Crack Theory with Singularities at the Boundary

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Abstract

We investigate crack problems, where the crack boundary has conical singularities. Elliptic operators with two-sided elliptic boundary conditions on the plus and minus sides of the crack will be interpreted as elements of a corner algebra of boundary value problems. The corresponding operators will be completed by extra edge conditions on the crack boundary to Fredholm operators in corner Sobolev spaces with double weights, and there are parametrices within the calculus.

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Introduction

This paper is aimed at studying elliptic operators in a domain of the form $G \setminus S$, where G is an open bounded subset of $\mathbb{R}^3$ with smooth boundary and $S \subset G$ a closed subset such that $\mathrm{int}S$ is an oriented smooth hypersurface of codimension 1 with conical singularities at the boundary. An example is $G = \{x \in \mathbb{R}^3 : |x| < 1\}$ with $S = \{x \in \mathbb{R}^3 : x_3 = 0, |x_1| + |x_2| \le \frac{1}{2}\}$. If the elliptic operator A in consideration is the Lamé system we have a situation of (linear) crack theory with S being a crack in a medium G. Boundary value problems for the operator A are given as

$$Au = f \quad \text{in} \quad \Omega \setminus S, \quad T_{\pm}u = g_{\pm} \quad \text{on} \quad \text{int}\, S_{\pm}, \quad Tu = g \quad \text{on} \quad \partial G. \qquad (1)$$

Here $S_{\pm}$ indicate the plus and minus sides of S, and the operators $T_{\pm}$ are of the form $T_{\pm} = \mathrm{r}_{\pm}B_{\pm}$ with differential operators $B_{\pm}$ in a neighbourhood of S with smooth coefficients and $\mathrm{r}_{\pm}$ denoting the operators of restriction to $\text{int}S_{\pm}$ from the respective sides. We assume that the operators $T_{\pm}$ satisfy the Shapiro-Lopatinskij condition with respect to A uniformly up to ∂S. Moreover, T is a boundary condition which satisfies the Shapiro-Lopatinskij condition with respect to A on ∂G.

A special case of our investigation is the 'quarter plane problem' which means a boundary value problem for the Laplace operator (or any other elliptic operator) in $\mathbb{R}^3 \setminus S$ for $S := \{x \in \mathbb{R}^3 : x_3 = 0, x_1 \geq 0, x_2 \geq 0\}$ with two sided elliptic boundary conditions on $S_{\pm}$ (i.e., for $x_3 \searrow 0$ and $x_3 \nearrow 0$). The specific problem consists of an adequate description of the behaviour of solutions near the origin which is a corner singularity. Outside the

origin on the positive x_i-axis, $i = 1, 2$, we have the typical behaviour of 'smooth crack theory', cf. [6]. For $x_i \to \infty$ on these parts of the crack boundary we have an effect from the calculus on manifolds with edges and exits to infinity (however, the latter aspect is not studied in the present article).

We will construct a pseudo-differential calculus containing the operators (1) together with the parametrices of elliptic elements. Another interesting point is the nature of weighted Sobolev spaces which encode elliptic regularity.

For our methods the assumptions on the dimensions are not essential, but in dimension 3 some elements of the calculus become easier. The crack theory (in arbitrary dimensions) within a pseudo-differential calculus for the case of a smooth crack boundary is systematically treated in the author's joint monograph with Kapanadze [6], see also the article [15]. Let us also note that the scenario has much in common with mixed problems, cf. the authors joint papers [5], [2] with Dines and Harutjunjan for elliptic operators, and Krainer and Zhou Xiaofang [9] for the parabolic case.

Mixed elliptic and parabolic problems as well as crack problems have been studied by many authors from different aspects before, see the bibliography of [6]. Let us mention, in particular, the work of Vishik and Eskin [21], [3] with a calculus of boundary value problems for pseudo-differential operators without the transmission property at the boundary, Rempel and Schulze [11], or a more recent paper jointly with Seiler [19] on the edge algebra structure of boundary value problems.

In the present note we want to demonstrate how the general calculus of the author's joint articles with Oliaro [10], De Donno [1] and Krainer [8] can be applied to crack problems with conical singularities at the crack boundary.

In order to illustrate the scenario we want to consider an example, namely the Laplace operator $A = \Delta$ and Dirichlet conditions on the minus side, Neumann conditions on the plus side of S and Dirichlet conditions on ∂G. Then the associated column matrix operator $\mathcal{A} := {}^{\mathrm{t}}(\Delta \;\, T_- \;\, T_+ \;\, T)$ (i.e., $T_-u := u|_{\text{int}S_-}$, $T_+u := \frac{\partial}{\partial n}u|_{\text{int}S_+}$ with $\frac{\partial}{\partial n}$ being the differentiation in normal direction to S, and $Tu := u|_{\partial G}$) induces a continuous operator

$$\mathcal{A} : H^s(G) \to \begin{array}{c} H^{s-2}(G) \\ \oplus \\ H^{s-\frac{1}{2}}(\mathrm{int}S_-) \\ \oplus \\ H^{s-\frac{3}{2}}(\mathrm{int}S_+) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial G) \end{array} \tag{2}$$

for every $s > \frac{3}{2}$. Here $H^2(G)$ is the standard Sobolev space of smoothness s on G (more precisely, $H^s(G) := \{u|_G : u \in H^s(\mathbb{R}^3)\}$); similarly we have $H^s(\partial G)$ in the standard sense on the manifold ∂G, and $H^s(\mathrm{int}S_\pm)$ means the restriction of $H^s_{\mathrm{loc}}(\widetilde{S})$ to $\mathrm{int}S$, where $\widetilde{S}$ is any C^∞ manifold (here of dimension 2) which contains S as an embedded manifold with conical singularities and boundary (see Section 1.4 below); subscripts '$\pm$' mean the interpretations as $+$ or $-$ sides of int S.

It is clear that the problem (1) cannot be solvable in the space $H^s(G)$ unless the data $g_\pm$ satisfy a corresponding compatibility condition. Instead of the standard Sobolev spaces we therefore employ certain weighted Sobolev spaces. We interpret $G \setminus S$ as a 'crack configuration' M_{crack}, cf. Section 2.3 below, introduce corner Sobolev spaces

$$\mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}), \quad \mathcal{V}^{s,(\gamma,\delta)}(S_\pm)$$

with double weights $(\gamma, \delta) \in \mathbb{R}^2$, cf. Section 2.1 below, and realise $\mathcal{A}$ as a continuous operator

$$\mathcal{A} : \mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}) \to \begin{array}{c} \widetilde{\mathcal{V}}^{s-2,(\gamma-2,\delta-2)}(M_{\mathrm{crack}}) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial G) \end{array} \tag{3}$$

for any $s > \frac{3}{2}$. Here

$$\widetilde{\mathcal{V}}^{s-2,(\gamma-2,\delta-2)}(M_{\mathrm{crack}}) := \begin{array}{c} \mathcal{V}^{s-2,(\gamma-2,\delta-2)}(M_{\mathrm{crack}}) \\ \oplus \\ \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(S_-) \\ \oplus \\ \mathcal{V}^{s-\frac{3}{2},(\gamma-\frac{3}{2},\delta-\frac{3}{2})}(S_+) \end{array} .$$

We then obtain that for all $\gamma \notin D$ for some discrete set D of real numbers and all $\delta \notin D_\gamma$ for another discrete set D_γ of reals (depending on γ) the operator $\mathcal{A}$ can be completed by additional trace and potential conditions along ∂S to a Fredholm operator

$$\begin{pmatrix} \mathcal{A} & \mathcal{K} \\ \mathcal{T} & \mathcal{Q} \end{pmatrix} : \begin{array}{c} \mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}) \\ \oplus \\ \mathcal{H}^{s-1,\delta-1}(\partial S, M) \end{array} \to \begin{array}{c} \widetilde{\mathcal{V}}^{s-2,(\gamma-2,\delta-2)}(M_{\mathrm{crack}}) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial G) \\ \oplus \\ \mathcal{H}^{s-3,\delta-3}(\partial S, M') \end{array} \tag{4}$$

for $s > \frac{3}{2}$, cf. Theorem 3.1 and Remark 3.2 below. The crack boundary ∂S is interpreted as a one-dimensional manifold with conical singularities, and $\mathcal{H}^{s,\delta}(\partial S, M)$ are the corresponding weighted Sobolev spaces on ∂S of distributional sections in the vector bundle M, cf. Section 1.3 below.

Our approach is completely general and can be applied to other conditions on int $S_\pm$ as well, e.g., Dirichlet or Neumann conditions on both sides, cf. Section 3.1 below. Also the elliptic operator A itself is arbitrary and may also be a system, e.g., Lamé's system, and also the dimension of G is unessential.

The ideas come from the theory of elliptic operators on manifolds with geometric singularities, here with corners and boundary. Comparing the results with [17] or [18] which consist of a corner calculus without boundary, here we treat the case of boundary value problems.

The technicalities are voluminous and cannot be given here in a selfcontained way. However, after the material of [16], [17], [6] and [2] it should be easy to complete the details. In the present exposition we mainly formulate the structure of operators $\mathcal{A}$ in suitable scales of weighted Sobolev spaces and a corresponding principal symbolic hierarchy

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A}), \sigma_\mathrm{c}(\mathcal{A})) \tag{5}$$

which determines ellipticity and parametrices. The first components $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$ are the principal interior and boundary symbols as they are known from the standard calculus of boundary value problems with the transmission property. The principal edge symbol $\sigma_\wedge(\mathcal{A})$ comes from the theory of pseudo-differential boundary value problems on a manifold with edges, cf. [6]. The conormal symbol $\sigma_\mathrm{c}(\mathcal{A})$ is the typical novelty compared with the crack theory for a smooth crack boundary.

Let us finally note that there is a similarity between our crack problems and mixed boundary value problems. The case with conical (or other) singularities at the interface seems not to be treated yet in the literature. An interesting problem is to characterise for relevant examples the number of extra interface (or crack) conditions (cf. also [2] for the case of smooth interfaces and Section 3.1 below) and to explicitly calculate the corner weights for which the operators are Fredholm (which is known in general up to a certain discrete set of weights).

1 Elements of the edge calculus

1.1 Edge Sobolev spaces and symbols

By assumption the crack boundary ∂S is smooth outside a finite set of conical singularities. For simplicity we assume that there is only one conical point $v \in \partial S$; the general case can be treated by similar arguments, using localisations and partitions of unity. The set $(\partial S)_\mathrm{reg} := \partial S \setminus \{v\}$ can be regarded as an edge and $G \setminus (\partial S)_\mathrm{reg}$ locally near any $y \in (\partial S)_\mathrm{reg}$ as the interior of a wedge $K \times \Omega$ for an open set $\Omega \subseteq \mathbb{R}^q$, $q = \dim(\partial S)_\mathrm{reg}$ $(= \dim G - 2)$, and $K = (\overline{\mathbb{R}}_+ \times I)/(\{0\} \times I)$, $I := [0, 2\pi]$, interpreted as the model cone of the wedge. The interval I corresponds to the unit circle in the normal plane $\mathbb{R}^2$ to $(\partial S)_\mathrm{reg}$ with distinguished end points which correspond to the plus and minus side of the crack. In other words, K is represented by $(\mathbb{R}^2 \setminus \mathbb{R}_+) \cup (\mathbb{R}_+^{(-)} \cup \mathbb{R}_+^{(+)})$ where $\mathbb{R}_+$ locally corresponds to the intersection of int$S$ with the 2-dimensional normal plane to ∂S at some $y \in (\partial S)_{\text{reg}}$, and $\text{int} S_\pm$ by two copies $\mathbb{R}_+^{(\pm)}$ of the half-axis that replace $\mathbb{R}_+$ by the $\pm$ sides of the slit. As is common, instead of $K \backslash \{v\}$ we often look at the stretched cone $I^\wedge = \mathbb{R}_+ \times I \ni (r, \phi)$. Then the domain (including plus and minus sides of the crack) near a point $y \in (\partial S)_{\text{reg}}$ is locally identified with a streched wedge $\overline{\mathbb{R}}_+ \times I \times \Omega$. This is just the starting point in [6]. We want to describe here some elements of the crack calculus for a smooth crack boundary for being able to organise the next step in the hierarchy of singularities, namely when the edge has conical singularities.

Note that pseudo-differential operators on manifolds with edges that have conical singularities are treated in [17] for the boundaryless case.

Let E be a Hilbert space equipped with a group $\kappa_\lambda : E \to E$, $\lambda \in \mathbb{R}_+$, of isomorphisms such that $\kappa_\lambda \kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \mathbb{R}_+$, strongly continuous in $\lambda \in \mathbb{R}_+$ (we then simply say that $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is a group action on E). An example is $E = H^s(\mathbb{R}_+^m) := H^s(\mathbb{R}^m)|_{\mathbb{R}_+^m}$ for $\mathbb{R}_+^m = \{x = (x_1, \ldots, x_m) : x_m > 0\}$, with $(\kappa_\lambda u)(x) = \lambda^{\frac{m}{2}} u(\lambda x)$, $\lambda \in \mathbb{R}_+$.

Definition 1.1. *The 'abstract' edge Sobolev space $\mathcal{W}^s(\mathbb{R}^q, E)$ of smoothness $s \in \mathbb{R}$ is defined as the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm $\| \langle\eta\rangle^s \| \kappa_{\langle\eta\rangle}^{-1} \hat{u}(\eta) \|_E \|_{L^2(\mathbb{R}^q_\eta)}$ ($\widehat{u}(\eta) = (F_{y \to \eta} u)(\eta)$ is the Fourier transform in $\mathbb{R}^q$).*

Remark 1.2. *For $E = H^s(\mathbb{R}_+^m)$ we have*

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}_+^m)) = H^s(\mathbb{R}^q \times \mathbb{R}_+^m)$$

for every $s \in \mathbb{R}$, in particular,

$$\mathcal{W}^s(\mathbb{R}^{m-1}, H^s(\mathbb{R}_+)) = H^s(\mathbb{R}_+^m).$$

Remark 1.3. *Let $E = \varprojlim_{j \in \mathbb{N}} E^j$ be a Fréchet space written as a projective limit of Hilbert spaces E^j with continuous embeddings $E^{j+1} \hookrightarrow E^j$, and let $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ be a group action on E^0 which restricts to group actions on E^j for every j (we then say that the Hilbert space E is endowed with a group action). In that case we have continuous embeddings $\mathcal{W}^s(\mathbb{R}^q, E^{j+1}) \hookrightarrow \mathcal{W}^s(\mathbb{R}^q, E^j)$, and we write*

$$\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{j \in \mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j).$$

Similarly to standard Sobolev spaces we also have 'comp' and 'loc' versions

$$\mathcal{W}^s_{\text{comp}}(\Omega, E) \quad \text{and} \quad \mathcal{W}^s_{\text{loc}}(\Omega, E)$$

for any open set $\Omega \subseteq \mathbb{R}^q$. More details on the nature of abstract edge Sobolev spaces may be found in [14] or [16].

Definition 1.4. (i) *Let E and $\widetilde{E}$ be Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ and $\{\widetilde{\kappa}_\lambda\}_{\lambda \in \mathbb{R}_+}$, respectively. Then the space of (operator-valued) symbols $S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ for an open set $U \subseteq \mathbb{R}^p$, $\mu \in \mathbb{R}$, is defined as the set of all $a(y, \eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ such that*

$$\sup \Big\{ \langle\eta\rangle^{-\mu+|\beta|} \| \widetilde{\kappa}_{\langle\eta\rangle}^{-1} \{ D_y^\alpha D_\eta^\beta a(y, \eta) \} \kappa_{\langle\eta\rangle} \|_{\mathcal{L}(E, \widetilde{E})} : (y, \eta) \in K \times \mathbb{R}^q \Big\} \quad (6)$$

is finite for every $K \subset\subset U$ and all multi-indices $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$.

(ii) $S^{(\mu)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \widetilde{E})$ *denotes the set of all* $f_{(\mu)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E, \widetilde{E}))$ *such that*

$$f_{(\mu)}(y, \lambda\eta) = \lambda^\mu \widetilde{\kappa}_\lambda f_{(\mu)}(y,\eta)\kappa_\lambda^{-1}$$

for all $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$, $\lambda \in \mathbb{R}_+$.

(iii) *The space* $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \widetilde{E})$ *of classical symbol is defined as the set of all* $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ *such that there are elements* $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \widetilde{E})$, $j \in \mathbb{N}$, *such that*

$$a(y,\eta) - \sum_{j=0}^{N} \chi(\eta) a_{(\mu-j)}(y,\eta) \in S^{\mu-N}(U \times \mathbb{R}^q; E, \widetilde{E})$$

for every $N \in \mathbb{N}$. *Here* $\chi(\eta)$ *is any excision function, i.e.,* $\chi \in C^\infty(\mathbb{R}^q)$, $\chi(\eta) = 0$ *for* $|\eta| < c_0$, $\chi(\eta) = 1$ *for* $|\eta| > c_1$ *for certain* $0 < c_0 < c_1$.

Remark 1.5. *Definition* 1.4 *reproduces standard scalar symbols when we insert* $E = \widetilde{E} = \mathbb{C}$ *with the trivial group actions.*

A symbol $a(x,\xi,\lambda) \in S^\mu_{\mathrm{cl}}(\Omega_{x'} \times \mathbb{R}_{x_n} \times \mathbb{R}^n_\xi \times \mathbb{R}^l_\lambda)$ for $\mu \in \mathbb{Z}$, $\Omega \subseteq \mathbb{R}^{n-1}$ open, $x = (x', x_n)$, $\xi = (\xi', \xi_n)$, is said to have the transmission property at $x_n = 0$, if

$$D^k_{x_n} D^\alpha_{\xi',\lambda}\big\{a_{(\mu-j)}(x', x_n, \xi', \xi_n, \lambda) - (-1)^{\mu-j} a_{(\mu-j)}(x', x_n', -\xi', -\xi_n, -\lambda)\big\} \quad (7)$$

vanishes on the set $\big\{(x', x_n, \xi', \xi_n, \lambda) : x' \in \Omega, x_n = 0, (\xi', \lambda) = 0, \xi_n \in \mathbb{R} \setminus \{0\}\big\}$ for all $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^{n-1+l}$ and all $j \in \mathbb{N}$.

Let $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^{n+l})_{\mathrm{tr}}$ denote the space of all symbols with the transmission property in that sense, and let $S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+l})_{\mathrm{tr}}$ be the space of restrictions of such symbols to $\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+l}$. With every $a(x,\xi,\lambda) \in S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+l})_{\mathrm{tr}}$ we associate a family of pseudo-differential operators on $\mathbb{R}_+$ by setting

$$\mathrm{op}^+(a)(x', \xi', \lambda) u(x_n) := \mathrm{r}^+ \mathrm{op}(a)(x', \xi', \lambda) \mathrm{e}^+ u(x_n), \quad (8)$$

where $\mathrm{op}(a)(x',\xi',\lambda)u(x_n) := \iint e^{i(x_n - \widetilde{x}_n)\xi_n} a(x', x_n, \xi', \lambda)\, u(\widetilde{x}_n) d\widetilde{x}_n đ\xi_n$, e^+ denotes the operator of extension of u from $\mathbb{R}_+ \ni x_n$ by zero to $\overline{\mathbb{R}}_-$ and $\mathrm{r}^+ u := u|_{\mathbb{R}_+}$, $đ\xi_n = (2\pi)^{-1} d\xi_n$. ($\mathrm{op}(a)$ basically refers to any extension of a to $\mathbb{R}$ with respect to x_n, but (8) is independent of this extension). In (7) we assume $u \in H^s(\mathbb{R}_+)$ for $s > -\frac{1}{2}$.

Remark 1.6. *For every* $a(x,\xi,\lambda) \in S^\mu_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n+l})_{\mathrm{tr}}$ *which is independent of* x_n *for large* x_n *we have*

$$\mathrm{op}^+(a)(x', \xi', \lambda) \in S^\mu(\Omega \times \mathbb{R}^{n-1+l}_{\xi',\lambda}; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+))$$

for every $s > -\frac{1}{2}$. *If* a *is independent of* x_n, *then* $\mathrm{op}^+(a)(x',\xi',\lambda)$ *is a classical operator-valued symbol in the sense of Definition* 1.4 (iii).

Definition 1.4 has a generalisation to pairs of Fréchet spaces E and $\widetilde{E}$ with group actions, cf. Remark 1.3. For instance, if E is a Hilbert space, $\widetilde{E}$ a Fréchet space, $\widetilde{E} = \varprojlim_j \widetilde{E}^j$, $\widetilde{E}^j$ Hilbert spaces, etc., then we have the symbol spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}^j)$ for all j and we then set

$$S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}) := \bigcap_{j \in \mathbb{N}} S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}^j).$$

Here '(cl)' means that the considerations are valid for classical and general symbols. The case when also E is Fréchet may be found, e.g., in [16].

Let $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \widetilde{E})$ be the subset of all symbols with constant coefficients, i.e., which are independent of y. Then we have

$$S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}) = C^\infty(U, S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \widetilde{E});$$

here we use the symbol spaces in their natural locally convex topologies which immediately follow from the definition.

Example 1.7. *Let us write the space $\mathcal{S}(\overline{\mathbb{R}}_+) := \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$ as a projective limit of Hilbert spaces*

$$\mathcal{S}(\overline{\mathbb{R}}_+) = \varprojlim_{k \in \mathbb{N}} \widetilde{E}^k$$

for $\widetilde{E}^k := \langle x_n \rangle^{-k} H^k(\mathbb{R}_+)$ with the group action $(\kappa_\lambda u)(x_n) = \lambda^{1/2} u(\lambda x_n)$, $\lambda > 0$. Then we have the space of symbols

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1} \times \mathbb{R}^l; L^2(\mathbb{R}_+) \oplus \mathbb{C}^m, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{m'}) \tag{9}$$

with the group actions $\mathrm{diag}(\kappa_\lambda, \mathrm{id})$ *on the respective direct sums.*

An element $g(x', \xi', \lambda) \in C^\infty(\Omega \times \mathbb{R}^{n-1+l}, \mathcal{L}(L^2(\mathbb{R}_+) \oplus \mathbb{C}^m, L^2(\mathbb{R}_+) \oplus \mathbb{C}^{m'})$ is said to be a Green symbol of order μ and type 0 *(of the calculus of boundary value problems with the transmission property at $x_n = 0$) if $g_0(x', \xi', \lambda) :=$* $\mathrm{diag}(1, \langle \xi', \lambda \rangle^{\frac{1}{2}}) g(x', \xi', \lambda) \mathrm{diag}(1, \langle \xi', \lambda \rangle^{-\frac{1}{2}})$ *and $g_0^*(x', \xi', \lambda)$ belong to be space* (9)*, where '$*$' indicates pointwise adjoints in the sense*

$$(g(x', \xi', \lambda)u, v)_{L^2(\mathbb{R}_+) \oplus \mathbb{C}^{m'}} = (u, g^*(x', \xi', \lambda)v)_{L^2(\mathbb{R}_+) \oplus \mathbb{C}^m}$$

for all $u \in L^2(\mathbb{R}_+) \oplus \mathbb{C}^m$, $v \in L^2(\mathbb{R}_+) \oplus \mathbb{C}^{m'}$. An operator family $g(x', \xi', \lambda)$ is called a Green symbol of type $\mathrm{d} \in \mathbb{N}$*, if it has the form*

$$g(x', \xi', \lambda) = g_0(x', \xi', \lambda) + \sum_{j=1}^{\mathrm{d}} g_j(x', \xi', \lambda) \mathrm{diag}\left(\frac{\partial^j}{\partial x_n^j}, 0 \right) \tag{10}$$

for Green symbols $g_j(x', \xi', \lambda)$ of order $\mu - j$ and type $0, i = 0, \ldots, \mathrm{d}$. *In this case we have*

$$g(x', \xi', \lambda) \in S^\mu_{\mathrm{cl}} \left(\Omega \times \mathbb{R}^{n-1+l}; H^s(\mathbb{R}_+) \oplus \mathbb{C}^m, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{m'} \right)$$

for every real $s > \mathrm{d} - \frac{1}{2}$.

Definition 1.8. *We set*

$$L^\mu_{(\mathrm{cl})}(\Omega; E, \widetilde{E}) := \left\{ \mathrm{Op}(a) : a(y, y', \eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; E, \widetilde{E}) \right\}$$

where $\mathrm{Op}(a)u(y) = \iint e^{i(y-y')\eta} a(y, y', \eta) u(y') dy' đ\eta$, $đ\eta := (2\pi)^{-q} d\eta$, $\Omega \subseteq \mathbb{R}^q$ *any open set.*

Operators $A \in L^{\mu}_{(\mathrm{cl})}(\Omega; E, \widetilde{E})$ are continuous in the sense

$$A : C_0^\infty(\Omega, E) \to C^\infty(\Omega, \widetilde{E})$$

and extend to continuous operators

$$A : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \widetilde{E}) \tag{11}$$

for all $s \in \mathbb{R}$.

1.2 Boundary value problems

The present section gives an outline of standard pseudo-differential boundary value problems with the transmission property at the boundary which depend on a parameter $\lambda \in \mathbb{R}^l$. First let X be a compact C^∞ manifold with boundary ∂X, and let $2X$ denote the double of X obtained by gluing together two copies X_+ and X_- of X along the common boundary such that $2X$ is a smooth and closed manifold. The given manifold X will be regarded as the plus side of $2X$.

If M is any closed compact C^∞ manifold we denote by $L^{\mu}_{\mathrm{cl}}(M; \mathbb{R}^l)$ the space of all classical parameter-dependent pseudo-differential operators on M, i.e., with local amplitude functions $a(x, \xi, \lambda)$ which are classical symbols in (ξ, λ) of order μ, and $L^{-\infty}(M; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(M))$, with $L^{-\infty}(M)$ being the space of all smoothing operators on M. More generally, for $E, F \in \mathrm{Vect}(M)$ (with $\mathrm{Vect}(.)$ denoting the set of all smooth complex vector bundles on the manifold in brackets) we have the space $L^{\mu}_{\mathrm{cl}}(M; E, F; \mathbb{R}^l)$ of all classical parameter-dependent pseudo-differential operators on M acting between Sobolev spaces $H^s(M, E)$ and $H^{s-\mu}(M, F)$ of distributional sections in the bundles.

On all smooth manifolds in consideration we fix Riemannian metrics; in the case of a C^∞ manifold X with boundary we choose a collar neighbourhood $\partial X \times [0, 1) \ni (x', x_n)$ and assume the Riemannian metric to be the product metric of a metric on ∂X and the standard metric on [0,1). The complex vector bundles in consideration are assumed to be equipped with Hermitian metrics.

Let $L^{\mu}_{\mathrm{cl}}(2X; \mathbb{R}^l)_{\mathrm{tr}}$ denote the subspace of all $A \in L^{\mu}_{\mathrm{cl}}(2X; \mathbb{R}^l)$ the local amplitude functions of which have the transmission property at the boundary (this concerns the charts intersecting ∂X). More generally, we have the spaces $L^{\mu}_{\mathrm{cl}}(2X; E, F; \mathbb{R}^l)_{\mathrm{tr}}$ for $E, F \in \mathrm{Vect}(2X)$.

An operator family $C_{11}(\lambda) : C^\infty(X) \to C^\infty(X)$ is called parameter-dependent smoothing and of type $\mathrm{d} \in \mathbb{N}$ if it has the form

$$C_{11}(\lambda) = \sum_{j=0}^{\mathrm{d}} G_j(\lambda) T^j \tag{12}$$

with $G_j(\lambda)$ having kernels in $\mathcal{S}(\mathbb{R}^l, C^\infty(X \times X))$ and a first order differential operator T on X which is locally near ∂X of the form $\partial/\partial x_n$. An operator $C_{21}(\lambda) : C^\infty(X) \to C^\infty(\partial X)$ is called parameter-dependent smoothing of type d if it has the form

$$C_{21}(\lambda) = \sum_{j=0}^{\mathrm{d}} B_j(\lambda) T^j$$

with $B_j(\lambda)$ having kernels in $\mathcal{S}(\mathbb{R}^l, C^\infty(\partial X \times X))$. Finally $C_{12}(\lambda) : C^\infty(\partial X) \to C^\infty(X)$ and $C_{22}(\lambda) : C^\infty(\partial X) \to C^\infty(X)$ are called parameter-dependent smoothing if the kernels belong to $\mathcal{S}(\mathbb{R}^l, C^\infty(X \times \partial X))$ and $\mathcal{S}(\mathbb{R}^l, C^\infty(\partial X \times \partial X))$, respectively.

All these notions have straightforward generalisations to operators between spaces of sections in smooth complex vector bundles on the corresponding manifold. In this sense by $\mathcal{B}^{-\infty,\mathrm{d}}(X; \boldsymbol{v}; \mathbb{R}^l)$ we denote the space of all smoothing operator families $\mathcal{C}(\lambda) = (C_{ij}(\lambda))_{i,j=1,2}$

$$\begin{array}{ccc} C^\infty(X, E) & & C^\infty(X, F) \\ \oplus & \to & \oplus \\ C^\infty(\partial X, H) & & C^\infty(\partial X, J) \end{array} \tag{13}$$

of type d; here $\boldsymbol{v} := (E, F; H, J)$ for $E, F \in \mathrm{Vect}(X)$ and $H, J \in \mathrm{Vect}(\partial X)$.

Let $V_1, \ldots, V_N$ be coordinate neighbourhoods on ∂X, let $\{\varphi_1, \ldots, \varphi_N\}$ be a subordinate partition of unity and $\{\psi_1, \ldots, \psi_N\}$ a system of functions $\psi_j \in C_0^\infty(V_j)$ which are equal to 1 on supp φ_j for all j, and let $\chi_j : V_j \to \Omega$ be charts, $\Omega \subseteq \mathbb{R}^{n-1}$ open, $n = \dim X$. Moreover, let $\omega \in C^\infty(X)$ be a function which is equal to 1 in a neighbourhood of ∂X and 0 outside $\partial X \times [0, \frac{1}{2})$. With symbols $g_k(x', \xi', \lambda)$ as in Example 1.7 we can associate operator families

$$\mathcal{G}_k(\lambda) = \mathrm{diag}(\omega\varphi_k, \varphi_k)(\chi_k^{-1})_* \mathrm{Op}_{x'}(g_k)(\lambda) \mathrm{diag}(\omega\psi_k, \psi_k), \tag{14}$$

where $\mathrm{Op}_{x'}(g_k)(\lambda)u(x') = \iint e^{i(x-x')\xi'} g_k(x', \xi', \lambda) u(x') dx' đ\xi'$ is interpreted as an action

$$\mathrm{Op}_{x'}(g_k)(\lambda) : \begin{array}{ccc} C_0^\infty(\Omega \times \overline{\mathbb{R}}_+) & & C^\infty(\Omega \times \overline{\mathbb{R}}_+) \\ \oplus & \to & \oplus \\ C_0^\infty(\Omega, \mathbb{C}^m) & & C^\infty(\Omega, \mathbb{C}^{m'}) \end{array}.$$

Such operators can easily be generalised to block matrices also in the upper left corners (i.e., when $L^2(\mathbb{R}_+)$ and $\mathcal{S}(\overline{\mathbb{R}}_+)$ in the formula (9) is replaced by $L^2(\mathbb{R}_+, \mathbb{C}^e)$ and $\mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^{e'})$ for some dimensions e and e', respectively). We then have invariance with respect to substituting transition maps of vector bundles. Then, if $\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{C}^e$, $\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{C}^{e'}$, $\Omega \times \mathbb{C}^m$ and $\Omega \times \mathbb{C}^{m'}$ are regarded as trivialisations of bundles E, F, H and J, respectively, we interpret the operator push forwards $(\chi_k^{-1})_*$ in the sense of maps between sections of bundles.

Let $\mathcal{B}_G^{\mu,\mathrm{d}}(X; \boldsymbol{v}; \mathbb{R}^l)$ for $\mu \in \mathbb{R}$, $\mathrm{d} \in \mathbb{N}$, be the space of all families $\mathcal{G}(\lambda)$ of operators (13) of the form

$$\mathcal{G}(\lambda) = \sum_{k=1}^{N} \mathcal{G}_k(\lambda) + \mathcal{C}(\lambda)$$

for arbitrary $\mathcal{G}_k(\lambda)$ of the form (14), and $\mathcal{C}(\lambda) \in \mathcal{B}^{-\infty,\mathrm{d}}(X; \boldsymbol{v}; \mathbb{R}^l)$.

Definition 1.9. *Let $\mathcal{B}^{\mu,\mathrm{d}}(X; \boldsymbol{v}; \mathbb{R}^l)$ for any $\mu \in \mathbb{Z}$, $\mathrm{d} \in \mathbb{N}$, $\boldsymbol{v} = (E, F; H, J)$, defined to be the space of all families $\mathcal{A}(\lambda) = (A_{ij}(\lambda))_{i,j=1,2}$ of operators* (13), *$\lambda \in \mathbb{R}^l$, which have the form*

$$\mathcal{A}(\lambda) = \mathrm{diag}(\mathrm{r}^+ A(\lambda) \mathrm{e}^+, 0) + \mathcal{G}(\lambda) \tag{15}$$

with $A(\lambda) \in L^\mu_{\mathrm{cl}}(2X; \widetilde{E}, \widetilde{F}; \mathbb{R}^l)_{\mathrm{tr}}$ for elements $\widetilde{E}, \widetilde{F} \in \mathrm{Vect}(2X)$ such that $E = \widetilde{E}|_X$, $F = \widetilde{F}|_X$, and e^+ denoting the operator of extension by zero to $2X \setminus X$, r^+ the operator of restriction to $\mathrm{int}X$, and $\mathcal{G}(\lambda) \in \mathcal{B}_G^{\mu,\mathrm{d}}(X; \boldsymbol{v}; \mathbb{R}^l)$. For $l = 0$ we simply write $\mathcal{B}^{\mu,\mathrm{d}}(X; \boldsymbol{v})$.

Note that $\mathcal{A}(\lambda) \in \mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{R}^l)$ implies $\mathcal{A}(\lambda_0) \in \mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v})$ for every fixed $\lambda_0 \in \mathbb{R}^l$.

A standard property of operators in $\mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v})$ in the case of compact X is the following result:

Theorem 1.10. *Every $\mathcal{A} \in \mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v})$ extends to continuous operators*

$$\mathcal{A} : \begin{array}{c} H^s(X,E) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial X,H) \end{array} \to \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ H^{s-\frac{1}{2}-\mu}(\partial X,J) \end{array}$$

for every real $s > \mathrm{d} - \frac{1}{2}$.

The operator families $\mathcal{A} \in \mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{R}^l)$ have a principal symbolic structure consisting of two components

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})) \tag{16}$$

with the (parameter-dependent) homogeneous principal interior symbol of $\mathcal{A}$

$$\sigma_\psi(\mathcal{A}) : \pi_X^* E \to \pi_X^* F, \tag{17}$$

$\pi_X : T^*X \times \mathbb{R}^l \setminus 0 \to X$, which is the principal symbol of $A(\lambda)|_X$, and the (parameter-dependent) homogeneous principal boundary symbol of $\mathcal{A}$

$$\sigma_\partial(\mathcal{A}) : \pi_{\partial X}^* \begin{array}{c} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ H \end{array} \to \pi_{\partial X}^* \begin{array}{c} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J \end{array}, \tag{18}$$

$\pi_{\partial X} : T^*(\partial X) \times \mathbb{R}^l \setminus 0 \to \partial X$; here $E' := E|_{\partial X}$, $F' := F|_{\partial X}$. Concerning more explanations and notation in this context, cf. [16, Chapter 4]. An operator family $\mathcal{A}(\lambda) \in \mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{R}^l)$ is called (parameter-dependent) elliptic if both (17) and (18) are isomorphisms.

Remark 1.11. (i) *Definition* 1.9 *has an immediate generalisation to the case of a non-compact manifold X with C^∞ boundary; as before, the corresponding operator classes are denoted by $\mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{R}^l)$. For instance, if G and S are as in the introduction we can form $X := (\overline{G}\setminus S) \cup (\mathrm{ind} S_- \cup \mathrm{int} S_+)$, i.e., to $\overline{G} \setminus S$ we add two copies of $\mathrm{int} S$ corresponding to the plus and minus sides of S. In this case we have $\partial X = \partial \overline{G} \cup \mathrm{int} S_- \cup \mathrm{int} S_+$. Since ∂X has several connected components it may be necessary to indicate the vector bundles on the different components separately.*

(ii) *For operators in $\mathcal{B}_G^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{R}^l)$ we may admit arbitrary $\mu \in \mathbb{R}$.*

The principal symbolic structure and ellipticity also make sense for a non-compact manifold X with C^∞ boundary, or if ∂X has several connected components.

Another situation when ∂X has several connected components is the case $X := I$ for an interval $I = [\alpha, \beta]$ on the real line. The operator families of the space $\mathcal{B}^{\mu,\mathrm{d}}(I;\boldsymbol{v};\mathbb{R}^l)$ then have the form

$$\mathcal{A}(\lambda) : \begin{array}{c} H^s(I,\mathbb{C}^e) \\ \oplus \\ \mathbb{C}^{n_-} \oplus \mathbb{C}^{n_+} \end{array} \to \begin{array}{c} H^{s-\mu}(I,\mathbb{C}^{e'}) \\ \oplus \\ \mathbb{C}^{n'_-} \oplus \mathbb{C}^{n'_+} \end{array},$$

continuous for $s > \mathrm{d} - \frac{1}{2}$; in this case $\boldsymbol{v}$ consists of the tuple of dimension data $(e, e'; n_-, n_+, n'_-, n'_+)$.

Since the latter case is basic for this exposition we want to formulate the classes of operator families for the case $e' = e = n_- = n_+ = n'_- = n'_+ = 1$ independently. The generalisation to arbitrary dimensions is then straightforward. Also for the case of different orders in the entries we can easily define corresponding operators if we first have formulated the operators for the same order μ in all entries. We will define the spaces

$$B^{\mu,\mathrm{d}}(I; \mathbb{R}^l)$$

for $\mu \in \mathbb{Z}, \mathrm{d} \in \mathbb{N}$ and

$$\mathcal{B}_G^{\mu,\mathrm{d}}(I; \mathbb{R}^l)$$

for arbitrary $\mu \in \mathbb{R}$.

Let $\mathcal{B}_G^{-\infty,0}(I)$ defined to be the space of all 3×3 block matrix operators

$$g = (g_{ij})_{i,j=1,2,3} : \begin{array}{c} H^s(I) \\ \oplus \\ \mathbb{C}^2 \end{array} \to \begin{array}{c} C^\infty(I) \\ \oplus \\ \mathbb{C}^2 \end{array},$$

$s > -\frac{1}{2}$, where g_{11} is an integral operator with kernel in $C^\infty(I \times I)$, $g_{1j}c := f_{1j}(\phi)c$ for $j = 2, 3$, $c \in \mathbb{C}$, $g_{i1}u = \int_I f_{i1}(\phi)u(\phi)d\phi$ for $i = 2, 3$, with arbitrary functions $f_{1j}, f_{i1} \in C^\infty(I)$ for $i, j = 2, 3$, and an arbitrary 2×2 matrix $(g_{ij})_{i,j=2,3}$ with entries in $\mathbb{C}$. The components of $c = (c_\alpha, c_\beta) \in \mathbb{C}^2$ are related to the end points $\{\alpha\}$ and $\{\beta\}$ of the interval I. The space $\mathcal{B}_G^{-\infty,0}(I)$ is Fréchet in a natural way (as a direct sum of its 9 components), and we set $\mathcal{B}_G^{-\infty,0}(I; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, \mathcal{B}_G^{-\infty,0}(I))$. Moreover let $\mathcal{B}_G^{-\infty,\mathrm{d}}(I; \mathbb{R}^l)$ for arbitrary $\mathrm{d} \in \mathbb{N}$ be the space of all operator families $g(\lambda) := g_0(\lambda) + \sum_{j=1}^{\mathrm{d}} G_j(\lambda)\mathrm{diag}(\partial_\phi^j, 0, 0)$ for arbitrary $g_j \in \mathcal{B}_G^{-\infty,0}(I; \mathbb{R}^l)$.

Let us now consider 2×2 block matrix symbols $g(\lambda)$ of the class

$$S^\mu_{\mathrm{cl}}(\mathbb{R}^l; L^2(\mathbb{R}_+) \oplus \mathbb{C},\ \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}), \tag{19}$$

where the group actions on $L^2(\mathbb{R}_+) \oplus \mathbb{C}$ or $\mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}$ are defined by $u(\phi) \oplus c \to \lambda^{\frac{1}{2}} u(\lambda\phi) \oplus c$, $\lambda \in \mathbb{R}_+$ such that the pointwise adjoint $g^*(\lambda)$ with respect to the $L^2(\mathbb{R}_+) \oplus \mathbb{C}$ scalar product belongs to the space (19).

With every such $g(\lambda)$ we can associate an operator family

$$a(\lambda) := \omega g(\lambda)\widetilde{\omega} : \begin{array}{c} H^s(I) \\ \oplus \\ \mathbb{C} \end{array} \to \begin{array}{c} C^\infty(I) \\ \oplus \\ \mathbb{C} \end{array}, \tag{20}$$

$s > -\frac{1}{2}$, for any fixed choice of functions $\omega, \widetilde{\omega} \in C^\infty(I)$ which are equal to 1 near $\phi = \alpha$ and vanish in a neighbourhood of the end point β. In a similar manner we can form operators

$$b(\lambda) := \chi_*(\omega g(\lambda)\omega) : \begin{array}{c} H^s(I) \\ \oplus \\ \mathbb{C} \end{array} \to \begin{array}{c} C^\infty(I) \\ \oplus \\ \mathbb{C} \end{array} \tag{21}$$

where $\chi : I \to I$ is the diffeomorphism defined by $\chi(\phi) := -\phi + \alpha + \beta$ which interchanges the role of α and β. In other words, the direct summands $\mathbb{C}$ in the spaces of (20) belong to the end point α, those in the spaces of (21) to the end point β. Writing (20) and (21) as block matrices with entries a_{ij} and b_{ij}, respectively, we now form

$$g(\lambda) := \begin{pmatrix} a_{11}+b_{11} & a_{12} & b_{12} \\ a_{21} & a_{22} & 0 \\ b_{21} & 0 & b_{22} \end{pmatrix} : \begin{array}{c} H^s(I) \\ \oplus \\ \mathbb{C} \\ \oplus \\ \mathbb{C} \end{array} \to \begin{array}{c} C^\infty(I) \\ \oplus \\ \mathbb{C} \\ \oplus \\ \mathbb{C} \end{array} . \tag{22}$$

More generally, we consider the operator families

$$g(\lambda) = g_0(\lambda) + \sum_{j=1}^{\mathrm{d}} g_j(\lambda)\mathrm{diag}(\partial_\phi^j, 0, 0) \tag{23}$$

for any $\mathrm{d} \in \mathbb{N}$, where $g_j(\lambda)$ are of the kind (22), of order $\mu - j$.

The space $\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^l)$ for $\mu \in \mathbb{R}$, $\mathrm{d} \in \mathbb{N}$ is defined to be the set of all operator functions $g(\lambda) + c(\lambda)$ for arbitrary families of the form (23) and $c(\lambda) \in \mathcal{B}_G^{-\infty,d}(I;\mathbb{R}^l)$. Let $B_G^{\mu,d}(I;\mathbb{R}^l)$ denote the space of upper left corners of elements of $\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^l)$.

Remark 1.12. *The space $\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^l)$ has a natural Fréchet topology. So we can form spaces of the kind $C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^l))$ or $\mathcal{A}(D, \mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^l))$; here $\mathcal{A}(D,E)$ for an open set $D \subseteq \mathbb{C}$ and a Fréchet space E denotes the space of all holomorphic functions in D with values in E.*

Let $S_{\mathrm{cl}}^\mu(I \times \mathbb{R}_\vartheta \times \mathbb{R}_\lambda^l)_{\mathrm{tr}}$ denote the space of all classical symbols of order $\mu \in \mathbb{Z}$ in the variable ϕ and covariables (ϑ, λ) (with ϑ being the covariable to ϕ). Recall that the transmission property at the end points of the interval I (for instance, at $\phi = \alpha$) of a symbol $a(\phi, \vartheta, \lambda)$ means that the homogeneous components $a_{(\mu-j)}(\phi,\vartheta,\lambda)$ of order $\mu - j$ satisfy the conditions

$$D_\phi^k D_\lambda^\gamma \{a_{(\mu-j)}(\phi,\vartheta,\lambda) - (-1)^{\mu-j} a_{(\mu-j)}(\phi,-\vartheta,-\lambda)\} = 0$$

on the set $\{(\phi,\vartheta,\lambda) : \phi = \alpha,\ \vartheta \in \mathbb{R}\setminus\{0\}, \lambda = 0\}$ for all $k \in \mathbb{N}$, $\gamma \in \mathbb{N}^l$, and all $j \in \mathbb{N}$. A similar condition is imposed at $\phi = \beta$.

Given a symbol $a \in S_{\mathrm{cl}}^\mu(I \times \mathbb{R}_{\vartheta,\lambda}^{1+l})_{\mathrm{tr}}$ we set

$$\mathrm{op}^I(a)(\lambda)u(\phi) := \mathrm{r}\,\mathrm{op}(\tilde{a}(\lambda)\,\mathrm{e}u(\phi)) \tag{24}$$

where $\tilde{a}(\phi,\vartheta,\lambda) \in S_{\mathrm{cl}}^\mu(\mathbb{R} \times \mathbb{R}_{\vartheta,\lambda}^{1+l})$ is any symbol such that $a = \tilde{a}|_{I \times \mathbb{R}_{\vartheta,\lambda}^{1+l}}$; here e is the operator of extension by zero from I to $\mathbb{R}\setminus I$ and r the restriction to $\mathrm{int} I$. We then have continuous operators $\mathrm{op}^I(a)(\lambda) : H^s(I) \to H^{s-\mu}(I)$ for all reals $s > -\frac{1}{2}$ (clearly the operators do not depend on the choice of $\tilde{a}$).

Definition 1.13. *The space $B^{\mu,\mathrm{d}}(I;\mathbb{R}_\lambda^l)$ for $\mu \in \mathbb{Z}$, $\mathrm{d} \in \mathbb{N}$, is defined to be the set of all operator families of the form $\mathrm{op}^I(a)(\lambda) + g(\lambda)$ for arbitrary $a \in S_{\mathrm{cl}}^\mu(I \times \mathbb{R}_{\vartheta,\lambda}^{1+l})_{\mathrm{tr}}$ and $g \in B_G^{\mu,\mathrm{d}}(I;\mathbb{R}^l)$. Moreover, we set*

$$\mathcal{B}^{\mu,\mathrm{d}}(I;\mathbb{R}^l) := \{\mathrm{diag}(p,0,0) + g : p \in B^{\mu,\mathrm{d}}(I;\mathbb{R}^l), g \in \mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^l)\}.$$

In the case $p \neq 0$ we assume $\mu \in \mathbb{Z}$, otherwise we admit $\mu \in \mathbb{R}$.

The space $\mathcal{B}^{\mu,\mathrm{d}}(I;\mathbb{R}^l)$ is Fréchet in a natural way.

1.3 Mellin operators and weighted spaces

For the calculus near (regular) geometric singularities such as conical points, edges and corners it is convenient to employ pseudo-differential operators based on the Mellin transform on the half-axis. This has been observed by many authors before, see, e.g., Kondratyev [7]. Introducing polar coordinates $(r, x) \in \mathbb{R}_+ \times S^n$ in $\mathbb{R}^{n+1}_{\tilde{x}} \setminus \{0\}$ a differential operator

$$\widetilde{A}(\tilde{x}, y, D_{\tilde{x}}, D_y) = \sum_{|\alpha| \le \mu} c_\alpha(\tilde{x}, y) D^\alpha_{\tilde{x},y}$$

in $\mathbb{R}^{n+1} \times \Omega$ with coefficients $c_\alpha(\tilde{x}, y) \in C^\infty(\mathbb{R}^{n+1} \times \Omega)$, $\Omega \subseteq \mathbb{R}^q$ open, takes the form

$$A(r, x, y, D_r, D_x, D_y) = r^{-\mu} \sum_{j+|\beta| \le \mu} a_{j\beta}(r, y) \left(-r \frac{\partial}{\partial r} \right)^j (r D_y)^\beta \tag{25}$$

with coefficients $a_{j\beta} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\beta|)}(S^n))$. Here $\mathrm{Diff}^\nu(.)$ denotes the space of all differential operators of order ν on the manifold in the brackets, with smooth coefficients. The typical Fuchs type differentiations $-r\frac{\partial}{\partial r}$ in (25) can be regarded as 'Mellin' operators with symbol z, i.e.,

$$-r \frac{\partial}{\partial r} = M^{-1} z M$$

where $Mu(z) = \int_0^\infty r^{z-1} u(r) dr$ is the Mellin transform, first on $C_0^\infty(\mathbb{R}_+)$ (with z varying in the complex plane) and later on extended to more general function and distribution spaces. Then the variable z will often vary on a 'weight line'

$$\Gamma_\beta = \{ z \in \mathbb{C} : \mathrm{Re} z = \beta \}$$

for some $\beta \in \mathbb{R}$. Mellin pseudo-differential operators with respect to some weight $\gamma \in \mathbb{R}$ have the form

$$\mathrm{op}_M^\gamma(f)(\lambda) u(r) := \int \int_0^\infty \left(\frac{r}{r'} \right)^{-(\frac{1}{2} - \gamma + i\rho)} f(r, r', \frac{1}{2} - \gamma + i\rho, \lambda) u(r') \frac{dr'}{r'} đ\varrho,$$

where $f(r, r', z, \lambda)$ is a parameter-dependent amplitude function with covariables $(z, \lambda) \in \Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^l$. In our applications f will be smooth in $(r, r') \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$ and takes values in $\mathcal{B}^{\mu,\mathrm{d}}(X; \boldsymbol{v}; \mathbb{R}_\rho \times \mathbb{R}^l)$ for a compact C^∞ manifold X with boundary, e.g., $X = I = [0, 2\pi]$, cf. the preceding section.

We want to specify this kind of Mellin symbols with respect to the dependence on z. If $U \subseteq \mathbb{C}$ is an open set, E a Fréchet space, by $\mathcal{A}(U, E)$ we denote the space of all holomorphic functions in U with values in E.

Let $\mathcal{B}^{\mu,\mathrm{d}}(X; \boldsymbol{v}; \mathbb{C} \times \mathbb{R}^l)$ denote the space of all operator functions $h(z, \lambda) \in \mathcal{A}(\mathbb{C}, \mathcal{B}^{\mu,\mathrm{d}}(X; \boldsymbol{v}; \mathbb{R}^l))$ such that

$$h(\beta + i\rho, \lambda) \in \mathcal{B}^{\mu,\mathrm{d}}(X; \boldsymbol{v}; \mathbb{R}_\rho \times \mathbb{R}^l_\lambda)$$

for every $\beta \in \mathbb{R}$, uniformly in $\beta \in [c', c'']$ for every $c' \le c''$. We also write

$$\mathcal{B}^{\mu,\mathrm{d}}(X; \boldsymbol{v}; \Gamma_\beta \times \mathbb{R}^l)$$

for the parameter-dependent space of boundary value problems on X if the parameter (z, λ) varies on $\Gamma_\beta \times \mathbb{R}^l$ (i.e., $(\mathrm{Im} z, \lambda)$ plays the role of the parameter).

Theorem 1.14. *For every* $f(z,\lambda) \in \mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\Gamma_\beta \times \mathbb{R}^l)$ *there exists an element*

$$h(z,\lambda) \in \mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{C} \times \mathbb{R}^l)$$

such that

$$h(\beta + i\varrho,\lambda) = f(\beta + i\varrho,\lambda) \mod \mathcal{B}^{-\infty,\mathrm{d}}(X;\boldsymbol{v};\mathbb{R}^{1+l}_{\varrho,\lambda}).$$

Starting from functions $\widetilde{f}(r,\widetilde{\rho},\widetilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+,\mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{R}^{1+l}_{\widetilde{\rho},\widetilde{\lambda}}))$ we can construct Mellin symbols $\widetilde{h}(r,z,\widetilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+,\mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{C}\times\mathbb{R}^l))$ such that, if we set

$$f(r,\rho,\lambda) := \widetilde{f}(r,r\rho,r\lambda), \quad h(r,z,\lambda) := \widetilde{h}(r,z,r\lambda),$$
$$f_0(r,\rho,\lambda) := \widetilde{f}(0,r\rho,r\lambda), \quad h_0(r,z,\lambda) := \widetilde{h}(0,z,r\lambda)$$

we obtain

$$\mathrm{op}_M^\gamma(h)(\lambda) = \mathrm{op}_r(f)(\lambda) \mod \mathcal{B}^{-\infty,\mathrm{d}}(X^\wedge;\boldsymbol{v};\mathbb{R}^l) \tag{26}$$

and

$$\mathrm{op}_M^\gamma(h_0)(\lambda) = \mathrm{op}_r(f_0)(\lambda) \mod \mathcal{B}^{-\infty,\mathrm{d}}(X^\wedge;\boldsymbol{v};\mathbb{R}^l) \tag{27}$$

for all $\gamma \in \mathbb{R}$. Here $\mathrm{op}_r(.)$ means the pseudo-differential action with respect to the Fourier transform in r. The relations (26) and (27) are interpreted as equations in $\mathcal{B}^{\mu,\mathrm{d}}(X^\wedge;\boldsymbol{v};\mathbb{R}^l)$, i.e., $X^\wedge$ is regarded as a non-compact C^∞ manifold with boundary, and operators are first applied to $u(r,x) \in C_0^\infty(\mathbb{R}_+,C^\infty(X))$.

Relations of the kind (26), (27) are of a similar structure as the corresponding ones for parameter-dependent pseudo-differential operators on a closed compact C^∞ manifold, cf. [16]. The case of boundary value problems is treated in [10]. For the calculus below it will also be useful to define the subspace

$$\mathcal{B}^{-\infty,\mathrm{d}}(X;\boldsymbol{v};\Gamma_\beta \times \mathbb{R}^l)_\varepsilon \tag{28}$$

of all (so called smoothing Mellin symbols of the cone algebra with weight control in an ε-strip around the weight line Γ_β) $f(z,\lambda) \in \mathcal{A}(\{\beta - \varepsilon < \mathrm{Re}z < \beta + \varepsilon\},\mathcal{B}^{-\infty,\mathrm{d}}(X;\boldsymbol{v};\mathbb{R}^l))$ for any $\varepsilon > 0$ which satisfy the condition

$$f(\delta + i\rho,\lambda) \in \mathcal{S}(\mathbb{R}^{1+l}_{\rho,\lambda},\mathcal{B}^{-\infty,\mathrm{d}}(X;\boldsymbol{v}))$$

for every $\delta \in (\beta-\varepsilon,\beta+\varepsilon)$ and uniformly in δ for every compact subinterval. In the considerations below it will be sufficient to have such smoothing Mellin symbols for the case $l = 0$.

Starting from operator families

$$\widetilde{f}(r,y,\widetilde{\rho},\widetilde{\eta},\widetilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega,\mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{R}^{1+q+l}))$$

for an open set $\Omega \subseteq \mathbb{R}^q$ we can find similarly as before an element

$$\widetilde{h}(r,y,z,\widetilde{\eta},\widetilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega,\mathcal{B}^{\mu,\mathrm{d}}(X;\boldsymbol{v};\mathbb{C}\times\mathbb{R}^{q+l}))$$

such that, if we set

$$f(r,y,\rho,\eta,\lambda) := \widetilde{f}(r,y,r\rho,r\eta,r\lambda), h(r,y,z,\eta,\lambda) := \widetilde{h}(r,y,z,r\eta,r\lambda)$$

we have

$$\mathrm{op}_M^{\gamma}(h)(y,\eta,\lambda) = \mathrm{op}_r(f)(y,\eta,\lambda) \mod C^\infty(\Omega, \mathcal{B}^{-\infty,\mathrm{d}}(X^\wedge; \boldsymbol{v}; \mathbb{R}^{q+l})).$$

A relation similar to (27) also holds for the corresponding families with subscript 0.

By a cut-off function in this paper we understand any real-valued function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ which is equal to 1 near $r = 0$. Let us choose cut-off functions ω_0, ω_1 and ω_2 such that $\omega_1 \equiv 1$ on supp ω_0, $\omega_0 \equiv 1$ on supp ω_2, and set

$$\begin{aligned} p_M(y,\eta,\lambda) &:= r^{-\mu}\omega_0(r[\eta,\lambda])\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta,\lambda)\omega_1(r[\eta,\lambda]), \\ p_\psi(y,\eta,\lambda) &:= r^{-\mu}(1-\omega_0(r[\eta,\lambda]))\mathrm{op}_r(f)(y,\eta,\lambda)(1-\omega_2(r[\eta,\lambda])). \end{aligned}$$

Here $[\eta,\lambda]$ is any strictly positive C^∞ function in $\mathbb{R}^{q+l}$ which is equal to $|\eta,\lambda|$ for $|\eta,\lambda| \geq C$ for some constant $C > 0$.

Moreover, for arbitrary cut-off functions $\sigma(r)$ and $\widetilde{\sigma}(r)$ we set

$$p(y,\eta,\lambda) := \sigma(r)\{p_M(y,\eta,\lambda) + p_\psi(y,\eta,\lambda)\}\widetilde{\sigma}(r). \tag{29}$$

Set

$$\begin{aligned} \sigma_\wedge(p)(y,\eta,\lambda) := {} & r^{-\mu}\omega_0(r|\eta,\lambda|)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(y,\eta,\lambda)\omega_1(r|\eta,\lambda|) \\ & + r^{-\mu}(1-\omega_0(r|\eta,\lambda|))\mathrm{op}_r(f_0)(y,\eta,\lambda)(1-\omega_2(r|\eta,\lambda|)). \end{aligned} \tag{30}$$

Let us now introduce weighted Sobolev spaces on an infinite stretched cone with base N, first for the case that N is a closed compact C^∞ manifold. We use the fact that for every $\mu \in \mathbb{R}$ there exists a parameter-dependent elliptic operator family $R^\mu(\lambda) \in L_{\mathrm{cl}}^\mu(N;\mathbb{R}^l)$ which induces isomorphisms

$$R^\mu(\lambda) : H^s(N) \to H^{s-\mu}(N)$$

for every $s \in \mathbb{R}$, $\lambda \in \mathbb{R}^l$. Let us apply this to $l = 1$.

By $\mathcal{H}^{s,\gamma}(N^\wedge)$ for $N^\wedge := \mathbb{R}_+ \times N$, $s, \gamma \in \mathbb{R}$ we denote the completion of $C_0^\infty(N^\wedge)$ with respect to the norm $\left\{(2\pi i)^{-1}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(\mathrm{Im} z)(Mu)(z)\|_{L^2(N)}^2\, dz\right\}^{\frac{1}{2}}$, $n = \dim N$. We then define the space

$$\mathcal{K}^{s,\gamma}(N^\wedge) := \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(N^\wedge), \quad v \in H_{\mathrm{cone}}^s(N^\wedge)\}.$$

Here $H_{\mathrm{cone}}^s(N^\wedge)$ denotes the subspace of all $v = \widetilde{v}|_{N^\wedge}$, $\widetilde{v} \in H_{\mathrm{loc}}^s(\mathbb{R} \times N)$, such that for every coordinate neighbourhood U on N, every diffeomorphism $\chi : U \to \widetilde{U}$ to an open set of S^n, $\chi(x) = \widetilde{x}$, and every $\varphi \in C_0^\infty(U)$ the function $\varphi(\chi^{-1}(\widetilde{x}))(1-\omega(r))v(r,\chi^{-1}(\widetilde{x}))$ belongs to the space $H^s(\mathbb{R}^{n+1})$ (where $(r,\widetilde{x})$ has the meaning of polar coordinates in $\mathbb{R}^{n+1} \setminus \{0\}$). The spaces $\mathcal{K}^{s,\gamma}(N^\wedge)$ are independent of the specific choice of ω. They are Hilbert spaces with natural scalar products which we choose for $s = \gamma = 0$ in such a way that $\mathcal{K}^{0,0}(N^\wedge) = H^{0,0}(N^\wedge) = r^{-\frac{n}{2}}L^2(\mathbb{R}_+ \times N)$ with $L^2(\mathbb{R}_+ \times N)$ referring to $drdx$.

For the case $N = 2X$ for a compact C^∞ manifold X with boundary we set

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{u|_{\mathrm{int}X^\wedge} : u \in \mathcal{K}^{s,\gamma}((2X)^\wedge)\} \tag{31}$$

with the quotient topology from $\mathcal{K}^{s,\gamma}(X^\wedge) \cong \mathcal{K}^{s,\gamma}((2X)^\wedge)/\mathcal{K}^{s,\gamma}(X_-^\wedge)_0$, where $\mathcal{K}^{s,\gamma}(X_-^\wedge)_0$ denotes the subspace of all $w \in \mathcal{K}^{s,\gamma}\ ((2X)^\wedge)$ which vanish on $\mathrm{int}X^\wedge$.

Let us finally define the Fréchet space

$$\mathcal{S}^{\gamma}_{\varepsilon}(N^{\wedge}) := \varprojlim_{k\in\mathbb{N}} \langle r\rangle^{-k}\mathcal{K}^{k,\gamma+\varepsilon-(1+k)^{-1}}(N^{\wedge})$$

for every weight $\gamma \in \mathbb{R}$ and $\varepsilon > 0$.

Analogous constructions are possible for the case of distributional sections in vector bundles, i.e., we have spaces of the kind

$$\mathcal{H}^{s,\gamma}(N^{\wedge}, E), \quad \mathcal{K}^{s,\gamma}(N^{\wedge}, E), \quad \mathcal{S}^{\gamma}_{\varepsilon}(N^{\wedge}, E),$$

$E \in \mathrm{Vect}(N^{\wedge})$, both for the case of closed compact C^{∞} manifolds N as well as for compact C^{∞} manifolds N with boundary.

Similarly, if B is a compact manifold with conical singularities, first without boundary, and $\mathbb{B}$ its stretched manifold, cf. Section 1.4 below, for every $J \in \mathrm{Vect}(\mathbb{B})$ we have the weighted spaces

$$\mathcal{H}^{s,\gamma}(\mathbb{B}, J) \tag{32}$$

defined by $\omega u \in \mathcal{H}^{s,\gamma}(N^{\wedge}, J)$, $(1-\omega)u \in H^{s}_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B}, J)$, where $\overline{\mathbb{R}}_{+} \times N$ is the local model of $\mathbb{B}$ near $\partial\mathbb{B} \cong N$ and ω a cut-off function on $\mathbb{B}$ which is supported in a collar neighbourhood of $\partial\mathbb{B}$ and equal to 1 near $\partial\mathbb{B}$. For brevity we denoted by J also the bundle associated with that in (32) as the pull back of $J|_{\partial\mathbb{B}}$ under the canonical projection $\overline{\mathbb{R}}_{+} \times N \to N$. Analogously, if D is a compact manifold with conical singularities and boundary, $\mathbb{D}$ its stretched manifold and $2\mathbb{D}$ the stretched manifold associated with $2D$, cf. Section 1.4 below, for every $\widetilde{E} \in \mathrm{Vect}(2\mathbb{D})$ we have the spaces $\mathcal{H}^{s,\gamma}(2\mathbb{D}, \widetilde{E})$ and $\mathcal{H}^{s,\gamma}(\mathbb{D}_{-}, \widetilde{E})_{0}$ (the closed subspace of all $w \in \mathcal{H}^{s,\gamma}(2\mathbb{D}, \widetilde{E})$ supported by the 'negative' copy of $\mathbb{D}$ in $2\mathbb{D}$). Then we form

$$\mathcal{H}^{s,\gamma}(\mathbb{D}, E) := \{u|_{\mathrm{int}\mathbb{D}_{\mathrm{reg}}} : u \in \mathcal{H}^{s,\gamma}(2\mathbb{D}, \widetilde{E})\}$$

for $E := \widetilde{E}|_{\mathbb{D}}$ (with $\mathbb{D}$ being the 'positive' copy of $\mathbb{D}$ in $2\mathbb{D}$) with the quotient topology of $\mathcal{H}^{s,\gamma}(2\mathbb{D}, \widetilde{E})/\mathcal{H}^{s,\gamma}(\mathbb{D}_{-}, \widetilde{E})_{0}$.

Theorem 1.15. *The family of operators* (29) *is a symbol of the class* $\mathcal{S}^{\mu}(\Omega \times \mathbb{R}^{q+l}_{\eta,\lambda}; \mathcal{E}, \widetilde{\mathcal{E}})$ *for*

$$\mathcal{E} := \mathcal{K}^{s,\gamma}(X^{\wedge}, E) \oplus \mathcal{K}^{s,\gamma-\frac{1}{2}}(\partial X, H), \tag{33}$$

$$\widetilde{\mathcal{E}} := \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}, F) \oplus \mathcal{K}^{s-\frac{1}{2}-\mu,\gamma-\frac{1}{2}-\mu}(\partial X, J) \tag{34}$$

for every $s > -\frac{1}{2}$, *as well as for*

$$\mathcal{E} := \mathcal{S}^{\gamma}_{\varepsilon}(X^{\wedge}, E) \oplus \mathcal{S}^{\gamma-\frac{1}{2}}_{\varepsilon}(\partial X, H), \tag{35}$$

$$\widetilde{\mathcal{E}} := \mathcal{S}^{\gamma-\mu}_{\varepsilon}(X^{\wedge}, F) \oplus \mathcal{S}^{\gamma-\frac{1}{2}-\mu}_{\varepsilon}(\partial X, J) \tag{36}$$

for every $\varepsilon > 0$.

There are other important categories of operator-valued amplitude functions of the edge calculus, the so called smoothing Melling symbols and the Green symbols. The smoothing Mellin symbols are of the form

$$m(y,\eta,\lambda) := r^{-\mu}\omega(r[\eta,\lambda])\mathrm{op}_{M}^{\gamma-\frac{n}{2}}(f)(y)\tilde{\omega}(r[\eta,\lambda]) \tag{37}$$

for some cut-off functions $\omega, \tilde{\omega}$ and a Mellin symbol

$$f(y) \in C^\infty(\Omega, \mathcal{B}^{-\infty,\mathrm{d}}(X; \boldsymbol{v}; \Gamma_{\frac{n+1}{2}-\gamma})_\varepsilon) \tag{38}$$

for some $\varepsilon > 0$. These are elements of $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{q+l}; \mathcal{E}, \widetilde{\mathcal{E}})$ for the spaces (33), (34), $s > -\frac{1}{2}$, as well as for (35), (36). Let us set

$$\sigma_\wedge(m)(y, \eta, \lambda) := r^{-\mu}\omega(r|\eta, \lambda|)\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(y)\tilde{\omega}(r|\eta, \lambda|). \tag{39}$$

Green symbols (of order $\mu \in \mathbb{R}$ and type 0) are defined as operator families

$$\mathfrak{g}(y, \eta, \lambda) : \mathcal{E} \to \mathcal{S}_\varepsilon, \quad (y, \eta, \lambda) \in \Omega \times \mathbb{R}^{q+l}, \tag{40}$$

for

$$\begin{aligned}
\mathcal{E} &:= \mathcal{K}^{s,\gamma}(X^\wedge, E) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, H) \oplus \mathbb{C}^m, \\
\mathcal{S}_\varepsilon &:= \mathcal{S}^{\gamma-\mu}_\varepsilon(X^\wedge, F) \oplus \mathcal{S}^{\gamma-\mu-\frac{1}{2}}_\varepsilon((\partial X)^\wedge, J) \oplus \mathbb{C}^{m'}
\end{aligned}$$

for certain dimensions m, m' which refer to the number of additional conditions on the edge, with some $\varepsilon > 0$ that depends on $\mathfrak{g}$, such that

$$\mathfrak{g}_0(y, \eta, \lambda) := \mathrm{diag}(1, \langle \eta, \lambda \rangle^{\frac{1}{2}}, \langle \eta, \lambda \rangle)\mathfrak{g}(y, \eta, \lambda)\mathrm{diag}(1, \langle \eta, \lambda \rangle^{-\frac{1}{2}}, \langle \eta, \lambda \rangle^{-1})$$

is an element of $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{q+l}; \mathcal{E}, \mathcal{S}_\varepsilon)$ and $\mathfrak{g}_0^*(y, \eta, \lambda)$ (the pointwise formal adjoint) an element of $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{q+l}; \widetilde{\mathcal{E}}, \widetilde{\mathcal{S}}_\varepsilon)$ for

$$\begin{aligned}
\widetilde{\mathcal{E}} &:= \mathcal{K}^{s,-\gamma+\mu}(X^\wedge, F) \oplus \mathcal{K}^{s-\frac{1}{2},-\gamma+\mu-\frac{1}{2}}((\partial X)^\wedge, H) \oplus \mathbb{C}^{m'}, \\
\widetilde{\mathcal{S}}_\varepsilon &:= \mathcal{S}^{-\gamma}_\varepsilon(X^\wedge, E) \oplus \mathcal{S}^{-\gamma-\frac{1}{2}}_\varepsilon((\partial X)^\wedge, J) \oplus \mathbb{C}^m
\end{aligned}$$

for every $s > -\frac{1}{2}$. A family of operators (40) is said to be a Green symbol of order $\mu \in \mathbb{R}$ and type $\mathrm{d} \in \mathbb{N}$ if it has the form

$$\mathfrak{g}(y, \eta, \lambda) = \mathfrak{g}_0(y, \eta, \lambda) + \sum_{j=1}^{\mathrm{d}} \mathfrak{g}_j(y, \eta, \lambda)\mathrm{diag}(T^j, 0, 0) \tag{41}$$

for a differential operator T similarly as in (12) (here operating in sections of the bundle E) and Green symbols $\mathfrak{g}_j(y, \eta, \lambda)$ of order μ and type 0, $j = 1, \ldots, \mathrm{d}$.

An operator function (41) represents a 3×3 matrix of classical operator-valued symbols with a corresponding matrix of orders

$$\begin{pmatrix} \mu & \mu - \frac{1}{2} & \mu - 1 \\ \mu + \frac{1}{2} & \mu & \mu - \frac{1}{2} \\ \mu + 1 & \mu + \frac{1}{2} & \mu. \end{pmatrix}. \tag{42}$$

Let $\sigma_\wedge(\mathfrak{g})(y, \eta, \lambda)$ denote the matrix of homogeneous principal symbols.

Now the parameter-dependent amplitude functions of the edge pseudo-differential calculus of boundary value problems are defined as

$$\mathfrak{a}(y, \eta, \lambda) = \mathfrak{p}(y, \eta, \lambda) + \mathfrak{m}(y, \eta, \lambda) + \mathfrak{g}(y, \eta, \lambda) \tag{43}$$

for $\mathfrak{p}(y,\eta,\lambda) = \begin{pmatrix} p(y,\eta,\lambda) & 0 \\ 0 & 0 \end{pmatrix}$, $\mathfrak{m}(y,\eta,\lambda) = \begin{pmatrix} m(y,\eta,\lambda) & 0 \\ 0 & 0 \end{pmatrix}$ with the corresponding 2×2 upper left corners (29) and (37), respectively, and a Green symbol $\mathfrak{g}(y,\eta,\lambda)$ of order μ and type d.

Let us set

$$\sigma_\wedge(\mathfrak{a})(y,\eta,\lambda) := \begin{pmatrix} \sigma_\wedge(p)(y,\eta,\lambda) + \sigma_\wedge(m)(y,\eta,\lambda) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\wedge(\mathfrak{g})(y,\eta,\lambda), \quad (44)$$

regarded as a family of operators

$$\sigma_\wedge(\mathfrak{a})(y,\eta,\lambda) : \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge, E) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, H) \\ \oplus \\ \mathbb{C}^m \end{matrix} \to \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge, F) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2}-\mu,\gamma-\frac{1}{2}-\mu}((\partial X)^\wedge, J), \\ \oplus \\ \mathbb{C}^{m'} \end{matrix}$$

$(y,\eta,\lambda) \in T^*\Omega \times \mathbb{R}^l \setminus 0$, $s > \mathrm{d} - \frac{1}{2}$. We then have the homogeneity

$$\begin{aligned} &\sigma_\wedge(\mathfrak{a})(y,\delta\eta,\delta\lambda) \\ &\quad = \delta^\mu \mathrm{diag}(\kappa_\delta^{(n)}, \delta^{\frac{1}{2}}\kappa_\delta^{(n-1)}, \delta)\sigma_\wedge(\mathfrak{a})(y,\eta,\lambda)\mathrm{diag}(\kappa_\delta^{(n)}, \delta^{\frac{1}{2}}\kappa_\delta^{(n-1)}, \delta)^{-1} \end{aligned}$$

for all $\delta \in \mathbb{R}_+$; $(\kappa_\delta^{(n)}u)(r,x) := \delta^{\frac{n+1}{2}}u(\delta r, x)$, $(\kappa_\delta^{(n-1)}v)(r,x') := \delta^{\frac{n}{2}}v(\delta r, x')$.

For crack problems we modify this for the case when X is an interval I with two end points (this gives then corresponding 4×4 matrices because of the two components of the boundary). In addition in the final calculus below we consider the entries separately and allow them to have different orders, according to the 'realistic' boundary value problems for differential operators and their parametrices.

Let $\mathcal{R}^{\mu,\mathrm{d}}(\Omega \times \mathbb{R}^{q+l}; \boldsymbol{w})$ for $\boldsymbol{w} = (E,F;H,J;m,m')$ denote the space of all operator families (43). The weight γ is given in connection with every element of that space. We employ this for the case $X = I$, $q = 0$ (then also Ω disappears) and $l = 1$. Then we have the class $\mathcal{R}^{\mu,\mathrm{d}}(\mathbb{R}; \boldsymbol{w})$. There is a natural version of holomorphic families $\mathcal{R}^{\mu,\mathrm{d}}(\mathbb{C}; \boldsymbol{w})$ studied in the author's joint papers with Oliaro [10] and De Donno [1]. The elements $h(w) \in \mathcal{R}^{\mu,d}(\mathbb{C}; \boldsymbol{w})$ are characterised by holomorphy in $w \in \mathbb{C}$ together with the property

$$h(\delta + i\tau) \in \mathcal{R}^{\mu,\mathrm{d}}(\mathbb{R}_\tau; \boldsymbol{w})$$

for every $\delta \in \mathbb{R}$, uniformly in compact δ-intervals.

1.4 Parameter-dependent cone boundary value problems

Let $\mathbb{A}$ be a C^∞ manifold with compact boundary $\partial\mathbb{A}$. Then $\mathbb{A}$ can be regarded as the stretched space of a manifold $A = \mathbb{A}/\partial\mathbb{A}$ with conical singularity represented by $\partial\mathbb{A} =: X$ collapsed to a point V. Then $\mathbb{A}$ is called the stretched manifold associated with A. Isomorphisms in the category of manifolds with conical singularities can be defined via diffeomorphisms of the associated stretched manifolds as C^∞ manifolds with boundary.

Let us set

$$\mathbb{A}_{\mathrm{sing}} := \partial\mathbb{A}, \quad \mathbb{A}_{\mathrm{reg}} := \mathbb{A} \setminus \mathbb{A}_{\mathrm{sing}}.$$

Another equivalent definition of a manifold A with conical singularity starts from a topological space A with a chosen point $v \in A$ such that $A \setminus \{v\}$ is a C^∞ manifold, and there is a neighbourhood V of v in A with a homeomorphism $\varphi : V \to (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X) =: X^\Delta$ for a closed compact manifold X, such that

$$\varphi|_{V\setminus\{v\}} : V \setminus \{v\} \to \mathbb{R}_+ \times X =: X^\wedge \tag{45}$$

is a diffeomorphism. By attaching X to $\mathbb{R}_+ \times X$ we obtain $\overline{\mathbb{R}}_+ \times X$; this allows us to attach X also to $V \setminus \{v\}$ which gives the stretched version $\mathbb{V}$ of the neighbourhood V. Under a suitable restriction on the nature of transition maps $X^\Delta \to X^\Delta$ for different representations of V we obtain the definition in an invariant way which gives us globally the stretched manifold $\mathbb{A}$ associated with A. Analogous definitions make sense for manifolds with more than one conical singularity; for simplicity we consider the case of one conical singularity. These definitions concern the case of a manifold A with conical singularity and 'without boundary'. Let us generalise this to the case with boundary.

A topological space D is called a manifold with conical singularity $v \in D$ and boundary if $D \setminus \{v\}$ is a C^∞ manifold with boundary, and there is a neighbourhood V of v in D with a homeomorphism $\varphi : V \to X^\Delta$ for a compact C^∞ manifold X with boundary, such that (45) is a diffeomorphism in the sense of C^∞ manifolds with boundary. Again under a natural condition for the transition maps for different choices of φ we can attach a copy of X to $D \setminus \{v\}$ in an invariant way to obtain the stretched manifold $\mathbb{D}$ associated with D.

There is a natural notion of doubling up D to a manifold $A := 2D$ with conical singularity and closed $\partial\mathbb{A} = 2X$. We then set

$$\mathbb{D}_{\mathrm{sing}} := \mathbb{A}_{\mathrm{sing}} \cap \mathbb{D}, \quad \mathbb{D}_{\mathrm{reg}} := \mathbb{D} \setminus \mathbb{D}_{\mathrm{sing}}.$$

It follows that $\mathbb{D}_{\mathrm{sing}}$ is diffeomorphic to X, and $\mathbb{D}_{\mathrm{reg}}$ is a C^∞ manifold with boundary. Moreover, $\mathbb{B} := \partial\mathbb{D}_{\mathrm{reg}} \cup \partial X$ is the stretched manifold of the manifold $B = \partial D$ with conical singularity v and without boundary.

Recall from [12], [13], or [6] that there are spaces $\mathcal{C}^{\mu,\mathrm{d}}(\mathbb{D}; \boldsymbol{v})$ of pseudo-differential boundary value problems of order μ and type d on $\mathbb{D}$ which constitute the corresponding cone algebra on $\mathbb{D}$. Here $\boldsymbol{v} := (E, F; H, J)$ is a tuple of smooth complex vector bundles $E, F \in \mathrm{Vect}(\mathbb{D})$, $H, J \in \mathrm{Vect}(\mathbb{B})$. The operators $\mathcal{A}$ in the cone algebra are continuous between weighted Sobolev spaces

$$\mathcal{A} : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{D}, E) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{B}, H) \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{D}, F) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2}-\mu,\gamma-\frac{1}{2}-\mu}(\mathbb{B}, J) \end{array}$$

for all $s \in \mathbb{R}$, $s > \mathrm{d} - \frac{1}{2}$. The weight $\gamma \in \mathbb{R}$ is given together with the operator $\mathcal{A}$ (the weight belongs to weight data $(\gamma, \gamma-\mu, \vartheta_\varepsilon)$ for a weight interval $\vartheta_\varepsilon = (-\varepsilon, 0]$, $\varepsilon = \varepsilon(\mathcal{A}) > 0$, which defines strips

$$\left\{z \in \mathbb{C} : \frac{n+1}{2} - \beta - \varepsilon < \mathrm{Re} z \leq \frac{n+1}{2} - \beta\right\}$$

for $\beta = \gamma$ and $\beta = \gamma - \mu$, referring to the involved smoothing Mellin and Green operators, cf. Section 3.2 below, and $n = \dim X$.

In this paper we need a parameter-dependent calculus of such cone operators. The notion of parameter-dependence in this case is not so straightforward as for pseudo-differential operators on smooth manifolds. On manifolds with conical singularities without boundary the parameter-dependent calculus was introduced in [18] in connection with operators on manifolds with corners. This was later on applied by Gil [4] for studying heat trace expansions for cone operators. The specific aspect in the parameter-dependent cone theory is that the parameter plays the role of an edge covariable of a corresponding edge calculus. For the infinite (stretched) cone $X^\wedge$ with a base X with boundary such edge amplitude functions have the form (43), see also the author's joint papers with Oliaro [10], De Donno [1], and Krainer [8].

For the applications to the corner theory which is also the point in the present theory we have to pass to block matrices with extra finite-dimensional entries. In other words, instead of $\boldsymbol{v} = (E, F; H, J)$ we consider

$$\boldsymbol{w} := (E, F; H, J; m, m')$$

for any $m, m' \in \mathbb{N}$.

The space $\mathcal{C}^{\mu,\mathrm{d}}(\mathbb{D}; \boldsymbol{w}; \mathbb{R}^l)$ of pseudo-differential boundary value problems of order μ and type d on $\mathbb{D}$, with parameters $\lambda \in \mathbb{R}^l$, is defined to be the set of all operator families of the form

$$\mathcal{A}(\lambda) + \mathcal{C}(\lambda) : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{D}, E) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{B}, H) \\ \oplus \\ \mathbb{C}^m \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{D}, F) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2}-\mu,\gamma-\frac{1}{2}-\mu}(\mathbb{B}, J) \\ \oplus \\ \mathbb{C}^{m'} \end{array}$$

such that, if we write $\mathcal{A}(\lambda) = (\mathcal{A}_{ij}(\lambda))_{i,j=1,2,3}$, we have the following properties:

(i) $\mathcal{A}_{11}(\lambda)$ restricted to int $\mathbb{D}_{\mathrm{reg}}$ belongs to $L^\mu_{\mathrm{cl}}(\mathrm{int}\,\mathbb{D}_{\mathrm{reg}}; E, F; \mathbb{R}^l)$,

(ii) $(\mathcal{A}_{ij}(\lambda))_{i,j=1,2}$ restricted to $\mathbb{D}_{\mathrm{reg}}$ belongs to $\mathcal{B}^{\mu,\mathrm{d}}(\mathbb{D}_{\mathrm{reg}}; \boldsymbol{v}; \mathbb{R}^l)$;

(iii) $\mathcal{A}(\lambda)$ restricted to a neighbourhood of $\mathbb{D}_{\mathrm{sing}}$ belongs to the space $\mathcal{R}^{\mu,\mathrm{d}}(\mathbb{R}^l_\lambda; \boldsymbol{w})$;

(iv) $\mathcal{C}(\lambda)$ is a Schwartz function in $\lambda \in \mathbb{R}^l$ with values in $\mathcal{C}^{-\infty,\mathrm{d}}(\mathbb{D}; \boldsymbol{w})$.

The latter space for $\mathrm{d} = 0$ is defined to be the set of all operators

$$\mathcal{C} : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{D}, E) \\ \oplus \\ \mathcal{H}^{s',\gamma-\frac{1}{2}}(\mathbb{B}, H) \\ \oplus \\ \mathbb{C}^m \end{array} \to \begin{array}{c} \mathcal{H}^{\infty,\gamma-\mu+\varepsilon}(\mathbb{D}, F) \\ \oplus \\ \mathcal{H}^{\infty,\gamma-\frac{1}{2}-\mu+\varepsilon}(\mathbb{B}, J) \\ \oplus \\ \mathbb{C}^{m'} \end{array}$$

for some $\varepsilon = \varepsilon(\mathcal{C}) > 0$, continuous for all $s, s' \in \mathbb{R}$, $s > -\frac{1}{2}$, such that the formal adjoint (referring to the scalar products in the corresponding spaces for $s = s' = \gamma = 0$) satisfy an analogous condition with the corresponding opposite weights. The space $\mathcal{C}^{-\infty,\mathrm{d}}(\mathbb{D}; \boldsymbol{w})$ for arbitrary $\mathrm{d} \in \mathbb{N}$ is defined to be the set of all sums

$$\mathcal{C} = \mathcal{C}_0 + \sum_{j=1}^{\mathrm{d}} \mathcal{C}_j \mathrm{diag}(T^j, 0, 0)$$

for arbitrary $\mathcal{C}_j \in \mathcal{C}^{-\infty,0}(\mathbb{D};\boldsymbol{w})$, $j=0,\dots,\mathrm{d}$, and a differential operator T of first order which differentiates transversally to the boundary.

The parameter-dependent principal symbolic structure of elements $\mathcal{A} \in \mathcal{C}^{\mu,\mathrm{d}}(\mathbb{D};\boldsymbol{w};\mathbb{R}^l)$ consists of triples

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A}))$$

with the interior principal symbol $\sigma_\psi(\mathcal{A}) := \sigma_\psi(\mathcal{A}_{11})$ in the sense of

$$L^\mu_{\mathrm{cl}}(\mathrm{int}\mathbb{D}_{\mathrm{reg}};E,F;\mathbb{R}^l),$$

the boundary symbol $\sigma_\partial(\mathcal{A}) := \sigma_\partial((\mathcal{A}_{ij})_{i,j=1,2})$ in the sense of $\mathcal{B}^{\mu,\mathrm{d}}(\mathbb{D}_{\mathrm{reg}};\boldsymbol{v};\mathbb{R}^l)$ and the 'edge' symbol $\sigma_\wedge(\mathcal{A})$ which is locally given by (44) (only depending on λ; the variables (y,η) do not occur in this case).

2 Corner boundary value problems

2.1 Corner Sobolev spaces

We now introduce corner Sobolev spaces with double weights in the two axial directions $r \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$. Let H be a Hilbert space which is endowed with a strongly continuous group of isomorphisms $\kappa_\lambda : H \to H$, $\lambda \in \mathbb{R}_+$, such that $\kappa_{\lambda\lambda'} = \kappa_\lambda\kappa_{\lambda'}$ for all $\lambda,\lambda' \in \mathbb{R}_+$; in this case we say that $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a group action on H.

Recall from Definition 1.1 that the space $\mathcal{W}^s(\mathbb{R}^q,H)$ is the completion of $\mathcal{S}(\mathbb{R}^q,H)$ with respect to the norm $\left\{\int \|\langle\eta\rangle^s\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H d\eta\right\}^{\frac{1}{2}}$. More generally, if H is a Fréchet space, written as a projective limit of Hilbert spaces H^j, $j \in \mathbb{N}$, with continuous embeddings $H^{j+1} \hookrightarrow H^j$ for all j, such that a group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, first given on the space H^0, restricts to a group action on H^j for every j, we say that $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a group action on H. Then we define $\mathcal{W}^s(\mathbb{R}^q,H)$ as the projective limit of the spaces $\mathcal{W}^s(\mathbb{R}^q,H^j)$ over $j \in \mathbb{N}$.

This construction will be applied, in particular, to $H = \mathcal{K}^{s,\gamma}(N^\wedge)$ for some compact C^∞ manifold N (with or without boundary), where the group action is defined by $(\kappa_\lambda u)(r,x) = \lambda^{\frac{n+1}{2}}u(\lambda r,x)$, $\lambda \in \mathbb{R}_+$; $n = \dim N$. The resulting spaces

$$\mathcal{W}^{s,\gamma}(N^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{s,\gamma}(N^\wedge)) \tag{46}$$

are called weighted edge Sobolev spaces. Spaces of this kind can also be globally defined on a manifold W with edge Y, locally near points of Y modelled on wedges $N^\triangle \times \Omega$, where $y \in \Omega \subseteq \mathbb{R}^q$ are local coordinates on Y and $N^\triangle = (\overline{\mathbb{R}}_+ \times N)/(\{0\}\times N)$. We shall work with the stretched manifold $\mathbb{W}$ of W which is obtained from $W \setminus Y$ by attaching the base manifold N over points of Y such that $\mathbb{W}$ is locally near the singular subset $\mathbb{W}_{\mathrm{sing}}$ of the form $\overline{\mathbb{R}}_+ \times N$ while $\mathbb{W}\setminus\mathbb{W}_{\mathrm{sing}} =: \mathbb{W}_{\mathrm{reg}}$ is diffeomorphic to the manifold $W\setminus Y$ (as smooth manifolds with boundary in the case of N with boundary).

Abstract corner Sobolev spaces $\mathcal{V}^{s,\delta}(\mathbb{R}_+ \times \mathbb{R}^q,H)$ of smoothness $s \in \mathbb{R}$ and corner weight $\delta \in \mathbb{R}$ (first for a Hilbert space H with group action) are defined as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^q,H)$ with respect to the norm

$$\left\{(2\pi i)^{-1}\int_{\Gamma_{\frac{m+1}{2}-\delta}}\int_{\mathbb{R}^q}\langle w,\eta\rangle^{2s}\|\kappa^{-1}_{\langle w,\eta\rangle}(M_{t\to w}F_{y\to\eta}u)(w,\eta)\|^2_H dw đ\eta\right\}^{1/2}$$

with $M_{t\to w}$ being the Mellin transform in the corner axis variable $t \in \mathbb{R}_+$ and $w \in \mathbb{C}$ the corresponding covariable, $F_{y\to\eta}$ the Fourier transform in $y \in \mathbb{R}^q$. The number $m \in \mathbb{N}$ is an information which is given together with the space H; in the case $H = \mathcal{K}^{s,\gamma}(N^\wedge)$ we set $m = n+1$ for $n = \dim N$. Corner spaces of that kind have been introduced in [18], see also [17].

In analogy of (46) we form the spaces

$$\mathcal{V}^{s,\delta}(\mathbb{R}_+ \times \mathbb{R}^q, \mathcal{K}^{s,\gamma}(N^\wedge))$$

and the corresponding global variants $\mathcal{V}^{s,\delta}(\mathbb{R}_+ \times Y; \mathcal{K}^{s,\gamma}(N^\wedge))$ by using charts on Y and the invariance of the local definition under transition diffeomorphisms. Finally, if $\mathbb{W}$ is a compact (stretched) manifold with edge Y we form the spaces

$$\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{W}^\wedge) := \{\omega v + \chi h : v \in \mathcal{V}^{s,\delta}(\mathbb{R}_+ \times Y, \mathcal{K}^{s,\gamma}(N^\wedge)), \quad h \in \mathcal{H}^{s,\delta}(M^\wedge)\}, \tag{47}$$

where M denotes the double of $\mathbb{W}$ which is a smooth compact manifold (with boundary if so is the base N of the local model cone for $\mathbb{W}$); moreover, ω is a cut-off function on $\mathbb{W}$ (i.e., C^∞ and supported in a neighbourhood of $\mathbb{W}_{\mathrm{sing}}$ and equal to 1 near $\mathbb{W}_{\mathrm{sing}}$), and $\chi = 1 - \omega$.

Note, in particular, that the construction of spaces of the kind (47) also works for a (stretched) manifold $\mathbb{D}$ with conical singularities and boundary, i.e., there are also the spaces $\mathcal{V}^{s,(\gamma,\delta)}(\mathbb{D}^\wedge)$. In the applications below we employ generalisations of these spaces to distributional sections of vector bundles.

2.2 Operators in the model corner near singular crack points

The next step of the construction of the crack operator calculus is to build up the corresponding operator-valued amplitude functions near the singular point v of the crack boundary ∂S.

First we have the space $\mathcal{B}^{\mu,\mathrm{d}}(I; \boldsymbol{v}; \mathbb{R}^l)$ of parameter-dependent boundary value problems on an interval $I = [0, 2\pi]$, with parameter $\lambda \in \mathbb{R}^l$. These operator families form the raw material of the operator-valued amplitude functions near the smooth part of the crack boundary, cf. [6]. They give us local descriptions of the crack operators represented by the given problem with two-sided boundary conditions and additional conditions of trace and potential type along the smooth crack boundary. In a neighbourhood of the conical point of the crack boundary we apply the corner calculus of boundary value problems which is a version of the theory from [18] for the case of boundary value problems instead of a 'closed' corner manifold. Here, for simplicity, we assume that our crack configuration is of dimension 3, i.e., the crack boundary is of dimension 1. By introducing suitable local coordinates we may assume that S locally near the conical point v is represented by a two-dimensional cone in $\mathbb{R}^3$ of the form

$$\{x \in \mathbb{R}^3 : x = 0 \quad \text{or} \quad x/|x| \in Z\} \tag{48}$$

where Z is a (closed) smooth curve on S^2 of finite length, with two end points ι_0 and ι_1 and without self-intersection.

The sphere S^2 with the embedded curve Z can be interpreted as a two-dimensional crack situation, where on the crack Z we impose boundary conditions from both sides which satisfy the Shapiro-Lopatinskij conditions with respect to an elliptic operator given on S^2. The calculus of elliptic boundary value problems on $S^2 \backslash Z$ with such conditions is a special case of the calculus of boundary value problems on a manifold with conical singularities and boundary. The conical singularities here are the points ι_0 and ι_1, the boundary consists of two copies $Z_\pm$ of Z (where the respective points ι_k from the $\pm$ sides are identified) and the local model of the configuration near ι_k is $(\overline{\mathbb{R}}_+ \times [0, 2\pi])/(\{0\} \times [0, 2\pi])$, $k = 0, 1$.

Let S^2_{crack} denote the configuration consisting of $(S^2 \setminus \text{int}Z) \cup \text{int}Z_- \cup \text{int}Z_+$ where the two sides of $\text{int}Z$ are distinguished as different parts of the corresponding boundary which has the conical points ι_0, ι_1. For a general (stretched) manifold $\mathbb{D}$ with conical singularities and boundary we have the parameter-dependent calculus of operators of the space $\mathcal{C}^{\mu,\mathrm{d}}(\mathbb{D}; \boldsymbol{w}; \mathbb{R}^l_\lambda)$, the cone algebra of boundary value problems on $\mathbb{D}$, with the parameter λ. Here $\boldsymbol{w} := (E, F; H, J; m, m')$ are bundle data in the corresponding operators, i.e., $E, F \in \text{Vect}(\mathbb{D})$, $H, J \in \text{Vect}(\mathbb{B})$ where $\mathbb{B}$ is the stretched manifold belonging to the boundary of $\mathbb{D}$.

We can specify this to $\mathbb{D} := S^2_{\text{crack}}$ with ι_0 and ι_1 as conical points and therefore have $\mathcal{C}^{\mu,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}; \mathbb{R}^l)$. We will only need the cases $l = 0$ or $l = 1$. The bundle data $\boldsymbol{w}$ are as follows. Let $\mathbb{Z}$ denote the configuration which consists of two copies Z_- and Z_+ of the curve Z where the respective end points are identified. Then we set $\boldsymbol{w} = (E, F; H, J; m, m')$ for $E, F \in \text{Vect}(S^2_{\text{crack}})$, $H, J \in \text{Vect}(\mathbb{Z}_{\text{reg}})$ (for $\mathbb{Z}_{\text{reg}} = \text{int}\mathbb{Z}_- \cup \text{int}\mathbb{Z}_+$). The set $\mathbb{Z}_{\text{reg}}$ consists of two connected components; so the elements H and J consist of pairs of bundles (which are of course trivial in this case), namely $H_\pm$ and $J_\pm$, the restrictions of H and J, respectively, to $\text{int}Z_\pm$. The stretched cone $\mathbb{Z}^\wedge = \mathbb{R}_+ \times \mathbb{Z}$ contains two border lines $\mathbb{I}^\wedge := \mathbb{R}_+ \cup \mathbb{R}_+$ (disjoint union belonging to the end points ι_0 and ι_1.)

Let $\mathcal{C}^{\mu,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}; \mathbb{C})$ be the space of all $h(w) \in \mathcal{A}(\mathbb{C}, \mathcal{C}^{\mu,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}))$ such that

$$h(\delta + i\tau) \in \mathcal{C}^{\mu,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}; \mathbb{R}^1_\tau)$$

for every $\delta \in \mathbb{R}$, uniformly in $\delta \in [c', c'']$ for every $c' \leq c''$. We also write $\mathcal{C}^{\mu,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}; \Gamma_\delta)$ for the space of all $f(w) \in \mathcal{C}^{\mu,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}; \mathbb{R}^1_\tau)$ if (τ, λ) for $\tau = \text{Im}\ w$ plays the role of the parameter.

Theorem 2.1. *For every $f(w) \in \mathcal{C}^{\mu,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}; \Gamma_\delta)$ there exists an element*

$$h(w) \in \mathcal{C}^{\mu,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}; \mathbb{C})$$

such that

$$h(\delta + i\tau) = f(\delta + i\tau) \mod \mathcal{C}^{-\infty,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}; \mathbb{R}^1_\tau).$$

Applying the construction of (47) to $\mathbb{W} = S^2_{\text{crack}}$ (modified for distributional sections of a vector bundle E on $(S^2_{\text{crack}})^\wedge$ we obtain the spaces

$$\mathcal{V}^{s,(\gamma,\delta)}((S^2_{\text{crack}})^\wedge, E).$$

Proposition 2.2. *Given an element $f(w) \in \mathcal{C}^{\mu,\mathrm{d}}(S^2_{\text{crack}}; \boldsymbol{w}; \Gamma_{1-\delta})$ the operator $\text{op}_M^{1-\delta}(f)$ induces continuous maps*

$$\text{op}_M^{1-\delta}(f) : \begin{array}{c} \mathcal{V}^{s,(\gamma,\delta)}((S^2_{\text{crack}})^\wedge, E) \\ \oplus \\ \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{Z}^\wedge, H) \\ \oplus \\ \mathcal{H}^{s-1,\delta-1}(\mathbb{I}^\wedge, \mathbb{C}^m) \end{array} \to \begin{array}{c} \mathcal{V}^{s-\mu,(\gamma-\mu,\delta)}((S^2_{\text{crack}})^\wedge, F) \\ \oplus \\ \mathcal{V}^{s-\mu-\frac{1}{2},(\gamma-\frac{1}{2}-\mu,\delta-\frac{1}{2}-\mu)}(\mathbb{Z}^\wedge, J) \\ \oplus \\ \mathcal{H}^{s-1-\mu,\delta-1-\mu}(\mathbb{I}^\wedge, \mathbb{C}^{m'}) \end{array} \tag{49}$$

for every real $s > \mathrm{d} - \frac{1}{2}$.

Similarly as the spaces of smoothing Mellin symbols (38) in the cone algebra of boundary value problems we need smoothing Mellin symbols in the corner operator algebra on $(S^2_{\mathrm{crack}})^\wedge$ with a control of weights in a prescribed ε-strip around a given weight line Γ_β. Let

$$\mathcal{C}^{-\infty,\mathrm{d}}(S^2_{\mathrm{crack}};\boldsymbol{w};\Gamma_\beta)_\varepsilon \tag{50}$$

for some $\varepsilon > 0$ denote the space of all

$$f(w) \in \mathcal{A}(\{\beta-\varepsilon < \mathrm{Re} w < \beta+\varepsilon\},\ \mathcal{C}^{-\infty,\mathrm{d}}(S^2_{\mathrm{crack}};\boldsymbol{w}))$$

such that

$$f(\delta+i\tau) \in \mathcal{C}^{-\infty,\mathrm{d}}(S^2_{\mathrm{crack}};\boldsymbol{w};\mathbb{R}_\tau)$$

for every $\delta \in (\beta-\varepsilon,\beta+\varepsilon)$, and uniformly in δ for every compact subinterval.

A Green operator $\mathcal{G}$ in the corner algebra on $(S^2_{\mathrm{crack}})^\wedge$ belonging to the weight data $((\gamma,\delta);(\gamma-\mu,\delta-\mu))$ of type 0 is an operator which is continuous as

$$\mathcal{G}: \begin{array}{c} \mathcal{V}^{s,(\gamma,\delta)}((S^2_{\mathrm{crack}})^\wedge,E) \\ \oplus \\ \mathcal{V}^{s',(\gamma-\frac12,\delta-\frac12)}(\mathbb{Z}^\wedge,H) \\ \oplus \\ \mathcal{H}^{s'',\delta-1}(\mathbb{I}^\wedge,\mathbb{C}^m) \end{array} \to \begin{array}{c} \mathcal{V}^{\infty,(\gamma-\mu+\varepsilon,\delta-\mu+\varepsilon)}((S^2_{\mathrm{crack}})^\wedge,F) \\ \oplus \\ \mathcal{V}^{\infty,(\gamma-\frac12-\mu+\varepsilon,\delta-\frac12-\mu+\varepsilon)}(\mathbb{Z}^\wedge,J) \\ \oplus \\ \mathcal{H}^{\infty,\delta-1-\mu+\varepsilon}(\mathbb{I}^\wedge,\mathbb{C}^{m'}) \end{array}$$

for all $s,s',s'' \in \mathbb{R}$, $s > -\frac12$, and for some $\varepsilon = \varepsilon(\mathcal{G}) > 0$, such that the formal adjoint $\mathcal{G}^*$ has analogous mapping properties. Here the formal adjoint refers to the scalar products in the spaces $\mathcal{V}^{0,(0,0)}((S^2_{\mathrm{crack}})^\wedge,\cdot) \oplus \mathcal{V}^{0,(-\frac12,-\frac12)}(\mathbb{Z}^\wedge,\cdot) \oplus \mathcal{H}^{0,-1}(\mathbb{I}^\wedge,\cdot)$. A Green operator on $(S^2_{\mathrm{crack}})^\wedge$ of type $\mathrm{d} \in \mathbb{N}$ is an operator of the form

$$\mathcal{G} = \mathcal{G}_0 + \sum_{j=1}^{\mathrm{d}} \mathcal{G}_j \mathrm{diag}(T^j,0,0)$$

for arbitrary Green operators $\mathcal{G}_j$ on $(S^2_{\mathrm{crack}})^\wedge$ of type 0 in the former sense and a differential operator T of first order which differentiates transversally to the boundary components $(\mathrm{int} Z_\pm)^\wedge$.

Let us interpret for the moment $(S^2_{\mathrm{crack}})^\wedge = \mathbb{R}_+ \times S^2_{\mathrm{crack}}$ as a cylinder C_{crack} without any special attention for $t \to 0$. This cylinder is then a crack configuration with a crack $\mathbb{B} := \mathbb{R}_+ \times \mathbb{Z}$ with smooth boundary $\mathbb{J}$ consisting of two copies of $\mathbb{R}_+$ (although not compact). The crack theory from [6, Chapter 5] then gives rise to a corresponding crack operator algebra constituted by spaces $\mathcal{C}^{\mu,\mathrm{d}}(C_{\mathrm{crack}};\boldsymbol{w})$ of continuous operators

$$\mathcal{A}: \begin{array}{c} \mathcal{W}^{s,\gamma}_{\mathrm{comp}}(C_{\mathrm{crack}},E) \\ \oplus \\ \mathcal{W}^{s-\frac12,\gamma-\frac12}_{\mathrm{comp}}(\mathbb{B},H) \\ \oplus \\ H^{s-1}_{\mathrm{comp}}(\mathbb{J},\mathbb{C}^m) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathrm{loc}}(C_{\mathrm{crack}},F) \\ \oplus \\ \mathcal{W}^{s-\frac12-\mu,\gamma-\frac12-\mu}_{\mathrm{loc}}(\mathbb{B},J) \\ \oplus \\ H^{s-1-\mu}_{\mathrm{loc}}(\mathbb{J},\mathbb{C}^{m'}) \end{array}.$$

The ‘comp’ and ‘loc’ notation in the $\mathcal{W}^{s,\gamma}$-spaces admits elements to have their support ‘up to’ the edge and the boundary, however, only for corresponding compact sets.

The corner algebra on $(S^2_{\mathrm{crack}})^\wedge$ is the union of all spaces $\mathcal{C}^{\mu,\mathrm{d}}((S^2_{\mathrm{crack}})^\wedge;\boldsymbol{w})$ over $\mu\in\mathbb{Z}$, $\mathrm{d}\in\mathbb{N}$, defined as follows: $\mathcal{C}^{\mu,\mathrm{d}}((S^2_{\mathrm{crack}})^\wedge;\boldsymbol{w})$ is the subspace of all operators

$$\mathcal{A}\in\mathcal{C}^{\mu,\mathrm{d}}(C_{\mathrm{crack}};\boldsymbol{w}),$$

such that for arbitrary cut-off functions $\omega(t),\tilde{\omega}(t)$, we have

$$\omega\mathcal{A}\tilde{\omega}=\omega r^{-\mu}\mathrm{op}_M^{\delta-1}(h)\tilde{\omega}+\omega r^{-\mu}\mathrm{op}_M^{\delta-1}(f)\tilde{\omega}+\mathcal{G} \tag{51}$$

for certain elements

$$h(t,w)\in C^\infty(\overline{\mathbb{R}}_+,\mathcal{C}^{\mu,\mathrm{d}}(S^2_{\mathrm{crack}};\boldsymbol{w};\mathbb{C})),\, f(w)\in\mathcal{C}^{-\infty,\mathrm{d}}(S^2_{\mathrm{crack}};\boldsymbol{w};\Gamma_{1-\delta})_\varepsilon$$

and a Green operator $\mathcal{G}$ in the abovementioned sense, for some $\varepsilon=\varepsilon(\mathcal{A})>0$.

The principal symbolic hierarchy of elements

$$\mathcal{A}=(\mathcal{C}_{ij})_{i,j=1,2,3}\in\mathcal{C}^{\mu,\mathrm{d}}((S^2_{\mathrm{crack}})^\wedge;\boldsymbol{w})$$

consists of tuples

$$\sigma(\mathcal{A})=(\sigma_\psi(\mathcal{A}),\sigma_\partial(\mathcal{A}),\sigma_\wedge(\mathcal{A}),\sigma_{\mathrm{c}}(\mathcal{A})).$$

Here $\sigma_\psi(\mathcal{A})=\sigma_\psi(\mathcal{A}_{11})$ is the principal symbol of $\mathcal{A}_{11}$ as a classical (corner degenerate) pseudo-differential operator of order μ. Moreover, $\sigma_\partial(\mathcal{A})=\sigma_\partial((\mathcal{A}_{ij})_{i,j=1,2})$ is the principal boundary symbol of the 2×2 upper left corner of $\mathcal{A}$; it refers to the two-sided boundary conditions on $\mathbb{R}_+\times\mathrm{int}Z_\pm$; thus $\sigma_\partial(\mathcal{A})$ consists of two components $\sigma_{\partial,\pm}(\mathcal{A})$ belonging to the corresponding $\pm$ sides. In the general definition of the space $\mathcal{C}^{\mu,\mathrm{d}}((S^2_{\mathrm{crack}})^\wedge;\boldsymbol{w})$ we have assumed, for convenience, that the orders of these conditions are the same on both sides. In the examples of Section 3.1 below we admit independent orders. Furthermore, $\sigma_\wedge(\mathcal{A})$ is the principal edge symbol, expressed in the sense of edge boundary value problems of the class $\mathcal{C}^{\mu,\mathrm{d}}(C_{\mathrm{crack}};\boldsymbol{w})$, belonging to the edges $\mathbb{R}_+\times\{\iota_k\}$, $k=0,1$. Finally, $\sigma_{\mathrm{c}}(\mathcal{A})$ is the corner conormal symbol of the operator $\mathcal{A}$, cf. also [8].

The components $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$ are as usual in the calculus of boundary value problems, see also Section 3.1 below. The principal edge symbol consists of a pair of families of operators $\sigma_\wedge(\mathcal{A})=(\sigma_{\wedge,0}(\mathcal{A}),\ \sigma_{\wedge,1}(\mathcal{A}))$ with $\sigma_{\wedge,k}(\mathcal{A})$ belonging to $\mathbb{R}_+\times\{\iota_k\}$, $k=0,1$, which are of the form

$$\sigma_{\wedge,k}(\mathcal{A})(t,\tau):\begin{array}{c}\mathcal{K}^{s,\gamma}(\mathbb{R}^2\setminus\mathbb{R}_+,E)\\ \oplus\\ \mathcal{K}^{s-\frac12,\gamma-\frac12}(\mathbb{R}_+,H_-)\\ \oplus\\ \mathcal{K}^{s-\frac12,\gamma-\frac12}(\mathbb{R}_+,H_+)\\ \oplus\\ \mathbb{C}^m\end{array}\to\begin{array}{c}\mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}^2\setminus\mathbb{R}_+,F)\\ \oplus\\ \mathcal{K}^{s-\frac12-\mu,\gamma-\frac12-\mu}(\mathbb{R}_+,J_-)\\ \oplus\\ \mathcal{K}^{s-\frac12-\mu,\gamma-\frac12-\mu}(\mathbb{R}_+,J_+)\\ \oplus\\ \mathbb{C}^{m'}\end{array}, \tag{52}$$

$s>\mathrm{d}-\frac12$, $(t,\tau)\in\mathbb{R}_+\times(\mathbb{R}\setminus\{0\})$.

Here, in abuse of notation, we wrote the bundles E,F, etc., also in the $\mathcal{K}^{s,\gamma}$-spaces, although these bundles are suitable restrictions of the original ones, combined with pull backs to infinite cylinders; the subscripts '$\pm$' at H and J indicate the '$\pm$ sides' of the boundary of the infinite cone $\mathbb{R}^2\setminus\mathbb{R}_+$ (consisting of two copies of $\mathbb{R}_+$ belonging in $\mathbb{R}^2 \setminus \mathbb{R}_+$ to the angles 0 and 2π in polar coordinates).

Let us point out once again that $\mathbb{R}_+ \times \{\iota_k\}$ are the two edges starting from the corner point, while $\mathbb{R}_+$ in the spaces in the formula (52) represents the intersection of the crack with the two-dimensional normal plane to $\mathbb{R}_+ \times \{\iota_k\}$.

The corner conormal symbol of the operator $\mathcal{A}$ is defined as the operator family

$$\sigma_{\mathrm{c}}(\mathcal{A})(w) := h(0,w) + f(w)$$

with the Mellin symbols contained in (51), and w running on the weight line $\Gamma_{\frac{3}{2}-\delta} \ni w$ (see also the explanations in Section 3.1 below),

$$\sigma_{\mathrm{c}}(\mathcal{A})(w) : \begin{array}{c} \mathcal{H}^{s,\gamma}(S^2_{\text{crack}}, E) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{Z}, H) \\ \oplus \\ \mathbb{C}^{2m} \end{array} \to \begin{array}{c} \mathcal{H}^{s-2,\gamma-2}(S^2_{\text{crack}}, F) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2}-\mu,\gamma-\frac{1}{2}-\mu}(\mathbb{Z}, J), \\ \oplus \\ \mathbb{C}^{2m'} \end{array}$$

$s > \mathrm{d} - \frac{1}{2}$. Again the notation with the bundles is to be interpreted in the right way in terms of restrictions of the originally given bundles. The weighted Sobolev spaces $\mathcal{H}^{s-\gamma}(S^2_{\text{crack}}, \cdot)$ and $\mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{Z}, \cdot)$ refer to the interpretation of S^2_{crack} as a manifold with conical singularities $\{\iota_0, \iota_1\}$ with boundary and $\mathbb{Z}$ (consisting of two copies $Z_{\mp}$ of Z that are pasted together at ι_0 and ι_1) as a one-dimensional manifold with conical singularities $\{\iota_0, \iota_1\}$.

An element $\mathcal{A} \in \mathcal{C}^{\mu,\mathrm{d}}((S^2_{\text{crack}})^\wedge; \boldsymbol{w})$ is called elliptic, if all components of $\sigma(\mathcal{A})$ are bijective; for $\sigma_\wedge(\mathcal{A})$ that means bijectivity in the representation of (52) in the form $\sigma_{\wedge,k}(\mathcal{A})(t,\tau) = t^{-\mu}\tilde{\sigma}_{\wedge,k}(t,\tilde{\tau})|_{\tilde{\tau}=t\tau}$, where $\tilde{\sigma}_{\wedge,k}(t,\tilde{\tau})$ is bijective for $\tilde{\tau} \neq 0$ up to $t = 0$.

2.3 The crack operator algebra

By the crack operator algebra we understand a calculus of (pseudo-differential) boundary value problems which contains the original crack problems with two-sided boundary conditions together with the parametrices in the elliptic case. The typical novelty here compared with [6] is the corner geometry near the singular points of the crack boundary.

Let us denote by M_{crack} the crack configuration described in Section 1.1. That means M_{crack} is equal to $(\overline{G} \setminus \mathrm{int}S) \cup \mathrm{int}S_- \cup \mathrm{int}S_+$ which is a space with singularities, the boundary ∂G and the crack $\mathbb{S}$ which is defined by two copies $S_\pm$ of S where the boundaries $\partial S_\pm$ are identified. Then $\mathrm{int}\mathbb{S} = \mathrm{int}S_- \cup \mathrm{int}S_+$. The space $M_{\text{crack}} \setminus \partial S$ is a C^∞ manifold with boundary $\partial G \cup \mathrm{int}S_- \cup \mathrm{int}S_+$. Thus on $M_{\text{crack}} \setminus \partial S$ we have the calculus of boundary value problems of the class

$$\mathcal{B}^{\mu,\mathrm{d}}(M_{\text{crack}} \setminus \partial S; \boldsymbol{c})$$

of order μ and type d, $\boldsymbol{c} := (E,F;H,J;K,L)$ for $E,F \in \mathrm{Vect}(\overline{G}), H,J \in \mathrm{Vect}(\mathbb{S}), K,L \in \mathrm{Vect}(\partial G)$. The elements $\mathcal{A}_{\text{int}}$ of that space represent continuous operators

$$\mathcal{A}_{\mathrm{int}} : \begin{array}{c} H^{s}_{\mathrm{comp}}(\overline{G}\setminus S,E) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\mathrm{int}\mathbb{S},H) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\partial G,K) \end{array} \to \begin{array}{c} H^{s-\mu}_{\mathrm{loc}}(\overline{G}\setminus S,F) \\ \oplus \\ H^{s-\frac{1}{2}-\mu}_{\mathrm{loc}}(\mathrm{int}\mathbb{S},J) \\ \oplus \\ H^{s-\frac{1}{2}-\mu}_{\mathrm{loc}}(\partial G,L) \end{array}$$

for all $s > d - \frac{1}{2}$. The principal symbolic hierarchy of these operators consists of tuples

$$\sigma(\mathcal{A}_{\mathrm{int}}) = (\sigma_{\psi}(\mathcal{A}_{\mathrm{int}}),\ \sigma_{\partial,\mathbb{S}}(\mathcal{A}_{\mathrm{int}}),\ \sigma_{\partial,\partial G}(\mathcal{A}_{\mathrm{int}})), \tag{53}$$

where $\sigma_{\psi}(\cdot)$ denotes the usual interior symbol, $\sigma_{\partial,\mathbb{S}}(\cdot)$ the pair of boundary symbols on the $\pm$-sides of $\mathrm{int}\mathbb{S}$, and $\sigma_{\partial,\partial G}(\cdot)$ the boundary symbol on the boundary ∂G of the domain. The picture is analogous to (16); the only difference here is that the boundary symbols split into components according to the parts $S_{\pm}$ and ∂G of the boundary. The ellipticity of an operator $\mathcal{A}_{\mathrm{int}} \in \mathcal{B}^{\mu,\mathrm{d}}(M_{\mathrm{crack}} \setminus \partial S; \boldsymbol{c})$ is determined by the bijectivity of the components of (53).

Furthermore, if $v \in S$ denotes the conical point of the crack, the space $M_{\mathrm{crack}} \setminus \{v\}$ is a manifold with 'smooth' crack $S \setminus \{v\}$ in the sense of [6, Chapter 5]. This gives rise to the corresponding crack algebra $\mathcal{C}^{\mu,\mathrm{d}}(M_{\mathrm{crack}} \setminus \{v\}; \boldsymbol{b})$ for $\boldsymbol{b} := (E,F;H,J;K,L;M,M')$. The elements $\mathcal{A}_{\mathrm{reg}}$ in this calculus represent continuous operators

$$\mathcal{A}_{\mathrm{reg}} : \begin{array}{c} \mathcal{W}^{s,\gamma}_{\mathrm{comp}}(M_{\mathrm{crack}}\setminus\{v\},E) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{\mathrm{comp}}(\mathbb{S}_{\mathrm{reg}},H) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\partial G,K) \\ \oplus \\ H^{s-1}_{\mathrm{comp}}(\partial S\setminus\{v\},M) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathrm{loc}}(M_{\mathrm{crack}}\setminus\{v\},F) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2}-\mu,\gamma-\frac{1}{2}-\mu}_{\mathrm{loc}}(\mathbb{S}_{\mathrm{reg}},J) \\ \oplus \\ H^{s-\frac{1}{2}-\mu}_{\mathrm{loc}}(\partial G,L) \\ \oplus \\ H^{s-1-\mu}_{\mathrm{loc}}(\partial S\setminus\{v\},M') \end{array}$$

for $s > \mathrm{d} - \frac{1}{2}$ and a chosen weight $\gamma \in \mathbb{R}$ which is given and fixed in connection with the operator.

The principal symbolic hierarchy of the operators $\mathcal{A}_{\mathrm{reg}}$ consists of tuples

$$\sigma(\mathcal{A}_{\mathrm{reg}}) = (\sigma_{\psi}(\mathcal{A}_{\mathrm{reg}}),\ \sigma_{\partial,\mathbb{S}}(\mathcal{A}_{\mathrm{reg}}),\ \sigma_{\partial,\partial G}(\mathcal{A}_{\mathrm{reg}}),\ \sigma_{\wedge}(\mathcal{A}_{\mathrm{reg}})). \tag{54}$$

The meaning of the first three components is similar to (52). In the global situation we do not make a difference between the edge symbols for different components of $\sigma_{\wedge}(\cdot)$, so we do not have an extra index k as in (52). The ellipticity of an operator $\mathcal{A}_{\mathrm{reg}} \in \mathcal{C}^{\mu,\mathrm{d}}(M_{\mathrm{crack}} \setminus \{v\}; \boldsymbol{b})$ is defined as the bijectivity of all components of (54) in the same sense as in [6].

The space M_{crack} is a manifold with corner $\{v\}$ and boundary, locally near v modelled on $(S^2_{\mathrm{crack}})^{\wedge}$, locally far from $\{v\}$ on a manifold with edge $\partial S \setminus \{v\}$ and boundary, and locally far from ∂S modelled on a C^{∞} manifold with boundary. We then have the weighted corner spaces $\mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}, E)$ given by $\mathcal{V}^{s,(\gamma,\delta)}((S^2_{\mathrm{crack}})^{\wedge}, E)$ near v, moreover, locally far from $\{v\}$ by the space $\mathcal{W}^{s,\gamma}_{\mathrm{loc}}(M_{\mathrm{crack}} \setminus \{v\}, E)$ and locally far from ∂S by $H^{s}_{\mathrm{loc}}(\overline{G} \setminus S, E)$.

The crack operator space $\mathcal{C}^{\mu,\mathrm{d}}(M_{\mathrm{crack}}; \boldsymbol{b})$ on M_{crack} of order μ and type d and with bundle data

$$\boldsymbol{b} = (E,F;H,J;K,L;M,M')$$

on M_{crack} is defined to be the set of all operators

$$\mathcal{A}+\mathcal{C}: \begin{array}{c} \mathcal{V}^{s,(\gamma,\delta)}(M_{\text{crack}},E) \\ \oplus \\ \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{S},H) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial G,K) \\ \oplus \\ \mathcal{H}^{s-1,\delta-1}(\partial S,M) \end{array} \to \begin{array}{c} \mathcal{V}^{s-\mu,(\gamma-\mu,\delta-\mu)}(M_{\text{crack}},F) \\ \oplus \\ \mathcal{V}^{s-\frac{1}{2}-\mu,(\gamma-\frac{1}{2}-\mu,\delta-\frac{1}{2}-\mu)}(\mathbb{S},J) \\ \oplus \\ H^{s-\frac{1}{2}-\mu}(\partial G,L) \\ \oplus \\ \mathcal{H}^{s-1-\mu,\delta-1-\mu}(\partial S,M') \end{array}, \tag{55}$$

continuous for $s > \mathrm{d}-\frac{1}{2}$ such that if we write $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,\ldots,4}$ the operator $(\mathcal{A}_{ij})_{i,j=1,2,3}$ restricts far from ∂S to an element $\mathcal{A}_{\text{int}}$ in $\mathcal{B}^{\mu,\mathrm{d}}(M_{\text{crack}} \setminus \partial S; \boldsymbol{c})$, moreover $\mathcal{A}$ restricts far from the point $v \in \partial S$ to an element $\mathcal{A}_{\text{reg}}$ in $\mathcal{C}^{\mu,\mathrm{d}}(M_{\text{crack}} \setminus \{v\}; \boldsymbol{b})$, and $(\mathcal{A}_{ij})_{i,j=1,2,4}$ localised near v defines a corner boundary value problem

$$\mathcal{A}_{\text{corner}}: \begin{array}{c} \mathcal{V}^{s,(\gamma,\delta)}((S^2_{\text{crack}})^\wedge,E) \\ \oplus \\ \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{Z}^\wedge,H) \\ \oplus \\ \mathcal{H}^{s-1,\delta-1}(\mathbb{I}^\wedge,M) \end{array} \to \begin{array}{c} \mathcal{V}^{s-\mu,(\gamma-\mu,\delta-\mu)}((S^2_{\text{crack}})^\wedge,F) \\ \oplus \\ \mathcal{V}^{s-\frac{1}{2}-\mu,(\gamma-\frac{1}{2}-\mu,\delta-\frac{1}{2}-\mu)}(\mathbb{Z}^\wedge,J) \\ \oplus \\ \mathcal{H}^{s-1-\mu,\delta-1-\mu}(\mathbb{I}^\wedge,M') \end{array}$$

belonging to the space $\mathcal{C}^{\mu,\mathrm{d}}((S^2_{\text{crack}})^\wedge; \boldsymbol{w})$ in the sense of Section 2.2; $\boldsymbol{w} = (E,F;H,J;M,M')$. These operators are continuous for all $s > \mathrm{d}-\frac{1}{2}$.

Finally, $\mathcal{C}$ is a smoothing operator of type d, which means the following. The operator $\mathcal{C}$ is a sum

$$\mathcal{C} = \mathcal{C}_0 + \sum_{j=1}^{\mathrm{d}} \mathcal{C}_j \operatorname{diag}(T^j,0,0,0)$$

where T is a first order differential operator in $\overline{G}$ which differentiates transversally to ∂G as well as to S (cf., similarly, the formula (12)), and the operators $\mathcal{C}_j$, $j = 0,\ldots,\mathrm{d}$, are smoothing and of type 0. The latter property of an operator $\mathcal{C}$ means the continuity of the map

$$\mathcal{C}: \begin{array}{c} \mathcal{V}^{s,(\delta,\gamma)}(M_{\text{crack}},E) \\ \oplus \\ \mathcal{V}^{s',(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{S},H) \\ \oplus \\ H^{s''}(\partial G,K) \\ \oplus \\ \mathcal{H}^{s''',\delta-1}(\partial S,M) \end{array} \to \begin{array}{c} \mathcal{V}^{\infty,(\delta-\mu+\varepsilon,\gamma-\mu+\varepsilon)}(M_{\text{crack}},F) \\ \oplus \\ \mathcal{V}^{\infty,(\gamma-\frac{1}{2}-\mu+\varepsilon,\delta-\frac{1}{2}-\mu+\varepsilon)}(\mathbb{S},J) \\ \oplus \\ H^{\infty}(\partial G,L) \\ \oplus \\ \mathcal{H}^{\infty,\delta-1-\mu+\varepsilon}(\partial S,M') \end{array} \tag{56}$$

for all $s,s',s'',s''' \in \mathbb{R}$, $s > -\frac{1}{2}$ and some $\varepsilon = \varepsilon(\mathcal{C}) > 0$ and a similar condition for the formal adjoint of $\mathcal{C}$ (referring to the scalar products in the corresponding spaces for $s = s' = s'' = s''' = 0$ and $\delta = \gamma = 0$).

The principal symbolic hierarchy of operators $\mathcal{A} \in \mathcal{C}^{\mu,\mathrm{d}}(M_{\text{crack}}; \boldsymbol{b})$ consists of tuples (5) with $\sigma_\partial(\mathcal{A}) = (\sigma_{\partial,\mathbb{S}}(\mathcal{A}),\ \sigma_{\partial,\partial G}(\mathcal{A}))$.

Recall that the domain G is assumed to be bounded. Then

$$\mathcal{A} \in \mathcal{C}^{\mu,\mathrm{d}}(M_{\text{crack}}; \boldsymbol{b}),\ \tilde{\mathcal{A}} \in \mathcal{C}^{\tilde{\mu},\tilde{\mathrm{d}}}(M_{\text{crack}}; \tilde{\boldsymbol{b}})$$

implies $\mathcal{A}\tilde{\mathcal{A}} \in \mathcal{C}^{\mu+\tilde{\mu},\max(\tilde{\mu}+\mathrm{d},\tilde{\mathrm{d}})}(M_{\mathrm{crack}}; \boldsymbol{b} \circ \tilde{\boldsymbol{b}})$, provided that the weights in the image of $\widetilde{\mathcal{A}}$ fit to those in the domain of $\mathcal{A}$. Here $\tilde{\boldsymbol{b}} = (\widetilde{E}, E; \widetilde{H}, H; \widetilde{K}, K; \widetilde{M}, M)$ and $\boldsymbol{b} \circ \tilde{\boldsymbol{b}} = (\widetilde{E}, F; \widetilde{H}, J; \widetilde{K}, L; \widetilde{M}, M')$, and we have

$$\sigma(\mathcal{A}\widetilde{\mathcal{A}}) = \sigma(\widetilde{\mathcal{A}})\sigma(\widetilde{\mathcal{A}})$$

with componentwise composition, with the rule $(\sigma_{\mathrm{c}}(\mathcal{A})\sigma_{\mathrm{c}}(\widetilde{\mathcal{A}}))(w) := \sigma_{\mathrm{c}}(\mathcal{A})(w + \widetilde{\mu})\sigma_{\mathrm{c}}(\widetilde{\mathcal{A}})(w)$.

2.4 Ellipticity and regularity of solutions

We now turn to ellipticity with extra conditions along the crack boundary and to the regularity of solutions in weighted corner Sobolev spaces.

Let

$$\boldsymbol{b} := (E, F; H, J; K, L; M, M'), \boldsymbol{c} := (E, F; H, J; K, L), \boldsymbol{w} := (E, F; H, J; M, M').$$

Definition 2.3. *An operator $\mathcal{A} \in \mathcal{C}^{\mu,\mathrm{d}}(M_{\mathrm{crack}}; \boldsymbol{b})$, is called elliptic if*

(i) $\mathcal{A}_{\mathrm{int}}$ *is elliptic in* $\mathcal{B}^{\mu,\mathrm{d}}(M_{\mathrm{crack}} \setminus \partial S; \boldsymbol{c})$,

(ii) $\mathcal{A}_{\mathrm{reg}}$ *is elliptic in* $\mathcal{C}^{\mu,\mathrm{d}}(M_{\mathrm{crack}} \setminus \{v\}; \boldsymbol{b})$,

(iii) $\mathcal{A}_{\mathrm{corner}}$ *is elliptic in* $\mathcal{C}^{\mu,\mathrm{d}}((S^2_{\mathrm{crack}})^\wedge; \boldsymbol{w})$.

Remark 2.4. *Note that* (ii) *in Definition* 2.3 *depends on the chosen weight $\gamma \in \mathbb{R}$ and the fibre dimensions $m(\gamma)$ and $m'(\gamma)$ of the bundles $M = M(\gamma)$ and $M' = M'(\gamma)$ may depend on γ. In fact, the cone conormal symbol $\sigma_M\sigma_\wedge(\mathcal{A}_{\mathrm{reg}})(z)$ is required to be bijective on the weight line $\Gamma_{1-\gamma}$ in the z-plane; the dimension of the base of the model cone* (*which is an interval*) *is equal to* 1. *Analogously, the condition* (iii) *in Definition* 2.3 *depends on the weight $\delta \in \mathbb{R}$, because the corner conormal symbol $\sigma_{\mathrm{c}}(\mathcal{A})(w) = \sigma_{\mathrm{c}}(\mathcal{A}_{\mathrm{corner}})(w)$ has to be bijective on the weight line $\Gamma_{\frac{3}{2}-\delta}$ in the w-plane; the dimension of the corner base S^2_{crack} is equal to* 2.

In the following theorem we set $\nu^+ = \max(\nu, 0)$ for some real ν, and

$$\boldsymbol{b}^{-1} := (F, E; J, H; L, K; M', M),$$

$\boldsymbol{b}_l := (E, E; H, H; K, K; M, M)$, $\boldsymbol{b}_r := (F, F; J, J; L, L; M', M')$.

Theorem 2.5. *An elliptic operator $\mathcal{A} \in \mathcal{C}^{\mu,\mathrm{d}}(M_{\mathrm{crack}}; \boldsymbol{b})$ has a parametrix $\mathcal{P} \in \mathcal{C}^{-\mu,(\mathrm{d}-\mu)^+}(M_{\mathrm{crack}}; \boldsymbol{b}^{-1})$ in the sense that $\mathcal{C}_l := \mathcal{I} - \mathcal{P}\mathcal{A}$ and $\mathcal{C}_r := \mathcal{I} - \mathcal{A}\mathcal{P}$ belong to $\mathcal{C}^{-\infty,\mathrm{d}_l}(M_{\mathrm{crack}}; \boldsymbol{v}_l)$ and $\mathcal{C}^{-\infty;\mathrm{d}_r}(M_{\mathrm{crack}}; \boldsymbol{b}_r)$, respectively; here $\mathrm{d}_l = \max(\mu, \mathrm{d}), \mathrm{d}_r = (\mathrm{d} - \mu)^+$. Moreover, an elliptic operator $\mathcal{A} \in \mathcal{C}^{\mu,\mathrm{d}}(M_{\mathrm{crack}}; \boldsymbol{b})$ defines a Fredholm operator*

$$\mathcal{A} : \begin{array}{c} \mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}, E) \\ \oplus \\ \mathcal{V}^{s-\frac{1}{2},(\gamma-\frac{1}{2},\delta-\frac{1}{2})}(\mathbb{S}, H) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial G, K) \\ \oplus \\ \mathcal{H}^{s-1,\delta-1}(\partial S, M) \end{array} \rightarrow \begin{array}{c} \mathcal{V}^{s-\mu,(\gamma-\mu,\delta-\mu)}(M_{\mathrm{crack}}, F) \\ \oplus \\ \mathcal{V}^{s-\frac{1}{2}-\mu,(\gamma-\frac{1}{2}-\mu,\delta-\frac{1}{2}-\mu)}(\mathbb{S}, J) \\ \oplus \\ H^{s-\frac{1}{2}-\mu}(\partial G, L) \\ \oplus \\ \mathcal{H}^{s-1-\mu,\delta-1-\mu}(\partial S, M') \end{array} \qquad (57)$$

for every $s > \max(\mu, \mathrm{d}) - \frac{1}{2}$.

Corollary 2.6. *Let $\mathcal{A} \in \mathcal{C}^{\mu,\mathrm{d}}(M_{\mathrm{crack}}; \boldsymbol{b})$ be an elliptic operator. Then $\mathcal{A}u = f$ with f being in the space on the right of* (57) *and u in the space on the left for some $r > \max(\mu, \mathrm{d}) - \frac{1}{2}$ in place of s entails that u belongs to the space on the left of* (57).

3 Examples and Remarks

3.1 Examples

We now specify our results on general pseudo-differential crack problems for the case of differential operators with differential boundary and crack conditions.

Let A be an elliptic differential operator of order μ,

$$A : H^s(G, E) \to H^{s-\mu}(G, F)$$

for vector bundles $E, F \in \mathrm{Vect}(\overline{G})$. Then A also induces continuous operators

$$A : \mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}, E) \to \mathcal{V}^{s-\mu,(\gamma-\mu,\delta-\mu)}(M_{\mathrm{crack}}, F)$$

for every $s, \gamma, \delta \in \mathbb{R}$ (as before we employ the same notation for bundles over different spaces when they are linked to each other in a natural way; in the case of the Lamé system we simply have a 3×3 system, i.e., the bundles are trivial and of fibre dimension 3). Moreover, consider vectors of trace conditions

$$T_\pm = (T_{\pm,i})_{i=1,\ldots,I}, \quad T_{\pm,i} = r_\pm B_{\pm,i},$$

with differential operators $B_{\pm,i}$ of order $\mu_{\pm,i}$, in a neighbourhood of S, mapping (distributional) sections of E to sections in bundles $\widetilde{J}_{\pm,i}$ (in that neighbourhood). Then, setting $J_{\pm,i} = \widetilde{J}_{\pm,i}|_{\mathrm{int}\, S_\pm}$ we obtain continuous operators

$$T_{\pm,i} : \mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}, E) \to \mathcal{V}^{s-\frac{1}{2}-\mu_{\mu_{\pm,i}},(\gamma-\frac{1}{2}-\mu_{\pm,i},\delta-\frac{1}{2}-\mu_{\pm,i})}(S_\pm, J_{\pm,i})$$

for all $s > \max\{\mu_{\pm,i} + \frac{1}{2} : i = 1, \ldots, I\}$. We assume that the operators $(T_{\pm,i})_{i=1,\ldots,I}$ satisfy the Shapiro-Lopatinskij condition on $S_\pm$ (uniformly up to ∂S from the respective sides). Moreover, let T be a vector of boundary conditions on ∂S, also satisfying the Shapiro-Lopatinskij condition with respect to A. For convenience, after a (pseudo-differential) reduction of orders we assume that T induces continuous operators

$$T : H^s(G, E) \to H^{s-\frac{1}{2}-\mu}(\partial G, L)$$

for $\mu = \mathrm{ord}\, A$, $s > \mu + \frac{1}{2}$, for some $L \in \mathrm{Vect}(\partial G)$. From the definition of the spaces $\mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}, E)$ (which are equal to $H^s(G, E)$ in a neigbourhood of ∂G) it follows that T also induces continuous operators

$$T : \mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}, E) \to H^{s-\frac{1}{2}-\mu}(\partial G, L)$$

for $s > \mu + \frac{1}{2}$. In other words the operator A together with the boundary conditions gives rise to a continuous operator

$$\mathcal{A}_1 : \mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}, E) \to \begin{array}{c} \mathcal{V}^{s-\mu,(\gamma-\mu,\delta-\mu)}(M_{\mathrm{crack}}, F) \\ \oplus \\ \bigoplus_{i=1}^{I} \mathcal{V}^{s-\frac{1}{2}-\mu_{+,i},(\gamma-\frac{1}{2}-\mu_{+,i},\delta-\frac{1}{2}-\mu_{+,i})}(S_+, J_{+,i}) \\ \oplus \\ \bigoplus_{i=1}^{I} \mathcal{V}^{s-\frac{1}{2}-\mu_{-,i},(\gamma-\frac{1}{2}-\mu_{-,i},\delta-\frac{1}{2}-\mu_{-,i})}(S_-, J_{-,i}) \\ \oplus \\ H^{s-\frac{1}{2}-\mu}(\partial G, L) \end{array} \tag{58}$$

for every $s > \max\{\mu_{\pm,i} + \frac{1}{2}, \mu + \frac{1}{2}\}$. Compared with the 3×3 upper left corners of operators (55) in the present case of differential operators we do not need potential operators. Recall that in (55) we have assumed unified orders μ (except for the shift by $\frac{1}{2}$) also in the boundary operators belonging to $S_\pm$ (therefore, it was adequate to represent $S_+ \cup S_-$ by $\mathbb{S}$).

The corner pseudo-differential calculus on $S_\pm$ also contains elliptic reductions of orders. In particular, there are isomorphisms

$$\mathcal{V}^{s-\frac{1}{2}-\mu_{\pm,i},(\gamma-\frac{1}{2}-\mu_{\pm,i},\delta-\frac{1}{2}-\mu_{\pm,i})}(S_\pm, J_{\pm,i}) \to \mathcal{V}^{s-\frac{1}{2}-\mu,(\gamma-\frac{1}{2}-\mu,\delta-\frac{1}{2}-\mu)}(S_\pm, J_{\pm,i}) \tag{59}$$

within the corner algebra on $S_\pm$.

By composing (58) from the left by a corresponding diagonal matrix of order reductions (59) we can pass to the situation of (51). The construction of such isomorphisms is voluminous.

Therefore, it is preferable to avoid such reductions of orders in the concrete examples and to slightly modify the general calculus for the case of different orders in the trace (and also potential) operators. In other words we tacitly employ the crack calculus of Sections 2.3 and 2.4 in a version of different orders as they are generated in (58).

Under the ellipticity assumptions on the operators A and $T_\pm, T$ the operator (58) satisfies the condition (i) of Definition 2.3. For (ii) we have to impose additional conditions of trace and potential type along ∂S (it may happen that only trace or only potential conditions are necessary, or no conditions at all). The existence of such extra conditions is not always guaranteed. The crack boundary plays the role of an edge, and there is a topological obstruction for the existence of edge conditions, cf. [20]. In the present case the edge ∂S has conical singularities. However in this situation the condition is very similar; the only modification is that we have to replace locally near $t = 0$ the edge covariable τ by $t\tau$ ($t \in \mathbb{R}_+$ is the corner axis variable in the notation of Section 2.2 with the dual variable τ). From now on we assume that the abovementioned topological obstruction vanishes for the operator (58) in consideration which is the case in the examples below. Then, according to a variant of a result of [20] for boundary value problems, similarly as (51) there exists an operator

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{K} \\ \mathcal{T} & \mathcal{Q} \end{pmatrix} : \begin{array}{c} \mathcal{V}^{s,(\gamma,\delta)}(M_{\mathrm{crack}}, E) \\ \oplus \\ \mathcal{H}^{s-1,\delta-1}(\partial S, M) \end{array} \to \begin{array}{c} \widetilde{\mathcal{V}}^{s-\mu,(\gamma-\mu,\delta-\mu)}(M_{\mathrm{crack}}, \widetilde{F}) \\ \oplus \\ H^{s-\frac{1}{2}-\mu}(\partial G, L) \\ \oplus \\ \mathcal{H}^{s-1-\mu,\delta-1-\mu}(\partial S, M') \end{array} \tag{60}$$

for

$$\widetilde{\mathcal{V}}^{s-\mu,(\gamma-\mu,\delta-\mu)}(M_{\text{crack}},\widetilde{F}) := \begin{array}{c} \mathcal{V}^{s-\mu,(\gamma-\mu,\delta-\mu)}(M_{\text{crack}},F) \\ \oplus \\ \bigoplus_{i=1}^{I} \mathcal{V}^{s-\frac{1}{2}-\mu_{+,i},(\gamma-\frac{1}{2}-\mu_{+,i},\delta-\frac{1}{2}-\mu_{+,i})}(S_+,J_{+,i}) \\ \oplus \\ \bigoplus_{i=1}^{I} \mathcal{V}^{s-\frac{1}{2}-\mu_{-,i},(\gamma-\frac{1}{2}-\mu_{-,i},\delta-\frac{1}{2}-\mu_{-,i})}(S_-,J_{-,i}) \end{array}$$

for a suitable choice of the weight δ such that $\mathcal{A}$ belongs to the crack calculus and satisfies the condition (ii) of Definition 2.3 for all weights δ outside some discrete set of reals. The explanation for the latter effect is a follows. First we can generate the extra crack boundary conditions near the conical singularity $v \in \partial S$ on the level of operator-valued Mellin amplitude functions with ellipticity referring to the edge symbolic component $\sigma_\wedge$. Then, by applying a kernel cut-off argument to the Mellin symbols we can pass to holomorphic amplitude functions in the complex covariable w. This has the consequence that the ellipticity of the crack conditions holds for all weights δ outside a discrete set, since the associated conormal symbol is a holomorphic family of Fredholm operators between spaces on S^2_{crack} and takes values in isomorphisms outside some discrete set. This allows us to apply the corresponding modification of Theorem 2.5 to the operator (60), i.e., we obtain that (60) is a Fredholm operator and has a parametrix in our crack operator calculus.

Let us now have a look at the example (3) for the Laplace operator in 3 dimensions, with Dirichlet conditions T_- and Neumann conditions T_+ on the respective sides of S. In contrast to the unified choice of orders of the operator on ∂G, cf. the formula (60), we will take the order as in (3).

Theorem 3.1. *For every $\gamma \notin \frac{1}{2}(\mathbb{Z}+\frac{1}{2})$ there exists a discrete set $D_\gamma \subset \mathbb{R}$ such that for every $\delta \in \mathbb{R} \setminus D_\gamma$ the operator* (3) *can be completed to an elliptic operator* (4) *in the corner algebra which defines a Fredholm operator for every $s > \frac{3}{2}$, and there is a parametrix of* (4) *in the corner algebra.*

The existence of a parametrix is a special case of Theorem 2.5 (up to the trivial modification of orders on ∂G). The construction of extra edge conditions on $\partial S \setminus \{v\}$ in abstract terms is nothing other than a corresponding construction of the edge calculus; this is possible, provided that the abovementioned topological obstruction vanishes. This is true in the present problem. In fact, from the point of view of the index of Fredholm families the crack situation with Dirichlet/Neumann conditions is homotopy equivalent to the Zaremba problem as is treated in [2].

To see this we compare the present operator family represented by the principal edge symbol

$$\sigma_\wedge(\mathcal{A}_1)(t,\tau) : \mathcal{K}^{s,\gamma_1}(\mathbb{R}^2 \setminus \mathbb{R}_+) \to \begin{array}{c} \mathcal{K}^{s-2,\gamma_1-2}(\mathbb{R}^2 \setminus \mathbb{R}_+) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2},\gamma_1-\frac{1}{2}}(\mathbb{R}_+) \\ \oplus \\ \mathcal{K}^{s-\frac{3}{2},\gamma_1-\frac{3}{2}}(\mathbb{R}_+) \end{array} \tag{61}$$

for some weight $\gamma_1 \in \mathbb{R}$ with the corresponding principal edge symbol belonging to the Zaremba problem

$$\sigma_\wedge(\mathcal{A}_0)(t,\tau):\mathcal{K}^{s,\gamma_0}(\mathbb{R}^2_+\setminus\{0\})\to \begin{array}{c}\mathcal{K}^{s-2,\gamma_0-2}(\mathbb{R}^2_+\setminus\{0\})\\ \oplus\\ \mathcal{K}^{s-\frac{1}{2},\gamma_0-\frac{1}{2}}(\mathbb{R}_+)\\ \oplus\\ \mathcal{K}^{s-\frac{3}{2},\gamma_0-\frac{3}{2}}(\mathbb{R}_+)\end{array} \tag{62}$$

for another weight $\gamma_0\in\mathbb{R}$, cf. [2, formula (47), where the half-axis for the Dirichlet side was denoted by $\mathbb{R}_-$]. It follows that there is a homotopy through Fredholm families

$$\sigma_\wedge(\mathcal{A}_r)(t,\tau):\mathcal{K}^{s,\gamma_r}(K_r)\to \begin{array}{c}\mathcal{K}^{s-2,\gamma_r-2}(K_r)\\ \oplus\\ \mathcal{K}^{s-\frac{1}{2},\gamma_r-\frac{1}{2}}(\mathbb{R}_+)\\ \oplus\\ \mathcal{K}^{s-\frac{3}{2},\gamma_r-\frac{3}{2}}(\mathbb{R}_+)\end{array} \tag{63}$$

when we choose a weight

$$\gamma_0\notin\mathbb{Z}+\frac{1}{2},\quad \gamma_0\in(\frac{1}{2}-k,\frac{3}{2}-k)$$

for some $k\in\mathbb{Z}$ and set

$$\gamma_r=\frac{\gamma_0}{1+r}\in\frac{1}{1+r}(\frac{1}{2}-k,\frac{3}{2}-k),\ 0\le r\le 1, \tag{64}$$

(the notation in the formula (64) means that the end points of the interval are multiplied by $(1+r)^{-1}$); here $K_r:=\{(x_1,x_2)\in\mathbb{R}^2\setminus\{0\}:(x_1,x_2)=|x_1,x_2|e^{i\phi},\ 0\le\phi\le\phi_r:=(1+r)\pi\}$.

For $r=1$ we distinguish the angles 0 and 2π such that $\mathcal{K}^{s,\gamma}(K_1)\cong\mathcal{K}^{s,\gamma}(\mathbb{R}^2\setminus\mathbb{R}_+)$.

Then, as a corollary of [2, formula (55)] it follows that

$$\mathrm{ind}\sigma_\wedge(\mathcal{A}_r)=k\quad\text{for all}\quad 0\le r\le 1,$$

in particular, ind $\sigma_\wedge(\mathcal{A}_1)=k$ for

$$\gamma_1\in\frac{1}{2}(\frac{1}{2}-k,\frac{3}{2}-k). \tag{65}$$

In fact, the weights γ_r for which the operators (63) are Fredholm for $\tau\neq 0$ are determined by the non-bijectivity points $z\in\mathbb{C}$ of the subordinate conormal symbol

$$\sigma_M\sigma_\wedge(\mathcal{A}_r)(z)=\begin{pmatrix}\sigma_M\sigma_\wedge(\Delta)\\ \sigma_M\sigma_\wedge(T_-)\\ \sigma_M\sigma_\wedge(T_+)\end{pmatrix}(z):H^s(I_r)\to\begin{array}{c}H^{s-2}(I_r)\\ \oplus\\ \mathbb{C}\oplus\mathbb{C}\end{array}$$

for $I_r:=\{\phi:0<\phi<(1+r)\pi\}$. Here $\sigma_M\sigma_\wedge(\Delta)(z)=\frac{\partial^2}{\partial\phi^2}+z^2$, and

$$\sigma_M\sigma_\wedge(T_-)u=u|_{\phi=0},\quad \sigma_M\sigma_\wedge(T_+)u=\frac{\partial u}{\partial\phi}\big|_{\phi=(1+r)\pi}.$$

Writing $z = a+ib$ we easily see that for $z \neq 0$ the kernel of $\sigma_M \sigma_\wedge(\Delta)(z)$ consists of all functions

$$u(\phi) = \{c_1 e^{-b\phi} e^{ia\phi} + c_2 e^{b\phi} e^{-ia\phi} : c_1, c_2 \in \mathbb{C}\}.$$

Now $u(0) = 0$ for such a function implies $c := c_1 = -c_2$. Moreover, we have

$$u'(\phi) = c(-b+ia)(e^{-b\phi} e^{ia\phi} + e^{b\phi} e^{-ia\phi}).$$

Assuming $c \neq 0$ (otherwise we have $u \equiv 0$) from $u'(\phi_r) = 0$ we obtain the condition

$$\begin{aligned} & e^{-b\phi_r}\{\cos(a\phi_r) + i\sin(a\phi_r)\} + e^{b\phi_r}\{\cos(-a\phi_r) + i\sin(-a\phi_r)\} \\ = \; & (e^{-b\phi_r} + e^{b\phi_r})\cos(a\phi_r) + i(e^{-b\phi_r} - e^{b\phi_r})\sin(a\phi_r) = 0. \end{aligned}$$

Since $e^{-b\phi} + e^{b\phi}$ never vanishes we obtain $\cos(a\phi_r) = 0$; then $\sin(a\phi_r) \neq 0$ yields $e^{-b\phi_r} - e^{b\phi_r} = 0$ and then $b = 0$. This gives us $a\phi_r = (k+\frac{1}{2})\pi$, $k \in \mathbb{Z}$, i.e., $a = (1+r)^{-1}(k+\frac{1}{2})$. In other words, the non-bijectivity points of $\sigma_M \sigma_\wedge(\mathcal{A}_r)(z)$ are

$$\{z \in \mathbb{C} : \operatorname{Im} z = 0, \quad \operatorname{Re} z = \frac{1}{1+r}(k+\frac{1}{2}),\ k \in \mathbb{Z}\}.$$

Remark 3.2. *Weights $\gamma = \gamma_1 \in \mathbb{R}$ which satisfy the condition* (65) *are an admissible choice for the result of Theorem* 3.1*, and the vector bundles M and M' are trivial and of fibre dimension $m(\gamma)$ and $m'(\gamma)$, respectively where*

$$k = m'(\gamma) - m(\gamma) \quad \textit{for} \quad \gamma \in \frac{1}{2}(\frac{1}{2} - k, \frac{3}{2} - k).$$

In other words, we have calculated the number of the additional trace and potential conditions on the crack boundary, more precisely, the difference of these numbers.

In a similar manner we can treat the case when instead of Dirichlet conditions on one side, Neumann conditions on the other we have Dirichlet or Neumann conditions on both sides.

3.2 Asymptotics

In this section we make some concluding remarks on the role of the 'weight improvement' parameter $\varepsilon > 0$ and refinements of the calculus with a control of asymptotic data.

The method which has been applied here may be regarded as a 'confication' of the edge algebra of boundary value problems. The meaning of the notation 'edge algebra' depends on some details concerning the choice of amplitude functions and asymptotic data. The edge amplitude functions are operator families with values in the cone algebra of boundary value problems on the infinite model cone. As such they contain so called smoothing Mellin and Green edge amplitude functions. Those are connected with the chosen control of asymptotics in the underlying cone calculus. This control can either mean, for instance, that the conormal symbols are meromorphic operator functions with values in boundary value problems on the base of the model cone

(here a compact C^∞ manifold with boundary), or that they are only given in an open weight strip around the weight line $\Gamma_{\frac{n+1}{2}-\gamma}$ we are looking at. The latter point of view gives rise to a more general calculus, where the Green operators, in contrast to those with a complete control of asymptotics, 'only' map weighted Sobolev distributions to smooth weighted functions with a weight impovement $\varepsilon > 0$, cf. the formula (40).

In other words, what we are doing here in our calculus, is ignoring the possible asymptotic information for $r \to 0$ on the model cones of wedges. For the conified edge theory, i.e., in the corner axis direction $t \in \mathbb{R}_+$, we do the same, i.e., we do not observe asymptotics for $t \to 0$. Also the corner conormal symbols are controlled in an ε-weight strip, cf. the formula (50).

The full asymptotic information for corner boundary value problems of the present type could be analysed as well. The program would be analogous to that of the article [17] which treats iterated corner asymptotics for $r \to 0$ and $t \to 0$ for the case of a closed compact base of the model cones. This is a voluminous program and goes beyond the scope of the present exposition.

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