GREEN FORMULAE FOR CONE DIFFERENTIAL OPERATORS

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ABSTRACT. Green formulae for elliptic cone differential operators are established. This is achieved by an accurate description of the maximal domain of an elliptic cone differential operator and its formal adjoint; thereby utilizing the concept of a discrete asymptotic type. From this description, the singular coefficients replacing the boundary traces in classical Green formulas are deduced.

CONTENTS

1. INTRODUCTION

1.1. **The main result.** Let X be a compact C^∞–manifold with non-empty boundary, ∂X. On the interior $X^\circ := X \setminus \partial X$, we consider differential operators A which on $U \setminus \partial X$ for some collar neighborhood $U \cong [0,1) \times Y$ of ∂X, with coordinates (t, y) and Y being diffeomorphic to ∂X, take the form

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t, y, D_y)(-t\partial_t)^j, \tag{1.1}$$

where $a_j \in C^\infty([0,1); \mathrm{Diff}^{\mu-j}(Y))$ for $0 \le j \le \mu$. Such differential operators A are called cone-degenerate, or being of Fuchs type; written as $A \in \mathrm{Diff}^{\mu}_{\mathrm{cone}}(X)$. They arise, e.g., when polar coordinates are introduced near a conical point.

Date: September 10, 2003.

2000 *Mathematics Subject Classification.* Primary: 35J70; Secondary: 34B05, 41A58.

Key words and phrases. Cone differential operators, discrete asymptotic types, function spaces with asymptotics, complete conormal symbols, Keldysh's formula, Green formula.

Throughout, we shall fix some reference weight $\delta \in \mathbb{R}$. This means that we will be working in the weighted L^2–space $\mathbb{H}^{0,\delta}(X)$ as basic function space, cf. (1.12) and also Appendix B. Let $A^* \in \mathrm{Diff}^{\mu}_{\mathrm{cone}}(X)$ be the formal adjoint to A, i.e,

$$(Au, v) = (u, A^* v)\,, \quad u,\, v \in C^{\infty}_{\mathrm{comp}}(X^{\circ}), \tag{1.2}$$

where (,) denotes the scalar product in $\mathbb{H}^{0,\delta}(X)$. Then it is customary to consider the maximal and minimal domains of A,

$$D(A_{\mathrm{max}}) := \{u \in \mathbb{H}^{0,\delta}(X) \mid Au \in \mathbb{H}^{0,\delta}(X)\},$$

and $D(A_{\mathrm{min}})$ is the closure of $C^{\infty}_{\mathrm{comp}}(X^{\circ})$ in $D(A_{\mathrm{max}})$ with respect to the graph norm. Similarly for $D(A^*_{\mathrm{max}})$, $D(A^*_{\mathrm{min}})$.

Our basic object of study is the *boundary sesquilinear form*

$$[u, v]_A := (Au, v) - (u, A^* v)\,, \quad u \in D(A_{\mathrm{max}}),\, v \in D(A^*_{\mathrm{max}}). \tag{1.3}$$

By virtue of (1.2), $[u, v]_A = 0$ if $u \in D(A_{\mathrm{min}})$ or $v \in D(A^*_{\mathrm{min}})$. Therefore, the boundary sesquilinear form $[\ ,\ ]_A$ descents to a sesquilinear form

$$[\ ,\ ]_A \colon D(A_{\mathrm{max}})/D(A_{\mathrm{min}}) \times D(A^*_{\mathrm{max}})/D(A^*_{\mathrm{min}}) \to \mathbb{C}, \tag{1.4}$$

denoted in the same manner.

Our basic task consists in computing (1.4). The result will be called a *Green formula* in analogy to the classical situation arising in mathematical analysis. Assuming ellipticity for A, cf. Definition 3.1, what we will actually do is to compute the value of $[\ ,\ ]_A$ on distinguished linear bases of $D(A_{\mathrm{max}})/D(A_{\mathrm{min}})$ and $D(A^*_{\mathrm{max}})/D(A^*_{\mathrm{min}})$, respectively.

The starting point is as follows: Assuming ellipticity for A, any solution $u = u(x)$ to the equation $Au = f(x)$ on X° possesses an asymptotic expansion

$$u(x) \sim \sum_{p} \sum_{k+l=m_p-1} \frac{(-1)^k}{k!}\, t^{-p} \log^k t\, \phi^{(p)}_l(y) \quad \text{as } t \to +0, \tag{1.5}$$

where the set $\{p \in \mathbb{C} \,|\, m_p \geq 1\}$ is discrete, $\mathrm{Re}\, p \to -\infty$ as $|p| \to \infty$ on this set, and $\phi^{(p)}_l \in C^{\infty}(Y)$ for all p, l, provided that the right-hand side f possesses a similar expansion. Introduce the linear operator T acting on the space of all formal asymptotic expansions of the form (1.5) by

$$\sum_{p} \sum_{k+l=\underline{m_p-1}} \frac{(-1)^k}{k!}\, t^{-p} \log^k t\, \phi^{(p)}_l(y) \mapsto \sum_{p} \sum_{k+l=\underline{m_p-2}} \frac{(-1)^k}{k!}\, t^{-p} \log^k t\, \phi^{(p)}_l(y). \tag{1.6}$$

As will be seen,

- the quotient $D(A_{\mathrm{max}})/D(A_{\mathrm{min}})$ is finite-dimensional,
- it consists of finite sums of the form (1.5), where $\dim X/2 - \delta - \mu < \mathrm{Re}\, p < \dim X/2 - \delta$,
- it is left invariant under the action of T.

In particular, T as acting on $D(A_{\mathrm{max}})/D(A_{\mathrm{min}})$ is nilpotent. Similarly for $D(A^*_{\mathrm{max}})/D(A^*_{\mathrm{min}})$.

Theorem 1.1. *Let $A \in \mathrm{Diff}^{\mu}_{\mathrm{cone}}(X)$ be elliptic. Then, to each Jordan basis*

$$\Phi_1, T\Phi_1, \dots, T^{m_1-1}\Phi_1, \dots, \Phi_e, T\Phi_e, \dots, T^{m_e-1}\Phi_e \tag{1.7}$$

of $D(A_{\mathrm{max}})/D(A_{\mathrm{min}})$, there exists a unique Jordan basis

$$\Psi_1, T\Psi_1, \dots, T^{m_1-1}\Psi_1, \dots, \Psi_e, T\Psi_e, \dots, T^{m_e-1}\Psi_e \tag{1.8}$$

of $D(A^*_{\mathrm{max}})/D(A^*_{\mathrm{min}})$ such that, for all i, j, k, l,

$$[T^r\Phi_i, T^s\Psi_j]_A = \begin{cases} (-1)^{s+1} & \text{if } i = j,\, r + s = m_i - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

Corollary 1.2. (a) Both spaces $D(A_{\max})/D(A_{\min})$ and $D(A^*_{\max})/D(A^*_{\min})$ have the same Jordan structure (with respect to T).

(b) The sesquilinear form $[\ ,\ ]_A$ in (1.4) is non-degenerate.

(c) The operator T is skew-adjoint with respect to $[\ ,\ ]_A$, i.e.,

$$[T\Phi, \Psi]_A + [\Phi, T\Psi]_A = 0 \tag{1.10}$$

for all $\Phi \in D(A_{\max})/D(A_{\min})$, $\Psi \in D(A^*_{\max})/D(A^*_{\min})$.

Remark 1.3. The conjugate Jordan basis $\Psi_1, T\Psi_1, \dots, T^{m_1-1}\Psi_1, \dots, \Psi_e, T\Psi_e, \dots, T^{m_e-1}\Psi_e$ in (1.8) can be found, at least in principal, once one controls the first μ conormal symbols $\sigma_c^{\mu-j}(A)(z)$ for $j = 0, 1, \dots, \mu-1$ of A, see (3.1). More precisely, let $\{\mathrm{t}^{-\mu-k}(z);\ k \in \mathbb{N}_0\}$ be the inverse to the complete conormal symbol $\{\sigma_c^{\mu-j}(z);\ j \in \mathbb{N}_0\}$ of A under the Mellin translation product, see (3.5). In particular, $\mathrm{t}^{-\mu-k}(z)$ for $k = 0, 1, 2, \dots$ is a meromorphic function on $\mathbb{C}$ taking values in the space $L^{-\mu}_{\mathrm{cl}}(Y)$ of classical pseudodifferential operators of order $-\mu$ on Y. Then the Jordan basis in (1.8) can be computed from the Jordan basis in (1.7) and the principal parts of the Laurent expansions of $\mathrm{t}^{-\mu-k}(z+\mu)$ around the poles in the strip $\dim X/2 - \delta - \mu + k < \operatorname{Re} z < \dim X/2 - \delta$ for $k = 0, 1, \dots, \mu-1$, see Theorem 4.1.

1.2. **Description of the content.** In Section 2, we discuss discrete asymptotic types for conormal asymptotics of the form (1.5). The central notions are properness of an asymptotic type and complete characteristic bases for proper asymptotic types. In Section 3, we study complete Mellin symbols that form an algebra unter the Mellin translation product. Here, the main result due to LIU–WITT [11] states that the type for the asymptotics annihilated by an elliptic, holomorphic complete Mellin symbol is proper; thus linking to cone differential operators, cf. Theorem 3.10. Then in Theorem 4.1, in Section 4, we establish a formula for the principal parts of the Laurent expansions around the poles of the inverses to holomorphic complete Mellin symbols under the Mellin translation product. This formula involves a complete characteristic basis and its conjugate complete characteristic basis, similar to the situation arising in Theorem 1.1. In fact, Theorem 4.1 is one of the two main technical results of this paper from which Theorem 1.1 is easily deduced. The other one is Theorem 5.1 in Section 5, where certain "bi-orthogonality" relations between the two complete characteristic bases of Theorem 4.1 are established. Theorem 1.1 is proved in Section 6. We start with a formula for the boundary sesquilinear form $[\ ,\ ]_A$ taken from GIL–MENDOZA [3], cf. Theorem 6.1. The proof of Theorem 1.1 then consists in evaluating this formula, where the latter basically means to "take the residue" of the formula from Theorem 5.1.

In Section 7, we discuss two examples showing how one can proceed from the "Green formula" (1.9) to genuine Green formulas in concrete situations. The two appendices are not mandatory for the main text, but improve understanding. In Appendix A, we are concerned with local asymptotic types, i.e., asymptotic types at some fixed singular exponent p from (1.5). Already here, all the ingredients from the main text of the paper occur in embryonic form. For instance, the forerunner of Theorem 5.1 is a famous formula due to KELDYSH [6], see Remark 5.2 (b). An analogue of the boundary sesquilinear form $[\ ,\ ]_A$ is also provided, cf. (A.3) and Proposition A.4. In Appendix B, we describe $D(A_{\max})$, $D(A_{\min})$ as function spaces with asymptotics. Among others, this gives a concise way of identifying the quotient $D(A_{\max})/D(A_{\min})$.

Let us mention some related work: Green formulae have been under investigation for a long period, see, e.g., CODDINGTON–LEVINSON [1] for O.D.E. and LIONS–MAGENES [10] for P.D.E. For singular situations, see, e.g., NAZAROV–PLAMENEVSKIJ [12]. Our approach to cone-degenerate differential operators is built upon work of SCHULZE [14, 15]. For instance, the fact that the quotient $D(A_{\max})/D(A_{\min})$ is finite-dimensional and consists of formal asymptotic expansions of finite length is an easy consequence, see also LESCH [9, Section 1.3]. Recently, GIL–MENDOZA [3] received results similar to ours. Without reaching the final formula (1.9), they studied much of the structure of the boundary sesquilinear form (1.4). In case A is symmetric, they applied their results to describe the self-adjoint extensions of A. Keldysh's formula was thoroughly discussed in KOZLOV–MAZ'YA [8, Appendix A].

1.3. **Notation.** Notation introduced here will be used without further comments.

• Scalar products on $L^2(Y)$ are given by

$$(\phi, \psi) := \int_Y \phi(y)\overline{\psi(y)}\, d\mu(y), \quad \langle \phi, \psi \rangle := \int_Y \phi(y)\psi(y)\, d\mu(y), \tag{1.11}$$

where $d\mu$ is a fixed positive C^∞–density $d\mu$ on Y. For an operator A on $C^\infty(Y)$, its formal adjoint A^* is defined with respect to the scalar product $(\ ,\ )$, while the transpose A^t is defined with respect to $\langle\ ,\ \rangle$. In particular,

$$\overline{A^*\phi} = A^t\bar{\phi}, \quad \phi \in C^\infty(Y).$$

For u, $v \in \mathbb{H}^{0,\delta}(X)$ supported in the collar neighborhood U of ∂X,

$$(u, v) := \int_{(0,1)\times Y} t^{\dim X - 2\delta - 1}\, u(t, y)\overline{v(t, y)}\, dt d\mu(y), \tag{1.12}$$

cf. (1.2), (1.3). There should be no ambiguity of usage the same symbol $(\ ,\ )$ in the two different situations (1.11), (1.12).

• Let J be a finite-dimensional linear space and T be a nilpotent operator acting on J. Then $\Phi_1, \dots, \Phi_e$ is called a *characteristic basis* of J (with respect to T, where the latter is often understood from the context) if

$$\Phi_1, T\Phi_1, \dots, T^{m_1-1}\Phi_1, \dots, \Phi_e, T\Phi_e, \dots, T^{m_e-1}\Phi_e \tag{1.13}$$

for certain integers $m_1, \dots, m_e \geq 1$ form a linear basis of J. The matrix of T with respect to such a linear basis has Jordan form. Therefore, a characteristic basis $\Phi_1, \dots, \Phi_e$ always exists, the integers $m_1, \dots, m_e$ are uniquely determined (up to permutation) and equal the sizes of the Jordan blocks, and e is the number of the Jordan blocks. The tuple $(m_1, \dots, m_e)$ is called the *characteristic* of J (or of the characteristic basis $\Phi_1, \dots, \Phi_e$).

• Let E be either the space $C^\infty(Y)$ (in Section 2) or a Banach space (in Appendix A). Then $E^\infty := \bigcup_{m\in\mathbb{N}} E^m$ denotes the space of finite sequences in E, where we identify E^m as linear subspace of E^{m+1} through the map $(\phi_0, \dots, \phi_{m-1}) \mapsto (0, \phi_0, \dots, \phi_{m-1})$, i.e., by adding a leading zero. For $\Phi \in E^\infty$, let $m(\Phi)$ be the least integer m so that $\Phi \in E^m$. The right shift operator T sending $(\phi_0, \phi_1, \dots, \phi_{m-1})$ to $(\phi_0, \phi_1, \dots, \phi_{m-2})$ acts on E^∞. In particular, $T^{m(\Phi)}\Phi = 0$, while $T^i\Phi \neq 0$ for $0 \leq i \leq m(\Phi) - 1$. (In case $E = C^\infty(Y)$, the operator T is directly related to the operator T in (1.6), see Remark 2.2.)

• For E as above, $p \in \mathbb{C}$, let $\mathcal{M}_p(E)$ be the space of germs E–valued meromorphic functions and $\mathcal{A}_p(E)$ be the space of germs of E–valued holomorphic functions at $z = p$. We identify the quotient $\mathcal{M}_p(E)/\mathcal{A}_p(E)$ with the space E^∞ through the map

$$\frac{\phi_0}{(z-p)^m} + \frac{\phi_1}{(z-p)^{m-1}} + \dots + \frac{\phi_{m-1}}{z-p} \mapsto (\phi_0, \phi_1, \dots, \phi_{m-1}).$$

Then T corresponds to multiplication by $z - p$. For $F \in \mathcal{M}_p(E)$, let $[F(z)]_p^*$ denote the principal part of $F(z)$ at $z = p$. For $\Phi = (\phi_0, \phi_1, \dots, \phi_{m-1}) \in E^\infty$, we set

$$\Phi[z-p] := \frac{\phi_0}{(z-p)^m} + \frac{\phi_1}{(z-p)^{m-1}} + \dots + \frac{\phi_{m-1}}{z-p} \in \mathcal{M}_p(E).$$

• Now let $E = C^\infty(Y)$. For ϕ, $\psi \in C^\infty(Y)$, let $\phi \otimes \psi$ be the rank-one operator $C^\infty(Y) \ni h \mapsto (h, \psi)\phi \in C^\infty(Y)$. More generally, for F, $G \in \mathcal{M}_p(C^\infty(Y))$, we introduce the meromorphic operator family $F(z) \otimes G(z)$ by

$$F(z) \otimes G(z)h := \big(h, G(\bar{z})\big)F(z), \quad h \in C^\infty(Y),$$

where $\big(h, G(\bar{z})\big) = \big\langle h, \bar{G}(z)\big\rangle$ and $\bar{G}(z) := \overline{G(\bar{z})}$. For $\Phi,\ \Psi \in [C^\infty(Y)]^\infty$, we further set

$$(\Phi \otimes \Psi)[z-p] := \frac{\phi_0 \otimes \psi_0}{(z-p)^m} + \frac{\phi_0 \otimes \psi_1 + \phi_1 \otimes \psi_0}{(z-p)^{m-1}} + \dots \frac{\phi_0 \otimes \psi_{m-1} + \dots + \phi_{m-1} \otimes \psi_0}{z-p}, \quad (1.14)$$

where $\Phi = (\phi_0, \dots, \phi_{m-1})$, $\Psi = (\psi_0, \dots, \psi_{m-1})$. In particular,

$$(\Phi \otimes \Psi)[z-p] = (z-p)^m \left[\Phi[z-p] \otimes \Psi[z-p]\right]_p^*, \quad (1.15)$$

where $m = \max\{m(\Phi), m(\Psi)\}$.

- If E is a Banach space, then we use the same notation, but with $(\ ,\ )$ replaced by the dual pairing $\langle\ ,\ \rangle$ between E and E'. This is due to the circumstance that in this situation we are working with the dual instead of the anti-dual to E. In particular, $\phi \otimes \psi$ for $\phi \in E$, $\psi \in E'$ means the rank-one operator $E \ni h \to \langle h, \psi\rangle\phi \in E$, while (1.14), (1.15) are formally unchanged.
- On $[C^\infty(Y)]^\infty$, we introduce three commuting involutions by

$$\begin{aligned} \boldsymbol{C}\Phi &:= (\bar{\phi}_0, \bar{\phi}_1, \dots, \bar{\phi}_{m-2}, \bar{\phi}_{m-1}), \\ \boldsymbol{I}\Phi &:= ((-1)^m \bar{\phi}_0, (-1)^{m-1}\bar{\phi}_1, \dots, \bar{\phi}_{m-2}, -\bar{\phi}_{m-1}), \\ \boldsymbol{J}\Phi &:= ((-1)^m \phi_0, (-1)^{m-1}\phi_1, \dots, \phi_{m-2}, -\phi_{m-1}), \end{aligned}$$

where $\Phi = (\phi_0, \phi_1, \dots, \phi_{m-2}, \phi_{m-1})$. Note that $\boldsymbol{I} = \boldsymbol{C}\boldsymbol{J}$, $T\boldsymbol{C} = \boldsymbol{C}T$, $\boldsymbol{I}T + T\boldsymbol{I} = 0$, and $\boldsymbol{J}T + T\boldsymbol{J} = 0$.
- The cut-off function $\omega \in C^\infty_{\text{comp}}(\overline{\mathbb{R}}_+)$ satisfies $\omega(t) = 1$ if $t \le 1/2$, $\omega(t) = 0$ if $t \ge 1$. It is used to localize into the collar neighborhood $U \cong [0,1) \times Y$ of ∂X.

2. Discrete asymptotic types

The notion of discrete asymptotic type for conormal cone asymptotics goes back to Rempel–Schulze [13] in the one-dimensional and Schulze [14] in the higher-dimensional case. It allows to integrate asymptotic information into a functional-analytic setting, cf. also Appendix B. The refinements of this notion presented here are due to Liu–Witt [11].

2.1. **Preliminaries.** Let $C^{\infty,\delta}_{\text{as}}(X)$ be the space of all $u \in C^\infty(X^\circ)$ possessing an asymptotic expansion as in (1.5), as $x \to \partial X$, where, additionally, $\operatorname{Re} p < \dim X/2 - \delta$ if $m_p \ge 1$. Moreover, let $C^\infty_{\mathcal{O}}(X)$ be the space of all $u \in C^\infty(X^\circ)$ vanishing to the infinite order on ∂X (i.e., $\phi_l^{(p)}(y) = 0$ for all p, l in (1.5)).

Henceforth, we shall fix a splitting $U \setminus \partial X \cong (0,1) \times Y$, $x \mapsto (t,y)$ of coordinates near ∂X. It turns out, however, that our constructions are independent of this chosen splitting of coordinates, cf. Remarks 2.2, 2.4, and 2.10 and Proposition B.4.

Definition 2.1. (a) A discrete subset $V \subset \mathbb{C}$ is called a *carrier of asymptotics* if $|\operatorname{Re} p| \to \infty$ on V as $|p| \to \infty$. For $\delta \in \mathbb{R}$, we write $V \in \mathcal{C}^\delta$ if, in addition, $V \subset \{z \in \mathbb{C};\ \operatorname{Re} z < \dim X/2 - \delta\}$.

(b) For $V \in \mathcal{C}^\delta$, we define $\mathcal{E}^\delta_V(Y)$ to be the space of all mappings $\Phi\colon \mathbb{C} \to [C^\infty(Y)]^\infty$ satisfying $\{p \in \mathbb{C} |\ \Phi(p) \ne 0\} \subseteq V$. In particular, $\mathcal{E}^\delta_V(Y) = \prod_{p \in V} [C^\infty(Y)]^\infty_p$, where $[C^\infty(Y)]^\infty_p$ is an isomorphic copy of $[C^\infty(Y)]^\infty$. Moreover, we set $\mathcal{E}^\delta(Y) := \bigcup_{V \in \mathcal{C}^\delta} \mathcal{E}^\delta_V(Y)$.

The operations T, $\boldsymbol{C}$, $\boldsymbol{I}$, and $\boldsymbol{J}$ are point-wise defined on $\mathcal{E}^\delta(Y)$, cf. Section 1.3. For instance, $T\Phi(p) = T(\Phi(p))$ for $\Phi \in \mathcal{E}^\delta(Y)$, $p \in \mathbb{C}$. We also write $m^p(\Phi)$ instead of $m(\Phi(p))$.

We next provide an isomorphism

$$C^{\infty,\delta}_{\text{as}}(X)/C^\infty_{\mathcal{O}}(X) \to \mathcal{E}^\delta(Y) \quad (2.1)$$

that is non-canonical in the sense that it depends on the choice of splitting of coordinates near ∂X: With the vector $\Phi \in \mathcal{E}^\delta_V(Y)$, where $\Phi(p) = (\phi^{(p)}_0, \dots, \phi^{(p)}_{m_p-1})$ for $p \in V$, we associate the formal asymptotic expansion

$$u(x) \sim \sum_{p\in V} \sum_{k+l=m_p-1} \frac{(-1)^k}{k!}\, t^{-p} \log^k t\; \phi^{(p)}_l(y) \;\text{ as } t \to +0,$$

see (1.5). (To see that (2.1) is surjective needs to invoke a Borel-type argument.)

Remark 2.2. The operator T acting on the quotient $C^{\infty,\delta}_{\text{as}}(X)/C^\infty_{\mathcal{O}}(X)$, as introduced in (1.6), is well-defined, i.e., it is independent of the chosen splitting of coordinates near ∂X. Moreover, the isomorphism (2.1) intertwines this operator and the right-shift operator T acting on $\mathcal{E}^\delta(Y)$.

We need some further notation: For $\Phi \in \mathcal{E}^\delta(Y)$, we introduce

$$\text{c-ord}(\Phi) := \dim X/2 - \max\{\operatorname{Re} p;\ \Phi(p) \neq 0\}$$

(the "*conormal order*" of Φ understood in an L^2-sense). Note that $\text{c-ord}(\Phi) > \delta$ if $\Phi \in \mathcal{E}^\delta(Y)$, and $\text{c-ord}(T^k\Phi) \to \infty$ as $k \to \infty$. Note also that, for $\Phi_i \in \mathcal{E}^\delta(Y)$, $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \dots$ satisfying $\text{c-ord}(\Phi_i) \to \infty$ as $i \to \infty$, the series $\sum_{i=1}^\infty \alpha_i\Phi_i$ is explained in $\mathcal{E}^\delta(Y)$ in a natural fashion. In particular,

$$\sum_{i=1}^{\infty} \alpha_i\Phi_i = 0 \quad\Longleftrightarrow\quad \text{c-ord}\Big(\sum_{i=1}^{i_0} \alpha_i\Phi_i\Big) \to \infty \quad \text{as } i_0 \to \infty.$$

Furthermore, $\Phi \in \mathcal{E}^\delta(Y)$ is called a *special vector* if $\Phi \in \mathcal{E}^\delta_{p-\mathbb{N}_0}(Y)$ for some $p \in \mathbb{C}$. If $\Phi \neq 0$, then p is uniquely determined by Φ and the additional requirement that $\Phi(p) \neq 0$. This complex number p is denoted by $\gamma(\Phi)$.

2.2. **Definition of discrete asymptotic types.** Discrete asymptotic types are certain linear subspace of the space $C^{\infty,\delta}_{\text{as}}(X)/C^\infty_{\mathcal{O}}(X)$ of all formal asymptotic expansions.

Definition 2.3. A *discrete asymptotic type*, P, for conormal cone asymptotics as $x \to \partial X$, of conormal order at least δ, is a linear subspace of $C^{\infty,\delta}_{\text{as}}(X)/C^\infty_{\mathcal{O}}(X)$ that is represented, in the given splitting $U \setminus \partial X \cong (0,1)\times Y$, $x \mapsto (t,y)$ of coordinates near ∂X, through the isomorphism (2.1) by a linear subspace $J \subset \mathcal{E}^\delta_V(Y)$ for some $V \in \mathcal{C}^\delta$ satisfying the following conditions:

(i) $TJ \subseteq J$.

(ii) $\dim J^{\delta+j} < \infty$ for all $j \in \mathbb{N}_0$, where $J^{\delta'} := J/(J \cap \mathcal{E}^{\delta'}(Y))$ for $\delta' \geq \delta$.

(iii) There is a sequence $\{p_i;\ 1 \leq j < e+1\} \subset \mathbb{C}$, where $e \in \mathbb{N}_0 \cup \{\infty\}$, such that $\operatorname{Re} p_i < \dim X/2 - \delta$ for all i, $\operatorname{Re} p_i \to -\infty$ as $i \to \infty$ if $e = \infty$, $V \subseteq \bigcup_{i=1}^e \{p_i\} - \mathbb{N}_0$, and

$$J = \bigoplus_{i=1}^{e} \Big(J \cap \mathcal{E}^\delta_{p_i-\mathbb{N}_0}(Y)\Big)\,.$$

The *empty asymptotic type*, $\mathcal{O}$, is represented by the trivial subspace $\{0\} \subset \mathcal{E}^\delta(Y)$. The set of all asymptotic types of conormal order at least δ is denoted by $\underline{\text{As}}^\delta(Y)$.

Remark 2.4. It can be shown that this notion of discrete asymptotic type is independent of the splitting of coordinates near ∂X. The latter means that changing coordinates $P \subset C^{\infty,\delta}_{\text{as}}(X)/C^\infty_{\mathcal{O}}(X)$ is represented by another linear subspace $J' \subset \mathcal{E}^\delta_{V'}(Y)$ for some $V' \in \mathcal{C}^\delta$ that also satisfies (i) to (iii) of Definition 2.3.

Let P, $P' \in \underline{\text{As}}^\delta$ be represented by J, $J' \subset \mathcal{E}^\delta(Y)$, respectively. For $\delta' \geq \delta$, we say that P, P' *coincide up to the conormal order* δ' if $J^{\delta'} = J'^{\delta'}$ as subspaces of $\mathcal{E}^\delta(Y)/\mathcal{E}^{\delta'}(Y)$. Similarly, for $\delta' > \delta$, we say that P, P' coincide up to the conormal order $\delta' - 0$ if P, P' coincide up to the conormal order $\delta' - \varepsilon$ for all $0 < \varepsilon \leq \delta' - \delta$.

It is important to observe that the set $\underline{\mathrm{As}}^\delta(X)$ of asymptotic types is partially ordered by inclusion of the representing spaces. This partial order on $\underline{\mathrm{As}}^\delta(X)$ will be denoted by $\preccurlyeq$. One of the fundamental principles in constructing asymptotic types obeying certain prescribed properties ensues from the following:

Proposition 2.5. $(\underline{\mathrm{As}}^\delta(X), \preccurlyeq)$ *is a lattice with the property that each non-empty subset* (*resp. each bounded subset*) *possesses a greatest lower bound* (*resp. a least upper bound*)*.*

As example, consider $P \in \underline{\mathrm{As}}^\delta$ and let $\delta' \geq \delta$. Then let $P^{\delta'} \preccurlyeq P$ denote the largest asymptotic type that coincides with the empty asymptotic type, $\mathcal{O}$, up to the conormal order δ. Similarly, for $\delta' > \delta$, let $P^{\delta'-0} \preccurlyeq P$ denote the largest asymptotic type that coincides with the empty asymptotic type up to the conormal order $\delta' - 0$. Of course, in this situation it is easy to provide representing spaces for $P^{\delta'}$ and $P^{\delta'-0}$, respectively, but in more involved situations such a task might be not that simple.

Remark 2.6. There is an abstract concept of introducing asymptotic types if a unital algebra $\mathfrak{M}$ acting on some linear space $\mathfrak{F}$ "modulo a distinguished linear subspace $\mathfrak{F}_0$ in the image" is given. In our case, $\mathfrak{M} = \bigcup_{\mu\in\mathbb{Z}} \mathrm{Symb}^\mu_M(Y)$ is the algebra of complete Mellin symbols, cf. Section 3, $\mathfrak{F} = C^{\infty,\delta}_{\mathrm{as}}(X)$, and $\mathfrak{F}_0 = C^\infty_{\mathcal{O}}(X)$. See WITT [18].

2.3. **Proper discrete asymptotic types.** Here we investigate properties of linear subspaces $J \subset \mathcal{E}^\delta_V(Y)$ satisfying (i) to (iii) of Definition 2.3.

Proposition 2.7. *Let $J \subset \mathcal{E}^\delta_V(Y)$ be a linear subspace for some $V \in \mathcal{C}^\delta$. Then there are an $e \in \mathbb{N}_0 \cup \{\infty\}$ and a sequence $\{\Phi_i;\ 1 \leq i < e+1\}$ of special vectors satisfying $\text{c-ord}(\Phi_i) \to \infty$ as $i \to \infty$ if $e = \infty$ such that the vectors $T^k\Phi_i$ for $1 \leq i < e+1$, $k \in \mathbb{N}_0$ span the space J if and only if* (i) *to* (iii) *of* Definition 2.3 *are fulfilled.*

For the rest of this section, assume that $J \subset \mathcal{E}^\delta_V(Y)$ is a linear subspace satisfying (i) to (iii) of Definition 2.3. Let $\Pi_j \colon J \to J^{\delta+j}$ be the canonical surjection. For $j' > j$, there is a natural surjective map $\Pi_{jj'} \colon J^{\delta+j'} \to J^{\delta+j}$ such that $\Pi_{jj''} = \Pi_{jj'}\Pi_{j'j''}$ for $j'' > j' > j$ and

$$\big(J, \Pi_j\big) = \varprojlim \big(J^{\delta+j}, \Pi_{jj'}\big).$$

Note that $T \colon J^{\delta+j} \to J^{\delta+j}$ is nilpotent, where the operator T is induced by $T \colon J \to J$. Let $(m^j_1, \dots, m^j_{e_j})$ be the characteristic of $J^{\delta+j}$, cf. (1.13) and thereafter.

The sequence $\{\Phi_i;\ 1 \leq i < e+1\} \subset J$ is said to be a *characteristic basis* of J if there are numbers $m_i \in \mathbb{N}_0 \cup \{\infty\}$, $m_i \geq 1$, such that $T^{m_i}\Phi_i = 0$ if $m_i < \infty$, while the sequence $\{T^k\Phi_i;\ 1 \leq i < e+1,\ 0 \leq k < m_i\}$ forms a basis of J.

Proposition 2.8. *Let $J \subset \mathcal{E}^\delta_V(Y)$ be a linear subspace as above and assume that $\{\Phi_i;\ 1 \leq i < e+1\}$ is a characteristic basis of J. Then the following conditions are equivalent:*

(a) *For each j, $\{\Pi_j\Phi_1, \dots, \Pi_j\Phi^j_{e_j}\}$ is a characteristic basis of $J^{\delta+j}$.*

(b) *For each j, $T^{m^j_1-1}\Phi_1, \dots, T^{m_{e_j}-1}\Phi_{e_j}$ are linearly independent over the space $\mathcal{E}^{\delta+j}(Y)$, while $T^k\Phi_i \in \mathcal{E}^{\delta+j}(Y)$, where either $1 \leq i \leq e_j$, $k \geq m^j_i$ or $i > e_j$.*

If these conditions hold, then the numbering within the tuples $(m^j_1, \dots, m^j_{e_j})$ can be chosen in such a way that, for each $j \geq 1$, there is a characteristic basis $\Phi^j_1, \dots, \Phi^j_{e_j}$ of $J^{\delta+j}$ such that, for all $j' > j$,

$$\Pi_{jj'}\Phi^{j'}_i = \begin{cases} \Phi^j_i & \text{if } 1 \leq i \leq e_j, \\ 0 & \text{if } e_j + 1 \leq i \leq e_{j'}. \end{cases}$$

Furthermore, the scheme

$$
\begin{array}{r l}
e_1 \text{ rows} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. &
\begin{array}{ccccc}
m_1^1 & m_1^2 & m_1^3 & m_1^4 & \dots \\
\dots & \dots & \dots & \dots & \dots \\
m_{e_1}^1 & m_{e_1}^2 & m_{e_1}^3 & m_{e_1}^4 & \dots
\end{array} \\
e_2 - e_1 \text{ rows} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. &
\begin{array}{ccccc}
 & m_{e_1+1}^2 & m_{e_1+1}^3 & m_{e_1+1}^4 & \dots \\
 & \dots & \dots & \dots & \dots \\
 & m_{e_2}^2 & m_{e_2}^3 & m_{e_2}^4 & \dots
\end{array} \\
e_3 - e_2 \text{ rows} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. &
\begin{array}{ccccc}
 & & m_{e_2+1}^3 & m_{e_2+1}^4 & \dots \\
 & & \dots & \dots & \dots \\
 & & m_{e_3}^3 & m_{e_3}^4 & \dots
\end{array} \\
 & \begin{array}{ccccc} & & & \vdots & \ddots, \end{array}
\end{array}
\tag{2.2}
$$

where in the jth column the characteristic of the space $J^{\delta+j}$ appears, is uniquely determined up to permutation of the kth and the k'th row, where $e_j + 1 \le k,\ k' \le e_{j+1}$ for some j ($e_0 = 0$).

Definition 2.9. An asymptotic type P is said to be *proper* if its representing space J possesses a characteristic basis $\{\Phi_i;\ 1 \le i < e+1\}$ consisting of special vectors that fulfill the equivalent conditions of Proposition 2.8. If the tuples $(m_1^j, \dots, m_{e_j}^j)$ are re-ordered according to this proposition, then the sequence

$$\left\{ \left(\gamma(\Phi_i); m_i^{j_i}, m_i^{j_i+1}, m_i^{j_i+2}, \dots\right);\ 1 \le i < e+1 \right\} \tag{2.3}$$

is called the *characteristic* of P.

Remark 2.10. The characteristic of a proper asymptotic type $P \in \underline{\mathrm{As}}^{\delta}(Y)$ is independent of the splitting of coordinates near ∂X.

An asymptotic type need not be proper. For an example, see LIU–WITT [11, Example 2.23]. However, we have the following result, which will be generalized in Theorem 3.10 below:

Theorem 2.11. *Let $A \in \mathrm{Diff}^{\mu}_{\mathrm{cone}}(X)$ be an elliptic cone-degenerate differential operator. Then*

$$\left\{ u \in C^{\infty,\delta}_{\mathrm{as}}(X) \;\middle|\; Au \in C^{\infty}_{\mathcal{O}}(X) \right\} / C^{\infty}_{\mathcal{O}}(X) \tag{2.4}$$

is a proper asymptotic type.

3. THE ALGEBRA OF COMPLETE MELLIN SYMBOLS

We study the algebra of complete Mellin symbols under the Mellin translation product. Furthermore, we introduce the important notion of a complete characteristic basis for the asymptotics annihilated by a holomorphic complete Mellin symbol.

3.1. **Cone differential operators.** Recall that we have fixed a splitting of coordinates $U \to [0,1) \times Y$, $x \mapsto (t,y)$ near ∂X, U being a collar neighborhood of ∂X. Let (τ, η) be the covariables to (t,y). The compressed covariable $t\tau$ to t is denoted by $\tilde\tau$, i.e., $(\tilde\tau, \eta)$ is the linear variable in the fiber of the compressed cotangent bundle $\tilde T^* U$.

For $A \in \mathrm{Diff}^{\mu}_{\mathrm{cone}}(X)$ as given in (1.1), we denote by $\sigma^{\mu}_{\psi}(A)$ its principal symbol, by $\tilde\sigma^{\mu}_{\psi}(A)$ its compressed principal symbol defined on $\tilde T^* U$ by

$$\sigma^{\mu}_{\psi}(A)(t,y,\tau,\eta) = t^{-\mu} \tilde\sigma^{\mu}_{\psi}(A)(t,y,t\tau,\eta), \quad (t,y,\tau,\eta) \in T^*(U \setminus \partial X) \setminus 0,$$

and by $\sigma^{\mu-j}_{c}(A)(z)$ for $j = 0, 1, 2, \dots$ its *jth conormal symbol*,

$$\sigma^{\mu-j}_{c}(A)(z) := \sum_{k=0}^{\mu} \frac{1}{j!} \frac{\partial^j a_k}{\partial t^j}(0, y, D_y) z^k, \quad z \in \mathbb{C}. \tag{3.1}$$

$\tilde{\sigma}_\psi^\mu(A)(t,y,\tilde{\tau},\eta)$ is smooth up to $t=0$ and $\sigma_c^{\mu-j}(z)$ for $j=0,1,2,\ldots$ is a holomorphic function in z taking values in $\mathrm{Diff}^\mu(Y)$.

Furthermore, if $A\in\mathrm{Diff}^\mu_{\text{cone}}(X)$, $B\in\mathrm{Diff}^\nu_{\text{cone}}(X)$, then $AB\in\mathrm{Diff}^{\mu+\nu}_{\text{cone}}(X)$ and

$$\sigma_c^{\mu+\nu-l}(AB)(z)=\sum_{j+k=l}\sigma_c^{\mu-j}(A)(z+\nu-k)\sigma_c^{\nu-k}(B)(z)$$

for $l=0,1,2,\ldots$ This formula is called the *Mellin translation product*.

Definition 3.1. (a) The operator $A\in\mathrm{Diff}^\mu_{\text{cone}}(X)$ is called *elliptic* if A is an elliptic differential operator on X° and

$$\tilde{\sigma}_\psi^\mu(A)(t,y,\tilde{\tau},\eta)\neq 0,\quad (t,y,\tilde{\tau},\eta)\in\tilde{T}^*U\setminus 0. \tag{3.2}$$

(b) The operator $A\in\mathrm{Diff}^\mu_{\text{cone}}(X)$ is called *elliptic with respect to the weight* $\delta\in\mathbb{R}$ if A is elliptic in the sense of (a) and, in addition,

$$\sigma_c^\mu(A)(z)\colon H^s(Y)\to H^{s-\mu}(Y),\quad \operatorname{Re} z=\dim X/2-\delta, \tag{3.3}$$

is invertible for some $s\in\mathbb{R}$ (and then for all $s\in\mathbb{R}$).

Proposition 3.2. *If $A\in\mathrm{Diff}^\mu_{\text{cone}}(X)$ is elliptic, then the set*

$$\bigl\{z\in\mathbb{C}\bigm| \sigma_c^\mu(A)(z) \text{ regarded as operator in (3.3) is not invertible}\bigr\}$$

is a carrier of asymptotics. In particular, there is a discrete set $D\subset\mathbb{R}$ such that A is elliptic with respect to the weight δ for all $\delta\in\mathbb{R}\setminus D$.

3.2. **Meromorphic Mellin symbols.** We consider the class of meromorphic operator-valued functions arising in point-wise inverting elliptic conormal symbols $\sigma_c^\mu(A)(z)$. For further details, see SCHULZE [15].

Definition 3.3. For $\mu\in\mathbb{Z}\cup\{-\infty\}$, the space $\mathcal{M}^\mu_{\text{as}}(Y)$ of *Mellin symbols* of order μ is defined as follows:

(a) The space $\mathcal{M}^\mu_{\mathcal{O}}(Y)$ of *holomorphic Mellin symbols* of order μ is the space of all $L^\mu_{\text{cl}}(Y)$–valued holomorphic functions $\mathfrak{m}(z)$ on $\mathbb{C}$ such that $\mathfrak{m}(z)\big|_{z=\beta+i\tau}\in L^\mu_{\text{cl}}(Y;\mathbb{R}_\tau)$ uniformly in $\beta\in[\beta_0,\beta_1]$, where $-\infty<\beta_0<\beta_1<\infty$.

(b) $\mathcal{M}^{-\infty}_{\text{as}}(Y)$ is the space of all meromorphic functions $\mathfrak{m}(z)$ on $\mathbb{C}$ taking values in $L^{-\infty}(Y)$ satisfying the following conditions:

(i) The Laurent expansion around each pole $z=p$ of $\mathfrak{m}(z)$ has the form

$$\mathfrak{m}(z)=\frac{\mathfrak{m}_0}{(z-p)^\nu}+\frac{\mathfrak{m}_1}{(z-p)^{\nu-1}}+\cdots+\frac{\mathfrak{m}_{\nu-1}}{z-p}+\sum_{j\geq 0}\mathfrak{m}_{\nu+j}(z-p)^j, \tag{3.4}$$

where $\mathfrak{m}_0,\mathfrak{m}_1,\ldots,\mathfrak{m}_{\nu-1}\in L^{-\infty}(Y)$ are finite-rank operators.

(ii) If the poles of $\mathfrak{m}(z)$ are numbered in a certain way, $p_1,p_2,p_3,\ldots$, then $|\operatorname{Re} p_j|\to\infty$ as $j\to\infty$ if the number of poles is infinite.

(iii) For any function $\chi(z)\in C^\infty(\mathbb{C})$ such that $\chi(z)=0$ if $\operatorname{dist}(z,\bigcup_j\{p_j\})\leq 1/2$ and $\chi(z)=1$ if $\operatorname{dist}(z,\bigcup_j\{p_j\})\geq 1$, we have $\chi(z)\mathfrak{m}(z)\big|_{z=\beta+i\tau}\in L^{-\infty}(Y;\mathbb{R}_\tau)$ uniformly in $\beta\in[\beta_0,\beta_1]$, where $-\infty<\beta_0<\beta_1<\infty$.

(c) We eventually set $\mathcal{M}^\mu_{\text{as}}(Y):=\mathcal{M}^\mu_{\mathcal{O}}(Y)+\mathcal{M}^{-\infty}_{\text{as}}(Y)$.

Write $\mathfrak{m}\in\mathcal{M}^\mu_{\text{as}}(Y)$ for $\mu\in\mathbb{Z}$ as $\mathfrak{m}(z)=\mathfrak{m}_0(z)+\mathfrak{m}_1(z)$, where $\mathfrak{m}_0\in\mathcal{M}^\mu_{\mathcal{O}}(Y)$, $\mathfrak{m}_1\in\mathcal{M}^{-\infty}_{\text{as}}(Y)$. Then the (parameter-dependent) principal symbol $\sigma_\psi^\mu\bigl(\mathfrak{m}_0(z)\big|_{z=\beta+i\tau}\bigr)\in S^{(\mu)}_{\text{cl}}((T^*Y\times\mathbb{R}_\tau)\setminus 0)$ is independent of the choice of the decomposition of $\mathfrak{m}(z)$ and also independent of $\beta\in\mathbb{R}$.

Definition 3.4. $\mathfrak{m}\in\mathcal{M}^\mu_{\text{as}}(Y)$ for $\mu\in\mathbb{Z}$ is called *elliptic* if $\sigma_\psi^\mu\bigl(\mathfrak{m}_0(z)\big|_{z=\beta+i\tau}\bigr)\neq 0$ everywhere.

Proposition 3.5. (a) $\bigcup_{\mu\in\mathbb{Z}} \mathcal{M}^{\mu}_{\text{as}}(Y)$ *is a filtered algebra with respect to the point-wise product as multiplication.*

(b) $\mathfrak{m} \in \mathcal{M}^{\mu}_{\text{as}}(Y)$ *is invertible within this algebra, i.e., with its inverse belonging to* $\mathcal{M}^{-\mu}_{\text{as}}(Y)$, *if and only if* $\mathfrak{m}(z)$ *is elliptic.*

We further introduce the algebra $\mathrm{Symb}^{\mu}_{M}(Y)$ of *complete Mellin symbols*.

Definition 3.6. For $\mu \in \mathbb{Z} \cup \{-\infty\}$, the space $\mathrm{Symb}^{\mu}_{M}(Y)$ consists of all sequences $\mathfrak{S}^{\mu} = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}_0\} \subset \mathcal{M}^{\mu}_{\text{as}}(Y)$. Moreover, an element $\mathfrak{S}^{\mu} \in \mathrm{Symb}^{\mu}_{M}(Y)$ is called *holomorphic* if $\mathfrak{S}^{\mu} = \{\mathfrak{s}^{\mu-j}(z); j \in \mathbb{N}_0\} \subset \mathcal{M}^{\mu}_{\mathcal{O}}(Y)$.

Proposition 3.7. (a) $\bigcup_{\mu\in\mathbb{Z}} \mathrm{Symb}^{\mu}_{M}(Y)$ *is a filtered algebra with involution with respect to the following operations:*

(i) *The Mellin translation product* $\mathfrak{S}^{\mu} \circ_M \mathfrak{T}^{\nu} = \{\mathfrak{u}^{\mu+\nu-l}(z);\, l \in \mathbb{N}_0\} \in \mathrm{Symb}^{\mu+\nu}_{M}(Y)$ *for* $\mathfrak{S}^{\mu} = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}_0\} \in \mathrm{Symb}^{\mu}_{M}(Y)$, $\mathfrak{T}^{\nu} = \{\mathfrak{t}^{\nu-k}(z);\, k \in \mathbb{N}_0\} \in \mathrm{Symb}^{\nu}_{M}(Y)$, *where*

$$\mathfrak{u}^{\mu+\nu-l}(z) = \sum_{j+k=l} \mathfrak{s}^{\mu-j}(z+\nu-k)\mathfrak{t}^{\nu-k}(z), \quad l = 0,1,2,\dots, \tag{3.5}$$

as multiplication.

(ii) *The operation* $(\mathfrak{S}^{\mu})^{*_M} = \{\mathfrak{r}^{\mu-j}(z);\, j \in \mathbb{N}_0\} \in \mathrm{Symb}^{\mu}_{M}(Y)$ *for* $\mathfrak{S}^{\mu} = \{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}_0\} \in \mathrm{Symb}^{\mu}_{M}(Y)$, *where*

$$\mathfrak{r}^{\mu-j}(z) = \mathfrak{s}^{\mu-j}(\dim X - 2\delta - \bar{z} - \mu + j)^*, \quad j = 0,1,2,\dots, \tag{3.6}$$

as involution.

(b) *The complete Mellin symbol* $\{\mathfrak{s}^{\mu-j}(z);\, j \in \mathbb{N}_0\} \in \mathrm{Symb}^{\mu}_{M}(Y)$ *is invertible within the filtered algebra* $\bigcup_{\mu\in\mathbb{Z}} \mathrm{Symb}^{\mu}_{M}(Y)$, *i.e., with its inverse belonging to* $\mathrm{Symb}^{-\mu}_{M}(Y)$, *if and only if* $\mathfrak{s}^{\mu}(z)$ *is elliptic in the sense of* Definition 3.4.

(c) *The map*

$$\bigcup_{\mu\in\mathbb{N}_0} \mathrm{Diff}^{\mu}_{\text{cone}}(X) \to \bigcup_{\mu\in\mathbb{Z}} \mathrm{Symb}^{\mu}_{M}(Y), \quad A \mapsto \{\sigma^{\mu-j}_c(A);\, j \in \mathbb{N}_0\}, \tag{3.7}$$

is a homomorphism of filtered algebras with involution.

3.3. **The space $L^{\delta}_{\mathfrak{S}^{\mu}}$.** For a complete Mellin symbol $\mathfrak{S}^{\mu}$, we introduce a special notation for the representing space of the “asymptotics annihilated” by $\mathfrak{S}^{\mu}$. We actually restrict ourselves to holomorphic complete Mellin symbol (although the definition can be generalized to meromorphic complete Mellin symbol by taking into account the possible “production of asymptotics,” cf. LIU–WITT [11]). The reason for that is that Theorem 3.10, in general, fails to hold without assuming holomorphy.

Definition 3.8. Let $\mathfrak{S}^{\mu} \in \mathrm{Symb}^{\mu}_{M}(Y)$ be holomorphic. Then the linear space $L^{\delta}_{\mathfrak{S}^{\mu}} \subset \mathcal{E}^{\delta}(Y)$ is spanned by all special vectors $\Phi \in \mathcal{E}^{\delta}(Y)$ satisfying

$$\sum_{j+k=l} \mathfrak{s}^{\mu-k}(z+k)\Phi(p-j)[z-p+l] \in \mathcal{A}_{p-l}(C^{\infty}(Y)) \tag{3.8}$$

for $l = 0,1,2,\dots$, where $p = \gamma(\Phi)$.

We also introduce a notation for the expression occuring on the left-hand side of (3.8):

$$\mathbf{\Theta}_l(\Phi)[z] = \mathbf{\Theta}_l(\Phi;\mathfrak{S}^{\mu})[z] := \sum_{j+k=l} \mathfrak{s}^{\mu-k}(z+k)\Phi(p-j)[z-p+l]. \tag{3.9}$$

Proposition 3.9. *For* $A \in \mathrm{Diff}^{\mu}_{\text{cone}}(Y)$, *the subspace of* $C^{\infty,\delta}_{\text{as}}(Y)/C^{\infty}_{\mathcal{O}}(X)$ *from* (2.4) *is represented by the linear space* $L^{\delta}_{\mathfrak{S}^{\mu}}$, *where* $\mathfrak{S}^{\mu} = \{\sigma^{\mu-j}_c(z);\, j \in \mathbb{N}_0\}$.

In this situation we sometimes write L^δ_A instead of $L^\delta_{\mathfrak{S}^\mu}$.

Generalizing Theorem 2.11, we have:

Theorem 3.10 (LIU–WITT [11, Theorem 2.31]). *For an elliptic, holomorphic $\mathfrak{S}^\mu \in \mathrm{Symb}^\mu_M(Y)$, $L^\delta_{\mathfrak{S}^\mu}$ represents a proper asymptotic type.*

We also need:

Lemma 3.11. *The adjoint relation to* (3.8),

$$\sum_{j+k=l} \mathfrak{r}^{\mu-k}(z+k)\Psi(q-j)[z-q+l] \in \mathcal{A}_{q-l}(C^\infty(Y)),$$

where $\mathfrak{R}^\mu \in \mathrm{Symb}^\mu_M(Y)$ is as in (3.6)*, is equivalent to*

$$\sum_{j+k=l} \mathfrak{s}^{\mu-k}(z)^t\, \boldsymbol{I}\Psi(q-j)[z-p-l] \in \mathcal{A}_{p+l}(C^\infty(Y)), \tag{3.10}$$

where $q = \dim X - 2\delta - \bar{p} - \mu$.

3.4. **Complete characteristic bases.** The control of asymptotics of the form (1.5), of conormal order at least δ, is equivalent to the control of the conormal symbols $\sigma_c^{\mu-j}(A)(z)$ for $j = 0, 1, 2, \dots$ in the half-spaces $\mathrm{Re}\, z < \dim X/2 - \delta - j$. We now investigate what is going on as $\delta \to -\infty$.

Let

$$\mathcal{E}(Y) := \bigcup_{\delta\in\mathbb{R}} \mathcal{E}^\delta(Y), \tag{3.11}$$

and $L_{\mathfrak{S}^\mu} := \bigcup_{\delta\in\mathbb{R}} L^\delta_{\mathfrak{S}^\mu}$ for a holomorphic $\mathfrak{S}^\mu \in \mathrm{Symb}^\mu_M(Y)$.

Definition 3.12. A *complete characteristic basis* of $L_{\mathfrak{S}^\mu}$ is the inductive limit

$$\varinjlim \left(\{\Phi^\delta_h;\ h \in \mathcal{I}^\delta\}, \tau_{\delta'\delta}\right)$$

of the following inductive system:

(a) For each $\delta \in \mathbb{R}$, $\{\Phi^\delta_h;\ h \in \mathcal{I}^\delta\}$ is a characteristic basis of $L^\delta_{\mathfrak{S}^\mu}$ of characteristic $\left(\gamma(\Phi^\delta_h); m_h^{j_h,\delta}, m_h^{j_h+1,\delta}, \dots\right)$ satisfying conditions (a), (b) of Proposition 2.8.

(b) For all $\delta > \delta'$, $\tau_{\delta'\delta} \colon \mathcal{I}^\delta \to \mathcal{I}^{\delta'}$ is an injection such that, for any $h \in \mathcal{I}^\delta$,

(i) $\gamma(\Phi^{\delta'}_{h'}) = \gamma(\Phi^\delta_h) + a$ for some $a \in \mathbb{N}_0$,

(ii) $\Phi^\delta_h = T^{m_{h'}^{j_{h'}+a-1,\delta'}} \Phi^{\delta'}_{h'}$,

where $h' = \tau_{\delta'\delta}(h)$ (as well as $\tau_{\delta''\delta} = \tau_{\delta''\delta'}\tau_{\delta'\delta}$ for $\delta > \delta' > \delta''$.)

We write $\mathcal{I} := \varinjlim \left(\mathcal{I}^\delta, \tau_{\delta'\delta}\right)$ with injections $\tau_\delta \colon \mathcal{I}^\delta \to \mathcal{I}$ (such that $\tau_\delta = \tau_{\delta'}\tau_{\delta'\delta}$ for $\delta > \delta'$) and

$$\{\Phi_h;\ h \in \mathcal{I}\} = \varinjlim \left(\{\Phi^\delta_h;\ h \in \mathcal{I}^\delta\}, \tau_{\delta'\delta}\right),$$

where each Φ_h for $h \in \mathcal{I}$ is the collection $\left\{\Phi^\delta_{h^\delta}\right\}$ with $h = \left\{h^\delta\right\}$.

The proof of (a) in the next proposition relies on the finite-dimensionality of the spaces $\mathcal{J}^{p+j}$:

Proposition 3.13. (a) *For each holomorphic $\mathfrak{S}^\mu \in \mathrm{Symb}^\mu_M(Y)$, $L_{\mathfrak{S}^\mu}$ possesses a complete characteristic basis.*

(b) *For any complete characteristic basis $\{\Phi_h;\ h \in \mathcal{I}\}$ of $L_{\mathfrak{S}^\mu}$, the expression*

$$T^{m^{p+l}(\Phi_h)}\Phi_h(p) := T^{m^{p+l}(\Phi^\delta_{h^\delta})}\Phi^\delta_{h^\delta}(p),$$

where $p \in \mathbb{C}$, $l \in \mathbb{N}_0$, $h^\delta = \tau_\delta(h)$, and δ is chosen in such a way that

$$\delta \leq \max\big\{\min\big\{\delta' \in \mathbb{R} \bigm| \gamma(\Phi^{\delta'}_{h^{\delta'}}) - \operatorname{Re} p \in \mathbb{N}_0\big\}, \dim X/2 - (\operatorname{Re} p + l)\big\},$$

is well-defined. (*If there is no δ such that $\gamma(\Phi^{\delta}_{h^\delta}) - \operatorname{Re} p \in \mathbb{N}_0$, then $T^{m^{p+l}(\Phi_h)}\Phi_h(p) := 0$.*)

4. Singularity structure of inverses

In the sequel, let $\mathfrak{S}^\mu = \{\mathfrak{s}^{\mu-j}(z);\ j \in \mathbb{N}_0\} \in \operatorname{Symb}^\mu_c(Y)$ be holomorphic and elliptic. The inverse to $\mathfrak{S}^\mu$ will then be denoted by $\mathfrak{T}^{-\mu} = \{\mathfrak{t}^{-\mu-k}(z);\ k \in \mathbb{N}_0\} \in \operatorname{Symb}^{-\mu}_M(Y)$, cf. Proposition 3.7 (b). In particular,

$$\sum_{j+k=l} \mathfrak{t}^{-\mu-j}(z+\mu)\mathfrak{s}^{\mu-k}(z+k) = \delta_{0l}\,\mathrm{id}, \quad l = 0,1,2,\dots$$

Before stating Theorems 4.1 and 5.1, we simplify the situation to be considered in their proofs. Due to the facts that

- *in the process of inverting $\mathfrak{S}^\mu$ with respect to the Mellin translation product, the "production of singularities" of $\mathfrak{t}^{-\mu-j}(z+\mu)$ at $z = p$ and $\mathfrak{t}^{-\mu-j'}(z+\mu)$ at $z = p'$, respectively, influences each other only if $p - p' \in \mathbb{Z}$,*
- *control on the singularity structure of $\mathfrak{t}^{-\mu-j}(z+\mu)$ in the half spaces $\operatorname{Re} z < \dim X/2 - \delta$ for each $\delta \in \mathbb{R}$ provides control on the singularity structure of $\mathfrak{t}^{-\mu-j}(z+\mu)$ in the whole of $\mathbb{C}$,*

we are allowed to assume the following *model situation*: The complete characteristic basis of $L_{\mathfrak{S}^\mu}$ under consideration consists of special vectors Φ_i for $1 \leq i < e+1$, where $e \in \mathbb{N}_0 \cup \{\infty\}$, $\gamma(\Phi_1) = p$, and

$$\gamma(\Phi_i) = p - l, \quad e_l + 1 \leq i \leq e_{l+1} \tag{4.1}$$

Then $0 = e_0 < e_1 \leq e_2 \leq \dots$ and $e = \max\{e_l \mid l \in \mathbb{N}_0\}$. When referring to this model situation, we denote

$$m^{l+1}_i := m^{p-l}(\Phi_i).$$

Theorem 4.1. *Let $\mathfrak{S}^\mu \in \operatorname{Symb}^\mu_c(Y)$ be holomorphic, elliptic. Then, for each complete characteristic basis $\{\Phi_h;\ h \in \mathcal{I}\}$ of $L_{\mathfrak{S}^\mu}$, there is a unique complete characteristic basis $\{\Psi_{h^*};\ h^* \in \mathcal{I}^*\}$ of $L_{\mathfrak{R}^\mu}$, where $\mathfrak{R}^\mu$ is given by* (3.6), *and a bijection $\tau^*\colon \mathcal{I} \to \mathcal{I}^*$ such that, for all $p \in \mathbb{C}$ and $j = 0,1,2,\dots$,*

$$\big[\mathfrak{t}^{-\mu-j}(z+\mu)\big]^*_p = \sum_h T^{m^{p+1}_h}\Phi_h(p-j) \otimes T^{m^{q+j+1}_{h^*}} \boldsymbol{J}\Psi_{h^*}(q)[z-p], \tag{4.2}$$

where $q = \dim X - 2\delta - \bar{p} - \mu$, $h^* = \tau^*(h)$, $m^{p+1}_h = m^{p+1}(\Phi_h)$, and $m^{q+j+1}_{h^*} = m^{q+j+1}(\Psi_{h^*})$.

Proof. We assume the model situation (4.1).

Step 1. The elements Φ of $L_{\mathfrak{S}^\mu}$ ($= L^\delta_{\mathfrak{S}^\mu}$) are given by the relations

$$\Phi(p-l)[z-p+l] = \sum_{j=0}^{l} \Big[\mathfrak{t}^{-\mu-j}(z+\mu)\phi^{(p-l+j)}(z)\Big]^*_{p-l+j}, \tag{4.3}$$

for $l = 0,1,2,\dots$, where $\phi^{(p-j)} = \sum_{r=0}^\infty \phi^{(p-j)}_r (z-p+j)^r \in \mathcal{A}_{p-j}(C^\infty(Y))$ for $j = 0,1,2,\dots$, cf. Liu–Witt [11]. In fact, the Taylor coefficients $\phi^{(p-j)}_r \in C^\infty(Y)$ can be chosen arbitrarily, since only a finite number of them enters the computation of $\Phi(p-l)$.

Since $\Phi(p-l) \in \operatorname{span}\big\{\Phi_i(p-l);\, 1 \le i \le e_{l+1}\big\}$ for $\Phi \in L_{\mathfrak{S}^\mu}$, we conclude that

$$\big[\mathrm{t}^{-\mu-j}(z+\mu)\big]^*_{p-l+j} = \sum_{i=1}^{e_{l+1}} \Phi_i(p-l) \otimes H_i^{(jl)}[z-p+l-j] \tag{4.4}$$

for certain $H_i^{(jl)} = \left(h_{0i}^{(jl)}, h_{1i}^{(jl)}, \dots, h_{m_i^{l+1}-1,i}^{(jl)}\right) \in [C^\infty(Y)]^\infty$, which are *a priori* of length m_i^{l+1}. Employing (4.4), we rewrite (4.3) as

$$\Phi(p-l) = \sum_{i=1}^{e_{l+1}} \sum_{r=0}^{m_i^{l+1}-1} \left(\sum_{j=0}^{l} \sum_{s=0}^{r} \big(\phi_{r-s}^{(p-l+j)}, h_{si}^{(jl)}\big) \right) T^r \Phi_i(p-l) \tag{4.5}$$

for $l = 0, 1, 2, \dots$

Step 2. We are going to show that (4.5) provides the unique representation of Φ as linear combination of the vectors $T^r\Phi_i$. More precisely, by induction on $l = 0, 1, 2, \dots$, we construct functions $h_{ri}^{(l)} \in C^\infty(Y)$ for $r \ge m_i^l$ such that, for each l,

$$h_{m_i^l,i}^{(l)} \text{ for all } i \text{ satisfying } m_i^l < m_i^{l+1} \text{ are linearly independent} \tag{4.6}$$

and

$$h_{ri}^{(jl)} = \begin{cases} h_{ri}^{(l-j)} & \text{if } r \ge m_i^{l-j}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

This means that the coefficient in front of $T^r\Phi_i$ equals $\sum_{j=0}^{l} \sum_{s=m_i^j}^{r} \big(\phi_{r-s}^{(p-j)}, h_{si}^{(j)}\big)$ provided that $m_i^l \le r \le m_i^{l+1} - 1$.

After the lth step, $h_{ri}^{(l')}$ will have been constructed for all l', i, r satisfying $l' \le l$, $1 \le i \le e_{l+1}$, $m_i^{l'} \le r \le m_i^{l+1} - 1$. Moreover, (4.7) will have been proved for all i, r satisfying $1 \le i \le e_{l+1}$, $0 \le r \le m_i^{l+1} - 1$.

Base of induction $l = 0$: We set $h_{ri}^{(0)} := h_{ri}^{(00)}$ for $1 \le i \le e_1$, $0 \le r \le m_i^1 - 1$.

Induction step $l' < l \to l$: We write (4.5) as

$$\Phi(p-l) = \sum_{i=1}^{e_l} \sum_{r=0}^{m_i^l-1} \left(\sum_{j=0}^{l} \sum_{s=0}^{r} \big(\phi_{r-s}^{(p-l+j)}, h_{si}^{(jl)}\big) \right) T^r \Phi_i(p-l) + \sum_{i=1}^{e_{l+1}} \sum_{r=m_i^l}^{m_i^{l+1}-1} \left(\sum_{j=0}^{l} \sum_{s=0}^{r} \big(\phi_{r-s}^{(p-l+j)}, h_{si}^{(jl)}\big) \right) T^r \Phi_i(p-l).$$

Φ regarded as a vector in $L^\delta_{\mathfrak{S}^\mu}$ modulo $L^{\delta'}_{\mathfrak{S}^\mu}$ for some δ' satisfying $\dim X/2 - \delta' < p - l < \dim X/2 - \delta' + 1$ is a unique linear combination of the vectors $T^r\Phi_i$ for $1 \le i \le e_{l+1}$, $0 \le r \le m_i^{l+1} - 1$. The coefficient in front of $T^r\Phi_i$ for $1 \le i \le e_l$, $0 \le r \le m_i^l - 1$ is known if one knows Φ modulo $L^{\delta'-1}_{\mathfrak{S}^\mu}$. By inductive hypothesis, this coefficient equals $\sum_{j=0}^{l-1} \sum_{s=m_i^j}^{r} \big(\phi_{r-s}^{(p-j)}, h_{si}^{(j)}\big)$. Thus, we obtain (4.7) for $1 \le i \le e_l$, $0 \le r \le m_i^l - 1$, since the functions $\phi_r^{(p-j)} \in C^\infty(Y)$ are arbitrary. It remains to set $h_{ri}^{(l')} := h_{ri}^{(l-l',l)}$ for $0 \le l' \le l$, $m_i^l \le r \le m_i^{l+1} - 1$.

In particular,

$$H_i^{(jl)} = \left(h_{m_i^{l-j},i}^{(l-j)}, h_{m_i^{l-j}+1i}^{(l-j)}, \dots, h_{m_i^{l+1}-1,i}^{(l-j)} \right)$$

is actually of length $m_i^{l+1} - m_i^{l-j}$.

Step 3. We now fix some $a \in \mathbb{N}_0$ and set

$$H_i^{(l)} = H_{i;a}^{(l)} := \left(h^{(l)}_{m_i^l,i}, h^{(l)}_{m_i^{l-j}+1i}, \dots, h^{(l)}_{m_i^a-1,i} \right)$$

for $0 \le l \le a-1$. Then $H_i^{(jl)} = T^{m_i^a - m_i^{l+1}} H_i^{(l-j)}$ and

$$\begin{aligned} &\left[\mathfrak{t}^{-\mu-j}(z+\mu-k)\right]^*_{p-b} \\ &\qquad = \sum_{i=1}^{e_{l+b+1}} T^{m_i^{k+b}} \Phi_i(p-l-b) \otimes T^{m_i^a - m_i^{l+b+1}} H_i^{(k+b)}[z-p+b] \end{aligned} \tag{4.8}$$

for all $j,\ k,\ l,\ b$ satisfying $j+k=l$, $l+b<a$. We shall employ (4.8) to show that the vectors $\Psi_1, \dots, \Psi_{e_a}$ defined by

$$\Psi_i(q+l) := \boldsymbol{J} H_i^{(l)}, \quad 0 \le l \le a-1, \tag{4.9}$$

where $q = \dim X - 2\delta - \bar{p} - \mu$, form a characteristic basis of $L^{\delta+\mu-a}_{\mathfrak{R}^\mu}$ modulo $L^{\delta+\mu}_{\mathfrak{R}^\mu}$.

In view of (4.6), $\Psi_1, \dots, \Psi_{e_a}$ form a characteristic basis of the T–invariant subspace of $\mathcal{E}^\delta(Y)$ modulo $\mathcal{E}^{\delta+a}(Y)$ generated by these vectors, of characteristic

$$\left\{ \left(q+l; m_i^a - m_i^l, m_i^a - m_i^{l-1}, \dots, m_i^a - m_i^1, m_i^a\right);\ 1 \le i \le e_a \right\},$$

where, for a given i, l is the least integer such that $m_i^{l+1} = m_i^a$. In particular, the dimension of this space equals $\dim L^\delta_{\mathfrak{S}^\mu} \big/ L^{\delta+a}_{\mathfrak{S}^\mu} = \sum_{i=1}^{e_a} m_i^a$. Invoking a duality argument, we see that it suffices to prove that each Ψ_i belongs to $L^{\delta+\mu-a}_{\mathfrak{R}^\mu}$ modulo $L^{\delta+\mu}_{\mathfrak{R}^\mu}$.

Step 4. By virtue of (3.10), we have to show that

$$\sum_{j+k=l} \mathfrak{s}^{\mu-k}(z)^t \boldsymbol{C} H_i^{(a-j-1)}[z-\tilde{p}-l] = O(1) \quad \text{as } z \to \tilde{p}+l \tag{4.10}$$

for $1 \le i \le e_a$, $0 \le l \le a-1$, where $\tilde{p} := p-a+1$.

For $l+b = a-1$, we infer from (4.8)

$$\begin{aligned} \delta_{0l}\,\mathrm{id} &= \sum_{j+k=l} \mathfrak{t}^{-\mu-j}(z+\mu-k)\mathfrak{s}^{\mu-k}(z) \\ &= \sum_{j+k=l} \sum_{i=1}^{e_a} (z-\tilde{p}-l)^{m_i^a - m_i^{a-j-1}} \left(T^{m_i^{a-j-1}} \Phi_i(\tilde{p})[z-\tilde{p}-l] \right) \\ &\qquad \otimes \left(\mathfrak{s}^{\mu-k}(z)^* H_i^{(a-j-1)}[z-\tilde{p}-l] \right) + O(1) \quad \text{as } z \to \tilde{p}+l. \end{aligned}$$

Since the leading entries of the vectors $T^{m_i^{a-j-1}} \Phi_i(\tilde{p})$ (if there are any) for different i are linearly independent, we arrive at (4.10).

Step 5. Returning to the notation $H_i^{(l)} = H_{i;a}^{(l)}$, we see that the Ψ_i defined by (4.9) for $1 \le i < e+1$ as $a \to \infty$ constitute a complete characteristic basis of $L_{\mathfrak{R}^\mu}$ modulo $L^{\delta+\mu}_{\mathfrak{R}^\mu}$. Furthermore, the considerations also show uniqueness for the complete characteristic basis of $L_{\mathfrak{R}^\mu}$ modulo $L^{\delta+\mu}_{\mathfrak{R}^\mu}$ just constructed. □

5. Generalization of Keldysh's formula

Conjugacy of complete characteristic bases in the sense of Theorem 4.1 forces certain *bilinear relations* between the bases elements to hold, as for local asymptotic types. We are now going to derive these relations keeping the notations of the previous section.

Theorem 5.1. *For all p, h, h^*, l, j satisfying $j \leq l$,*

$$\sum_{r=j}^{l} \Big\langle \boldsymbol{\Theta}_{l-r}(T^{m_h^{p+l+1}} \Phi_h)[z+r], T^{m_{h^*}^{q+1}} \boldsymbol{I}\Psi_{h^*}(q-r)[z-p] \Big\rangle$$
$$= \delta_{hh^*}(z-p)^{-(m_h^p - m_h^{p+l+1})} + O((z-p)^{-(m_h^p - m_h^{p+j})}) \quad \text{as } z \to p, \quad (5.1)$$

where $q = \dim X - 2\delta - \bar{p} - \mu$, $m_h^{p+j} = m^{p+j}(\Phi_h)$, $m_{h^*}^{q+1} = m^{q+1}(\Psi_{h^*})$,

$$\delta_{hh^*} = \begin{cases} 1 & \text{if } h^* = \tau^*(h), \\ 0 & \text{otherwise,} \end{cases}$$

and $\boldsymbol{\Theta}_{l-r}(T^{m_h^{p+l+1}} \Phi_h)[z+r]$ was defined in (3.9).

Remark 5.2. (a) (5.1) constitutes an asymptotic expansion formula, with $j = l$ being the basic case and further correction terms added as j is getting smaller. In Section 7, we will be in need of the most refined case $j = 0$.

(b) In case $l = 0$, we recover Keldysh's formula

$$\Big\langle \mathfrak{s}^{\mu}(z)\, T^{m_h^{p+1}} \Phi_h(p)[z-p], T^{m_{h^*}^{q+1}} \boldsymbol{I}\Psi_{h^*}(q)[z-p] \Big\rangle$$
$$= \delta_{hh^*}(z-p)^{-(m_h^p - m_h^{p+1})} + O(1) \quad \text{as } z \to p,$$

cf. (A.2).

To prove Theorem 5.1 we need the following simple result:

Lemma 5.3. *Assume the model situation* (4.1). *Let $a_{ij}(z) \in \mathcal{A}_{p-l}(\mathbb{C})$ for some $l \in \mathbb{N}_0$. Then*

$$\sum_{i=1}^{e_l+1} \sum_{j=0}^{l} a_{ij}(z)\, T^{m_i^j} \Phi_i(p-l)[z-p+l] \in \mathcal{A}_{p-l}(C^\infty(Y))$$

if and only if

$$\sum_{r=0}^{j} (z-p+l)^{m_i^r} a_{ir}(z) = O((z-p+l)^{m_i^{j+1}})$$

for all $1 \leq i \leq e_l + 1$, $0 \leq j \leq l$.

Proof of Theorem 5.1. We again assume the model situation (4.1).

We reenter the scene at formulas (4.8). Using these formulas, we write

$$\mathfrak{t}^{-\mu-j}(z+\mu)$$
$$= \sum_{b=0}^{a-j-1} \sum_{i=1}^{e_{j+b+1}} (T^{m_i^b} \Phi_i(p-j-b)) \otimes (T^{m_i^a - m_i^{j+b+1}} H_i^{(b)})[z-p+b] + G_j(z)$$

for $0 \leq j \leq a-1$, where $G_j(z)$ is holomorphic on the strip $\dim X/2 - \delta - a + j < \operatorname{Re} z < \dim X/2 - \delta$. For any $0 \leq l' \leq a-1$, we get

$$\delta_{0l'}\, \mathrm{id} = \sum_{j+k=l'} \mathfrak{t}^{-\mu-j}(z+\mu+j)\mathfrak{s}^{\mu-k}(z+l')$$
$$= \sum_{j+k=l'} \sum_{b=0}^{a-l'-1} \sum_{i=1}^{e_{l'+b+1}} (z - p + l' + b)^{m_i^{l'+b+1} - m_i^{k+b}} \Big(T^{m_i^{k+b}} \Phi_i(p-l'-b)[z-p+l'+b]\Big)$$

$$\otimes\left(\mathfrak{s}^{\mu-k}(z+l')^*T^{m_i^a-m_i^{l'+b+1}}H_i^{(k+b)}[z-p+l'+b]\right)+\sum_{j+k=l'}G_j(z+j)\mathfrak{s}^{\mu-k}(z+l') \tag{5.2}$$

We now apply the operator (5.2) to $T^{m_{i'}^{\tilde b-l}}\Phi_{i'}(p-\tilde b+l')[z-p+\tilde b]$, where $l\le\tilde b\le a-1$, and then sum up for l' from 0 to l. Since

$$\begin{aligned}\sum_{j+k\le l}&G_j(z+j)\mathfrak{s}^{\mu-k}(z+j+k)T^{m_{i'}^{\tilde b-l}}\Phi_{i'}(p-\tilde b+j+k)[z-p+\tilde b]\\&=\sum_{j=0}^{l}G_j(z+j)\left(\sum_{k=0}^{l-j}\mathfrak{s}^{\mu-k}(z+j+k)T^{m_{i'}^{\tilde b-l}}\Phi_{i'}(p-\tilde b+j+k)[z-p+\tilde b]\right)\\&=\sum_{j=0}^{l}G_j(z+j)\mathbf{\Theta}_{l-j}(T^{m_{i'}^{\tilde b-l}}\Phi_{i'})[z+j]\in\mathcal{A}_{p-\tilde b}(C^\infty(Y)).\end{aligned}$$

taking the principal value at $z=p-\tilde b$ on both sides of the resulting equation, we obtain

$$\begin{aligned}&T^{m_{i'}^{\tilde b-l}}\Phi_{i'}(p-\tilde b)[z-p+\tilde b]=\sum_{j+k\le l}\sum_{i=1}^{e_{\tilde b+1}}(z-p+\tilde b)^{m_i^{\tilde b+1}-m_i^{k+b}}\\&\left\langle\mathfrak{s}^{\mu-k}(z+j+k)\,T^{m_{i'}^{\tilde b-l}}\Phi_{i'}(p-b)[z-p+\tilde b],T^{m_i^a-m_i^{\tilde b+1}}\boldsymbol{C}H_i^{(k+b)}[z-p+\tilde b]\right\rangle\\&\qquad\qquad T^{m_i^{k+b}}\Phi_i(p-\tilde b)[z-p+\tilde b]+O(1)\quad\text{as } z\to p-\tilde b,\end{aligned}$$

where $\tilde b=b+j+k$. We get

$$\begin{aligned}T^{m_{i'}^{\tilde b-l}}\Phi_{i'}(p-\tilde b)[z-p+\tilde b]&=\sum_{i=1}^{e_{\tilde b}+1}\sum_{j=0}^{l}a_{ij}(z)\,T^{m_i^{\tilde b-j}}\Phi_i(p-\tilde b)[z-p+\tilde b]+O(1)\\&=\sum_{i=1}^{e_{\tilde b}+1}\sum_{j=\tilde b-l}^{\tilde b}a_{i,\tilde b-j}(z)\,T^{m_i^j}\Phi_i(p-\tilde b)[z-p+\tilde b]+O(1)\quad\text{as } z\to p-\tilde b,\end{aligned}$$

where

$$\begin{aligned}a_{ij}(z)&=(z-p+\tilde b)^{m_i^{\tilde b+1}-m_i^{\tilde b-j}}\\&\qquad\times\left\langle\mathbf{\Theta}_{l-j}(T^{m_{i'}^{\tilde b-l}}\Phi_{i'})[z+j],T^{m_i^a-m_i^{\tilde b+1}}\boldsymbol{C}H_i^{(\tilde b-j)}[z-p+\tilde b]\right\rangle.\end{aligned}$$

By virtue of Lemma 5.3, we conclude that, for all $0\le j\le l$,

$$\begin{aligned}&\sum_{r=j}^{l}(z-p+l)^{m_i^{\tilde b-r}}a_{ir}(z)\\&\qquad=\delta_{ii'}(z-p+l)^{m_i^{\tilde b-l}}+O((z-p+l)^{m_i^{\tilde b-j+1}})\quad\text{as } z\to p-l,\end{aligned}$$

i.e.,

$$\begin{aligned}&\sum_{r=j}^{l}\left\langle\mathbf{\Theta}_{l-r}(T^{m_{i'}^{\tilde b-l}}\Phi_{i'})[z+r],T^{m_i^a-m_i^{\tilde b+1}}\boldsymbol{C}H_i^{(\tilde b-r)}[z-p+\tilde b]\right\rangle\\&\quad=\delta_{ii'}(z-p+l)^{-(m_i^{\tilde b+1}-m_i^{\tilde b-l})}+O((z-p+l)^{-(m_i^{\tilde b+1}-m_i^{\tilde b-j+1})})\quad\text{as } z\to p-l.\end{aligned}$$

In view of (4.9), the latter is (5.1) in the model situation (4.1). □

6. The boundary sesquilinear form

In this section, we shall prove Theorem 1.1. From GIL–MENDOZA [3, Theorem 7.11], we first quote:

Theorem 6.1. *For all $u \in D(A_{\max})$, $v \in D(A^*_{\max})$,*

$$[u,v]_A = -\sum_{j=0}^{\mu-1} \sum_{\dim X/2-\delta-\mu+j<\operatorname{Re} p<\dim X/2-\delta} \operatorname{Res}_{z=p}\big(\sigma_c^{\mu-j}(A)(z)(\omega u)\tilde{}(z),(\omega v)\tilde{}(\dim X-2\delta-\bar z-\mu+j)\big), \quad (6.1)$$

where $\omega(t)$ is a cut-off function and $(\omega u)\tilde{}(z,\cdot)=M_{t\to z}\{(\omega u)(t,\cdot)\}$ denotes the Mellin transform, see before (B.1).

Proof of Theorem 1.1. We divide the proof into several steps.

Step 1. Because of $D(A_{\max}) = \mathbb{H}^{\mu,\delta}_{P^\delta_A}(X)$ and $D(A_{\min}) = \mathbb{H}^{\mu,\delta}_{P^{\delta+\mu-0}_A}(X)$, see Theorem B.6, and similarly for $D(A^*_{\max})$, $D(A^*_{\min})$ with P^δ_A replaced with $P^\delta_{A^*}$, we have to compute the induced sesquilinear form

$$[\ ,\ ]_A\colon L^\delta_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}\times L^\delta_{\mathfrak{R}^\mu}/L^{\delta+\mu-0}_{\mathfrak{R}^\mu}\to\mathbb{C}. \quad (6.2)$$

Here $\mathfrak{S}^\mu=\{\mathfrak{s}^{\mu-j}(z);\ j\in\mathbb{N}_0\}$, where $\mathfrak{s}^{\mu-j}(z)=\sigma_c^{\mu-j}(A)(z)$, and $\mathfrak{R}^\mu=\{\mathfrak{r}^{\mu-j}(z);\ j\in\mathbb{N}_0\}$, where $\mathfrak{r}^{\mu-j}(z)=\sigma_c^{\mu-j}(A^*)(z)$. For the relation between $\mathfrak{s}^{\mu-j}(z)$, $\mathfrak{r}^{\mu-j}(z)$, see (3.6). We will evaluate the sesquilinear form (6.2) using formula (6.1).

From this description, it is also seen that the spaces $D(A_{\max})/D(A_{\min})$, $D(A^*_{\max})/D(A^*_{\min})$ are invariant under the action of the operator T from (1.6). (This result is implicitly contained in Theorem 2.11.)

Step 2. It suffices to prove (1.9) for an arbitrary characteristic basis $\Phi_1,\dots,\Phi_e$ of the quotient $L^\delta_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}$. For then non-degeneracy of the sesquilinear form (6.2) and also property (1.10) follow, where the latter holds for all $\Phi\in L^\delta_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}$, $\Psi\in L^\delta_{\mathfrak{R}^\mu}/L^{\delta+\mu-0}_{\mathfrak{R}^\mu}$. If $\Phi'_1,\dots,\Phi'_e$ another characteristic basis of the quotient $L^\delta_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}$, we have (after a possible renumbering)

$$\Phi'_i = C\,\Phi_i,\quad 1\le i\le e$$

for some linear invertible operator $C\colon L^\delta_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}\to L^\delta_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}$ that commutes with T. Denoting by C^* the adjoint to C with respect to the non-degenerate sesquilinear form (6.2) (C^* also commutes with T), the conjugate characteristic basis $\Psi'_1,\dots,\Psi'_e$ to $\Phi'_1,\dots,\Phi'_e$ is given by

$$\Psi'_i=(C^*)^{-1}\,\Psi_i,\quad 1\le i\le e,$$

where $\Psi_1,\dots,\Psi_e$ is the conjugate characteristic basis to $\Phi_1,\dots,\Phi_e$.

Step 3. Let $\Phi_1,\dots,\Phi_e$ be a characteristic basis of $L^\delta_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}$, of characteristic $(m_1,\dots,m_e)$ say, and let $\Psi_1,\dots,\Psi_e$ be the conjugate characteristic basis of $L^\delta_{\mathfrak{R}^\mu}/L^{\delta+\mu-0}_{\mathfrak{R}^\mu}$ according to Theorem 4.1. The latter means that there are corresponding versions of Theorems 4.1, 5.1 valid for the Mellin symbols $\mathfrak{t}^{-\mu-j}(z)$ for $0\le j<\mu$ of the strip $\dim X/2-\delta-\mu+j<\operatorname{Re}z<\dim X/2-\delta-\mu$, where now in (4.2), (5.1) elements of the quotients $L^\delta_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}$, $L^\delta_{\mathfrak{R}^\mu}/L^{\delta+\mu-0}_{\mathfrak{R}^\mu}$ enter. Likewise, we may assume that $\Phi_1,\dots,\Phi_e$ stem (by projection) from a characteristic basis of $L^\delta_{\mathfrak{S}^\mu}$ that can be extended to a complete characteristic basis of $L_{\mathfrak{S}^\mu}$.

We will make this latter assumption to keep the notation from Theorems 4.1, 5.1.

For $\Phi \in L^{\delta}_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}$, $\Psi \in L^{\delta}_{\mathfrak{R}^\mu}/L^{\delta+\mu-0}_{\mathfrak{R}^\mu}$, rewrite (6.1) as

$$[\Phi,\Psi]_A = -\sum_{k=0}^{\mu-1} \sum_{\dim X/2-\delta-\mu+k<\mathrm{Re}\,p<\dim X/2-\delta} \times \,\mathrm{Res}_{z=p}\Big\langle \mathfrak{s}^{\mu-k}(z)\Phi(p)[z-p], \boldsymbol{I}\Psi(q+k)[z-p]\Big\rangle, \quad (6.3)$$

where $q = \dim X - 2\delta - \bar{p} - \mu$.

Now choose Φ belonging to the Jordan basis $\Phi_1,\dots,T^{m_1-1}\Phi_1,\dots,\Phi_e,\dots,T^{m_e-1}\Phi_e$ of the quotient $L^{\delta}_{\mathfrak{S}^\mu}/L^{\delta+\mu-0}_{\mathfrak{S}^\mu}$ and Ψ belonging to the conjugate Jordan basis $\Psi_1,\dots,T^{m_1-1}\Psi_1,\dots,$ $\Psi_e,\dots,T^{m_e-1}\Psi_e$ of the quotient $L^{\delta}_{\mathfrak{R}^\mu}/L^{\delta+\mu-0}_{\mathfrak{R}^\mu}$. That means that

$$\Phi = T^iT^{m_h^{p+l+1}}\Phi_h$$

for some h, p, l, i, where $\dim X/2-\delta-\mu < \mathrm{Re}\,p \le \dim X/2-\delta-(\mu-1)$, $\mathrm{Re}\,p+l \le \dim X/2-\delta$, and $0 \le i < m_h^{p+l} - m_h^{p+l+1}$. We may further assume that

$$\Psi = T^jT^{m_{h^*}^{q+1}}\Psi_{h^*},$$

where $q = \dim X - 2\delta - \bar{p} - \mu$ and $0 \le j < m_{h^*}^{q+l+1} - m_{h^*}^{q+1}$, since otherwise $[\Phi,\Psi]_A = 0$.

Under these hypotheses, in (6.3) there are non-zero residues at most at $z = p+r$ for $r = 0,\dots,l$, i.e.,

$$\begin{aligned}
[\Phi,\Psi]_A &= (-1)^{j+1}\sum_{k=0}^{l}\sum_{r=k}^{l}\mathrm{Res}_{z=p+r}\Big\langle \mathfrak{s}^{\mu-k}(z)T^iT^{m_h^{p+l+1}}\Phi_h(p+r)[z-p-r],\\
&\qquad T^j\boldsymbol{I}T^{m_{h^*}^{q+1}}\Psi_{h^*}(q-r+k)[z-p-r]\Big\rangle\\
&= (-1)^{j+1}\sum_{k=0}^{l}\sum_{r=k}^{l}\mathrm{Res}_{z=p+r-k}\\
&\qquad \Big\langle \mathfrak{s}^{\mu-k}(z+k)T^iT^{m_h^{p+l+1}}\Phi_h(p+r)[z-p-r+k],\\
&\qquad T^j\boldsymbol{I}T^{m_{h^*}^{q+1}}\Psi_{h^*}(q-r+k)[z-p-r+k]\Big\rangle\\
&= (-1)^{j+1}\sum_{k=0}^{l}\sum_{r=0}^{l-k}\mathrm{Res}_{z=p+r}\\
&\qquad \Big\langle \mathfrak{s}^{\mu-k}(z+k)T^iT^{m_h^{p+l+1}}\Phi_h(p+r+k)[z-p-r],\\
&\qquad T^j\boldsymbol{I}T^{m_{h^*}^{q+1}}\Psi_{h^*}(q-r)[z-p-r]\Big\rangle\\
&= (-1)^{j+1}\sum_{r=0}^{l}\mathrm{Res}_{z=p+r}\\
&\qquad \Big\langle \sum_{k=0}^{l-r}\mathfrak{s}^{\mu-k}(z+k)T^iT^{m_h^{p+l+1}}\Phi_h(p+r+k)[z-p-r],\\
&\qquad T^j\boldsymbol{I}T^{m_{h^*}^{q+1}}\Psi_{h^*}(q-r)[z-p-r]\Big\rangle\\
&= (-1)^{j+1}\sum_{r=0}^{l}\mathrm{Res}_{z=p+r}(z-p-r)^{i+j}\Big\langle \boldsymbol{\Theta}_{l-r}(T^{m_h^{p+l+1}}\Phi_h)[z],\\
&\qquad \boldsymbol{I}T^{m_{h^*}^{q+1}}\Psi_{h^*}(q-r)[z-p-r]\Big\rangle
\end{aligned}$$

$$= (-1)^{j+1} \operatorname{Res}_{z=p} (z-p)^{i+j} \sum_{r=0}^{l} \Big\langle \boldsymbol{\Theta}_{l-r} (T^{m_h^{p+l+1}} \Phi_h)[z+r], \\ \boldsymbol{I} T^{m_{h^*}^{q+1}} \Psi_{h^*}(q-r)[z-p] \Big\rangle$$

Therefore,

$$[\Phi, \Psi]_A = \begin{cases} (-1)^{j+1} & \text{if } \tau^*(h) = h^*,\, i+j = m_h^p - m_h^{p+l+1} - 1, \\ 0 & \text{otherwise,} \end{cases}$$

by virtue of Theorem 5.1.

This completes the proof. □

7. Examples

We discuss two examples of ordinary differential operators on the half-line $\mathbb{R}_+$. The first example demonstrates the usage of Theorem 1.1 for the computation of the boundary sequilinear form, while in the second example it is shown how our fundamental formulas like (4.2) can be independently verified.

7.1. **First example.** This example concerns the cone-degenerate third-order operator

$$A = \partial_t^3 + t^{-1} \partial_t^2 \quad \text{on } \mathbb{R}_+.$$

The conormal symbols are

$$\sigma_c^3(A)(z) = -z(z+1)^2$$

and $\sigma_c^{3-j}(A)(z) = 0$ for $j \geq 1$. Thus, 1, $t \log t$, and t are exact solutions to the equation $Au = 0$. A complete characteristic basis Φ_1, Φ_2 of L_A is given by

$$\Phi_1(0) = (1), \quad \Phi_2(-1) = (1, 0),$$

and $\Phi_1(p) = 0$ for $p \neq 0$, $\Phi_2(p) = 0$ for $p \neq -1$.

We choose $\underline{\delta = -1}$. Then we have $A^* = -\partial_t^3 - 5t^{-1}\partial_t^2 - 4t^{-2}\partial_t$, $\sigma_c^3(A^*)(z) = z(z-1)^2$, and $\sigma_c^{3-j}(A^*)(z) = 0$ for $j \geq 1$. From

$$-\frac{1}{z(z+1)^2} = \frac{1}{(z+1)^2} + \frac{1}{z+1} - \frac{1}{z}$$

we infer that the complete characteristic basis Ψ_1, Ψ_2 of L_{A^*} that is conjugate to Φ_1, Φ_2 is given by

$$\Psi_1(1) = (1, -1), \quad \Psi_2(0) = (1),$$

and $\Psi_1(p) = 0$ for $p \neq 1$, $\Psi_2(p) = 0$ for $p \neq 0$, where $\tau^*(1) = 2$, $\tau^*(2) = 1$.

Writing

$$u(t) = \omega(t)\,(\alpha + \beta_0 t \log t + \beta_1 t) + u_0(t),$$
$$v(t) = \omega(t)\,\big(\gamma_0 t^{-1} \log t + \gamma_1 t^{-1} + \delta\big) + v_0(t)$$

for α, β_0, β_1, γ_0, γ_1, $\delta \in \mathbb{C}$, where $\omega(t)$ is a cut-off function and $u_0 \in D(A_{\min})$, $v_0 \in D(A^*_{\min})$, we then obtain

$$[u, v]_A = -\alpha\bar{\delta} + \beta_0\bar{\gamma}_0 + \beta_0\bar{\gamma}_1 - \beta_1\bar{\gamma}_0.$$

7.2. **Second example.** We consider the non-degenerate, second-order, constant coefficient operator

$$A = \partial_t^2 + a\partial_t + b \quad \text{on } \mathbb{R}_+,$$

where $a,\, b \in \mathbb{C}$. We have $A^* = \partial_t^2 - \bar{a}\partial_t + \bar{b}$ (with $\underline{\delta = 0}$) and the Green formula is directly checked to be

$$[u,v]_A = u(0)\bar{v}'(0) - u'(0)\bar{v}(0) - a\,u(0)\bar{v}(0), \quad u,\, v \in \mathcal{S}(\overline{\mathbb{R}}_+). \tag{7.1}$$

The space L^0_A has characteristic $\{(-j;1,1,\dots);\, j = 0,1\}$. Therefore, $L^{2-0}_A = \{0\}$, $D(A_{\text{max}})/D(A_{\text{min}}) \cong L^0_A$ is two-dimensional, and the elements of $D(A_{\text{max}})/D(A_{\text{min}})$ are in one-to-one correspondence with the (in fact, analytic) solutions $u(t) = u(t;\alpha,\beta)$ for $\alpha,\, \beta \in \mathbb{C}$ to

$$Au = 0, \quad u(0) = \alpha, \quad u'(0) = \beta.$$

In the following, we shall make this identification.

A complete characteristic basis of $L_A = L^0_A$ is given by

$$u_1(t) = u(t;1,0), \quad u_2(t) = u(t;0,1).$$

We look at (7.1) to find the conjugate complete characteristic basis of L_{A^*} to be

$$v_1(t) = u(t;1,\bar{a}), \quad v_2(t) = v(t;0,-1),$$

where $v(t) = v(t;\alpha,\beta)$ is the solution to $A^*v = 0$, $v(0) = \alpha$, $v'(0) = \beta$.

Proposition 7.1. (a) *We have*

$$u_1(t) = 1 + \sum_{j\ge 2} (-1)^{j-1}\, \frac{b\Pi_{j-2}(a,b)}{j!}\, t^j, \quad u_2(t) = \sum_{j\ge 1} (-1)^{j-1}\, \frac{\Pi_{j-1}(a,b)}{j!}\, t^j$$

and

$$v_1(t) = \sum_{j\ge 0} \frac{\Pi_j(\bar{a},\bar{b})}{j!}\, t^j, \quad v_2(t) = -\sum_{j\ge 1} \frac{\Pi_{j-1}(\bar{a},\bar{b})}{j!}\, t^j,$$

where $\Pi_0(a,b) = 1$, $\Pi_1(a,b) = a$,

$$\Pi_j(a,b) = a\Pi_{j-1}(a,b) - b\Pi_{j-2}(a,b), \quad j = 2,3,\dots$$

(i.e., $\Pi_2(a,b) = a^2 - b$, $\Pi_3(a,b) = a^3 - 2ab$, $\Pi_4(a,b) = a^4 - 3a^2b + b^2$, *etc.).*

(b) *We also have*

$$\mathfrak{t}^{-k-2}(z+2) = \frac{\Pi_k(a,b)}{(z-k)(z-k+1)\dots z(z+1)}, \quad k = 0,1,2,\dots,$$

where $\mathfrak{t}^{-k-2}(z)$ *has the same meaning as before. In particular, the poles of* $\mathfrak{t}^{-k-2}(z)$ *are simple and, for* $l = -1,0,1,\dots,k$,

$$\operatorname{Res}_{z=l} \mathfrak{t}^{-k-2}(z+2) = \frac{(-1)^{k-l}\Pi_k(a,b)}{(k-l)!(l+1)!}.$$

The key in re-proving formulas like (4.2) is:

Lemma 7.2. (i) *For* $l \ge 2$, $0 \le j \le l-2$,

$$\Pi_l(a,b) = \Pi_{j+1}(a,b)\Pi_{l-j-1}(a,b) - b\Pi_j(a,b)\Pi_{l-j-2}(a,b).$$

(ii) *For* $j \ge 0$,

$$\Pi_j(-\bar{a},\bar{b}) = (-1)^j\, \overline{\Pi_j(a,b)}.$$

APPENDIX A. LOCAL ASYMPTOTIC TYPES

We briefly discuss the notion of local asymptotic type, i.e., asymptotic types at one singular exponent $p \in \mathbb{C}$ in (1.5). Moreover, we investigate an analogue of the boundary sesquilinear form in this simpler situation, see (A.3). Most of the material is taken from WITT [17].

Let E be a Banach space, E' be its topological dual, and let $\langle\ ,\ \rangle$ denote the dual pairing between E, E'. Pick $p \in \mathbb{C}$. For notations like $\mathcal{M}_p(E)$, $\mathcal{A}_p(E)$, E^∞, the right-shift operator T acting on E^∞, $\Phi \otimes \Psi[z-p]$ for $\Phi \in E^\infty$, $\Psi \in E'^\infty$, and the identification $\mathcal{M}_p(E)\big/\mathcal{A}_p(E) \cong E^\infty$, see Section 1.3.

Let $\mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$ be the space of germs of $\mathcal{L}(E)$–valued finitely meromorphic functions $F(z)$ at $z = p$, i.e.,

$$F(z) = \frac{F_0}{(z-p)^\nu} + \frac{F_1}{(z-p)^{\nu-1}} + \cdots + \frac{F_{\nu-1}}{z-p} + \sum_{j \geq 0} F_j (z-p)^j, \tag{A.1}$$

where $F_0, F_1, \ldots, F_{\nu-1} \in \mathcal{L}(E)$ are finite-rank operators. Let $\mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$ be the space of germs of $\mathcal{L}(E)$-valued normally meromorphic functions $F(z)$ at $z = p$, i.e., the space of finitely meromorphic functions $F(z)$, where, in addition, $F(z)$ for $z \neq p$ close to p is invertible and $F_\nu \in \mathcal{L}(E)$ is a Fredholm operator. $\mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$ is the group of invertible elements of the algebra $\mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$.

For $F \in \mathcal{A}_p(\mathcal{L}(E))$, let L_F denote the space of all $(\phi_0, \phi_1, \ldots, \phi_{m-1}) \in E^\infty$ such that

$$F(z) \left(\frac{\phi_0}{(z-p)^m} + \frac{\phi_1}{(z-p)^{m-1}} + \cdots + \frac{\phi_{m-1}}{(z-p)} \right) \in \mathcal{A}_p(E).$$

Remark A.1. The theory can be developed for $F \in \mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$ upon an appropriate modification of the definition of L_F. For $F \in \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$, L_F is again an asymptotic type, and Propositions A.4, A.5, and A.7 continue to hold in this case. See WITT [17].

Definition A.2. A local asymptotic type $J \subset E^\infty$ is a finite-dimensional linear subspace that is invariant under the action of the right shift operator T. The set of all local asymptotic types is denoted by $\mathcal{J}(E)$.

Note that T as acting on J is nilpotent. The characteristic $(m_1, \ldots, m_e)$ of T on J is called the characteristic of the asymptotic type.

Proposition A.3. *We have*

$$\mathcal{J}(E) = \left\{ L_F \;\middle|\; F \in \mathcal{A}_p(\mathcal{L}(E)) \cap \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E)) \right\}.$$

Proposition A.4. *For $F \in \mathcal{A}_p(\mathcal{L}(E)) \cap \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$, we have $F^t \in \mathcal{A}_p(\mathcal{L}(E)) \cap \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E'))$. Moreover, for each characteristic basis $\Phi_1, \ldots, \Phi_e$ of L_F, there exists a unique characteristic basis $\Psi_1, \ldots, \Psi_e$ of L_{F^t} such that*

$$\left[F^{-1}(z)\right]_p^* = \sum_{i=1}^{e} (\Phi_i \otimes \Psi_i)[z-p].$$

In particular, both asymptotic types L_F, L_{F^t} have the same characteristic.

The next result is Keldysh's formula, cf. KELDYSH [6], KOZLOV–MAZ'YA [8].

Proposition A.5. *For $\Phi_1, \ldots, \Phi_e$ and $\Psi_1, \ldots, \Psi_e$ as in* Proposition A.4,

$$\left\langle F(z)\Phi_i[z-p], \Psi_j[z-p] \right\rangle = \delta_{ij}(z-p)^{-m_i} + O(1) \quad \text{as } z \to p. \tag{A.2}$$

Remark A.6. Writing $F(z)$ as in (A.1), (A.2) can be rewritten as

$$\sum_{\nu+r+s=m_i+l} \langle F_\nu \phi_r^{(i)}, \psi_s^{(j)} \rangle = \delta_{ij}\delta_{0l}$$

for $0 \le l \le m_j - 1$, where summation is restricted to the range $0 \le r \le m_i - 1$, $0 \le s \le m_j - 1$ and $\Phi_i = \big(\phi_0^{(i)}, \phi_1^{(i)}, \dots, \phi_{m_i-1}^{(i)}\big)$, $\Psi_j = \big(\psi_0^{(j)}, \psi_1^{(j)}, \dots, \psi_{m_j-1}^{(j)}\big)$.

For $F \in \mathcal{A}_p(\mathcal{L}(E)) \cap \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, we then consider the bilinear form $[\ ,\ ]_F$ defined on the space $L_F \times L_{F^t}$ by

$$[\Phi, \Psi]_F := \operatorname{Res}_{z=p} \big\langle F(z)\Phi[z-p], \Psi[z-p]\big\rangle. \tag{A.3}$$

Proposition A.7. *Evaluated on the bases $T^r\Phi_i$ for $1 \le i \le e$, $0 \le r \le m_i - 1$ of L_F and $T^s\Psi_j$ for $1 \le j \le e$, $0 \le s \le m_j - 1$ of L_{F^t},*

$$\big[T^r\Phi_i, T^s\Psi_j\big]_F = \begin{cases} 1 & \text{if } i = j, r + s = m_i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows immediately from Proposition A.5. □

APPENDIX B. FUNCTION SPACES WITH ASYMPTOTICS

The maximal and minimal domains of cone-degenerate elliptic differential operators are cone Sobolev spaces with asymptotics, as we are going to demonstrate now. We refer to SCHULZE [14, 15] for more on function spaces with asymptotics, where, however, asymptotics are observed on so-called "half-open weight intervals," a setting leading to Fréchet spaces. The present setting due to LIU–WITT [11], where asymptotics are observed on "closed weight intervals," provides a scale of Hilbert spaces.

B.1. **Weighted cone Sobolev spaces.** Let $Mu(z) = \tilde{u}(z) = \int_0^\infty t^{z-1}u(t)\,dt$ for $z \in \mathbb{C}$ (or subsets thereof) be the Mellin transformation, suitably extended to certain distribution classes. Recall that

$$M\colon L^2(\mathbb{R}_+, t^{-2\delta}dt) \to L^2(\Gamma_{1/2-\delta}; (2\pi i)^{-1}dz), \tag{B.1}$$

is an isometry, where $\Gamma_\gamma := \{z \in \mathbb{C} \mid \operatorname{Re} z = \gamma\}$ for $\gamma \in \mathbb{R}$. Moreover,

$$M_{t\to z}\big\{(-t\partial_t)u\big\}(z) = z\,\tilde{u}(z),$$
$$M_{t\to z}\big\{t^{-p}u\big\}(z) = \tilde{u}(z-p), \quad p \in \mathbb{C}.$$

The function

$$\mathfrak{m}_{p,k}(z,y) := M_{t\to z}\left\{\frac{(-1)^k}{k!}\,\omega(t)t^{-p}\log^k t\,\phi(y)\right\},$$

where $p \in \mathbb{C}$, $k \in \mathbb{N}_0$, $\phi \in C^\infty(Y)$, and $\omega(t)$ is a cut-off function, belongs to $\mathcal{M}_{\mathrm{as}}^{-\infty}(Y)$. Furthermore,

$$\mathfrak{m}_{p,k}(z) - \frac{\phi(y)}{(z-p)^{k+1}} \in \mathcal{A}(\mathbb{C}; C^\infty(Y)).$$

For s, $\delta \in \mathbb{R}$, the space $\mathcal{H}^{s,\delta}(X)$ consists of all $u \in H^s_{\mathrm{loc}}(X^\circ)$ such that $M_{t\to z}\{\omega u\}(z) \in L^2_{\mathrm{loc}}\big(\Gamma_{\dim X/2-\delta}; H^s(Y)\big)$ and

$$\frac{1}{2\pi i}\int_{\Gamma_{\dim X/2-\delta}} \big\|R^s(z)M_{t\to z}\{\omega u\}(z)\big\|^2_{L^2(Y)}\,dz < \infty.$$

Here, $R^s(z) \in L^s(Y; \Gamma_{\dim X/2-\delta})$ is an order-reducing family, i.e., $R^s(z)$ is parameter-dependent elliptic and $R^s(z)\colon H^{s+s'}(Y) \to H^{s'}(Y)$ is invertible for all $s' \in \mathbb{R}$, $z \in \Gamma_{\dim X/2-\delta}$. For instance, if $\mathfrak{m}(z) \in \mathcal{M}^s_{\text{as}}(Y)$ is elliptic and the line $\Gamma_{\dim X/2-\delta}$ is free of poles of $\mathfrak{m}(z)$, then $\mathfrak{m}(z)\big|_{\Gamma_{\dim X/2-\delta}}$ is such an order-reduction.

B.2. **Cone Sobolev spaces with asymptotics.** The starting point is the following observation:

Theorem B.1 (LIU–WITT [11, Theorem 2.43])**.** *Let $s, \delta \in \mathbb{R}$ and $P \in \underline{\text{As}}^\delta(Y)$ be a proper asymptotic type. Then there exists an elliptic Mellin symbol $\mathfrak{m}^s_P(z) \in \mathcal{M}^s_{\mathcal{O}}(Y)$ such that the line $\Gamma_{\dim X/2-\delta}$ is free of poles of $\mathfrak{m}^s_P(z)^{-1}$ and, for $\mathfrak{S}^s = \{\mathfrak{m}^s_P(z), 0, 0, \dots\} \in \text{Symb}^s_M(Y)$, $L^\delta_{\mathfrak{S}^s}$ represents the asymptotic type P.*

Definition B.2. Let $s \geq 0$, $\delta \in \mathbb{R}$, and $P \in \underline{\text{As}}^\delta(Y)$ be proper. Then the space $\mathbb{H}^{s,\delta}_P(X)$ consists of all functions $u \in \mathcal{H}^{s,\delta}(X)$ such that $M_{t\to z}\{\omega u\}(z)$ is meromorphic for $\operatorname{Re} z > \dim X/2 - \delta - s$ with values in $H^s(Y)$,

$$\mathfrak{m}^s_P(z) M_{t\to z}\{\omega u\}(z) \in \mathcal{A}\big(\{z \in \mathbb{C} \mid \operatorname{Re} z > \dim X/2 - \delta - s\}; L^2(Y)\big),$$

where $\mathfrak{m}^s_P(z)$ is as in Theorem B.1, and

$$\sup_{0<s'<s} \frac{1}{2\pi i} \int_{\Gamma_{\dim X/2-\delta-s'}} \left\| \mathfrak{m}^s_P(z) M_{t\to z}\{\omega u\}(z) \right\|^2_{L^2(Y)} dz < \infty.$$

We list some properties of the spaces $\mathbb{H}^{s,\delta}_P(X)$:

Proposition B.3. (a) $\{\mathbb{H}^{s,\delta}_P(X);\ s \geq 0\}$ *is an interpolation scale of Hilbert spaces with respect to the complex interpolation method.*

(b) $\mathbb{H}^{s,\delta}_{\mathcal{O}}(X) = \mathcal{H}^{s,\delta+s}(X)$.

(c) *We have*

$$\mathbb{H}^{s,\delta}_P(X) = \mathbb{H}^{s,\delta}_{\mathcal{O}}(X) \oplus \Big\{ \omega(t) \sum_{\operatorname{Re} p > \dim X/2-\delta-s} \ \sum_{k+l=m_p-1} \frac{(-1)^k}{k!} t^{-p} \log^k t\, \phi^{(p)}_l(y) \ \Big| \ \Phi(p) = (\phi^{(p)}_0, \dots, \phi^{(p)}_{m_p-1}) \text{ for some } \Phi \in J \Big\},$$

where the linear space $J \subset \mathcal{E}^\delta_V(Y)$ represents the asymptotic type P, provided that

$$\operatorname{Re} p \neq \dim X/2 - \delta - s, \quad p \in V.$$

(d) $\mathbb{H}^{s,\delta}_P(X) \subseteq \mathbb{H}^{s',\delta'}_{P'}(X)$ *if and only if $s \geq s'$, $\delta + s \geq \delta' + s'$, and $P \preccurlyeq P'$ up to the conormal order $\delta' + s'$.*

(e) $C^\infty_P(X) := \bigcap_{s\geq 0} \mathbb{H}^{s,\delta}_P(X)$ *is dense in* $\mathbb{H}^{s,\delta}_P(X)$.

Proposition B.4. *The spaces $\mathbb{H}^{s,\delta}_P(X)$ are invariant under coordinate changes in the sense explained in* Remark 2.4.

B.3. **Mapping properties and elliptic regularity.** Here we are concerned with the regularity and asymptotics of solutions u to the equation

$$Au(x) = f(x) \quad \text{on } X^\circ, \tag{B.2}$$

where $A \in \text{Diff}^\mu_{\text{cone}}(X)$ is elliptic. Assuming $u \in \mathbb{H}^{0,\delta}(X)$ and $f \in \mathbb{H}^{s,\delta}_Q(X)$, where $s \geq 0$ and $Q \in \underline{\text{As}}^\delta(Y)$, we are going to show that $u \in \mathbb{H}^{s+\mu,\delta}_P(X)$ for some resulting $P \in \underline{\text{As}}^\delta(Y)$. By interior elliptic regularity, we already know that $u \in H^{s+\mu}_{\text{loc}}(X^\circ)$. So we are left with the behavior of $u = u(x)$ as $x \to \partial X$.

Let P^δ_A be the asymptotic type represented by L^δ_A. Similarly, let $P^{\delta+\mu-0}_A \preccurlyeq P^\delta_A$ be the asymptotic type represented by $L^{\delta+\mu-0}_A$. Then $P^{\delta+\mu-0}_A$ is the largest asymptotic type that coincides with the empty asymptotic type, $\mathcal{O}$, up to the conormal order $\delta+\mu-0$. Note that, for each $P \in \underline{\mathrm{As}}^{\delta}(Y)$ satisfying $P \preccurlyeq P_A^{\delta}$ up to the conormal order $\delta+\mu$, there is a $Q \in \underline{\mathrm{As}}^{\delta}(Y)$ such that

$$A\colon \mathbb{H}_P^{s+\mu,\delta}(X) \to \mathbb{H}_Q^{s,\delta}(X)$$

for all $s \geq 0$. The minimal such $Q \in \underline{\mathrm{As}}^{\delta}(Y)$ is denoted by $Q^{\delta}(P;A)$. In particular, $Q^{\delta}(P_A;A) = Q^{\delta}(\mathcal{O};A) = \mathcal{O}$.

The question raised for equation (B.2) is answered by the next result:

Proposition B.5. *Let* $A \in \mathrm{Diff}^{\mu}_{\mathrm{cone}}(X)$ *be elliptic. Then*:

(a) *The map*

$$\big\{P \in \underline{\mathrm{As}}^{\delta}(Y)\colon P \succcurlyeq P_A^{\delta},\ P \text{ coincides with } P_A^{\delta} \text{ up to the conormal order } \delta+\mu\big\} \to \underline{\mathrm{As}}^{\delta}(Y), \quad P \mapsto Q^{\delta}(P;A) \tag{B.3}$$

is an order-preserving bijection.

(b) *For any solution* u *to* (B.2), $u \in \mathbb{H}^{0,\delta}(X)$ *and* $f \in \mathbb{H}_Q^{s,\delta}(X)$ *implies* $u \in \mathbb{H}^{s+\mu,\delta}_{P^{\delta}(Q;A)}(X)$, *where* $Q \mapsto P^{\delta}(Q;A)$ *is the inverse to* (B.3).

Note that $Q^{\delta}(P^{\delta}(Q;A);A) = Q$. Therefore, $P \mapsto P^{\delta}(Q^{\delta}(P;A);A)$ is a hull operation. Note also that both maps $P \to Q^{\delta}(P;A)$ and $Q \mapsto P^{\delta}(Q;A)$ to (B.3) can be computed purely on the level of the complete conormal symbols $\{\sigma_c^{\mu-j}(A)(z);\ j \in \mathbb{N}_0\}$.

Theorem B.6. *Let* $A \in \mathrm{Diff}^{\mu}_{\mathrm{cone}}(X)$ *be elliptic. Then*

$$D(A_{\mathrm{max}}) = \mathbb{H}^{\mu,\delta}_{P_A^{\delta}}(X), \quad D(A_{\mathrm{min}}) = \mathbb{H}^{\mu,\delta}_{P_A^{\delta+\mu-0}}(X). \tag{B.4}$$

In particular,

$$D(A_{\mathrm{max}})/D(A_{\mathrm{min}}) \cong L_A^{\delta}/L_A^{\delta+\mu-0}. \tag{B.5}$$

Proof. (B.4) is a consequence of elliptic regularity, while (B.5) follows from the description given in Proposition B.3 (c) and interpolation. □

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