

# Pseudodifferential Operators<sup>1</sup>

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# **Differential Operators on Manifolds with Singularities**

**Analysis and Topology**

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## Chapter 4

# Pseudodifferential Operators

### 4.1. Preliminary Remarks

Pseudodifferential operators ( $\psi$ DO) are one of the main analytic tools of elliptic theory on smooth manifolds as well as manifolds with singularities. Although it is the latter that are of main interest to us, we speak only about  $\psi$ DO on smooth manifolds in this chapter. On the one hand, this permits us to avoid additional technical complications in describing the main ideas, which will help us motivate the constructions of  $\psi$ DO on manifolds with singularities in Part II. On the other hand, the calculus of  $\psi$ DO on smooth manifolds serves as building material for more complicated calculi; here the most important role is played by  $\psi$ DO acting in spaces of sections of *infinite-dimensional bundles*, which will be clearly seen in Part II.

First of all, we would like to recall some basic facts about pseudodifferential operators and their importance in elliptic theory. This will be done in this section. The subsequent two sections deal with classical  $\psi$ DO on smooth manifolds and  $\psi$ DO with operator-valued symbols ( $\psi$ DO in sections of Hilbert bundles). The index theorem is the subject of the last section.

To avoid misunderstanding, note that  $\psi$ DO have a wide variety of applications and we do not even attempt to give a comprehensive coverage of any aspect of the theory (even with the stipulation that we have focused our attention on the needs of elliptic theory). We refer the readers interested in deep subtleties of  $\psi$ DO theory to Hörmander's classical treatise (Hörmander 1983a)–(Hörmander 1985b).

#### 4.1.1. Why are pseudodifferential operators needed?

Pseudodifferential operators arose at the dawn of elliptic theory as a natural class of operators containing the (almost) inverses of elliptic differential operators. The fact that the solutions of elliptic differential equations with constant coefficients are given by the convolutions of the right-hand sides with appropriate *potentials*, i.e., functions smooth away from zero and having a singularity of certain type at zero had been known in partial differential equations from time immemorial. For equations with variable coefficients, it had also been known that the solutions are given by similar convolutions but with kernels that have an additional (smooth) dependence on the observation point; the corresponding operators were called (and are called up to now) *singular integral operators*. These operators apparently had a nature very different from that of differential operators whose inverses they were, so that an explicit description of the algebra combining both types of operators seemed to be a non-awarding and complicated task. The creators of  $\psi$ DO theory observed that, being rewritten in terms of the Fourier transform, the definition of singular integral operators acquires a very simple form equally suitable for the representation of differential operators (Kohn and Nirenberg 1965), (Hörmander 1965). Operators of this form were called *pseudodifferential*. Furthermore, the notion of the symbol for  $\psi$ DO has a clear, transparent meaning (in

contrast to the seemingly artificial definition of the symbol for singular integral operators), and the product of operators corresponds (modulo lower-order operators) to the product of symbols. If we note that negative-order operators on compact manifolds are compact, then it becomes clear that the introduction of  $\psi$ DO makes the finiteness theorem for elliptic operators almost tautological: if an operator is elliptic (i.e., its principal symbol is invertible), then any  $\psi$ DO whose principal symbol is the inverse of the principal symbol of the original operator is an almost inverse of the latter, whence the Fredholm property follows. Thus, the first fact determining the importance of  $\psi$ DO in elliptic theory is the following:

1. *The composition law for pseudodifferential operators trivializes the finiteness theory for elliptic operators.*

However, the usefulness of  $\psi$ DO in elliptic theory would be still at doubt if this were the only reason. All in all, they had been able to prove the finiteness theorem (even though in a more complicated manner) without resorting to pseudodifferential operators. The genuine value of  $\psi$ DO reveals itself in topological issues of elliptic theory, in particular, in index theory. The proof of the Atiyah–Singer index theorem (e.g., see (Palais 1965)) is based on the reduction of elliptic symbols, which are invertible functions on the cosphere bundle of the manifold in question, to the simplest form by (stable) homotopies, in other words, on the *homotopy classification* of elliptic symbols. Unfortunately, even starting from the symbol of a differential operator, one cannot carry out such a homotopy in the class of symbols of differential operators (i.e., restrictions of polynomial symbols to the cosphere bundle). To conclude that the indices of the operators with symbols corresponding to the beginning and end of the homotopy are the same, we should therefore be able to lift a homotopy of arbitrary elliptic symbols to a homotopy of Fredholm operators. The  $\psi$ DO calculus solves this problem: the symbol of a  $\psi$ DO can be an arbitrary smooth function on the cosphere bundle, and the invertibility of a symbol implies the Fredholm property of the corresponding operator by virtue of the finiteness theorem. Thus we see that

2. *Pseudodifferential operator naturally arise in homotopies used in the classification of elliptic operators.*

Note also that

3. *Pseudodifferential operator naturally arise in the reduction of elliptic boundary value problems to the boundary.*

Finally,  $\psi$ DO often prove useful from a purely technical point of view. For example, in many cases it is more convenient to deal with bounded operators in  $L^2$  than unbounded operators in the same space or even bounded operators in the Sobolev scale. One can reduce an unbounded operator to a bounded one by multiplying it by an appropriate negative power of the Laplace operator. The resulting operators are no longer differential; they are  $\psi$ DO of order zero. Thus

4. *Pseudodifferential operators provide a natural framework for the order reduction procedure.*

The list could readily be continued, but even now it should be clear that pseudodifferential operators make their entrance in elliptic theory not by mere occasion.

#### 4.1.2. What is a pseudodifferential operator?

Now that we are sure that considering  $\psi$ DO is indeed useful, let us briefly recall their construction and main properties. For now, we carry out all constructions locally, in a given coordinate system  $(x_1, \dots, x_n)$  (or, if you wish, in the space  $\mathbb{R}^n$ ).

Let  $\hat{H}$  be a differential operator of order  $\leq m$  of the form

$$\hat{H} = \sum_{|\alpha| \leq m} a_\alpha(x) \left( -i \frac{\partial}{\partial x} \right)^\alpha, \quad (4.1)$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multiindex with integer nonnegative components, the coefficients  $a_\alpha(x) = a_\alpha(x_1, \dots, x_n)$  are smooth functions, and

$$\left( -i \frac{\partial}{\partial x} \right)^\alpha = \left( -i \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( -i \frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$

The operator (4.1) can be treated as a *function*

$$\hat{H} = H \left( x, -i \frac{\partial}{\partial x} \right) \quad (4.2)$$

of the operator of differentiation  $-i\partial/\partial x$  and multiplication by the independent variables  $x$  with symbol

$$H(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \quad (4.3)$$

a polynomial of degree  $\leq m$  in the variables  $\xi$ . (The numbers over operators in (4.2) denote, according to (Maslov 1973), the order in which the operators act: first the function to be acted upon by  $\hat{H}$  is differentiated, and then the results are multiplied by the coefficients and summed. In what follows we usually adopt this standard ordering of operator arguments and omit the numbers over operators.) However, it is not immediate from (4.2), (4.3) how to generalize this definition to nonpolynomial symbols  $H(x, \xi)$ . There are at least two ways to do this, which however lead to equivalent results. First, one can use *noncommutative analysis*, the general theory of functions of noncommuting operators (Maslov 1973) (see also a popular exposition in (Nazaikinskii, Sternin and Shatalov 1995)). In this framework, one represents the symbol  $H(x, \xi)$  in the form

$$H(x, \xi) = \left( \frac{-i}{2\pi} \right)^{n/2} \int \tilde{H}(x, y) e^{-iy\xi} dy, \quad (4.4)$$

where  $\tilde{H}(x, y)$  is the inverse Fourier transform (in the sense of distributions) of the function  $H(x, \xi)$  with respect to the variable  $\xi$  and then replaces  $\xi$  on the right-hand side in (4.4) by the vector operator  $-i\partial/\partial x$  (with regard to the adopted ordering of operators). Thus

$$H \left( x, -i \frac{\partial}{\partial x} \right) \stackrel{\text{def}}{=} \left( \frac{-i}{2\pi} \right)^{n/2} \int \tilde{H}(x, y) e^{-y \frac{\partial}{\partial x}} dy. \quad (4.5)$$

The expression on the right-hand side in (4.5) is meaningful, since the operator exponential is well defined as the translation operator

$$\left[ e^{-y \frac{\partial}{\partial x}} \right] u(x) = u(x - y).$$

The integrand in (4.5) is strongly continuous in each Sobolev space, and the integral is treated in the sense of distributions. (For the case of an operator-valued integrand, this is known as the *Pettis integral*.)

EXERCISE 4.1. Show that the formulas (4.1) and (4.5) give the same result for polynomial symbols (4.3).

◀ *Hint.* The Fourier transform of a polynomial is a linear combination of the delta function at zero and its derivatives. ▶

Second, one can recall the relationship between the Fourier transforms of a function and its derivatives and rewrite (4.1) in the equivalent form

$$\left[ H \left( x, -i \frac{\partial}{\partial x} \right) u \right] (x) = \left( \frac{i}{2\pi} \right)^{n/2} \int H(x, \xi) e^{ix\xi} \tilde{u}(\xi) d\xi, \quad (4.6)$$

where  $\tilde{u}(\xi)$  is the Fourier transform of  $u$ . The right-hand side of (4.6) is well defined without the assumption that  $H(x, \xi)$  is a polynomial.

EXERCISE 4.2. Prove that the formulas (4.5) and (4.6) give the same result if the function  $H(x, \xi)$  sufficiently rapidly decays together with sufficiently many derivatives as  $|\xi| \rightarrow \infty$  (so that all integrals in question converge absolutely).

◀ In fact, these formulas are well defined and give the same result for a much wider symbol class, in particular, for *tempered symbols*, which grow at infinity together with all derivatives no faster than some power of  $|\xi|$ , depending on the symbol (see (Maslov 1973)). ▶

Let us state a boundedness theory for pseudodifferential operators without proof.

THEOREM 4.3. Suppose that for some  $m \in \mathbb{R}$  the symbol  $H(x, \xi)$  satisfies the estimates

$$\left| \frac{\partial^{|\alpha|+|\beta|} H(x, \xi)}{\partial x^\alpha \partial \xi^\beta} \right| \leq C_{\alpha\beta} (1 + |\xi|)^{m-|\beta|}, \quad (x, \xi) \in \mathbb{R}^{2n}, \quad (4.7)$$

for arbitrary multiindices  $\alpha$  and  $\beta$  with constants  $C_{\alpha,\beta}$  that depend only on  $\alpha$  and  $\beta$ . Then the operator  $H(x, -i\partial/\partial x)$  is well defined by either of the formulas (4.5) and (4.6) and continuous in the Sobolev spaces

$$H \left( x, -i \frac{\partial}{\partial x} \right) : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n), \quad s \in \mathbb{R}. \quad (4.8)$$

Thus operators with polynomially growing symbols are continuous in the Sobolev scale. We denote the class of symbols satisfying the estimates (4.7) by  $S^m(\mathbb{R}^{2n})$ .

One of the main results of  $\psi$ DO theory is the *composition formula*, which states that the product of two  $\psi$ DO is again a  $\psi$ DO and permits one to compute the symbol of the latter.

THEOREM 4.4. Let  $H_1(x, \xi) \in S^m(\mathbb{R}^{2n})$  and  $H_2(x, \xi) \in S^l(\mathbb{R}^{2n})$  be given symbols. The product

$$A = H_1\left(x, -i\frac{\partial}{\partial x}\right) \circ H_2\left(x, -i\frac{\partial}{\partial x}\right) \quad (4.9)$$

of pseudodifferential operators is again a pseudodifferential operator, and its symbol  $A(x, \xi)$  has the following asymptotic expansion as  $|\xi| \rightarrow \infty$ :

$$A(x, \xi) \simeq \sum_{k=0}^{\infty} (-i)^k \sum_{|\alpha|=k} \frac{\partial^\alpha H_1(x, \xi)}{\partial \xi^\alpha} \frac{\partial^\alpha H_2(x, \xi)}{\partial x^\alpha}. \quad (4.10)$$

Thus, in the leading term ( $k = 0$ ) the symbol of a product of  $\psi$ DO is just the product of the symbols of the factors.

EXERCISE 4.5. Verify the formula (4.10) for the case of differential operators.

Hörmander's class  $S^m(\mathbb{R}^{2n}) \equiv S_{1,0}^m(\mathbb{R}^{2n})$  of symbols satisfying the estimates (4.7) is unnecessarily wide for most problems in elliptic theory, and one usually exploits only the subset  $\mathcal{S}_{cl}^m(\mathbb{R}^{2n})$  of *classical* symbols possessing an asymptotic expansion in homogeneous functions as  $|\xi| \rightarrow \infty$ :

$$H(x, \xi) \simeq \sum_{j=0}^{\infty} H_j(x, \xi), \quad (4.11)$$

where  $H_j(x, \xi)$  is homogeneous of degree  $m - j$  in  $\xi$ :

$$H_j(x, \lambda\xi) = \lambda^{m-j} H_j(x, \xi), \quad \lambda > 0, \quad |\xi| \neq 0.$$

In particular, almost inverses of elliptic differential operators are *classical  $\psi$ DO*, i.e.,  $\psi$ DO with classical symbols.

The function  $H_0(x, \xi)$  is called the *principal symbol* of the operator  $H(x, -i\partial/\partial x)$ . In the following, by writing  $H_0(x, -i\partial/\partial x)$ , where  $H_0$  is a function homogeneous of degree  $m$  in  $\xi$ , we mean an (arbitrary) operator  $H(x, -i\partial/\partial x)$ , where  $H(x, \xi)$  is some symbol<sup>1</sup> in  $S_{cl}^m(\mathbb{R}^{2n})$  whose principal part is equal to  $H_0$ .

We shall see in the next section how one can transfer the  $\psi$ DO theory from Euclidean space to manifolds. And now let us find out what one should strive for in the construction of a  $\psi$ DO calculus.

### 4.1.3. What properties should the pseudodifferential calculus possess?

Let us state the properties necessary for a  $\psi$ DO calculus to be useful in elliptic theory. To simplify the statements, we restrict ourselves to zero-order operators<sup>2</sup> and use an abstract algebraic language.

A general algebra  $\mathcal{A}$  of zero-order  $\psi$ DO should be a subalgebra of the algebra of bounded operators in a Hilbert space. There should be a *symbol mapping* defined on this algebra, i.e., a homomorphism  $\sigma : \mathcal{A} \rightarrow \mathcal{S}$  into a unital topological algebra  $\mathcal{S}$  such that

<sup>1</sup>One can define such a symbol, say, by the formula  $H(x, \xi) = \psi(\xi)H_0(x, \xi)$ , where  $\psi(\xi)$  is an excision function vanishing near zero and equal to unity at infinity.

<sup>2</sup>As a rule, once the calculus of zero-order  $\psi$ DO has been constructed, the general case is reduced to this by order reduction.



$\sigma(A) = 0$  if and only if  $A$  belongs to the ideal  $\mathcal{K}$  of compact operators.

In other words, the homomorphism  $\sigma$  generates a well defined *monomorphism*

$$\tilde{\sigma} : \mathcal{A}/(\mathcal{K} \cap \mathcal{A}) \longrightarrow \mathcal{S}$$

of the Calkin algebra<sup>3</sup>  $\mathcal{A}/(\mathcal{K} \cap \mathcal{A})$  into  $\mathcal{S}$ . Next, there should be a continuous linear mapping

$$Q : \mathcal{S} \longrightarrow \mathcal{A}$$

such that

$$\sigma Q = \text{id} : \mathcal{S} \longrightarrow \mathcal{S} \text{ (i.e., } Q \text{ is the right inverse of } \sigma).$$

DEFINITION 4.6. Under the above-mentioned conditions, we refer to  $\mathcal{S}$  as the *algebra of (principal) symbols* of elements of the algebra  $\mathcal{A}$ . The element  $\sigma(A)$ , where  $A \in \mathcal{A}$ , will be called the *symbol* of the operator  $A$ . The mapping  $Q$  will be called *quantization*.

DEFINITION 4.7. An element  $A \in \mathcal{A}$  is said to be *elliptic* if its symbol  $\sigma(A)$  is an invertible element of the algebra  $\mathcal{S}$ . Invertible symbols are also called elliptic.

The main analytic fact of elliptic theory is the relationship between ellipticity and the Fredholm property: an operator is Fredholm if and only if its symbol is invertible. Assertions of this kind are known as *finiteness theorems*. The following (entirely trivial) assertion shows that the finiteness theorem is valid in the abstract framework of Definition 4.6.

PROPOSITION 4.8. 1) *Under the above-mentioned conditions, the mapping  $Q$  is a left almost inverse of  $\sigma$  and an almost algebra homomorphism, i.e., a homomorphism modulo compact operators:*

$$Q(\sigma(A)) - A, Q(a)Q(b) - Q(ab) \in \mathcal{K} \quad \text{for any } A \in \mathcal{A}, a, b \in \mathcal{S}. \quad (4.12)$$

2) *An operator  $A \in \mathcal{A}$  is Fredholm if and only if it is elliptic (i.e., if its symbol  $\sigma(A)$  is invertible).*

EXERCISE 4.9. Prove this proposition.

The ordinary invariance properties of the index are also preserved in this abstract framework.

PROPOSITION 4.10. 1) *The index of an elliptic element  $A \in \mathcal{A}$  is completely determined by the principal symbol  $\sigma(A)$ .*

2) *Let  $A_0, A_1 \in \mathcal{A}$  be elements such that their symbols  $\sigma(A_0)$  and  $\sigma(A_1)$  are homotopic in the class of elliptic symbols (i.e., can be joined by a continuous curve  $\sigma_t$ ,  $t \in [0, 1]$ , consisting of elliptic symbols). Then*

$$\text{ind } A_0 = \text{ind } A_1.$$

---

<sup>3</sup>Recall that if  $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$  is a subalgebra of the algebra of bounded operators in a Hilbert space  $\mathcal{H}$ , then the quotient algebra  $\mathcal{A}/\mathcal{A} \cap \mathcal{K}$ , where  $\mathcal{K} \subset \mathcal{B}(\mathcal{H})$  is the ideal of compact operators, is called the Calkin algebra corresponding to  $\mathcal{A}$ .

*Proof.* Indeed, the operator  $A - Q(\sigma(A))$  is compact, so that

$$\text{ind } A = \text{ind } Q(\sigma(A)),$$

whence assertion 1) follows. To prove assertion 2), it suffices to join  $A_0$  and  $A_1$  by a continuous curve  $A_t$  consisting of Fredholm operators, since the index of a Fredholm operator is homotopy invariant. In view of 1), we can assume that  $A_j = Q(\sigma(A_j))$ ,  $j = 0, 1$ , and then the desired curve can be taken in the form  $A_t = Q(\sigma_t)$ .  $\square$

Thus the solution of the index problem for an abstract algebra  $\psi$ DO  $\mathcal{A}$  can be carried out, for example, as follows:

- 1) one obtains a homotopy classification of the set of elliptic symbols;
- 2) one constructs an “index functional”  $\text{ind}_t$  on the set of homotopy classes of elliptic symbols such that the index formula

$$\text{ind } A = \text{ind}_t[\sigma(A)]$$

holds, where  $[\sigma(A)]$  is the homotopy class of an elliptic symbol  $\sigma(A)$ .

*Remark 4.11.* In fact, we have somewhat roughened the situation: the genuine homotopy classification is usually intractable, and one deals with a *stable homotopy classification*, where the equivalence relation on symbols is generated by homotopies *and* by the addition of “trivial” direct summands (for which the index of the corresponding operators is a priori zero). In various implementations of this scheme, the class of trivial symbols is described in different ways.

## 4.2. Classical Pseudodifferential Operators on Smooth Manifolds

### 4.2.1. Definition of pseudodifferential operators on a manifold

The construction of  $\psi$ DO in  $\mathbb{R}^n$  was briefly described in Sec. 4.1.2. Now assume that we wish to consider  $\psi$ DO on a smooth compact manifold  $M$  rather than in  $\mathbb{R}^n$ . How we define them in this case? Let us proceed as follows. We cover the manifold  $M$  by coordinate neighborhoods and define  $\psi$ DO by formulas like (4.6) in each neighborhood. Let us paste the local definitions together. A key point in this pasting is the study of the behavior of  $\psi$ DO under changes of coordinates. It turns out (e.g., see (Hörmander 1965)) that changes of variables take pseudodifferential operators to pseudodifferential operators modulo operators with smooth kernel, and moreover, the leading part of the symbol (the leading term in the asymptotic expansion (4.11) for classical  $\psi$ DO, which will be referred to as the *principal symbol* in what follows) behaves as an invariantly defined function on the cotangent bundle  $T_0^*M$  of  $M$  without the zero section. More precisely, the following assertion holds.

**PROPOSITION 4.12.** *Let  $U, V \subset \mathbb{R}^n$  be given domains, and let  $\psi : U \rightarrow V$  be a diffeomorphism. Next, let  $H(x, \xi) \in S^m(\mathbb{R}^{2n})$  be a symbol such that<sup>4</sup>  $\text{supp}_x H \Subset U$ . Take an arbitrary function  $\chi(x) \in C_0^\infty(U)$  such that  $\chi(x)H(x, \xi) = H(x, \xi)$ . Then the operator*

$$\widehat{G} = (\psi^*)^{-1} \circ H\left(x, -i\frac{\partial}{\partial x}\right) \circ \chi(x) \circ \psi^* \quad (4.13)$$

<sup>4</sup>Here  $\text{supp}_x$  is the projection of the support on the  $x$ -space.

is also a pseudodifferential operator of order  $\leq m$ , i.e.,

$$\widehat{G} = G\left(x, -i\frac{\partial}{\partial x}\right), \quad G \in S^m(\mathbb{R}^{2n}), \quad (4.14)$$

and moreover, the symbol  $G(x, \xi)$  satisfies the relation

$$G\left(\psi(x), \left(\frac{t\partial\psi}{\partial x}\right)^{-1} \xi\right) - H(x, \xi) \in S^{m-1}(\mathbb{R}^{2n}). \quad (4.15)$$

In particular, the formula (4.15) acquires an especially simple form for classical symbols admitting the expansion (4.11) into homogeneous functions:

$$G_0\left(\psi(x), \left(\frac{t\partial\psi}{\partial x}\right)^{-1} \xi\right) = H_0(x, \xi). \quad (4.16)$$

We see that the principal symbols indeed are transformed under changes of variables as functions on the cotangent bundle.

Thus we can give the following definition of a pseudodifferential operator on  $M$ .

**DEFINITION 4.13.** A pseudodifferential operator with principal symbol  $D$  on a smooth compact manifold  $M$  is an operator  $\widehat{D}$  in the scale of Sobolev spaces on  $M$  such that

- 1)  $\varphi\widehat{D}\psi$  is an integral operator with smooth kernel on  $M$  whenever  $\varphi$  and  $\psi$  are smooth functions on  $M$  with disjoint supports;
- 2) if  $\varphi$  and  $\psi$  are smooth functions on  $M$  supported in the same coordinate neighborhood  $U$ , then  $\varphi\widehat{D}\psi$  is a pseudodifferential operator in  $\mathbb{R}^n$  (here we identify  $U$  with a subset in  $\mathbb{R}^n$  using the coordinate mapping) with principal symbol  $\varphi(x)\psi(x)D(x, \xi)$ .

An operator with a given symbol can be constructed, e.g., as follows. Cover  $M$  with coordinate neighborhoods  $U_j$ ,  $j = 1, \dots, N$ , and consider a smooth partition of unity

$$1 = \sum_{j=1}^N \chi_j(x)^2$$

subordinate to this cover. We define a pseudodifferential operator with symbol  $D(x, \xi)$  by the formula

$$D\left(x, -i\frac{\partial}{\partial x}\right) \stackrel{\text{def}}{=} \sum_{j=1}^N (\chi_j D)\left(x, -i\frac{\partial}{\partial x}\right) \circ \chi_j(x),$$

where the  $j$ th summand is determined in the local coordinates of the chart  $U_j$  as the composition of the operator of multiplication by  $\chi_j(x)$  (which localizes the function to be acted upon by the  $\psi$ DO into the chart  $U_j$ ) and a pseudodifferential operator with symbol  $\chi_j(x)D(x, \xi)$  in  $\mathbb{R}^n$ . (Here the symbol is expressed via the canonical coordinates  $T^*U_j$ , which we denote by the same letters  $(x, \xi)$ .)

**EXERCISE 4.14.** Check that the operator thus constructed satisfies Definition 4.13.

◀ *Hint.* Use theorems on the composition of  $\psi$ DO and their behavior under changes of coordinates. ▶

*Remark 4.15.* The operator is not uniquely determined by the symbol. For example, the operator given by the above-mentioned construction depends also on the choice of the cover and the partition of unity. However, all operators with the same symbol differ by lower-order operators.

EXERCISE 4.16. Prove that  $\psi$ DO on  $M$  obey the following composition law similar to Theorem 4.4.

THEOREM 4.17. *The principal symbol of the product of pseudodifferential operators is equal to the product of their principal symbols.*

### 4.2.2. Hörmander's definition of pseudodifferential operators

Along with the above approach to the definition of  $\psi$ DO, which is due to Kohn and Nirenberg (Kohn and Nirenberg 1965), there is another approach devised by Hörmander (Hörmander 1965). In this approach, classical pseudodifferential operators are defined (from the very beginning, on a manifold  $M$ !) in an invariant way as continuous linear operators

$$P : C^\infty(M) \longrightarrow C^\infty(M)$$

such that if  $f$  and  $g$  are smooth functions,  $f$  is compactly supported,  $g$  is real-valued, and  $dg \neq 0$  on  $\text{supp } f$ , then as  $\lambda \rightarrow \infty$  one has an asymptotic expansion

$$e^{-i\lambda g} P(f e^{i\lambda g}) \sim \sum_{j=0}^{\infty} P_j(f, g) \lambda^{s_j}, \quad (4.17)$$

locally uniform in  $g$ , for some monotone decreasing sequence of real numbers  $s_j \rightarrow -\infty$ , where the coefficients  $P_j(f, g)$  are smooth functions on  $M$ .

PROPOSITION 4.18 ((Hörmander 1965)). *One has*

$$P_0(f, g) = H(dg)f,$$

where  $H(\xi)$  is a homogeneous function of degree  $s_0$  on the cotangent bundle of  $M$  without the zero section.

The function  $H(\xi)$  is called the *principal symbol* of the  $\psi$ DO in question.

EXERCISE 4.19. Prove that in local coordinates both definitions (for  $s_j = m - j$ ) result in the same class of classical  $\psi$ DO of order  $m$ , and moreover, the principal symbol (of order  $m$ ) of a  $\psi$ DO coincides in local coordinates with the function  $H_0(x, \xi)$  in the asymptotic expansion (4.11).

Hörmander's definition possesses a number of interesting properties. In particular,

- 1) pseudodifferential operators are defined intrinsically (i.e., as operators possessing certain properties) rather than constructively (i.e., by some recipe allowing one to construct them);
- 2) the definition uses no coordinate systems, so that the invariant nature of the principal symbol is immediate.

### 4.2.3. Basic properties of pseudodifferential operators

Now let us summarize the main properties of the theory of elliptic  $\psi$ DO on smooth compact manifolds.

Let  $M$  be a smooth compact manifold. Classical  $\psi$ DO of integer order on  $M$  form the *filtered algebra*

$$L^\infty(M) = \bigcup_{k=-\infty}^{\infty} L^k(M),$$

where  $L^k(M)$  is the space of  $\psi$ DO of order  $\leq k$ . For an operator  $D \in L^k(M)$ , there is a well defined *principal symbol* (or, more precisely, a principal symbol of order  $k$ )  $\sigma(D)$ , which is a positively homogeneous function of degree  $k$  on the cotangent bundle  $T_0^*M$  of  $M$  without the zero section. Next, for each  $k \in \mathbb{Z}$  there is an exact sequence

$$0 \longrightarrow L^{k-1}(M) \longrightarrow L^k(M) \xrightarrow{\sigma} \mathcal{O}^k(T^*M) \longrightarrow 0,$$

where  $\mathcal{O}^k(T^*M)$  is the space of homogeneous functions of degree  $k$  on  $T_0^*M$ .

Furthermore,

$$L^0(M) \cap \mathcal{K}_s = L^{-1}(M)$$

for all  $s$ , where  $\mathcal{K}_s$  is the space of compact operators in the Sobolev space  $H^s(M)$ . Thus the space

$$\mathcal{O}^0(T^*M) \cong L^0(M)/L^{-1}(M) = L^0(M)/L^0(M) \cap \mathcal{K}_s$$

of principal symbols is just the Calkin algebra corresponding to the algebra  $L^0(M)$ .

We see that the  $\psi$ DO calculus on a smooth compact manifold possesses all properties stated in Sec. 4.1.3. In particular, the almost invertibility (invertibility modulo compact operators) is equivalent to the invertibility of the principal symbol in the Calkin algebra, whence the main role of the principal symbol in elliptic theory on smooth compact manifolds.

*Remark 4.20.* The notion of the principal symbol can be introduced for nonclassical  $\psi$ DO as well. However, the principal symbols are no longer functions on  $T_0^*M$  in this case; instead, they are elements of the quotient space  $S^m(T^*M)/S^{m-1}(T^*M)$  (where  $S^m(T^*M)$  is the space of smooth functions on  $T^*M$  satisfying the estimates (4.7) in local coordinates).

*Remark 4.21.* So far we have considered operators with scalar symbols, i.e., operators acting in function spaces on the manifold  $M$ . In a similar way, one constructs pseudodifferential operators acting in spaces of sections of finite-dimensional vector bundles over  $M$ . In trivializing neighborhoods of these bundles, the symbol of a  $\psi$ DO is represented by a  $k \times l$  matrix function, where  $k$  and  $l$  are the dimensions of the bundles in whose sections the operator acts.

EXERCISE 4.22. Prove that the principal symbol of a  $\psi$ DO

$$A : C^\infty(M, E) \longrightarrow C^\infty(M, F)$$

acting in sections of vector bundles  $E$  and  $F$  over a manifold  $M$  is naturally interpreted as a homomorphism

$$\sigma(A) : \pi^*E \longrightarrow \pi^*F$$

of the lifts of these bundles to the cotangent bundle of  $M$  without the zero section. Here

$$\pi : T_0^*M \longrightarrow M$$

is the natural projection.

## 4.3. Pseudodifferential Operators in Sections of Hilbert Bundles

### 4.3.1. Hilbert bundles

We shall consider pseudodifferential operators on a smooth compact manifold  $X$  without boundary acting in sections of Hilbert bundles over  $X$ .

DEFINITION 4.23. A *Hilbert bundle* over a smooth compact manifold  $X$  is a vector bundle  $E \rightarrow X$  whose fiber is a Hilbert space  $\mathcal{H}$  and whose transition functions are operator norm smooth mappings defined on intersections of trivializing neighborhoods and ranging in the group  $\mathcal{U}(\mathcal{H})$  of unitary operators in  $\mathcal{H}$ .

In the following, we consider only the case in which the space  $\mathcal{H}$  is infinite-dimensional (and separable). Then the group  $\mathcal{U}(\mathcal{H})$  is contractible by Kuiper's theorem (e.g., see (Atiyah 1989)), and consequently, any Hilbert bundle with fiber  $\mathcal{H}$  is trivial. Hence we assume without loss of generality that the bundles in question have the form  $X \times \mathcal{H} \rightarrow X$ .

### 4.3.2. Operator-valued symbols. Specific features of the infinite-dimensional case

At first glance, the problem of constructing a calculus of  $\psi$ DO acting in sections of Hilbert bundles seems to be pretty trivial. Indeed, one should only assume in the general definition (4.6) that the symbol  $H(x, \xi)$  is not a scalar function but ranges in the algebra  $\mathcal{B}(\mathcal{H})$  of bounded operators in  $\mathcal{H}$  (an operator-valued symbol). If this function is smooth and satisfies, say, the estimates (4.7) (where the operator norm should be used instead of the absolute value on the left-hand side), then the corresponding operator will be well defined in the Sobolev spaces:

$$H\left(x, -i\frac{\partial}{\partial x}\right) : H^s(\mathbb{R}^n, \mathcal{H}) \rightarrow H^{s-m}(\mathbb{R}^n, \mathcal{H}), \quad s \in \mathbb{R}. \quad (4.18)$$

Under changes of coordinates, the symbol still behaves “well,” i.e., is transformed (modulo lower-order symbols) as a function on the cotangent bundle. Thus, one can define  $\psi$ DO globally on the manifold, and a pseudodifferential operator is uniquely determined by its principal symbol modulo lower-order operators. However, there are two facts showing that this straightforward generalization of the definitions to the operator-values case is not completely adequate.

1) In contrast to the scalar case (and the case of finite-dimensional bundles), the embeddings

$$H^s(X, \mathcal{H}) \subset H^t(X, \mathcal{H}), \quad s > t,$$

of Sobolev spaces *are not compact*. Consequently, neither are negative-order operators. Thus a pseudodifferential operator in this setting *is not uniquely determined by its principal symbol modulo compact operators*. Accordingly, one cannot verify whether an operator is Fredholm if only the principal symbol is known. It follows that such a calculus will be useless in elliptic theory.

EXERCISE 4.24. Check that an operator of order  $-1$  with principal symbol  $|\xi|^{-1}I$ , where  $I$  is the identity operator in  $\mathcal{H}$ , is not compact in  $L^2(X, \mathcal{H})$ .

2) Even the simplest examples show that it is of no interest to consider classical (i.e., asymptotically homogeneous in the momentum variables  $\xi$ ) symbols. Indeed, such an example already arises if we consider the ordinary  $\psi$ DO on the product of two manifolds. Let  $M = X \times Y$ , where  $X$  and  $Y$  are smooth compact manifolds without boundary. Then pseudodifferential operators on  $M$  can be treated as  $\psi$ DO on  $X$  whose symbols are, in turn, pseudodifferential operators on  $Y$ . In local coordinates, we can write

$$H\left(x, y, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right) = G\left(x, -i\frac{\partial}{\partial x}\right),$$

where

$$G(x, \xi) = H\left(x, y, \xi, -i\frac{\partial}{\partial y}\right).$$

Even if the symbol  $H(x, y, \xi, \eta)$  is asymptotically homogeneous with respect to the pair  $(\xi, \eta)$ , the symbol  $G(x, \xi)$  cannot be asymptotically homogeneous in  $\xi$ !

Thus, the “naïve” approach described above should be revised. We should abandon the homogeneity requirement for the symbols. On the other hand, we need to subject the symbols to some additional conditions guaranteeing that an operator with symbol of negative order is compact. Such additional conditions indeed exist; they are known as the *compact fiber variation* conditions. Starting from the next subsection, we present the theorem of  $\psi$ DO with symbols of compact fiber variation, restricting ourselves to the case of symbols (and operators) of zero order, for which (as already noted in (Luke 1972)) the theory can be stated in the most natural way.

### 4.3.3. Symbols of compact fiber variation

Let  $\mathcal{B}(\mathcal{H})$  be the Banach algebra of bounded operators in  $\mathcal{H}$ , and let  $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$  be the closed ideal of compact operators. The canonical coordinates on the cotangent bundle  $T^*X$  will be denoted by  $(x, \xi)$ . For convenience, we fix some smooth norm  $|\xi|$  in the fibers of  $T^*X$ .

DEFINITION 4.25. By  $S_{CV}^0(T^*X)$  we denote the space of infinitely differentiable (in the uniform operator topology) functions

$$f : T^*X \longrightarrow \mathcal{B}(\mathcal{H})$$

that satisfy the estimates

$$\left\| \frac{\partial^{\alpha+\beta} f(x, \xi)}{\partial x^\alpha \partial \xi^\beta} \right\|_{\mathcal{B}(\mathcal{H})} \leq C_{\alpha\beta} (1 + |\xi|)^{-|\beta|}, \quad |\alpha| + |\beta| = 0, 1, 2, \dots, \quad (4.19)$$

in any canonical coordinate system  $(x, \xi)$  on  $T^*X$  and have the property of *compact fiber variation*

$$f(x, \xi) - f(x, \tilde{\xi}) \in \mathcal{K}(\mathcal{H}) \quad \text{for } \xi, \tilde{\xi} \in T_x^*X. \quad (4.20)$$

In applications one often deals with symbols with similar properties defined only outside the zero section of the cotangent bundle. Hence we introduce a space of such symbols.

Namely, by  $S_{CV}^0(T_0^*X)$  we denote the space of functions

$$f : T_0^*X \longrightarrow \mathcal{B}(\mathcal{H})$$

that satisfy the estimates (4.19) for  $|\xi| > \varepsilon$  for each  $\varepsilon > 0$  with constants  $C_{\alpha\beta}$  depending on  $\varepsilon$  and have the property (4.20) for  $\xi, \tilde{\xi} \neq 0$ .

Elements of the spaces  $S_{CV}^0(T^*X)$  and  $S_{CV}^0(T_0^*X)$  will be called *symbols* (of degree 0) of *compact fiber variation* on  $T^*X$  (respectively, on  $T_0^*X$ ).

*Remark 4.26.* Condition (4.20) is equivalent to the requirement

$$\frac{\partial f(x, \xi)}{\partial \xi} \in \mathcal{K}(\mathcal{H}). \quad (4.21)$$

(The equivalence can be proved by integration from  $\xi$  to  $\tilde{\xi}$ .)

Both spaces  $S_{CV}^0(T^*X)$  and  $S_{CV}^0(T_0^*X)$  are obviously algebras (with respect to pointwise multiplication).

*Remark 4.27.* The algebra  $S_{CV}^0(T^*X)$  contains the ideal  $J_K^{-1}(T^*X)$  of compact-valued symbols

$$f : T^*X \longrightarrow \mathcal{K}(\mathcal{H})$$

satisfying the estimates

$$\left\| \frac{\partial^{\alpha+\beta} f(x, \xi)}{\partial x^\alpha \partial \xi^\beta} \right\|_{\mathcal{B}(\mathcal{H})} \leq C_{\alpha\beta} (1 + |\xi|)^{-1-|\beta|}, \quad |\alpha| + |\beta| = 0, 1, 2, \dots, \quad (4.22)$$

in any canonical coordinate system  $(x, \xi)$ .

The following lemma shows that a symbol of compact fiber variation defined outside the zero section of  $T^*X$  can always be filled up to an everywhere defined symbol of compact fiber variation and that the fill-up is essentially unique.

**LEMMA 4.28.** *For any symbol  $f \in S_{CV}^0(T_0^*X)$ , there exists a symbol  $\tilde{f} \in S_{CV}^0(T^*X)$  such that  $f(x, \xi) = \tilde{f}(x, \xi)$  for sufficiently large  $|\xi|$ . If  $\tilde{f} \in S_{CV}^0(T^*X)$  is another symbol with the same property, then the difference  $\tilde{f} - \tilde{f}$  is compact-valued and has a compact support.*

*Proof.* Let us present a proof of this lemma, which, although technical, is from our point of view typical in dealing with symbols of this class.

1. The existence of  $\tilde{f}$ . On the unit spheres

$$\mathbb{S}^{n-1}(x) = \{|\xi| = 1\} \subset T_x^*X,$$

we take an arbitrary smooth measure  $d\mu(\xi)$  smoothly depending on  $x$  with the property

$$\int_{\mathbb{S}^{n-1}(x)} d\mu(\xi) = 1$$

and set

$$\tilde{f}(x, \xi) = \chi(|\xi|)f(x, \xi) + (1 - \chi(|\xi|)) \int_{\mathbb{S}^{n-1}(x)} f(x, \xi') d\mu(\xi'), \quad (4.23)$$



where  $\chi(t) \in C^\infty(\mathbb{R}_+)$  is an excision function that vanishes near zero and is equal to unity in a neighborhood of infinity. (For  $\xi = 0$ , we define  $f(x, \xi)$  on the right-hand side of the formula to be zero.) Then, obviously,  $f(x, \xi) = \tilde{f}(x, \xi)$  for sufficiently large  $|\xi|$ . Moreover,  $f(x, \xi)$  is infinitely differentiable and satisfies the estimates (4.19). Let us verify the compact fiber variation property (it suffices to do this for  $\xi = 0$ ):

$$\tilde{f}(x, \xi) - \tilde{f}(x, 0) = \chi(|\xi|) \int_{\mathbb{S}^{n-1}(x)} (f(x, \xi) - f(x, \xi')) d\mu(\xi') \in \mathcal{K}(\mathcal{H})$$

for each  $\xi \in T_x^*X$ , since for  $\xi = 0$  the right-hand side is zero and for  $\xi \neq 0$  the integrand is compact (and continuously depends on  $\xi'$ ).

2. Uniqueness of  $\tilde{f}$  modulo compactly supported compact remainders. If  $\tilde{f}$  and  $\tilde{\tilde{f}}$  are two symbols with these properties, then both coincide with  $f$  for sufficiently large  $\xi$ , so that the difference is compactly supported. Since both are of compact fiber variation, we have

$$\begin{aligned} \tilde{f}(x, \xi) - \tilde{\tilde{f}}(x, \xi) &= [\tilde{f}(x, \xi) - \tilde{f}(x, \xi_0)] - [\tilde{\tilde{f}}(x, \xi) - \tilde{\tilde{f}}(x, \xi_0)] \\ &= [\tilde{f}(x, \xi) - \tilde{\tilde{f}}(x, \xi_0)] - [\tilde{\tilde{f}}(x, \xi) - \tilde{\tilde{f}}(x, \xi_0)] \in \mathcal{K}(\mathcal{H}) \end{aligned}$$

once the norm of  $\xi_0 \in T_x^*X$  is large enough that  $\tilde{f}(x, \xi_0) = \tilde{\tilde{f}}(x, \xi_0)$ .

□

The symbol  $\tilde{f}$  constructed in Lemma 4.28 will be called a *fill-up* of  $f$ . It is symbols from the space  $S_{CV}^0(T_0^*X)$  that play the main role for us, since such symbols (undefined for  $\xi = \emptyset$ ) arise in applications.

#### 4.3.4. Definition of pseudodifferential operators

For operator-valued symbols of compact fiber variation, pseudodifferential operators are defined in a standard way (see Sec. 4.2.1). The construction is literally the same as in the finite-dimensional case. For the reader's convenience, let us reproduce the corresponding formulas.

Let

$$f(x, \xi) : \mathcal{H} \longrightarrow \mathcal{H}, \quad (x, \xi) \in T^*X,$$

be an operator-valued symbol on  $X$ . We cover  $X$  by coordinate neighborhoods  $U_j$ ,  $j = 1, \dots, N$ , and consider a smooth partition of unity

$$1 = \sum_{j=1}^N \chi_j(x)^2$$

subordinate to the cover. A pseudodifferential operator with symbol  $f(x, \xi)$  can be defined by the formula

$$f\left(x, -i\frac{\partial}{\partial x}\right) \stackrel{\text{def}}{=} \sum_{j=1}^N (\chi_j f)\left(x, -i\frac{\partial}{\partial x}\right) \circ \chi_j(x), \quad (4.24)$$

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<sup>5</sup>Or even on a compact set

where the  $j$ th term in the sum is defined in the local coordinates of the chart  $U_j$  as the composition of the operator of multiplication by the function  $\chi_j(x)$  (which localizes the function into the chart  $U_j$ ) and a pseudodifferential operator with symbol  $\chi_j(x)f(x, \xi)$  in  $\mathbb{R}^n$  (here the symbol is expressed via the canonical coordinates in  $T^*U_j$ , which we denote by the same letters  $(x, \xi)$ ), given with the help of the Fourier transform:

$$(\chi_j f) \left( x, -i \frac{\partial}{\partial x} \right) u(x) = \left( \frac{1}{2\pi} \right)^{n/2} \int e^{ix\xi} \chi_j(x) f(x, \xi) \tilde{u}(\xi) d\xi, \quad u \in C_0^\infty(\mathbb{R}^n), \quad (4.25)$$

where  $\tilde{u}(\xi)$  is the Fourier transform of  $u(x)$ .

PROPOSITION 4.29. *Let  $f \in S_{CV}^0(T^*X)$ . The formulas (4.24) and (4.25) define a continuous operator*

$$f \left( x, -i \frac{\partial}{\partial x} \right) : L^2(X, \mathcal{H}) \longrightarrow L^2(X, \mathcal{H}), \quad (4.26)$$

which is independent of the choice of the atlas  $U_j$  and the subordinate partition of unity modulo compact operators.

EXERCISE 4.30. Prove this proposition.

◀ *Hint.* Use the standard scheme to prove the invariance of the definition of  $\psi$ DO ((Hörmander 1965), (Kohn and Nirenberg 1965)). It is based on the composition and change of variable formulas for  $\psi$ DO in  $\mathbb{R}^n$ , which can be transferred word for word from the finite-dimensional case. In addition, one should only prove the compactness of remainders in these formula in the infinite-dimensional case. This can be done on the basis of Lemma 4.31 below, the proof of which can be found in (Luke 1972, Proposition 2.1). ▶

LEMMA 4.31. *If a symbol  $g(x, \xi) \in S_{CV}^0(\mathbb{R}^{2n})$  is compactly supported in  $x$ , is compact-valued, and satisfies the condition*

$$(1 + |\xi|)^\varepsilon g(x, \xi) \in S_{CV}^0(\mathbb{R}^{2n}) \quad \text{for some } \varepsilon > 0,$$

then the operator  $g(x, -i\partial/\partial x)$  is compact in  $L^2(\mathbb{R}^n, \mathcal{H})$ .

Proposition 4.29 permits one to give the following definition.

DEFINITION 4.32. Let  $f \in S_{CV}^0(T^*X)$ . A pseudodifferential operator with symbol  $f$  is an operator (4.26) given modulo compact operators by the formulas (4.24)–(4.25).

Let  $f \in S_{CV}^0(T_0^*X)$ . A pseudodifferential operator with symbol  $f$  is the operator

$$\tilde{f} \left( x, -i \frac{\partial}{\partial x} \right) : L^2(X, \mathcal{H}) \longrightarrow L^2(X, \mathcal{H}),$$

where  $\tilde{f}(x, \xi) \in S_{CV}^0(T^*X)$  is an arbitrary fill-up of  $f(x, \xi)$ .

By Lemma 4.28, the symbol  $\tilde{f}$  exists and is uniquely determined modulo compact-valued compactly supported symbols, which correspond to compact  $\psi$ DO by Lemma 4.31. Thus  $\psi$ DO with symbol  $f \in S_{CV}^0(T_0^*X)$  is well defined modulo compact operators. By abuse of notation, we denote such an operator by

$$f \left( x, -i \frac{\partial}{\partial x} \right).$$

### 4.3.5. The composition theorem

The operators with operator-valued symbols thus defined form an algebra. More precisely, the following assertion holds.

**PROPOSITION 4.33.** *Let  $p, q \in S_{CV}^0(T_0^*X)$ , and let  $P$  and  $Q$  be  $\psi$ DO with symbols  $p$  and  $q$ , respectively. Then  $PQ$  is a  $\psi$ DO with symbol  $pq$ .*

This theorem essentially deals with the principal symbol (since we have defined pseudodifferential operators modulo compact operators). But by fixing the arbitrary elements in the construction of  $\psi$ DO (the cover of  $X$  by charts, the partition of unity, the fill-up, etc.) one can obtain more precise composition formulas, which have the standard form in local coordinates (e.g., see (Kohn and Nirenberg 1965)). We shall use such formulas below in the proof of the index theorem in the operator-valued case.

### 4.3.6. Ellipticity

The composition formula modulo compact operators given by Proposition 4.33 permits one to give a natural definition of ellipticity and prove the finiteness theorem.

Since we do not impose any homogeneity condition on the symbols, we must be careful in the definition of ellipticity.

**DEFINITION 4.34.** A symbol  $p \in S_{CV}^0(T_0^*X)$  is said to be *elliptic* if there exists an  $R > 0$  such that  $p(x, \xi)$  is invertible for  $|\xi| > R$  and the inverse satisfies the estimate

$$\|p(x, \xi)^{-1}\| \leq C, \quad |\xi| > R, \quad (4.27)$$

for some constant  $C$ .

**EXERCISE 4.35.** Show that if a symbol  $p$  is elliptic, then it is Fredholm for all  $(x, \xi) \in T_0^*X$  and its arbitrary fill-up  $\tilde{p}$  is Fredholm on  $T^*X$ .

◀ Use the fact that the addition of a compact operator to a Fredholm operator does not destroy the Fredholm property. ▶

**PROPOSITION 4.36.** *A symbol  $p \in S_{CV}^0(T_0^*X)$  is elliptic if and only if there exists a symbol  $q \in S_{CV}^0(T^*X)$  such that the symbols  $q\tilde{p} - 1$  and  $\tilde{p}q - 1$  are compactly supported and compact-valued for any fill-up  $\tilde{p} \in S_{CV}^0(T^*X)$  of the symbol  $p$ .*

*Proof.* We define  $q(x, \xi)$  for  $|\xi| > R$  by the formula

$$q(x, \xi) = p(x, \xi)^{-1}.$$

Outside the balls of radius  $R$ , this symbol satisfies the estimates (4.19) and has a compact fiber variation. We extend it into the interior of the above-mentioned balls using the construction in the proof of Lemma 4.28 with an excision function  $\chi(t)$  equal to zero for  $t \leq R$ . By construction, the resulting symbol lies in  $S_{CV}^0(T^*X)$ ; we again denote it by  $q(x, \xi)$ . The fact that  $q\tilde{p} - 1$  and  $\tilde{p}q - 1$  are compactly supported and compact-valued is then obvious.  $\square$

**Remark 4.37.** This property shows that the ellipticity of a symbol  $p \in S_{CV}^0(T_0^*X)$  is equivalent to the invertibility of the corresponding “principal symbol,” that is, the element generated by  $p$  in the quotient algebra  $S_{CV}^0(T_0^*X)/J_K^{-1}(T^*X)$ .

### 4.3.7. The finiteness theorem

Now we can state and prove the main analytic theorem of elliptic theory for the case of operator-valued symbols.

**THEOREM 4.38** (the finiteness theorem). *If  $p \in S_{CV}^0(T_0^*X)$  is an elliptic symbol, then the operator  $P = p(x, -i\partial/\partial x)$  is Fredholm.*

*Proof.* Indeed, the operator  $Q = q(x, -i\partial/\partial x)$  with the symbol  $q(x, \xi)$  constructed in Proposition 4.36 is a two-sided almost inverse of  $P$ .  $\square$

**EXERCISE 4.39.** Show that this result is a special case of the general scheme of the abstract finiteness theorem in Sec. 4.1.3.

## 4.4. The Index Theorem

### 4.4.1. The Atiyah–Singer index theorem

We have already note that the index of a classical elliptic  $\psi$ DO on a smooth compact manifold is completely determined by its principal symbol and is a homotopy invariant of the latter. More precisely, since the index of an operator with unit principal symbol is zero, it follows that the index of an elliptic operator is a stable homotopy invariant of the principal symbol. In more detail, we say that two elliptic symbols  $\sigma_1$  and  $\sigma_2$  over a manifold  $M$  are *stably homotopic* if for some nonnegative numbers  $N_1$  and  $N_2$  the symbols  $\sigma_1 \oplus \mathbf{1}_{N_1}$  and  $\sigma_2 \oplus \mathbf{1}_{N_2}$  are homotopic in the class of elliptic symbols. The set of equivalence classes of elliptic symbols with respect to this equivalence relation is denoted by  $\text{Ell}(M)$ . It is an abelian group with respect to the operation induced by the direct sum of elliptic operators.

One can give a different description of the group  $\text{Ell}(M)$  in terms of vector bundles and  $K$ -theory. An elliptic symbol

$$\sigma : \pi^*E \longrightarrow \pi^*F$$

over a manifold  $M$  specifies an isomorphism of the bundles  $\pi^*E$  and  $\pi^*F$  over  $T^*M$  outside the zero section of  $T^*M$ . In other words, we have a triple  $(\pi^*E, \pi^*F, \sigma)$  consisting of two bundles over the space  $T^*M$  and an isomorphism of these bundles outside a compact set. By definition, such a triple specifies an element of the  $K$ -group with compact supports of the space  $T^*M$ . We denote this element by  $[\sigma] \in K_c(T^*M)$ .

◀ Recall that the  $K$ -group with compact supports of a space  $X$  is defined as the set of equivalence classes of triples  $(E, F, \sigma)$ , where  $E$  and  $F$  are bundles over  $X$  and  $\sigma$  is an isomorphism of these bundles outside some compact subset of  $X$ , by the following equivalence relation: two triples  $(E_1, F_1, \sigma_1)$  and  $(E_2, F_2, \sigma_2)$  are said to be equivalent if they become isomorphic after the addition of some trivial triples as direct summands. Moreover, a triple  $(E, F, \sigma)$  is said to be trivial if  $\sigma$  can be extended to an isomorphism defined everywhere on  $X$ . ▶

The following remarkable theorem holds.

**THEOREM 4.40** (Atiyah–Singer). *The mapping  $\sigma \mapsto [\sigma]$  is a well-defined abelian group homomorphism*

$$\chi : \text{Ell}(M) \longrightarrow K_c(T^*M). \quad (4.28)$$

*Moreover, this homomorphism is an isomorphism.*

The homomorphism  $\chi$  is called the *difference construction*, and the element  $[\sigma]$  is referred to as the difference element corresponding to the elliptic symbol  $\sigma$ . Informally speaking, the word “difference” means here that the element  $[\sigma]$  is represented as the difference  $\pi^*E - \pi^*F$  of some vector bundles over  $T^*X$  isomorphic (i.e., canceling in some sense) at infinity.

Thus the difference construction takes each symbol of an elliptic operator on  $M$  to an element of the  $K$ -group with compact supports of the cotangent bundle of  $M$ . It is clear from the preceding that the index of an elliptic operator  $A$  is determined by the difference element  $[\sigma(A)]$  of its symbol.

Atiyah and Singer presented a homomorphism that computes the index in these terms. Namely, if  $p : X \rightarrow Y$  is a smooth mapping of closed manifolds, then in  $K$ -theory there is a well-defined *direct image homomorphism*

$$p_! : K_c(T^*X) \rightarrow K_c(T^*Y)$$

(see (Atiyah 1989)). In particular, if  $Y = pt$  is a point and  $p$  is a constant mapping, we obtain the homomorphism

$$p_! : K_c(T^*X) \rightarrow K_c(pt) \cong \mathbb{Z}.$$

**THEOREM 4.41** (the Atiyah–Singer index theorem). *For elliptic operators on  $M$ , the index formula*

$$\text{ind } A = p_!([\sigma(A)]) \tag{4.29}$$

*is valid.*

#### 4.4.2. The index theorem for pseudodifferential operators in sections of Hilbert bundles

Let us now state an analog of the Atiyah–Singer theorem for elliptic operators with operator-value symbols. Let  $p \in S_{CV}^0(T_0^*X)$  be an elliptic symbol. Recall that this symbol (more precisely, its fill-up) is a family of Fredholm operators on the cotangent bundle  $T^*X$  invertible at infinity. It turns out that symbols of this form, by analogy with the finite-dimensional case considered in the preceding subsection, determine elements  $[p] \in K_c(T^*X)$  in the  $K$ -group with compact supports of the cotangent bundle.

**The difference construction and the families index.** The index of a single Fredholm operator is an integer. Now if we have a family  $T_y$  of Fredholm operators parametrized by a compact set  $Y$ , then the “index” of this family is naturally defined as an element of the  $K$ -group of the parameter space. Indeed, if  $\dim \ker T_y = \text{const}$  (which can always be achieved by a small perturbation of the family), then the kernels and cokernels of operators in the family form vector bundles over the parameter space, and the *index of the family  $T$*  is defined as the difference of these two bundles:

$$\text{ind } T = [\ker T] - [\text{coker } T] \in K(Y). \tag{4.30}$$

The relationship with the ordinary integer index is obvious: for a one point space  $Y = pt$ , we have  $K(pt) = \mathbb{Z}$  and  $\text{ind } T = \dim \ker T - \dim \text{coker } T$ .

It is known that the families index in the general case is not only independent of a perturbation making the family of kernels a vector bundle but also remains unchanged under continuous deformations of the family. Let  $[Y, \text{Fred}]$  be the set of homotopy classes of Fredholm families. The famous Atiyah–Jänich theorem (Atiyah 1989) states that the families index defines an isomorphism

$$[Y, \text{Fred}] \stackrel{\text{ind}}{\cong} K(Y)$$

of this set with the  $K$ -group.

For a noncompact parameter space, it is natural to consider the  $K$ -group  $K_c(Y)$  with compact supports. It turns out that if a Fredholm family is invertible for parameter values outside a compact set, then it also has an index that is an element of the  $K$ -group with compact supports:

$$\text{ind } T \in K_c(Y).$$

To define this element, one uses an appropriate perturbation of the original family by a family of finite-dimensional operators that vanish at infinity.

◀ More precisely, the perturbation is added to the direct sum of the original family with the zero endomorphism of the trivial bundle  $\mathbb{C}^N$  of sufficiently large dimension  $N$ . Since the perturbation is compactly supported, it follows that both the kernel and the cokernel of the perturbed family at infinity coincide with  $\mathbb{C}^N$  and hence we have an isomorphism between them outside a compact set. If, moreover, the kernel and the cokernel are vector bundles (which is guaranteed by the choice of the perturbation), then we have a triple defining an element of the  $K$ -group with compact supports. ▶

EXERCISE 4.42. Construct such a perturbation.

One can readily establish the following properties of the families index:

- 1) for a compact parameter space, the index coincides with the expression (4.30);
- 2) the index of a family of finite-dimensional operators coincides with the difference construction in the preceding subsection;
- 3) the index is invariant under deformations: if  $T_{y,t}$  is a homotopy of Fredholm families, then  $\text{ind } T_t = \text{ind } T_0 \in K_c(Y)$ ; here it is assumed that  $T_{y,t}$  is uniformly continuous on some compact set  $K \subset Y$  and is always invertible outside  $K$ .

**The index theorem.** Thus for an elliptic symbol  $p$  there is a well-defined index  $\text{ind } p \in K_c(T^*X)$ .

**THEOREM 4.43.** *Let  $p \in S_{CV}^0(T_0^*X)$  be an elliptic symbol. Then the index of the corresponding elliptic operator is expressed as the direct image in  $K$ -theory:*

$$\text{ind } p \left( x, -i \frac{\partial}{\partial x} \right) = \pi_* \text{ind } p, \quad (4.31)$$

where  $\pi : X \rightarrow pt$  is the mapping of the manifold  $X$  into a point.

#### 4.4.3. Proof of the index theorem

Without loss of generality, we can assume from the very beginning that  $p \in S_{CV}^0(T^*X)$  (i.e., the fill-up has already been chosen). The proof of the index theorem splits into two stages.

**A special case: symbols homogeneous for large  $|\xi|$ .** Suppose that the symbol  $p$  satisfies the condition

$$p(x, t\xi) = p(x, \xi) \quad \text{for } \xi \geq R \text{ and } t > 1, \quad (4.32)$$

i.e., is homogeneous of degree zero at infinity. The proof for this case is given in (Luke 1972). The main idea is to deform the operator to the direct sum of an operator induced by a vector bundle isomorphism and an elliptic operator with finite-dimensional symbol.

EXERCISE 4.44 (for advanced readers). Try to construct such a homotopy.

◀ With this homotopy, it suffices to verify the index formula for the resulting direct sum. The invertible operator does not contribute to (4.31), and for the finite-dimensional symbol the desired formula coincides with the Atiyah–Singer formula. ▶

**The general case: reduction to homogeneous symbols.** In this case, we use a method due to Hörmander (Hörmander 1985a, Theorem 19.2.3) to reduce the case of general elliptic symbols to the homogeneous case. Hörmander’s theorem gives the desired reduction for symbols acting in finite-dimensional bundles, and our main task is to verify that all constructions remain valid in the infinite-dimensional case.

The reduction is based on the following assertion, which we give in a special case convenient to us.

PROPOSITION 4.45 ((Hörmander 1985a, Theorem 19.1.10)). *Let  $T_\varepsilon, S_\varepsilon \in \mathcal{B}(\mathcal{H})$ ,  $\varepsilon \in [0, \varepsilon_0]$ , be two strongly continuous operator families such that the families  $S_\varepsilon T_\varepsilon - 1$  and  $T_\varepsilon S_\varepsilon - 1$  are uniformly compact. Then  $\text{ind } S_\varepsilon = -\text{ind } T_\varepsilon$  is independent of  $\varepsilon$ .*

◀ An operator family  $Q_\varepsilon \in \mathcal{B}(\mathcal{H})$  is said to be *uniformly compact* if the union of images of the unit ball under the action of all  $Q_\varepsilon$  is precompact, i.e., has a compact closure. ▶

Let  $q \in S_{CV}^0(T^*X)$  be a symbol such that  $pq = qp = 1$  for  $|\xi| > R$ .

Let  $\psi(t)$ ,  $t \geq 0$ , be a smooth real function such that

$$\psi(t) = \begin{cases} 1 & \text{for } t < 1, \\ 1/t & \text{for } t > 2. \end{cases} \quad (4.33)$$

Consider the symbols

$$p_\varepsilon(x, \xi) = p(x, \xi\psi(\varepsilon\xi)), \quad q_\varepsilon(x, \xi) = q(x, \xi\psi(\varepsilon\xi)). \quad (4.34)$$

They have the following properties:

- 1)  $p_0 = p$  and  $q_0 = q$ ;
- 2)  $p_\varepsilon$  and  $q_\varepsilon$  are uniformly bounded in  $S_{CV}^0(T^*X)$  for  $\varepsilon \in [0, 1]$ ;
- 3) for  $\varepsilon > 0$ , the symbols  $p_\varepsilon$  and  $q_\varepsilon$  satisfy condition (4.32), and for sufficiently small  $\varepsilon$  they are elliptic;
- 4) for sufficiently small  $\varepsilon > 0$ , the compactly supported compact-valued symbols

$$r_1 = p_\varepsilon q_\varepsilon - 1, \quad r_2 = q_\varepsilon p_\varepsilon - 1 \quad (4.35)$$

are independent of  $\varepsilon$ .

(The proof coincides with the one given in (Hörmander 1985a, Theorem 19.2.3) for the finite-dimensional case word for word.) We define operators

$$P_\varepsilon = p_\varepsilon \left( x, -i \frac{\partial}{\partial x} \right), \quad Q_\varepsilon = q_\varepsilon \left( x, -i \frac{\partial}{\partial x} \right) \quad (4.36)$$

by formulas of the form (4.24)–(4.26) with coordinate neighborhoods and partition of unity independent of  $\varepsilon$ . We claim that (for sufficiently small  $\varepsilon$ )

- (a)  $\text{ind } p_\varepsilon \in K_c(T^*X)$  is independent of  $\varepsilon$ ;
- (b) the families  $P_\varepsilon$  and  $Q_\varepsilon$  satisfy the assumptions of Proposition 4.45.

This implies the index theorem, since, on the one hand, the symbol  $p_\varepsilon$  is homogeneous at infinity for  $\varepsilon > 0$ , so that  $\text{ind } P_\varepsilon = \pi! \text{ind } p_\varepsilon$  (by the first part of the proof), and on the other hand, the passage to the limit as  $\varepsilon \rightarrow 0$  is possible by Proposition 4.45. Hence it suffices to prove assertions (a) and (b).

The former follows from the homotopy invariance of the difference element, since for sufficiently small  $\varepsilon$  the symbol  $p_\varepsilon$  varies with  $\varepsilon$  only outside a sufficiently large ball  $\{|\xi| > R\}$ , where  $R \simeq 1/\varepsilon$ , and remains invertible in the exterior of the ball.

EXERCISE 4.46. Prove assertion (b).

◀ *Hint.*

1) The families  $P_\varepsilon$  and  $Q_\varepsilon$  are strongly continuous, since they are uniformly bounded and each term occurring in their definition via the sum (4.24) over coordinate charts on  $X$  is strongly continuous on the set of functions whose Fourier transform is compactly supported.

2) The operators  $P_\varepsilon Q_\varepsilon - 1$  and  $Q_\varepsilon P_\varepsilon - 1$  are compact and continuously depend on  $\varepsilon$ . Indeed, the compactness is obvious, while the continuity follows from the fact that their complete symbols in local coordinate systems, as well as their derivatives, continuously depend on  $\varepsilon$  on compact subsets of values of  $\xi$ , are uniformly bounded and uniformly decay as  $\xi \rightarrow \infty$ , and hence are uniformly continuous in  $\varepsilon$  for all  $\xi$ .

It follows from 2) that the families  $P_\varepsilon Q_\varepsilon - 1$  and  $Q_\varepsilon P_\varepsilon - 1$  are uniformly compact. ▶

Thus the proof of the theorem is complete.

## 4.5. Bibliographical Remarks

A detailed construction (including composition formulas and changes of variables) of pseudodifferential operators in  $\mathbb{R}^n$  or on a closed manifold can be found in the classical papers (Kohn and Nirenberg 1965), (Hörmander 1965) or the book (Hörmander 1985a). The compositions formulas and boundedness theorems without additional assumptions such as stabilization at infinity can be found in numerous places. In particular, they can be obtained by methods of noncommutative analysis (Maslov 1973) (see also the elementary exposition in (Nazaikinskii, Sternin and Shatalov 1995)). Pseudodifferential operators in Hilbert bundles were described in (Luke 1972), where the proof of the Index Theorem 4.43 was only given for homogeneous symbols. We give the proof of this formula in the general case.

Finally, the reader can find the required facts of  $K$ -theory, say, in (Atiyah 1989) and also in (Luke and Mishchenko 1998).



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