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A COVERING PROPERTY OF THE RIEMANN ZETA-FUNCTION

P. M. GAUTHIER AND N. TARKHANOV

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ABSTRACT. For each compact subset K of the complex plane $\mathbb{C}$ which does not surround zero, the Riemann surface S_ζ of the Riemann zeta function restricted to the critical half-strip $0 < \Re s < 1/2$ contains infinitely many schlicht copies of K lying 'over' K. If S_ζ also contains at least one such copy, for some K which surrounds zero, then the Riemann hypothesis fails.

Contents

1. An universality theorem

The critical strip for the Riemann zeta-function is the strip $0 < \Re s < 1$. Because of the symmetry properties of the zeta-function, one often restricts one's attention to the strip $S = \{s \in \mathbb{C} : 1/2 < \Re s < 1\}$, which we shall call the critical half-strip. If f is a function defined on S and τ is a real number, we denote by f_τ the vertical translate of f given by $f_\tau(s) = f(s+i\tau)$. Let K be a closed disc centered at $3/4$ and contained in the half-strip S. Voronin has shown that the Riemann zeta-function has the following remarkable universality property: the vertical translates of ζ are dense in the space of functions holomorphic on K which have no zeros.

Recently [1] it has been shown that most holomorphic functions have a similar universality property (without the restriction on zeros). However, it is difficult to provide other explicit examples than the Riemann zeta function.

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It is not known whether one can remove the restrictions on zeros in Voronin's theorem, but doing so would negate the Riemann hypothesis. Indeed, suppose the function $s - 3/4$ could be uniformly approximated on the disc K by vertical translates ζ_τ of the Riemann zeta-function. Then, by Rouché's theorem, some such translate would have a zero in K. Thus, the zeta-function would have a zero in the translate $K_\tau = K + i\tau$ of K. Since K_τ is disjoint from the critical line $\Re s = 1/2$, this would violate the Riemann hypothesis.

The following universality theorem of Bagchi improves Voronin's theorem.

Theorem 1.1 (Bagchi)**.** *For each compact subset K of the critical half-strip S with $\mathbb{C} \setminus K$ connected, for each function f holomorphic on K and having no zeros, and for each $\varepsilon > 0$, there is a vertical translate ζ_τ of the Riemann zeta-function, such that $|\zeta_\tau - f| < \varepsilon$ on K.*

For references to these and other results regarding the universality property of the Riemann zeta-function, we refer to [4].

The following universality theorem is essentially a reformulation of the theorem of Bagchi.

Theorem 1.2. *For each function f holomorphic on the critical half-strip S and having no zeros, there is a sequence of vertical translates of the Riemann zeta-function which converges to f. That is, there is a sequence $\{\tau_j\}$ of real numbers, such that $\zeta_{\tau_j} \to f$ uniformly on compact subsets of S.*

Of course, this follows immediately from Bagchi's theorem, since each compact subset of S is contained in one whose complement is connected. The converse implication also holds. That is, Bagchi's theorem follows from Theorem 1.1 and the following lemma.

Lemma 1.3. *Let K be a compact subset of $\mathbb{C}$ with connected complement. Then, each function holomorphic and having no zeros on K can be uniformly approximated by entire functions having no zeros.*

Proof. Let f be holomorphic on K and without zeros. Then, f is an exponential. That is, $f = \exp F$ for some function F holomorphic on K. By Runge's theorem, there is a sequence of polynomials P_n, such that $P_n \to F$ uniformly on K. Set $g_n = \exp P_n$. Then, $g_n \to f$ uniformly on K. □

2. Schlicht copies

If Y_1 and Y_2 are two Riemann surfaces, we shall use the expressions 'Y_1 and Y_2 are conformally equivalent' and 'Y_1 and Y_2 are biholomorphic' interchangeably.

Let us use the notation $\rho : Y \to \mathbb{C}$ to signify a Riemann surface spread (with possible branching) over $\mathbb{C}$. That is, Y is a Riemann surface and ρ is a holomorphic function on Y. We say that a point $p \in Y$ lies 'over' $\rho(p)$ and we think of ρ as a projection. The Riemann surface Y has a natural metric which is the pullback by ρ of the euclidean metric on the target space $\mathbb{C}$. Let A be a subset of the target $\mathbb{C}$. We say that a subset $\widetilde{A}$ of Y is a schlicht copy of A lying over A, if ρ restricted to $\widetilde{A}$ is a homeomorphism of $\widetilde{A}$ onto A. If moreover A is open, then $\widetilde{A}$ is a sheet of Y and it is not only homeomorphic but in fact conformally isometric to A. In general, we shall say that $\widetilde{A}$ is a conformally isometric copy of A lying over A, if some open neighborhood of $\widetilde{A}$ is a conformally isometric copy of A lying over A.

The most important example of a Riemann surface spread over $\mathbb{C}$ is the Riemann surface G_f of a holomorphic function f defined on a domain G of $\mathbb{C}$. In this case, the function f lifts to a biholomorphic mapping $\widetilde{f} : G \to G_f$ and $f = \rho \circ \widetilde{f}$. Although the domain G and the Riemann surface G_f are equivalent as Riemann surfaces, they are usually different as metric spaces, where G_f has the natural metric defined above and G has the natural euclidean metric which it inherits as a subset of $\mathbb{C}$.

Suppose, for $j = 1, 2$, that $\rho_j : Y_j \to \mathbb{C}$ are Riemann surfaces spread over $\mathbb{C}$. If A_j are respectively compact subsets of Y_j, we shall say that A_1 lies *schlicht* over A_2 if there is a homeomorphism $\varphi : A_1 \to A_2$ such that $\rho_1 = \rho_2 \circ \varphi$. Of course, this relation is symmetric. That is, A_1 lies schlicht over A_2 if and only if A_2 lies schlicht over A_1. If A_1 is schlicht over A_2, then, by the Brouwer theorem, A_1 is open in Y_1 if and only if A_2 is open in Y_2. In this case, φ is biholomorphic and A_1 and A_2 are conformally isometric. For general sets A_1 and A_2, we say that they are conformally isometric if they are respectively contained in open subsets of Y_1 and Y_2 which are conformally isometric.

3. A covering property

Let Y be any simply connected Riemann surface endowed with a conformal metric. For example, the Riemann surface G_g of a holomorphic function g defined on a simply connected domain G is conformally equivalent to G and hence is also simply connected. If Y is hyperbolic, S_ζ and Y are conformally equivalent, and if we take the conformal metric on S_ζ induced by the conformal metric on Y, then S_ζ and Y are conformally isometric. On the other hand, if Y is not hyperbolic, and K is a compact proper subset of Y, let Y_K be any simply connected neighborhood of K, which is not all of Y. Then, Y_K is hyperbolic and hence conformally equivalent to S_ζ. With respect to the conformal metric induced on S_ζ by the conformal metric on Y_K inherited from Y, S_ζ is conformally isometric to Y_K and, in particular, contains a conformally isometric copy of K. But now, the metric on S_ζ depends not only on Y, but even on K. Moreover, we are only assured that S_ζ contains *one* conformally isometric copy of K in this manner. We wish to find *infinitely many* conformally isometric copies of K in S_ζ, and moreover we wish to always use the *same* metric on S_ζ, namely, the natural metric of S_ζ as a surface spread over $\mathbb{C}$.

If Y is a Riemann surface spread over $\mathbb{C}$ and $\mu \in \mathbb{C}$, we can define in an obvious manner the translate $Y + \mu$ of Y, which is also a Riemann surface spread over $\mathbb{C}$ and there is a natural biholomorphism from Y to $Y + \mu$. For each subset $K \subset Y$ we denote by $K + \mu$ the subset of $Y + \mu$ which corresponds to K by this biholomorphism and we call $K + \mu$ the translate of K in $Y + \mu$.

Theorem 3.1. *For each Riemann surface Y spread without ramification over $\mathbb{C}$ and for each closed Jordan domain $K \subset Y$, there is a $\mu_K \in \mathbb{C}$ such that, for each $|\mu| > |\mu_K|$, the Riemann surface S_ζ of the Riemann zeta-function restricted to the critical half-strip S contains infinitely many conformally isometric copies over the translate $K + \mu$ of K in the translated Riemann surface $Y + \mu$ of Y.*

Proof. Suppose, first, that $Y \subset \mathbb{C}$, that is, that ρ is just the inclusion mapping. Suppose, moreover, that $0 \notin K$. Let W_0 be a Jordan domain such that

$$K \subset W_0 \subset \overline{W}_0 \subset \mathbb{C} \setminus \{0\}.$$

Choose $0 < r_0 < 1/4$ and let D_0 be the disc centered at $3/4$ of radius r_0. By the Riemann mapping theorem, there is a biholomorphic mapping f from D_0 onto W and by the Osgood-Carathéodory theorem, f extends to a homeomorphism of $\overline{D}_0$ onto $\overline{W}_0$. Let $\mathcal{K}$ be the inverse image of K and choose $0 < r_3 < r_2 < r_1 < r_0$, such that $\mathcal{K} \subset D_3$, where D_j denotes the disc centered at $3/4$ of radius r_j. Denote by W_j the image $f(D_j)$ of D_j.

The function f is the uniform limit on $\overline{D}_0$ of functions holomorphic on $\overline{D}_0$ and so by Voronin's universality theorem, f is the uniform limit on $\overline{D}_0$ of vertical translates of the Riemann zeta-function. In particular, setting m equal to the distance between ∂W_2 and ∂W_1, there is a vertical translate ζ_τ of the zeta-function, such that $|f - \zeta_\tau| < m$ on $\overline{D}_0$. By Rouché's theorem, for each $w \in W_2$, the functions $f - w$ and $\zeta_\tau - w$ have the same number of zeros (counting multiplicities) in D_1. But, for each $w \in W_2$, the function $f - w$ has precisely one zero in D_1. Thus, the function ζ_τ assumes each value $w \in W_2$ precisely once in D_1. Let h denote the restriction of ζ_τ to D_1. Then, ζ_τ maps $h^{-1}(W_2)$ biholomorphically onto W_2. Now denote by W_2^{-1} the vertical translate of $h^{-1}(W_2)$ by $i\tau$. Then, ζ maps W_2^{-1} biholomorphically onto W_2. Denote by $\widetilde{W}_2$ the biholomorphic copy of W_2^{-1} in the Riemann surface S_ζ of the function ζ restricted to the critical half-strip S. Then, $\widetilde{W}_2$ is a conformally isometric copy of W_2 lying over W_2. Of course, since $K \subset W_2$, the set $\widetilde{W}_2$ contains a conformally isometric copy $\widetilde{K}$ of K lying over K. We have assumed that Y is a plane domain and that $0 \notin K$.

Now let $\rho : Y \to \mathbb{C}$ be a Riemann surface spread over $\mathbb{C}$ without ramification and let K be a closed Jordan domain in Y. Suppose, moreover, that $0 \notin \rho(K)$. Let $\overline{W}_0$ be a closed Jordan domain in Y with interior W_0 such that $K \subset W_0$ and $0 \notin \rho(\overline{W}_0)$. By the uniformization theorem, there is a biholomorphic mapping f from D_0 onto W_0 and by (a generalization of) the Osgood-Carathéodory theorem, f extends to a homeomorphism of $\overline{D}_0$ onto $\overline{W}_0$. Define D_j as before.

In order to adapt the proof we gave for Y, a plane domain, to the present situation, where Y is a Riemann surface, we should require a Rouché theorem for holomorphic mappings taking their values in a Riemann surface. This problem can be circumvented in the following way. By the universality theorem, we can obtain a translate ζ_τ of the Riemann zeta-function which approximates $\rho \circ f$ so well on $\overline{D}_0$ that for each $s \in \overline{D}_1$, there is an open neighborhood N_s of s in D_0 on which $\rho \circ f$ is injective and such that $\zeta_\tau(s)$ lies in $(\rho \circ f)(N_s)$. Now fix a point $s \in D_2$. Let ρ^{-1} be the germ at $\zeta_\tau(s)$ of the inverse of ρ restricted to $f(N_s)$. The germ $f^{-1} \circ \rho^{-1} \circ \zeta_\tau$ can be continued holomorphically along all paths in D_1 and, by the Monodromy Theorem, gives rise to a holomorphic function on D_1 which, by abuse of notation, we denote by $f^{-1} \circ \rho^{-1} \circ \zeta_\tau$. By approximating $\rho \circ f$ sufficiently well on W_0 by translates ζ_τ of the Riemann zeta-function, we may approximate the identity $f^{-1} \circ f$ as well as we please on W_2 by functions $f^{-1} \circ \rho^{-1} \circ \zeta_\tau$. By Rouché's theorem, if the approximations are sufficiently good, then for each $w \in \overline{D}_3$, the identity function and $f^{-1} \circ \rho^{-1} \circ \zeta_\tau$ assume the value w the same number of times in D_2. Thus, $f^{-1} \circ \rho^{-1} \circ \zeta_\tau$ assumes each value in $\overline{D}_3$ precisely once in D_2. Denote by W_3^{-1} the image of D_3 by the inverse of $f^{-1} \circ \rho^{-1} \circ \zeta_\tau$ restricted to D_2. Then, $f^{-1} \circ \rho^{-1} \circ \zeta_\tau$ maps W_3^{-1} biholomorphically onto D_3. We conclude from this that the holomorphic continuation $\rho^{-1} \circ \zeta_\tau$ is well defined and maps W_3^{-1} biholomorphically onto W_3. Writing

$$\rho_\zeta(\tilde{\zeta}(s + i\tau)) = \zeta(s + i\tau) = \zeta_\tau(s),$$

we have that $\rho^{-1} \circ \rho_\zeta \circ \tilde{\zeta}$ maps $W_3^{-1} + i\tau$ biholomorphically onto W_3. Denote by $\widetilde{W}_3$ the image of $W_3^{-1} + i\tau$ in S_ζ by the biholomorphic mapping $\tilde{\zeta}$. Then $\widetilde{W}_3$ is a conformally isometric copy of W_3 lying over W_3 in the Riemann surface S_ζ of the Riemann zeta-function restricted to S. Since S_ζ has a conformally isometric copy $\widetilde{W}_3$ of W_3 over W_3, of course $\widetilde{W}_3$ contains a conformally isometric copy $\widetilde{K}$ of K over K.

One can revisit the above proof, to obtain, not only one, but in fact infinitely many conformally isometric copies $\widetilde{K}_j$, $j = 1, 2, \ldots$, of K lying over K. In fact, by the universality Theorem 1.2, there is a sequence ζ_{τ_j} of translates of the zeta-function which converges to $\rho \circ f$ on $\overline{D}_0$. Obviously, we can suppose that $\rho \circ f$ is not itself the Riemann zeta-function, for example, by choosing W_0 with non-analytic boundary. Since $\rho \circ f$ is not the Riemann zeta-function, it follows that $\tau_j \to \infty$. Hence, we may suppose that the vertical translates of $\overline{D}_0$ by $i\tau_j$ are disjoint. Thus, the τ_j give rise to disjoint conformally isometric copies of W_3 lying over W_3. These contain respectively infinitely many conformally isometric copies of K over K.

We have assumed that $0 \notin \rho(K)$. To conclude the proof, we merely note that if $0 \in \rho(K)$, we may choose $\mu_K \in \mathbb{C}$ such that, for each $|\mu| > |\mu_K|$, the projection of $K + \mu$ does include 0. We then apply the previous arguments to the new set $K + \mu$ in the Riemann surface $Y + \mu$. □

Corollary 3.2. *For each Riemann surface Y spread without ramification over $\mathbb{C}$ and for each closed Jordan domain $K \subset Y$ having no points over 0, the Riemann surface S_ζ of the Riemann zeta-function restricted to the critical half-strip S contains infinitely many conformally isometric copies of K over K.*

Proof. In proving the theorem, this is precisely what we showed, before the last paragraph of the proof, since, up to that point, we were working under the assumption that $0 \notin \rho(K)$. □

Corollary 3.3. *For each Riemann surface Y spread without ramification over $\mathbb{C}$ and for each closed Jordan domain $K \subset Y$, the Riemann surface S_ζ of the Riemann zeta-function restricted to the critical half-strip S contains infinitely many conformally isometric copies of K.*

Proof. Choose an admissible μ. Then, the Riemann surface S_ζ contains infinitely many conformally isometric copies of $K + \mu$ over $K + \mu$. Now these copies may not be over K, but they are still conformally isometric to K since $K + \mu$ is conformally isometric to K. □

4. Domains with ramification

Let us say that K is a closed Jordan domain spread over $\mathbb{C}$ if K is a closed Jordan domain in some Riemann surface spread over $\mathbb{C}$. In the above covering theorem, we have considered only Jordan domains spread over $\mathbb{C}$ without ramification. Let us now consider the situation when ramification is present. The following theorem shows that we cannot always cover Jordan domains having a simple ramification, but we can always cover certain translates thereof.

Theorem 4.1. *For each Riemann surface X spread over $\mathbb{C}$, in particular for the Riemann surface S_ζ of the Riemann zeta-function restricted to the critical half-strip S, there exists a closed Jordan domain K over $\mathbb{C}$ having precisely one ramification* *point which moreover is simple, and X contains no conformally isometric copy of K over K itself.*

Proof. Consider the Riemann surface $\mathbb{C}_a$ of the function $h(z) = a + z^2$, defined on $\mathbb{C}$. This is a two-sheeted surface spread over $\mathbb{C}$ and having a simple ramification point over the point a. This surface is simply connected, since it is biholomorphic to the domain of h, namely $\mathbb{C}$. We denote the projection which spreads $\mathbb{C}_a$ over $\mathbb{C}$ by ρ_a. Let Q be a closed disc in the domain $\mathbb{C}$, and $\tilde{h}(Q) = K_a$ the closed Jordan domain in the surface $\mathbb{C}_a$. Then $\rho(K_a) = h(Q)$ is a closed disc of center a.

Let $\rho : X \to \mathbb{C}$ be any Riemann surface spread over $\mathbb{C}$, and suppose X contains a conformally isometric copy $\tilde{K}_a$ of K_a over K_a. Then, since $\rho^{-1} \circ \rho_a$ maps $\tilde{K}_a$ onto K_a bijectively, K_a is also a two-sheeted covering of the same disc $\overline{D}_a$, having a ramification point x_a over a. Thus $\rho(x_a) = a$. Since ramification points are isolated, the set R_X of ramification points of the surface X is at most countable. These lie over the countable set $\rho(R_X)$. Choose $a \notin \rho(R_X)$. Then, the above discussion shows that the Riemann surface X does not contain a conformally isometric copy of K_a over K_a. This proves the theorem. □

Theorem 4.2. *For each closed Jordan domain K over $\mathbb{C}$, having precisely one ramification point which moreover is simple, the surface S_ζ contains* infinitely *many conformally isometric copies of K over translates of K.*

Proof. Let K be any Jordan domain over $\mathbb{C}$ having precisely one ramification point which moreover is simple. In order to prove that S_ζ contains conformally isometric copies of K over translates of K, we need only make a slight modification of the proof of Theorem 3.1. We may assume (by translation of K) that $0 \notin \rho(K)$. From our definition of a Jordan domain spread over $\mathbb{C}$, we know that K lies in some Riemann surface Y spread over $\mathbb{C}$. Let $\overline{W}_0$ be a closed Jordan domain in Y with interior W_0, such that $K \subset W_0$ and $0 \notin \rho(\overline{W}_0)$. We may assume that the unique ramification point of K is also the unique ramification point of $\overline{W}_0$. Following the notation in the proof of Theorem 3.1, we may assume that ζ_τ approximates $\rho \circ f$ so well on $\overline{D}_0$ that the derivative of ζ_τ has only one simple zero in $\overline{D}_0$ and that it is as near as we wish to the unique zero of $(\rho \circ f)'$ in $\overline{D}_0$. We may also assume that not only these critical points are closed, but also that the corresponding critical values are close. Thus, by post-composing with a small shift by μ, we may obtain that $\rho \circ f + \mu$ and ζ_τ have the same unique critical value. This means that $\tilde{\zeta}_\tau(D_0)$ has a unique ramification point which is precisely over the unique ramification point of $K + \mu$ and of the same order.

Let a be the unique zero of $(\rho \circ f)'$ in D_0. For some $\epsilon > 0$, the portion of W_0 lying over the disc $|w - f(a)| < \epsilon$ is the two sheeted covering of this disc with ramification point over the branch point $f(a)$. Let N_a be f^{-1} of the disc $|w - f(a)| < \epsilon/2$. Since both $f(D_0)$ and $\tilde{\zeta}_\tau(D_0)$ have simple ramification points over $(\rho \circ f)(a) = \zeta_\tau(a)$, we may assume that the approximation is so good that the composition $\rho^{-1} \circ \rho_\zeta$ is biholomorphic on $\tilde{\zeta}_\tau(N_a)$.

We may also assume that the approximation is so good that for each $s \in \overline{D}_1 \setminus N_a$ there is an open neighbourhood N_s of s in D_0 on which $\rho \circ f$ is injective and such that $\zeta_\tau(s)$ lies in $(\rho \circ f)(N_s)$. Now fix a point $s \in D_2 \setminus N_a$. Let ρ^{-1} be the germ at $\zeta_\tau(s)$ of the inverse of ρ restricted to $\tilde{f}(N_s)$. The germ $f^{-1} \circ \rho^{-1} \circ \zeta_\tau$ can be continued holomorphically along all paths in D_1 and by the Monodromy Theorem gives rise to a holomorphic function on D_1 which, by abuse of notation, we denote by $f^{-1}\circ\rho^{-1}\circ\zeta_\tau$. By approximating $\rho\circ f$ sufficiently well on $\overline{D}_0$ by translates ζ_τ of the Riemann zeta-function, we may approximate the identity $f^{-1}\circ f$ as well as we please on $\overline{D}_2\setminus N_a$ by functions $f^{-1}\circ\rho^{-1}\circ\zeta_\tau$. Since $\rho^{-1}\circ\rho_\zeta$ is an isometry near the ramification point and in particular on $\tilde{\zeta}_\tau(N_a)$, we may thus approximate the identity function as well as we please on all of $\overline{D}_2$ by functions $f^{-1}\circ\rho^{-1}\circ\zeta_\tau$.

The rest of the proof that S_ζ contains a conformally isometric copy of K over $K+\tau$ is the same as in the proof of Theorem 3.1. We omit the details. This proves the theorem. □

Moreover, there exist Jordan domains with two simple ramification points, for which even translates cannot be covered.

Theorem 4.3. *For each Riemann surface X spread over $\mathbb{C}$, in particular for S_ζ, there exists a closed Jordan domain K over $\mathbb{C}$ having precisely two ramification points which moreover are simple and X contains* no *conformally isometric copy of K (not even over translates of K).*

Proof. Let X be any Riemann surface spread over $\mathbb{C}$. Since X has at most countably many ramification points, the distance between any two such ramification points can only assume countably many values. Now let K be a Jordan domain over $\mathbb{C}$ with two ramification points, such that the distance between these two points is different from these countably many values. Then, X cannot contain a conformally isometric copy of K. □

5. Compact sets not capturing zero

The topological hull of a set $A\subset\mathbb{C}$ is the union of A with the bounded complementary components. Let us say that a subset A of $\mathbb{C}$ *captures* zero, if zero is contained in the hull of A. A compact set K fails to capture zero if and only if there is a path from zero to infinity in the complement of K.

Theorem 5.1. *Over each compact subset K of $\mathbb{C}$ which does not capture zero, the Riemann surface S_ζ of the Riemann zeta-function restricted to the critical half-strip S contains infinitely many conformally isometric copies of K. Moreover, if the Riemann surface S_ζ contains a schlicht copy of some compact set which does capture zero, then the Riemann hypothesis fails.*

Proof. Let K be a compact subset of $\mathbb{C}$ which does not capture zero. There exist a Jordan domain J such that

$$K\subset J\subset\overline{J}\subset\mathbb{C}\setminus\{0\}.$$

The first part of the theorem follows immediately by applying the previous theorem to the closed Jordan domain $\overline{J}$, noting that in the proof translations of $\overline{J}$ are not required, since $0\notin\rho(\overline{J})$.

The second assertion follows from the following claim. Let f be a holomorphic function in a simply connected domain G, whose restriction to some compact subset Q of G is a homeomorphism of Q onto its image, which we denote by K. Then $f(G)$ contains the topological hull $\widehat{K}$ of K. Now we have only to verify the claim.

Fact 1: Homeomorphic compact subsets of $\mathbb{C}$ have the same number of complementary components. This follows from the following observations (see [2]). Firstly, if K is a compact subset of $\mathbb{C}$, then, for any cohomology theory, the number of complementary components of K is given by the zeroth cohomology group of $\mathbb{C}\setminus K$

with real coefficients. Namely, $H^0(\mathbb{C}\setminus K,\mathbb{R}) = \oplus_1^n \mathbb{R}$, where n is the number of components of $\mathbb{C}\setminus K$. Secondly, by Alexander duality, $H^0(\mathbb{C}\setminus K,\mathbb{R}) = H_1(K,\mathbb{R})\oplus\mathbb{R}$. Thirdly, $H_1(K,\mathbb{R})$ is a topological invariant.

Let us define a *Jordan set* $\mathit{\Gamma}$ in $\mathbb{C}$ to be a compact subset which satisfies the conclusion of the Jordan curve theorem. That is, $\mathbb{C}\setminus\mathit{\Gamma}$ has precisely two components and each point of $\mathit{\Gamma}$ lies on the boundary of each of these components.

Fact 2: If K is a compact subset of $\mathbb{C}$ whose complement is not connected, then K contains a Jordan set $\mathit{\Gamma}$. We can prove this in the following way. Let V be a bounded complementary component of K. Let $K_1 = \overline{V}$ and let $\mathit{\Gamma} = \partial\widehat{K}_1$. We show that $\mathbb{C}\setminus\mathit{\Gamma}$ has precisely two complementary components. The complement of $\mathit{\Gamma}$ consists of the complement W of $\widehat{K}_1$ and the interior $\widehat{K}_1^\circ$ of $\widehat{K}_1$. From the definition of the topological hull, it follows that W is connected. We show that $\widehat{K}_1^\circ$ is also connected. Any component of the interior of a compact set having connected complement also has connected complement. In particular, if U is the component of $\widehat{K}_1^\circ$ which contains V, then, the complement of U is connected from which it follows that the complement of $\overline{U}$ is also connected. Since $\overline{V}\subset\overline{U}\subset\widehat{K}_1$, it follows that $\overline{U}$ is the topological hull $\widehat{K}_1$ of $K_1 = \overline{V}$. Thus, U is dense in $\widehat{K}_1$ and so U is the unique component of $\widehat{K}_1^\circ$. Hence, the complement of the boundary $\mathit{\Gamma}$ of $\widehat{K}_1$ has precisely two components, U and W. Since U is dense in $\widehat{K}_1$, we have $\partial U = \mathit{\Gamma}$. The boundary of any set is the same as the boundary of its complement. Hence, $\mathit{\Gamma} = \partial W$. We have established that $\mathit{\Gamma}$ is a Jordan set.

Fact 3: Let f be a homeomorphism of a compact subset $Q\subset\mathbb{C}$ onto the compact set $K\subset\mathbb{C}$. Then, Q is a Jordan set if and only if K is a Jordan set. Indeed, suppose K is a Jordan set and let U and W be the complementary components of K. By Fact 1, the complement of Q has two components and so, by Fact 2, Q contains a Jordan set Q_0. Set $K_0 = f(Q_0)$. We wish to show that $K_0 = K$. Clearly $K_0\subset K$. We have only to show that if $w\notin K_0$ then $w\notin K$. By Fact 1, K_0 has precisely two complementary components, say U_0 and W_0. Each of these contains a complementary component of K and so they must contain different complementary components of K. We may assume that $U\subset U_0$ and $W\subset W_0$. If $w\notin K_0$, then w has an open neighborhood N disjoint from K_0. Thus, N lies entirely in U_0 or W_0 and consequently N is disjoint from W or from U. Thus, $w\notin\partial W$ or $w\notin\partial U$. Since K is a Jordan set, this means that $w\notin K$. This completes the proof that $K = K_0$ and consequently, $Q = Q_0$. Therefore, Q is a Jordan set. We have shown that if K is a Jordan set then so is Q. The argument also goes the other way, since f^{-1} too is a homeomorphism.

Let f be a function holomorphic in a simply connected domain G and let f map a Jordan subset $\mathit{\Gamma}'$ of G homeomorphically onto (the Jordan set) $\mathit{\Gamma}$. Let U' and W' be respectively the bounded and the unbounded complementary components of $\mathit{\Gamma}'$. Similarly, let U and W be respectively the bounded and the unbounded complementary components of $\mathit{\Gamma}$. Since G is simply connected, it is not hard to see that $U'\subset G$.

Fact 4: $f(U')\cap U\neq\emptyset$. To prove this conjecture, suppose $f(U')\cap U=\emptyset$. Then $f(U')\subset W$. Choose a point in $f(U')$. There is a path σ from this point to ∞ in W. Since $f(U')$ is bounded, σ must contain a boundary point of $f(U')$. Now,

$$\partial f(U')\subset f(\partial U') = f(\mathit{\Gamma}') = \mathit{\Gamma} = \partial W,$$

where the first inclusion is because f is an open mapping. This contradicts the fact that σ lies in W.

We now show that $f(G)$ contains *every* point of U. Let b be a point of U. By Fact 4, there is a point a in U which is the image by f of some point α in U'. Since f is open, we may assume that $a \neq b$. Let U_j be a decreasing sequence of Jordan domains which contain $\overline{U}$ and which converge to $\overline{U}$. Also, let U_∞ be a Jordan domain which contains the points a and b and whose closure is contained in U. By the Riemann mapping theorem, there is a biholomorphic mapping h_j of the unit disc onto U_j such that $h_j(0) = b$ and $h_j'(0) > 0$. By Montel's theorem, the family h_j is normal and so we may assume that h_j converges to a holomorphic mapping h of the unit disc into $\overline{U}$ with $h(0) = b$. By the Schwarz lemma, $h_j'(0) > h_\infty{}'(0)$. Thus h is non-constant and in fact biholomorphic onto its image V. Since $V \subset \overline{U}$ and h is non-constant, V contains no point of $\partial\overline{U}$. Now, $\partial\overline{U} = \partial U$, since K is a Jordan set. This implies that $|h_j^{-1}(w)| \to 1$ uniformly on ∂U. On the other hand, by Pick's lemma applied to $h_j^{-1} \circ h_\infty$, we see that h_j^{-1} is bounded away from 1 on U_∞. Thus, U_∞ is contained in V. In particular, $a \in V$. We may fix j so large that $|h_j^{-1}(a)| < |h_j^{-1}(w)|$ for each $w \in \partial U$. By Rouché's theorem (which holds even though U' may not be smoothly bounded [1]), the functions $h_j^{-1} \circ f$ and $(h_j^{-1} \circ f) - h_j^{-1}(a)$ have the same number of zeros in U'. Now $(h_j^{-1} \circ f) - h_j^{-1}(a)$ has a zero at the point $\alpha \in U'$ and hence $h_j^{-1} \circ f$ assumes the value 0 at some point $\beta \in U'$. Thus, $f(\beta) = h(0) = b$, which completes the proof of the claim.

The previous paragraph can perhaps be slightly simplified by invoking the Carathéodory kernel theorem. □

6. Bloch radius

Let X be a Riemann surface, let f be a holomorphic function on X, and let G_f be the Riemann surface of f (possibly branched) over $\mathbb{C}$. We say that G_f contains a schlicht disc of radius r over a point $w \in \mathbb{C}$ if there is a domain in X which is mapped by f biholomorphically onto the disc of radius r centered at w. We define the Bloch radius of f over w, denoted by $b(w)$ to be the radius of the 'largest' schlicht disc over w. More precisely,

$$b(w) := \sup\{r : \text{ there is a schlicht disc of radius } r \text{ over } w\}.$$

If there is no schlicht disc over w, we set $b(w) = 0$.

For the Riemann zeta-function we get the following covering theorem.

Theorem 6.1. *For each $w \in \mathbb{C}$ the Bloch radius $b(w)$ of the Riemann zeta-function over w has the following lower bound:*

$$b(w) \geq |w|.$$

Moreover, if the Riemann hypothesis holds, we have equality.

Proof. The first part is an immediate consequence of Theorem 5.1. Indeed, for $w = 0$ the theorem is trivial. For $w \neq 0$, let K be an arbitrary closed disc centered at w and of radius $r < |w|$. Then K does not capture zero, and so by Theorem 5.1 there is a schlicht copy $\tilde{K}$ of K lying over K in the Riemann surface S_ζ. The set $\tilde{K}$ is a schlicht disc of radius r over w. This gives the first part.

The Riemann hypothesis would imply $b(w) \leq |w|$, which together with the first part yields the second part. □

7. Cyclicity of the Riemann zeta-function

The remarkable universality theorem of Voronin improved by Bagchi (Theorem 1.1) yields approximation by translates of the Riemann zeta-function. If we allow not only translates but also linear combinations thereof, we obtain much more with much less effort. In fact, as the following theorem shows, we can approximate on arbitrary compact subsets, not just compact subsets of the fundamental half-strip having connected complement. Nor do we need to assume that the functions to be approximated have no zeros. We can approximate any holomorphic function on any compact set.

Theorem 7.1 (Runge type). *For each compact subset K of $\mathbb{C}$, for each function f holomorphic on K and for each ϵ, there are finitely many values $a_1, a_2, \ldots, a_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that*

$$|\sum_{j=1}^{n} \lambda_j \, \zeta_{a_j} - f| < \epsilon$$

on K.

In functional analytic terminology, the previous theorem implies that the Riemann zeta-function is a *cyclic vector* for the translation operator in the space of entire functions.

Proof. Let f be holomorphic on K. We may assume that $f \in C_0^\infty$. Since the function $\Phi(s) = \pi^{-1}\zeta(s+1)$ is a fundamental solution for the Cauchy-Riemann operator,

$$\begin{aligned} f(s) &= (\bar{\partial} f * \Phi)(s) \\ &= \iint (\bar{\partial} f)(v)\, \Phi(s-v)\, dxdy. \end{aligned}$$

If s lies outside the support of $\bar{\partial} f$, then $f(s)$ can be approximated by Riemann sums. Such sums have the form

$$\sum_{j=1}^{n} \lambda_j \, \zeta_{a_j}.$$

Moreover, the approximation is uniform on compact subsets disjoint from the support of $\bar{\partial} f$, in particular on K. □

More refined approximations are possible. For example, we have the following result.

Theorem 7.2. *Let $K \subset \mathbb{C}$ be a compact set and σ a subset of $\mathbb{C} \setminus K$, such that any holomorphic function on $\mathbb{C} \setminus K$ vanishing up to order A on σ is identically 0. Then for each function f holomorphic on K and for each ϵ, there are finitely many $a_1, a_2, \ldots, a_n \in \sigma$ and $\lambda_{1,k}, \lambda_{2,k}, \ldots, \lambda_{n,k} \in \mathbb{C}$, $0 \le k \le N < A$, such that*

$$|\sum_{j=1}^{n} \sum_{k=0}^{N} \lambda_{j,k} \, \zeta^{(k)}(\cdot - a_j + 1) - f| < \epsilon$$

on K.

Proof. This is a particular case of Theorem 5.3.2 in [3]. □

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(Paul M. Gauthier) Département de mathématiques et de statistique, Université de Montréal, CP 6128 Centre Ville, Montréal, Québec H3C 3J7, Canada

E-mail address: gauthier@dms.umontreal.ca

(Nikolai Tarkhanov) Universität Potsdam, Institut für Mathematik, Postfach 60 15 53, 14415 Potsdam, Germany

E-mail address: tarkhanov@math.uni-potsdam.de