our case compact) C^∞ manifold with boundary, $n = \dim X$, and $\Omega \subseteq \mathbb{R}^q$ is an open set which corresponds to a chart on Z. More precisely, $W \setminus Z$ is a C^∞ manifold with boundary, Z is a C^∞ manifold, $q = \dim Z$. We assume that the transition maps between different local wedges

$$X^\Delta \times \Omega \to X^\Delta \times \widetilde{\Omega} \tag{1.1}$$

restrict to diffeomorphisms

$$\mathbb{R}_+ \times X \times \Omega \to \mathbb{R}_+ \times X \times \widetilde{\Omega}; \tag{1.2}$$

those are assumed to be restrictions of diffeomorphisms $\mathbb{R} \times X \times \Omega \to \mathbb{R} \times X \times \widetilde{\Omega}$ (in the sense of C^∞ manifolds with boundary) to $\mathbb{R}_+ \times X \times \Omega$. We then obtain transition maps

$$\overline{\mathbb{R}}_+ \times X \times \Omega \to \overline{\mathbb{R}}_+ \times X \times \widetilde{\Omega} \tag{1.3}$$

which induce diffeomorphisms $\{0\} \times X \times \Omega \to \{0\} \times X \times \widetilde{\Omega}$; they represent the transition maps of an X-bundle over Z.

Similarly to the process of doubling up a C^∞ manifold X with boundary by gluing together two copies X_+, X_- of X along the common boundary, we can double up W to a manifold $2W$ without boundary and edge Z. The local wedges for $2W$ are $(2X)^\Delta \times \Omega$, where $2X$ is the double of X (where we identify X with X_+). The transition maps in the version for $2X$ are $(2X)^\Delta \times \Omega \to (2X)^\Delta \times \widetilde{\Omega}$; the corresponding diffeomorphisms $\mathbb{R}_+ \times (2X) \times \Omega \to \mathbb{R}_+ \times (2X) \times \widetilde{\Omega}$ are restrictions of diffeomorphisms $\mathbb{R} \times (2X) \times \Omega \to \mathbb{R} \times (2X) \times \widetilde{\Omega}$ to $\mathbb{R}_+ \times (2X) \times \Omega$. All these maps restrict to the corresponding ones for $X = X_+$ itself.

We have the cocycle of maps $\{0\} \times (2X) \times \Omega \to \{0\} \times (2X) \times \widetilde{\Omega}$ which admits to attach $2X \times \Omega$ to $\mathbb{R}_+ \times 2X \times \Omega$ which yields $\overline{\mathbb{R}}_+ \times (2X) \times \Omega$. This is an invariant construction. The cocycle of maps $\overline{\mathbb{R}}_+ \times (2X) \times \Omega \to \overline{\mathbb{R}}_+ \times (2X) \times \widetilde{\Omega}$ allows us to pass to the so called stretched manifold $2\mathbb{W}$ which is a C^∞ manifold with boundary $\partial(2\mathbb{W})$ that has the structure of a $2X$-bundle over Z. There is then a canonical continuous map

$$2\mathbb{W} \to 2W \tag{1.4}$$

which restricts to a diffeomorphism $(2\mathbb{W})_{\mathrm{reg}} \to (2W) \setminus Z$ for $(2\mathbb{W})_{\mathrm{reg}} := 2\mathbb{W} \setminus \partial(2\mathbb{W})$. Let us set $(2\mathbb{W})_{\mathrm{sing}} := \partial(2\mathbb{W})$.

For W itself we then define the stretched manifold $\mathbb{W}$ to be the preimage of W under the projection (1.4), and we set

$$\mathbb{W}_{\mathrm{reg}} := \mathbb{W} \cap (2\mathbb{W})_{\mathrm{reg}}, \ \ \mathbb{W}_{\mathrm{sing}} := \mathbb{W} \cap (2\mathbb{W})_{\mathrm{sing}}.$$

For convenience we assume that the transition maps (1.3) (as well as the analogues for the double) are independent of r near zero and that $\mathbb{W}_{\mathrm{sing}}$ $((2\mathbb{W})_{\mathrm{sing}})$ is a trivial X $((2X))$-bundle over Z. The second assumption is not essential for our calculus; the first one can considerably be weakened (though this will not be necessary for us).

Let us now pass to the definition of weighted spaces, first on an infinite stretched cone $N^\wedge = \mathbb{R}_+ \times N$ for a closed C^∞ manifold N, $n = \mathrm{dim} N$ (later on we apply this for $N = 2X$).

Let $L^\mu_{\mathrm{cl}}(N; \mathbb{R}^l)$ denote the space of all classical pseudo-differential operators of order μ on N with parameters $\lambda \in \mathbb{R}^l$ (i.e., the local amplitude functions $a(x, \xi, \lambda)$ are classical symbols in (ξ, λ), and $L^{-\infty}(N; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(N))$, where $L^{-\infty}(N)$ is the space of smoothing operators on N). We use the fact that for every $\nu \in \mathbb{R}$ there is a parameter-dependent elliptic element $R^\nu(\lambda) \in L^\nu_{\mathrm{cl}}(N; \mathbb{R}^l)$ which induces isomorphisms $R^\nu(\lambda) : H^s(N) \to H^{s-\nu}(N)$ for all $\nu \in \mathbb{R}$, $\lambda \in \mathbb{R}^l$. Let M denote the Mellin transform on $\mathbb{R}_+$, $Mu(w) = \int_0^\infty r^{w-1} u(r) dr$, first for $u \in C_0^\infty(\mathbb{R}_+)$ (with the covariable w varying on $\mathbb{C}$), and then extended to more general distribution spaces, also vector-valued ones. In the latter case we often consider w on a weight line $\Gamma_\beta := \{w \in \mathbb{C} : \mathrm{Re}\, w = \beta\}$ for some real β.

Let $\mathcal{H}^{s,\gamma}(N^\wedge)$ denote the completion of $C_0^\infty(\mathbb{R}_+, C^\infty(N))$ with respect to the norm

$$\left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \| R^s(\mathrm{Im}\, w)(Mu(w)) \|^2_{L^2(N)} dw \right\}^{1/2}.$$

We will also need a slight modification of these spaces for $r \to \infty$. To this end we consider charts $\chi_1 : U \to V$ on N to open sets $V \subset S^n$ and define $\chi : \mathbb{R}_+ \times U \to \{\tilde{x} \in \mathbb{R}^{n+1} : \frac{\tilde{x}}{|\tilde{x}|} \in V\}$ by

$\chi(r,x) = r\chi_1(x)$ for $r \in \mathbb{R}_+$. Then $H^s_{\mathrm{cone}}(N^\wedge)$ denotes the subspace of all $v \in H^s_{\mathrm{loc}}(\mathbb{R} \times N)|_{\mathbb{R}_+ \times N}$ such that for every $\varphi \in C_0^\infty(U)$ we have $(1-\omega)\varphi v \circ \chi^{-1} \in H^s(\mathbb{R}^{n+1}_{\tilde{x}})$ for every χ and any cut-off function $\omega(r)$ (in this paper a cut-off function on the half axis will be any $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ that is equal to 1 near the origin). We then define

$$\mathcal{K}^{s,\gamma}(N^\wedge) := \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(N^\wedge), v \in H^s_{\mathrm{cone}}(N^\wedge)\}, \tag{1.5}$$

$s, \gamma \in \mathbb{R}$. The spaces (1.5) are Hilbert spaces with suitable scalar products; for $s = \gamma = 0$ we have $\mathcal{K}^{0,0}(N^\wedge) = r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times N)$ (where L^2 refers to the product measure $drdx$ for some Riemannian metric on N).

For $N = 2X$ we define the spaces

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{u|_{\mathrm{int} X^\wedge} : u \in \mathcal{K}^{s,\gamma}((2X)^\wedge)\}$$

with the Hilbert space structure of the quotient space $\mathcal{K}^{s,\gamma}((2X)^\wedge)/\sim$, where $u \sim v$ means $u|_{\mathrm{int} X^\wedge} = v|_{\mathrm{int} X^\wedge}$.

On the spaces $\mathcal{K}^{s,\gamma}(N^\wedge)$ we have a strongly continuous group of isomorphisms

$$\kappa_\lambda : \mathcal{K}^{s,\gamma}(N^\wedge) \to \mathcal{K}^{s,\gamma}(N^\wedge) \tag{1.6}$$

when we set $\kappa_\lambda u(r,x) = r^{\frac{n+1}{2}} u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$. For purposes below we set

$$\mathcal{S}^\gamma(N^\wedge) := \omega \mathcal{K}^{\infty,\gamma}(N^\wedge) + (1-\omega)\mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(N))$$

for any cut-off function, and

$$\mathcal{S}^\gamma(X^\wedge) := \{u|_{\mathrm{int} X^\wedge} : u \in \mathcal{S}^\gamma((2X)^\wedge)\}.$$

If a Hilbert space E is endowed with a strongly continuous group $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : E \to E$, such that $\kappa_{\lambda\delta} = \kappa_\lambda \kappa_\delta$ for all $\lambda, \delta \in \mathbb{R}_+$, we say that E is endowed with a group action. There is then a scale $\mathcal{W}^s(\mathbb{R}^q, E)$ of so called abstract edge Sobolev spaces of smoothness $s \in \mathbb{R}$, defined as the completion of $\mathcal{S}(\mathbb{R}^q, E)$ in the norm $\{\int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \hat{u}(\eta)\|^2_E d\eta\}^{1/2}$; here $\hat{u}$ means the Fourier transform (of E-valued functions) in $\mathbb{R}^q$. The space $\mathcal{W}^s(\mathbb{R}^q, E)$ is a Hilbert space with the scalar product

$$(u,v)_{\mathcal{W}^s(\mathbb{R}^q,E)} = \int \langle\eta\rangle^{2s} (\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta), \kappa^{-1}_{\langle\eta\rangle}\hat{v}(\eta))^2_E d\eta$$

where $(\cdot,\cdot)_E$ denotes the scalar product in E. If $\Omega \subseteq \mathbb{R}^q$ is an open set, $\mathcal{W}^s_{\mathrm{comp}(z)}(\Omega, E)$ denotes the set of all $u \in \mathcal{W}^s(\Omega, E)$ that have compact support in $z \in \Omega$; moreover, $\mathcal{W}^s_{\mathrm{loc}(z)}(\Omega, E)$ is the space of all E-valued distributions on E such that $\varphi u \in \mathcal{W}^s_{\mathrm{comp}(z)}(\Omega, E)$ for every $\varphi \in C_0^\infty(\Omega)$.

Applying this definition to $E = \mathcal{K}^{s,\gamma}(N^\wedge)$ with the group action (1.6) we obtain weighted edge Sobolev spaces

$$\mathcal{W}^{s,\gamma}(N^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(N^\wedge))$$

which are Hilbert spaces. Note that $\mathcal{W}^{0,0}(N^\wedge \times \mathbb{R}^q) = r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times N \times \mathbb{R}^q)$.

If W is a (say, compact) manifold with edge Z (first without boundary) and $\mathbb{W}$ its stretched manifold we define $\mathcal{W}^{s,\gamma}(\mathbb{W})$ as the subspace of all $u \in H^s_{\mathrm{loc}}(\mathbb{W}_{\mathrm{reg}})$ such that in the splitting of variables $(r,x,z) \in N^\wedge \times \mathbb{R}^q$ near $\mathbb{W}_{\mathrm{sing}}$ the function $\omega\varphi u$ belongs to $\mathcal{W}^{s,\gamma}(N^\wedge \times \mathbb{R}^q)$ for any cut-off function $\omega(r)$ and $\varphi \in C_0^\infty(Z)$ that localises u in a neighborhood of a point $z \in Z$. Also in $\mathcal{W}^{s,\gamma}(\mathbb{W})$ we can introduce Hilbert space scalar products for all s, γ by using corresponding local scalar products and a partition of unity.

For the case of a (compact) manifold W with edge Z and boundary we first consider the double $2W$ and the associated stretched manifolds $\mathbb{W}$ and $2\mathbb{W}$, respectively. By construction, $2\mathbb{W}$ also consists of two copies $\mathbb{W}_\pm$ of $\mathbb{W}$, and we identify $\mathbb{W}_+$ with $\mathbb{W}$. Then we set

$$\mathcal{W}^{s,\gamma}(\mathbb{W}) = \{u|_{\mathrm{int}\mathbb{W}_{\mathrm{reg}}} : u \in \mathcal{W}^{s,\gamma}(2\mathbb{W})\},$$

endowed with the quotient topology from the isomorphism $\mathcal{W}^{s,\gamma}(\mathbb{W}) \cong \mathcal{W}^{s,\gamma}(2\mathbb{W})/\sim$, with the equivalence relation $u \sim v \Leftrightarrow u|_{\mathrm{int}\mathbb{W}_{\mathrm{reg}}} = v|_{\mathrm{int}\mathbb{W}_{\mathrm{reg}}}$. This gives us a Hilbert space structure in the space $\mathcal{W}^{s,\gamma}(\mathbb{W})$ for the case when W is a manifold with edge and boundary.

Remark 1.1.1. *Let W be a manifold with boundary and edge Z. Then $V := \{\partial(W \setminus Z)\} \cup Z$ is a manifold without boundary and with edge Z. If $\mathbb{W}$ and $\mathbb{V}$ are the associated stretched manifolds we have $\partial\mathbb{W}_{\mathrm{reg}} = \mathbb{V}_{\mathrm{reg}}$, and $\mathbb{V} = \partial\mathbb{W}_{\mathrm{reg}} \cup \partial\mathbb{W}_{\mathrm{sing}}$.*

An example is the wedge $W = X^\Delta \times \Omega$ with $Z = \Omega$ and a C^∞ manifold X with boundary ∂X; then $V = (\partial X)^\Delta \times \Omega$, $\mathbb{W} = \overline{\mathbb{R}}_+ \times X \times \Omega$, and $\mathbb{V} = \overline{\mathbb{R}}_+ \times \partial X \times \Omega$.

Our spaces may be generalised to the case of distributional sections in a (smooth complex) vector bundle. In this case we write

$$\mathcal{W}^{s,\gamma}(\mathbb{W}, F) \text{ and } \mathcal{W}^{s,\gamma}(\mathbb{V}, G),$$

where F and G are the corresponding bundles over $\mathbb{W}$ and $\mathbb{V}$, respectively; more details on those spaces will be given below. Our calculus will be formulated for $F = \mathbb{W} \times \mathbb{C}$ (the trivial bundle of fibre dimension 1). Concerning G, in general, it is necessary to admit non-trivial bundles, although in the first part of the paper for simplicity those bundles are assumed to be trivial, too. The generalisation to the case of non-trivial F and G is easy and left to the reader. For references in Chapter 4 below, we prepare some notation for general G. First, if G_1 is a vector bundle on $X^\wedge$ we have a straightforward definition of the space $\mathcal{K}^{s,\gamma}(X^\wedge, G_1)$. Since every G_1 can be regarded as the pull back of a bundle G_2 over X under the canonical projection $X^\wedge \to X$ we will often employ the same notation for G_1 and G_2. Analogously, every bundle G_3 over $X^\wedge \times \mathbb{R}^q$ is the pull back of some G_2 over X under $X^\wedge \times \mathbb{R}^q \to X$; also here we often take the same letters. Now if G is a bundle on $\mathbb{V}$ we can localise it near $\mathbb{V}_{\mathrm{sing}}$ to a singular chart to $X^\wedge \times \mathbb{R}^q$. Then $\mathcal{W}^{s,\gamma}(\mathbb{V}, G)$ is defined as the subspace of all $u \in H^s_{\mathrm{loc}}(\mathbb{V}_{\mathrm{reg}}, G)$ which are locally near $\mathbb{V}_{\mathrm{sing}}$ a pull back of an element in $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge, G_1))$ where G_1 over $X^\wedge$ corresponds to the restriction of G to that singular chart. We will simply write G for all involved bundles G, G_1, G_2 (this should not cause confusion).

1.2 Boundary value problems

Let W be a manifold with boundary and edge Z. Let $\mathbb{W}$ denote its stretched manifold, and $2\mathbb{W}$ its double as explained before.

A differential operator A of order μ on $\mathbb{W}_{\mathrm{reg}}$ (with smooth coefficients up to $\partial\mathbb{W}_{\mathrm{reg}}$) is said to be edge-degenerate if it has in the splitting of variables (r, x, z) near $\mathbb{W}_{\mathrm{sing}}$ the form

$$A = r^{-\mu} \sum_{k+|\alpha| \le \mu} a_{k\alpha}(r, z)\big(-r\frac{\partial}{\partial r}\big)^k (rD_z)^\alpha \tag{1.7}$$

with coefficients $a_{k\alpha}(r, z) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(k+|\alpha|)}(X))$. Here X is the base of the model cone which is a C^∞ manifold with boundary ∂X, and $\mathrm{Diff}^\nu(X)$ means the Fréchet space of all differential operators on X of order ν with smooth coefficients up to the boundary.

A (differential) boundary value problem for the operator A in the category of edge-degenerate operators is formulated as

$$Au = f \text{ in } \mathbb{W}_{\mathrm{reg}},\ Tu = g \text{ on } \partial\mathbb{W}_{\mathrm{reg}} \tag{1.8}$$

with a column vector $T = {}^{\mathrm{t}}(T_1, \ldots, T_N)$ of trace operators of the form

$$T_j u = B_j u|_{\partial\mathbb{W}_{\mathrm{reg}}}, \;\; j = 1, \ldots, N, \tag{1.9}$$

where B_j are edge-degenerate differential operators of order μ_j, as explained before, i.e., in a neighbourhood of $\overline{\partial\mathbb{W}_{\mathrm{reg}}}$ in $2\mathbb{W}$ near $\mathbb{W}_{\mathrm{sing}}$ of the form

$$B_j u = r^{-\mu_j} \sum_{k+|\beta|\le\mu_j} b_{j,k\beta}(r,z)\big(-r\frac{\partial}{\partial r}\big)^k (rD_z)^\beta,$$

with coefficients $b_{j,k\beta}(r,z) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu_j-(k+|\beta|)}(X))$ for all k, β and j.

Assume W to be compact. Then $\mathcal{A} := \begin{pmatrix} A \\ T \end{pmatrix}$ induces a continuous operator

$$\mathcal{A} : \mathcal{W}^{s,\gamma}(\mathbb{W}) \to \boldsymbol{\mathcal{W}}^{s-\mu,\gamma-\mu}(\mathbb{W}) \tag{1.10}$$

for $s > \max\{\mu_j + \frac{1}{2} : j = 1, \ldots, N\}$, $\gamma \in \mathbb{R}$, where $\boldsymbol{\mathcal{W}}^{s-\mu,\gamma-\mu}(\mathbb{W}) := \begin{matrix} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}) \\ \oplus \\ \oplus_{j=1}^N \mathcal{W}^{s-\mu_j-\frac{1}{2},\gamma-\mu_j-\frac{1}{2}}(\mathbb{V}) \end{matrix}$ (in the non-compact case we have a similar continuity between the 'comp' and 'loc' versions of edge Sobolev spaces).

Occasionally, local coordinates on $\mathbb{W}_{\mathrm{reg}}$ will be denoted by y with covariables η. Close to $\partial\mathbb{W}_{\mathrm{reg}}$ we then write $y = (y', y_{n+q})$ with y' being tangent and y_{n+q} normal to the boundary; the splitting of covariables is then $\eta = (\eta', \eta_{n+q})$. Finally, close to $r = 0$ we have the splitting $y = (r, x, z)$ with $\eta = (\rho, \xi, \zeta)$ and $y' = (r, x', z)$, $\eta' = (\rho, \xi', \zeta)$.

Let us introduce the principal symbolic hierarchy of operators (1.10)

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A})). \tag{1.11}$$

First we have the standard homogeneous principal symbol $\sigma_\psi(\mathcal{A}) := \sigma_\psi(A) \in C^\infty(T^*\mathbb{W}_{\mathrm{reg}} \setminus 0)$. Using the edge-degeneracy we can form a 'compressed' version of $\sigma_\psi(\mathcal{A})$ near $\mathbb{W}_{\mathrm{sing}}$, defined (with respect to local coordinates x on X and z on Z) by

$$\tilde{\sigma}_\psi(\mathcal{A})(r, x, z, \rho, \xi, \zeta) := r^\mu \sigma_\psi(\mathcal{A})(r, x, z, r^{-1}\rho, \xi, r^{-1}\zeta), \tag{1.12}$$

which is homogeneous of order μ in $(\rho, \xi, \zeta) \ne 0$ and smooth in r up to zero.

The second component $\sigma_\partial(\mathcal{A})$ is called the boundary symbol. In future, r_M will denote the operator of restriction to the set M. On $\mathbb{W}_{\mathrm{reg}}$ near $\partial\mathbb{W}_{\mathrm{reg}}$ in the splitting of variables into $(r, x, z) \in \mathbb{R}_+ \times \overline{\mathbb{R}}^n_+ \times \Omega$, $x = (x', x_n)$ (with x' being tangent to ∂X and x_n normal to ∂X) and the covariables (ρ, ξ, ζ), $\xi = (\xi', \xi_n)$ we have

$$\sigma_\partial(\mathcal{A})(r, x', z, \rho, \xi', \zeta) := \begin{pmatrix} \sigma_\psi(A)(r, x', 0, z, \rho, \xi', D_{x_n}, \zeta) \\ {}^{\mathrm{t}}\big(\mathrm{r}_{\{x_n=0\}} \sigma_\psi(B_j)(r, x', 0, \rho, \xi', D_{x_n}, \zeta)_{j=1,\ldots,N}\big) \end{pmatrix}$$

which is an operator family

$$\sigma_\partial(\mathcal{A})(r, x', z, \rho, \xi', \zeta) : H^s(\mathbb{R}_+) \longrightarrow \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^N \end{matrix} \tag{1.13}$$

parametrised by $(r, x', z, \rho, \xi', \zeta) \in T^*\partial\mathbb{W}_{\mathrm{reg}} \setminus 0$. Similarly as before near $\mathbb{W}_{\mathrm{sing}}$ in the splitting of variables (r, x, z), $x = (x', x_n)$ (with respect to local coordinates on X near ∂X and z on Z) we have a compressed version, namely,

$$\tilde{\sigma}_\partial(\mathcal{A})(r, x', z, \rho, \xi', \zeta) := \mathrm{diag}\big(r^\mu, (r^{\mu_j})_{j=1,\ldots,N}\big)\sigma_\partial(\mathcal{A})(r, x', z, r^{-1}\rho, \xi', r^{-1}\zeta) \tag{1.14}$$

$(\rho, \xi', \zeta) \neq 0$, which is smooth up to $r = 0$.

Finally there is the so called principal edge symbol

$$\sigma_\wedge(\mathcal{A})(z,\zeta) := \begin{pmatrix} \sigma_\wedge(A)(z,\zeta) \\ {}^{\mathrm{t}}\big(\sigma_\wedge(T_j)(z,\zeta)\big)_{j=1,\dots,N} \end{pmatrix}$$

for

$$\sigma_\wedge(A)(z,\zeta) = r^{-\mu} \sum_{k+|\alpha|\leq\mu} a_{k\alpha}(0,z)(-r\frac{\partial}{\partial r})^k (r\zeta)^\alpha, \tag{1.15}$$

$$\sigma_\wedge(T_j)(z,\zeta) = \mathrm{r}_{\partial X} r^{-\mu_j} \sum_{k+|\beta|\leq\mu_j} b_{j,k\beta}(0,z)(-r\frac{\partial}{\partial r})^k (r\zeta)^\beta. \tag{1.16}$$

The edge symbol $\sigma_\wedge(\mathcal{A})$ represents a family of continuous operators

$$\sigma_\wedge(\mathcal{A})(z,\zeta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \boldsymbol{\mathcal{K}}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{1.17}$$

parametrised by $(z,\zeta) \in T^*Z \setminus 0$, where $\boldsymbol{\mathcal{K}}^{s-\mu,\gamma-\mu}(X^\wedge) := \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ \oplus_{j=1}^N \mathcal{K}^{s-\mu_j-\frac{1}{2},\gamma-\mu_j-\frac{1}{2}}((\partial X)^\wedge) \end{matrix}$. The homogeneity of the component of $\sigma_\wedge(\mathcal{A})$ is expressed in terms of group actions on the respective spaces

$$\kappa_\lambda^{(n)} : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge), \;\; \kappa_\lambda^{(n-1)} : \mathcal{K}^{s,\gamma}((\partial X)^\wedge) \to \mathcal{K}^{s,\gamma}((\partial X)^\wedge),$$

$s, \gamma \in \mathbb{R}$, defined by $(\kappa_\lambda^{(k)} u)(r,\cdot) := \lambda^{\frac{k+1}{2}} u(\lambda r, \cdot)$ for $\lambda \in \mathbb{R}_+$. In fact, we then have

$$\sigma_\wedge(A)(z,\lambda\zeta) = \lambda^\mu \kappa_\lambda^{(n)} \sigma_\wedge(A)(z,\zeta)(\kappa_\lambda^{(n)})^{-1}, \sigma_\wedge(T_j)(z,\lambda\zeta) = \lambda^{\mu_j+\frac{1}{2}} \kappa_\lambda^{(n-1)} \sigma_\wedge(T_j)(z,\zeta)(\kappa_\lambda^{(n)})^{-1},$$

for all $\lambda \in \mathbb{R}_+$, $j = 1, \dots, N$.

Remark 1.2.1. *The operator* (1.10) *is compact if* $\sigma(\mathcal{A}) = 0$.

Definition 1.2.2. The operator $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ is said to be $(\sigma_\psi, \sigma_\partial)$-elliptic in the edge-degenerate sense if
(i) $\mathcal{A}$ is elliptic on $\mathbb{W}_{\mathrm{reg}}$ in the standard sense, i.e., $\sigma_\psi(\mathcal{A})(y,\eta) \neq 0$ for all $(y,\eta) \in T^*\mathbb{W}_{\mathrm{reg}} \setminus 0$ and $\sigma_\partial(\mathcal{A})(y',\eta')$ bijective as a family of maps (1.13) for all $(y',\eta') \in T^*\partial\mathbb{W}_{\mathrm{reg}} \setminus 0$ (and sufficiently large s);
(ii) $\mathcal{A}$ is elliptic in the compressed sense, i.e., in the corresponding splitting of variables close to $\mathbb{W}_{\mathrm{sing}}$ we require $\tilde\sigma_\psi(\mathcal{A})(r,x,z,\rho,\xi,\zeta) \neq 0$ up to $r = 0$ and $\tilde\sigma_\partial(\mathcal{A})(r,x',z,\rho,\xi',\zeta)$ bijective, both up to $r = 0$ and for non-vanishing covectors (for sufficiently large s).

Theorem 1.2.3. *Let* $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ *be* $(\sigma_\psi, \sigma_\partial)$*-elliptic in the sense of* Definition 1.2.2. *Then for every* $z \in Z$ *there exists a countable set* $D(z) \subset \mathbb{C}$ *with finite intersection* $D(z) \cap \{c \leq \mathrm{Re}\, w \leq c'\}$ *for every* $c \leq c'$*, such that* (1.17) *are Fredholm operators for all* $\zeta \neq 0$ *and all* $\gamma \in \mathbb{R}$ *with*

$$\Gamma_{\frac{n+1}{2}} \cap D(z) = \emptyset \tag{1.18}$$

(*and all sufficiently large* s).

A proof of Theorem 1.2.3 is given in [14]. Recall that $D(z)$ is equal to the set of those $w \in \mathbb{C}$ such that the subordinate principal conormal symbol

$$\sigma_M \sigma_\wedge(\mathcal{A})(z,w) := \begin{pmatrix} \sum_{k=0}^{\mu} a_{k0}(0,z) w^k \\ \mathrm{r}_{\partial X}\, {}^{\mathrm{t}}\big(\sum_{k=0}^{\mu_j} b_{j,k0}(0,z) w^k\big)_{j=1,\dots,N} \end{pmatrix}$$

as a family of Fredholm operators

$$\sigma_M\sigma_\wedge(\mathcal{A})(z,w) : H^s(X) \longrightarrow \begin{array}{c} H^{s-\mu}(X) \\ \oplus \\ \oplus_{j=1}^N H^{s-\mu_j-\frac{1}{2}}(\partial X) \end{array}$$

(for s large) is not bijective (the exceptional set $D(z)$ is just as described before in Theorem 1.2.3). In the sequel we assume that for some weight $\gamma \in \mathbb{R}$ the condition (1.18) is satisfied for all $z \in Z$.

Theorem 1.2.4. *Let* $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ *be a* $(\sigma_\psi, \sigma_\partial)$*-elliptic operator and* $\gamma \in \mathbb{R}$ *such that the condition* (1.18) *holds. Then* (1.17) *is a family of Fredholm operators.*

A proof of Theorem 1.2.4 may be found in [14]. The operator A is globally defined on $\mathbb{W}$ by local expressions, combined with a partition of unity, the coefficients $a_{k\alpha}(r,z)$ of (1.7) may assumed to be independent of r for $r > R$ for some $R > 0$. A similar property is assumed on the coefficients $b_{j,k\beta}(r,z)$ involved in the boundary operators. Let us set

$$a(z,\zeta) := r^{-\mu} \sum_{k+|\alpha|\le\mu} a_{k\alpha}(r,z)\big(-r\frac{\partial}{\partial r}\big)^k (r\zeta)^\alpha$$

and

$$b_j(z,\zeta) := r^{-\mu_j} \sum_{k+|\beta|\le\mu_j} b_{j,k\beta}(r,z)\big(-r\frac{\partial}{\partial r}\big)^k (r\zeta)^\beta.$$

Then $a(z,\zeta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$, $\mathrm{r}_{\partial X} b_j(z,\zeta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu_j-\frac{1}{2},\gamma-\mu_j-\frac{1}{2}}((\partial X)^\wedge)$ are families of continuous operators, C^∞ in $(z,\zeta) \in \Omega \times \mathbb{R}^q$ and they are operator-valued symbols in the sense of the following general definition.

Definition 1.2.5. (i) Let E and $\tilde{E}$ be Hilbert spaces with group actions $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+}$, respectively. Then $S^\mu(\Omega\times\mathbb{R}^q; E, \tilde{E})$ for $\mu \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^p$ open, denotes the subspace of all $a(z,\zeta) \in C^\infty(\Omega\times\mathbb{R}^q, \mathcal{L}(E,\tilde{E}))$ such that

$$\sup_{z\in K,\zeta\in\mathbb{R}^q} \langle\zeta\rangle^{-\mu+|\beta|} \|\tilde{\kappa}^{-1}_{\langle\zeta\rangle}\{D_z^\alpha D_\zeta^\beta a(z,\zeta)\}\kappa_{\langle\zeta\rangle}\|_{\mathcal{L}(E,\tilde{E})}$$

is finite for every $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$ and arbitrary $K \Subset \Omega$.

(ii) Let $S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$ be the space of all $a_{(\mu)}(z,\zeta) \in C^\infty(\Omega \times (\mathbb{R}^q\setminus\{0\}); \mathcal{L}(E,\tilde{E}))$ such that

$$a_{(\mu)}(z,\delta\zeta) = \delta^\mu \tilde{\kappa}_\delta a_{(\mu)}(z,\zeta)\kappa_\delta^{-1}$$

for all $\delta \in \mathbb{R}_+$, $(z,\zeta) \in \Omega \times (\mathbb{R}^q\setminus\{0\}))$. Then $S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q; E,\tilde{E})$ is defined to be the set of all $a(z,\zeta) \in S^\mu(\Omega\times\mathbb{R}^q; E,\tilde{E})$ such that there are elements $a_{(\mu-j)}(z,\zeta) \in S^{(\mu-j)}(\Omega\times(\mathbb{R}^q\setminus\{0\}); E,\tilde{E})$ with $a - \chi\sum_{j=0}^N a_{(\mu-j)} \in S^{\mu-(N+1)}(\Omega\times\mathbb{R}^q; E,\tilde{E})$ for all $N \in \mathbb{N}$, where $\chi(\zeta)$ is any excision function in $\mathbb{R}^q$.

Pseudo-differential operators with amplitude functions $a(z,\zeta)$ of that kind (for $\Omega \subseteq \mathbb{R}^q$ open) are given by

$$\mathrm{Op}_z(a)u(z) = \iint e^{i(z-z')\zeta} a(z,\zeta)u(z')dz'đ\zeta, \;\; đ\zeta = (2\pi)^{-q}d\zeta;$$

occasionally we also write $\mathrm{Op}(\cdot)$. For $q = 1$ we also write $\mathrm{op}_z(\cdot)$ rather than $\mathrm{Op}(\cdot)$ (this will play a role for z replaced by the cone axis variable r). Pseudo-differential operators in $r \in \mathbb{R}_+$ will also occur in connection with the Mellin transform. We then set

$$\mathrm{op}_M^\gamma(f)u(r) := \iint \big(\frac{r'}{r}\big)^{\frac{1}{2}-\gamma+i\rho} f(r, \frac{1}{2}-\gamma+i\rho)u(r')\frac{dr'}{r'}đ\rho, \tag{1.19}$$

$\gamma \in \mathbb{R}$, for an amplitude function $f(r,w)$ on $\mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma}$. In the application below $f(r,w)$ takes values in suitable operator spaces and the argument function is vector-valued. (1.19) is interpreted as a weighted Mellin pseudo-differential operator (with weight γ.

Proposition 1.2.6. $\mathrm{Op}_z(a) : \mathcal{W}^s_{\mathrm{comp}(z)}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}(z)}(\Omega, \tilde{E})$ *is continuous for all* s.

Proofs of this result in different generality concerning the space $E, \tilde{E}$ are given in [30] or [31]. The general case is treated in [35].

Remark 1.2.7. *We have*

$$a(z,\zeta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

and

$$\mathrm{r}_{\partial X} b_j(z,\zeta) \in S^{\mu_j+\frac{1}{2}}\big(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu_j-\frac{1}{2},\gamma-\mu_j-\frac{1}{2}}((\partial X)^\wedge)\big)$$

for all sufficiently large s *and all* $\gamma \in \mathbb{R}$.

Since the dimensions p and q in Definition 1.2.5 are independent we also have the symbol spaces $S^\mu(\Omega \times \mathbb{R}^{q+l}; E, \tilde{E})$ with the covariables (ζ, λ), where $\lambda \in \mathbb{R}^l$ is interpreted as a parameter. It is often sufficient to consider the case $\Omega = \mathbb{R}^q$, since Ω plays the role of local coordinates on a manifold. It is then convenient to give the symbol spaces $S^\mu(\mathbb{R}^q \times \mathbb{R}^{q+l}; E, \tilde{E})$ a slightly different meaning compared with Definition 1.2.5 by requiring

$$\sup_{(\zeta,\lambda)\in\mathbb{R}^{q+l}} \langle \zeta, \lambda\rangle^{-\mu+|\beta|} \|\tilde{\kappa}^{-1}_{\langle\zeta,\lambda\rangle}\{D^\alpha_z D^\beta_{\zeta,\lambda} a(z,\zeta,\lambda)\}\kappa_{\langle\zeta,\lambda\rangle}\|_{\mathcal{L}(E,\tilde{E})} < \infty$$

for all $\alpha \in \mathbb{N}^q$, $\beta \in \mathbb{N}^{q+l}$.

Remark 1.2.8. *For* $L^\mu(\mathbb{R}^q; E, \tilde{E}; \mathbb{R}^l) := \{\mathrm{Op}(a)(\lambda) : a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^{q+l}; E, \tilde{E})\}$, *we then have a bijection*

$$\mathrm{Op}(\cdot)(\lambda) : S^\mu(\mathbb{R}^q \times \mathbb{R}^{q+l}; E, \tilde{E}) \to L^\mu(\mathbb{R}^q; E, \tilde{E}; \mathbb{R}^l) \tag{1.20}$$

and $\mathrm{Op}(a)(\lambda)\,\mathrm{Op}(b)(\lambda) = \mathrm{Op}(a\#b)(\lambda)$ *for every two symbols* a *and* b *of order* μ *and* ν, *respectively* (*where the spaces in the middle are assumed to fit together*), *with a unique symbol* $a\#b$ *or order* $\mu+\nu$ (*the Leibniz product between* a *and* b), *where*

$$a\#b = ab \text{ modulo a symbol of order } \mu+\nu-1. \tag{1.21}$$

Let $S^{-\infty}(\mathbb{R}^q \times \mathbb{R}^{q+l}; E, \tilde{E}) := \bigcap_{\mu\in\mathbb{R}} S^\mu(\mathbb{R}^q \times \mathbb{R}^{q+l}; E, \tilde{E})$. Then the bijection (1.20) holds including $\mu = -\infty$ (where $L^{-\infty}(\dots) := \bigcap_{\mu\in\mathbb{R}} L^\mu(\dots)$).

1.3 Ellipticity with edge conditions

Ellipticity of an operator $\mathcal{A}$ on a manifold W with edge Z and boundary should mean the bijectivity of all components of the principal symbolic hierarchy (1.11). The condition for $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$ is given in Definition 1.2.2. However this only entails the Fredholm property of $\sigma_\wedge(\mathcal{A})$ in the sense of Theorem 1.2.3. For the bijectivity of $\sigma_\wedge(\mathcal{A})$ we have to enlarge the operator $\mathcal{A}$ to a block matrix

$$\mathfrak{A} := \begin{pmatrix} \mathcal{A} & \mathcal{K} \\ \mathcal{T} & \mathcal{Q} \end{pmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(\mathbb{W}) \\ \oplus \\ H^s(Z, J_-) \end{matrix} \longrightarrow \begin{matrix} \boldsymbol{\mathcal{W}}^{s-\mu,\gamma-\mu}(\mathbb{W}) \\ \oplus \\ H^{s-\mu}(Z, J_+) \end{matrix} \tag{1.22}$$

by additional edge conditions. Here $\mathcal{T}$ has the meaning of a trace operator, $\mathcal{K}$ of a potential operator with respect to the edge, while $\mathcal{Q}$ is a classical pseudo-differential operator acting between distributional sections of suitable (smooth complex) vector bundles J_- and J_+ over Z.

In order to formulate the extra operators we enlarge the Fredholm family (1.17) to a family of isomorphisms

$$\sigma_\wedge(\mathfrak{A})(z,\zeta) := \begin{pmatrix} \sigma_\wedge(\mathcal{A}) & \sigma_\wedge(\mathcal{K}) \\ \sigma_\wedge(\mathcal{T}) & \sigma_\wedge(\mathcal{Q}) \end{pmatrix}(z,\zeta) : \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ J_{-,z} \end{matrix} \longrightarrow \begin{matrix} \boldsymbol{\mathcal{K}}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ J_{+,z} \end{matrix} \tag{1.23}$$

by additional finite-dimensional entries $\sigma_\wedge(\mathcal{K})$, $\sigma_\wedge(\mathcal{T})$ and $\sigma_\wedge(\mathcal{Q})$. Here $J_{\pm,z}$ are the fibres of $J_\pm$ over the point $z \in Z$. We may construct (1.23) first for $(z,\zeta) \in S^*Z$ (the unit cosphere bundle induced by T^*Z) and then extend the operators by 'κ_δ-homogeneity' to all $(z,\zeta) \in T^*Z \setminus 0$. The homogeneities of the entries are determined by the spaces in the first components of (1.23) and the chosen order shift in the spaces on Z in the formula (1.22). For instance, we require

$$\sigma_\wedge(\mathcal{T})(z,\delta\zeta) = \delta^\mu \sigma_\wedge(\mathcal{T})(z,\zeta)(\kappa_\delta^{(n)})^{-1}$$

for $\delta \in \mathbb{R}_+$, $(z,\zeta) \in T^*Z \setminus 0$, or, for the component of $\sigma_\wedge(\mathcal{K})(z,\zeta)$ for $\mathcal{K} := (\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_N)$

$$\sigma_\wedge(\mathcal{K}_0)(z,\delta\zeta) = \delta^\mu \kappa_\delta^{(n)} \sigma_\wedge(\mathcal{K}_0)(z,\zeta), \;\; \sigma_\wedge(\mathcal{K}_j)(z,\delta\zeta) = \delta^{\mu_j+\frac{1}{2}} \kappa_\delta^{(n-1)} \sigma_\wedge(\mathcal{K}_j)(z,\zeta),$$

$j = 1, \ldots, N$.

Recall that the construction of the additional entries (first for $(z,\zeta) \in S^*Z$) is close to a corresponding idea in boundary value problems, cf. [5]. The first step is to find the potential part $\sigma_\wedge(\mathcal{K})$ such that the first row of (1.23) is surjective for all $(z,\zeta) \in S^*Z$. In this construction, if we do not take care of a minimal choice of the fibre dimension j_- of J_-, we can simply take $J_- := Z \times \mathbb{C}^{j_-}$ (as is well known there always exists a j_- such that $\big(\sigma_\wedge(\mathcal{A}) \;\; \sigma_\wedge(\mathcal{K})\big)$ is surjective for all $(z,\zeta) \in S^*Z$). The operators (1.17) belong to the cone algebra of boundary value problems on the infinite stretched cone, cf. [26], and they are exit elliptic for $r \to \infty$, cf. [14]. This admits to choose $\sigma_\wedge(\mathcal{K})(z,\zeta)$ in such a way that

$$\sigma_\wedge(\mathcal{K})(z,\zeta) : J_{-,z} \longrightarrow \begin{matrix} \mathcal{S}^{\gamma-\mu+\varepsilon}(X^\wedge) \\ \oplus \\ \oplus_{j=0}^N \mathcal{S}^{\gamma-\mu_j-\frac{1}{2}+\varepsilon}((\partial X)^\wedge) \end{matrix}$$

for some $\varepsilon > 0$ (we may even replace the spaces in the latter relation by $C_0^\infty(X^\wedge)$ and $C_0^\infty((\partial X)^\wedge)$, respectively, because these spaces are dense in the weighted Sobolev spaces).

In order to complete the first row of (1.23) to an isomorphism, the second row necessarily has to map $\ker\big(\sigma_\wedge(\mathcal{A}) \;\; \sigma_\wedge(\mathcal{K})\big)(z,\zeta)$ isomorphically to $J_{+,z}$. This can be organised by a projection of $\begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ J_{-,z} \end{matrix}$ to that kernel, composed with an isomorphism to $J_{+,z}$. Recall from the general ('elementary') K-theory, cf. [3], that such constructions have a functional analytic background. If M is a compact topological space (say, arcwise connected), H, $\tilde{H}$ Hilbert spaces, $\mathcal{F}(H,\tilde{H})$ the set of all Fredholm operators $H \to \tilde{H}$ in the topology induced by $\mathcal{L}(H,\tilde{H})$, and $a \in C(M, \mathcal{F}(H,\tilde{H}))$ a continuous family, there always exists a continuous family of isomorphisms

$$\begin{pmatrix} a(m) & k(m) \\ t(m) & q(m) \end{pmatrix} : \begin{matrix} H \\ \oplus \\ G_{-,m} \end{matrix} \to \begin{matrix} \tilde{H} \\ \oplus \\ G_{+,m} \end{matrix}$$

for suitable (continuous, complex) vector bundles $G_\pm$ over M. Then the pair (G_+, G_-) represents an element $[G_+] - [G_-]$ in $K(M)$, the K-group on M, called the index of the operator family a, written

$$\operatorname{ind}_M a \in K(M).$$

As is known, this element is independent of the choice of $G_\pm$ and the families k, t, q. In the present case we have $M = S^*Z$, $H = \mathcal{K}^{s,\gamma}(X^\wedge)$, $\tilde{H} = \boldsymbol{\mathcal{K}}^{s-\mu,\gamma-\mu}(X^\wedge)$ for any fixed (sufficiently large) s, and we may talk about C^∞ vector bundles on the respective manifold. Moreover, we want to interpret $G_\pm$ as pull backs of bundles $J_\pm$ on Z with respect to the canonical projection $\pi : S^*Z \to Z$.

Remark 1.3.1. *The property*

$$\operatorname{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}) \in \pi^*K(Z) \tag{1.24}$$

is a necessary and sufficient condition for the existence of operators (1.22) *which have* (1.23) *as the homogeneous principal edge symbol. It may happen that* (1.24) *is violated. The property* (1.24) *is a topological obstruction for the existence of* (1.22) *in the edge calculus.*

Similar phenomena are known in the theory of elliptic boundary value problems, cf. Atiyah, Bott [2], or [5]. Recall that boundary value problems have much in common with edge problems, cf. [33, 34]. Let us also note that when the obstruction (1.24) is non-vanishing, there are pseudo-differential (boundary or edge) calculi with global projection conditions, cf. [32, 34], which generalise those of Atiyah, Patodi and Singer [4].

Let us make a few further remarks about the nature of the edge symbols of the additional operators $\mathcal{K}$, $\mathcal{T}$, $\mathcal{Q}$. As noted before the operator $\mathcal{Q}$ is a classical pseudo-differential operator on Z. Concerning $\sigma_\wedge(\mathcal{T})(z,\zeta)$ and $\sigma_\wedge(\mathcal{K})(z,\zeta)$ which can be generated by the above construction in local coordinates $(z,\zeta) \in T^*\Omega \setminus 0$, $\Omega \subseteq \mathbb{R}^q$ open, we pass to

$$t(z,\zeta) := \chi(\zeta)\sigma_\wedge(\mathcal{T})(z,\zeta), \;\; k(z,\zeta) := \chi(\zeta)\sigma_\wedge(\mathcal{K})(z,\zeta)$$

for any excision function $\chi(\zeta)$ in $\mathbb{R}^q$. This gives us elements

$$t(z,\zeta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathbb{C}^{j_+})$$

and

$$k_0(z,\zeta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathbb{C}^{j_-}, \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)),$$

$$k_j(z,\zeta) \in S^{\mu_j+\frac{1}{2}}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathbb{C}^{j_-}, \mathcal{K}^{s-\mu_j-\frac{1}{2},\gamma-\mu_j-\frac{1}{2}}((\partial X)^\wedge)),$$

$j = 1,\dots,N$, cf. Definition 1.2.5 (ii). Locally over Ω we then form the pseudo-differential operators $\mathrm{Op}_z(t)\omega(r)$ and $\omega(r)\,\mathrm{Op}_z(k_j)$, $j = 0,\dots,N$, for some cut-off function $\omega(r)$. Then the global operators $\mathcal{T}$ and $\mathcal{K}_j$ are defined as finite sums of operators of the kind

$$\varphi'\,\mathrm{Op}_z(t)\psi'\omega, \;\text{ and } \omega\varphi'\,\mathrm{Op}_z(k_j)\psi',$$

(of course, combined with pull backs to the manifold) where φ' is coming from a partition of unity on Z, and ψ' is a C_0^∞ function in a corresponding coordinate neighbourhood that is equal to 1 on $\operatorname{supp}\varphi'$. This is an invariant construction globally on our manifold with boundary and edge Z, including the transition behaviour of the bundles $J_\pm$. Applying Proposition 1.2.6 we obtain the continuity of the operators between the weighted edge spaces in (1.22).

Theorem 1.3.2. *The topological obstruction for existence of elliptic edge conditions is independent of the choice of the weight γ for which* (1.17) *is a Fredholm family.*

Proof. Let us give a proof for the situation of elliptic operators on a manifold W with edge Z and without boundary. It will then become clear how to argue in the case of boundary value problems. By assumption our elliptic operator A on W is edge-degenerate near Z in stretched coordinates $(r,x,z) \in \mathbb{R}_+ \times X \times \Omega$, $\Omega \subseteq \mathbb{R}^q$, cf. the formula (1.7). In the present case the base X of the cone is closed and compact. The principal edge symbol has the form

$$\sigma_\wedge(A)(z,\zeta) = r^{-\mu} \sum_{k+|\alpha|\le\mu} a_{k\alpha}(0,z)\big(-r\frac{\partial}{\partial r}\big)^k (r\zeta)^\alpha. \tag{1.25}$$

Assume that $\sigma_\wedge(A)$ induces families of Fredholm operators

$$\sigma_\wedge(A)_{\gamma_i}(z,\zeta):\mathcal{K}^{s,\gamma_i}(X^\wedge)\to\mathcal{K}^{s-\mu,\gamma_i-\mu}(X^\wedge),$$

for different weights $\gamma_i\in\mathbb{R}$, $i=0,1$. We then have index elements $\mathrm{ind}_{S^*Z}\sigma_\wedge(A)_{\gamma_i}\in K(S^*Z)$, $i=0,1$. We will show

$$\mathrm{ind}_{S^*Z}\sigma_\wedge(A)_{\gamma_0}\in\pi^*K(Z)\Leftrightarrow\mathrm{ind}_{S^*Z}\sigma_\wedge(A)_{\gamma_1}\in\pi^*K(Z). \tag{1.26}$$

Choose cut-off functions $\omega(r)$, $\omega_0(r)$, $\omega_1(r)$ such that $\omega_0(r)\equiv 1$ on $\operatorname{supp}\omega$, $\omega\equiv 1$ on $\operatorname{supp}\omega_1$, and write

$$\sigma_\wedge(A)_{\gamma_i}(z,\zeta)=r^{-\mu}\omega(r|\zeta|)\,\mathrm{op}_M^{\gamma_i-\frac{n}{2}}(h)(z,\zeta)\omega_0(r|\zeta|)+r^{-\mu}(1-\omega(r|\zeta|))a(z,\zeta)(1-\omega_1(r|\zeta|))$$

where $h(r,z,w,\zeta):=\sum_{k+|\alpha|\le\mu}a_{k\alpha}(0,z)w^k(r\zeta)^\alpha$ and $a(z,\zeta):=\sigma_\wedge(A)(z,\zeta)$ (the latter notation is justified since $a(z,\zeta)$ is multiplied by localisations far from $r=0$ such that there is no relation to the weights at 0). Choose a strictly positive function $\boldsymbol{k}\in C^\infty(\mathbb{R}_+)$ such that $\boldsymbol{k}(r)=r$ for $0<r<c_0$, $\boldsymbol{k}(r)\equiv 1$ for $c_1<r$, for certain $0<c_0<c_1$. In particular, fix c_0 in such a way that $\omega\equiv 1$, $\omega_0\equiv 1$ for all $0<r\le c_0$. Setting

$$a_1(z,\zeta):=\boldsymbol{k}^{\gamma_0-\gamma_1}\sigma_\wedge(A)_{\gamma_1}(z,\zeta)\boldsymbol{k}^{-\gamma_0+\gamma_1}:\mathcal{K}^{s,\gamma_0}(X^\wedge)\to\mathcal{K}^{s-\mu,\gamma_0-\mu}(X^\wedge);$$

we obtain $\mathrm{ind}_{S^*Z}\sigma_\wedge(A)_{\gamma_1}=\mathrm{ind}_{S^*Z}a_1$. Moreover, let $a_0(z,\zeta)=\sigma_\wedge(A)_{\gamma_0}(z,\zeta)$. We have (writing from now on $\omega=\omega(r|\zeta|)$, etc.)

$$a_1(z,\zeta)=r^{-\mu}\omega\boldsymbol{k}^{\gamma_0-\gamma_1}\,\mathrm{op}_M^{\gamma_1-\frac{n}{2}}(h)(z,\zeta)\boldsymbol{k}^{-\gamma_0+\gamma_1}\omega_0+(1-\omega)\boldsymbol{k}^{\gamma_0+\gamma_1}a(z,\zeta)\boldsymbol{k}^{-\gamma_0+\gamma_1}(1-\omega_1).$$

Let us write $h(r,z,w,\zeta)=f(z,w)+g(r,z,w,\zeta)$ for $f(z,w)=\sum_{k=0}^{\mu}a_{k0}(0,z)w^k$, $g(r,z,w,\zeta)=\sum_{\substack{k+|\alpha|\le\mu\\|\alpha|>0}}a_{k\alpha}(0,z)w^k(r\zeta)^\alpha$. Using the identity

$$\omega\boldsymbol{k}^{\gamma_0-\gamma_1}\,\mathrm{op}_M^{\gamma_1-\frac{n}{2}}(f)(z)\boldsymbol{k}^{-\gamma_0+\gamma_1}\omega_0=\omega\,\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(T^{\gamma_0-\gamma_1}f)(z)\omega_0$$

(since $\boldsymbol{k}(r)=r$ on the support of ω and ω_0) we obtain

$$a_1(z,\zeta)=r^{-\mu}\omega\,\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(T^{\gamma_0-\gamma_1}f)(z)\omega_0+c_1(z,\zeta)+(1-\omega)\boldsymbol{k}^{\gamma_0+\gamma_1}a(z,\zeta)\boldsymbol{k}^{-\gamma_0+\gamma_1}(1-\omega_1)$$

for $c_1(z,\zeta):=r^{-\mu}\omega\,\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(T^{\gamma_0-\gamma_1}g)(z,\zeta)\omega_0$. Let us also reformulate $a_0(z,\zeta)$ as

$$a_0(z,\zeta)=r^{-\mu}\omega\,\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(f)(z)\omega_0+c_0(z,\zeta)+(1-\omega)a(z,\zeta)(1-\omega_1)$$

for $c_0(z,\zeta):=r^{-\mu}\omega\,\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(g)(z,\zeta)\omega_0$. The operator families $c_i(z,\zeta)$ are compact operator-valued; so they do not affect the index elements and may be ignored. Moreover, the difference

$$(1-\omega)\{\boldsymbol{k}^{\gamma_0+\gamma_1}a(z,\zeta)\boldsymbol{k}^{-\gamma_0+\gamma_1}-a(z,\zeta)\}(1-\omega_1)$$

is also compact operator-valued; so it is admitted to compare the index elements of

$$p_1(z,\zeta):=r^{-\mu}\omega\,\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(T^{\gamma_0-\gamma_1}f)(z)\omega_0+\psi a(z,\zeta)\psi_1,$$
$$p_0(z,\zeta):=r^{-\mu}\omega\,\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(f)(z)\omega_0+\psi a(z,\zeta)\psi_1$$

for $\psi:=1-\omega$, $\psi_1:=1-\omega_1$. Set for abbreviation

$$\mathrm{op}_M(f_0)(z):=\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(f)(z),\ \ \mathrm{op}_M(f_1)(z):=\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(T^{\gamma_0-\gamma_1}f)(z).$$

Then we have

$$p_0(z,\zeta) = r^{-\mu}\omega \operatorname{op}_M(f_0)(z)\omega_0 + \psi a(z,\zeta)\psi_1, \ \ p_1(z,\zeta) = r^{-\mu}\omega \operatorname{op}_M(f_1)(z)\omega_0 + \psi a(z,\zeta)\psi_1.$$

Let $a^{-1}(z,\zeta)$ denote a parametrix of $a(z,\zeta)$ on $\mathbb{R}_+ \times X$; then

$$\psi a^{-1}(z,\zeta)\psi_1\psi a(z,\zeta)\psi_1 \sim \psi^2 \tag{1.27}$$

where '$\sim$' means equality modulo a family of compact operators between the respective spaces. Then a parametrix of $p_1(z,\zeta)$ has the form

$$p_1^{-1}(z,\zeta) = \omega \operatorname{op}_M(f_1^{-1})(z)r^{\mu}\omega_0 + \psi a^{-1}(z,\zeta)\psi_1$$

where $f_1^{-1}(z,w)$ is the usual inverse of $f_1(z,w)$ on the weight line $\Gamma_{\frac{n+1}{2}-\gamma_0}$. We obtain

$$\begin{aligned} p_1^{-1}p_0 &= \{\omega \operatorname{op}_M(f_1^{-1})r^{\mu}\omega_0 + \psi a^{-1}\psi_1\}\{r^{-\mu}\omega \operatorname{op}_M(f_0)\omega_0 + \psi a\psi_1\} \\ &= \omega \operatorname{op}_M(f_1^{-1})\omega \operatorname{op}_M(f_0)\omega_0 + \psi a^{-1}\psi_1\psi a\psi_1 &(1.28)\\ &+ \omega \operatorname{op}_M(f_1^{-1})r^{\mu}\omega_0\psi a\psi_1 + \psi a^{-1}\psi_1 r^{-\mu}\omega \operatorname{op}_M(f_0)\omega_0. &(1.29)\end{aligned}$$

The second summand of (1.28) is characterised in (1.27). For the first summand we write

$$\omega \operatorname{op}_M(f_1^{-1})\omega \operatorname{op}_M(f_0)\omega_0 = \omega^2 \operatorname{op}_M(f_1^{-1}f_0)\omega_0 + d(z)$$

for a family $d(z)$ of continuous operators (not necessarily compact). Writing $f_1^{-1}f_0 = 1-(1-f_1^{-1}f_0)$ we obtain $\omega^2 \operatorname{op}_M(f_1^{-1})\omega \operatorname{op}_M(f_0)\omega_0 = \omega^2 - \omega^2 \operatorname{op}_M(1-f_1^{-1}f_0)\omega_0$ which gives us

$$\omega \operatorname{op}_M(f_1^{-1})\omega \operatorname{op}_M(f_0)\omega_0 = \omega^2 + d_1$$

for another family $d_1(z)$ of continuous operators.

For the first term in (1.29) we write

$$\omega \operatorname{op}_M(f_1^{-1})r^{\mu}\omega_0\psi a\psi_1 = \omega\tilde{\omega} \operatorname{op}_M(f_1^{-1})r^{\mu}\omega_0\psi a\psi_1 + \omega(1-\tilde{\omega}) \operatorname{op}_M(f_1^{-1})r^{\mu}\omega_0\psi a\psi_1 \tag{1.30}$$

with a cut-off function $\tilde{\omega}$ such that $\tilde{\omega} \equiv 0$ on $\operatorname{supp}\omega_0\psi$ (noting that $\omega_0\psi \in C_0^{\infty}(\mathbb{R}_+)$). The operators

$$\omega\tilde{\omega} \operatorname{op}_M(f_1^{-1})r^{\mu}\omega_0\psi : \mathcal{K}^{s-\mu,\gamma_0-\mu}(X^\wedge) \to \mathcal{K}^{s,\gamma_0}(X^\wedge)$$

are compact since $\tilde{\omega}\psi$ vanishes on $\operatorname{supp}\omega_0\psi$; in fact, as is known from the cone calculus, cf. [31], the operator is of Green type. Thus the first summand on the right of (1.30) is compact. The second summand vanishes in a neighbourhood of 0. Let us write $\psi_1 = \psi_1\tilde{\tilde{\omega}} + \psi_1(1-\tilde{\tilde{\omega}})$ for a cut-off function $\tilde{\tilde{\omega}}$ such that the support of $\omega_0\psi$ and $1-\tilde{\tilde{\omega}}$ are disjoint. Then the operators $\omega_0\psi a\psi_1(1-\tilde{\tilde{\omega}})$ are compact as is known from the calculus on manifolds with conical exit to infinity, here $X^\wedge$. Then the second summand is equal to

$$\omega(1-\tilde{\omega}) \operatorname{op}_M(f_1^{-1})r^{\mu}\omega_0\psi a\psi_1\tilde{\tilde{\omega}} \tag{1.31}$$

modulo compact operators. Now (1.31) is localised far from $r=0$ and $r=\infty$ and has the structure $\alpha P\beta Q\gamma$ for $\alpha := \omega(1-\tilde{\omega})$, $\beta := \omega_0\psi$, $\gamma := \psi_1\tilde{\tilde{\omega}} \in C_0^{\infty}(\mathbb{R}_+)$, and elliptic operators $P := \operatorname{op}_M(f_1^{-1})r^{\mu}$, $Q := a$ such that P is a parametrix of Q. In such a situation we have $\alpha P\beta Q\gamma \sim \alpha\beta\gamma$. In other words, (1.31)$\sim \omega(1-\tilde{\omega})\omega_0\psi\psi_1\tilde{\tilde{\omega}} = \omega\psi$, i.e.,

$$\omega \operatorname{op}_M(f_1^{-1})r^{\mu}\omega_0\psi a\psi_1 \sim \omega\psi.$$

In a similar manner we can prove

$$\psi a^{-1}\psi_1 r^{-\mu}\omega \operatorname{op}_M(f_0)\omega_0 \sim \omega\psi.$$

This gives us altogether

$$p_1^{-1}p_0 \sim \omega^2 + 2\omega\psi + \psi^2 + d_1 = 1 + d_1.$$

For the index elements we obtain

$$\mathrm{ind}_{S^*Z}(p_1^{-1}p_0) = \mathrm{ind}_{S^*Z}p_1^{-1} + \mathrm{ind}_{S^*Z}p_0 = \mathrm{ind}_{S^*Z}(1+d_1). \tag{1.32}$$

By hypotheses we have $\mathrm{ind}_{S^*Z}p_0 \in \pi^*K(Z)$; moreover, $\mathrm{ind}_{S^*Z}(1+d_1) \in \pi^*K(Z)$, since $1+d_1$ is independent of ζ. From (1.32) it follows that $\mathrm{ind}_{S^*Z}p_1^{-1} \in \pi^*K(Z)$. By virtue of

$$\mathrm{ind}_{S^*Z}(p_1^{-1}p_1) = \mathrm{ind}_{S^*Z}p_1^{-1} + \mathrm{ind}_{S^*Z}p_1 = \mathrm{ind}_{S^*Z}1 \in \pi^*K(Z)$$

we finally obtain $\mathrm{ind}_{S^*Z}p_1 \in \pi^*K(Z)$. Summing up we proved the relation (1.26).

In the case of an edge symbol

$$\sigma_\wedge(\mathcal{A})(z,\zeta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \boldsymbol{\mathcal{K}}^{s-\mu,\gamma-\mu}(X^\wedge)$$

associated with an elliptic boundary value problem we can argue in the same manner, because all conclusions in terms of Mellin calculus near zero or calculus near a conical exit to infinity have direct analogues in the case of boundary value problems, cf. [27]. □

Definition 1.3.3. The operator (1.22) is called elliptic in the calculus of boundary value problems on the manifold W with edge if the three components of (1.11) are bijective (i.e., the first two in the sense of Definition 1.2.2 and the third as a family of isomorphisms (1.23) for all $(z,\zeta) \in T^*Z \setminus 0$ and any sufficiently large s).

The bijectivity of (1.23) is an analogue of the Shapiro-Lopatinskij condition in standard boundary value problems (it is just the bijectivity of the boundary symbol $\sigma_\partial(\cdot)$).

Remark 1.3.4. *Given a* $(\sigma_\psi, \sigma_\partial)$*-elliptic operator* $\mathcal{A}$*, it may be very difficult to compute* (1.24) *and* (*in order to construct* (1.23)) *the dimensions of kernels and cokernels. It belongs just to the program of the present paper to analyse a sufficient large class of examples where this information can be derived. Also the condition* $\mathrm{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}) \in \pi^*K(Z)$ *is far from being trivial. Our examples will also satisfy this condition.*

Theorem 1.3.5. *Let* $\mathfrak{A}$ *be elliptic in the sense of* Definition 1.3.3*. Then* (1.22) *is a Fredholm operator for every sufficiently large* s*.*

A proof of this result may be found in [14]. We shall outline a few details in Section 4.3, also in connection with the existence of corresponding pseudo-differential parametrices in the edge calculus.

2 Model problems in the half-space

2.1 Edge characterisations of Sobolev spaces

Our manifold M, $\varrho = \dim M$, with boundary ∂M and edge $Z \subset \partial M$, $q = \dim Z$, will locally near a point $z \in Z$ be described by the half-space

$$\overline{\mathbb{R}}^\varrho_+ = \{y = (y_1, \ldots, y_\varrho) \in \mathbb{R}^\varrho : y_\varrho \geq 0\}$$

such that ∂M corresponds to $\mathbb{R}^{\varrho-1}$ and Z to the hyperplane $x_1 = \cdots = x_d = 0$ for $d := \varrho - q$. Let us also write $\tilde{x} := (y_1, \ldots, y_d) = (\tilde{x}_1, \ldots, \tilde{x}_d)$, $\tilde{x}' = (\tilde{x}_1, \ldots, \tilde{x}_{d-1})$, and $z := (y_{d+1}, \ldots, y_\varrho) = (z_1, \ldots, z_q)$. The corresponding covariables will be denoted by $\tilde{\eta}$, $\tilde{\eta}'$, and ζ, respectively.

We form the following operator-valued symbols

$$t(\zeta,\lambda) := {}^{\mathrm{t}}\big(t^{\alpha}(\zeta,\lambda) : |\alpha| < s - \frac{d}{2}\big)$$

and

$$k(\zeta,\lambda) := \big(k^{\alpha}(\zeta,\lambda) : |\alpha| < s - \frac{d}{2}\big)$$

for every $s > \frac{d}{2}$, $s - \frac{d}{2} \notin \mathbb{N}$, where $\lambda \in \mathbb{R}^l$ is a parameter and

$$t^{\alpha}(\zeta,\lambda) : H^s(\mathbb{R}^d_+) \to \mathbb{C},\ \ k^{\alpha}(\zeta,\lambda) : \mathbb{C} \to H^{\infty}(\mathbb{R}^d_+)$$

are defined as follows:

$$t^{\alpha}(\zeta,\lambda)u := [\zeta,\lambda]^{-\frac{d}{2}-|\alpha|}(D^{\alpha}_{\tilde{x}}u)(0),\ \ k^{\alpha}(\zeta,\lambda)c := [\zeta,\lambda]^{\frac{d}{2}}\frac{1}{\alpha!}([\zeta,\lambda]\tilde{x})^{\alpha}\omega_{+}([\zeta,\lambda]\tilde{x})$$

for any fixed $\omega \in C^{\infty}_0(\mathbb{R}^d)$ which is equal to 1 in a neighbourhood of $\tilde{x} = 0$, and $\omega_+ := \omega|_{\mathbb{R}^d_+}$. Setting

$$\iota(s,d) := \sharp\{\alpha : \alpha \in \mathbb{N}^d, |\alpha| < s - \frac{d}{2}\},$$

we obtain operator-valued symbols (with constant coefficients)

$$t(\zeta,\lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; H^s(\mathbb{R}^d_+), \mathbb{C}^{\iota(s,d)}),\ \ k(\zeta,\lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; \mathbb{C}^{\iota(s,d)}, H^s(\mathbb{R}^d_+)). \tag{2.1}$$

Observe that we even have $k(\zeta,\lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; \mathbb{C}^{\iota(s,d)}, \mathcal{S}(\overline{\mathbb{R}}^d_+))$ for $\mathcal{S}(\overline{\mathbb{R}}^d_+) := \mathcal{S}(\mathbb{R}^d)|_{\mathbb{R}^d_+}$. The pointwise composition gives us

$$t(\zeta,\lambda)k(\zeta,\lambda) = \mathrm{id}_{\mathbb{C}^{\iota(s,d)}}$$

for all $(\zeta,\lambda) \in \mathbb{R}^{d+l}$. Moreover,

$$p(\zeta,\lambda) := \mathrm{id} - k(\zeta,\lambda)t(\zeta,\lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; H^s(\mathbb{R}^d_+), H^s_0(\mathbb{R}^d_+)) \tag{2.2}$$

takes values in the space of projections to the space

$$H^s_0(\mathbb{R}^d_+) := \{u \in H^s(\mathbb{R}^d_+) : (D^{\alpha}_{\tilde{x}}u)(0) = 0 \text{ for all } |\alpha| < s - \frac{d}{2}\}. \tag{2.3}$$

Observe that, when we endow $H^s_0(\mathbb{R}^d_+)$ with the group action $\kappa_\delta : u(\tilde{x}) \to \delta^{d/2}u(\delta\tilde{x})$, $\delta \in \mathbb{R}_+$, we have

$$\mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^d_+)) = \{f(\tilde{x},z) \in H^s(\mathbb{R}^{d+q}_+) : (D^{\alpha}_{\tilde{x}}f)(0,z) = 0 \text{ for all } |\alpha| < s - \frac{d}{2}\}.$$

In addition, using polar coordinates $\tilde{x} \to (r,\phi)$, $\overline{\mathbb{R}}^d_+ \setminus \{0\} \to (S^{d-1}_+)^{\wedge} = \mathbb{R}_+ \times S^{d-1}_+$ for $S^{d-1}_+ := \overline{\mathbb{R}}^d_+ \cap S^{d-1}$ we have the indentifications

$$H^s_0(\overline{\mathbb{R}}^d_+) = \mathcal{K}^{s,s}((S^{d-1}_+)^{\wedge}),\ \ \mathcal{W}^{s,s}(\mathbb{R}^q, H^s_0(\mathbb{R}^d_+)) = \mathcal{W}^{s,s}((S^{d-1}_+)^{\wedge} \times \mathbb{R}^q)$$

for all $s > \frac{d}{2}$, $s - \frac{d}{2} \notin \mathbb{N}$. Incidentally (if we nevertheless have in mind the representation of operators in $\tilde{x}$ rather than (r,ϕ)) it will be convenient to write

$$\mathcal{K}^{s,s}((S^{d-1}_+)^{\wedge}) =: \mathcal{K}^{s,s}(\mathbb{R}^d_+ \setminus \{0\}),\ \ \mathcal{W}^{s,s}((S^{d-1}_+)^{\wedge} \times \mathbb{R}^q) =: \mathcal{W}^{s,s}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q).$$

By construction

$$\begin{pmatrix} p(\zeta,\lambda) \\ t(\zeta,\lambda) \end{pmatrix} : H^s(\mathbb{R}^d_+) \to \begin{matrix} H^s_0(\mathbb{R}^d_+) \\ \oplus \\ \mathbb{C}^{\iota(s,d)} \end{matrix} \tag{2.4}$$

is a family of isomorphisms for all $s > \frac{d}{2}$, $s - \frac{d}{2} \notin \mathbb{N}$, $(\zeta, \lambda) \in \mathbb{R}^{q+l}$, and (2.4) belongs to

$$S^0_{\mathrm{cl}}\left(\mathbb{R}^{q+l}; H^s(\mathbb{R}^d_+), \begin{array}{c} H^s_0(\mathbb{R}^d_+) \\ \oplus \\ \mathbb{C}^{\iota(s,d)} \end{array}\right). \tag{2.5}$$

Then for

$$P(\lambda) := \mathrm{Op}_z(p)(\lambda), \;\; T(\lambda) := \mathrm{Op}_z(t)(\lambda),$$

the operators

$$\begin{pmatrix} P(\lambda) \\ T(\lambda) \end{pmatrix} : H^s(\mathbb{R}^{d+q}_+) \to \begin{array}{c} \mathcal{W}^{s,s}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \\ \oplus \\ H^s(\mathbb{R}^q, \mathbb{C}^{\iota(s,d)}) \end{array}$$

define a family of isomorphisms, and we have

$$\begin{pmatrix} P(\lambda) \\ T(\lambda) \end{pmatrix}^{-1} = (E \;\; K(\lambda))$$

where $E : \mathcal{W}^{s,s}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \to H^s(\mathbb{R}^{d+q}_+)$ is the canonical embedding and $K(\lambda) = \mathrm{Op}(k)(\lambda)$. This is a consequence of (2.4) and (2.5).

In a similar manner, for $s > \frac{d-1}{2}$, $s - \frac{d-1}{2} \notin \mathbb{N}$, we obtain isomorphisms

$$\begin{pmatrix} P'(\lambda) \\ T'(\lambda) \end{pmatrix} : H^s(\mathbb{R}^{d+q-1}) \to \begin{array}{c} \mathcal{W}^{s,s}((\mathbb{R}^{d-1} \setminus \{0\}) \times \mathbb{R}^q) \\ \oplus \\ H^s(\mathbb{R}^q, \mathbb{C}^{\iota(s,d-1)}) \end{array}$$

with the inverse $(E' \;\; K'(\lambda))$, where the notation is of analogous meaning as before for the half-space case (in the expression for $K'(\lambda)$ we employ a function $\omega' \in C^\infty_0(\mathbb{R}^{d-1})$ which is equal to 1 near the origin). If necessary we write the smoothness as subscript, e.g., $P_s(\lambda)$, $T_s(\lambda)$, etc. Observe that

$$\ker T_s(\lambda) = E_s \mathcal{W}^{s,s}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \text{ for } s > \frac{d}{2},\, s - \frac{d}{2} \notin \mathbb{N}, \tag{2.6}$$

and similarly,

$$\ker T'_{s-\frac{1}{2}}(\lambda) = E'_{s-\frac{1}{2}} \mathcal{W}^{s-\frac{1}{2}, s-\frac{1}{2}}((\mathbb{R}^{d-1} \setminus \{0\}) \times \mathbb{R}^q) \text{ for } s > \frac{d}{2},\, s - \frac{d}{2} \notin \mathbb{N} \tag{2.7}$$

for all $\lambda \in \mathbb{R}^l$.

2.2 Examples for ellipticity with additional edge conditions

We consider the Dirichlet problem for the Laplace operator Δ in $\mathbb{R}^{d+q}_+$ which represents a continuous operator

$$\mathcal{A} := \begin{pmatrix} \Delta \\ D \end{pmatrix} : H^s(\mathbb{R}^{d+q}_+) \to \begin{array}{c} H^{s-2}(\mathbb{R}^{d+q}_+) \\ \oplus \\ H^{s-1/2}(\mathbb{R}^{d+q-1}) \end{array} \tag{2.8}$$

for any fixed choice of $s > \frac{3}{2}$ (in order that $s - 2 > -\frac{1}{2}$). To illustrate the nature of additional edge conditions we reformulate (2.8) into an operator

$$\mathfrak{A}^s : \begin{array}{c} \mathcal{W}^{s,s}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \\ \oplus \\ H^s(\mathbb{R}^q, \mathbb{C}^{\iota(s,d)}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-2,s-2}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \\ \oplus \\ \mathcal{W}^{s-1/2,s-1/2}((\mathbb{R}^{d-1} \setminus \{0\}) \times \mathbb{R}^q) \\ \oplus \\ H^{s-2}(\mathbb{R}^q, \mathbb{C}^{\iota(s-2,d)}) \\ \oplus \\ H^{s-1/2}(\mathbb{R}^q, \mathbb{C}^{\iota(s-1/2,d-1)}). \end{array} \tag{2.9}$$

First of all, observe that $\mathcal{A}$ itself induces (by restriction) a continuous operator

$$\mathcal{A}^s : \mathcal{W}^{s,s}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \to \begin{array}{c} \mathcal{W}^{s-2,s-2}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \\ \oplus \\ \mathcal{W}^{s-1/2,s-1/2}((\mathbb{R}^{d-1}_+ \setminus \{0\}) \times \mathbb{R}^q). \end{array} \tag{2.10}$$

We now apply the isomorphisms from the preceding section for an arbitrary fixed λ (which we omit for the moment). In this way, composing (2.8) from the right with the operator $(E_s \;\; K_s)$ for $s > d/2$, $s - d/2 \notin \mathbb{N}$, we obtain

$$\begin{pmatrix} \Delta E_s & \Delta K_s \\ DE_s & DK_s \end{pmatrix} : \begin{array}{c} \mathcal{W}^{s,s}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \\ \oplus \\ H^s(\mathbb{R}^q, \mathbb{C}^{\iota(s,d)}) \end{array} \longrightarrow \begin{array}{c} H^{s-2}(\mathbb{R}^{d+q}_+) \\ \oplus \\ H^{s-1/2}(\mathbb{R}^{d+q-1}). \end{array}$$

Then we pass to the composition

$$\begin{pmatrix} P_{s-2} & 0 \\ 0 & P'_{s-1/2} \\ T_{s-2} & 0 \\ 0 & T'_{s-1/2} \end{pmatrix} \begin{pmatrix} \Delta E_s & \Delta K_s \\ DE_s & DK_s \end{pmatrix} = \begin{pmatrix} P_{s-2}\Delta E_s & P_{s-2}\Delta K_s \\ P'_{s-1/2}DE_s & P'_{s-1/2}DK_s \\ T_{s-2}\Delta E_s & T_{s-2}\Delta K_s \\ T'_{s-1/2}DE_s & T'_{s-1/2}DK_s \end{pmatrix} =: \mathfrak{A}^s, \tag{2.11}$$

for $s - 2 > \frac{d}{2}$, $s - 2 - \frac{d}{2} \notin \mathbb{N}$ and $s > \frac{d}{2}$, $s - \frac{d}{2} \notin \mathbb{N}$.

Note that the operator (2.10) just coincides with the 2×1 upper left corner of $\mathfrak{A}^s$. Let us write, for abbreviation,

$$\mathfrak{A}^s = \begin{pmatrix} \mathcal{A}^s & \mathcal{K}^s \\ \mathcal{T}^s & \mathcal{Q}^s \end{pmatrix} \tag{2.12}$$

for

$$\mathcal{K}^s := {}^{\mathrm{t}}(P_{s-2}\Delta K_s \;\; P'_{s-1/2}DK_s), \;\; \mathcal{T}^s := {}^{\mathrm{t}}(T_{s-2}\Delta E_s \;\; T'_{s-1/2}DE_s) \tag{2.13}$$

and $\mathcal{Q}^s = {}^{\mathrm{t}}(T_{s-2}\Delta K_s \;\; T'_{s-1/2}DK_s)$.

Remark 2.2.1. *We have $\mathcal{T}^s \equiv 0$. In fact, Δ and D induces maps*

$$\Delta : E_s\mathcal{W}^{s,s}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \to E_{s-2}\mathcal{W}^{s-2,s-2}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q)$$

and

$$D : E_s\mathcal{W}^{s,s}((\mathbb{R}^d_+ \setminus \{0\}) \times \mathbb{R}^q) \to E'_{s-\frac{1}{2}}\mathcal{W}^{s-\frac{1}{2},s-\frac{1}{2}}((\mathbb{R}^{d-1} \setminus \{0\}) \times \mathbb{R}^q),$$

respectively. It suffices then to apply the relations (2.6) *and* (2.7)*.*

We want to interpret the operator $\mathfrak{A}^s$ as an edge-boundary value problem on the (stretched) manifold with edge and boundary

$$(\overline{\mathbb{R}}_+ \times S^{d-1}_+) \times \mathbb{R}^q \ni (r, \phi, z);$$

in this interpretation $\mathbb{R}^q$ is the edge, and the (stretched) boundary is equal to $\mathbb{R}^q \times S^{d-1}_+$ which has itself a boundary $\mathbb{R}^q \times \partial S^{d-1}_+ = \mathbb{R}^q \times S^{d-2}$. We first rewrite (2.8) in polar coordinates with respect to the variables $\tilde{x} \in \mathbb{R}^d_+ \setminus \{0\}$, $\tilde{x} \mapsto (r, \phi)$. For purposes below we write Δ more generally as

$$A = r^{-2} \sum_{j+|\alpha| \leq 2} a_{j\alpha}(r, z)(-r\frac{\partial}{\partial r})^j (rD_z)^\alpha$$

with coefficients $a_{j\alpha}(r, z) \in C^\infty(\mathbb{R}^q \times \overline{\mathbb{R}}_+, \mathrm{Diff}^{2-(j+|\alpha|)}(S^{d-1}_+))$ (which is the form of an arbitrary differential operator A of order 2 with smooth coefficients in polar coordinates).

In the present case for the Laplacian the coefficients $a_{j\alpha}(r,z)$ are independent of r,z. The function

$$a(z,\zeta) := r^{-2} \sum_{j+|\alpha|\le 2} a_{j\alpha}(r,z)(-r\frac{\partial}{\partial r})^j (r\zeta)^\alpha$$

represents an element of $S^2(\mathbb{R}^q \times \mathbb{R}^q; E, \tilde{E})$ for

$$E = \mathcal{K}^{s,\gamma}((S_+^{d-1})^\wedge),\ \ \tilde{E} = \mathcal{K}^{s-2,\gamma-2}((S_+^{d-1})^\wedge).$$

Then we have $A = \mathrm{Op}_z(a)$. Moreover, the operator D of restriction to the boundary has the interpretation of $D = \mathrm{Op}(a')$ for

$$a' : \mathcal{K}^{s,\gamma}((S_+^{d-1})^\wedge) \to \mathcal{K}^{s-1/2,\gamma-1/2}((S^{d-2})^\wedge),$$

$a'u := u|_{(S^{d-2})^\wedge}$, which is also an operator-valued symbol (although it is independent of variables and covariables) of order $\frac{1}{2}$, namely, $a' \in S^{1/2}(\mathbb{R}^q_\zeta; E, \mathbb{C})$. Applying this for $s = \gamma$ we obtain altogether $\mathcal{A}^s = \mathrm{Op}_z \begin{pmatrix} a \\ a' \end{pmatrix} = \begin{pmatrix} \mathrm{Op}(a) \\ \mathrm{Op}(a') \end{pmatrix}$ which is just the explanation of (2.10). Clearly in a similar manner we can generalise this to an operator $A(\gamma)$ by interpreting a and a' as operator-valued symbols with the weight γ in place of s; this yields a continuity similar to (2.10) between the Sobolev spaces with arbitrary weight $\gamma \in \mathbb{R}$.

Let us also consider the homogeneous principal edge symbol of (2.10) which is a column vector

$$\sigma_\wedge(\mathcal{A}^s)(z,\zeta) : \mathcal{K}^{s,s}((S_+^{d-1})^\wedge) \to \begin{matrix} \mathcal{K}^{s-2,s-2}((S_+^{d-1})^\wedge) \\ \oplus \\ \mathcal{K}^{s-1/2,s-1/2}((S^{d-2})^\wedge) \end{matrix} \tag{2.14}$$

for $\zeta \neq 0$. The second entry of (2.14) is the operator of restriction to the boundary, the first one has the form

$$r^{-2} \sum_{j+|\alpha|\le 2} a_{j\alpha}(0,z)(-r\frac{\partial}{\partial r})^j (r\zeta)^\alpha$$

(recall that in our example there is no dependence on z; so we omit it again).

For every fixed $\zeta \neq 0$ the operator (2.14) represents a Dirichlet problem for the operator

$$\sigma_\wedge(\Delta)(\zeta) := -|\zeta|^2 + \sum_{j=1}^{d} \frac{\partial^2}{\partial \tilde{x}_j^2} : H^s(\mathbb{R}^d_+) \to H^{s-2}(\mathbb{R}^d_+)$$

with respect to the boundary $\mathbb{R}^{d-1}$.

Theorem 2.2.2. *For every $\zeta \neq 0$ the operator* (2.14) *is Fredholm, and we have*

$$\operatorname{ind} \sigma_\wedge(\mathcal{A}^s) = \iota(s-2,d) + \iota(s-\frac{1}{2}, d-1) - \iota(s,d)$$

for every $s > \frac{3}{2}$, $s-2 > \frac{d}{2}$, $s-2-\frac{d}{2} \notin \mathbb{N}$, $s > \frac{d}{2}$, $s - \frac{d}{2} \notin \mathbb{N}$.

Proof. The construction which produced the operator (2.9) from (2.8) can be carried out on the level of ζ-depending operator families

$$\sigma_\wedge(\mathcal{A})(\zeta) := \begin{pmatrix} \sigma_\wedge(\Delta)(\zeta) \\ D \end{pmatrix} : H^s(\mathbb{R}^d_+) \to \begin{matrix} H^{s-2}(\mathbb{R}^d_+) \\ \oplus \\ H^{s-1/2}(\mathbb{R}^{d-1}) \end{matrix} \tag{2.15}$$

under the same condition on $s \in \mathbb{R}$ as before (and D in the meaning of the operator of restriction to $\tilde{x}=0$). Instead of K_s, P_{s-2}, etc., we take the operator families

$$\sigma_\wedge(K_s)(\zeta): c \to \left(|\zeta|^{d/2}\frac{1}{\alpha!}(|\zeta|\tilde{x})^\alpha\omega_+(|\zeta|\tilde{x})c_\alpha\right)_{|\alpha|<s-d/2}$$

for $c=(c_\alpha)_{|\alpha|<s-d/2} \in \mathbb{C}^{\iota(s,d)}$, for

$$\sigma_\wedge(P_{s-2})(\zeta) = \mathrm{id} - \sigma_\wedge(K_{s-2})(\zeta)\sigma_\wedge(T_{s-2})(\zeta),$$

$$\sigma_\wedge(T_{s-2})(\zeta): u \to \left(|\zeta|^{-d/2-|\alpha|}(D^\alpha_{\tilde{x}}u)(0)\right)_{|\alpha|<s-2-d/2}$$

and

$$\sigma_\wedge(P'_{s-1/2})(\zeta) := \mathrm{id} - \sigma_\wedge(K'_{s-1/2})(\zeta)\sigma_\wedge(T'_{s-1/2})(\zeta)$$

with $\sigma_\wedge(K'_{s-1/2})(\zeta)$ and $\sigma_\wedge(T'_{s-1/2})(\zeta)$ being defined in an analogous manner as the expressions before, now for $d-1$ instead of d.

Now a similar composition as (2.11) gives us an operator family

$$\sigma_\wedge(\mathfrak{A}^s)(\zeta) := \begin{pmatrix}\sigma_\wedge(\mathcal{A}^s) & \sigma_\wedge(\mathcal{K}^s)\\ \sigma_\wedge(\mathcal{T}^s) & \sigma_\wedge(\mathcal{Q}^s)\end{pmatrix}(\zeta) \tag{2.16}$$

for $\sigma_\wedge(\mathcal{K}^s) = {}^{\mathrm{t}}(\sigma_\wedge(P_{s-2})\sigma_\wedge(\Delta)\sigma_\wedge(K_s)\;\; \sigma_\wedge(P'_{s-1/2})\sigma_\wedge(D)\sigma_\wedge(K_s))$, etc., cf. the formulas (2.12) and (2.13),

$$\sigma_\wedge(\mathfrak{A}^s)(\zeta): \begin{array}{c}\mathcal{K}^{s,s}(\mathbb{R}^d_+\setminus\{0\})\\ \oplus\\ \mathbb{C}^{\iota(s,d)}\end{array} \longrightarrow \begin{array}{c}\mathcal{K}^{s-2,s-2}(\mathbb{R}^d_+\setminus\{0\})\\ \oplus\\ \mathcal{K}^{s-1/2,s-2/2}(\mathbb{R}^{d-1}\setminus\{0\})\\ \oplus\\ \mathbb{C}^{\iota(s-2,d)}\\ \oplus\\ \mathbb{C}^{\iota(s-1/2,d-1)}\end{array}. \tag{2.17}$$

We will show, cf. Lemma 2.2.3 below, that (2.15) is a family of isomorphisms. Then, since the factors in the composition are isomorphisms, also (2.17) consists of isomorphisms. This implies that the upper left corner of (2.16) is a family of Fredholm operators, and the index is just the difference of the dimensions in the finite-dimensional components of (2.17). □

Lemma 2.2.3. *For every $\zeta \neq 0$ the operators* (2.15) *are isomorphisms for all $s > \frac{3}{2}$.*

Proof. The half-space $\mathbb{R}^d_+$ may be regarded as a manifold with exit to infinity and boundary. Let us set $\mathcal{A}_\zeta := \sigma_\wedge(\mathcal{A})(\zeta)$, cf. the formula (2.15). The ellipticity of boundary value problems in such a situation is determined by a principal symbolic hierarchy

$$(\sigma_\psi(\mathcal{A}_\zeta), \sigma_e(\mathcal{A}_\zeta), \sigma_{\psi,e}(\mathcal{A}_\zeta); \sigma_\partial(\mathcal{A}_\zeta), \sigma_{e'}(\mathcal{A}_\zeta), \sigma_{\partial,e'}(\mathcal{A}_\zeta)), \tag{2.18}$$

where $\sigma_\psi(\mathcal{A}_\zeta) = -|\tilde{\xi}|^2$, $\sigma_e(\mathcal{A}_\zeta) = -|\zeta|^2 - |\tilde{\xi}|^2$, $\sigma_{\psi,e}(\mathcal{A}_\zeta) = -|\tilde{\xi}|^2$, and $\sigma_\partial(\mathcal{A}_\zeta)(\tilde{\xi}') = \sigma_\partial(\mathcal{A}_\zeta)(\tilde{\xi}') = -|\tilde{\xi}'|^2 + \frac{\partial^2}{\partial\tilde{x}_d^2}$,

$$\sigma_{e'}(\mathcal{A}_\zeta)(\zeta,\tilde{\xi}') = -|\zeta|^2 - |\tilde{\xi}'|^2 + \frac{\partial^2}{\partial\tilde{x}_d^2} : H^s(\mathbb{R}_+) \to \begin{array}{c}H^{s-2}(\mathbb{R}_+)\\ \oplus\\ \mathbb{C}\end{array},$$

and $\sigma_{\partial,e'}(\mathcal{A}_\zeta)(\tilde{\xi}') = \sigma_\partial(\mathcal{A}_\zeta)(\tilde{\xi}')$. Ellipticity means that $\sigma_\psi(\mathcal{A}_\zeta) \neq 0$ on $\overline{\mathbb{R}}^d_+ \times (\mathbb{R}^d \setminus \{0\})$, $\sigma_e(\mathcal{A}_\zeta) \neq 0$ on $(\overline{\mathbb{R}}^d_+ \setminus \{0\}) \times \mathbb{R}^d$, $\sigma_{\psi,e}(\mathcal{A}_\zeta) \neq 0$ on $(\overline{\mathbb{R}}^d_+ \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\})$, and bijectivity of $\sigma_\partial(\mathcal{A}_\zeta)$ on $\mathbb{R}^{d-1} \times (\mathbb{R}^{d-1}\setminus\{0\})$, $\sigma_{e'}(\mathcal{A}_\zeta)$ on $(\mathbb{R}^{d-1}\setminus\{0\}) \times \mathbb{R}^{d-1}$, and $\sigma_{\partial,e'}(\mathcal{A}_\zeta)$ on $(\mathbb{R}^{d-1}\setminus\{0\}) \times (\mathbb{R}^{d-1}\setminus\{0\})$. Now for $\zeta \neq 0$ we have $\mathcal{A}_\zeta = \mathrm{Op}_{\tilde{x}'}(\sigma_{\partial,e'}(\mathcal{A}_\zeta)(\tilde{\xi}'))$ and $\mathcal{A}_\zeta^{-1} = \mathrm{Op}_{\tilde{x}'}(\sigma^{-1}_{\partial,e'}(\mathcal{A}_\zeta)(\tilde{\xi}'))$. □

Theorem 2.2.4. *The operator $\mathfrak{A}^s$, given by* (2.9) *is elliptic in the edge calculus when s satisfies the conditions of* Theorem 2.2.2.

We shall prove this theorem in more general form below, cf. Theorem 2.3.2.

2.3 The general case

Let $A = \sum_{|\alpha|\le 2m} a_\alpha(y) D_y^\alpha$ be an elliptic differential operator in $\overline{\mathbb{R}}^\varrho_+$ with smooth coefficients $a_\alpha \in C^\infty(\overline{\mathbb{R}}^\varrho_+)$ that are assumed to be independent of y for large $|y|$. Consider an elliptic boundary value problem for the operator A in the half-space

$$Au = f \text{ in } \mathbb{R}^\varrho_+, \quad Tu = g \text{ on } \mathbb{R}^{\varrho-1}. \tag{2.19}$$

The trace operator T is assumed to be of the form $T = {}^{\mathrm{t}}(T_1, \dots, T_m)$, $T_j u = B_j u|_{\mathbb{R}^{\varrho-1}}$, where B_j is a differential operator of order μ_j with smooth coefficients that are also independent of y for large $|y|$. Ellipticity of the operator $\mathcal{A} := \begin{pmatrix} A \\ T \end{pmatrix}$ refers to the principal symbolic hierarchy

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$$

of boundary value problems, where $\sigma_\psi(\mathcal{A}) := \sigma_\psi(A)$ is the homogeneous principal symbol of order $2m$ of A itself, while

$$\sigma_\partial(\mathcal{A})(y', \eta') := \begin{pmatrix} \sigma_\psi(A)(y', 0, \eta', D_{y_\varrho}) \\ {}^{\mathrm{t}}\big(\mathrm{r}_{\{y_\varrho = 0\}} \sigma_\psi(B_j)(y', 0, \eta', D_{y_\varrho})\big)_{j=1,\dots,m} \end{pmatrix} \tag{2.20}$$

is the principal boundary symbol of $\mathcal{A}$. Ellipticity of $\mathcal{A}$ is defined by the following conditions:
(i)

$$\sigma_\psi(\mathcal{A}) \neq 0 \text{ on } T^*\overline{\mathbb{R}}^\varrho_+ \setminus 0, \tag{2.21}$$

(ii) (2.20) as an operator function

$$\sigma_\partial(\mathcal{A})(y', \eta') : H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-2m}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^m \end{matrix} \tag{2.22}$$

is a family of isomorphisms for any sufficiently large s and all $(y', \eta') \in T^*\mathbb{R}^{\varrho-1} \setminus 0$.

Similarly as in the preceding section we rephrase the continuous operator

$$\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix} : H^s(\mathbb{R}^\varrho_+) \to \begin{matrix} H^{s-2m}(\mathbb{R}^\varrho_+) \\ \oplus \\ \oplus_{j=1}^m H^{s-\mu_j - 1/2}(\mathbb{R}^{\varrho-1}) \end{matrix} \tag{2.23}$$

as an operator in edge representation with respect to the splitting of variables $y = (\tilde{x}, z)$, $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_d) \in \mathbb{R}^d_+$, $z = (x_1, \dots, x_q) \in \mathbb{R}^q$, $\varrho = q + d$. We fix $s \in \mathbb{R}$ sufficiently large, i.e., $s > \max_{1\le j\le m} \left\{ 2m - \frac{1}{2}, \mu_j + \frac{1}{2} \right\}$. We also form the family of operators

$$\sigma_\wedge(\mathcal{A})(z, \zeta) : H^s(\mathbb{R}^d_+) \longrightarrow \begin{matrix} H^{s-2m}(\mathbb{R}^d_+) \\ \oplus \\ \oplus_{j=1}^m H^{s-\mu_j-\frac{1}{2}}(\mathbb{R}^{d-1}) \end{matrix} \tag{2.24}$$

defined by the column matrix consisting of

$$\sigma_\psi(A)(0,z,D_{\tilde{x}},\zeta) : H^s(\mathbb{R}^d_+) \to H^{s-2m}(\mathbb{R}^d_+) \tag{2.25}$$

and

$$\mathrm{r}_{\{\tilde{x}_d=0\}}\sigma_\psi(B_j)(0,z,D_{\tilde{x}},\zeta) : H^s(\mathbb{R}^d_+) \to H^{s-\mu_j-\frac{1}{2}}(\mathbb{R}^{d-1}), \tag{2.26}$$

for $j=1,\dots,m$.

Then a procedure similar to that which gave us (2.9) allows us to rewrite the operator $\mathcal{A}$ in the form

$$\mathfrak{A}^s : \begin{array}{c} \mathcal{W}^{s,s}\big((\mathbb{R}^d_+\setminus\{0\})\times\mathbb{R}^q\big) \\ \oplus \\ H^s(\mathbb{R}^q,\mathbb{C}^{\iota(s,d)}) \end{array} \longrightarrow \begin{array}{c} \boldsymbol{\mathcal{W}}^{s-2m,s-2m}\big((\mathbb{R}^d_+\setminus\{0\})\times\mathbb{R}^q\big) \\ \oplus \\ \boldsymbol{H}^{s-2m}(\mathbb{R}^q,\mathbb{C}^{\boldsymbol{\iota}(s-2m,d)}) \end{array}. \tag{2.27}$$

for $s-2m>\frac{d}{2}$, $s-2m-\frac{d}{2}\notin\mathbb{N}$ and $s-\mu_j>\frac{d}{2}$, $s-\mu_j-\frac{d}{2}\notin\mathbb{N}$. Here $\boldsymbol{\mathcal{W}}^{s-2m,s-2m}\big((\mathbb{R}^d_+\setminus\{0\})\times\mathbb{R}^q\big) := \begin{array}{c} \mathcal{W}^{s-2m,s-2m}\big((\mathbb{R}^d_+\setminus\{0\})\times\mathbb{R}^q\big) \\ \oplus \\ \oplus_{j=1}^m\mathcal{W}^{s-\mu_j-\frac{1}{2},s-\mu_j-\frac{1}{2}}\big((\mathbb{R}^{d-1}\setminus\{0\})\times\mathbb{R}^q\big) \end{array}$ and $\boldsymbol{H}^{s-2m}(\mathbb{R}^q,\mathbb{C}^{\boldsymbol{\iota}(s-2m,d)}) := \begin{array}{c} H^{s-2m}(\mathbb{R}^q,\mathbb{C}^{\iota(s-2m,d)}) \\ \oplus \\ \oplus_{j=1}^m H^{s-\mu_j-\frac{1}{2}}(\mathbb{R}^q,\mathbb{C}^{\iota(s-\mu_j-\frac{1}{2},d-1)}) \end{array}$.
In fact, we first have isomorphisms

$$\big(E_s \quad K_s(\lambda)\big) : \begin{array}{c} \mathcal{W}^{s,s}((\mathbb{R}^d_+\setminus\{0\})\times\mathbb{R}^q) \\ \oplus \\ H^s(\mathbb{R}^q,\mathbb{C}^{\iota(s,d)}) \end{array} \longrightarrow H^s(\mathbb{R}^\varrho_+), \tag{2.28}$$

$$\begin{pmatrix} P_{s-2m}(\lambda) \\ T_{s-2m}(\lambda) \end{pmatrix} : H^{s-2m}(\mathbb{R}^\varrho_+) \longrightarrow \begin{array}{c} \mathcal{W}^{s-2m,s-2m}((\mathbb{R}^d_+\setminus\{0\})\times\mathbb{R}^q) \\ \oplus \\ H^{s-2m}(\mathbb{R}^q,\mathbb{C}^{\iota(s-2m,d)}) \end{array}, \tag{2.29}$$

and

$$\begin{pmatrix} P'(\lambda) \\ T'(\lambda) \end{pmatrix} : \oplus_{j=1}^m H^{s-\mu_j-\frac{1}{2}}(\mathbb{R}^{\varrho-1}_+) \longrightarrow \begin{array}{c} \oplus_{j=1}^m\mathcal{W}^{s-\mu_j-\frac{1}{2},s-\mu_j-\frac{1}{2}}((\mathbb{R}^{d-1}\setminus\{0\})\times\mathbb{R}^q) \\ \oplus \\ \oplus_{j=1}^m H^{s-\mu_j-\frac{1}{2}}(\mathbb{R}^q,\mathbb{C}^{\iota(s-\mu_j-\frac{1}{2},d-1)}) \end{array}.$$

Then it follows that (when we again fix λ and then omit it)

$$\mathfrak{A}^s = \begin{pmatrix} P_{s-2m}AE_s & P_{s-2m}AK_s \\ P'TE_s & P'TK_s \\ 0 & T_{s-2m}AK_s \\ 0 & T'TK_s \end{pmatrix}. \tag{2.30}$$

In the latter expression we employed an analogue of Remark 2.2.1.

Let $\mathcal{A}^s$ denote the $(m+1)\times 1$ upper left corner of $\mathfrak{A}^s$,

$$\mathcal{A}^s =: \begin{pmatrix} A^s \\ T^s \end{pmatrix} : \mathcal{W}^{s,s}\big((\mathbb{R}^d_+\setminus\{0\})\times\mathbb{R}^q\big) \to \boldsymbol{\mathcal{W}}^{s-2m,s-2m}\big((\mathbb{R}^d_+\setminus\{0\})\times\mathbb{R}^q\big),$$

where $A^s = P_{s-2m}AE_s$, $T^s = P'TE_s = {}^{\mathrm{t}}(T^s_1,\dots,T^s_m)$ for $T^s_j = P'T_jE_s$. We then form

$$\sigma_\wedge(\mathcal{A}^s)(z,\zeta) := \begin{pmatrix} \sigma_\wedge(A^s) \\ \sigma_\wedge(T^s) \end{pmatrix}(z,\zeta)$$

where

$$\sigma_\wedge(A^s)(z,\zeta) := \sigma_\psi(A)(0,z,D_{\tilde{x}},\zeta) : \mathcal{K}^{s,s}(\mathbb{R}^d_+\setminus\{0\}) \to \mathcal{K}^{s-2m,s-2m}(\mathbb{R}^d_+\setminus\{0\}), \tag{2.31}$$

$$\sigma_\wedge(T^s_j)(z,\zeta) = \mathrm{r}_{\{\tilde{x}_d=0\}}\sigma_\psi(B_j)(0,z,D_{\tilde{x}},\zeta) : \mathcal{K}^{s,s}(\mathbb{R}^d_+\setminus\{0\}) \to \mathcal{K}^{s-\mu_j-\frac{1}{2},s-\mu_j-\frac{1}{2}}(\mathbb{R}^{d-1}\setminus\{0\}),$$

for $j=1,\dots,m$, are just the restriction of the operators (2.25) and (2.26) to the subspace $\mathcal{K}^{s,s}(\mathbb{R}^d_+\setminus\{0\})$.

Theorem 2.3.1. *Let* $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ *be associated with elliptic boundary problem* (2.19). *Then*

$$\sigma_\wedge(\mathcal{A}^s)(z,\zeta) : \mathcal{K}^{s,s}(\mathbb{R}^d_+ \setminus \{0\}) \to \boldsymbol{\mathcal{K}}^{s-2m,s-2m}(\mathbb{R}^d_+ \setminus \{0\}) \tag{2.32}$$

is a family of Fredholm operators for all $(z,\zeta) \in T^*\mathbb{R}^q \setminus 0$ *and all* $s - 2m > \frac{d}{2}$, $s - 2m - \frac{d}{2} \notin \mathbb{N}$ *and* $s - \mu_j > \frac{d}{2}$, $s - \mu_j - \frac{d}{2} \notin \mathbb{N}$, *and we have*

$$\operatorname{ind} \sigma_\wedge(\mathcal{A}^s)(z,\zeta) = \iota(s-2m,d) + \sum_{j=1}^{m} \iota(s - \mu_j - \frac{1}{2}, d-1) - \iota(s,d). \tag{2.33}$$

Proof. Let us show that (2.24) for every $(z,\zeta) \in T^*\mathbb{R}^q \setminus 0$ is a family of isomorphisms. In fact, $\sigma_\wedge(\mathcal{A})(z,\zeta)$ is a family of elliptic boundary value problems in $\overline{\mathbb{R}}^d_+$, regarded as a manifold with boundary and exit to infinity. The ellipticity here is defined in terms of the principal symbolic hierarchy

$$\sigma(\cdot) = (\sigma_\psi(\cdot), \sigma_e(\cdot), \sigma_{\psi,e}(\cdot); \sigma_\partial(\cdot), \sigma_{e'}(\cdot), \sigma_{\partial,e'}(\cdot)).$$

Let us set $\mathcal{A}_{z,\zeta} := \sigma_\wedge(\mathcal{A})(z,\zeta)$, cf. the formula (2.24). As in the special case of operators in the preceding section the Shapiro-Lopatinskij ellipticity of our original problem in $\mathbb{R}^\varrho_+$ entails the invertibility of $\sigma_{\partial,e'}(\mathcal{A}_{z,\zeta})(\tilde{\xi}')$ for all $z \in \mathbb{R}^q$, $\tilde{\xi}' \in \mathbb{R}^{d-1}$. In fact, $(\tilde{\xi}',\zeta)$ corresponds to η', and the bijectivity of $\sigma_\partial(\mathcal{A})(y',\eta')$ in the sense of (2.22) for all $\xi' \neq 0$ entails the invertibility of $\sigma_\partial(\mathcal{A}_{z,\zeta})(\tilde{\xi}')$ for all $(z,\zeta) \in T^*\mathbb{R}^q \setminus 0$ and $\tilde{\xi}' \in \mathbb{R}^{d-1}$. Then we have

$$\mathcal{A}_{z,\zeta} = \mathrm{Op}_{\tilde{x}'}(\sigma_{\partial,e'}(\mathcal{A}_{z,\zeta})(\tilde{\xi}'))$$

and

$$\mathcal{A}^{-1}_{z,\zeta} = \mathrm{Op}_{\tilde{x}'}(\sigma^{-1}_{\partial,e'}(\mathcal{A}_{z,\zeta})(\tilde{\xi}')),$$

is just the inverse of $\mathcal{A}_{z,\zeta}$ as a family of maps (2.24).

Now (2.24) can be reformulated in the form

$$\sigma_\wedge(\mathfrak{A}^s)(z,\zeta) : \begin{array}{c} \mathcal{K}^{s,s}(\mathbb{R}^d_+ \setminus \{0\}) \\ \oplus \\ \mathbb{C}^{\iota(s,d)} \end{array} \longrightarrow \begin{array}{c} \boldsymbol{\mathcal{K}}^{s-2m,s-2m}(\mathbb{R}^d_+ \setminus \{0\}) \\ \oplus \\ \mathbb{C}^{\iota(s-2m,d)} \\ \oplus \\ \oplus_{j=1}^m \mathbb{C}^{\iota(s-\mu_j-\frac{1}{2},d-1)} \end{array} \tag{2.34}$$

and $\mathcal{A}_{z,\zeta}$ defined by (2.32) is just the $(m+1) \times 1$ upper left corner for $(z,\zeta) \in T^*\mathbb{R}^q \setminus 0$. Therefore, this is Fredholm, and (2.33) is just a consequence of the isomorphism (2.34). □

We now complete (2.34) to the principal symbolic hierarchy

$$\sigma(\mathfrak{A}^s) = (\sigma_\psi(\mathfrak{A}^s), \sigma_\partial(\mathfrak{A}^s), \sigma_\wedge(\mathfrak{A}^s))$$

of the operator $\mathfrak{A}^s$. Similarly as in the preceding section we express the involved operators in polar coordinates with respect to the variables $\tilde{x} \in \overline{\mathbb{R}}^d_+ \setminus \{0\}$, $\tilde{x} \mapsto (r,\phi) \in \mathbb{R}_+ \times S^{d-1}_+$. For the operator A we obtain

$$A = r^{-2m} \sum_{k+|\alpha| \leq 2m} a_{k\alpha}(r,z)(-r\frac{\partial}{\partial r})^k (rD_z)^\alpha$$

with coefficients $a_{k\alpha}(r,z) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \mathrm{Diff}^{2m-(k+|\alpha|)}(S^{d-1}_+))$, and the trace operator takes the form

$$T_j = \mathrm{r}_{\{\tilde{x}_d = 0\}} r^{-\mu_j} \sum_{k+|\beta| \leq \mu_j} b_{j,k\beta}(r,z)(-r\frac{\partial}{\partial r})^k (rD_z)^\beta$$

with $b_{j,k\beta}(r,z) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \mathrm{Diff}^{\mu_j-(k+|\beta|)}(S_+^{d-1}))$, $j = 1,\dots,m$. The homogeneous principal symbol $\sigma_\psi(\mathfrak{A}^s)$ is then defined as

$$\sigma_\psi(\mathfrak{A}^s)(r,\phi,z,\rho,\theta,\zeta) := \sigma_\psi(A)(r,\phi,z,\rho,\theta,\zeta).$$

Similarly as (1.12) we form

$$\tilde{\sigma}_\psi(\mathfrak{A}^s)(r,\phi,z,\rho,\theta,\zeta) := r^{2m}\sigma_\psi(\mathfrak{A}^s)(r,\phi,z,r^{-1}\rho,\theta,r^{-1}\zeta)$$

which is also homogeneous of order $2m$ in the covariables $(\rho,\theta,\zeta) \neq 0$ and a smooth function in (r,ϕ,z) up to $r = 0$. Finally we set

$$\sigma_\partial(\mathfrak{A}^s)(r,\phi',z,\rho,\theta',\zeta) := \begin{pmatrix} \sigma_\partial(A)(r,\phi',z,\rho,\theta',\zeta) \\ {}^{\mathrm{t}}\big(\sigma_\partial(T_j)(r,\phi',z,\rho,\theta',\zeta)\big)_{j=1,\dots,m} \end{pmatrix}$$

for

$$\sigma_\partial(A)(r,\phi',z,\rho,\theta',\zeta) := \sigma_\psi(A)(r,\phi',0,z,\rho,\theta',D_{\phi_{d-1}},\zeta) : H^s(\mathbb{R}_+) \to H^{s-2m}(\mathbb{R}_+),$$

and

$$\sigma_\partial(T_j)(r,\phi',z,\rho,\theta',\zeta) := \mathrm{r}_{\{\phi_{d-1}=0\}}\sigma_\psi(B_j)(r,\phi',0,z,\rho,\theta',D_{\phi_{d-1}},\zeta) : H^s(\mathbb{R}_+) \to \mathbb{C}.$$

We also have the expressed boundary symbol

$$\tilde{\sigma}_\partial(\mathfrak{A}^s)(r,\phi',z,\rho,\theta',\zeta) := \mathrm{diag}(r^{2m},(r^{\mu_j})_{j=1,\dots,m})\sigma_\partial(\mathfrak{A}^s)(r,\phi',z,r^{-1}\rho,\theta',r^{-1}\zeta)$$

with components that are κ_λ-homogeneous in the covariables $(\rho,\theta',\zeta) \neq 0$ and smooth in (r,ϕ',z) up to $r = 0$.

Theorem 2.3.2. *The operator* (2.27) *is elliptic in the calculus of boundary value problems on the manifold* $\overline{\mathbb{R}}_+^\varrho$ *with edge* $\mathbb{R}^q$*, i.e., the components of the principal symbolic hierarchy* $\sigma(\mathfrak{A}^s) = (\sigma_\psi(\mathfrak{A}^s), \sigma_\partial(\mathfrak{A}^s), \sigma_\wedge(\mathfrak{A}^s))$ *are bijective in the sense of Definition* 1.2.2.

Proof. The condition (2.21) implies that $\sigma_\psi(\mathfrak{A}^s)(r,\phi,z,\rho,\theta,\zeta) \neq 0$ for all $(\rho,\theta,\zeta) \neq 0$ and in addition $\tilde{\sigma}_\psi(\mathfrak{A}^s)(r,\phi,z,\rho,\theta,\zeta) \neq 0$ up to $r = 0$.

In a similar manner, the bijectivity of (2.22) implies the bijectivity of the boundary symbol

$$\sigma_\partial(\mathfrak{A}^s)(r,\phi',z,\rho,\theta',\zeta) : H^t(\mathbb{R}_+) \to \begin{array}{c} H^{t-2m}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^m \end{array}$$

for all $(\rho,\theta',\zeta) \neq 0$ and sufficiently large t, and also the bijectivity of $\tilde{\sigma}_\partial(\mathfrak{A}^s)(r,\phi',z,\rho,\theta',\zeta)$ up to $r = 0$.

The principal edge symbol $\sigma_\wedge(\mathfrak{A}^s)(z,\zeta)$ is the same as (2.34) and hence bijective as noted in the proof of Theorem 2.3.1. Summing up we proved the ellipticity of $\mathfrak{A}^s$ with respect to $(\sigma_\psi,\sigma_\partial,\sigma_\wedge)$. ☐

Observe that in polar coordinates $\tilde{x} \mapsto (r,\phi)$ we have

$$\sigma_\wedge(\mathfrak{A}^s)(z,\zeta) : \begin{array}{c} \mathcal{K}^{s,s}\big((S_+^{d-1})^\wedge\big) \\ \oplus \\ \mathbb{C}^{\iota(s,d)} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-2m,s-2m}\big((S_+^{d-1})^\wedge\big) \\ \oplus \\ \mathbb{C}^{\iota(s-2m,d)} \\ \oplus \\ \oplus_{j=1}^m \mathbb{C}^{\iota(s-\mu_j-\frac{1}{2},d-1)} \end{array} \tag{2.35}$$

(cf. also the abbreviation in the formula (1.17)) for $(z,\zeta) \in T^*\mathbb{R}^q \setminus 0$, with the $(m+1)\times 1$ upper left corner

$$\sigma_\wedge(\mathcal{A}^s)(z,\zeta) = \begin{pmatrix} r^{-2m} \sum\limits_{k+|\alpha|\le 2m} a_{k\alpha}(0,z)(-r\dfrac{\partial}{\partial r})^k (r\zeta)^\alpha \\ \big(\mathrm{r}_{\{\tilde{x}_d=0\}} r^{-\mu_j} \sum\limits_{k+|\beta|\le \mu_j} b_{j,k\beta}(0,z)(-r\dfrac{\partial}{\partial r})^k (r\zeta)^\beta\big)_{j=1,\dots,m} \end{pmatrix} :$$

$$\mathcal{K}^{s,s}\big((S_+^{d-1})^\wedge\big) \to \boldsymbol{\mathcal{K}}^{s-2m,s-2m}\big((S_+^{d-1})^\wedge\big) \quad (2.36)$$

cf. the formula (2.32). The operators (2.36) represent a parameter-dependent family in the cone calculus of boundary value problems on the infinite (stretched) cone $(S_+^{d-1})^\wedge$ with boundary $(S^{d-2})^\wedge$.

Remark 2.3.3. *Note that s in the spaces $\mathcal{K}^{s,s}\big((S_+^{d-1})^\wedge\big)$, etc., plays the role both of smoothness and weight. Many relations remain true if we replace the smoothness index by t (sufficiently large). In particular, instead of* (2.36) *we can write*

$$\sigma_\wedge(\mathcal{A}^s)(z,\zeta) : \mathcal{K}^{t,s}\big((S_+^{d-1})^\wedge\big) \to \boldsymbol{\mathcal{K}}^{t-2m,s-2m}\big((S_+^{d-1})^\wedge\big).$$

Remark 2.3.4. *The principal conormal symbol of $\sigma_\wedge(\mathfrak{A}^s)(z,\zeta)$ (in the sense of the cone calculus) represents a family of continuous operators*

$$\sigma_M\sigma_\wedge(\mathfrak{A}^s)(z,w) : H^t(S_+^{d-1}) \longrightarrow \begin{matrix} H^{t-2m}(S_+^{d-1}) \\ \oplus \\ \oplus_{j=1}^m H^{t-\mu_j-\frac{1}{2}}(S^{d-2}) \end{matrix} =: \boldsymbol{H}^{t-2m}(S_+^{d-1}) \quad (2.37)$$

parametrised by the weight line $\Gamma_{\frac{d}{2}-s} \ni w$ and $z \in \mathbb{R}^q$. From the formulas (2.35) *and* (2.36) *we have*

$$\sigma_M\sigma_\wedge(\mathfrak{A}^s)(z,w) = \begin{pmatrix} \sum\limits_{k=0}^{2m} a_{k0}(0,z)w^k \\ \mathrm{r}_{S^{d-2}}{}^{\mathrm{t}}\big(\sum\limits_{k=0}^{\mu_j} b_{j,k0}(0,z)w^k\big)_{j=1,\dots,m} \end{pmatrix}. \quad (2.38)$$

The conormal symbol (2.37) *is a subordinate symbolic structure of the calculus of edge boundary value problems. The ellipticity of $\mathfrak{A}^s$, more precisely, the Fredholm property of* (2.36) *implies that the operators* (2.37) *are a family of isomorphisms for all $w \in \Gamma_{\frac{d}{2}-s}$, $z \in \mathbb{R}^q$.*

3 Global constructions

3.1 Edge representation of boundary problems

Let M be a compact C^∞ manifold with boundary ∂M, and let $Z \subset \partial M$ be a compact C^∞ submanifold of codimension $d > 1$. For simplicity, we let Z have a trivial normal bundle in ∂M (this assumption is not really essential).

We now interpret M as a manifold W with edge Z and boundary, and ∂M as a manifold V with edge Z (and without boundary). Locally near Z the manifold W is modelled on $\overline{\mathbb{R}}_+^d \times \mathbb{R}^q$, and the associated stretched manifold $\mathbb{W}$ is locally described by $\overline{\mathbb{R}}_+ \times S_+^{d-1} \times \mathbb{R}^q$ (with the splitting of variables (r,ϕ,z)); moreover, V is locally modelled on $\mathbb{R}^{d-1}\times\mathbb{R}^q$ and the associated stretched manifold $\mathbb{V}$ on $\overline{\mathbb{R}}_+ \times S^{d-2} \times \mathbb{R}^q$ (with the splitting of variables (r,ϕ',z)).

On M we have the standard Sobolev spaces $H^s(M) = \{u|_{\mathrm{int}M} : u \in H^s(2M)\}$. For $s \geq 0$, $s - \frac{d}{2} \notin \mathbb{N}$, we form the subspaces

$$H_0^s(M) = \big\{u \in H^s(M) : D_{\tilde{x}}^\alpha u(0,z) = 0 \text{ locally near } Z \text{ for all } |\alpha| < s - \frac{d}{2}\big\}.$$

This refers to the splitting of variables $(\tilde{x}, z) \in \overline{\mathbb{R}}_+^d \times \mathbb{R}^q$ on M near Z. Similarly, for $s \geq 0$, $s - \frac{d-1}{2} \notin \mathbb{N}$, we form

$$H_0^s(\partial M) = \big\{v \in H^s(\partial M) : D_{\tilde{x}'}^\alpha v(0,z) = 0 \text{ locally near } Z \text{ for all } |\alpha| < s - \frac{d-1}{2}\big\},$$

which refers to the splitting of variables $(\tilde{x}', z) \in \mathbb{R}^{d-1} \times \mathbb{R}^q$ on ∂M near Z.

Motivated by the equivalent descriptions

$$\begin{aligned} H_0^s(M) = \big\{u \in H^s(M) :\ & \varphi u \in \mathcal{W}^{s,s}((\mathbb{R}_+^d \setminus \{0\}) \times \mathbb{R}^q) \text{ for every } \varphi \in C_0^\infty(M) \\ & \text{supported in a coordinate neighbourhood near } \tilde{x} = 0\big\}. \end{aligned}$$

and

$$\begin{aligned} H_0^s(\partial M) = \big\{u \in H^s(\partial M) :\ & \varphi' v \in \mathcal{W}^{s,s}((\mathbb{R}^{d-1} \setminus \{0\}) \times \mathbb{R}^q) \text{ for every } \varphi' \in C_0^\infty(\partial M) \\ & \text{supported in a coordinate neighbourhood near } \tilde{x}' = 0\big\}, \end{aligned}$$

we also write $\mathcal{W}^{s,s}(\mathbb{W})$ and $\mathcal{W}^{s,s}(\mathbb{V})$ instead of $H_0^s(M)$ and $H_0^s(\partial M)$, respectively.

Let A be a differential operators on M of order $2m$ (with smooth coefficients up to ∂M), regarded as a continuous map

$$H^s(M) \to H^{s-2m}(M).$$

Moreover, let B_j be differential operators of order μ_j, given in a collar neighbourhood of ∂M in M (with smooth coefficients up to ∂M), and form the continuous operators $T_j u := B_j u|_{\partial M}$, $j = 1, \ldots, m$. Then $T = {}^{\mathrm{t}}(T_1, \ldots, T_m)$ together with A represents a global boundary value problem for A

$$\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix} : H^s(M) \to \boldsymbol{H}^{s-2m}(M), \tag{3.1}$$

concerning notation, cf. the formula (2.37).

Our next objective is to reformulate $\mathcal{A}$ as an edge problem in the sense of Section 1.2. To this end we employ the following theorem.

Theorem 3.1.1. *For every fixed $s \geq 0$, $s - \frac{d}{2} \notin \mathbb{N}$ there is a family of isomorphisms*

$$\begin{pmatrix} E & K(\lambda) \end{pmatrix} : \begin{matrix} \mathcal{W}^{s,s}(\mathbb{W}) \\ \oplus \\ H^s(Z, \mathbb{C}^{\iota(s,d)}) \end{matrix} \longrightarrow H^s(M) \tag{3.2}$$

for $|\lambda|$ sufficiently large, where (3.2) *localises near Z to the operators* (2.28)*, and the inverse $\begin{pmatrix} P(\lambda) \\ T(\lambda) \end{pmatrix} := \begin{pmatrix} E & K(\lambda) \end{pmatrix}^{-1}$ localises to* (2.29) *(both modulo lower order terms, to be explained in the construction below).*

The proof of Theorem 3.1.1 will be given in Section 3.2 below.

Applying an analogous construction for the case without boundary, for every $s \geq 0$, $s-\mu_j > \frac{d}{2}$, $s-\mu_j-\frac{d-1}{2} \notin \mathbb{N}$, we obtain a family of isomorphisms

$$\begin{pmatrix} E' & K'(\lambda) \end{pmatrix} : \begin{array}{c} \oplus_{j=1}^m \mathcal{W}^{s-\mu_j-\frac{1}{2}, s-\mu_j-\frac{1}{2}}(\mathbb{V}) \\ \oplus \\ \oplus_{j=1}^m H^{s-\mu_j-\frac{1}{2}}(Z, \mathbb{C}^{\iota(s-\mu_j-\frac{1}{2}, d-1)}) \end{array} \longrightarrow \oplus_{j=1}^m H^{s-\mu_j-\frac{1}{2}}(\partial M). \tag{3.3}$$

for $|\lambda|$ large, with the inverse $\begin{pmatrix} P'(\lambda) \\ T'(\lambda) \end{pmatrix}$ which localises near Z to the operators as constructed in Section 2.3 before (modulo corresponding lower order terms).

Remark 3.1.2. *The operator* $\begin{pmatrix} A \\ T \end{pmatrix}$ *combined with the isomorphisms* (3.2) *and* (3.3) (*for sufficiently large* $|\lambda|$, *where the parameter* λ *is fixed and then omitted*) *gives us a block matrix* $\mathfrak{A}^s$ *of analogous structure as* (2.30), *which is an operator in the edge calculus on* $\mathbb{W}$. *The conditions on* s, *i.e.,*

$$s-2m > \frac{d}{2},\ s-2m-\frac{d}{2} \notin \mathbb{N},\ s-\mu_j > \frac{d}{2},\ s-\mu_j-\frac{d}{2} \notin \mathbb{N} \text{ for } j=1,\dots,m, \tag{3.4}$$

are weight conditions for the ellipticity. The operator $\mathfrak{A}^s$ *itself is continuous in the sense*

$$\mathfrak{A}^s : \begin{array}{c} \mathcal{W}^{t,\gamma}(\mathbb{W}) \\ \oplus \\ H^t(Z, \mathbb{C}^{\iota(s,d)}) \end{array} \longrightarrow \begin{array}{c} \boldsymbol{\mathcal{W}}^{t-2m,\gamma-2m}(\mathbb{W}) \\ \oplus \\ \boldsymbol{H}^{t-2m}(Z, \mathbb{C}^{\boldsymbol{\iota}(s-2m,d)}) \end{array} \tag{3.5}$$

for all $t \in \mathbb{R}$, $t > \mu_j + \frac{1}{2}$, $j = 1, \dots, m$, *and for all* $\gamma \in \mathbb{R}$, *cf. the formula* (1.10).

We now assume that $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ is elliptic, i.e., A is an elliptic differential operator on M and the boundary operator T satisfy the Shapiro-Lopatinskij condition, that is, in local representation near ∂M the boundary symbol (1.13) is a family of isomorphisms. Then as a consequence of Theorem 2.3.2 we obtain the following result:

Theorem 3.1.3. *Let* $\mathcal{A}$ *be an elliptic boundary value problem on* M, *and let* s *satisfy the conditions* (3.4). *Then the operator* $\mathfrak{A}^s$ *is elliptic in the edge calculus on* $\mathbb{W}$ *with respect to the principal symbolic hierarchy* (1.11), *cf.* Definition 1.3.3, *here for the weight* $\gamma = s$.

Corollary 3.1.4. Under the conditions of Theorem 3.1.3, the operator (3.5) is Fredholm for $\gamma = s$ and $t > \mu_j + \frac{1}{2}$, $j = 1, \dots, m$, and we have

$$\operatorname{ind} \mathcal{A} = \operatorname{ind} \mathfrak{A}^s.$$

This will follow from the existence of a parametrix in the edge calculus, cf. Section 4.3 below.

Remark 3.1.5. *Recall that the number of elliptic edge trace and potential conditions rapidly changes if we change* s; *the difference of the number of these conditions is just equal to* $\operatorname{ind} \sigma_\wedge(\mathcal{A}^s)$, *calculated in* Theorem 2.3.1.

3.2 The construction of global isomorphisms

In this section we give the proof of Theorem 3.1.1.

We first consider the case that M is closed and compact. The proof for a manifold with boundary is then straightforward and left to the reader. We fix a covering of M by coordinate neighbourhoods $U_1, \ldots, U_L, U_{L+1}, \ldots, U_N$ such that $U_j \cap Z \neq \emptyset$ for $1 \leq j \leq L$, $U_j \cap Z = \emptyset$ for $L+1 \leq j \leq N$. The charts $\chi_j : U_j \to \chi_j(U_j)$ map to open sets of $\mathbb{R}^d \times \mathbb{R}^q$ and we assume that for $U'_j := U_j \cap Z$, $1 \leq j \leq L$, $\chi'_j := \chi_j|_{U'_j} : U'_j \to \chi'_j(U'_j)$ are charts on Z to open sets of $\mathbb{R}^q$. In addition (without loss of generality) we assume that $U_j \cup U_k$ is again a coordinate neighbourhood of our system. Moreover, we fix a subordinate partition of unity $\{\varphi_1, \ldots, \varphi_N\}$, $\varphi_j \in C_0^\infty(U_j)$, and a system of functions $\{\psi_1, \ldots, \psi_N\}$, $\psi_j \in C_0^\infty(U_j)$, such that $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$ for all j. We make the following special choice of these functions such that when $\tilde{U}_j$ for $1 \leq j \leq L$ denotes the connected component of U_j that intersects Z and $\tilde{\varphi}_j := \varphi_j|_{\tilde{U}_j}$ in local coordinates we have $\tilde{\varphi}_j := \sigma\vartheta\varphi'_j$, for $\varphi'_j := \varphi_j|_{U'_j}$, where $\sigma \equiv 1$ in a neighbourhood of $\tilde{x}' = 0$. Similarly, for $1 \leq j \leq L$ we assume on $\tilde{\psi}_j := \psi_j|_{\tilde{U}_j}$ that $\tilde{\psi}_j := \tau\delta\psi'_j$ for $\psi'_j := \psi_j|_{U'_j}$, where $\tau \equiv 1$ in a neighbourhood of $\tilde{x}' = 0$, and $\tau \equiv 1$ on $\operatorname{supp} \sigma$.

We construct the analogues of the operators (3.2) for the closed case in the form

$$\begin{pmatrix} E & K(\lambda) \end{pmatrix} := \sum_{j=1}^{L} \begin{pmatrix} \varphi_j E \psi_j & \varphi_j K(\lambda) \psi'_j \end{pmatrix} + \sum_{j=L+1}^{N} \begin{pmatrix} \varphi_j & 0 \end{pmatrix} \tag{3.6}$$

where $E : \mathcal{W}^{s,s}(\mathbb{W}) \to H^s(M)$ is the canonical embedding, and

$$K(\lambda)v(z) = \operatorname{Op}(k)(\lambda)v(z),$$

$$k(\zeta, \lambda) := \big(k^\alpha(\zeta, \lambda) : |\alpha| < s - \frac{d}{2}\big) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; \mathbb{C}^{\iota(s,d)}, H^s(\mathbb{R}^d)).$$

The inverse of $\begin{pmatrix} E & K(\lambda) \end{pmatrix}$ will be approximated by

$$\begin{pmatrix} P_0(\lambda) \\ T_0(\lambda) \end{pmatrix} := \sum_{k=1}^{L} \begin{pmatrix} \psi_k P(\lambda) \varphi_k \\ \psi'_k T(\lambda) \varphi_k \end{pmatrix} + \sum_{k=L+1}^{N} \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \tag{3.7}$$

for large λ.

Observe that the operator of multiplication by a function $\varphi(\tilde{x}, z) \in C_0^\infty(\mathbb{R}^{d+q})$ represents an operator-valued symbol $\varphi \in S^0(\mathbb{R}^{q+l}; E, E)$ for $E = H^s(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$ or $H^s_0(\mathbb{R}^d)$. It does not depend on the covariables but it is not classical in the sense of Definition 1.2.5. Relations of that kind belong to the tools for the calculations below.

Remark 3.2.1. *Another aspect is that for every* $\varphi \in C_0^\infty(\mathbb{R}^{d+q})$, $\beta \in C_0^\infty(\mathbb{R}^q)$ *and* $\operatorname{supp} \varphi \cap \{\tilde{x} = 0\} = \emptyset$ *implies that*

$$\varphi \operatorname{Op}(k)(\lambda)\beta = \operatorname{Op}(c)(\lambda)$$

for a $c(z, \zeta, \lambda) \in S^{-\infty}(\mathbb{R}^q \times \mathbb{R}^{q+l}; \mathbb{C}^{\iota(s,d)}, \mathcal{S}(\mathbb{R}^d))$. *In fact, we can write*

$$\varphi \operatorname{Op}(k)(\lambda)\beta = \varphi|\tilde{x}|^{-2K}[\zeta]^{-2K} \operatorname{Op}(|[\zeta]\tilde{x}|^{2K} k)(\lambda)\beta.$$

By virtue of $\varphi\,|\tilde{x}|^{-2K} \in C_0^\infty(\mathbb{R}^{d+q})$ *for any* $K \in \mathbb{N}$ *the operator of multiplication by* $\varphi\,|\tilde{x}|^{-2K}$ *belongs to* $S^{-\infty}(\mathbb{R}^q \times \mathbb{R}^{q+l}; \mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$. *We have* $\big|[\zeta]\tilde{x}\big|^{2K} k(\zeta, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; \mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$, *i.e.,*

$$k_{-2K}(\zeta, \lambda) := [\zeta]^{-2K}\big(\big|[\zeta]\tilde{x}\big|^{2K} k(\zeta, \lambda)\big) \in S^{-2K}_{\mathrm{cl}}(\mathbb{R}^{q+l}; \mathbb{C}^{\iota(s,d)}, \mathcal{S}(\mathbb{R}^d)).$$

Applying Remark 1.2.8 *we see that* $\varphi|\tilde{x}|^{-2K} \operatorname{Op}(k_{-2K})(\lambda)\beta = \operatorname{Op}(r)(\lambda)$ *for an symbol* $r(z, \zeta, \lambda) \in S^{-2K}(\mathbb{R}^{q+l}; \mathbb{C}^{\iota(s,d)}, \mathcal{S}(\mathbb{R}^d))$. *Since this holds for all* $K \in \mathbb{N}$, *the relation* (1.20) *for* $\mu = -\infty$ *gives us the assertion.*

Writing for abbreviation

$$\mathcal{E}_j := \begin{cases} \varphi_j E \psi_j & \text{for } 1 \le j \le L, \\ \varphi_j & \text{for } L+1 \le j \le N, \end{cases} \quad \mathcal{C}_j(\lambda) := \begin{cases} \varphi_j K(\lambda) \psi'_j & \text{for } 1 \le j \le L, \\ 0 & \text{for } L+1 \le j \le N, \end{cases}$$

$$\mathcal{P}_k(\lambda) := \begin{cases} \psi_k P(\lambda) \varphi_k & \text{for } 1 \le k \le L, \\ \varphi_k & \text{for } L+1 \le k \le N, \end{cases} \quad \mathcal{T}_k(\lambda) := \begin{cases} \psi'_k T(\lambda) \varphi_k & \text{for } 1 \le k \le L, \\ 0 & \text{for } L+1 \le k \le N, \end{cases}$$

we have

$$\begin{pmatrix} E & K(\lambda) \end{pmatrix} = \sum_{j=1}^{N} \begin{pmatrix} \mathcal{E}_j & \mathcal{C}_j(\lambda) \end{pmatrix}, \quad \begin{pmatrix} P_0(\lambda) \\ T_0(\lambda) \end{pmatrix} = \sum_{k=1}^{N} \begin{pmatrix} \mathcal{P}_k(\lambda) \\ \mathcal{T}_k(\lambda) \end{pmatrix}.$$

We then consider

$$\begin{pmatrix} P_0(\lambda) \\ T_0(\lambda) \end{pmatrix} \begin{pmatrix} E & K(\lambda) \end{pmatrix} = \sum_{j,k=1}^{N} \begin{pmatrix} \mathcal{P}_k(\lambda)\mathcal{E}_j & \mathcal{P}_k(\lambda)\mathcal{C}_j(\lambda) \\ \mathcal{T}_k(\lambda)\mathcal{E}_j & \mathcal{T}_k(\lambda)\mathcal{C}_j(\lambda) \end{pmatrix} \tag{3.8}$$

and

$$\begin{pmatrix} E & K(\lambda) \end{pmatrix} \begin{pmatrix} P_0(\lambda) \\ T_0(\lambda) \end{pmatrix} = \sum_{j,k=1}^{N} \{\mathcal{E}_j \mathcal{P}_k(\lambda) + \mathcal{C}_j(\lambda)\mathcal{T}_k(\lambda)\}. \tag{3.9}$$

Let us first characterise the entries of (3.8). For $1 \le j, k \le L$ we have

$$\mathcal{P}_k(\lambda)\mathcal{E}_j = \psi_k P(\lambda)\varphi_k \varphi_j E \psi_j, \;\; \mathcal{P}_k(\lambda)\mathcal{C}_j(\lambda) = \psi_k P(\lambda)\varphi_k\varphi_j K(\lambda)\psi'_j,$$

$$\mathcal{T}_k(\lambda)\mathcal{E}_j = \psi'_k T(\lambda)\varphi_k \varphi_j E \psi_j, \;\; \mathcal{T}_k(\lambda)\mathcal{C}_j(\lambda) = \psi'_k T(\lambda)\varphi_k\varphi_j K(\lambda)\psi'_j.$$

For $u(z) \in \mathcal{W}^s(\mathbb{R}^q, H_0^s(\mathbb{R}^d))$ we have

$$\mathcal{P}_k(\lambda)\mathcal{E}_j u(z) = \{\varphi_k\varphi_j - \psi_k \operatorname{Op}(kt)(\lambda)\varphi_k\varphi_j E\psi_j\}u(z),$$

$$\psi_k \operatorname{Op}(kt)(\lambda)\varphi_k\varphi_j E\psi_j u(z) = \psi'_k \tau \operatorname{Op}(kt)(\lambda)\varphi'_k\sigma\varphi'_j\sigma\psi'_j\tau u(z) + D_{kj}(\lambda)u(z), \tag{3.10}$$

where, according to Remark 3.2.1, $D_{kj}(\lambda) = \operatorname{Op}(d_{kj}(z,\zeta,\lambda))$ for a symbol $d_{kj}(z,\zeta,\lambda) \in S^{-\infty}(\mathbb{R}^q \times \mathbb{R}^{q+l}; H_0^s(\mathbb{R}^d), H_0^s(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d))$. Since $\varphi'_k\sigma\varphi'_j\sigma\psi'_j\tau u(z)$ takes values in $\ker t(\zeta,\lambda)$ we see that the first summand of (3.10) vanishes, i.e., we have $\mathcal{P}_k(\lambda)\mathcal{E}_j = \varphi_k\varphi_j + D_{kj}(\lambda)$ for $1 \le j, k \le L$.

For $v(z) \in H^s(\mathbb{R}^q, \mathbb{C}^{\iota(s,d)})$ we obtain

$$\begin{aligned} \mathcal{P}_k(\lambda)\mathcal{C}_j(\lambda)v(z) &= \psi_k \operatorname{Op}(p)(\lambda)\varphi_k\varphi_j \operatorname{Op}(k)(\lambda)\psi'_j v(z) \\ &= \psi_k \operatorname{Op}(p)(\lambda)\beta(z)\operatorname{Op}(k)(\lambda)\psi'_j v(z) + C_{kj}(\lambda)v(z) \end{aligned} \tag{3.11}$$

for $\beta(z) = \varphi'_k(z)\varphi'_j(z)$ where we used the fact that (by an appropriate choice of the cut-off function ω involved in $k(\zeta,\lambda)$) $\sigma^2 k(\zeta,\lambda) = k(\zeta,\lambda)$, and $C_{kj}(\lambda) = \operatorname{Op}(c_{kj})(\lambda)$ for a corresponding symbol of order $-\infty$, again by Remark 3.2.1. In order to shorten notation in the rest of the proof we write '$\sim$' when equalities hold modulo term of order $-\infty$ in λ. Applying Remark 1.2.8 we obtain

$$\operatorname{Op}(p)(\lambda)\beta(z) = \operatorname{Op}(p\#\beta)(\lambda) = \beta\operatorname{Op}(p)(\lambda) + \operatorname{Op}(r)(\lambda)$$

for a symbol $r(z,\zeta,\lambda)$ of order -1 and we obtain

$$\mathcal{P}_k(\lambda)\mathcal{C}_j(\lambda) \sim \psi_k \operatorname{Op}(pk)(\lambda)\psi'_j + \psi_k\operatorname{Op}(rk)(\lambda)\psi'_j.$$

Now we have $pk = 0$. By Remark 1.2.8 the second summand on the right can be reformulated as $R_{kj}(\lambda) = \operatorname{Op}(r_{kj})(\lambda)$ for a symbol $r_{kj}(z,\zeta,\lambda) \in S_{\mathrm{cl}}^{-1}(\mathbb{R}^q \times \mathbb{R}^{q+l}_{\zeta,\lambda}; \mathbb{C}^{\iota(s,d)}, H_0^s(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d))$. In other words, we have

$$\mathcal{P}_k(\lambda)\mathcal{C}_j(\lambda) \sim R_{kj}(\lambda), \;\; 1 \le j, k \le L. \tag{3.12}$$

Moreover,

$$\mathcal{T}_k(\lambda)\mathcal{E}_j u(z) = \psi_k' \operatorname{Op}(t)(\lambda)\varphi_k\varphi_j E\psi_j u(z) \equiv 0, \ \ 1 \leq j,k \leq L, \tag{3.13}$$

since $\varphi_k\varphi_j E\psi_j u(z)$ takes values in $\ker t(\zeta,\lambda)$. Furthermore, again using Remark 1.2.8 from the pseudo-differential calculus we obtain

$$\mathcal{T}_k(\lambda)\mathcal{C}_j(\lambda)v(z) \sim \psi_k' \operatorname{Op}(t)(\lambda)\varphi_k\varphi_j \operatorname{Op}(k)(\lambda)\psi_j' v(z) = \varphi_k'\varphi_j' v(z) + R_{kj}'(\lambda)v(z)$$

for $R_{kj}'(\lambda) = \psi_k' \operatorname{Op}(r_{kj}')(\lambda)\psi_j'$, with a symbol $r_{kj}'(z,\zeta,\lambda) \in S_{\mathrm{cl}}^{-1}(\mathbb{R}^q \times \mathbb{R}^{q+l}; \mathbb{C}^{\iota(s,d)}, \mathbb{C}^{\iota(s,d)})$.

For $L+1 \leq j \leq N$, $1 \leq k \leq L$ we have

$$\mathcal{P}_k(\lambda)\mathcal{E}_j \sim \psi_k P(\lambda)\varphi_k\varphi_j = \varphi_k\varphi_j, \ \ \mathcal{P}_k(\lambda)\mathcal{C}_j(\lambda) = \mathcal{T}_k(\lambda)\mathcal{C}_j(\lambda) = 0,$$

and $\mathcal{T}_k(\lambda)\mathcal{E}_j = \psi_k' T(\lambda)\varphi_k\varphi_j = 0$, since $\operatorname{supp}\varphi_j \cap Z = \emptyset$, i.e., $\varphi_k\varphi_j \in \ker T(\lambda)$.

Moreover, for $1 \leq j \leq L$, $L+1 \leq k \leq N$,

$$\mathcal{P}_k(\lambda)\mathcal{E}_j = \varphi_k\varphi_j E\psi_j = \varphi_k\varphi_j, \ \ \mathcal{T}_k(\lambda)\mathcal{E}_j = \mathcal{T}_k(\lambda)\mathcal{C}_j(\lambda) = 0,$$

and $\mathcal{P}_k(\lambda)\mathcal{C}_j(\lambda) = \varphi_k\varphi_j K(\lambda)\psi_j' := C_{kj}(\lambda)$ is of order $-\infty$ in $\lambda \in \mathbb{R}^l$, cf. Remark 3.2.1.

Finally, for $L+1 \leq j,k \leq N$ it follows that

$$\mathcal{P}_k(\lambda)\mathcal{E}_j = \varphi_k\varphi_j, \ \ \mathcal{P}_k(\lambda)\mathcal{C}_j(\lambda) = \mathcal{T}_k(\lambda)\mathcal{E}_j = \mathcal{T}_k(\lambda)\mathcal{C}_j(\lambda) = 0.$$

For the expression (3.9) we first assume $1 \leq j,k \leq L$. Then

$$\begin{aligned}\mathcal{E}_j\mathcal{P}_k(\lambda) + \mathcal{C}_j(\lambda)\mathcal{T}_k(\lambda) &= \varphi_j E\psi_j\psi_k P(\lambda)\varphi_k + \varphi_j K(\lambda)\psi_j'\psi_k' T(\lambda)\varphi_k \\ &= \varphi_j\varphi_k - \varphi_j\psi_k K(\lambda)T(\lambda)\varphi_k + \varphi_j K(\lambda)\psi_j'\psi_k' T(\lambda)\varphi_k \sim \varphi_j\varphi_k + \tilde{R}_{jk}(\lambda)\end{aligned}$$

for $\tilde{R}_{jk}(\lambda) = -\varphi_j\psi_k \operatorname{Op}(kt)(\lambda)\varphi_k + \varphi_j \operatorname{Op}(k)(\lambda)\operatorname{Op}(\psi_j'\psi_k' t)(\lambda)\varphi_k$. From Remark 1.2.8 we obtain

$$\begin{aligned}\varphi_j \operatorname{Op}(k)(\lambda)\operatorname{Op}(\psi_j'\psi_k' t)(\lambda)\varphi_k &\sim \varphi_j\psi_j'\psi_k' \operatorname{Op}(kt)(\lambda)\varphi_k + \varphi_j \operatorname{Op}(\tilde{r}_{jk})(\lambda)\varphi_k \\ &\sim \varphi_j\psi_k' \operatorname{Op}(kt)(\lambda)\varphi_k + \varphi_j \operatorname{Op}(\tilde{r}_{jk})(\lambda)\varphi_k\end{aligned}$$

for a symbol $\tilde{r}_{jk}(z,\zeta,\lambda) \in S_{\mathrm{cl}}^{-1}(\mathbb{R}^q \times \mathbb{R}^{q+l}; H^s(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$. In the latter relation we employed $\psi_j = \psi_j'\tau$, $\varphi_j\psi_j = \varphi_j$, and $\tau k(\zeta,\lambda) = k(\zeta,\lambda)$.

For $L+1 \leq j \leq N$, $1 \leq k \leq L$ we have

$$\mathcal{E}_j\mathcal{P}_k(\lambda) + \mathcal{C}_j(\lambda)\mathcal{T}_k(\lambda) = \varphi_j\psi_k P(\lambda)\varphi_k = \varphi_j\psi_k(1 - \operatorname{Op}(kt)(\lambda))\varphi_k = \varphi_j\varphi_k + \tilde{C}_{jk}(\lambda)$$

for $\tilde{C}_{jk}(\lambda) = -\varphi_j\psi_k \operatorname{Op}(kt)(\lambda)\varphi_k = \operatorname{Op}(\tilde{c}_{jk})(\lambda)$ for a symbol $\tilde{c}_{jk}(z,\zeta,\lambda) \in S^{-\infty}(\mathbb{R}^q \times \mathbb{R}^{q+l}; H^s(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$. The latter relation can be obtained by similar arguments as in Remark 3.2.1.

For $1 \leq j \leq L$, $L+1 \leq k \leq N$ we have

$$\mathcal{E}_j\mathcal{P}_k(\lambda) + \mathcal{C}_j(\lambda)\mathcal{T}_k(\lambda) = \varphi_j E\psi_j\varphi_k = \varphi_j\varphi_k.$$

Finally, in the case $L+1 \leq j,k \leq N$ we have

$$\mathcal{E}_j\mathcal{P}_k(\lambda) + \mathcal{C}_j(\lambda)\mathcal{T}_k(\lambda) = \varphi_j\varphi_k.$$

From these calculations we obtain the following result (with 1 denoting identity operators in different spaces):

Proposition 3.2.2. *We have*

$$\begin{pmatrix} P_0(\lambda) \\ T_0(\lambda) \end{pmatrix} \begin{pmatrix} E & K(\lambda) \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & R(\lambda) \\ 0 & R'(\lambda) \end{pmatrix}, \tag{3.14}$$

and

$$\begin{pmatrix} E & K(\lambda) \end{pmatrix} \begin{pmatrix} P_0(\lambda) \\ T_0(\lambda) \end{pmatrix} \sim 1 + \tilde{R}(\lambda) \tag{3.15}$$

with remainders $R(\lambda)$, $R'(\lambda)$, $\tilde{R}(\lambda)$ of order -1 in $\lambda \in \mathbb{R}^l$ in the sense of the pseudo-differential calculus along Z with corresponding operator-valued symbols.

Proof. We obtained

$$\begin{pmatrix} \mathcal{P}_k(\lambda)\mathcal{E}_j & \mathcal{P}_k(\lambda)\mathcal{C}_j(\lambda) \\ \mathcal{T}_k(\lambda)\mathcal{E}_j & \mathcal{T}_k(\lambda)\mathcal{C}_j(\lambda) \end{pmatrix} \sim \begin{pmatrix} \varphi_j\varphi_k & R_{kj}(\lambda) \\ 0 & \varphi'_k\varphi'_j + R'_{kj}(\lambda) \end{pmatrix}$$

and

$$\mathcal{E}_j\mathcal{P}_k(\lambda) + \mathcal{C}_j(\lambda)\mathcal{T}_k(\lambda) \sim \varphi_j\varphi_k + \tilde{R}_{jk}(\lambda),$$

where $R_{kj}(\lambda)$, $R'_{kj}(\lambda)$, $\tilde{R}_{jk}(\lambda)$ are of order -1, concentrated near Z. Taking the sums over $j, k = 1, \ldots, N$ we just obtain the relations (3.14) and (3.15), respectively. □

Summing up we have

$$\begin{pmatrix} P_0(\lambda) \\ T_0(\lambda) \end{pmatrix} \begin{pmatrix} E & K(\lambda) \end{pmatrix} = 1 + \mathcal{R}(\lambda)$$

modulo a Schwartz function in λ; where 1 is the 2×2 identity matrix and $\mathcal{R}(\lambda) := \begin{pmatrix} 0 & R(\lambda) \\ 0 & R'(\lambda) \end{pmatrix}$ is locally on Z in coordinates $z \in \mathbb{R}^q$ a parameter-dependent pseudo-differential operator with symbol in

$$S^{-1}\left(\mathbb{R}^q \times \mathbb{R}^{q+l}; \begin{matrix} H_0^s(\mathbb{R}^d) \\ \oplus \\ \mathbb{C}^{\iota(s,d)} \end{matrix}, \begin{matrix} H_0^s(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d) \\ \oplus \\ \mathbb{C}^{\iota(s,d)} \end{matrix} \right).$$

There is asymptotic summation in symbols and associated operators of that kind. This allows us to form

$$\sum_{j=0}^{\infty} (-1)^j \mathcal{R}^j(\lambda) \begin{pmatrix} P_0(\lambda) \\ T_0(\lambda) \end{pmatrix} =: \begin{pmatrix} P_1(\lambda) \\ T_1(\lambda) \end{pmatrix}$$

and we then have $\begin{pmatrix} P_1(\lambda) \\ T_1(\lambda) \end{pmatrix} \begin{pmatrix} E & K(\lambda) \end{pmatrix} = 1 + \mathcal{C}(\lambda)$ for an operator function $\mathcal{C}(\lambda)$ which is Schwartz in $\lambda \in \mathbb{R}^l$. Since $1 + \mathcal{C}(\lambda)$ is invertible for $|\lambda|$ large enough we conclude that $\begin{pmatrix} E & K(\lambda) \end{pmatrix}$ is invertible for those λ, and the inverse of $\begin{pmatrix} E & K(\lambda) \end{pmatrix}$ has the form

$$\begin{pmatrix} P(\lambda) \\ T(\lambda) \end{pmatrix} = (1 + \mathcal{C}(\lambda))^{-1} \begin{pmatrix} P_1(\lambda) \\ T_1(\lambda) \end{pmatrix}.$$

This gives us an analogue of Theorem 3.1.1 for the case without boundary.

Remark 3.2.3. *Replacing λ in this conclusion by $(\lambda_0, \lambda) \in \mathbb{R}^{1+l}$ we obtain the isomorphism result for $|\lambda_0|$ large and all $\lambda \in \mathbb{R}^l$. So we obtain invertibility of such operator families for all $\lambda \in \mathbb{R}^l$.*

Proof of Theorem 3.1.1. An inspection of the proof of Proposition 3.2.2 shows that the calculations remain valid for the case with boundary when we reinterpret the meaning of the involved operators in an evident manner. First the functions $(\varphi_{L+1}, \ldots, \varphi_N)$ in the second sums of (3.6) and (3.7) are chosen as in the beginning of this section. The local operator-valued symbols involved in $T(\lambda)$, $K(\lambda)$ and $P(\lambda)$ are now given by (2.1) and (2.2), respectively. The generalisation of the calculations of the proof of Proposition 3.2.2 to the case with boundary is now straightforward. □

4 Edge operators for arbitrary weights

4.1 Relative index relations

As we saw in Chapter 1, ellipticity in the calculus on a manifold W with edges makes sense in weighted edge Sobolev spaces of arbitrary weights, provided that the corresponding principal edge symbols are bijective in the sense of (1.23). In Chapter 3 we constructed elliptic edge problems for the case when the weight is equal to the smoothness s, cf. Theorem 3.1.3. According to Remark 3.1.2 we now realise the upper left corner of $\mathfrak{A}^s$ as a continuous operator

$$\mathcal{A}^\gamma : \mathcal{W}^{t,\gamma}(\mathbb{W}) \to \boldsymbol{\mathcal{W}}^{t-2m,\gamma-2m}(\mathbb{W}) \tag{4.1}$$

for sufficiently large t and any real γ (we hope this will not cause confusion compared with (3.1); the operator $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ in polar coordinates induces continuous maps (4.1) for all reals γ and sufficiently large t).

Assuming the Shapiro-Lopatinskij ellipticity of $\mathcal{A}$, the question is now to calculate the number of extra edge conditions for arbitrary $\gamma \in \mathbb{R}$ (possibly up to a discrete set of exceptional values) such that a corresponding operator $\mathfrak{A}^\gamma$ in the edge calculus having (4.1) as the upper left corner is Fredholm. This will be done in terms of relative index results for the associated edge symbols.

Proposition 4.1.1. *If a weight $\gamma \in \mathbb{R}$ satisfies the condition*

$$\gamma - 2m > \frac{d}{2}, \gamma - 2m - \frac{d}{2} \notin \mathbb{N}, \gamma - \mu_j > \frac{d}{2}, \gamma - \mu_j - \frac{d}{2} \notin \mathbb{N}, \text{ for } j = 1, \dots, m, \tag{4.2}$$

the operator family

$$\sigma_M \sigma_\wedge(\mathcal{A}^\gamma)(z, w) : H^t(S_+^{d-1}) \to \boldsymbol{H}^{t-2m}(S_+^{d-1})$$

is a family of isomorphisms for all $w \in \Gamma_{\frac{d}{2}-\gamma}$ and all $z \in Z$ (concerning the spaces, cf. the formula (2.37)).

Proof. From the ellipticity of $\mathfrak{A}^s$ for the weights as in (3.4) we have the bijectivity of $\sigma_\wedge(\mathfrak{A}^s)(z,\zeta)$ on $T^*Z \setminus 0$ which entails the Fredholm property of $\sigma_\wedge(\mathcal{A}^s)(z,\zeta)$ in $\mathcal{K}^{s,s}$-spaces, according to the upper left $(m+1) \times 1$ corner of (2.35). A necessary condition for that is that $\sigma_M \sigma_\wedge(\mathcal{A}^s)(z, w)$ has no non-bijectivity points on the line $\Gamma_{\frac{d}{2}-s}$; clearly the operators $\sigma_M \sigma_\wedge(\mathcal{A}^s)(z, w)$ are independent of s, cf. the right hand side of (2.38). This allows us to interpret s as our weight γ and to apply this information for $\sigma_M \sigma_\wedge(\mathcal{A}^\gamma)(z, w)$. $\square$

In the following consideration we admit arbitrary weights γ, β. Assume that the operators

$$\sigma_\wedge(\mathcal{A}^\gamma)(z,\zeta) : \mathcal{K}^{t,\gamma}\big((S_+^{d-1})^\wedge\big) \to \boldsymbol{\mathcal{K}}^{t-2m,\gamma-2m}\big((S_+^{d-1})^\wedge\big) \tag{4.3}$$

are Fredholm for all $(z,\zeta) \in T^*Z \setminus 0$ and t large enough, cf. Remark 2.3.3. Recall that (4.3) is Fredholm if and only if the conormal symbol $\sigma_M \sigma_\wedge(\mathcal{A}^\gamma)(z, w)$ which has the form

$$a(z,w) := \begin{pmatrix} \displaystyle\sum_{k=0}^{2m} a_{k0}(0,z) w^k \\ \Big(\mathrm{r}_{S^{d-2}} \displaystyle\sum_{k=0}^{\mu_j} b_{j,k0}(0,z) w^k\Big)_{j=1,\dots,m} \end{pmatrix} : H^t(S_+^{d-1}) \to \boldsymbol{H}^{t-2m}(S_+^{d-1}) \tag{4.4}$$

is bijective for all $w \in \Gamma_{\frac{d}{2}-\gamma}$, $z \in Z$.

Let us fix (z,ζ) and form the operator

$$K^\gamma := \operatorname{diag}(r^{2m}, (r^{\mu_j})_{j=1,\dots,m})\sigma_\wedge(\mathcal{A}^\gamma)(z,\zeta),$$

$$K^\gamma : \mathcal{K}^{t,\gamma} \to \tilde{\boldsymbol{\mathcal{K}}}^{t-2m,\gamma} \tag{4.5}$$

for $\mathcal{K}^{t,\gamma} := \mathcal{K}^{t,\gamma}\big((S_+^{d-1})^\wedge\big)$, $\tilde{\boldsymbol{\mathcal{K}}}^{t-2m,\gamma} := \begin{array}{c} r^{2m}\mathcal{K}^{t-2m,\gamma-2m}\big((S_+^{d-1})^\wedge\big) \\ \oplus \\ \oplus_{j=1}^m r^{\mu_j}\mathcal{K}^{t-\mu_j-\frac{1}{2},\gamma-\frac{1}{2}}\big((S^{d-2})^\wedge\big)\end{array}$ for any fixed sufficiently large t (the index is independent of t). Analogously, we consider the Fredholm operator

$$K^\beta : \mathcal{K}^{t,\beta} \to \tilde{\boldsymbol{\mathcal{K}}}^{t-2m,\beta} \tag{4.6}$$

for another weight $\beta \in \mathbb{R}$ (the subordinate principal conormal symbol $\sigma_M\sigma_\wedge(\mathcal{A}^\beta)(z,w)$ is then bijective for all $w \in \Gamma_{\frac{d}{2}-\beta}$, $z \in Z$). Choose a cut-off function $\omega(r)$ and form the operator

$$B^\gamma := \operatorname{op}_M^{\gamma-\frac{d-1}{2}}(a) + \omega \begin{pmatrix} \displaystyle\sum_{\substack{k+|\alpha|\le 2m \\ |\alpha|>0}} a_{k\alpha}(0,z)\big(-r\frac{\partial}{\partial r}\big)^k (r\zeta)^\alpha \\ \displaystyle\big(\mathrm{r}_{(S^{d-2})^\wedge}\sum_{\substack{k+|\beta|\le \mu_j \\ |\beta|>0}} b_{j,k\beta}(0,z)\big(-r\frac{\partial}{\partial r}\big)^k (r\zeta)^\beta\big)_{j=1,\dots,m}\end{pmatrix},$$

$$B^\gamma : \mathcal{H}^{t,(\gamma,\delta)} \to \tilde{\boldsymbol{\mathcal{H}}}^{t-2m,(r,\delta)}, \tag{4.7}$$

where $\mathcal{H}^{t,(\gamma,\delta)} := \omega_1\mathcal{H}^{t,\gamma}\big((S_+^{d-1})^\wedge\big) + (1-\omega_1)\mathcal{H}^{t,\delta}\big((S_+^{d-1})^\wedge\big)$ for weights $\gamma < \delta$ and a cut-off function ω_1,

$$\tilde{\boldsymbol{\mathcal{H}}}^{t-2m,(\gamma,\delta)} := \omega_1 \begin{pmatrix} \mathcal{H}^{t-2m,\gamma}\big((S_+^{d-1})^\wedge\big) \\ \oplus \\ \oplus_{j=1}^m \mathcal{H}^{t-\mu_j-\frac{1}{2},\gamma-\frac{1}{2}}\big((S^{d-2})^\wedge\big)\end{pmatrix} + (1-\omega_1)\begin{pmatrix} \mathcal{H}^{t-2m,\delta}\big(\big((S_+^{d-1})^\wedge\big) \\ \oplus \\ \oplus_{j=1}^m \mathcal{H}^{t-\mu_j-\frac{1}{2},\delta-\frac{1}{2}}\big((S^{d-2})^\wedge\big)\end{pmatrix}.$$

In a similar manner we consider the operator

$$B^\beta : \mathcal{H}^{t,(\beta,\delta)} \to \tilde{\boldsymbol{\mathcal{H}}}^{t-2m,(\beta,\delta)} \tag{4.8}$$

for another weight $\beta < \delta$.

For every $R > 0$ such that $\omega = 1$ on $[0,R)$ we have $K^\gamma|_{0<r<R} = B^\gamma|_{0<r<R}$, $K^\beta|_{0<r<R} = B^\beta|_{0<r<R}$, and $K^\gamma|_{R<r<\infty} = K^\beta|_{R<r<\infty}$, $B^\gamma|_{R<r<\infty} = B^\beta|_{R<r<\infty}$ (modulo some compact operators in the respective spaces). In the following proposition, without loss of generality, we assume $\gamma < \beta$.

Proposition 4.1.2. *Let $\gamma < \beta < \delta$ be arbitrary weights such that $\Gamma_{\frac{d}{2}-\gamma}$, $\Gamma_{\frac{d}{2}-\beta}$, and $\Gamma_{\frac{d}{2}-\delta}$ have no non-bijectivity points of* (4.4) *(for any fixed $z \in Z$ and sufficiently large t). Then the operators* (4.5)*,* (4.6)*,* (4.7) *and* (4.8) *are Fredholm, and we have*

$$\operatorname{ind} K^\gamma - \operatorname{ind} K^\beta = \operatorname{ind} B^\gamma - \operatorname{ind} B^\beta = \boldsymbol{n}(\beta,\gamma), \tag{4.9}$$

where $\boldsymbol{n}(\beta,\gamma)$ is the sum of null-multiplicities of the non-bijectivity points of $a(z,w)$ in the strip $\{w \in \mathbb{C} : \frac{d}{2}-\beta < \operatorname{Re} w < \frac{d}{2}-\gamma\}$ (in the sense of Gohberg and Sigal [12]*).*

Proof. The Fredholm property of K^γ and K^β is satisfied when $\sigma_\wedge(\mathcal{A}^\gamma)$ and $\sigma_\wedge(\mathcal{A}^\beta)$ are Fredholm. From the structure of these operators, cf. the formula (2.36), we know that the ellipticity conditions for $r > 0$ (up to $r = \infty$ in the sense of exit ellipticity) are satisfied. The Fredholm property is then equivalent to the bijectivity of $\sigma_M\sigma_\wedge(\mathcal{A}^\gamma)$ and $\sigma_M\sigma_\wedge(\mathcal{A}^\beta)$ on the lines $\Gamma_{\frac{d}{2}-\gamma}$ and $\Gamma_{\frac{d}{2}-\beta}$, respectively. Concerning the operators B^γ and B^β we are in the situation of [13], up to a transformation of the infinite (stretched) cone $\mathbb{R}_+ \times S_+^{d-1}$ to the infinite cylinder $\mathbb{R} \times S_+^{d-1}$. From that we know that when $a(z,w)$ has no non-bijectivity points on $\Gamma_{\frac{d}{2}-\gamma}$ and $\Gamma_{\frac{d}{2}-\delta}$ for $\gamma < \delta$ the operator

$$\mathrm{op}_M^{\gamma-\frac{d-1}{2}}(a) : \mathcal{H}^{t,(\gamma,\delta)} \to \widetilde{\boldsymbol{\mathcal{H}}}^{t-2m,(\gamma,\delta)}$$

is Fredholm, and we have $\mathrm{ind}\,\mathrm{op}_M^{\gamma-\frac{d-1}{2}}(a) = \boldsymbol{n}(\delta,\gamma)$. Since $B^\gamma = \mathrm{op}_M^{\gamma-\frac{d-1}{2}}(a)$ modulo a compact operator it follows that $\mathrm{ind}\,B^\gamma = \boldsymbol{n}(\delta,\gamma)$. Analogous arguments yield the Fredholm property of B^β and $\mathrm{ind}\,B^\beta = \boldsymbol{n}(\delta,\beta)$. This yields

$$\mathrm{ind}\,B^\gamma - \mathrm{ind}\,B^\beta = \boldsymbol{n}(\delta,\gamma) - \boldsymbol{n}(\delta,\beta) = \boldsymbol{n}(\beta,\gamma). \tag{4.10}$$

Because of the above mentioned compatibility conditions between the operators K^γ, K^β and B^γ, B^β over corresponding subregions of $\mathbb{R}_+ \times S_+^{d-1}$ and corresponding compatibilities of the respective spaces we can apply a relative index result of [22] which says

$$\mathrm{ind}\,K^\gamma - \mathrm{ind}\,K^\beta = \mathrm{ind}\,B^\gamma - \mathrm{ind}\,B^\beta. \tag{4.11}$$

Combining (4.10) and (4.11) gives us the relation (4.9). □

Corollary 4.1.3. (i) Let $\sigma_M\sigma_\wedge(\mathcal{A}^\gamma)(z,w)$ and $\sigma_M\sigma_\wedge(\mathcal{A}^\beta)(z,w)$ have no non-bijectivity points on the lines $\Gamma_{\frac{d}{2}-\gamma}$ and $\Gamma_{\frac{d}{2}-\beta}$, respectively. Then

$$\mathrm{ind}\,\sigma_\wedge(\mathcal{A}^\gamma)(z,\zeta) - \mathrm{ind}\,\sigma_\wedge(\mathcal{A}^\beta)(z,\zeta) = \boldsymbol{n}(\beta,\gamma). \tag{4.12}$$

(ii) If the weights γ and β both satisfy the condition (4.2), $\gamma < \beta$, then we have

$$\begin{aligned}\boldsymbol{n}(\beta,\gamma) = \{\iota(\beta-2m,d) + \sum_{j=1}^{m} \iota(\beta-\mu_j-\frac{1}{2}, d-1) - \iota(\beta,d)\}\\ - \{\iota(\gamma-2m,d) + \sum_{j=1}^{m} \iota(\gamma-\mu_j-\frac{1}{2}, d-1) - \iota(\gamma,d)\}.\end{aligned}$$

The consideration so far concerns index shifts for fixed $(z,\zeta) \in S^*Z$. If we associate the Fredholm family $\sigma_\wedge(\mathcal{A}^\gamma)(z,\zeta)$ and $\sigma_\wedge(\mathcal{A}^\beta)(z,\zeta)$ with the K-theoretic index elements

$$\mathrm{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}^\gamma),\ \ \mathrm{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}^\beta) \in K(S^*Z),$$

by Theorem 1.3.2 we then have

$$\mathrm{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}^\gamma) \in \pi^*K(Z) \Leftrightarrow \mathrm{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}^\beta) \in \pi^*K(Z). \tag{4.13}$$

4.2 Construction of edge conditions

Let $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ be an elliptic boundary problem on M. As we know, for every $\gamma \in \mathbb{R}$ we can realise $\mathcal{A}$ as a continuous operator

$$\mathcal{A}^\gamma : \mathcal{W}^{t,\gamma}(\mathbb{W}) \to \boldsymbol{\mathcal{W}}^{t-2m,\gamma-2m}(\mathbb{W}) \tag{4.14}$$

for every sufficiently large t, cf. the notation in the formula (1.22). As soon as γ satisfies the condition (4.2) we can complete $\mathcal{A}^\gamma$ to a block matrix operator $\mathfrak{A}^\gamma$

$$\mathfrak{A}^\gamma : \begin{pmatrix} \mathcal{W}^{t,\gamma}(\mathbb{W}) \\ \oplus \\ H^t(Z, \mathbb{C}^{\iota(s,d)}) \end{pmatrix} \longrightarrow \begin{matrix} \boldsymbol{\mathcal{W}}^{t-2m,\gamma-2m}(\mathbb{W}) \\ \oplus \\ \boldsymbol{H}^{t-2m}(Z, \mathbb{C}^{\boldsymbol{\iota}(\gamma-2m,d)}) \end{matrix}, \tag{4.15}$$

which is Fredholm and belongs to the edge calculus of boundary value problems on $\mathbb{W}$.

The case which is not automatically covered by this construction are small weights β. To construct edge conditions which complete (4.14) to a corresponding Fredholm operator $\mathfrak{A}^\beta$ we need two assumptions:
(i) The conormal symbol

$$\sigma_M \sigma_\wedge(\mathcal{A}^\beta)(z,w) : H^t(S_+^{d-1}) \to \boldsymbol{H}^{t-2m}(S_+^{d-1}) \tag{4.16}$$

has no non-bijectivity points on $\Gamma_{\frac{d}{2}-\beta}$ for all $z \in Z$;
(ii) we have

$$\mathrm{ind}_{S^*Z} \sigma_\wedge(\mathcal{A}^\beta) \in \pi^* K(Z). \tag{4.17}$$

Proposition 4.2.1. *If $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ is an elliptic boundary problem on M. Then we have*

$$\mathrm{ind}_{S^*Z} \sigma_\wedge(\mathcal{A}^\gamma) \in \pi^* K(Z)$$

for every $\gamma \in \mathbb{R}$ such that $\sigma_M \sigma_\wedge(\mathcal{A}^\gamma)(z,w)$ has no non-bijectivity points on $\Gamma_{\frac{d}{2}-\gamma}$ for all $z \in Z$.

Proof. For large γ as in (4.2) we know that

$$\mathrm{ind}_{S^*Z} \sigma_\wedge(\mathcal{A}^\gamma) = [\mathbb{C}^{j_+}] - [\mathbb{C}^{j_-}],$$

for $j_+ := \iota(\gamma, d)$, $j_- := \iota(\gamma - 2m, d) + \sum_{j=1}^m \iota(\gamma - \mu_j - \frac{1}{2}, d-1)$ (here $[\mathbb{C}^j]$ means the equivalence class in $K(Z)$ generated by the trivial bundle $Z \times \mathbb{C}^j$). This shows that (4.17) holds for those γ. Then it suffices to apply the relation (4.13). □

If $\gamma \in \mathbb{R}$ is a weight satisfying the condition (4.2) we explicitly know the number of additional edge conditions such that the corresponding edge problem (4.15) is Fredholm. For small γ (for instance, $\gamma < 0$) we do not have any information of that kind, but our relative index result gives us the following theorem:

Theorem 4.2.2. *Let $\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix}$ be an elliptic boundary value problem on M, and let $\beta \in \mathbb{R}$ be an arbitrary weight such that the conormal symbol (4.16) has no non-bijectivity points on $\Gamma_{\frac{d}{2}-\beta}$ for all $z \in Z$. Then the operator $\mathcal{A}^\beta : \mathcal{W}^{t,\beta}(\mathbb{W}) \to \boldsymbol{\mathcal{W}}^{t-2m,\beta-2m}(\mathbb{W})$ can be completed by additional edge conditions to a Fredholm operator*

$$\mathfrak{A}^\beta : \begin{matrix} \mathcal{W}^{t,\beta}(\mathbb{W}) \\ \oplus \\ H^t(Z, J_-) \end{matrix} \longrightarrow \begin{matrix} \boldsymbol{\mathcal{W}}^{t-2m,\beta-2m}(\mathbb{W}) \\ \oplus \\ H^{t-2m}(Z, J_+) \end{matrix}$$

for suitable vector bundles $J_\pm$ over Z, where the fibre dimensions $j_\pm$ of $J_\pm$ satisfy the relation

$$j_+ - j_- = \{\iota(\gamma - 2m, d) + \sum_{j=1}^m \iota(\gamma - \mu_j - \frac{1}{2}, d-1) - \iota(\gamma, d)\} - \boldsymbol{n}(\beta, \gamma) \tag{4.18}$$

for any $\gamma \in \mathbb{R}$ as in (4.2) (the right hand side of (4.18) is independent of γ).

Proof. By assumption the conormal symbol (4.16) has no non-bijectivity points on $\Gamma_{\frac{d}{2}-\beta}$ for all $z \in Z$. Thus

$$\sigma_\wedge(\mathcal{A}^\beta)(z,\zeta) : \mathcal{K}^{t,\beta}\big((S_+^{d-1})^\wedge\big) \to \mathcal{K}^{t-2m,\beta-2m}\big((S_+^{d-1})^\wedge\big) \tag{4.19}$$

is a family of Fredholm operators. The main step of the construction of $\mathfrak{A}^\beta$ is to fill up (4.19) to a family of isomorphisms. By virtue of (4.13) there exist elements $J_\pm \in \mathrm{Vect}(Z)$ such that

$$\mathrm{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}^\beta) = [J_+] - [J_-] \in \pi^* K(Z).$$

Applying Remark 1.3.1 we find a family of isomorphisms

$$\sigma_\wedge(\mathfrak{A}^\beta) = \begin{pmatrix} \sigma_\wedge(\mathcal{A}^\beta) & \sigma_\wedge(\mathcal{K}^\beta) \\ \sigma_\wedge(\mathcal{T}^\beta) & \sigma_\wedge(\mathcal{Q}^\beta) \end{pmatrix}(z,\zeta) : \begin{matrix} \mathcal{K}^{t,\beta}\big((S_+^{d-1})^\wedge\big) \\ \oplus \\ J_{-,z} \end{matrix} \to \begin{matrix} \mathcal{K}^{t-2m,\beta-2m}\big((S_+^{d-1})^\wedge\big) \\ \oplus \\ J_{+,z} \end{matrix}$$

smoothly depending on $(z,\zeta) \in S^*(S^{d-2})$. In order to verify (4.18) we first note that the bijectivity of $\sigma_\wedge(\mathfrak{A}^\beta)$ entails the relation $\mathrm{ind}\,\sigma_\wedge(\mathcal{A}^\beta)(z,\zeta) = j_+ - j_-$. Moreover, we have a relation for $\mathrm{ind}\,\sigma_\wedge(\mathcal{A}^\gamma)(z,\zeta)$ by (2.33) for $s := \gamma$. Then (4.18) is a consequence of (4.12). $\square$

4.3 Edge parametrices

Ellipticity for edge operators $\mathfrak{A}$ has been studied for the case of differential operators $\mathcal{A}$ in the upper left corner and differential boundary conditions (both edge-degenerate). In order to show the Fredholm property from the ellipticity with respect to $(\sigma_\psi, \sigma_\partial, \sigma_\wedge)$ we should construct parametrices in the pseudo-differential edge calculus. In this section we outline the most important steps of that construction. The general background will be Boutet de Monvel's calculus of pseudo-differental boundary value problems with the transmission property at the boundary.

First recall that for every (not necessary compact) C^∞ manifold X with boundary there is the space $\mathcal{B}^{\mu,d}(X;\mathbb{R}^l)$ of all parameter-dependent boundary value problems of Boutet de Monvel's calculus, of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}$. The elements are 2×2 block matrices of operators

$$\mathcal{A}(\lambda) : \begin{matrix} H^s_{\mathrm{comp}}(X) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\partial X, G_-) \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}_{\mathrm{loc}}(X) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}_{\mathrm{loc}}(\partial X, G_+) \end{matrix} \tag{4.20}$$

for (smooth complex) vector bundles $G_\pm$ over ∂X (in the first component we take, for simplicity, scalar operators). Here $H^s_{\mathrm{comp}}(X) := H^s_{\mathrm{comp}}(2X)|_{\mathrm{int}X}$, and, similarly, with 'loc', while 'comp' and 'loc' in the second component of the expression (4.20) is used in the standard meaning. In (4.20) we assume $s > -\frac{1}{2}$.

Let us first give a definition of $\mathcal{B}^{-\infty,d}(X;\mathbb{R}^l)$, the parameter-dependent smoothing operators of type $d \in \mathbb{N}$. For $d = 0$ and $l = 0$ this space is defined to be the set of all 2×2 block matrix $\mathcal{A}$ that are (for the case of trivial $G_\pm$ of fibre dimension 1) integral operators with kernels in $C^\infty(X \times X)$, $C^\infty(X \times \partial X)$, $C^\infty(\partial X \times X)$ and $C^\infty(\partial X \times \partial X)$, respectively (the generalisation to non-trivial $G_\pm$ is straightforward). The space $\mathcal{B}^{-\infty,0}(X)$ then has a Fréchet topology, and we set $\mathcal{B}^{-\infty,0}(X;\mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, \mathcal{B}^{-\infty,0}(X))$. Moreover, $\mathcal{B}^{-\infty,d}(X;\mathbb{R}^l)$ is the space of all operator families of the form

$$\mathcal{C}(\lambda) = \mathcal{C}_0(\lambda) + \sum_{j=1}^{d} \mathcal{C}_j(\lambda)\mathrm{diag}(D^j, 0)$$

for elements $\mathcal{C}_j(\lambda) \in \mathcal{B}^{-\infty,0}(X;\mathbb{R}^l)$ and any differential operator D of first order on X that is equal to ∂_{x_n} in a collar neighbourhood of ∂X.

The space $\mathcal{B}^{\mu,d}(X;\mathbb{R}^l)$ of parameter-dependent pseudo-differential boundary value problems on X with the transmission property of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}$ is defined to be the set of all

$$\mathcal{A}(\lambda) = \operatorname{diag}(A(\lambda), 0) + \mathcal{G}(\lambda) + \mathcal{C}(\lambda) \tag{4.21}$$

for arbitrary $\mathcal{C}(\lambda) \in \mathcal{B}^{-\infty,d}(X;\mathbb{R}^l)$, a Green operator family $\mathcal{G}(\lambda)$ (cf. the definition below) and $A(\lambda) = \mathrm{r}^+\tilde{A}(\lambda)\mathrm{e}^+$ for an element $\tilde{A}(\lambda) \in L^\mu_{\mathrm{cl}}(2X;\mathbb{R}^l)$ that has the transmission property at the boundary. Here e^+ is the operator of extension by zero from $\mathrm{int}X$ to the double $2X$, and r^+ the restriction from $2X$ to $\mathrm{int}X$. Concerning the notation $L^\mu_{\mathrm{cl}}(M;\mathbb{R}^l)$, cf. Section 1.1. Finally, $\mathcal{G}(\lambda)$ is locally near the boundary in coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ of the form $\mathrm{Op}_{x'}(g)(\lambda)$ for a $g(x', \xi', \lambda)$ of Green symbol of order μ and type $d \in \mathbb{N}$. To recall the definition, a Green symbol of order μ and type 0 is of the form

$$g_0(x', \xi', \lambda) = \operatorname{diag}(1, \langle \xi', \lambda \rangle^{\frac{1}{2}})\tilde{g}_0(x', \xi', \lambda)\operatorname{diag}(1, \langle \xi', \lambda \rangle^{-\frac{1}{2}})$$

for a $\tilde{g}_0(x', \xi', \lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^{n-1+l}_{\xi',\lambda}; L^2(\mathbb{R}_+) \oplus \mathbb{C}^{g_-}; \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{g_+})$, such that the pointwise formal adjoint $\tilde{g}_0^*(x', \xi', \lambda)$ is a symbol of the same kind with interchanged $g_\pm$. A Green symbol of order μ and type $d \in \mathbb{N}$ is then a sum

$$g(x', \xi', \lambda) = g_0(x', \xi', \lambda) + \sum_{j=1}^{d} g_j(x', \xi', \lambda)\operatorname{diag}(\frac{\partial^j}{\partial x_n^j}, 0)$$

for Green symbols $g_j(x', \xi', \lambda)$ of order $\mu - j$ and type 0.

Remark 4.3.1. *The space of all $\mathcal{A}(\lambda) \in \mathcal{B}^{\mu,d}(X;\mathbb{R}^l)$ for fixed $G_\pm$ is a Fréchet space in a natural way. More details on Boutet de Monvel's calculus of boundary value problems may be found in* [5, 31].

Similarly as in boundary value problems for differential operators, cf. Section 1.2, every $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\mathbb{R}^l)$ has a pair of parameter-dependent principal symbols

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})). \tag{4.22}$$

We only need this here for $l = 0$. In particular, if we replace X by $\mathbb{W}_{\mathrm{reg}}$, we also have a pair of stretched symbols locally near $\mathbb{W}_{\mathrm{sing}}$, namely, $\tilde{\sigma}(\mathcal{A}) = (\tilde{\sigma}_\psi(\mathcal{A}), \tilde{\sigma}_\partial(\mathcal{A}))$, cf. (1.12) and (1.14). There is then a straightforward analogue of Definition 1.2.2 for $\mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}})$.

Theorem 4.3.2. *There is a (so-called) Mellin quantization*

$$C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\zeta}})) \to C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\mathbb{C} \times \mathbb{R}^q_{\tilde{\zeta}})),\ \ \tilde{p}(r, z, \tilde{\rho}, \tilde{\zeta}) \mapsto \tilde{h}(r, z, w, \tilde{\zeta})$$

such that $\mathrm{op}_r(p)(z, \zeta) = \mathrm{op}^\gamma_M(h)(z, \zeta) \mod C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge;\mathbb{R}^q_\zeta))$ *for*

$$p(r, z, \rho, \zeta) := \tilde{p}(r, z, r\rho, r\zeta),\ \ h(r, z, w, \zeta) := \tilde{h}(r, z, w, r\zeta).$$

In addition, setting

$$p_0(r, z, \rho, \zeta) := \tilde{p}(0, z, r\rho, r\zeta),\ \ h_0(r, z, w, \zeta) := \tilde{h}(0, z, w, r\zeta),$$

we also have $\mathrm{op}_r(p_0)(z, \zeta) = \mathrm{op}^\gamma_M(h_0)(z, \zeta) \mod C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge;\mathbb{R}^q_\zeta))$.

A proof of Theorem 4.3.2 is given in [14].

Let W be a compact manifold with edge Z and boundary, $V = \partial(W \setminus Z) \cup Z$, and $\mathbb{W}$ and $\mathbb{V}$ be the associated stretched manifolds, cf. the notation in Remark 1.1.1. The parametrices which we want to express belong to the edge algebra $\mathfrak{Y}^{\mu,d}(\mathbb{W})$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, of boundary value problems on $\mathbb{W}$. Set $\boldsymbol{\mathcal{W}}^{s,\gamma}(\mathbb{W};G) := \mathcal{W}^{s,\gamma}(\mathbb{W}) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{V},G)$ for any vector bundle G on $\mathbb{V}$. Operators in the edge algebra are 3×3-block matrices

$$\mathfrak{A} = (\mathfrak{A}_{ij})_{i,j=1,2,3} : \begin{array}{c} \boldsymbol{\mathcal{W}}^{s,\gamma}(\mathbb{W};G_-) \\ \oplus \\ H^s(Z,J_-) \end{array} \longrightarrow \begin{array}{c} \boldsymbol{\mathcal{W}}^{s-\mu,\gamma-\mu}(\mathbb{W};G_+) \\ \oplus \\ H^{s-\mu}(Z,J_+) \end{array} \tag{4.23}$$

with vector bundles $G_\pm$ on $\mathbb{V}$ and $J_\pm$ on Z, where $(\mathfrak{A}_{ij})_{i,j=1,2} \in \mathcal{B}^{\mu,d}(\mathbb{W}_{\text{reg}})$. The continuity of (4.23) will be a consequence of Definition 4.3.3 below.

Global smoothing operators $\mathfrak{C}$ in the edge calculus for type $d = 0$ are characterised by the mapping property

$$\mathfrak{C} : \begin{array}{c} \boldsymbol{\mathcal{W}}^{s,\gamma}(\mathbb{W};G_-) \\ \oplus \\ H^s(Z,J_-) \end{array} \longrightarrow \begin{array}{c} \boldsymbol{\mathcal{W}}^{\infty,\gamma-\mu+\varepsilon}(\mathbb{W};G_+) \\ \oplus \\ H^{\infty}(Z,J_+) \end{array}$$

for some $\varepsilon > 0$, together with a similar mapping property for the formal adjoint $\mathfrak{C}^*$, and an $\varepsilon(\mathfrak{C}) > 0$. For arbitrary type $d \in \mathbb{N}$ the global smoothing operators have the form

$$\mathfrak{C} = \mathfrak{C}_0 + \sum_{j=1}^{d} \mathfrak{C}_j \operatorname{diag}(D^j, 0, 0) \tag{4.24}$$

with D being of similar meaning as in Boutet de Monvel's calculus on a smooth manifold with boundary, and $\mathfrak{C}_j$ global smoothing operators of type 0. Let $\mathfrak{Y}^{-\infty,d}(\mathbb{W})$ denote the space of all such operators $\mathfrak{C}$.

Let us fix cut-off functions $\omega(r)$, $\omega_0(r)$, $\omega_1(r)$, $\sigma(r)$, $\sigma_0(r)$, where $\omega_0 \equiv 1$ on $\operatorname{supp}\omega$, $\omega \equiv 1$ on $\operatorname{supp}\omega_1$, and form the 2×2-block matrix operator

$$\begin{aligned} a(z,\zeta) := r^{-\mu}\sigma(r)\{&\omega(r[\zeta]) \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(z,\zeta)\omega_0(r[\zeta]) \\ &+ (1-\omega(r[\zeta])) \operatorname{op}_r(p)(z,\zeta)(1-\omega_1(r[\zeta]))\}\sigma_0(r), \end{aligned} \tag{4.25}$$

where p is related to h via the Mellin quantisation. We then have

$$a(z,\zeta) = \operatorname{diag}(1, \langle\zeta\rangle^{\frac{1}{2}})\tilde{a}(z,\zeta)\operatorname{diag}(1, \langle\zeta\rangle^{-\frac{1}{2}})$$

for a symbol $\tilde{a}(z,\zeta) \in S^{\mu}_{\text{cl}}(\Omega \times \mathbb{R}^q; \boldsymbol{\mathcal{K}}^{s,\gamma}(X^\wedge;G_-), \boldsymbol{\mathcal{K}}^{s-\mu,\gamma-\mu}(X^\wedge;G_+))$; the spaces are defined by

$$\boldsymbol{\mathcal{K}}^{s,\gamma}(X^\wedge;G) := \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge;G) \end{array}$$

for any vector bundle G over $X^\wedge$ with the group action $\operatorname{diag}(\{\kappa_\delta^{(n)}\}_{\delta\in\mathbb{R}_+}, \{\kappa_\delta^{(n-1)}\}_{\delta\in\mathbb{R}_+})$. Moreover, let $\mathcal{M}^{-\infty,d}(X;\Gamma_\beta)$ denote the space of all functions $f(w)$ that are holomorphic in $\beta - \varepsilon < \operatorname{Re} w < \beta + \varepsilon$ taking values in the space $\mathcal{B}^{-\infty,d}(X)$ for some $\varepsilon(f) > 0$, such that $f(\delta + i\rho) \in \mathcal{S}(\mathbb{R}_\rho, \mathcal{B}^{-\infty,d}(X))$ for every $\beta - \varepsilon < \delta < \beta + \varepsilon$, uniformly in compact subintervals. Then, for any $f(z,w) \in C^\infty(\Omega, \mathcal{M}^{-\infty,d}(X;\Gamma_{\frac{n+1}{2}-\gamma}))$ we form the operator family

$$m(z,\zeta) := r^{-\mu}\omega(r[\zeta]) \operatorname{op}_M^{\gamma-\frac{n}{2}}(f)(z)\omega_0(r[\zeta]) \tag{4.26}$$

for any fixed choice of cut-off functions ω, ω_0. If we assume f to take values in smoothing operators of Boutet de Monvel's calculus referring to bundles $G_\pm$ on the boundary, we have $m(z,\zeta) = \operatorname{diag}(1, \langle\zeta\rangle^{\frac{1}{2}})\tilde{m}(z,\zeta)\operatorname{diag}(1, \langle\zeta\rangle^{-\frac{1}{2}})$ for a symbol

$$\tilde{m}(z,\zeta) \in S^{\mu}_{\text{cl}}(\Omega \times \mathbb{R}^q; \boldsymbol{\mathcal{K}}^{s,\gamma}(X^\wedge;G_-), \boldsymbol{\mathcal{K}}^{\infty,\gamma-\mu}(X^\wedge;G_+)).$$

It remains to explain the contribution of trace (and potential) operators to the edge calculus on the symbolic level. For parametrices it turns out that (similarly as in boundary value problems) we also need an analogue of Green operators. These can be formulated together with the trace and potential operators at the same time. For the definition we set $\boldsymbol{\mathcal{S}}^\gamma(X^\wedge;G) := \mathcal{S}^\gamma(X^\wedge)\oplus\mathcal{S}^{\gamma-\frac{1}{2}}((\partial X)^\wedge,G)$. A Green symbol $\mathfrak{g}(z,\zeta)$ of order $\mu\in\mathbb{R}$ and type 0 is an operator-valued symbol $\mathfrak{g}(z,\zeta)=\mathrm{diag}(1,\langle\zeta\rangle^{\frac{1}{2}},1)\tilde{\mathfrak{g}}(z,\zeta)\mathrm{diag}(1,\langle\zeta\rangle^{-\frac{1}{2}},1)$ for a symbol $\tilde{\mathfrak{g}}(z,\zeta)$ that belongs to $S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;E,\tilde{E})$ with spaces $E:=\boldsymbol{\mathcal{K}}^{s,\gamma}(X^\wedge;G_-)\oplus\mathbb{C}^{j_-}$ for arbitrary $s>-\frac{1}{2}$, and $\tilde{E}:=\boldsymbol{\mathcal{S}}^{\gamma-\mu+\varepsilon}(X^\wedge;G_+)\oplus\mathbb{C}^{j_+}$ for some $\varepsilon(\tilde{\mathfrak{g}})>0$, such that the pointwise formal adjoint $\mathfrak{g}^*(z,\zeta)$ is of analogous structure. Moreover, a Green symbol $\mathfrak{g}(z,\zeta)$ of order $\mu\in\mathbb{R}$ and type $d\in\mathbb{N}$ is defined as

$$\mathfrak{g}(z,\zeta)=\mathfrak{g}_0(z,\zeta)+\sum_{j=1}^{d}\mathfrak{g}_j(z,\zeta)\mathrm{diag}(D^j,0,0) \tag{4.27}$$

for Green symbols $\mathfrak{g}_j(z,\zeta)$ of order μ and type 0, where D is of similar meaning as in (4.24).

Definition 4.3.3. Let W be a compact manifold with edge Z and boundary, and $\mathbb{W}$ be the associated stretched manifold. A 3×3-block matrix operator $\mathfrak{A}$ is said to belong to the space $\mathfrak{Y}^{\mu,d}(\mathbb{W})$ of edge boundary value problems of order $\mu\in\mathbb{Z}$ and type $d\in\mathbb{N}$ if $\mathfrak{A}$ is modulo an operator in $\mathfrak{Y}^{-\infty,d}(\mathbb{W})$ of the following structure:
(i) $(\mathfrak{A}_{ij})_{i,j=1,2}\in\mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}})$;
(ii) locally near $\mathbb{W}_{\mathrm{sing}}$ in the splitting of variables into $(r,x,z)\in X^\wedge\times\Omega$ we have $\mathfrak{A}=\mathrm{Op}_z(\mathfrak{a})$, where $\mathfrak{a}(z,\zeta)$ is an amplitude function of order μ of the form

$$\mathfrak{a}(z,\zeta)=\begin{pmatrix}a(z,\zeta)+m(z,\zeta) & 0\\ 0 & 0\end{pmatrix}+\mathfrak{g}(z,\zeta)$$

for symbols (4.25), (4.26) and (4.27), respectively.

Operators $\mathfrak{A}\in\mathfrak{Y}^{\mu,d}(\mathbb{W})$ induce continuous operators (4.23) for $s\in\mathbb{R}$, $s>d-\frac{1}{2}$. This is an easy consequence of the continuity of local operators with amplitude functions $\mathfrak{a}(z,\zeta)$ in edge Sobolev spaces (together with the continuity of $(\mathfrak{A}_{ij})_{i,j=1,2}$ in standard (local) Sobolev spaces on $\mathbb{W}_{\mathrm{reg}}$). Modulo compact operators an operator $\mathfrak{A}\in\mathfrak{Y}^{\mu,d}(\mathbb{W})$ is determined by its principal symbol

$$\sigma(\mathfrak{A})=\big(\sigma_\psi(\mathfrak{A}),\sigma_\partial(\mathfrak{A}),\sigma_\wedge(\mathfrak{A})\big),$$

similarly defined as (1.11). The interior symbol $\sigma_\psi(\mathfrak{A})$ is nothing other than the homogeneous symbol of $\mathfrak{A}_{11}\in L^\mu_{\mathrm{cl}}(\mathrm{int}\mathbb{W}_{\mathrm{reg}})$. In the local splitting of variables (r,x,z) near $\mathbb{W}_{\mathrm{sing}}$ we can write

$$\sigma_\psi(\mathfrak{A})(r,x,z,\rho,\xi,\zeta)=r^{-\mu}\tilde{\sigma}_\psi(\mathfrak{A})(r,x,z,r\rho,\xi,r\zeta)$$

for the compressed interior symbol $\tilde{\sigma}_\psi(\mathfrak{A})(r,x,z,\tilde{\rho},\xi,\tilde{\zeta})$ which is smooth up to $r=0$. Moreover, from $(\mathfrak{A}_{ij})_{i,j=1,2}\in\mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}})$ we have the homogeneous boundary symbol

$$\sigma_\partial(\mathfrak{A}):=\sigma_\partial((\mathfrak{A}_{ij})_{i,j=1,2}).$$

In the local variables (r,x',z) near $\mathbb{V}_{\mathrm{sing}}$ we have

$$\sigma_\partial(\mathfrak{A})(r,x',z,\rho,\xi',\zeta)=r^{-\mu}\tilde{\sigma}_\partial(\mathfrak{A})(r,x',z,r\rho,\xi',r\zeta)$$

for the compressed boundary symbol $\tilde{\sigma}_\partial(\mathfrak{A})(r,x',z,\tilde{\rho},\xi',\tilde{\zeta})$ which is smooth up to $r=0$.

Finally, we have the homogeneous principal edge symbol

$$\sigma_\wedge(\mathfrak{A})(z,\zeta):\begin{array}{c}\boldsymbol{\mathcal{K}}^{s,\gamma}(X^\wedge;G_-)\\ \oplus\\ J_{-,z}\end{array}\longrightarrow\begin{array}{c}\boldsymbol{\mathcal{K}}^{s-\mu,\gamma-\mu}(X^\wedge;G_+)\\ \oplus\\ J_{+,z}\end{array} \tag{4.28}$$

for $(z,\zeta) \in T^*Z \setminus 0$. Here

$$\sigma_\wedge(\mathfrak{A})(z,\zeta) := \begin{pmatrix} \sigma_\wedge(a+m)(z,\zeta) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\wedge(\mathfrak{g})(z,\zeta)$$

for

$$\sigma_\wedge(a)(z,\zeta) = r^{-\mu}\{\omega(r|\zeta|)\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(z,\zeta)\omega_0(r|\zeta|) + (1-\omega(r|\zeta|))\,\mathrm{op}_r(p_0)(z,\zeta)(1-\omega_1(r|\zeta|))\},$$

$$\sigma_\wedge(m)(z,\zeta) = r^{-\mu}\omega(r|\zeta|)\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(z)\omega_0(r|\zeta|),$$

and $\sigma_\wedge(\mathfrak{g})(z,\zeta)$ is the homogeneous principal part of $\mathfrak{g}$ as a classical operator-valued symbol.

Remark 4.3.4. *Let $\mathfrak{A} \in \mathfrak{Y}^{\mu,d}(\mathbb{W})$, $\mathfrak{B} \in \mathfrak{Y}^{\nu,e}(\mathbb{W})$ and assume that the bundles in the range of $\mathfrak{B}$ fit to the ones in the domain of $\mathfrak{A}$. Then we have $\mathfrak{A}\mathfrak{B} \in \mathfrak{Y}^{\mu+\nu,h}(\mathbb{W})$ for $h = \max(d+\nu, e)$ and $\sigma(\mathfrak{A}\mathfrak{B}) = \sigma(\mathfrak{A})\sigma(\mathfrak{B})$ with componentwise composition.*

Definition 4.3.5. An $\mathfrak{A} \in \mathfrak{Y}^{\mu,d}(\mathbb{W})$ is said to be elliptic if
(i) $\mathcal{A} := (\mathfrak{A}_{ij})_{i,j=1,2} \in \mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}})$ is elliptic in the sense of Definition 1.2.2 (here in the corresponding pseudo-differential set-up, cf. the notation (4.22));
(ii) (4.28) is a family of isomorphisms for $(z,\zeta) \in T^*Z \setminus 0$ and $s > \max(\mu,d) - \frac{1}{2}$.

Theorem 4.3.6. *The ellipticity of an operator $\mathfrak{A} \in \mathfrak{Y}^{\mu,d}(\mathbb{W})$ entails the existence of a parametrix $\mathfrak{P} \in \mathfrak{Y}^{-\mu,(d-\mu)^+}(\mathbb{W})$ such that $\mathfrak{A}\mathfrak{P}-\mathfrak{I}$ and $\mathfrak{P}\mathfrak{A}-\mathfrak{I}$ are smoothing and of type $(d-\mu)^+$ and $\max(\mu,d)$, respectively (with $\mathfrak{I}$ being the respective identity operators). Moreover, an elliptic operator $\mathfrak{A}$ induces a Fredholm operator (4.28) for every $s > \max(\mu,d) - \frac{1}{2}$ (here $\nu^+ := \max(\nu,0)$ for any $\nu \in \mathbb{R}$).*

The existence of a parametrix can be proved in a similar manner as a corresponding result in [14] which concerns subclasses with (discrete or continuous) asymptotics.

4.4 Concluding remarks

In the general calculus of the preceding section we made a few simplifying technical assumptions on the orders of the operators referring to $\partial\mathbb{W}_{\mathrm{reg}}$ and the edge Z. They are not really essential, because there are order and weight reducing isomorphisms within the edge calculus on the boundary. First observe that our space $\mathfrak{Y}^{\mu,d}(\mathbb{W})$ of edge operators contains a subspace of 2×2-lower right corners which may be identified with the space $\mathfrak{Y}^{\mu}(\mathbb{V})$ of edge operators of order μ on $\mathbb{V}$ (recall that $\mathbb{V}$ is the stretched manifold belonging to $V = \partial(W \setminus Z) \cup Z$ which is a (compact) manifold without boundary and with edge Z). Ellipticity in $\mathfrak{Y}^{\mu}(\mathbb{V})$ is induced by the one on $\mathfrak{Y}^{\mu,d}(\mathbb{W})$.

In order to reduce orders it suffices to employ the following result:
For every ν, $\gamma \in \mathbb{R}$ there exists an elliptic element $\mathfrak{R}^\nu \in \mathfrak{Y}^{\nu}(\mathbb{V})$ which induces isomorphisms

$$\mathfrak{R}^\nu : \mathcal{W}^{s,\gamma}(\mathbb{V},G) \to \mathcal{W}^{s-\nu,\gamma-\nu}(\mathbb{V},G) \tag{4.29}$$

for every vector bundle G over $\mathbb{V}$. Then, in order to reduce our operators $\mathfrak{A}^\gamma$ of the kind (4.15) to the set-up of Definition 4.3.3 it suffices to compose $\mathfrak{A}^\gamma$ with diagonal matrices of operators (4.29) for suitable ν, plus (in the $\oplus$ sense) the identity in the upper left corner, and diagonal matrices of standard reductions of orders on Z in the space of classical pseudo-differential operators on a C^∞ manifold. Here we also use Remark 4.3.4.

Theorem 4.4.1. *An elliptic operator of the form $\mathfrak{A}^\beta$, cf.* Theorem 4.2.2*, has a parametrix in the edge pseudo-differential calculus on $\mathbb{W}$, modified by reductions of orders on $\mathbb{V}$ and Z, respectively.*

This is an immediate consequence of Theorem 4.3.6.

Remark 4.4.2. Theorem 4.4.1 *can be regarded as a result on Mellin quantisation of an element of Boutet de Monvel's calculus on a smooth manifold with boundary, relative to a 'fictitious' edge which is an embedded compact* C^∞ *manifold* Z *on the boundary. This gives us wide classes of examples of elliptic edge boundary value problems in weighted edge Sobolev spaces, where ellipticity refers to all three principal symbolic components, including the evaluation of the* (*difference of the*) *number of trace and potential edge conditions.*

Let $\mathcal{Y}^{\mu,d}(\mathbb{W})$ denote the space of all 2×2 upper left corners of operators in $\mathfrak{Y}^{\mu,d}(\mathbb{W})$; by definition we then have $\mathcal{Y}^{\mu,d}(\mathbb{W}) \subset \mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}})$. In $\mathcal{Y}^{\mu,d}(\mathbb{W})$ we also have the three components $(\sigma_\psi, \sigma_\partial, \sigma_\wedge)$.

Theorem 4.4.3. *Let* $\mathfrak{A}^\gamma \in \mathcal{Y}^{\mu,d}(\mathbb{W})$ *be elliptic with respect to* $(\sigma_\psi, \sigma_\partial)$ (*which induces the corresponding conditions with respect to* $(\tilde{\sigma}_\psi, \tilde{\sigma}_\partial)$ *near* $\mathbb{W}_{\mathrm{sing}}$). *Then for every* $z \in Z$ *there is a countable set* $D(z) \subset \mathbb{C}$ *as in* Theorem 1.2.3 *such that*

$$\sigma_\wedge(\mathcal{A}^\gamma)(z,\zeta) : \mathcal{K}^{s,\gamma}(X^\wedge; G_-) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge; G_+)$$

is a Fredholm operator for every $\zeta \neq 0$ *and every* $\gamma \in \mathbb{R}$ *satisfying* (1.18).

This result is a direct generalisation of Theorem 1.2.3 to the operator classes of Definition 4.3.3.

Remark 4.4.4. Theorem 1.3.2 *can be generalised to the space* $\mathcal{Y}^{\mu,d}(\mathbb{W})$. *More precisely, if* $\gamma \in \mathbb{R}$ *is a weight such that* (1.18) *holds for all* $z \in Z$ *we can form an index element* $\mathrm{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}^\gamma) \in K(S^*Z)$. *Then if* $\mathcal{A}^\gamma$ *is an element of* $\mathcal{Y}^{\mu,d}(\mathbb{W})$ *with respect to different weights* γ_0, γ_1 (*which is not always the case*) *then we can prove again the relation* (1.26). *The property* (1.24) *is just a necessary and sufficient condition for being able to complete* $\mathcal{A}^\gamma$ *by additional edge conditions to an elliptic element* $\mathfrak{A}^\gamma \in \mathfrak{Y}^{\mu,d}(\mathbb{W})$.

Remark 4.4.5. *If* (1.26) *is satisfied the index elements* $\mathrm{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}^{\gamma_i})$ *for* $i = 0, 1$ *are related to each other by a similar relative index relation as formulated in* Corollary 4.1.3 *in parameter-dependent form. This is related to the parameter-dependent spectral flow of the conormal symbol, similarly as in* [20].

Remark 4.4.6. *The general edge calculus for boundary value problem, introduced in* Definition 4.3.3 *is more general than that treated in* [14] *which is a subcalculus with continuous* (*and also discrete*) *asymptotics near the edges. Let us also note that the calculus admits a generalisation to the case that* $\partial\mathbb{W}$ *is not necessarily a trivial* X*-bundle, similarly as in* [9] *for the case without boundary.*

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