A note on the ramified Cauchy problem

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ABSTRACT. – In this paper, the ramified Cauchy problem in $\mathbf{C}^2$ for operator with multiple characteristics of constant multiplicity and second member ramified around some analytic set is studied.

Keywords: Ramified Cauchy problem, Analytic continuation.

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1 Introduction

The aim of this paper is the study of the ramified Cauchy problem in $\mathbf{C}^2$ with a second member ramified around some analytic set. This problem was studied in $\mathbf{C}^{n+1}$ with a second member ramified around characteristic hypersurfaces by LEICHTNAM [5] and PONGÉRARD-WAGSCHAL [7].

First, we call back what kind of problem it is. In a neighbourhood of $0 \in \mathbf{C}^{n+1}$, the chart will be written $x = (x_0, \dots, x_n)$, this non-characteristic holomorphic Cauchy problem is studied

$$\begin{cases} a(x,D)u(x) = v(x), \\ D_0^h u(x) = w_h(x') \quad \text{for } x_0 = 0,\ 0 \leq h < m \end{cases} \tag{1}$$

where $x' = (x_1, \dots, x_n)$, $a(x,D)$ is a linear differential operator of order m with multiple characteristics of constant multiplicity. The Cauchy datas w_h are assumed ramified around the hyperplane $T : x_0 = x_1 = 0$ of $S : x_0 = 0$. We write K_i, $i = 1, \dots, d$, the characteristic hypersurfaces of the operator $a(x,D)$ which going out from T and we assume v ramified around $K = \bigcup_{i=1}^{d} K_i$. Under this assumptions, we have the LEICHTNAM's theorem. This theorem was proved by LEICHTNAM in 1990 [5] and PONGÉRARD-WAGSCHAL in 1998 [7].

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Theorem 1.1 *There is a connected open neighbourhood Ω of $0 \in \mathbf{C}^{n+1}$ such that the problem (1) admits an unique solution which is holomorphic on the universal covering $\mathcal{R}(\Omega \setminus K)$ of $\Omega \setminus K$.*

The monodromy's solution of such problems was studied by CAMALÈS [1, 2].

From point of view of methods, in [5] and [7], the authors build, in a first time, some holomorphic solutions which are ramified around each hypersurface K_i. This problem is solved by the resolution of an integro-differential problem. The two variables in this integro-differential problem are $x \in \mathbf{C}^{n+1}$ and t who lives in the universal covering of a connected open of $\mathbf{C}^2$ (in [7], it's a product of two connected opens of $\mathbf{C}$. This simplify the proof). Some gluing's technics of this solutions allow to solve the problem (1) and prove the theorem 1.1.

In this paper, we present a simpler proof of this result in the case $\mathbf{C}^2$. More generaly, an integral writting of the solution will be presented. This writting allows us the study of the analytic continuation of the solution when the second member is ramified around some analytic set. For this, we need the HAMADA-LERAY-WAGSCHAL's theorem [3] proved in 1976. This theorem is the LEICHTNAM's theorem in the case $v \equiv 0$. In this case, the solution could be written $\sum_{i=1}^{d} u_i$ where each u_i is ramified around the characteristic hypersurface K_i.

2 Notations and main theorem

We consider, in a neighbourhood of $0 \in \mathbf{C}^2$, with chart $x = (x_0, x_1)$, an holomorphic linear differential operator of order $m \geq 2$

$$a(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

where $\alpha \in \mathbf{N}^2$. We assume that the hyperplane $S : x_0 = 0$ is non-characteristic for $a(x, D)$ (so we can assume $a_{(m,0)} \equiv 1$). We note by Ω_0 a connected open neighbourhood of $0 \in \mathbf{C}^2$ such that all the functions a_α are defined and holomorphic on Ω_0. For the operator, we assume that it's a operator with multiple characteristics of constant multiplicity : this means that we can write his principal symbol under the form

$$g(x, \xi) = \prod_{i=1}^{d} (\xi_0 - \lambda_i(x)\xi_1)^{m_i} \qquad \text{for } \xi = (\xi_0, \xi_1) \tag{2}$$

with $\lambda_i(0) \neq \lambda_j(0)$ if $i \neq j$.

The resolution of following Cauchy problems of first order

$$\begin{cases} D_0 k_i(x) & = & \lambda_i(x) D_1 k_i(x), \\ k_i(x) & = & x_1 \quad \text{for } x_0 = 0 \end{cases} \tag{3}$$

allows us to define, in a neighbourhood of 0, the characteristic hypersurfaces K_i : $k_i(x) = 0$ containing 0. We note by $K = \bigcup_{i=1}^{d} K_i$ the union of these hypersurfaces. Moreover, with a change of chart, we can assume that $k_1(x) = x_1$ and then $K_1 : x_1 = 0$. The map $x \mapsto (x_0, k_i(x))$, for $i = 1, \ldots, d$ are diffeomorphisms in a neighbourhood of 0. We note $x \mapsto (x_0, l_i(x))$ the inverse of each diffeomorphism.

Remark 2.1 *With the definition of k_i and l_i, we get*

$$k_i(x_0, l_i(x)) = l_i(x_0, k_i(x)) = x_1 \qquad i = 1, \ldots, d.$$

Now, let L an analytic set such that $L \cap S = \{0\}$. We study the following Cauchy problem

$$\begin{cases} a(x, D)u(x) &= v(x), \\ \\ D_0^h u(x) &= 0 \qquad \text{for } x_0 = 0,\ 0 \leq h < m. \end{cases} \tag{4}$$

This is the assumption on the data v: let $\Omega \subset \Omega_0$ a connected open neighbourhood of $0 \in \mathbf{C}^2$ such that $\Omega \cap S$ is connected and a point $a \in \Omega \cap S \setminus \{0\}$. We assume that the germ v, holomorphic in a, has an analytic continuation along all paths with origin a and drawn in $\Omega \setminus L$. (Then v is a holomorphic function on $\mathcal{R}(\Omega \setminus L)$ where $\mathcal{R}(X)$ is the universal covering of the complex manifold X.)

This is now the aim theorem of this paper

Theorem 2.1 *The solution of problem (4) can be write, in a neighbourhood of $a \in S$ (a closed to 0),*

$$u(x) = \int_{S_m} \left(\int_\Gamma \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m)}$$

where

- $t = (t_1, \ldots, t_m) \in \mathbf{C}^m$,
- $dt_{(k)} = dt_1 \wedge \ldots \wedge dt_k$, *for* $k = 1 \ldots, m$,
- S_k *is the simplex defined by* $s \in \Delta_k \mapsto x_0 s$, Δ_k *being the standard euclidean simplex of* $\mathbf{R}^k$, $\{s \in \mathbf{R}^k;\ 0 \leq s_1 \leq \ldots \leq s_k \leq 1\}$, *for* $k = 1 \ldots, m$,
- Γ *is the circle of center* a *and radius* $|a|/2$,
- α *is a holomorphic function on* $\mathcal{R}(\Omega \setminus \bigcup_{i=1}^{d} \tilde{K}_i)$, *where* Ω *is a connected open neighbourhood of* $0 \in \mathbf{C}^m \times \mathbf{C}^2 \times \mathbf{C}$ *and*

$$\tilde{K}_i = \{(t, x, \zeta);\ l_i(t_m, k_i(x)) - \zeta = 0\}.$$

Remark 2.2 *With all our definitions and notations, we have*

$$u(x) = \int_0^{x_0} dt_m \int_0^{t_m} dt_{m-1} \ldots \int_0^{t_2} dt_1 \int_\Gamma \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta$$

where each inegral $\int_A^B \ldots dt_j$ *is the path integral along the segment* $[A,B]$ *in the complex plane* $\mathbf{C}_{t_j}$.

We will show, with the help of this theorem, how we can get, in $\mathbf{C}^2$, the theorem 1.1.

Corollary 2.1 *If* v *is ramified around* K*, then there is a connected open neighbourhood* Ω' *of* $0 \in \mathbf{C}^2$ *such that the problem (4) get a unique holomorphic solution on* $\mathcal{R}(\Omega' \setminus K)$.

3 Proof of theorem 2.1

We search the solution as indicated in theorem 2.1, this means that we search the solution under the form

$$u(x) = \int_{S_m} \left(\int_\Gamma \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m)}.$$

Lemma 3.1 *We get, for all* $k = 1, \ldots, m$,

$$\begin{aligned} D_0^k u(x) &= \sum_{h=1}^{k} \int_{S_{m-h}} \left(\int_\Gamma \Big(Q_h^k(D_0, D_{t_m})\alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-h)} \\ &\qquad + \int_{S_m} \left(\int_\Gamma D_0^k \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m)} \quad \text{if } k < m, \\ &= \sum_{h=1}^{m-1} \int_{S_{m-h}} \left(\int_\Gamma \Big(Q_h^m(D_0, D_{t_m})\alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-h)} \\ &\qquad + \int_{S_m} \left(\int_\Gamma D_0^m \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m)} \\ &\qquad + \int_\Gamma \alpha(x_0, x, \zeta) v(x_0, \zeta) d\zeta \quad \text{if } k = m \end{aligned}$$

where Q_h^k *is an operator of order* $k-h$ *and* $Q_k^k \equiv 1$.
Moreover, the hyperplane $S^{t_m} : x_0 = t_m$ *is non-characteristic, in* 0*, for the operators* Q_h^k.

We note, in the proof, $Q_h^k = 0$ if $h > k$ and $Q_0^k = D_0^k$.

Proof We show the first part by reccursive process.
For $k = 1$. We have

$$\begin{aligned} D_0 u(x) &= \int_{S_{m-1}} \left(\int_\Gamma \alpha(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-1)} \\ &\qquad + \int_{S_m} \left(\int_\Gamma D_0 \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m)}. \end{aligned}$$

This show the first part of the lemma for $k = 1$. Now, assume the lemma true for $k \leq m-1$.
We consider, in a first time, the case $k < m-1$. Then, we get, by reccursive process,

$$\begin{aligned} D_0^{k+1}u(x) &= \sum_{h=1}^{k} \int_{S_{m-h}} \left(\int_\Gamma \Big((D_0 + D_{t_m}) Q_h^k(D_0, D_{t_m}) \alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-h)} \\ &\quad + \int_{S_{m-(h+1)}} \left(\int_\Gamma \Big(Q_h^k(D_0, D_{t_m}) \alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-h-1)} \\ &\quad + \int_{S_{m-1}} \left(\int_\Gamma \Big(D_0^k \alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-1)} \\ &\quad + \int_{S_m} \left(\int_\Gamma D_0^{k+1} \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m)}. \end{aligned}$$

This means

$$\begin{aligned} D_0^{k+1}u(x) &= \sum_{h=1}^{k+1} \int_{S_{m-h}} \left(\int_\Gamma \Big((D_0 + D_{t_m}) Q_h^{k+1}(D_0, D_{t_m}) \alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-h)} \\ &\quad + \int_{S_m} \left(\int_\Gamma D_0^{k+1} \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m)} \end{aligned}$$

where $Q_h^{k+1} = (D_0 + D_{t_m}) Q_h^k + Q_{h-1}^k$. This is end the proof with the assumptions on the operators Q_h^k.
We consider, now, the case $k = m-1$. By reccursive process, we get,

$$\begin{aligned} D_0^m u(x) &= \sum_{h=1}^{m-1} \int_{S_{m-h}} \left(\int_\Gamma \Big((D_0 + D_{t_m}) Q_h^{m-1}(D_0, D_{t_m}) \alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-h)} \\ &\quad + \int_{S_{m-(h+1)}} \left(\int_\Gamma \Big(Q_h^{m-1}(D_0, D_{t_m}) \alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-h-1)} \\ &\quad + \int_{S_m} \left(\int_\Gamma D_0^m \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m)} \\ &\quad + \int_{S_{m-1}} \left(\int_\Gamma \Big(D_0^{m-1} \alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-1)} \\ &\quad + \int_{S_1} \left(\int_\Gamma \Big((D_0 + D_{t_m}) \alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(1)} \\ &\quad + \int_\Gamma \alpha(x_0, x, \zeta) v(x_0, \zeta) d\zeta. \end{aligned}$$

This means

$$\begin{aligned} D_0^m u(x) &= \sum_{h=1}^{m-1} \int_{S_{m-h}} \left(\int_\Gamma \Big(Q_h^m(D_0, D_{t_m}) \alpha \Big)(x_0, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m-h)} \\ &\quad + \int_{S_m} \left(\int_\Gamma D_0^m \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta \right) dt_{(m)} \\ &\quad + \int_\Gamma \alpha(x_0, x, \zeta) v(x_0, \zeta) d\zeta \end{aligned}$$

where $Q_h^m = (D_0 + D_{t_m})Q_h^{m-1} + Q_{h-1}^{m-1}$. This is end the proof with the assumptions on the operators Q_h^k.

Now, the fact that the hyperplane S^{t_m} is non-characteristic for operators Q_h^k $(0 \leq h \leq k \leq m)$ will be proved.
For $k = 1$, we get $Q_1^1 = 1$ and $Q_0^1 = D_0$, then S^{t_m} is non-characteristic for Q_1^1 and Q_0^1. So the result is true for $k = 1$
Assume the result true for all operators $Q_{h'}^{k'}$ with $h' \in [0, k']$, $k' \in [0, k]$ with $k < m$. We have $Q_{k+1}^{k+1} = 1$ and $Q_0^{k+1} = D_0^{k+1}$. So S^{t_m} is non-characteristic for Q_{k+1}^{k+1} and Q_0^{k+1}. Now choose $h \in [1, k]$. Then, with what have been said before, we have

$$Q_h^{k+1} = (D_0 + D_{t_m})Q_h^k + Q_{h-1}^k.$$

$(D_0 + D_{t_m})Q_h^k$ is an operator of order $k - h + 1$ and S^{t_m} is characteristic for this operator. Q_{h-1}^k is an operator of order $k - h + 1$ and, by reccursive process, S^{t_m} is non-characteristic for this operator. So, S^{t_m} is non-characteristic for the operator Q_h^{k+1}. This is end the proof of the second part of the lemma. ■

We deduce, from this lemma, that

$$\begin{aligned} a(x, D)u(x) = & \int_\Gamma \alpha(x_0, x, \zeta)v(x_0, \zeta)d\zeta \\ & + \sum_{h=1}^{m-1} \int_{S_h} \left(\int_\Gamma (P_h(x, D, D_{t_m})\alpha)(x_0, x, \zeta)v(t_1, \zeta)d\zeta \right) dt_{(h)} \\ & + \int_{S_m} \left(\int_\Gamma a(x, D)\alpha(t_m, x, \zeta)v(t_1, \zeta)d\zeta \right) dt_{(m)} \end{aligned}$$

where P_h is a differential operator of order h. Explicitly, we have

$$P_h(x, D, D_{t_m}) = \sum_{\alpha_0 = m-h}^{m} \sum_{\alpha_1 = 0}^{m-\alpha_0} a_\alpha(x) Q_{m-h}^{\alpha_0}(D_0, D_{t_m}) D_1^{\alpha_1}.$$

Moreover, the hyperplane S^{t_m} is non-characteristic for these operators in $\{0\}$. Indeed, if we note $s(t, x) = x_0 - t_m$, we have

$$\sigma(P_h)(0; 1, 0, -1) = a_{(m,0)}(0) Q_{m-h}^m(1, -1)$$

where $\sigma(P_h)$ is the principal symbol of the operator P_h. We know that $a_{(m,0)} \equiv 1$ and, by lemma 3.1, that $Q_{m-h}^m(1, -1) \neq 0$; so S^{t_m} is non-characteristic for operators $P_h(x, D, D_{t_m})$ in $\{0\}$.

Then, with the help of the representation integral formula of Cauchy, u will be solution of the Cauchy problem (4) if α is solution of the following Cauchy problem

$$\left\{ \begin{array}{rcll} a(x, D)\alpha(t_m, x, \zeta) & = & 0, & \\ P_h(x, D, D_{t_m})\alpha(t_m, x, \zeta) & = & 0 & \text{for } x_0 = t_m,\ 1 \leq h < m - 1, \\ \alpha(t_m, x, \zeta) & = & \dfrac{1}{\zeta - x_1} & \text{for } x_0 = t_m. \end{array} \right.$$

This problem is equivalent with this problem

$$\left\{\begin{array}{lcl} a(x,D)\alpha(t_m,x,\zeta) & = & 0, \\ & & \\ D_0^h\alpha(t_m,x,\zeta) & = & w_h(x_1-\zeta); \quad \text{for } x_0=t_m,\ 1\leq h<m, \end{array}\right. \tag{5}$$

where functions w_h present a polar singularity in 0.
By the change of chart : $(t,y,\zeta)=(t,x_0-t_m,x_1-\zeta,\zeta)$ (with $y=(y_0,y_1)$), the problem (5) can be write

$$\left\{\begin{array}{lcl} b(t,y,\zeta,D_y)\alpha(t_m,y,\zeta) & = & 0, \\ & & \\ eD_0^h\alpha(t_m,y,\zeta) & = & w_h(y_1); \quad \text{for } y_0=0,\ 1\leq h<m \end{array}\right. \tag{6}$$

where $b(t,y,\zeta,D_y)=a(y_0+t_m,y_1+\zeta,D_y)$. With the help of the writting (2) of the principal symbol of the operator $a(x,D)$, we deduce that the operator $b(t,y,\zeta,D_y)$ is an operator with multiple characteristics of constant multiplicity. Indeed, his principal symbol is

$$\prod_{i=1}^{d}(\xi_0-\lambda_i(y_0+t_m,y_1+\zeta)\xi_1)^{m_i}$$

and $\lambda_i(0)\neq\lambda_j(0)$ if $i\neq j$.

Now, we want to write, in a explicit way, the characteristics going out from the set $\{(t,y,\zeta);\ y_0=y_1=0\}$ of this operator. We write $\tilde{k}_i$, $i=1,\ldots,d$, the solution of the following Cauchy problem

$$\left\{\begin{array}{lcl} D_{y_0}\tilde{k}_i(t,y,\zeta) & = & \lambda_i(y_0+t_m,y_1+\zeta)D_{y_1}\tilde{k}_i(t,y,\zeta), \\ & & \\ \tilde{k}_i(t,y,\zeta) & = & y_1 \quad \text{for } y_0=0. \end{array}\right. \tag{7}$$

By the definition (3) of functions k_i and with the help of remark 2.1, we can easily see that the solution of problem (7) is given by

$$l_i(t_m,k_i(y_0+t_m,y_1+\zeta))-\zeta.$$

Then, characteristics of the operator $b(t,y,\zeta,D_y)$ containing $\{(t,y,\zeta);\ y_0=y_1=0\}$ are given by hypersurfaces $\tilde{K}_i$ whose equations are

$$\tilde{K}_i : l_i(t_m,k_i(y_0+t_m,y_1+\zeta))-\zeta=0.$$

We can deduce, with help of HAMADA-LERAY-WAGSCHAL's theorem, that there is a connected open neighbourhood Ω' of the origin of $\mathbf{C}^m\times\mathbf{C}^2\times\mathbf{C}$ such that the solution of problem (6) is holomorphic on $\mathcal{R}(\Omega'\setminus\tilde{K})$, where $\tilde{K}=\bigcup_{i=1}^d\tilde{K}_i$. Then, the solution of problem (5) is holomorphic sur $\mathcal{R}(\Omega'\setminus\tilde{K})$, each $\tilde{K}_i$ could be written, in the chart (t,x,ζ)

$$\tilde{K}_i : l_i(t_m,k_i(x))-\zeta=0 \qquad i=1,\ldots,d.$$

This is end the proof of theorem 2.1.

4 Proof of corrolary 2.1

We can, now, with help of theorem 2.1, prove the corollary 2.1. First, we note, $U(t,x)$ the germ at the point $(0,a)$ define by

$$U(t,x) = \int_\Gamma \alpha(t_m, x, \zeta) v(t_1, \zeta) d\zeta.$$

With the help of HAMADA-LERAY-WAGSCHAL's theorem, we can write this germ under the following form

$$U(t,x) = \sum_{i=1}^{d} U_i(t,x) = \sum_{i=1}^{d} \int_\Gamma \alpha_i(t_m, x, \zeta) v(t_1, \zeta) d\zeta$$

where α_i is only ramified around $\tilde{K}_i$. We call back that v is ramified around K. Then, there is a connected open neighbourhood Ω of $0 \in \mathbf{C}^m \times \mathbf{C}^2 \times \mathbf{C}$ such that the germ $v_i(t,x,\zeta) = \alpha_i(t_m, x, \zeta) v(t_1, \zeta)$ define a holomorphic function on $\mathcal{R}(\Omega \smallsetminus L_i)$ where L_i is given by the equation

$$L_i : \Big(l_i(t_m, k_i(x)) - \zeta \Big) \Big(\prod_{j=1}^{d} k_j(t_1, \zeta) \Big) = 0.$$

We call back the following result (see, for example, PHAM [6]). Let γ be a path of origin $(0,a) \in \mathbf{C}^m \times \mathbf{C}^2$, then the germ U_i at the point $(0,a)$ could be analyticly continued along γ if there is a family $\alpha_r : \mathcal{S}^1 \to \mathbf{C}$ $(r \in [0,1])$ of cycle such that

- $\alpha_0 = \Gamma$,
- $(\gamma(r), \alpha_r(s)) \in \Omega \smallsetminus L_i$,
- $\alpha : [0,1] \times \mathcal{S}^1 \ni (r,s) \mapsto \alpha_r(s)$ is a continuous map, $\mathcal{S}^1 = \{z \in \mathbf{C};\ |z| = 1\}$.

Lemma 4.1 *There is a connected open neighbourhood Ω' of the origin of $\mathbf{C}^m \times \mathbf{C}^2$ such that U_i is a holomorphic function on $\mathcal{R}(\Omega' \smallsetminus \tilde{L}_i)$ where*

$$\tilde{L}_i = \Big\{ (t,x) \in \Omega';\ \prod_{j=1}^{d} k_j(t_1, l_i(t_m, k_i(x))) = 0 \Big\}.$$

Proof Let $\gamma = (\gamma_{t_1}, \ldots, \gamma_{t_m}, \gamma_x)$ a path of origin $(0,a)$ drawn in $\Omega \smallsetminus \tilde{L}_i$. Our goal will be to build a circle of center $l_i(\gamma_{t_m}(r), k_i(\gamma_x(r)))$ and radius $\varepsilon(r)$ such that all the zeros of the equation $\prod_{j=1}^{d} k_j(\gamma_{t_1}(r), \zeta) = 0$ will be outside of the closed disk $\overline{D}\big(l_i(\gamma_{t_m}(r), k_i(\gamma_x(r))), \varepsilon(r)\big)$ (see Figure 1). The construction of such a family of cycles verifying conditions of deformation will be possible only for path γ drawn in $\Omega' \smallsetminus \tilde{L}_i$ where $\Omega' \subset \Omega$ is a connected open neighbourhood of the origin sufficiently small.

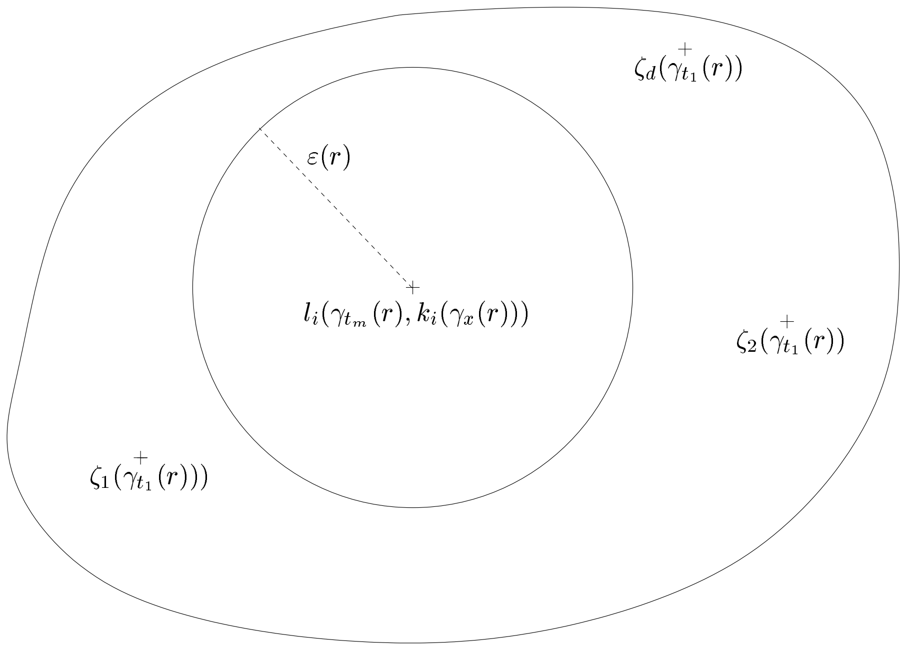

Figure 1: Geometric configuration

With the help of the preparation theorem of Weierstrass, the equation

$$\prod_{j=1}^{d} k_j(t_1, \zeta) = 0$$

admits d zeros. These zeros will be note by $\zeta_j(t_1)$, $j = 1, \ldots, d$ (remark that, for all j, $\zeta_j(0) = 0$). Then, by definition of $\tilde{L}_i$, $l_i(\gamma_{t_m}(r), k_i(\gamma_x(r))) \neq \zeta_j(\gamma_{t_1}(r))$ for all $j = 1, \ldots, d$. Now we note

$$\varepsilon(r) = \frac{1}{2} \min_{j=1,\ldots,d} \left\{ \left| l_i(\gamma_{t_m}(r), k_i(\gamma_x(r))) - \zeta_j(\gamma_{t_1}(r)) \right| \right\};$$

$\varepsilon(r)$ is a positive real with continuous dependance of r and such that $\varepsilon(0) = |a|/2$. Now, we define

$$\alpha_r : \mathcal{S}^1 \ni s \mapsto l_i(\gamma_{t_m}(r), k_i(\gamma_x(r))) + \varepsilon(r)s.$$

It is easy to see that for all paths from origin $(0, a)$ and drawn in $\Omega' \setminus L_i$ where Ω' is sufficiently small, and a sufficiently closed to the origin, the precedent family $(\alpha_r)_{r \in [0,1]}$ of cycles verify conditions of the deformation. This is end the proof of the lemma. ■

So now, we can say that the solution of problem (4) can be write

$$u(x) = \sum_{i=1}^{d} u_i(x) = \sum_{I=1}^{d} \int_{S_m} U_i(t, x) dt_{(m)}$$

where U_i is only ramified around $\tilde{L}_i = \left\{ (t, x) \in \Omega';\ \prod_{j=1}^{d} k_j(t_1, l_i(t_m, k_i(x))) = 0 \right\}$. To find the universal covering where are defined the germs u_i, we will use results on the analytic continuation of germs defined by integral. These results are exposed by Kobayashi in [4] or by Camalès in [1]. We call back now, the aim result on the analytic continuation of germs defined by integral.

Let $f : (t, x) \in \mathbf{C}_t^m \times \mathbf{C}_x^{n+1} \mapsto f(t, x)$ a holomorphic germ, in a neighbourhood of $(0; 0, a') = (0; a) \in \Omega = T \times X$. We assume that this germ define a holomorphic function on the universal covering $\mathcal{R}(\Omega \setminus L)$ (where L is an anlytic set of $\mathbf{C}_t^m \times \mathbf{C}_x^{n+1}$ such that $(0; a) \notin L$). We note for all $i = 1, \ldots, m+1$

$$W_i = \{(t, x);\ t_i = t_{i-1}\} \quad t_0 = 0, \quad t_{m+1} = x_0.$$

We define a germ u, at the point a, by

$$u(x) = \int_{S_m} f(t, x) dt_{(m)} = \int_{\Delta_m} x_0^m f(x_0 s, x) ds_{(m)}.$$

We note Δ_m^i, $i = 1, \ldots, m+1$, the faces of simplex Δ_m : this means

$$\Delta_m^i = \{s \in \Delta_m;\ s_i = s_{i-1}\} \quad s_0 = 0 \quad s_{m+1} = 1.$$

Then, the germ u could be analyticly continued along a path γ from a and drawn in $\mathbf{C}_x^{n+1}$ if the m-simplexe $\alpha_0 \equiv 0$ could be continuously deform all along γ. Precisely, if there is a family of m-simplexes $\alpha_r : \Delta_m \to \mathbf{C}_t^m$ for $r \in [0, 1]$ such that

- $\alpha_0 \equiv 0$,
- $(\alpha_r(s), \gamma(r)) \in \Omega \setminus L$,
- $(\alpha_r(s), \gamma(r)) \in W_i$ if $s \in \Delta_m^i$,
- $\alpha : (r, s) \in [0, 1] \times \Delta_m \mapsto \alpha_r(s) \in \mathbf{C}_t^m$ is a continuous map.

The simplest method for the determination of analytic continuation, in our case, is to compute the singularities for each integration. For example, in the case $d = 2$ and $i = 1$, we say (remember that we choose a chart such that $k_1(x) = x_1$ and then, with remark 2.1, $l_1(x) = x_1$)

$$u_1(x) = \int_0^{x_0} V_1(t_2, x) dt_2$$

where

$$V_1(t_2, x) = \int_0^{t_2} U_1(t, x) dt_1.$$

Then, we can show that V_1 is ramified around $\{(t_2, x) \in \mathbf{C} \times \mathbf{C}^2;\ x_1 k_2(t_2, x_1) = 0\}$ and u_1 is ramified around $\{x \in \mathbf{C}^2;\ x_1 k_2(x) = 0\} = K$.

More generaly, we can show that, for all m, all the germs u_i are ramified around K. Then u is ramified around K; this is end the proof of corollary 2.1. Then we have proved the Leichtnam's theorem in $\mathbf{C}^2$.

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