MIXED PROBLEMS WITH A PARAMETER

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ABSTRACT. Let X be a smooth n-dimensional manifold and D be an open connected set in X with smooth boundary ∂D. Perturbing the Cauchy problem for an elliptic system $Au = f$ in D with data on a closed set $\Gamma \subset \partial D$ we obtain a family of mixed problems depending on a small parameter $\varepsilon > 0$. Although the mixed problems are subject to a non-coercive boundary condition on $\partial D \backslash \Gamma$ in general, each of them is uniquely solvable in an appropriate Hilbert space $\mathcal{D}_T$ and the corresponding family $\{u_\varepsilon\}$ of solutions approximates the solution of the Cauchy problem in $\mathcal{D}_T$ whenever the solution exists. We also prove that the existence of a solution to the Cauchy problem in $\mathcal{D}_T$ is equivalent to the boundedness of the family $\{u_\varepsilon\}$. We thus derive a solvability condition for the Cauchy problem and an effective method of constructing its solution. Examples for Dirac operators in the Euclidean space $\mathbb{R}^n$ are considered. In the latter case we obtain a family of mixed boundary problems for the Helmholtz equation.

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INTRODUCTION

This paper is based on the following simple observation. Consider an operator equation $Tu = f$ with a bounded operator $T : H \to \tilde{H}$ in Hilbert spaces. If there is a $u \in H$ satisfying $Tu = f$ then f is orthogonal to the null-space of the adjoint operator T^* in $\tilde{H}$. On the other hand, for $f \in (\ker T^*)^\perp$ the equation $Tu = f$ is obviously equivalent to $T^*Tu = T^*f$. The latter need not have any solution, however, the slightly perturbed equation $T^*Tu + \varepsilon u = v$ is uniquely solvable for any $v \in H$, provided that $\varepsilon > 0$. Note that the solution of the equation can be

effectively constructed, for the operator $T^*T + \varepsilon$ is positive definite. We thus get a family

$$u_\varepsilon = (T^*T + \varepsilon)^{-1} T^* f$$

in H, whose limit is a good candidate for the solution of $Tu = f$ that is orthogonal to the null-space of T. Indeed, if $v \in H$ satisfies $Tv = 0$ then by Lemma 12.1.25 of [Tar95b] we get

$$
\begin{aligned}
(u_\varepsilon, v)_H &= (f, T(T^*T + \varepsilon)^{-1} v)_{\tilde{H}} \\
&= (f, (TT^* + \varepsilon)^{-1} Tv)_{\tilde{H}} \\
&= 0,
\end{aligned}
$$

as desired. If $f = Tu$ for some $u \in H$, then $u_\varepsilon = u - \varepsilon(T^*T + \varepsilon)^{-1} u$ is obviously bounded in H.

Conversely, if the norm $\|u_\varepsilon\|_H$ is bounded uniformly in $\varepsilon \ll 1$ then u_ε converges for $\varepsilon \searrow 0$ to the only solution $u \in H$ of $Tu = f$ that is orthogonal to $\ker T$.

In this way we derive a solvability condition and an approximate solution to the equation $Tu = f$ in H. We refer the reader to Section 12.1.5 of [Tar95b] for an extremal property of u_ε.

When applying the approach in the study of the Cauchy problem for solutions of an elliptic equation $Au = f$, one needs to complete it by refined analysis. By the above, the calculus of the Cauchy problems which are ill-posed by the very nature can be elaborated in the framework of the calculus of operators $T^*T + \varepsilon I$ depending on a parameter $\varepsilon > 0$. In order to avoid sophisticated adjoint operators one uses L^2-scalar products which necessarily leads to unbounded closed operators with dense domains. Hence, it requires much more efforts to make use of the construction described above.

The operator T is given the domain consisting of those functions u in D which are square integrable along with Au and whose Cauchy data with respect to A vanish on a closed set $\Gamma \subset \partial D$. Then the domain of the adjoint operator T^* is proved to consist of square integrable functions g on D, such that the Cauchy data of g with respect to A^* vanish in the complement of Γ. It follows that the natural domain of the Laplacian T^*T is a subspace of square integrable functions u on D, such that the Cauchy data of u with respect to A vanish on Γ and the Cauchy data of Tu with respect to A^* vanish on $\partial D \setminus \Gamma$. This gives rise to a mixed boundary value problem for the elliptic operator A^*A in D similar to the classical Zaremba problem [Zar10].

Our paper demonstrates rather strikingly that the calculus of Cauchy problems for solutions of elliptic equations just amounts to the calculus of mixed boundary value problems for elliptic equations with a parameter, cf. [Sim87]. While this observation seems to be of purely mathematical interest, the explicit solutions we construct by the classical Fourier method may be of practical importance in applications.

1. PRELIMINARIES

Let X be a C^∞ manifold of dimension n with a smooth boundary ∂X. We tacitly assume that it is embedded into a smooth closed manifold $\tilde{X}$ of the same dimension.

For any smooth $\mathbb{C}$-vector bundles E and F over X, we write $\mathrm{Diff}^m(X; E, F)$ for the space of all linear partial differential operators of order $\leq m$ between sections of E and F.

We denote by E^* the conjugate bundle of E. Any Hermitian metric $(.,.)_x$ on E gives rise to a sesquilinear bundle isomorphism $*_E : E \to E^*$ by the equality $\langle *_E v, u \rangle_x = (u, v)_x$ for all sections u and v of E.

Pick a volume form dx on X, thus identifying dual and conjugate bundles. For $A \in \mathrm{Diff}^m(X; E, F)$, denote by $A' \in \mathrm{Diff}^m(X; F^*, E^*)$ the transposed operator and by $A^* \in \mathrm{Diff}^m(X; F, E)$ the formal adjoint operator. We have $A^* = *_E^{-1} A' *_F$, cf. [Tar95b, 4.1.4] and elsewhere.

For an open set $O \subset X$, we write $L^2(O, E)$ for the Hilbert space of all measurable sections of E over O with a finite norm $(u, u)_{L^2(O,E)} = \int_O (u, u)_x dx$. We also denote by $H^s(O, E)$ the Sobolev space of distribution sections of E over O, whose weak derivatives up to order s belong to $L^2(O, E)$.

Given any open set O in X°, the interior of X, we let $\mathcal{S}_A(O)$ stand for the space of weak solutions to the equation $Au = 0$ in O. Obviously, the subspace of $H^s(O, E)$ consisting of all weak solutions to $Au = 0$ is closed.

Write $\sigma^m(A)$ for the principal homogeneous symbol of the operator A, $\sigma^m(A)$ living on the (real) cotangent bundle T^*X of X. From now on we assume that $\sigma^m(A)$ is injective away from the zero section of T^*X. Hence it follows that the Laplacian A^*A is an elliptic differential operator of order $2m$ on X.

If the dimensions of E and F are equal then A is elliptic, too. Otherwise we will call it overdetermined elliptic operator.

We can assume without restriction of generality that A is included into a compatibility complex of differential operators $A^i \in \mathrm{Diff}^{m_i}(X; E^i, E^{i+1})$ over X, where $i = 0, 1, \ldots, N$ and $A^0 = A$. This complex is elliptic in a natural way, see for instance [Tar95a, 4.1.2]). If A is elliptic then the compatibility complex is trivial, i.e., $A^i = 0$ for all $i > 0$.

Let D be a relatively compact domain in X° with a smooth boundary ∂D. For $u \in L^2(D, E)$ we always regard Au as a distribution section of F over D.

A large class of operators A possess the following property which is usually referred to as Unique Continuation Property,

$(U)_s$: Given any domain $D \subset X^\circ$, if $u \in \mathcal{S}_A(D)$ vanishes on a non-empty open subset of D then $u \equiv 0$ in all of D.

This property implies in particular the existence of a left fundamental solution for A in the interior of X.

Consider the Hermitian form

$$D(u, v) = (u, v)_{L^2(D,E)} + (Au, Av)_{L^2(D,F)}$$

on the space $C^\infty(\overline{D}, E)$ of all smooth sections of E over the closure of D. The functional $D(u) = \sqrt{D(u, u)}$ is usually called the Graph Norm related to the unbounded operator $A : L^2(D, E) \to L^2(D, F)$. Write $\mathcal{D}_A$ for the completion of $C^\infty(\overline{D}, E)$ with respect to $D(\cdot)$. Then $\mathcal{D}_A$ is a Hilbert space with the scalar product $D(.,.)$, and A maps $\mathcal{D}_A$ continuously to $L^2(D, F)$.

Note that if $A = \nabla$ is the gradient operator in $\mathbb{R}^n$ then $\mathcal{D}_A = H^1(D)$. Let us clarify what kind elements are in this space in the general case.

To this end we fix a Dirichlet system B_j, $j = 0, 1, \ldots, m - 1$, of order $m - 1$ on ∂D. More precisely, each B_j is a differential operator of type $E \to F_j$ and order

$m_j \leq m - 1$ in a neighbourhood U of ∂D, where $m_i \neq m_j$ for $i \neq j$. Moreover, the symbols $\sigma^{m_j}(B_j)$, if restricted to the conormal bundle of ∂D, have ranks equal to the dimensions of F_j. Set

$$t(u) = \oplus_{j=0}^{m-1} B_j u$$

for $u \in H^m(D, E)$.

For $s > 0$ we denote by $H^{-s}(\partial D, F_j)$ the dual of the space $H^s(\partial D, F_j)$ with respect to the pairing in $L^2(\partial D, F_j)$.

Lemma 1.1. *For every $u \in \mathcal{D}_A$, we have $u \in H^m_{\text{loc}}(D, E)$. Moreover $t(u)$ has weak boundary values on ∂D belonging to $\oplus_{j=0}^{m-1} H^{-m_j - 1/2}(\partial D, F_j)$.*

Proof. Fix an element $u \in \mathcal{D}_A$. Since A is elliptic we deduce from $Au \in L^2(D, F)$ that $u \in H^m_{\text{loc}}(D, E)$.

As usual, we denote by $H^{-m}(D, E)$ the completion of $C^\infty(\overline{D}, E)$ with respect to the norm

$$|u|_{-m} = \sup_{\substack{v \in C^\infty(\overline{D}, E) \\ t(v) = 0}} \frac{|(u, v)_{L^2(D,E)}|}{\|v\|_{H^m(D,E)}}.$$

Then we easily verify that A^* extends to a map of $L^2(D, F)$ to $H^{-m}(D, E)$, more explicitly,

$$(A^* f, v) := (f, Av)_{L^2(D,F)}$$

for each $f \in L^2(D, F)$ and $v \in \mathring{H}^m(D, E)$.

By the very definition, the distribution $A^* f$ is always orthogonal under the pairing in $L^2(D, E)$ to the null-space of the Dirichlet problem for $A^* A$. Therefore, for every $f \in L^2(D, F)$ there exists a section $Gf \in H^m(D, E)$ satisfying $A^* A\, Gf = A^* f$ in D and $t(Gf) = 0$ on ∂D, see for instance [SST03]. Any $u \in \mathcal{D}_A$ can be thus presented in the form

$$u = G\, Au + (u - G\, Au).$$

By the construction, we get $G\, Au \in \mathring{H}^m(D, E)$ and $u - G\, Au \in \mathcal{D}_A \cap \mathcal{S}_{A^* A}(D)$. As $u - G\, Au \in L^2(D, E)$ is of finite order growth near ∂D, we conclude by Lemma 9.4.4 of [Tar95b] that $t(u - G\, Au)$ has weak boundary values on ∂D belonging to $\oplus_{j=0}^{m-1} H^{-m_j - 1/2}(\partial D, F_j)$.

As $t(G\, Au) \in \oplus_{j=0}^{m-1} H^{m - m_j - 1/2}(\partial D, F_j)$ vanishes on the boundary even in the usual sense for Sobolev spaces, the proof is complete. $\qquad\square$

Let $\{C_j\}_{j=0}^{m-1}$ be the adjoint Dirichlet system for $\{B_j\}_{j=0}^{m-1}$ with respect to the Green formula for A (see for instance [Tar95b, Remark 9.2.6]). For $g \in H^m(D, F)$, we set

$$n(g) = \oplus_{j=0}^{m-1} C_j g.$$

Suppose Γ is a closed subset of ∂D. The cases $\Gamma = \emptyset$ and $\Gamma = \partial D$ are permitted, too. We write Γ° for the interior of Γ in the relative topology of ∂D.

Given any $u \in L^2(D, E)$ with $Au \in L^2(D, F)$, we say that $t(u) = 0$ on the set Γ if

$$(1.1) \qquad\qquad \int_D ((Au, g)_x - (u, A^* g)_x)\, dx = 0$$

for all sections $g \in C^\infty(\overline{D}, F)$ satisfying $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$.

Lemma 1.2. *If $u \in \mathcal{D}_A$ and $t(u) = 0$ on Γ then $u \in H^m_{\text{loc}}(D \cup \Gamma^\circ, E)$.*

In particular, $t(u)$ has zero boundary values on Γ° in the usual sense of Sobolev spaces.

Proof. The case $\Gamma = \emptyset$ has been already treated in Lemma 1.1. Assume that Γ is non-empty.

Choose a smooth real-valued function ϱ on X with the property that

$$(1.2) \qquad\qquad D = \{x \in X : \varrho(x) < 0\}$$

and $\nabla \varrho(x) \neq 0$ for all $x \in \partial D$. Set $D_\varepsilon = \{x \in X : \varrho(x) < \varepsilon\}$, then $D_{-\varepsilon} \Subset D \Subset D_\varepsilon$ for all sufficiently small $\varepsilon > 0$, and the boundary of $D_{\pm\varepsilon}$ is as smooth as the boundary of D.

We first show that the weak boundary values of $t(u)$ vanish on Γ in the sense that

$$\lim_{\varepsilon \to 0+} \int_{\partial D_{-\varepsilon}} \sum_{j=0}^{m-1} (B_j u, g_j)_x ds = 0$$

for all $g_j \in C^\infty(U, F_j)$, $j = 0, 1, \ldots, m-1$, satisfying $(\operatorname{supp} g_j) \cap \partial D \subset \Gamma$. To this end, choose a function $g \in C^\infty(\overline{D}, F)$, such that $n(g) = \oplus_{j=0}^{m-1} g_j$ on ∂D, cf. Lemma 9.3.5 in [Tar95b]. Since $u \in L^2(D, E)$ and $Au \in L^2(D, F)$, we obtain by the Green formula

$$
\begin{aligned}
\lim_{\varepsilon \to 0+} \int_{\partial D_{-\varepsilon}} \sum_{j=0}^{m-1} (B_j u, g_j)_x ds &= \lim_{\varepsilon \to 0+} \int_{D_{-\varepsilon}} \left((Au, g)_x - (u, A^* g)_x \right) dx \\
&= \int_D \left((Au, g)_x - (u, A^* g)_x \right) dx \\
&= 0
\end{aligned}
$$

because $t(u) = 0$ on Γ in the sense of (1.1) and $g \in C^\infty(\overline{D}, F)$ satisfies $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$.

We thus have $A^* A u \in H^{-m}(D, E)$ and the weak boundary values of $t(u)$ vanish on Γ. As $A^* A$ is an elliptic operator of order $2m$ and $u \mapsto t(u)$ is a Dirichlet system of order $m-1$, we conclude using the local regularity theorem for solutions of the Dirichlet problem for $A^* A$ that $u \in H^m_{\mathrm{loc}}(D \cup \Gamma^\circ)$ (see for instance Theorem 9.3.17 of [Tar95b]), as desired. $\qquad\square$

The proof actually shows that for sections $u \in L^2(D, E)$ with $Au \in L^2(D, F)$ the equality (1.1) just amounts to saying that the weak boundary values of $t(u)$ vanish on Γ°.

Let $\mathcal{D}_T$ stand for the completion of the space of all sections u in $C^\infty(\overline{D}, E)$, satisfying $t(u) = 0$ on Γ, with respect to the norm $u \mapsto D(u)$. By the very definition, $\mathcal{D}_T$ is a closed subspace in $\mathcal{D}_A$, and it is a Hilbert space itself with the induced Hilbert structure.

It is well known that if Γ is the whole boundary then $\mathcal{D}_T = \overset{\circ}{H}{}^m(D, E)$, the closure of $C^\infty_{\mathrm{comp}}(D, E)$ in $H^m(D, E)$.

Lemma 1.3. *If $u \in \mathcal{D}_T$ then $t(u) = 0$ on Γ in the sense of (1.1).*

Proof. If $u \in \mathcal{D}_T$ then there exists a sequence $\{u_k\}_{k \in \mathbb{N}}$ in $C^\infty(\overline{D}, E)$ satisfying $t(u_k) = 0$ on Γ, such that

$$\lim_{k \to \infty} D(u_k - u) = 0.$$

Hence

$$\int_D ((Au, g)_x - (u, A^*g)_x)\, dx = \lim_{k \to \infty} \int_D ((Au_k, g)_x - (u_k, A^*g)_x)\, dx$$

$$= \lim_{k \to \infty} \int_{\partial D} \sum_{j=0}^{m-1} (B_j u_k, C_j g)_x ds$$

$$= 0$$

for all $g \in C^\infty(\overline{D}, F)$ satisfying $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$, because $t(u_k) = 0$ on Γ. Therefore, $t(u) = 0$ on $\partial \Gamma$. $\qquad \square$

We are now in a position to characterise the space $\mathcal{D}_T$ in a much more convenient way.

Theorem 1.4. *As defined above, $\mathcal{D}_T$ is a closed subspace of $\mathcal{D}_A$ consisting of all $u \in \mathcal{D}_A$ satisfying $t(u) = 0$ on Γ.*

Proof. Write H for the subspace of $\mathcal{D}_A$ consisting of all $u \in \mathcal{D}_A$ satisfying $t(u) = 0$ on Γ. It is easy to see that H is a closed subspace of $\mathcal{D}_A$. Lemma 1.3 states that $\mathcal{D}_T$ is a subspace of H. Since $\mathcal{D}_T$ is complete by the very definition, we shall have established the theorem if we prove that the orthogonal complement $\mathcal{D}_T^\perp$ of $\mathcal{D}_T$ in H is zero.

To this end, pick a section $u \in H$ satisfying $D(u, v) = 0$ for all $v \in C^\infty(\overline{D}, E)$, such that $t(v) = 0$ on Γ. If moreover v fulfills $n(Av) = 0$ on $\partial D \setminus \Gamma^\circ$ then we readily get

$$(1.3) \qquad\qquad (u, (A^*A + I)v)_{L^2(D,E)} = 0,$$

which is due to (1.1).

We now observe that every $w \in C^\infty(\overline{D}, E)$ can be approximated in the $L^2(D, E)$-norm by sections of the form $(A^*A + 1)v$, where $v \in C^\infty(\overline{D}, E)$ satisfies $t(v) = 0$ on Γ and $n(Av) = 0$ on $\partial D \setminus \Gamma^\circ$. This latter is a consequence of the fact that the unbounded operator $T^*T + 1$ in $L^2(D, E)$ with domain $\mathcal{D}_{T^*T}$ is positive, and so invertible, see § 3 below. We thus deduce from (1.3) that $u = 0$. It follows that $\mathcal{D}_T^\perp = \{0\}$, as desired. $\qquad \square$

2. The Cauchy problem

A rough formulation of the Cauchy problem for the operator A in the domain D reads as follows: Given any sections f of F over D and u_0 of $\oplus_{j=0}^{m-1} F_j$ over Γ, find a section u of E over D, such that $Au = f$ in D and $t(u)$ has suitable limit values on Γ coinciding with u_0.

Note that some regularity of u up to Γ is needed for $t(u)$ to possess limit values on Γ. Moreover, we are going to use Hilbert space methods for the study of the Cauchy problem. Hence the space $\mathcal{D}_A$ seems to be a natural choice for posing the problem.

What is still lacking is a proper function space $B(\Gamma)$ for the Cauchy data u_0 on Γ. It is not difficult to introduce such a space in the case where Γ is the entire boundary, namely

$$B(\partial D) = \mathcal{D}_A / \mathring{H}^m(D, E).$$

By Lemma 1.1, this quotient space can be specified within $\oplus_{j=0}^{m-1} H^{-m_j-1/2}(\partial D, F_j)$ under t, although the norm of the former is essentially stronger than the norm of the latter.

Theorem 1.4 suggests us to set

$$B(\Gamma) = \frac{\mathcal{D}_A}{\mathcal{D}_T}$$

in general. Using the approach of [Tar95b, Ch. 1] one can specify $B(\Gamma)$ within $\oplus_{j=0}^{m-1} H^{-m_j-1/2}(\Gamma, F_j)$ under t. Of course, it is difficult to explicitly describe the elements of $B(\Gamma)$. However, for applications it suffices to know that there is a natural embedding

$$\oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\Gamma, F_j) \hookrightarrow B(\Gamma).$$

Using the spaces $B(\Gamma)$ allows one to reduce the Cauchy problem with non-zero Cauchy data on Γ to the Cauchy problem with homogeneous boundary data. Indeed, given $f \in L^2(D, F)$ and $u_0 \in B(\Gamma)$, we look for a section $u \in \mathcal{D}_A$ satisfying $Au = f$ in D and $t(u) = u_0$ on Γ. By the very definition of the space $B(\Gamma)$ there is a $U_0 \in \mathcal{D}_A$ with the property that $\tau(U_0) = u_0$ on Γ. This latter equality just amounts to saying that $U_0 - u_0 \in \mathcal{D}_T$. Set $u = U_0 + U$, then $u \in \mathcal{D}_A$ is equivalent to $U \in \mathcal{D}_A$. Furthermore, $t(u) = u_0$ on Γ is equivalent to $t(U) = 0$. Since $AU = f - AU_0$ and $AU_0 \in L^2(D, F)$, substituting $u = U_0 + U$ into the problem leads to the Cauchy problem with $u_0 = 0$.

Problem 2.1. Let $f \in L^2(D, F)$ be an arbitrary section. Find $u \in \mathcal{D}_T$ such that $Au = f$ in D.

If $\Gamma^\circ \neq \emptyset$ and the Unique Continuation Property $(U)_s$ holds for A then Problem 2.1 has at most one solution, cf. Theorem 10.3.5 of [Tar95b]. Otherwise we can not guarantee that the null-space $\mathcal{S}_A(D) \cap \mathcal{D}_T$ of this problem is trivial. It is well known that the Cauchy problem for elliptic equations is ill-posed in general. Moreover, if A is overdetermined then additional necessary conditions arise for the problem to be solvable. In fact, these conditions reflect the fact that the image of $\mathcal{D}_T$ by A may be not dense in $L^2(D, F)$.

Let us formulate this more precisely. To this end, we invoke as usual the boundary conditions which are adjoint for t with respect to the Green formula in D. Similarly to (1.1), for $g \in L^2(D, F)$ with $A^*g \in L^2(D, E)$, we say that $n(g) = 0$ on the set $\partial D \setminus \Gamma^\circ$ if

$$(2.1) \qquad \int_D \left((Au, g)_x - (u, A^*g)_x \right) dx = 0$$

for all sections $u \in C^\infty(\overline{D}, E)$ satisfying $t(u) = 0$ on Γ.

Recall that $A^1 \in \mathrm{Diff}^{m_1}(X; F, E^2)$ stands for a compatibility operator for A over X, i.e., A^1 is in a sense "smallest" differential operator with the property that $A^1 A \equiv 0$ on X. We make use of the Green formula for A^1 in the same way as above to introduce the relations "$n(v) = 0$ on $\partial D \setminus \Gamma^\circ$", for all sections $v \in L^2(D, E^2)$ with $A^{1*}v \in L^2(D, F)$, and "$t(f) = 0$ on Γ", for all sections $f \in L^2(D, F)$ with $A^1 f \in L^2(D, E^2)$.

The boundary equations $n(v) = 0$ for sections of E^2 and $t(f) = 0$ for sections of F are no longer induced by any Dirichlet system on ∂D as those at steps 1 and 0, respectively.

Lemma 2.1. *Assume that $f \in L^2(D, F)$ belongs to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$. Then*

1) *$A^1 f = 0$ in D in the sense of distributions;*
2) *$t(f) = 0$ on Γ;*
3) *$(f, g)_{L^2(D,F)} = 0$ for all $g \in L^2(D, F)$ satisfying $A^* g = 0$ in D and $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$.*

Proof.

1) Let f belong to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$. Then there is a sequence $\{u_k\}_{k \in \mathbb{N}}$ in $\mathcal{D}_T$, such that $\{Au_k\}_{k \in \mathbb{N}}$ converges to f in $L^2(D, F)$. Without loss of generality we may assume that each u_k is of class $C^\infty(\overline{D}, E)$, for such functions are dense in $\mathcal{D}_T$. As $A^1 A \equiv 0$, we get

$$
\begin{aligned}
(f, A^{1*}v)_{L^2(D,F)} &= \lim_{k \to \infty} (Au_k, A^{1*}v)_{L^2(D,F)} \\
&= \lim_{k \to \infty} (u_k, (A^1 A)^* v)_{L^2(D,E)} \\
&= \lim_{k \to \infty} 0 \\
&= 0
\end{aligned}
$$

for all $v \in C^\infty(\overline{D}, E^2)$ satisfying $n(A^{1*}v) = 0$ on $\partial D \setminus \Gamma^\circ$. In particular, this equality is fulfilled for all sections $v \in C^\infty(\overline{D}, E^2)$ of compact supports in D, which implies $A^1 f = 0$ in D.

2) Suppose $v \in C^\infty(\overline{D}, E^2)$ is any section satisfying $n(v) = 0$ on $\partial D \setminus \Gamma^\circ$. Then $n(A^{1*}v) = 0$ holds on $\partial D \setminus \Gamma^\circ$, too, which is a consequence of $A^* A^{1*} = 0$ and Stokes' formula. By 1), we get

$$
\begin{aligned}
-(f, A^{1*}v)_{L^2(D,F)} &= \int_D \left((A^1 f, v)_x - (f, A^{1*}v)_x \right) dx \\
&= 0,
\end{aligned}
$$

the first equality being a consequence of the fact that $A^1 f = 0$ in D. Hence it follows that $t(f) = 0$ on Γ.

3) Finally,

$$
\begin{aligned}
(f, g)_{L^2(D,F)} &= \lim_{k \to \infty} (Au_k, g)_{L^2(D,F)} \\
&= \lim_{k \to \infty} \int_D \left((Au_k, g)_x - (u_k, A^* g)_x \right) dx \\
&= \lim_{k \to \infty} 0 \\
&= 0
\end{aligned}
$$

provided that $g \in L^2(D, F)$ satisfies $A^* g = 0$ in D and $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$. This proves 3). $\qquad \square$

The condition 3) is not only necessary but also sufficient in order that f would belong to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$.

Lemma 2.2. *If f satisfies the condition 3) of Lemma 2.1 then f lies in the closure of $A\mathcal{D}_T$ in $L^2(D, F)$.*

Proof. Write V for the space of all $g \in L^2(D, F)$ satisfying $A^* g = 0$ in D and $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$. We shall have established the lemma if we show that V

coincides with the orthogonal complement of the image $A\mathcal{D}_T$ in $L^2(D, F)$. By definition, $g \in (A\mathcal{D}_T)^\perp$ if

$$(2.2) \qquad\qquad (g, Au)_{L^2(D,F)} = 0$$

for all $u \in \mathcal{D}_T$. Since $\mathcal{D}_T$ contains all smooth functions of compact support in D, we conclude that $(A\mathcal{D}_T)^\perp \subset \mathcal{S}_{A^*}(D)$. Then equality (2.2) imlplies that $(A\mathcal{D}_T)^\perp \subset V$ because

$$(g, Au)_{L^2(D,F)} = -\int_D ((A^*g, u)_x - (g, Au)_x)\, dx$$

for all $g \in V$.

On the other hand, the inclusion $V \subset (A\mathcal{D}_T)^\perp$ follows from (1.1) because each $u \in \mathcal{D}_T$ can be approximated in the norm $D(\cdot)$ by sections $u_k \in C^\infty(\overline{D}, E)$ satisfying $t(u_k) = 0$ on Γ. $\qquad\square$

Denote by $\mathcal{H}^1(D, \Gamma)$ the space of all $g \in L^2(D, F)$ satisfying $A^*g = A^1g = 0$ in D and $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$. Following [SST03] we call $\mathcal{H}^1(D, \Gamma)$ the harmonic space in the Cauchy problem with data on Γ. This is an analogue of the well-known harmonic spaces in the Neumann problem for the Laplace operator, cf. [Tar95a, 4.1].

Lemma 2.3. *When combined with*

    4) $(f, g)_{L^2(D,F)} = 0$ *for all* $g \in \mathcal{H}^1(D, \Gamma)$,

the condition 1) *of Lemma 2.1 implies that* f *belongs to the closure of* $A\mathcal{D}_T$ *in* $L^2(D, F)$.

Proof. Let the conditions 1) and 4) are fulfilled for $f \in L^2(D, F)$. The proof of Lemma 2.2 shows that

$$(2.3) \qquad\qquad f = f_1 + f_2,$$

where f_1 belongs to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$ and $f_2 \in V$. As $A^1f = 0$ in D, we deduce by Lemma 2.1 that $A^1f_2 = 0$ in D. This means $f_2 \in \mathcal{H}^1(D, \Gamma)$. Finally, 4) implies

$$\begin{aligned} 0 &= (f, f_2)_{L^2(D,F)} \\ &= (f_2, f_2)_{L^2(D,F)} \end{aligned}$$

whence $f_2 = 0$, and so f belongs to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$. $\qquad\square$

Obviously, if f belongs to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$ then it satisfies 4) by Lemma 2.1, 3). It follows that the condition 3) of Lemma 2.1 is equivalent to 1) + 4).

Lemma 2.4. *When combined with*

    5) $(f, g)_{L^2(D,F)} = 0$ *for all* $g \in \mathcal{H}^1(D, \Gamma)$ *satisfying* $t(g) = 0$ *on* Γ,

the conditions 1) *and* 2) *of Lemma 2.1 imply that* f *belongs to the closure of* $A\mathcal{D}_T$ *in* $L^2(D, F)$.

Proof. Let the conditions 1), 2) and 5) hold true for $f \in L^2(D, F)$. Taking into account Lemma 2.1 and decomposition (2.3) we readily conclude that $A^1f_2 = 0$ in D and $t(f_2) = 0$ on Γ. Finally, 5) implies

$$\begin{aligned} 0 &= (f, f_2)_{L^2(D,F)} \\ &= (f_2, f_2)_{L^2(D,F)} \end{aligned}$$

whence $f_2 = 0$. Thus, $f = f_1$ belongs to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$, as
desired.   □

Remark 2.1. Of course, if A is elliptic then $A^1 = 0$ and the conditions 1) and
2) are always fulfilled. As for the condition 3), one easily proves that each $g \in$
$L^2(D, F)$ satisfying $A^*g = 0$ in D and $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$ vanishes identically
in all of D, provided that A^* is elliptic, $\Gamma \neq \partial D$ and A^* possesses the Unique
Continuation Property $(U)_s$ in a neighbourhood of $\overline{D}$ (see, for instance, [Tar95b,
Theorem 10.3.5]). If A is overdetermined elliptic then the domain D should possess
some convexity property relative to A, in order that $\mathcal{H}^1(D, \Gamma)$ or $\{g \in \mathcal{H}^1(D, \Gamma) :$
$t(g) = 0$ on $\Gamma\}$ might be trivial. In the case $\Gamma = \emptyset$ we refer the reader to [Tar95a,
4.1.3] for more details.

We have thus described the closure of $A\mathcal{D}_T$ in $L^2(D, F)$. It is a more difficult
task to describe the image $A\mathcal{D}_T$ itself. The following lemma is the first step in this
direction.

Lemma 2.5. *Let $f \in L^2(D, F)$ belong to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$. Then
a section $u \in \mathcal{D}_T$ is a solution to Problem 2.1 if and only if*

$$(2.4) \qquad\qquad (Au, Av)_{L^2(D,F)} = (f, Av)_{L^2(D,F)}$$

for all $v \in \mathcal{D}_T$.

Proof. If Problem 2.1 is solvable and u is one of its solutions then (2.4) is obviously
satisfied.

Conversely, if (2.4) holds for an element $u \in \mathcal{D}_T$ then $A^*(Au - f) = 0$ in D
because the space $\mathcal{D}_T$ contains all smooth functions of compact support in D. It
follows that

$$\int_D \left((A^*(Au - f), v)_x - (Au - f, Av)_x \right) dx \;\; = \;\; -(Au - f, Av)_{L^2(D,F)}$$
$$= \;\; 0$$

for all $v \in C^\infty(\overline{D}, E)$ satisfying $t(v) = 0$ on Γ, which is due to (2.4). Hence
$n(Au - f) = 0$ on $\partial D \setminus \Gamma^\circ$. Finally, since both Au and f belong to the closure of
$A\mathcal{D}_T$ in $L^2(D, F)$, Lemma 2.1, 3) shows that

$$(Au - f, Au - f)_{L^2(D,F)} = 0,$$

i.e., $Au = f$ in D.   □

In conclusion of this section let us clarify the meaning of (2.4). Namely, this
equality amounts to saying that a solution $u \in \mathcal{D}_T$ of the Cauchy problem $Au = f$
is actually a solution to the mixed problem

$$(2.5) \qquad \begin{cases} A^*Au & = & A^*f & \text{in} & D; \\ t(u) & = & 0 & \text{on} & \Gamma, \\ n(Au) & = & n(f) & \text{on} & \partial D \setminus \Gamma^\circ. \end{cases}$$

Indeed, the proof of Lemma 2.5 shows that $A^*Au = A^*f$ in D in the sense of
distributions and $n(Au) = n(f)$ in the sense that $n(Au - f) = 0$ on $\partial D \setminus \Gamma^\circ$. In
particular, if $n(f)$ is well defined on $\partial D \setminus \Gamma^\circ$ then also $n(Au)$ is well defined on
$\partial D \setminus \Gamma^\circ$.

Of course, the mixed problem (2.5) considered in appropriate spaces gives noth-
ing but (2.4).

In the next sections we will systematically use the generalised setting (2.4) of Problem 2.1 in order to derive its solvability conditions.

3. A PERTURBATION

Equation (2.4) surprisingly shows that Problem 2.1 may be well posed in many cases. Namely, this is the case if the Hermitian form $(A\cdot, A\cdot)_{L^2(D,F)}$ is actually a scalar product on $\mathcal{D}_T$ inducing the same topology as the original scalar product $D(\cdot, \cdot)$. For example, not only the gradient operator ∇ in $\mathbb{R}^n$ meets this latter condition but also many other overdetermined elliptic operators A with finite-dimensional kernel $\mathcal{S}_A(D)$. Of course, $(A\cdot, A\cdot)_{L^2(D,F)}$ is always a scalar product on $\mathcal{D}_T$ if $\Gamma \neq \emptyset$ and A possesses the property $(U)_s$. However, the completion of $\mathcal{D}_T$ with respect to $(A\cdot, A\cdot)_{L^2(D,F)}$ may lead to a space with elements of arbitrary order of growth near ∂D.

This observation suggests us to perturb the Hermitian form $(A\cdot, A\cdot)_{L^2(D,F)}$ thus obtaining a "good" scalar product on $\mathcal{D}_T$. For this purpose let us introduce a family of Hermitian forms

$$(u, v)_\varepsilon = (Au, Av)_{L^2(D,F)} + \varepsilon \, (u, v)_{L^2(D,E)}$$

on $\mathcal{D}_T$, parametrised by $\varepsilon > 0$. For each fixed $\varepsilon > 0$, the corresponding norm $\|u\|_\varepsilon = \sqrt{(u, u)_\varepsilon}$ is equivalent to the graph norm $D(u)$ on $\mathcal{D}_T$. More precisely, we get

$$(3.1) \qquad \min\{1, \sqrt{\varepsilon}\} \, D(u) \leq \|u\|_\varepsilon \leq \max\{1, \sqrt{\varepsilon}\} \, D(u)$$

for all $u \in \mathcal{D}_A$.

Taking into account Lemma 2.5 we now consider the following perturbed Cauchy problem:

Problem 3.1. Given any $f \in L^2(D, F)$ and $h \in L^2(D, E)$, find an element $u_\varepsilon \in \mathcal{D}_T$ satisfying

$$(3.2) \qquad (Au_\varepsilon, Av)_{L^2(D,F)} + \varepsilon \, (u_\varepsilon, v)_{L^2(D,E)} = (f, Av)_{L^2(D,F)} + \varepsilon \, (h, v)_{L^2(D,E}$$

for all $v \in \mathcal{D}_T$.

Note that the equation (3.2) leads to a perturbation of mixed problem (2.5), more precisely,

$$(3.3) \qquad \begin{cases} A^*Au_\varepsilon + \varepsilon \, u_\varepsilon & = & A^*f + \varepsilon \, h & \text{in} & D; \\ t(u_\varepsilon) & = & 0 & \text{on} & \Gamma, \\ n(Au_\varepsilon) & = & n(f) & \text{on} & \partial D \setminus \Gamma^\circ. \end{cases}$$

Indeed, since the space $\mathcal{D}_T$ contains all smooth functions with compact support in D, (3.2) implies $A^*Au_\varepsilon + \varepsilon \, u_\varepsilon = A^*f + \varepsilon \, h$ in D in the sense of distributions. The boundary condition $t(u_\varepsilon) = 0$ on Γ follows from Lemma 1.3. Finally, $n(Au_\varepsilon) = n(f)$ holds in the sense that $n(Au_\varepsilon - f)$ on $\partial D \setminus \Gamma^\circ$ because

$$\begin{aligned} A^*(Au_\varepsilon - f) & = & \varepsilon(h - u_\varepsilon) \\ & \in & L^2(D, E) \end{aligned}$$

in D and

$$
\int_D \left((A^*(Au_\varepsilon - f), v)_x - (Au_\varepsilon - f, Av)_x \right) dx
$$
$$
= \varepsilon (h - u_\varepsilon, v)_{L^2(D,E)} - (Au_\varepsilon - f, Av)_{L^2(D,F)}
$$
$$
= 0
$$

for all $v \in C^\infty(\overline{D}, E)$ satisfying $t(v) = 0$ on Γ, the latter equality being due to (3.2). If the restriction of $n(f)$ to $\partial D \setminus \Gamma^\circ$ makes sense, then the restriction of $n(Au)$ does so.

If considered in appropriate function spaces, the mixed problem (3.3) gives certainly nothing but (3.2).

In general, mixed problems (2.5) and (3.3) have non-coercive boundary conditions on $\partial D \setminus \Gamma^\circ$. Hence they fail to be well-posed in the relevant weighted Sobolev spaces, cf. [HS01]. The principal difference between Problems 2.1 and 3.1 is that the last one is well-posed in $\mathcal{D}_T$.

Lemma 3.1. *For every $\varepsilon > 0$, $f \in L^2(D, F)$ and $h \in L^2(D, E)$ there exists a unique solution $u_\varepsilon(f, h) \in \mathcal{D}_T$ to Problem 3.1. Moreover, it satisfies*

$$
\|u_\varepsilon(f, h)\|_\varepsilon \le \|f\|_{L^2(D,F)} + \sqrt{\varepsilon}\, \|h\|_{L^2(D,E)}.
$$

Proof. Really, the estimates (3.1) imply that the vector space $\mathcal{D}_T$ endowed with the scalar product $(\cdot, \cdot)_\varepsilon$ is a Hilbert space. The Schwarz inequality yields

$$
\left| (f, Av)_{L^2(D,F)} + \varepsilon (h, v)_{L^2(D,E)} \right|
$$
$$
\le \|f\|_{L^2(D,F)} \|Av\|_{L^2(D,F)} + \varepsilon \|h\|_{L^2(D,F)} \|v\|_{L^2(D,E)}
$$
$$
\le \|f\|_{L^2(D,F)} \|v\|_\varepsilon + \sqrt{\varepsilon} \|h\|_{L^2(D,F)} \sqrt{\varepsilon \|v\|^2_{L^2(D,E)}}
$$
$$
\le c_\varepsilon(f, h)\, \|v\|_\varepsilon
$$

with

$$
c_\varepsilon(f, h) = \|f\|_{L^2(D,F)} + \sqrt{\varepsilon}\, \|h\|_{L^2(D,E)}.
$$

Hence the map

$$
v \mapsto (f, Av)_{L^2(D,F)} + \varepsilon (h, v)_{L^2(D,E)}
$$

defines a continuous linear functional $\mathcal{F}_{f,h}$ on $\mathcal{D}_T$, whose norm is majorised by $\|\mathcal{F}_{f,h}\| \le c_\varepsilon(f, h)$.

We now use the Riesz theorem to conclude that there exists a unique element $u_\varepsilon(f, h) \in \mathcal{D}_T$ with

$$
\mathcal{F}_{f,h}(v) = (u_\varepsilon(f, h), v)_\varepsilon
$$

for every $v \in \mathcal{D}_T$. Clearly, $u_\varepsilon(f, h)$ is a solution to Problem 3.1. Finally, by the Riesz theorem we get

$$
\|u_\varepsilon(f, h)\|_\varepsilon \le c_\varepsilon(f, h),
$$

as desired. $\square$

The equations (3.3) show that Lemma 3.1 gives information on the solvability of a mixed problem for the elliptic operator $A^*A + \varepsilon$ with very special data on D, Γ and $\partial D \setminus \Gamma^\circ$. Let us clarify what kind solvability theorems can be obtained for arbitrary data.

For a triple $w \in L^2(D, E)$ and

$$(3.4) \qquad \begin{aligned} u_0 &\in \ \oplus_{j=0}^{m-1} H^{2m-m_j-1/2}(\Gamma, F_j), \\ u_1 &\in \ \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\partial D \setminus \Gamma^\circ, F_j), \end{aligned}$$

we investigate the problem of finding a section u of the bundle E over D which satisfies

$$(3.5) \qquad \left\{ \begin{aligned} A^*Au + \varepsilon\, u &= w & \text{in} \quad D; \\ t(u) &= u_0 & \text{on} \quad \Gamma, \\ n(Au) &= u_1 & \text{on} \quad \partial D \setminus \Gamma^\circ, \end{aligned} \right.$$

the equations in D and on the boundary of D being understood in a proper sense. From what has already been proved it is clear what we mean by this proper sense, namely

$$\begin{aligned} (Au, g)_{L^2(D,F)} - (u, A^*g)_{L^2(D,E)} &= (u_0, n(g))_{\oplus L^2(\Gamma, F_j)}, \\ (u, v)_\varepsilon &= (w, v)_{L^2(D,E)} - (u_1, t(v))_{\oplus L^2(\partial D \setminus \Gamma^\circ, F_j)} \end{aligned}$$

$$(3.6)$$

for all $g \in C^\infty(\overline{D}, F)$ satisfying $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$, and for all $v \in C^\infty(\overline{D}, E)$ satisfying $t(v) = 0$ on Γ, respectively.

Theorem 3.2. *Let $(A^*A)^2$ possess the Unique Continuation Property $(U)_s$. Then, for every triple (w, u_0, u_1) there exists a unique solution $u \in \mathcal{D}_A \cap H^{2m}_{\mathrm{loc}}(D \cup \Gamma^\circ, E)$ to Problem 3.5. Moreover, there is a constant $C(\varepsilon) > 0$ which does not depend on (w, u_0, u_1), such that*

$$(3.7)$$
$$\|u\|_\varepsilon^2 \le C(\varepsilon) \Big(\|w\|_{L^2(D,E)}^2 + \|u_0\|_{\oplus H^{2m-m_j-1/2}(\Gamma, F_j)}^2 + \|u_1\|_{\oplus H^{m-m_j-1/2}(\partial D \setminus \Gamma^\circ, F_j)}^2 \Big).$$

Proof. Choose arbitrary u_0 and u_1 as in (3.4). Obviously, there are sections

$$\begin{aligned} U_0 &\in \ \oplus_{j=0}^{m-1} H^{2m-m_j-1/2}(\partial D, F_j), \\ U_1 &\in \ \oplus_{j=0}^{m-1} H^{m-m_j-1/2}(\partial D, F_j), \end{aligned}$$

such that $U_0 = u_0$ on Γ, $U_1 = u_1$ on $\partial D \setminus \Gamma^\circ$ and

$$\|U_0\|_{\oplus H^{2m-m_j-1/2}(\partial D, F_j)}^2 + \|U_1\|_{\oplus H^{m-m_j-1/2}(\partial D, F_j)}^2$$
$$\le 2 \Big(\|u_0\|_{\oplus H^{2m-m_j-1/2}(\Gamma, F_j)}^2 + \|u_1\|_{\oplus H^{m-m_j-1/2}(\partial D \setminus \Gamma^\circ, F_j)}^2 \Big).$$

$$(3.8)$$

As the pair $\{t, n \circ A\}$ is a Dirichlet system of order $2m - 1$ on ∂D, solving the Dirichlet problem for $(A^*A)^2$ yields a section $U' \in H^{2m}(D, E)$ with the following properties

$$(3.9) \qquad \left\{ \begin{aligned} (A^*A)^2 U' &= 0 & \text{in} \quad D; \\ t(U') &= U_0 & \text{on} \quad \partial D, \\ n(AU') &= U_1 & \text{on} \quad \partial D. \end{aligned} \right.$$

Moreover, there exists a positive constant $C > 0$ which is independent of U, such that

$$(3.10) \quad \|U'\|_{H^{2m}(D,E)}^2 \le C \Big(\|U_0\|_{\oplus H^{2m-m_j-1/2}(\partial D, F_j)}^2 + \|U_1\|_{\oplus H^{m-m_j-1/2}(\partial D, F_j)}^2 \Big),$$

see for instance [Tar95b].

According to Lemma 3.1 there exists a solution $U'' \in \mathcal{D}_T$ to Problem 3.1 with $f = 0$ and

$$
\begin{aligned}
h &= \frac{1}{\varepsilon}(w - A^*AU') - U' \\
&\in L^2(D, E).
\end{aligned}
$$

Set $u = U' + U''$. Then, integrating by parts and using Lemma 3.1 we easily obtain

$$
\begin{aligned}
(u, v)_\varepsilon &= ((A^*A + \varepsilon)U', v)_{L^2(D,E)} - (n(AU'), t(v))_{\oplus L^2(\partial D, F_j)} \\
&+ (w, v)_{L^2(D,E)} - ((A^*A + \varepsilon)U', v)_{L^2(D,E)} \\
&= (w, v)_{L^2(D,E)} - (u_1, t(v))_{\oplus L^2(\partial D \setminus \Gamma^\circ, F_j)}
\end{aligned}
$$

for every $v \in C^\infty(\overline{D}, E)$ satisfying $t(v) = 0$ on Γ, i.e., the second equality of (3.6) holds true.

On the other hand, for every $g \in C^\infty(\overline{D}, F)$ satisfying $n(g) = 0$ on $\partial D \setminus \Gamma^\circ$, we get

$$
\begin{aligned}
&(Au, g)_{L^2(D,F)} - (u, A^*g)_{L^2(D,E)} \\
&= (AU', g)_{L^2(D,F)} - (U', A^*g)_{L^2(D,E)} + (AU'', g)_{L^2(D,F)} - (U'', A^*g)_{L^2(D,E)} \\
&= (AU', g)_{L^2(D,F)} - (U', A^*g)_{L^2(D,E)}
\end{aligned}
$$

because $U'' \in \mathcal{D}_T$. Once again integrating by parts we obtain

$$
\begin{aligned}
(AU', g)_{L^2(D,F)} - (U', A^*g)_{L^2(D,E)} &= (t(U'), n(g))_{\oplus L^2(\partial D, F_j)} \\
&= (u_0, n(g))_{\oplus L^2(\Gamma, F_j)},
\end{aligned}
$$

i.e., the first equality of (3.6) is fulfilled.

By the elliptic regularity of the Dirichlet problem for the operator $A^*A + \varepsilon$ we deduce that $u \in H^{2m}_{\mathrm{loc}}(D \cup \Gamma^\circ, E)$.

If all of w and u_0, u_1 vanish then (3.6) and Theorem 1.4 imply that the corresponding solution u lies in $\mathcal{D}_T$. On the other hand, the second equality of (3.6) means that u is orthogonal to $\mathcal{D}_T$ with respect to $(\cdot, \cdot)_\varepsilon$, i.e., $u \equiv 0$ which proves the uniqueness.

Finally, according to Lemma 3.1 we get

$$
\begin{aligned}
\|u\|_\varepsilon &\leq \|U'\|_\varepsilon + \|U''\|_\varepsilon \\
&\leq c\|U'\|_{H^{2m}(D,E)} + \frac{1}{\sqrt{\varepsilon}}\Big(\|w\|_{L^2(D,E)} + \|A^*AU'\|_{L^2(D,E)}\Big) + \sqrt{\varepsilon}\|U'\|_{L^2(D,E)}.
\end{aligned}
$$

Combining this estimate with (3.8) and (3.10) we arrive at (3.7), as desired. $\square$

One sees that the regularity up to ∂D of the solution u in Theorem 3.2 fails to correspond to the smoothness of the data w and u_0, u_1. To justify this we recall that the boundary conditions $n \circ A$ on $\partial D \setminus \Gamma^\circ$ are not coercive in general. Were $n \circ A$ coercive we would have $u \in H^{2m}_{\mathrm{loc}}(\overline{D} \setminus \partial \Gamma, E)$. However, we could not guarantee even in this case that $u \in H^s(D, E)$ for some $s > m$ unless certain additional conditions were imposed on the triple (w, u_0, u_1) on $\partial \Gamma$. This is typical for the mixed problems, cf. [Esk73], [HS01] and elsewhere.

4. The main theorem

Set $u_\varepsilon(f) = u_\varepsilon(f, 0)$. The inequalities (3.1) and Lemma 3.1 give us a rough estimate for the family $\{u_\varepsilon(f)\}_{\varepsilon>0}$, namely

$$D(u_\varepsilon(f)) \leq \frac{1}{\sqrt{\varepsilon}} \|f\|_{L^2(D,F)}.$$

Thus, it might be unbounded while $\varepsilon \to 0+$.

Let us see how the behaviour of the family $\{u_\varepsilon(f)\}_{\varepsilon>0}$ reflects on the solvability of Problem 2.1.

Theorem 4.1. *The family $\{u_\varepsilon(f)\}_{\varepsilon>0}$ is bounded in $\mathcal{D}_T$ if and only if there exists $u \in \mathcal{D}_T$ satisfying (2.4).*

Proof. We first prove the following lemma.

Lemma 4.2. *Let there be a set $\Delta \subset (0, +\infty)$, such that*
  1) *zero is an accumulation point of Δ;*
  2) *the family $\{u_\delta(f)\}_{\delta\in\Delta}$ is bounded in $\mathcal{D}_T$.*

Then there exists $u \in \mathcal{D}_T$ satisfying (2.4).

Proof. Suppose zero is an accumulation point of Δ and the family $\{u_\delta(f)\}_{\delta\in\Delta}$ is bounded in $\mathcal{D}_T$. By (3.2), we have

$$(Au_\delta(f), Av)_{L^2(D,F)} + \delta\, (u_\delta(f), v)_{L^2(D,E)} = (f, Av)_{L^2(D,F)}$$

for all $v \in \mathcal{D}_T$. Passing to the limit, when $\Delta \ni \delta \to 0$, in the last equality and using the fact that $\{u_\delta(f)\}_{\delta\in\Delta}$ is bounded, we obtain

$$(4.1) \qquad \lim_{\delta\to 0+} (Au_\delta(f), Av)_{L^2(D,F)} = (f, Av)_{L^2(D,F)}$$

for all $v \in \mathcal{D}_T$.

It is well known that every bounded set in a Hilbert space is weakly compact. Hence there is a subsequence $\{u_{\delta_j}(f)\} \subset \mathcal{D}_T$ weakly convergent in $\mathcal{D}_T$ to an element $u \in \mathcal{D}_T$. Here, $\{\delta_j\}$ converges to 0 when $j \to \infty$.

Note that (3.2) implies

$$(u_\varepsilon(f), v)_{L^2(D,E)} = 0$$

for all $v \in \mathcal{D}_T \cap \mathcal{S}_A(D)$, i.e., both $\{u_{\delta_j}(f)\}$ and u are $L^2(D,E)$-orthogonal to $\mathcal{D}_T \cap \mathcal{S}_A(D)$. Let us show that $\{u_{\delta_j}(f)\}$ converges weakly to u in $L^2(D,E)$ when $j \to \infty$.

Given any $v \in L^2(D,E)$, the map $u \mapsto (u,v)_{L^2(D,E)}$ defines a continuous linear functional $\mathcal{F}_v$ on $\mathcal{D}_T$ with $\|\mathcal{F}_v\| \leq \|v\|_{L^2(D,E)}$. We now invoke the Riesz representation theorem to conclude that there exists a unique element $\tilde{v} \in \mathcal{D}_T$ with $D(u, \tilde{v}) = \mathcal{F}_v(u)$ for every $u \in \mathcal{D}_T$. Hence

$$
\begin{aligned}
\lim_{j\to\infty} (u_{\delta_j}(f), v)_{L^2(D,E)} &= \lim_{j\to\infty} D(u_{\delta_j}(f), \tilde{v}) \\
&= D(u, \tilde{v}) \\
&= (u, v)_{L^2(D,E)}.
\end{aligned}
$$

This exactly means that $\{u_{\delta_j}(f)\}$ converges weakly in $L^2(D,E)$.

Now we easily calculate

$$\lim_{\Delta \ni \delta \to 0+} (Au_\delta(f), Av)_{L^2(D,F)} = \lim_{\Delta \ni \delta \to 0+} \left(D(u_\delta(f), v) - (u_\delta(f), v)_{L^2(D,E)} \right)$$
$$= D(u, v) - (u, v)_{L^2(D,E)}$$
$$= (Au, Av)_{L^2(D,F)}$$

(4.2)

for all $v \in \mathcal{D}_T$. Combining (4.1) and (4.2) we see that (2.4) holds true for u. $\square$

Note that if (2.4) is solvable then there exists a solution u which is $L^2(D,E)$-orthogonal to $\mathcal{D}_T \cap \mathcal{S}_A(D)$.

We will have a stronger statement than Theorem 4.1 if we prove the following lemma.

Lemma 4.3. *If there exists $u \in \mathcal{D}_T$ satisfying (2.4) then the family $\{u_\varepsilon(f)\}_{\varepsilon>0}$ is bounded in $\mathcal{D}_T$ and*

$$\lim_{\varepsilon \to 0+} \|A(u_\varepsilon - u)\|_{L^2(D,F)} = 0.$$

Moreover, $\{u_\varepsilon(f)\}_{\varepsilon>0}$ converges weakly to $u \in \mathcal{D}_T$ as $\varepsilon \to 0+$, if u is $L^2(D,E)$-orthogonal to $\mathcal{D}_T \cap \mathcal{S}_A(D)$.

Proof. Let there exist $u \in \mathcal{D}_T$ satisfying (2.4). Set $R_\varepsilon = u_\varepsilon(f) - u$. Then (2.4) and (3.2) imply

(4.3) $(AR_\varepsilon, Av)_{L^2(D,F)} + \varepsilon \, (R_\varepsilon, v)_{L^2(D,E)} = -\varepsilon \, (u, v)_{L^2(D,E)}$

for all $v \in \mathcal{D}_T$, i.e., $R_\varepsilon = u_\varepsilon(0, -u)$ is the solution to Problem 3.1 with $f = 0$ and $h = -u$.

According to (3.1) and Lemma 3.1 we have

$$D(R_\varepsilon) \leq \frac{1}{\sqrt{\varepsilon}} \|R_\varepsilon\|_\varepsilon$$
$$\leq \frac{1}{\sqrt{\varepsilon}} \sqrt{\varepsilon} \|u\|_{L^2(D,E)}$$
$$= \|u\|_{L^2(D,E)}.$$

Therefore, the family $\{R_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathcal{D}_T$, and so the family $\{u_\varepsilon(f)\}_{\varepsilon>0}$ is bounded, too. Now (4.3) implies

$$\lim_{\varepsilon \to 0+} \|A(u_\varepsilon(f) - u)\|_{L^2(D,F)}^2 = \lim_{\varepsilon \to 0+} \|AR_\varepsilon\|_{L^2(D,F)}^2$$
$$= -\lim_{\varepsilon \to 0+} \varepsilon \left(\|R_\varepsilon\|_{L^2(D,E)}^2 + (u, R_\varepsilon)_{L^2(D,E)} \right)$$
$$= 0.$$

Finally, let us prove that $\{u_\varepsilon(f)\}_{\varepsilon>0}$ converges weakly to u in $\mathcal{D}_T$ as $\varepsilon \to 0+$, provided that u is $L^2(D,E)$-orthogonal to $\mathcal{D}_T \cap \mathcal{S}_A(D)$. We argue by contradiction. Indeed, if $\{u_\varepsilon(f)\}_{\varepsilon>0}$ does not converge weakly to u in $\mathcal{D}_T$ then there are $v \in \mathcal{D}_T$, $\gamma > 0$ and a sequence $\{\varepsilon_j\}$ tending to $0+$ as $j \to \infty$, such that

(4.4) $|D(u_{\varepsilon_j} - u, v)| \geq \gamma$

for every $j \in \mathbb{N}$. But the sequence $\{u_{\varepsilon_j}\}$ is bounded in the Hilbert space $\mathcal{D}_T$, and so it possesses a subsequence which converges weakly in $\mathcal{D}_T$. By abuse of notation we denote it again by $\{u_{\varepsilon_j}\}$. As we have already seen in the proof of Lemma 4.2,

the weak limit of $\{u_{\varepsilon_j}\}$ is u. This contradicts (4.4), and thus the assertion of the lemma is proved. $\qquad\square$

The proof of Theorem 4.1 is complete. $\qquad\square$

Note that if Problem 2.1 is solvable then there exists a unique solution u which is $L^2(D, E)$-orthogonal to $\mathcal{D}_T \cap \mathcal{S}_A(D)$.

Corollary 4.4. *Suppose f belongs to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$. Then the family $\{u_\varepsilon(f)\}_{\varepsilon > 0}$ is bounded in $\mathcal{D}_T$ if and only if Problem 2.1 is solvable. Moreover,*

$$\lim_{\varepsilon \to 0+} \|Au_\varepsilon(f) - f\|_{L^2(D,F)} = 0$$

and even $\{u_\varepsilon(f)\}_{\varepsilon > 0}$ converges weakly, when $\varepsilon \to 0+$, to the solution $u \in \mathcal{D}_T$ of Problem 2.1 which is $L^2(D, E)$-orthogonal to $\mathcal{D}_T \cap \mathcal{S}_A(D)$.

Proof. This follows from Theorem 4.1 and Lemmas 4.3 and 2.5. $\qquad\square$

Is it true that $\{u_\varepsilon(f)\}_{\varepsilon > 0}$ converges to u in the topology of $H^m_{\mathrm{loc}}(D \cup \Gamma^\circ, E)$ if $u \in \mathcal{D}_T$ is the solution to Problem 2.1 which is $L^2(D, E)$-orthogonal to $\mathcal{D}_T \cap \mathcal{S}_A(D)$? To answer this question we observe, by Lemma 4.3, that the family $\{u_\varepsilon(f) - u\}_{\varepsilon > 0}$ is bounded in $\mathcal{D}_T$ and

$$\lim_{\varepsilon \to 0+} \|A(u_\varepsilon(f) - u)\|_{L^2(D,F)} \;=\; 0,$$
$$t(u_\varepsilon(f) - u) \;=\; 0$$

on Γ for every $\varepsilon > 0$. Then, applying [Tar97, Theorem 7.2.6] we see that $\{u_\varepsilon(f)\}_{\varepsilon > 0}$ converges to u in $H^m_{\mathrm{loc}}(D \cup \Gamma^\circ, E)$.

5. The well-posed case

It is well known that a linear operator $T : H \to \tilde{H}$ in normed spaces has a continuous inverse if and only if $\|u\|_H \le c\,\|Tu\|_{\tilde{H}}$ for every $u \in H$, the constant $c > 0$ being independent of u. Hence, the (Cauchy) Problem 2.1 is well-posed if and only if there exists a constant $c > 0$ such that

$$(5.1) \qquad \|u\|_{L^2(D,E)} \le c\|Au\|_{L^2(D,F)}$$

for all $u \in \mathcal{D}_T$.

Theorem 5.1. *Let the (Cauchy) Problem 2.1 be well posed. Then for every $f \in L^2(D, F)$ there exists a limit*

$$u = \lim_{\varepsilon \to 0+} u_\varepsilon(f)$$

in $\mathcal{D}_T$. Moreover, u is the solution to Problem 2.1 if f belongs to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$.

Proof. Indeed, it follows from (5.1) that the Hermitian form

$$h(u, v) := (Au, Av)_{L^2(D,F)}$$

defines a scalar product on $\mathcal{D}_T$ inducing the same topology as the original one. We now use the Riesz representation theorem to see that for every $f \in L^2(D, F)$ there is a unique element $u \in \mathcal{D}_T$ satisfying (2.4).

Moreover, (5.1) yields

$$D(u_\varepsilon(f) - u) \le \sqrt{c + 1}\,\|u_\varepsilon(f) - u\|_\varepsilon.$$

Then using (4.3) and Lemma 3.1 we see that

$$\begin{aligned}
D(u_\varepsilon(f) - u) &\leq \sqrt{c+1}\,\|u_\varepsilon(0, -u)\|_\varepsilon \\
&\leq \sqrt{c+1}\,\sqrt{\varepsilon}\,\|u\|_{L^2(D,E)}.
\end{aligned}$$

Therefore, we get

$$\lim_{\varepsilon \to 0+} D(u_\varepsilon(f) - u) = 0,$$

and so Corollary 4.4 shows that u is a solution to Problem 2.1 provided f belongs to the closure of $A\mathcal{D}_T$ in $L^2(D, F)$. □

Apparently, if A is a differential operator with finite-dimensional kernel $\mathcal{S}_A(D)$ then the (Cauchy) Problem 2.1 is well posed for A.

Example 5.1. Let $X = \mathbb{R}$, $A = d/dx$, $D = (a, b)$ with $-\infty < a < b < \infty$, and $\Gamma = \{a\}$. Then $\mathcal{D}_A = H^1(D)$. The Cauchy problem

$$\begin{cases} u'(x) &= f(x) \quad \text{for} \quad x \in (a, b), \\ u(a) &= u_0, \end{cases}$$

with $u_0 \in \mathbb{R}$, is known to be well posed in Sobolev spaces as well as in spaces of smooth functions on $[a, b]$. Its solution can be easily found by the formula

$$u(x) = u_0 + \int_a^x f(y)\,dy.$$

Let us look at the corresponding family of mixed problems. In this case we have $A^* = -d/dx$ and $\partial D \setminus \Gamma^\circ = \{b\}$, hence the mixed problems are

$$\begin{cases} u_\varepsilon''(x) - \varepsilon\,u_\varepsilon(x) &= f'(x) \quad \text{for} \quad x \in (a, b), \\ u_\varepsilon(a) &= u_0, \\ u_\varepsilon'(b) &= f(b), \end{cases}$$

where $u_0 \in \mathbb{R}$ is arbitrary. One easily calculates that

$$u_\varepsilon(x) = u_0 + \int_a^x f(y)\cosh(\sqrt{\varepsilon}(x-y))\,dy + \frac{\sinh(\sqrt{\varepsilon}(x-a))}{\cosh(\sqrt{\varepsilon}(b-a))}\int_a^b f(y)\sinh(\sqrt{\varepsilon}(b-y))\,dy$$

and

$$\lim_{\varepsilon \to 0+} u_\varepsilon = u$$

even in the norm of $C^1[a, b]$, if $f \in C[a, b]$.

6. FINDING THE SOLUTION

Let us discuss the very important question of how to find the solution of Problem 3.1, and hence a solution to Problem 2.1. Of course, if an explicit orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ in the space $\mathcal{D}_T$ with the scalar product $(\cdot, \cdot)_\varepsilon$ is available, then one easily obtains

$$(6.1) \qquad\qquad u_\varepsilon(f, h) = \sum_{j=1}^\infty (u_\varepsilon(f, h), e_i)_\varepsilon\, e_i.$$

According to (3.2) we have

$$(6.2) \qquad\qquad (u_\varepsilon(f, h), e_i)_\varepsilon = (f, Ae_i)_{L^2(D,F)} + \varepsilon\,(h, e_i)_{L^2(D,E)},$$

hence (6.1) and (6.2) give us a complete description of the solution $u_\varepsilon(f, h)$ to Problem 3.1. Unfortunately, it is not an easy task to construct an explicit basis $\{e_i\}_{i \in \mathbb{N}}$.

Example 6.1. Let $\Gamma = \partial D \cap S$ where S is a sufficiently smooth hypersurface near ∂D. Choose a defining function $\delta(x)$ for S. Then we can start with a linearly independent system of the form $\{(\delta(x))^{m-1} P_i(x)\}$ in $\mathcal{D}_T$, where $P_i(x)$ are polynomials of increasing degree taking their values in E_x. Orthogonalising it by the standard Gram-Schmidt procedure we arrive at an orthonormal system in $\mathcal{D}_T$. In order to obtain a basis we have certainly to guarantee that the system $\{(\delta(x))^{m-1} P_i(x)\}$ be dense in $\mathcal{D}_T$. However, for applications it suffices to have merely a finite number of basis elements.

Let us describe an alternative way of finding the solution. Assume that the operator $A^* A + \varepsilon$ possesses the Unique Continuation property $(U)_s$ in a neighbourhood of $\overline{D}$. Then it has a two-sided fundamental solution there (see for instance [Tar95a]). Fix such a fundamental solution $\Phi_\varepsilon(x, y)$ for $A^* A + \varepsilon$. For each $s \geq 0$, it induces a continuous linear map $\Phi_\varepsilon : H^s(D, E) \to H^{s+2m}(D, E)$ by $u \mapsto r_+ \Phi_\varepsilon(e_+ u)$ where e_+ means the extension by zero to all of X and r_+ the restriction to D. This map actually extends to a continuous map $\Phi_\varepsilon : H^s(D, E) \to H^{s+2m}(D, E)$ for all $s \in \mathbb{R}$, being a right inverse of $A^* A + \varepsilon$. Every element $u \in \mathcal{D}_A$ may be thus written in the form

$$(6.3) \qquad u = U + \Phi_\varepsilon((A^* A + \varepsilon)u),$$

where $U \in \mathcal{D}_A \cap \mathcal{S}_{A^* A + \varepsilon}(D)$. Indeed, fix $u \in \mathcal{D}_A$. Since $Au \in L^2(D, F)$ we deduce that $A^* Au \in H^{-m}(D, E)$. It follows that

$$
\begin{aligned}
\Phi_\varepsilon((A^* A + \varepsilon)u) \quad &\in \quad H^m(D, E) \\
&\subset \quad \mathcal{D}_A.
\end{aligned}
$$

Setting $U = u - \Phi_\varepsilon((A^* A + \varepsilon)u)$ yields readily (6.3) with $U \in \mathcal{D}_A \cap \mathcal{S}_{A^* A + \varepsilon}(D)$, as desired.

In practice one usually has only a complete linearly independent system $\{U_i\}_{i \in \mathbb{N}}$ of solutions to $(A^* A + \varepsilon)U = 0$ on neighborhoods of $\overline{D}$, or even on all of X°.

Lemma 6.1. *Assume that $A^* A + \varepsilon$ possesses the Unique Continuation Property $(U)_s$. If $M \subset \mathcal{S}_{A^* A + \varepsilon}(\overline{D})$ is a dense set in $C^{m-1}(\overline{D}, E) \cap \mathcal{S}_{A^* A + \varepsilon}(D)$ then it is dense in $\mathcal{D}_A \cap \mathcal{S}_{A^* A + \varepsilon}(D)$.*

Proof. When endowed with the scalar product $(\cdot, \cdot)_\varepsilon$, $\mathcal{D}_A \cap \mathcal{S}_{A^* A + \varepsilon}(D)$ is a Hilbert space. Hence it suffices to prove that the orthogonal complement of M in this space is zero.

To this end, pick $u \in \mathcal{D}_A \cap \mathcal{S}_{A^* A + \varepsilon}(D)$. Since u belongs to $L^2(D, E)$ it has a finite order of growth near ∂D, cf. [Tar95b]. It follows that the expressions $t(u)$ and $n(Au)$ have weak boundary values u_0 and u_1 in the space of distributions on ∂D.

Let $v_0 \in \oplus_{j=0}^{m-1} C^\infty(\partial D, F_j)$. As t is a Dirichlet system of order $m-1$ on ∂D, there is a section $v \in C^\infty(\overline{D}, E)$ satisfying $t(v) = v_0$. Then

$$
< u_1, v_0 > =: \lim_{\delta \to 0-} \int_{\partial D_\delta} (n(Au), v)_x \, ds_\delta(x)
$$

and the definition does not depend on the particular choice of v. Since the Dirichlet problem for $A^*A + \varepsilon$ in D is uniquely solvable over the whole scale of Sobolev spaces, we can take $v \in C^\infty(\overline{D}, E) \cap \mathcal{S}_{A^*A+\varepsilon}(D)$.

If u is orthogonal to $M \subset \mathcal{S}_{A^*A+\varepsilon}(\overline{D})$ with respect to the scalar product $(\cdot, \cdot)_\varepsilon$ then

$$
\begin{aligned}
0 &= (u, v)_\varepsilon \\
&= \lim_{\delta \to 0-} \int_{D_\delta} (Au, Av)_x \, dx + \varepsilon \, (u, v)_{L^2(D,E)} \\
&= \lim_{\delta \to 0-} \left(\int_{\partial D_\delta} (n(Au), t(v))_x \, ds_\delta(x) + \int_{D_\delta} (A^*Au, v)_x \, dx \right) + \varepsilon \, (u, v)_{L^2(D,E)} \\
&= \lim_{\delta \to 0-} \int_{\partial D_\delta} (n(Au), t(v))_x \, ds_\delta(x)
\end{aligned}
$$

for all $v \in M$. As M is dense in $C^{m-1}(\overline{D}, E) \cap \mathcal{S}_{A^*A+\varepsilon}(D)$ it follows that $n(Au) = 0$ on ∂D.

On the other hand, since $u \in \mathcal{D}_A$ it can be approximated in the norm $D(\cdot)$ by a sequence $\{u_k\} \subset C^\infty(\overline{D}, E)$. Then

$$
\begin{aligned}
(u, u)_\varepsilon &= \lim_{k \to \infty} (u, u_k)_\varepsilon \\
&= \lim_{k \to \infty} \lim_{\delta \to 0-} \int_{\partial D_\delta} (n(Au), u_k)_x \, ds_\delta(x) \\
&= 0
\end{aligned}
$$

whence $u \equiv 0$ in D. $\square$

For $M = \mathcal{S}_{A^*A+\varepsilon}(X^\circ)$, the hypothesis of Lemma 6.1 is not too restrictive. It is fulfilled, e.g., if the complement of D has no compact components in X°, see [Tar95a]. In particular, this is the case if ∂D is connected.

Applying to $\{U_i\}_{i \in \mathbb{N}}$ the Gram-Schmidt orthogonalisation procedure with respect to the scalar product $(\cdot, \cdot)_\varepsilon$, we obtain an orthonormal basis $\{b_i = b_i(\varepsilon)\}_{i \in \mathbb{N}}$ in $\mathcal{D}_A \cap \mathcal{S}_{A^*A+\varepsilon}(D)$.

The equality (6.3) suggest us to look for solutions to mixed Problem 3.1 of the form

$$
(6.4) \qquad u_\varepsilon(f, h) = \Phi_\varepsilon(A^*f + \varepsilon h) + \sum_{i=1}^\infty c_i(\varepsilon) b_i(\varepsilon)
$$

where the series on the right-hand side converges in $\mathcal{D}_A$. The point is to find the coefficients $c_i(\varepsilon)$ through f and h. For this purpose, we denote by $\Pi_{\Gamma, \varepsilon}$ the orthogonal projection

$$
\mathcal{D}_A \cap \mathcal{S}_{A^*A+\varepsilon}(D) \to \mathcal{D}_T \cap \mathcal{S}_{A^*A+\varepsilon}(D)
$$

with respect to the scalar product $(\cdot, \cdot)_\varepsilon$.

Lemma 6.2. *Each solution $u_\varepsilon(f, h) \in \mathcal{D}_T$ of Problem 3.1 may be written as the series (6.4) where*

$$
c_i(\varepsilon) = (f, A\Pi_{\Gamma, \varepsilon} b_i)_{L^2(D,F)} + \varepsilon \, (h, \Pi_{\Gamma, \varepsilon} b_i)_{L^2(D,E)} - (\Phi_\varepsilon(A^*f + \varepsilon h), b_i)_\varepsilon.
$$

Proof. Indeed, let $u_\varepsilon \in \mathcal{D}_T$ be a solution of Problem 3.1. As we have seen in §3,

$$
(A^*A + \varepsilon) u_\varepsilon = A^*f + \varepsilon h
$$

in D. Using (6.3) we easily arrive at (6.4) with some uniquely defined coefficients $c_i(\varepsilon)$.

Write $\tilde{\Pi}_{\Gamma,\varepsilon}$ for the orthogonal projection $\mathcal{D}_A \to \mathcal{D}_T$ with respect to $(\cdot,\cdot)_\varepsilon$. Since $\tilde{\Pi}_{\Gamma,\varepsilon}$ is self-adjoint in $\mathcal{D}_A$, we get

$$
\begin{aligned}
(6.5) \qquad (u_\varepsilon, b_i)_\varepsilon &= (\tilde{\Pi}_{\Gamma,\varepsilon} u_\varepsilon, b_i)_\varepsilon \\
&= (u_\varepsilon, \tilde{\Pi}_{\Gamma,\varepsilon} b_i)_\varepsilon \\
&= (f, A\tilde{\Pi}_{\Gamma,\varepsilon} b_i)_{L^2(D,F)} + \varepsilon\,(h, \tilde{\Pi}_{\Gamma,\varepsilon} b_i)_{L^2(D,E)},
\end{aligned}
$$

the last equality being a consequence of (3.2).

Now (6.4) implies

$$
(u_\varepsilon, b_i)_\varepsilon = (\Phi_\varepsilon(A^* f + \varepsilon h), b_i)_\varepsilon + c_i(\varepsilon).
$$

Combining this with (6.5) yields

$$
c_i(\varepsilon) = (f, A\tilde{\Pi}_{\Gamma,\varepsilon} b_i)_{L^2(D,F)} + \varepsilon\,(h, \tilde{\Pi}_{\Gamma,\varepsilon} b_i)_{L^2(D,E)} - (\Phi_\varepsilon(A^* f + \varepsilon h), b_i)_\varepsilon.
$$

Finally, for every $v \in C^\infty_{\mathrm{comp}}(D, E)$ we get

$$
\begin{aligned}
(\tilde{\Pi}_{\Gamma,\varepsilon} b_i, (A^*A + \varepsilon)v)_{L^2(D,E)} &= (A\tilde{\Pi}_{\Gamma,\varepsilon} b_i, Av)_{L^2(D,F)} + \varepsilon\,(\tilde{\Pi}_{\Gamma,\varepsilon} b_i, v)_{L^2(D,E)} \\
&= (b_i, \tilde{\Pi}_{\Gamma,\varepsilon} v)_\varepsilon \\
&= (b_i, v)_\varepsilon \\
&= ((A^*A + \varepsilon) b_i, v)_{L^2(D,E)} \\
&= 0.
\end{aligned}
$$

This means that $\tilde{\Pi}_{\Gamma,\varepsilon} b_i$ belongs to $\mathcal{D}_T \cap \mathcal{S}_{A^*A+\varepsilon}(D)$ whence $\tilde{\Pi}_{\Gamma,\varepsilon} b_i = \Pi_{\Gamma,\varepsilon} b_i$, showing the lemma. $\qquad\square$

We have thus derived expressions for the coefficients $c_i(\varepsilon)$ through f and h. However, it is not an easy task to explicitly construct the family of projections $\{\Pi_{\Gamma,\varepsilon}\}$.

Lemma 6.3. *For every $u \in \mathcal{D}_A \cap \mathcal{S}_{A^*A+\varepsilon}(D)$, the projection $\Pi_{\Gamma,\varepsilon} u$ just amounts to the solution of Problem 3.1 with $f = Au$ and $h = u$.*

Proof. By the very definition, $\Pi_{\Gamma,\varepsilon} u \in \mathcal{D}_T \cap \mathcal{S}_{A^*A+\varepsilon}(D)$ and

$$
(u - \Pi_{\Gamma,\varepsilon} u, v)_\varepsilon = 0
$$

for all $v \in \mathcal{D}_T$ satisfying $(A^*A + \varepsilon)v = 0$ in D.

Further, the solution $u_\varepsilon = u_\varepsilon(Au, u)$ of Problem 3.1 with $f = Au$ and $h = u$ belongs to $\mathcal{D}_T \cap \mathcal{S}_{A^*A+\varepsilon}(D)$ because $A^* f + \varepsilon h = (A^*A + \varepsilon)u = 0$. Moreover, (3.3) gives

$$
(u - u_\varepsilon, v)_\varepsilon = 0
$$

for all $v \in \mathcal{D}_T$.

We wish to show that $\Pi_{\Gamma,\varepsilon} u = u_\varepsilon$, which is equivalent to $\|\Pi_{\Gamma,\varepsilon} u - u_\varepsilon\|_\varepsilon = 0$. To this end, write

$$
\begin{aligned}
(\Pi_{\Gamma,\varepsilon} u - u_\varepsilon, \Pi_{\Gamma,\varepsilon} u - u_\varepsilon)_\varepsilon &= -((u - \Pi_{\Gamma,\varepsilon} u) - (u - u_\varepsilon), \Pi_{\Gamma,\varepsilon} u - u_\varepsilon)_\varepsilon \\
&= -(u - \Pi_{\Gamma,\varepsilon} u, \Pi_{\Gamma,\varepsilon} u - u_\varepsilon)_\varepsilon - (u - u_\varepsilon, \Pi_{\Gamma,\varepsilon} u - u_\varepsilon)_\varepsilon.
\end{aligned}
$$

By the above, both summands on the right-hand side vanish because $\Pi_{\Gamma,\varepsilon} u - u_\varepsilon$ belongs to $\mathcal{D}_T \cap \mathcal{S}_{A^*A+\varepsilon}(D)$. $\qquad\square$

Of course, the lemma does not allow one to effectively determine the Fourier coefficients c_i. On the one hand, to find c_i we only need to know $\Pi_{\Gamma,\varepsilon} b_i$. On the other hand, this requires, by Lemma 6.3, a solution of Problem 3.1 with very special data f and h.

Let us now describe how to find solutions to Problem 3.1 for "good" data. For this purpose we introduce for $s \geq 2m$ the Hermitian form

$$h(u,v) = (t(u), t(v))_{\oplus H^{s-m_j-1/2}(\Gamma, F_j)} + (n(Au), n(Av))_{\oplus H^{s-m-m_j-1/2}(\partial D \setminus \Gamma^\circ, F_j)}$$

on the space H of all $u \in \mathcal{D}_A \cap \mathcal{S}_{A^*A+\varepsilon}(D)$ with the property that

$$\begin{aligned} t(u) &\in \oplus_{j=0}^{m-1} H^{s-m_j-1/2}(\Gamma, F_j), \\ n(Au) &\in \oplus_{j=0}^{m-1} H^{s-m-m_j-1/2}(\partial D \setminus \Gamma^\circ, F_j), \end{aligned}$$

the expressions $t(u)$ and $n(Au)$ being understood in the sense of weak boundary values.

Lemma 6.4. *Suppose $s \geq 2m$. When endowed with the scalar product $h(\cdot, \cdot)$, H is a Hilbert space.*

Proof. Indeed, (3.7) implies that $h(\cdot, \cdot)$ is a scalar product on H. Moreover, if $\{u_k\}$ is a Cauchy sequence in H then it a Cauchy sequence in $\mathcal{D}_A \cap \mathcal{S}_{A^*A+\varepsilon}(D)$. Since this latter is a Hilbert space, $\{u_k\}$ has a limit u in this space. Moreover, both $\{t(u_k)\}$ and $\{n(Au_k)\}$ converge to $t(u)$ and $n(Au)$ in the space of distributions on ∂D, or, more precisely, in $\oplus_{j=0}^{m-1} H^{-m_j-1/2}(\partial D, F_j)$ and $\oplus_{j=0}^{m-1} H^{-m-m_j-1/2}(\partial D, F_j)$, respectively. By assumption, $\{t(u_k)\}$ and $\{n(Au_k)\}$ are Cauchy sequences in the Hilbert spaces $\oplus_{j=0}^{m-1} H^{s-m_j-1/2}(\Gamma, F_j)$ and $\oplus_{j=0}^{m-1} H^{s-m-m_j-1/2}(\partial D \setminus \Gamma^\circ, F_j)$, respectively. Hence, they converge to elements u_0 and u_1 in these spaces. Finally, the uniqueness of a limit yields $t(u) = u_0$ on Γ and $n(Au) = u_1$ on $\partial D \setminus \Gamma^\circ$, i.e., $u \in H$, which completes the proof. $\square$

Let $\{U_i\}_{i \in \mathbb{N}}$ be a complete linearly independent system in H. Applying the Gram-Schmidt orthogonalisation to $\{U_i\}_{i \in \mathbb{N}}$ we get an orthonormal basis $\{B_i\}_{i \in \mathbb{N}}$ in H.

Theorem 6.5. *Let $s \geq 2m$. Then, for every $w \in H^{s-2m}(D, E)$ and*

$$\begin{aligned} u_0 &\in \oplus_{j=0}^{m-1} H^{s-m_j-1/2}(\Gamma, F_j), \\ u_1 &\in \oplus_{j=0}^{m-1} H^{s-m-m_j-1/2}(\partial D \setminus \Gamma^\circ, F_j), \end{aligned}$$

the series

$$u = \Phi_\varepsilon(w) + \sum_{i=1}^{\infty} k_i B_i$$

converges in $\mathcal{D}_A$ and satisfies (3.5), provided that

$$k_i = h(u - \Phi_\varepsilon(w), B_i).$$

Proof. This is a direct consequence of Theorem 3.2. Recall that the boundary equations $t(u) = u_0$ on Γ and $n(Au) = u_1$ on $\partial D \setminus \Gamma^\circ$ are interpreted in the sense of (3.6). $\square$

From this theorem we deduce, in particular, that

$$\Pi_{\Gamma,\varepsilon} b_i = \sum_{q=1}^{\infty} k_i B_i,$$

with the coefficients

$$k_i = (n(Ab_i)), n(AB_i))_{\oplus H^{s-m-m_j-1/2}(\partial D \setminus \Gamma^\circ, F_j)}.$$

Knowing $\Pi_{\Gamma,\varepsilon} b_i$ we can find, by Lemma 6.2, the solution of Problem 3.1 for any data $f \in L^2(D, F)$ and $h \in L^2(D, F)$. Of course, if both f and h are smooth enough, namely $f \in H^{s-m}(D, F)$ and $h \in H^{s-2m}(D, E)$ with $s \geq 2m$, then we can determine the solution of Problem 3.1 directly by Theorem 6.5.

One question still unanswered is whether a complete system $\{U_i\}_{i \in \mathbb{N}}$ in H may be chosen to consists of solutions to $(A^*A + \varepsilon)u = 0$ on neighbourhoods of $\overline{D}$. Analysis similar to that in the proof of Lemma 6.1 shows that this is always the case if ∂D is smooth enough, e.g., of class C^{2m-1}.

7. Dirac operators

Let $X = \mathbb{R}^n$, where $n \geq 2$, and $E = \mathbb{R}^n \times \mathbb{C}^k$, $F = \mathbb{R}^n \times \mathbb{C}^l$. The sections of E are functions of n real variables with values in $\mathbb{C}^k$, and similarly for F.

Let A be a Dirac operator, i.e., a homogeneous first order differential operator with constant coefficients in $\mathbb{R}^n$,

$$A = \sum_{j=1}^n A_j \frac{\partial}{\partial x_j},$$

such that

$$(7.1) \qquad (\sigma(A)(\xi))^* \sigma(A)(\xi) = |\xi|^2 E_k$$

for all $\xi \in \mathbb{R}^n$. Here, A_j are $(l \times k)$-matrices of complex numbers and E_k is the identity $(k \times k)$-matrix.

The Dirac operators satisfy $A^*A = -E_k \Delta$, where Δ is the usual Laplace operator in $\mathbb{R}^n$.

The perturbed mixed problem (3.3) reads as

$$\begin{cases} (-\Delta + \varepsilon)u_\varepsilon & = & A^*f & \text{in} & D; \\ t(u_\varepsilon) & = & 0 & \text{on} & \Gamma, \\ n(Au_\varepsilon) & = & n(f) & \text{on} & \partial D \setminus \Gamma^\circ, \end{cases}$$

where

$$n(f) = (\sigma(A)(\nabla \varrho))^* f$$

and ϱ is a defining function of D in the sense of (1.2). Thus, this is a family of mixed problems for the Helmholtz equation.

We are going to study the (Cauchy) Problem 2.1 on the unit ball $D = \mathbb{B}$ in $\mathbb{R}^n$. To this end, we pass to spherical coordinates $x = r\,S(\varphi)$ where φ are coordinates on the unit sphere $\partial D = \mathbb{S}$ in $\mathbb{R}^n$. The Laplace operator Δ in the spherical coordinates takes the form

$$(7.2) \qquad \Delta = \frac{1}{r^2}\left(\left(r\frac{\partial}{\partial r}\right)^2 + (n-2)\left(r\frac{\partial}{\partial r}\right) - \Delta_{\mathbb{S}}\right),$$

where $\Delta_{\mathbb{S}}$ is the Laplace-Beltrami operator on the unit sphere.

To solve the homogeneous equation $(-\Delta + \varepsilon)u_\varepsilon = 0$ we make use of the Fourier method of separation of variables. Writing $u_\varepsilon(r, \varphi) = g(r, \varepsilon)h(\varphi)$ we get two separate equations for g and h, namely

$$
\begin{aligned}
\left(\left(r\frac{\partial}{\partial r} \right)^2 + (n-2)\left(r\frac{\partial}{\partial r} \right) - \varepsilon r^2 \right) g &= c\,g \\
\Delta_{\mathbb{S}} h &= c\,h,
\end{aligned}
$$

c being an arbitrary constant.

The second equation has non-zero solutions if and only if c is an eigenvalue of $\Delta_{\mathbb{S}}$. These are well known to be $c = i(n + i - 2)$, for $i = 0, 1, \dots$ (see for instance [TS72]). The corresponding eigenfunctions of $\Delta_{\mathbb{S}}$ are spherical harmonics $h_i(\varphi)$ of degree i, i.e.,

$$(7.3) \qquad \Delta_{\mathbb{S}} h_i = i(n + i - 2)\, h_i.$$

Consider now the following ordinary differential equation with respect to the variable $r > 0$

$$(7.4) \qquad \left(\left(r\frac{\partial}{\partial r} \right)^2 + (n-2)\left(r\frac{\partial}{\partial r} \right) - \left(i(n + i - 2) + \varepsilon r^2 \right) \right) g(r, \varepsilon) = 0.$$

This is a version of the Bessel equation, and the space of its solutions is two-dimensional.

For example, if $\varepsilon = 0$ then $g(r, 0) = ar^i + br^{2-i-n}$ with arbitrary constants a and b is a general solution to (7.4). In this situation any function $r^i h_i(\varphi)$ is a homogeneous harmonic polynomial. In the general case the space of solutions to (7.4) contains a one-dimensional subspace of functions bounded at the point $r = 0$, cf. [TS72].

For $i = 0, 1, \dots$, fix a non-zero solution $g_i(r, \varepsilon)$ of (7.4) which is bounded at $r = 0$. Then

$$(7.5) \qquad (-\Delta + \varepsilon)\,(g_i(r, \varepsilon)h_i(\varphi)) = 0$$

on all of $\mathbb{R}^n$. Indeed, by (7.2), (7.3) and (7.4) we conclude that this equality holds in $\mathbb{R}^n \setminus \{0\}$. We now use the fact that $g_i(r, \varepsilon)h_i(\varphi)$ is bounded at the origin to see that (7.5) holds.

It is known that there are exactly

$$J(i) = \frac{(n + 2i - 2)(n + i - 3)!}{i!(n - 2)!}$$

linearly independent spherical harmonics of degree i. In [Shl96] a system

$$\{H_i^{(j)}(\varphi)\}_{\substack{i=0,1,\dots \\ j=1,\dots,k\,J(i)}}$$

of $\mathbb{C}^k$-valued functions is constructed, such that

1) the components of $H_i^{(j)}(\varphi)$ are spherical harmonics of degree i;
2) $\{H_i^{(j)}(\varphi)\}$ is an orthonormal basis in $L^2(\mathbb{S}, E)$;
3) $\{A\,(r^i H_i^{(j)}(\varphi))\}$ is an orthogonal system in $L^2(\mathbb{B}, F)$.

More precisely, this system $\{H_i^{(j)}(\varphi)\}$ consists of eigenfunctions of the operator $n \circ A$,

$$(7.6) \qquad (\sigma(A)(rS(\varphi)))^* A\left(r^i H_i^{(j)}(\varphi) \right) = \lambda_i^{(j)}\left(r^i H_i^{(j)}(\varphi) \right),$$

where $\lambda_i^{(j)} \geq 0$.

Lemma 7.1. *The system*

$$\{b_i^{(j)}(r, \varphi, \varepsilon) := g_i(r, \varepsilon)\, H_i^{(j)}(\varphi)\}_{\substack{i=0,1,\dots \\ j=1,\dots,k\, J(i)}}$$

is orthogonal with respect to both Hermitian forms $(\cdot, \cdot)_{L^2(\mathbb{B}, E)}$ *and* $(A\cdot, A\cdot)_{L^2(\mathbb{B}, F)}$.

Proof. Indeed, as $\{H_i^{(j)}\}$ is an orthonormal basis in the space $L^2(\mathbb{S}, E)$ on the unit sphere, the system $\{b_i^{(j)}\}$ is orthogonal in $L^2(\mathbb{B}, E)$ because

$$
\begin{aligned}
(b_i^{(j)}, b_p^{(q)})_{L^2(\mathbb{B}, E)} &= (H_i^{(j)}, H_p^{(q)})_{L^2(\mathbb{S}, E)} \int_0^1 r^{n-1} g_i(r, \varepsilon) \overline{g_p(r, \varepsilon)} \, dr \\
&= 0
\end{aligned}
$$

for $i \neq p$ or $j \neq q$.

Further, integrating by parts we get

(7.7) $\quad (Ab_i^{(j)}, Ab_p^{(q)})_{L^2(\mathbb{B}, F)} = -(b_i^{(j)}, \Delta b_p^{(q)})_{L^2(\mathbb{B}, E)} + g_i(1, \varepsilon)\, (H_i^{(j)}, n(Ab_p^{(q)}))_{L^2(\mathbb{S}, E)}.$

On the other hand, (7.5) implies

(7.8) $\qquad\qquad -(b_i^{(j)}, \Delta b_p^{(q)})_{L^2(\mathbb{B}, E)} + \varepsilon\, (b_i^{(j)}, b_p^{(q)})_{L^2(\mathbb{B}, E)} = 0$

for $i \neq p$ or $j \neq q$.

Let us write the expression $n \circ A$ in spherical coordinates. Denote by $S'(\varphi)$ the Jacobi matrix of $S(\varphi)$. Set

$$(S'(\varphi))^{-1} := \left((S'(\varphi))^T\, S'(\varphi)\right)^{-1} (S'(\varphi))^T.$$

Since the rank of $S'(\varphi)$ is equal to $n-1$, the inverse matrix of $(S'(\varphi))^T\, S'(\varphi)$ exists and is smooth. Moreover, $(S'(\varphi))^{-1}$ is a left inverse for $S'(\varphi)$. An easy calculation shows that

$$\frac{\partial}{\partial x_j} = S_j(\varphi)\, \frac{\partial}{\partial r} + \frac{1}{r} \sum_{i=1}^{n-1} (S'(\varphi))_{i,j}^{-1}\, \frac{\partial}{\partial \varphi_i}$$

where $(S'(\varphi))_{i,j}^{-1}$ is the (i, j)-entry of $(S'(\varphi))^{-1}$.

Now (7.1) implies

(7.9) $\qquad n \circ A = \sum_{k=1}^{n} A_k^*\, r S_k(\varphi) \sum_{j=1}^{n} A_j\, \frac{\partial}{\partial x_j} = r \frac{\partial}{\partial r} + R(\varphi, \partial_\varphi)$

where

$$R(\varphi, \partial_\varphi) = \sum_{k=1}^{n} A_k^*\, S_k(\varphi) \sum_{j=1}^{n} A_j \sum_{i=1}^{n-1} (S'(\varphi))_{i,j}^{-1}\, \frac{\partial}{\partial \varphi_i}.$$

Using (7.6) and (7.9) we conclude that

$$
\begin{aligned}
\lambda_i^{(j)} (r^i H_i^{(j)}(\varphi)) &= n(A(r^i H_i^{(j)}(\varphi))) \\
&= i\, r^i\, H_i^{(j)}(\varphi) + r^i\, R(\varphi, \partial_\varphi) H_i^{(j)}(\varphi).
\end{aligned}
$$

Hence

$$R(\varphi, \partial_\varphi) H_i^{(j)}(\varphi) = \left(\lambda_i^{(j)} - i\right) H_i^{(j)}(\varphi),$$

and so (7.9) yields

$$(7.10) \qquad \begin{aligned} n(Ab_i^{(j)}) &= r\, g_i'\, H_i^{(j)} + g_i\, R(\varphi, \partial_\varphi) H_i^{(j)} \\ &= \left(rg_i' + (\lambda_i^{(j)} - i)g_i \right) H_i^{(j)}. \end{aligned}$$

Therefore,

$$(7.11) \qquad (H_i^{(j)}, n(Ab_p^{(q)}))_{L^2(\mathbb{S}, E)} = 0$$

for $i \neq p$ or $j \neq q$.

Combining (7.7) (7.8) and (7.11) we see that the system $\{b_i^{(j)}\}$ is orthogonal with respect to $(A\cdot, A\cdot)_{L^2(\mathbb{B}, F)}$. $\qquad \square$

Remark 7.1. Note that $g_i'(1, \varepsilon) + (\lambda_i^{(j)} - i)g_i(1, \varepsilon) \neq 0$ for all $\varepsilon > 0$. Indeed, otherwise $n(Ab_i^{(j)}) = 0$ on $\mathbb{S}$ and (7.7), (7.8) would imply $b_i^{(j)} \equiv 0$, which is wrong.

Theorem 7.2. *For every $\delta > 0$, the system*

$$\{b_i^{(j)}(r, \varphi, \varepsilon)\}_{\substack{i=0,1,\dots \\ j=1,\dots,k\, J(i)}}$$

is an orthogonal basis in the space $\mathcal{D}_A \cap \mathcal{S}_{-\Delta + \varepsilon E_k}(\mathbb{B})$ with the scalar product $(\cdot, \cdot)_\delta$.

Proof. The orthogonality follows immediately from Lemma 7.1. As for the completeness of the system $\{b_i^{(j)}\}$ in $\mathcal{D}_A \cap \mathcal{S}_{-\Delta + \varepsilon E_k}(\mathbb{B})$, we observe that the estimates (3.1) guarantee that every scalar product $(\cdot, \cdot)_\delta$ with $\delta > 0$ induces in $\mathcal{D}_A$ the same topology as $D(\cdot, \cdot)$. Hence it is sufficient to prove the completeness for $\delta = 1$. Finally, since the system of harmonics $\{H_i^{(j)}\}$ is dense in $C^{m-1}(\mathbb{S}, E)$ we see that $\{b_i^{(j)}\}$ is dense in $C^{m-1}(\mathbb{S}, E) \cap \mathcal{S}_{-\Delta + \varepsilon E_k}(\mathbb{B})$. Then the completeness is a consequence of Lemma 6.1. $\qquad \square$

As a fundamental solution $\Phi_\varepsilon(x, y)$ of the operator $-\Delta + \varepsilon$ in $\mathbb{R}^3$ we may choose one of the standard kernels

$$\Phi_\varepsilon(x, y) = e^{\pm\sqrt{\varepsilon}|x-y|}.$$

In $\mathbb{R}^2$ we can take as $\Phi_\varepsilon(x, y)$ a Hankel function, see for instance [TS72].

Example 7.1. Let $A = \nabla$ be the gradient operator in $\mathbb{R}^n$. For every domain $D \subset\subset \mathbb{R}^n$, we have $\mathcal{D}_A = H^1(D)$. Since the estimate (5.1) holds true for ∇ (see [Mik76]), the (Cauchy) Problem 2.1 is well posed in $\mathcal{D}_T$. In this case $k = 1$, $l = n$, $A^* = -\mathrm{div}$ is a multiple of the divergence operator in $\mathbb{R}^n$ and

$$n \circ A = |x|\, \frac{\partial}{\partial n} = r\, \frac{\partial}{\partial r}$$

where $\partial/\partial n$ is the derivative along the outward unit normal vector to ∂D. In particular, this means that every homogeneous harmonic polynomial $r^i h_i$ is an eigenfunction of $n \circ A$ corresponding to the eigenvalue $\lambda_i = i$. For example, in $\mathbb{R}^2$ we can take

$$\begin{aligned} b_0^{(1)} &= \frac{1}{\sqrt{2\pi}}\, g_0(r, \varepsilon), \\ b_i^{(1)} &= \frac{1}{\sqrt{\pi}}\, g_i(r, \varepsilon) \cos(i\varphi), \\ b_i^{(2)} &= \frac{1}{\sqrt{\pi}}\, g_i(r, \varepsilon) \sin(i\varphi), \end{aligned}$$

where g_i are Hankel's functions. In the case $s = 5/2$ and $\Gamma = \{r = 1, \, \varphi \in [0, \pi]\}$ the Gram-Schmidt orthogonalisation in H gives

$$
\begin{aligned}
B_0^{(1)} &= \frac{g_0(r, \varepsilon)}{\sqrt{\pi} \sqrt{|g_0(1, \varepsilon)|^2 + |g_0'(1, \varepsilon)|^2}}, \\
B_1^{(1)} &= \frac{2\, g_1(r, \varepsilon) \cos \varphi}{\sqrt{\pi} \sqrt{|g_1(1, \varepsilon)|^2 + |g_1'(1, \varepsilon)|^2}}, \\
B_1^{(2)} &= \frac{2a g_0(r, \varepsilon) + \sqrt{\pi} g_1(r, \varepsilon) \sin \varphi}{\sqrt{b}},
\end{aligned}
$$

with

$$
\begin{aligned}
a &= g_0(1, \varepsilon) g_1(1, \varepsilon) - g_0'(1, \varepsilon) g_1'(1, \varepsilon), \\
b &= \frac{3}{2} \pi^2 + 4a(1 + |g_0(1, \varepsilon)|^2 + |g_0'(1, \varepsilon)|^2),
\end{aligned}
$$

and so on.

Example 7.2. Let $A := \partial_1 + \sqrt{-1} \partial_2$ be (2-multiple of) the Cauchy-Riemann operator in $\mathbb{C}$. Then the (Cauchy) Problem 2.1 is ill-posed in $\mathcal{D}_T$. In this case $k = l = 1$, $A^* = -\partial_1 + \sqrt{-1} \partial_2$ and

$$
n \circ A = \bar{z}\, \bar{\partial} = r \frac{\partial}{\partial r} + \sqrt{-1} \frac{\partial}{\partial \varphi}
$$

hold. The system $\{b_i^{(j)}\}$ may be chosen as follows

$$
\begin{aligned}
b_0^{(1)} &= \frac{1}{\sqrt{2\pi}}\, g_0(r, \varepsilon), \\
b_i^{(1)} &= \frac{1}{\sqrt{\pi}}\, g_i(r, \varepsilon) e^{\sqrt{-1}\, i\varphi}, \\
b_i^{(2)} &= \frac{1}{\sqrt{\pi}}\, g_i(r, \varepsilon) e^{-\sqrt{-1}\, i\varphi},
\end{aligned}
$$

with $\lambda_0^{(1)} = 0$, $\lambda_i^{(1)} = 0$ and $\lambda_i^{(2)} = 2i$.

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