Localization (Surgery) in Elliptic Theory[1]

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Differential Operators
on Manifolds with Singularities

Analysis and Topology

Contents

Chapter 1

Localization (Surgery) in Elliptic Theory

1.1. The Index Locality Principle

1.1.1. What is locality?

The index of an elliptic operator D on a smooth compact manifold M without boundary does not change if lower-order terms are added to the operator (since the index of a Fredholm operator is stable under compact perturbations) and hence depends only on the principal symbol $\sigma(D)$ of the operator D. In topological terms, the index is expressed via the principal symbol by the Atiyah–Singer formula (Atiyah and Singer 1963)

$$\operatorname{ind} D = p_![\sigma(D)],$$

where $[\sigma(D)]$ is the element of the K-group with compact supports of the cotangent bundle of M determined via the principal symbol by the difference construction and $p_!$ is the direct image in K-theory induced by the mapping $p : M \longrightarrow \{\mathrm{pt}\}$ of M into a operator. How can one describe, however, the *analytic* structure of the dependence of the index on the principal symbol? Apparently, the most detailed answer is given by the so-called "local index formula" (e.g., see (Gilkey 1995)): the index of an elliptic operator D on a closed manifold M is expressed by the formula

$$\operatorname{ind} D = \int_M \alpha(x), \tag{1.1}$$

where the "local density" $\alpha(x)$ at a point $x \in M$ depends only on the principal symbol $\sigma(D)$ and its derivatives in the fiber T_x^*M of the cotangent bundle over x, and moreover, one can write out explicit formulas for this density.

It is rather difficult to extend such a locality property of the index to more general situations (say, to the case of the index of elliptic operators on manifolds with singularities): in these situations, there are far more complicated (noncommutative) principal symbol structures, for which formulas like (1.1) are no longer valid. Hence the following coarser locality property (which directly follows from (1.1) in the case of elliptic operators on a smooth compact manifold without boundary) will be of interest to us.

If one changes the operator D (within the class of elliptic operators) on some open subset $V \subset M$, then the index changes by a number that depends only on the structure of the original and new operators on V and is independent of the structure of D outside V.

◀ Indeed, by (1.1) the variation of the index is equal to

$$\int_V \alpha(x) - \int_V \alpha'(x),$$

where $\alpha'(x)$ is the density corresponding to the new operator, for the integrals over $M \setminus V$ cancel out. ▶

This argument shows that the variation of the operator can be interpreted rather widely: for example, we can change not only the operator itself but also the bundles in whose sections it acts and even the manifold M itself. (More precisely, we can replace V by some other set V', leaving $M \setminus V$ unchanged.)

In the next subsection we show this locality property in the simplest case directly, without resorting to the local index formula (1.1). The idea in this example will help us later on to state the general index locality principle as well.

1.1.2. A pilot example

Let

$$D, D' : C^\infty(M, E) \longrightarrow C^\infty(M, F) \tag{1.2}$$

be two elliptic pseudodifferential operators (PDO) acting on the same smooth compact manifold M without boundary in sections of the same bundles and coinciding everywhere outside a compact subset $V \subset M$. In this situation, the index locality property can readily be proved by elementary means.

◀ For example, one can argue as follows. The difference of indices of D and D' is equal to

$$\operatorname{ind} D - \operatorname{ind} D' = \operatorname{ind} D(D')^{-1}, \tag{1.3}$$

where $(D')^{-1}$ is an almost inverse[1] of D'. In turn, the index of the elliptic PDO $D(D')^{-1}$ is completely determined by its principal symbol $\sigma = \sigma(D)\sigma(D')^{-1}$, which is equal to unity over $M \setminus V$ and hence actually depends only on the values of the principal symbols $\sigma(D)$ and $\sigma(D')$ over V. ▶

What is the essence of the argument given in this example? Obviously, the key point is that, for elliptic PDO, taking the product and passing to an almost inverse are local operations (modulo compact operators, which do not affect the index anyway). This follows from the (pseudo)locality property of (pseudo)differential operators. It is the locality[2] of these operations that implies that the operator $D(D')^{-1}$ is unit outside V and is determined in V only by the operators D and D' also in V. Moreover, we actually use only the locality with respect to the pair of sets V and $M \setminus V$: were the operators nonlocal in V and in $M \setminus V$ separately, our conclusions would remain in force.

Thus, the pilot example essentially shows that we deal with the implication

$$\boxed{\text{locality of elliptic PDO}} \implies \boxed{\text{locality of the index}}$$

The abstract index locality principle given in forthcoming subsections is just a generalization of this implication. To speak of local operators in the abstract case, we need the following:

[1] An inverse modulo compact operators.

[2] We omit the prefix “pseudo” in what follows.

- there should be an analog of the notion of *support* for elements of spaces where the operators will act;
- for such spaces, we should define an analog of the notion of an *operator whose integral kernel is localized near the diagonal.*

Such spaces, which will be called *collar spaces*, are introduced in the following subsection, and then we define an appropriate class of operators, referred to as *proper operators*, in such spaces.

1.1.3. Collar spaces

The support of a smooth section $u \in C^\infty(M, E)$ of a vector bundle E over a manifold M is defined as the closure of the set of points $x \in M$ such that $u(x) \neq 0$. It is important in this definition that u is a mapping defined on M. However, there is an equivalent definition that does not rely on this fact explicitly. Namely, since $C^\infty(M, E)$ is a module over the algebra $C^\infty(M)$ of smooth functions on M, one can define the support $\operatorname{supp} u$ as the intersection of the zero sets of all functions $\varphi \in C^\infty(M)$ that annihilate u. The latter definition is valid for an arbitrary $C^\infty(M)$-module regardless of its nature (i.e., of whether elements of this module are defined on M).

Collar spaces are just defined as modules over a function algebra, and so their supports are described by the above-mentioned construction. To state the index locality principle, we need not consider any general function algebras; it suffices to consider functions on the closed interval $[-1, 1]$. (Accordingly, the supports of elements of collar spaces will be subsets of this interval.)

Consider the unital topological algebra $C^\infty([-1, 1])$ of smooth functions $\varphi(t)$, $t \in [-1, 1]$, on the interval $[-1, 1]$.

DEFINITION 1.1. A *collar space* is a separable Hilbert space H equipped with the structure of a module over $C^\infty([-1, 1])$ such that the action of $C^\infty([-1, 1])$ is continuous in the uniform operator topology and the unit function $\mathbf{1} \in C^\infty([-1, 1])$ acts as the identity operator in H.

Let us give one of the main examples of collar spaces.

EXAMPLE 1.2. Let M be a compact C^∞ manifold without boundary, and let $\chi : M \longrightarrow [-1, 1]$ be a smooth mapping. Then each Sobolev space $H^s(M)$ can be made a collar space if we define a natural action of $C^\infty([-1, 1])$ on $H^s(M)$ by the formula

$$(\varphi f)(x) = \varphi(\chi(x)) f(x), \quad x \in M,$$

for any $\varphi \in C^\infty([-1, 1])$ and $f \in H^s(M)$. In this and similar more general cases, the subset

$$U = \overline{\chi^{-1}(-1, 1)} \subset M,$$

of the manifold M, where the bar stands for the closure, will be referred to as the *collar*. We represent this graphically in Fig. 1.1, where the collar U is dashed and the function χ takes the values $\chi = -1$ to the left of the collar (i.e., in M_-) and $\chi = +1$ to the right of the collar (i.e., in M_+).

For any element $h \in H$ of a collar space H, there is a naturally defined notion of *support*.

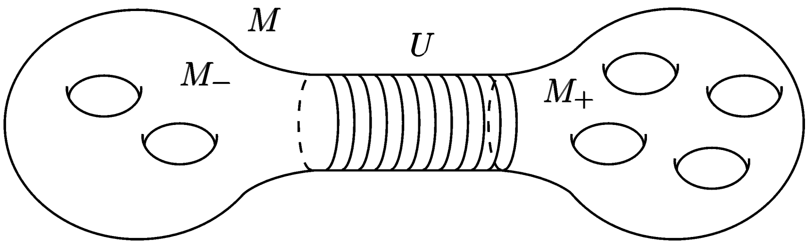

Figure 1.1. A manifold with a collar.

DEFINITION 1.3. The *support* of an element $h \in H$ is the subset $\operatorname{supp} h$ of the interval $[-1,1]$ given by the formula

$$\operatorname{supp} h = \bigcap \varphi^{-1}(0),$$

where the intersection is taken over all functions $\varphi \in C^\infty([-1,1])$ such that $\varphi h = 0$.

The supports of elements of a collar space have all the natural properties. Just to list a few, $\operatorname{supp} h$ is closed, $\operatorname{supp} h$ is empty if and only if $h = 0$, $\operatorname{supp}(h_1 + h_2) \subset \operatorname{supp} h_1 \cup \operatorname{supp} h_2$, etc. Note that in the situation of Example 1.2 the set $\operatorname{supp} h$ is just the closure of the image of the ordinary support of h under the mapping χ. For example, if $f \in H^s(M)$ is supported in M_-, then $\operatorname{supp} f = \{-1\}$.

One can single out subspaces of a collar space H by imposing conditions on the supports of elements. Let $F \subset [-1,1]$ be a given subset. Then the set

$$\overset{\circ}{H}_F = \{h \in H \mid \operatorname{supp} h \subset F\}$$

is a linear manifold (lineal) in H, which is closed provided that so is F. In the general case, we define a subspace $H_F \subset H$ as the *closure* of $\overset{\circ}{H}_F$. Not that it is not true in general that $H_F = H_{\bar{F}}$.

Just as in the case of ordinary supports, functions with disjoint supports are linearly independent. More precisely, if $F_1, \dots, F_m \subset [-1,1]$ are subsets such that $\bar{F}_j \cap \bar{F}_k = \varnothing$ for $j \neq k$, then

$$H_{F_1 \cup \dots \cup F_m} = H_{F_1} \oplus \dots \oplus H_{F_m},$$

where the sum is direct but not necessarily orthogonal.

1.1.4. Elliptic operators

Here we introduce the notion of an elliptic operator in a collar space. This notion inherits two characteristic properties of elliptic PDO, namely, pseudolocality and the Fredholm property. First, let us give an adequate statement of pseudolocality in the context of collar spaces.

DEFINITION 1.4. Let $A : H_1 \longrightarrow H_2$ be a continuous linear operator in collar spaces and $K \subset [-1,1] \times [-1,1]$ a closed subset. We say that *the support of A is contained in K* if

$$\operatorname{supp} Ah \subset K(\operatorname{supp} h) \tag{1.4}$$

for every $h \in H$, where K is treated as a multimapping of the interval $[-1, 1]$ into itself:

$$Kx \stackrel{\text{def}}{=} \{y \in [-1, 1] \mid (x, y) \in K\}.$$

The *support* $\operatorname{supp} A$ of the operator A is the intersection of all closed sets K with property (1.4).

It readily follows from the definition that if A and B are operators in collar spaces, then one has $\operatorname{supp}(BA) \subset \operatorname{supp} B \circ \operatorname{supp} A$, where the right-hand side is understood as the composition of multimappings.

In the theory of PDO, pseudolocality is understood as the fact that the integral kernel of a PDO is supported modulo smooth functions in an arbitrarily small neighborhood of the diagonal. Hence, using smooth cutoff functions concentrated near the diagonal, one can include an elliptic PDO in a continuous family of elliptic operators with the same symbol and with kernels supported in a shrinking neighborhood of the diagonal as the parameter of the family tends to zero. Let us transfer this property to the case of collar spaces. Let

$$\Delta = \{(x, x) | x \in [-1, 1]\} \subset [-1, 1] \times [-1, 1]$$

be the diagonal and

$$\Delta_\varepsilon = \{(x, y) \in [-1, 1] \times [-1, 1] \mid |x - y| < \varepsilon\} \tag{1.5}$$

its ε-neighborhood.

DEFINITION 1.5. A *proper operator* in collar spaces H_1 and H_2 is a family of continuous linear operators

$$A_\delta : H_1 \longrightarrow H_2$$

with parameter $\delta > 0$ such that

(i)A_δ continuously depends on δ in the uniform operator topology;

(ii) for each $\varepsilon > 0$ there is a $\delta_0 > 0$ such that

$$\operatorname{supp} A_\delta \subset \Delta_\varepsilon \quad \text{for} \quad \delta < \delta_0. \tag{1.6}$$

In the situation of Example 1.2, PDO in Sobolev spaces can naturally be viewed as proper operators (more precisely, included in appropriate families).

◀ Indeed, as was indicated above, a PDO can be included in a continuous family of PDO with the same symbol and with supports of the integral kernels shrinking to the diagonal (in $M \times M$). It remains to note that the preimage of the ε-neighborhood of the diagonal in $M \times M$ under the mapping $\chi \times \chi$ is contained in Δ_ε. ▶

Now we can give the definition of elliptic operators in collar spaces.

DEFINITION 1.6. An *elliptic operator* in collar spaces H and G is a proper operator

$$D_\delta : H \longrightarrow G$$

such that D_δ is Fredholm for each δ and has an almost inverse D_δ^{-1} such that the family D_δ^{-1} is also a proper operator.

Here, as usual, the almost inverse of a bounded operator A is defined as an operator A^{-1} such that the products AA^{-1} and $A^{-1}A$ differ from the identity operators by compact operators in the corresponding spaces.

Remark 1.7. The class of elliptic operators in collar spaces considered here is wider than Atiyah's class of *abstract* elliptic operators (Atiyah 1969), consisting of Fredholm operators $A : H \longrightarrow G$ acting in Hilbert $C(X)$-modules H and G (where $C(X)$ is the C^*-algebra of continuous functions on a compact set X) and commuting with the action of $C(X)$ modulo compact operators: the operator

$$\varphi A - A\varphi : H \longrightarrow G$$

is compact for every $\varphi \in C(X)$.

More precisely, every $C([-1,1])$-module is a collar space by virtue of the continuous embedding of algebras $C^\infty([-1,1]) \subset C([-1,1])$, and any abstract elliptic operator A on X can be embedded in a family that is an elliptic operator in collar spaces.

EXERCISE 1.8. Prove this.

However, the opposite is not true.

◀ Indeed, we define the structure of a collar space on $L_2(\mathbb{R})$ via the mapping

$$\chi : \mathbb{R} \longrightarrow [-1,1], \quad \chi(t) = \frac{2}{\pi} \tan^{-1} t.$$

The operator

$$\begin{aligned} A_\delta : L_2(\mathbb{R}) &\longrightarrow L_2(\mathbb{R}) \\ f(t) &\longmapsto f(t+\delta) \end{aligned}$$

is an elliptic operator in collar spaces but is not an abstract elliptic operator for any $\delta > 0$. ▶

1.1.5. Surgery and the relative index theorem

The relative index theorem stated in this subsection expresses the index locality principle at the abstract level of collar spaces. The *relative index* of two Fredholm operators D' and D is the difference

$$\operatorname{ind}(D', D) = \operatorname{ind} D' - \operatorname{ind} D.$$

In the locality property considered above for the case of a smooth compact manifold without boundary, the operator D' was obtained from an elliptic operator D on a manifold M by some change on a subset $V \subset M$. (Moreover, V itself could change, as far as $M \setminus V$ remained intact.) At the abstract level, we should first make it clear what changes of operators are to be considered. Admissible changes will be referred to as *surgeries*; they include surgeries of collar spaces themselves (which corresponds to the replacement of V by V') and associated surgeries of operators.

Let H_1 and H_2 be collar spaces.

DEFINITION 1.9. If for some $F \subset [-1,1]$ there is a given isomorphism (not necessarily isometric)

$$j : H_1(F) \cong H_2(F),$$

then we say that H_1 and H_2 *coincide* on F (or *are obtained from each other by surgery on* $[-1,1]\backslash F$). In this case, we write

$$H_1 \stackrel{F}{=} H_2 \text{ or } H_1 \xleftrightarrow{[-1,1]\backslash F} H_2.$$

We point out that the *specific form* of the isomorphism is important here (rather than the existence of some isomorphism, which is always the case if $H_1(F)$ and $H_2(F)$ have the same dimension).

Let us now extend the notion of surgery to operators. Since proper (and, in particular, elliptic) operators in collar spaces do not change the supports of elements "too much," one can reasonably speak of coincidence of such operators on a subset of $[-1,1]$ provided that the spaces themselves where the operators act coincide on the subset.

DEFINITION 1.10. Let $F \subset [-1,1]$ be a subset open in $[-1,1]$, and let $H_1 \overset{F}{=} H_2$ and $G_1 \overset{F}{=} G_2$ be collar spaces. We say that proper operators

$$A_1 : H_1 \to G_1,$$
$$A_2 : H_2 \to G_2$$

coincide on F if for each compact subset $K \subset F$ the following condition is satisfied: there is a number $\delta_0 = \delta_0(K) > 0$ such that

$$A_{1\delta} h = A_{2\delta} h \tag{1.7}$$

whenever $\delta < \delta_0$ and $\operatorname{supp} h \subset K$. Under this condition, we also say that A_1 is obtained from A_2 by *surgery* on $[-1,1]\backslash F$ and write

$$A_1 \overset{F}{=} A_2 \text{ or } A_1 \overset{[-1,1]\backslash F}{\longleftrightarrow} A_2.$$

We note that (1.7) is well defined, since $h \in H_1(K) \cong H_2(K)$ and for small δ one has $A_{1\delta} h$, $A_{2\delta} h \in G_1(F) \cong G_2(F)$; the latter inclusion follows from the fact that F is open.

In the following, we deal with *surgery diagrams* rather than individual surgeries, more precisely, with squares of the form

$$\begin{array}{ccc} H_1 & \overset{B}{\longleftrightarrow} & H_2 \\ {\scriptstyle A}\updownarrow & & \updownarrow{\scriptstyle C} \\ H_3 & \overset{D}{\longleftrightarrow} & H_4, \end{array}$$

where the H_i are collar spaces and $A, B, C, D \subset [-1,1]$. This square is said to *commute* if the diagram

$$\begin{array}{ccc} H_1(F) & \approx & H_2(F) \\ \wr\wr & & \wr\wr \\ H_3(F) & \approx & H_4(F), \end{array} \qquad F = [-1,1]\backslash\{A \cup B \cup C \cup D\}$$

of isomorphisms commutes, where the arrows are the restrictions to the relevant subspaces of the corresponding isomorphisms occurring in Definition 1.9. A similar square of modifications for *operators* is said to commute if the underlying squares of modifications of collar spaces commute.

Let us now state the main theorem of this section.

THEOREM 1.11. *Let the surgery diagram*

$$\begin{array}{ccc} D & \overset{-1}{\longleftrightarrow} & D_- \\ {\scriptstyle 1}\updownarrow & & \updownarrow{\scriptstyle 1} \\ D_+ & \overset{-1}{\longleftrightarrow} & D_\pm \end{array} \tag{1.8}$$

of elliptic operators in collar spaces commute. Then the relative indices of the operator occurring in the diagram satisfy the relation

$$\operatorname{ind}(D, D_-) = \operatorname{ind}(D_+, D_\pm).$$

The proof can be found in (Nazaikinskii and Sternin 2001).

The following two sections provide examples illustrating the index locality principle for elliptic operators on smooth manifolds and elliptic boundary value problems. Applications of this principle to elliptic problems on manifolds with singularities will be given in Part III.

1.2. Surgery in Index Theory on Smooth Manifolds

In this section we consider two examples, one of which pertains to compact manifolds and the other, to noncompact ones.

1.2.1. The Booß–Wojciechowski theorem

In this subsection, we study how the index of an elliptic PDO on a manifold changes under surgery on the manifold where the operator is defined and some associated surgery on the bundles in whose sections it acts. The index increment formula can naturally be treated as a *relative index formula*, for the symbol before and after the surgery is essentially the same but is realized differently depending on how the manifold (and the bundles) have been glued from pieces.

Let M be an orientable manifold (possibly, with boundary and/or singularities), and let $S \subset M$ be an embedded smooth compact two-sided submanifold of codimension 1 contained in the smooth "interior" part of M. Next, let U be a collar neighborhood of S contained in the smooth part of M. We choose and fix some trivialization $U = (-1, 1) \times S$ of this neighborhood and use there the coordinates (t, s), $t \in (0, 1)$, $s \in S$. Let

$$g \colon S \to S$$

be a given diffeomorphism. We perform the following operation: we cut M along S and glue together again, identifying each point $(-0, s)$ on the left coast of the cut with the corresponding point $(+0, g(s))$ on the right coast. The resulting smooth manifold (the smooth structure is well defined, since we have chosen and fixed the trivialization) will be denoted by M_g and called the *surgery of M via g*.

Let E be a vector bundle over M. Suppose that we are given an isomorphism of vector bundles

$$\mu \colon E|_S \to g^* \left(E|_S \right).$$

Then over M_g there is a naturally defined vector bundle $E_{g,\mu}$ (by clutching along S with the help of μ), which will be called the *surgery of E via the pair (g, μ)*.

Now let E and F be two vector bundles over M, and let

$$a : \pi^* E \to \pi^* F,$$

where $\pi : T_0^* M \to T_0^* M$ is the natural projection, be an elliptic symbol of some order m. By choosing the representation

$$E|_U = \widetilde{\pi}^{-1} \left(E|_S \right), \qquad F|_U = \widetilde{\pi}^{-1} \left(F|_S \right),$$

of the bundles E, F over $U = [0,1] \times S$, where $\widetilde{\pi}\colon [0,1] \times S \to S$ is the natural projection, and by passing to a homotopic symbol if necessary, we can assume that a is independent of the coordinate t in a sufficiently small neighborhood of S (that is, $a \equiv a_0$ in that neighborhood). Consider the mapping (denoted by the same letter)

$$a_0 \stackrel{\text{def}}{=} a\big|_{\pi^{-1}S}\colon \pi^* E|_{\pi^{-1}S} \longrightarrow \pi^* F|_{\pi^{-1}S}. \tag{1.9}$$

With regard to the trivialization chosen, this mapping can be rewritten in the form

$$a_0(p,s,\xi)\colon E_s \longrightarrow F_s, \qquad p^2 + |\xi|^2 \neq 0, \quad s \in S,$$

where p is the dual variable of t and ξ is a point in the fiber of T^*S over s.

Suppose that a surgery g of M and associated surgeries μ_E and μ_F of the bundles E and F are given. If the diagram

$$\begin{array}{ccc} E_s & \xrightarrow{\quad a_0(p,s,\xi) \quad} & F_s \\ \downarrow \mu_E(s) & & \downarrow \mu_F(s) \\ E_{g(s)} & \xrightarrow{\quad a_0(p,g(s),{}^t g_s(s)^{-1}\xi) \quad} & F_{g(s)} \end{array}, \tag{1.10}$$

where ${}^t g_s(s)$ is the transposed Jacobi matrix of the mapping g at the point s, commutes, then the surgery takes the original symbol a to a new (smooth) symbol $\widetilde{a}$ on the cotangent bundle $T_0^* M_g$. (The smoothness of the newly obtained symbol is guaranteed by the independence of a on the coordinate t in a neighborhood of S.) Our task is to establish how the surgery affects the index of the corresponding PDO. By the locality principle, this index increment (the relative index) depends only on the surgery on S, and so we use the relative index theorem to pass to a simpler model.

Thus, let A and $\widetilde{A}$ be operators with principal symbols a and $\widetilde{a}$ on the manifolds M and M_g, respectively, obtained from each other by the above-mentioned surgery. The problem is to find the relative index $\operatorname{ind} \widetilde{A} - \operatorname{ind} A$.

It follows from Theorem 1.11 that the relative index is independent of the structure of the manifolds and operators in question outside a small neighborhood of S. Hence we can use the simplest model for the computations. Namely, consider the manifold $M = S \times S^1$ and the elliptic pseudodifferential operator

$$A_0 : H^s(M,E) \to H^{s-m}(M,F)$$

of order m with principal symbol a_0 independent of $\varphi \in S^1$. (The bundles E and F are lifted to M with the help of the natural projection $M = S \times S^1 \to S$.) Next, let M_g be the surgery of M with the help of g, let E_{g,μ_E} and E_{g,μ_F} be the associated surgeries of the bundles E and F, and let

$$\widetilde{A}_0 : H^s(M_g, E_{g,\mu_E}) \to H^{s-m}(M_g, E_{g,\mu_F})$$

be the new elliptic pseudodifferential operator with principal symbol $\widetilde{a}_0$ coinciding with A_0 outside a neighborhood of the set S, where the surgery is done.

The operators A_0 and $\widetilde{A}_0$ are elliptic operators on compact manifolds, and their index can be calculated by the Atiyah–Singer theorem. Moreover, the index of A_0 is zero, since its symbol is independent of $\varphi \in S^1$. Hence in this model only one term in the expression for the relative index is nontrivial:

$$\operatorname{ind} \widetilde{A} - \operatorname{ind} A = \operatorname{ind} \widetilde{A}_0.$$

This is the assertion of the Booß–Wojciechowski theorem.

1.2.2. The Gromov–Lawson theorem

The locality principle in index theory on noncompact manifolds was apparently obtained for the first time (for the special case of Dirac operators on noncompact Riemannian manifolds) by Gromov and Lawson (Gromov and Lawson 1983), who obtained the corresponding relative index theorem. (Later Anghel (Anghel 1993) generalized their result to arbitrary self-adjoint elliptic first-order operators on a complete Riemannian manifold.) In this subsection, we briefly describe the result due to Gromov and Lawson and its relationship to the general index locality principle.

Let X_0 and X_1 be complete even-dimensional Riemannian manifolds, and let D_0 and D_1 be generalized Dirac operators on X_0 and X_1, respectively, acting on sections of vector bundles S_1 and S_2. We say that D_0 and D_1 *coincide at infinity* if there exist compact subsets $K_0 \subset X_0$ and $K_1 \subset X_1$, an isometry

$$F : (X_0 \backslash K_0) \overset{\approx}{\to} (X_1 \backslash K_1),$$

and an isometry

$$\widetilde{F} : S_0 \mid_{X_0 \backslash K_0} \to S_1 \mid_{X_1 \backslash K_1}$$

of vector bundles such that

$$D_1 = \widetilde{F} \circ D_0 \circ \widetilde{F}^{-1} \quad \text{on} \quad X_1 \backslash K_1.$$

To simplify the notation, we identify $X_0 \backslash K_0$ with $X_1 \backslash K_1$ and write

$$D_0 = D_1 \quad \text{on} \quad \Omega = X_0 \backslash K_0 \cong X_1 \backslash K_1.$$

In this situation we can define the *topological relative index* $\operatorname{ind}_t(D_1^+, D_0^+)$ of the operators

$$D_1^+ : \Gamma(S_1^+) \to \Gamma(S_1^-) \quad \text{and} \quad D_0^+ : \Gamma(S_0^+) \to \Gamma(S_0^-)$$

as follows. If X_0 and X_1 are compact, then we simply set

$$\operatorname{ind}_t(D_1^+, D_0^+) = \operatorname{index}(D_1^+) - \operatorname{index}(D_0^+).$$

If X_0 (and hence X_1) is noncompact, then we use the following procedure. We cut the manifolds X_0 and X_1 along some compact hypersurface $H \subset \Omega$ and compactify them by attaching some compact manifold with boundary H. The operators D_0^+ and D_1^+ can be extended to elliptic operators $\widetilde{D}_0^+$ and $\widetilde{D}_1^+$ on the compact manifolds thus obtained. Now we set

$$\operatorname{ind}_t(D_1^+, D_0^+) = \operatorname{ind}(\widetilde{D}_1^+) - \operatorname{ind}(\widetilde{D}_0^+). \tag{1.11}$$

It follows from the Relative Index Theorem 1.11 that the right-hand side of (1.11) is independent of the arbitrariness in the above construction.

Next, let the operators D_0^+ and D_1^+ be *positive at infinity* (the precise definition is given in (Gromov and Lawson 1983); roughly speaking, this condition means that the free terms in the operators $(D_0^+)^* D_0$ and $(D_1^+)^* D_1$ expressed via covariant derivatives are positive). Then the operators D_0^+ and D_1^+ are Fredholm, and one can define the *analytical relative index*

$$\operatorname{ind}_a(D_1^+, D_0^+) = \operatorname{ind}_a(D_1^+) - \operatorname{ind}_a(D_0^+). \tag{1.12}$$

The Gromov–Lawson relative index theorem states that *the topological and analytical relative indices coincide*:

$$\operatorname{ind}_a(D_1^+, D_0^+) = \operatorname{ind}_t(D_1^+, D_0^+). \tag{1.13}$$

In (Gromov and Lawson 1983) one can also find a more general theorem pertaining to the case in which the operators D_0 and D_1 coincide only on some of the "ends" of X_0 and X_1 at infinity. In this case, one again has a formula like (1.13), where the right-hand side is no longer the "topological relative index," but it is rather the analytical index of some elliptic Fredholm operator on a (generally speaking, noncompact) manifold obtained from X_0 and X_1 by cutting away the "common" ends along some hypersurface followed by gluing along that hypersurface. The proof uses the same technique.

We can conclude (as is easily seen from the second theorem) that the *topological index* actually has nothing to do with the Gromov–Lawson relative index theorem: this theorem states the equality of the *analytical* relative indices for two pairs of operators obtained from each other by simultaneous surgery on a part of the manifold where they coincide; the topological index occurs in the answer only if the newly obtained operators fall within the scope of the Atiyah–Singer theorem. (On the other hand, naturally, the *applications* of theorems of that type are just related to transforming the original operators to new operators such that the Atiyah–Singer theorem or any other theorem expressing the index in topological terms can be used.) As to the equality of analytic relative indices, it directly falls within the scope of Theorem 1.11.

1.3. Surgery for Boundary Value Problems

In this section, we describe some applications of the locality principle for the relative index to the theory of boundary value problems for elliptic differential operators.

1.3.1. Notation

Let X be a smooth compact n-dimensional manifold with boundary $\partial X = Y$ that is a smooth closed manifold of dimension $n-1$. We choose and fix a representation of some collar neighborhood U of the boundary in the form of a direct product

$$U \simeq Y \times [0,1), \tag{1.14}$$

where Y is taken to $Y \times \{0\}$ by the identity mapping. The coordinate on $[0,1)$ will be denoted by t, an the local coordinates on the boundary by $y = (y_1, \ldots, y_{n-1})$, so that local coordinates on X in U have the form

$$x = (x_1, \ldots, x_n) = (y, t).$$

If E is a vector bundle over X, then the restriction $E|_U$ is isomorphic to the lift to U of the restriction $E|_Y$ of the same bundle to the boundary:

$$E_U \simeq \pi_U^* E|_Y, \tag{1.15}$$

where $\pi_U : U \to Y$ is the projection naturally associated with the representation (1.14)

Now let

$$\widehat{D} : C^\infty(X, E_1) \to C^\infty(X, E_2) \tag{1.16}$$

be an elliptic differential operator of order m on X acting in sections of finite-dimensional vector bundles E_1 and E_2. Then, using the trivialization (1.14) and the associated representations (1.15) of E_1 and E_2 over U as the lifts of $E_1|_Y$ and $E_2|_Y$, we can represent the operator (1.16) in U in the form

$$\widehat{D} = \sum_{j=0}^{m} \widehat{D}_j(t)\left(-i\frac{\partial}{\partial t}\right)^j, \tag{1.17}$$

where

$$\widehat{D}_j(t) \,:\, C^\infty(Y, E_1|_Y) \to C^\infty(Y, E_2|_Y)$$

is a differential operator of order $m-j$ in sections of bundles over Y, depending on the parameter t. Next, the coefficient $\widehat{D}_m(t)$ is a differential operator of order 0, i.e., a bundle homomorphism, and since $\widehat{D}$ is elliptic, this coefficient is a bundle isomorphism. Dividing the operator $\widehat{D}$ in U by this coefficient on the left, we can assume without loss of generality that the bundles $E_1|_Y$ and $E_2|_Y$ coincide and the coefficient itself is the identity operator.

The operator family

$$\mathfrak{D}(p) = \sum_{j=0}^{m} \widehat{D}_j(0)p^j \,:\, H^s(Y) \to H^{s-m}(Y) \tag{1.18}$$

acting in Sobolev spaces[3] on Y and obtained from the representation (1.17) by freezing the coefficients at the boundary $t=0$ and by replacing the operator $-i\partial/\partial t$ with the variable p will be called the *conormal symbol* of the operator $\widehat{D}$.

If $u \in H^s(X)$ is an element of a Sobolev space on X, then for $s > m - 1/2$ by trace theorems we have a well-defined jet of order $m-1$ of u on Y. With regard to the identifications (1.14) and (1.15), it can be rewritten in the form

$$j_X^{m-1}u = \left(u\Big|_{t=0}, \frac{\partial u}{\partial t}\Big|_{t=0}, \dots, \frac{\partial^{m-1}u}{\partial t^{m-1}}\Big|_{t=o}\right) \in H^{s-1/2}(Y) \oplus \dots \oplus H^{s-m+1/2}(Y). \tag{1.19}$$

Boundary value problems for $\widehat{D}$ are stated in terms of the boundary jet (1.19) of u, to which one applies some differential or pseudodifferential operators. Since for $m > 1$ the space on the right-hand side in (1.19), which for brevity will be denoted by

$$\mathcal{H}_m^{s-1/2}(Y) = \bigoplus_{k=0}^{m-1} H^{s-k-1/2}(Y), \tag{1.20}$$

is a direct sum of Sobolev spaces of *distinct orders*, the orders of ΨDO in such spaces must be understood in the sense of Douglis–Nirenberg.

Recall that, for example, under this definition the operator

$$\widehat{B} = (\widehat{b}_{jk})_{j,k=0}^{m-1} \,:\, \mathcal{H}_m^s(Y) \to \mathcal{H}_m^{s-l}(Y) \tag{1.21}$$

given by a matrix $\widehat{b}_{jk}$ of PDO is of order $\le l$ if the orders (in the usual sense) of its entries satisfy the conditions

$$\operatorname{ord}\widehat{b}_{jk} \le l + j - k.$$

[3]In what follows, we usually omit the bundles in the notation of Sobolev spaces.

The *principal symbol* of the operator (1.21) of order l in the sense of Douglis–Nirenberg is given by the matrix

$$\sigma(\widehat{B}) = (\sigma_{l+j-k}(\widehat{b}_{jk}))_{j,k=0}^{m-1} \tag{1.22}$$

of principal symbols of the corresponding orders of the entries of the operator matrix (1.21).

1.3.2. General boundary value problems

Let $\widehat{D}$ be an elliptic differential operator (1.16) on a manifold X. A *general boundary value problem* for $\widehat{D}$ is a problem of the form

$$\left\{\begin{array}{rcl} \widehat{D}u & = & f \in H^{s-m}(X), \\ \widehat{B}j_Y^{m-1}u & = & g \in \mathcal{L}, \end{array}\right. \tag{1.23}$$

where $s > m - 1/2$, $u \in H^s(X)$, $\mathcal{L}$ is a Hilbert space, and $\widehat{B}$ is a continuous linear operator in the spaces

$$\widehat{B} : \mathcal{H}_m^{s-1/2}(Y) \to \mathcal{L}. \tag{1.24}$$

Ordinary boundary value problems are the special case in which $\mathcal{L}$ is a Sobolev space of sections of some vector bundle over the boundary and B is a (pseudo)differential operator. If $\widehat{D}$ is the Dirac operator on an even-dimensional manifold X, $\mathcal{L}$ is the positive spectral subspace of the tangential operator, and B is the orthogonal projection on $\mathcal{L}$, then we arrive at the Atiyah–Patodi–Singer problem (Atiyah, Patodi and Singer 1975), more precisely, a more general problem in which the nonlocal boundary data may be nonzero.

As shown by these examples, of main interest is the case in which $\mathcal{L}$ is not an abstract Hilbert space but rather a subspace of some Sobolev space on the boundary[4] and B is a PDO. More precisely, we shall consider only subspaces that are ranges of pseudodifferential projections. If $\widehat{P}$ is a pseudodifferential projection on some subspace $\widehat{L}$ of a Sobolev space of sections of some vector bundle F over Y, then the principal symbol $P = \sigma(\widehat{P})$ is a projection on a subbundle $L \subset \pi^*F$ over T_0^*Y, where $\pi : T_0^*Y \to Y$ is the natural projection. The subbundle L is called the *principal symbol* of $\widehat{L}$.(This is well defined, that is, independent of the choice of a pseudodifferential projection on $\widehat{L}$.) The pseudodifferential version of the general boundary value problem (1.23) for an unknown function $u \in H^s(X, E_1)$ has the form

$$\left\{\begin{array}{rcl} \widehat{D}u & = & f \in H^{s-m}(X, E_2), \\ \widehat{B}j_Y^{m-1}u & = & g \in \widehat{P}H(Y, F), \end{array}\right. \tag{1.25}$$

where $H(Y, F)$ is a Sobolev space of sections of a bundle F over the boundary (we intentionally omit the index on this space, since it can be a usual Sobolev space or a space of the form $\mathcal{H}_m^s(Y)$) and $\widehat{B} : \mathcal{H}_m^{s-1/2} \to H(Y, F)$ is a PDO such that $R(\widehat{B}) \subseteq R(\widehat{P})$. (The last inclusion necessarily implies that $R(B) \subseteq R(P) = L$.)

Of general boundary value problems (1.23), we single out problems that are a straightforward (nonhomogeneous) analog of the Atiyah–Patodi–Singer problem. Namely, let an operator $\widehat{D}$ of order m be given. On the basis of the conormal symbol (1.18) of $\widehat{D}$, we shall construct a pseudodifferential projection

$$\widehat{P}_+ : \mathcal{H}_m^s(Y) \to \mathcal{H}_m^s(Y) \tag{1.26}$$

in the Cauchy data space (1.20).

[4] In particular, the entire Sobolev space.

◀ This projection can be constructed as follows (see details in (Nazaikinskii, Schulze, Sternin and Shatalov 1998)). If the order m of the operator $\widehat{D}$ is equal to 1, then in the neighborhood U it is represented (up to multiplication by a bundle isomorphism) in the form

$$\widehat{D} = -i\frac{\partial}{\partial t} + A(t),$$

and as $\widehat{P}_+$ we take the spectral projection of the tangential operator $A(0)$ corresponding to the part of the spectrum lying in the right half-plane. If the order m is greater than one, then a standard trick reduces $\widehat{D}$ to a matrix operator $\widehat{D_1}$ of the first order with respect to $-i\frac{\partial}{\partial t}$, and as $\widehat{P}_+$ we take the spectral projection of the tangential operator for $\widehat{D_1}$. ▶

In the following, we also set

$$\widehat{P}_- \stackrel{\text{def}}{=} 1 - \widehat{P}_+. \tag{1.27}$$

Needless to say, if $\widehat{D}$ is the Dirac operator, then the projection $\widehat{P}_+$ thus introduced coincides with the Atiyah–Patodi–Singer spectral projection. The *spectral boundary value problem* is problem (1.23) of the special form

$$\begin{cases} \widehat{D}u = f \in \mathcal{H}^{s-u}(X) \\ \widehat{P}_+ j_Y^{m-1} u = g \in \widehat{P}_+ \mathcal{H}_m^{s-1/2}(Y). \end{cases} \tag{1.28}$$

Problem (1.28), which will be denoted by $(\widehat{D}, \widehat{P}_+)$ for brevity, is always Fredholm. The index of a general Fredholm boundary value problem (1.23), which will be denoted by $(\widehat{D}, \widehat{B})$, is expressed by the formula

$$\operatorname{ind}(\widehat{D}, \widehat{P}) = \operatorname{ind}(\widehat{D}, \widehat{P}_+) + \operatorname{ind}(\widehat{B} : \widehat{P}_+ \mathcal{H}_m^{s-1/2}(Y) \to \mathcal{L}). \tag{1.29}$$

Problem (1.25) is Fredholm if and only if the principal symbol B of the operator $\widehat{B}$ is an isomorphism between the principal symbol L_+ of the subspace $\widehat{L}_+ = \widehat{P}_+ \mathcal{H}_m^{s-1/2}(Y)$ and L. In this case, the above general index formula (1.29) is also valid.

1.3.3. A model boundary value problem on a cylinder

In this subsection, the simplest model problem that will later be used as a technical tool in the application of the locality principle to boundary value problems.

Let Y be a closed C^∞ manifold. On the cylinder

$$C = Y \times [-1, 1] \tag{1.30}$$

with boundary

$$\partial C = (Y \times \{-1\}) \cup (Y \times \{+1\})$$

consisting of two separate components (faces) $Y \times \{\pm 1\}$, we consider an elliptic differential operator D of order m with coefficients independent of the coordinate $t \in [-1, 1]$:

$$\widehat{D} = \left(-i\frac{\partial}{\partial t}\right)^m + \sum_{j=0}^{m-1} \widehat{D}_j \left(-i\frac{\partial}{\partial t}\right)^j. \tag{1.31}$$

Here $\widehat{D}_j$ is a differential operator of order $m-j$ on Y; in accordance with the preceding, we assume that $\widehat{D}_m$ (the coefficient of $(-i\partial/\partial t)^m$) is the identity operator.

The conormal symbol of $\widehat{D}$ on each of the faces has the form

$$\mathfrak{D}_{-1}(p) = p^m + \sum_{j=0}^{m-1} \widehat{D}_j p^j \quad \text{on} \quad Y \times \{-1\}, \tag{1.32}$$

$$\mathfrak{D}_1(p) = \mathfrak{D}_{-1}(-p) \equiv (-p)^m + \sum_{j=0}^{m-1} \widehat{D}_j (-p)^j \quad \text{on} \quad Y \times \{1\}. \tag{1.33}$$

We denote $\mathfrak{D}_{-1}(p)$ simply by $\mathfrak{D}(p)$ and the corresponding positive spectral projection in $\mathcal{H}^s_m(Y)$ by $\widehat{P}_+$. Then the positive spectral projection corresponding to $\mathfrak{D}_1(p)$ differs from $\widehat{P}_- = 1 - \widehat{P}_+$ by a finite-dimensional operator (and coincides with $\widehat{P}_-$ if $D(p)$ is invertible for all $p \in \mathbb{R}$).

Model problem 1 (a spectral problem)

$$\begin{cases} \widehat{D}u = f \in H^s(C), \\ \widehat{P}_+ j^{m-1}_{Y\times\{-1\}} u = g \in \widehat{P}_+ \mathcal{H}^{s-1/2}_m(Y), \\ \widehat{P}_- j^{m-1}_{Y\times\{1\}} u = h \in \widehat{P}_- \mathcal{H}^{s-1/2}_m(Y). \end{cases} \tag{1.34}$$

In this problem, the boundary conditions are determined by complementary projections ($\widehat{P}_+ + \widehat{P}_- = 1$) on the faces of the cylinder.

THEOREM 1.12. *The index of the model problem* (1.34) *is zero.*

◀ One can prove this, for example, as follows. For simplicity, assume that the operator $\widehat{D}$ is of the first order. The index of problem (1.34) does not change if we replace $\widehat{D}$ by $\widehat{D} - \varepsilon$, where ε is a small positive number. Hence without loss of generality we can assume that the conormal symbol $\mathfrak{D}(p)$ is invertible for all $p \in \mathbb{R}$, and then problem (1.34) is uniquely solvable for any right-hand sides and boundary conditions. ▶

1.3.4. The Agranovich–Dynin theorem

This theorem, as well as the "dual" Agranovich theorem considered in the next subsection, is one of the expressions of the locality principle as applied to boundary value problems.

THEOREM 1.13. *Let $\widehat{D}$ be an elliptic differential operator on a compact C^∞ manifold X with boundary $\partial X = Y$, and let $\widehat{B}_1$ and $\widehat{B}_2$ be two operators each of which specifies elliptic* (*in the sense of Shapiro–Lopatinskii* (Agranovich 1997)) *boundary conditions for the operator $\widehat{D}$. Then the relative index of the elliptic boundary value problems $(\widehat{D}, \widehat{B}_1)$ and $(\widehat{D}, \widehat{B}_2)$ is equal to*

$$\operatorname{ind}(\widehat{D}, \widehat{B}_1) - \operatorname{ind}(\widehat{D}, \widehat{B}_2) = \operatorname{ind}(\widehat{B_1 \circ B_2^{-1}}) \tag{1.35}$$

where $(\widehat{B_1 \circ B_2^{-1}})$ is an elliptic PDO on Y with principal symbol $(B_1 \circ B_2^{-1})$; here B_1 and B_2 are treated as the restrictions of the principal symbols of the operators $\widehat{B}_1$ and $\widehat{B}_2$ to the subbundle L_+, which is the principal symbol of the subspace $\widehat{L}_+$.

Proof. We shall derive this theorem from the locality principle for the relative index. We equip Sobolev spaces on X with the structure of collar spaces using a function $\chi : X \to [-1,1]$ equal to -1 in a neighborhood of Y, equal to 1 outside the collar neighborhood U of Y, and increasing from -1 to 1 in U. In various function spaces on Y, we also introduce the structure of collar spaces by setting

$$\varphi g \stackrel{\text{def}}{=} \varphi(-1)g \tag{1.36}$$

for any elements g of such spaces and any $\varphi \in C^\infty([-1,1])$. Then elliptic boundary value problems generate elliptic operators in collar spaces (this follows from the structure of parametrices of boundary value problems; e.g., see (Hörmander 1985)). Without loss of generality, we can assume that the coefficients of $\widehat{D}$ are independent of the collar variable t in U. Consider the surgery diagram shown in Fig. 1.2. Here $\widehat{D}_0$ in the right column of the diagram is the operator on the cylinder naturally obtained from $\widehat{D}$ by

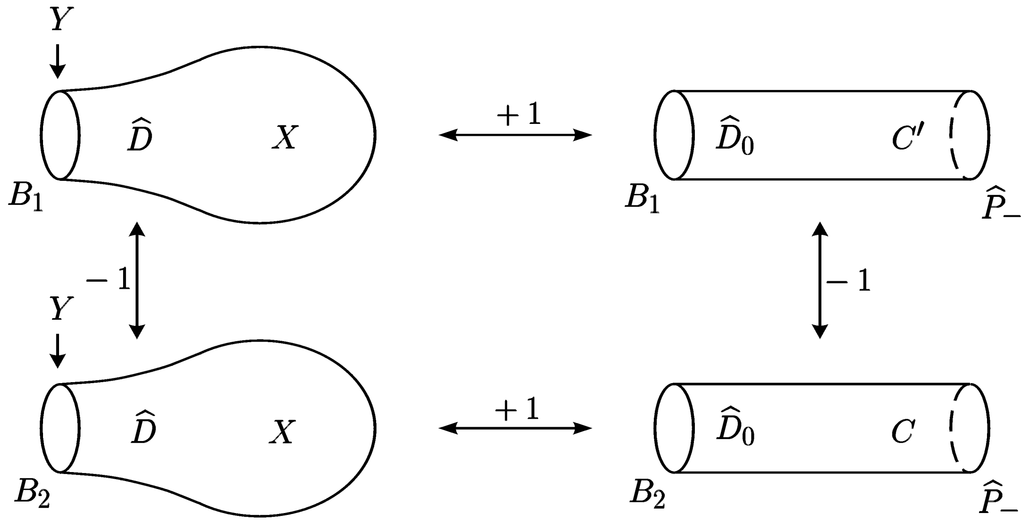

Figure 1.2. Surgery for boundary value problems.

freezing the coefficients on the boundary.

By the locality principle for the relative index, we have

$$\operatorname{ind}(\widehat{D},\widehat{B}_1) - \operatorname{ind}(\widehat{D},\widehat{B}_2) = \operatorname{ind}(\widehat{D}_0,\widehat{B}_1,\widehat{P}_-) - \operatorname{ind}(\widehat{D}_0,\widehat{B}_2,\widehat{P}_-). \tag{1.37}$$

The indices on the right-hand side can be computed by formula (1.29) with regard to the fact that the index of the problem $(\widehat{D}_0,\widehat{P}_+,\widehat{P}_-)$ is zero. We have

$$\operatorname{ind}(\widehat{D}_0,\widehat{B}_1,\widehat{P}_-) = \operatorname{ind}(\widehat{B}_1 : \widehat{L}_+ \to \mathcal{L}_1),$$
$$\operatorname{ind}(\widehat{D}_0,\widehat{B}_2,\widehat{P}_-) = \operatorname{ind}(\widehat{B}_2 : \widehat{L}_+ \to \mathcal{L}_2),$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the Sobolev spaces on Y in which the operators $\widehat{B}_1$ and $\widehat{B}_2$ act. Then

$$\operatorname{ind}(\widehat{D}_0,\widehat{B}_1,\widehat{P}_-) - \operatorname{ind}(\widehat{D}_0,\widehat{B}_2,\widehat{P}_-) = \operatorname{ind}(\widehat{B}_1\widehat{B}_2^{[-1]} : \mathcal{L}_2 \to \mathcal{L}_1) = \operatorname{ind}(\widehat{B_1B_2^{-1}}), \tag{1.38}$$

as desired. (As usual, by $\widehat{B}_2^{[-1]}$ we denote the almost inverse of $\widehat{B}_2$.) □

1.3.5. The Agranovich theorem

The Agranovich theorem deals in a sense with the opposite case.

THEOREM 1.14. *Let $\widehat{D}_1$ and $\widehat{D}_2$ be two elliptic dimensional operators on a compact C^∞ manifold X with boundary $\partial X = Y$ coinciding in a collar neighborhood of the boundary, and let $\widehat{B}$ be a boundary operator satisfying the Shapiro–Lopatinskii conditions with respect to $\widehat{D}_1$ (and hence with respect to $\widehat{D}_2$). Then the relative index of the problems $(\widehat{D}_1, \widehat{B})$ and $(\widehat{D}_2, \widehat{B})$ is equal to*

$$\operatorname{ind}(\widehat{D}_1, \widehat{B}) - \operatorname{ind}(\widehat{D}_2, \widehat{B}) = \operatorname{ind}(\widehat{D_1 D_2^{-1}}), \tag{1.39}$$

where $\widehat{D_1 D_2^{-1}}$ is a PDO on X with principal symbol $D_1 D_2^{-1}$ acting as the identity operator of functions supported in a sufficiently small neighborhood of the boundary.

Remark 1.15. Since the operator $D_1 D_2^{-1}$ acts as the operator of multiplication by the (unit) function in a neighborhood of the boundary, it obviously requires no boundary conditions.

Proof. The operators $\widehat{D}_1$ and $\widehat{D}_2$ can be extended to the double $2X = X \underset{Y}{\cup} X$ as elliptic operators (see (Seeley 1969)).

Since $\widehat{D}_1$ and $\widehat{D}_2$ coincide near the boundary, we can assume that the extensions coincide on the second copy of X. Let us denote these extensions by $\mathfrak{D}_1$ and $\mathfrak{D}_2$. Now consider the surgery diagram shown in Fig. 1.3.

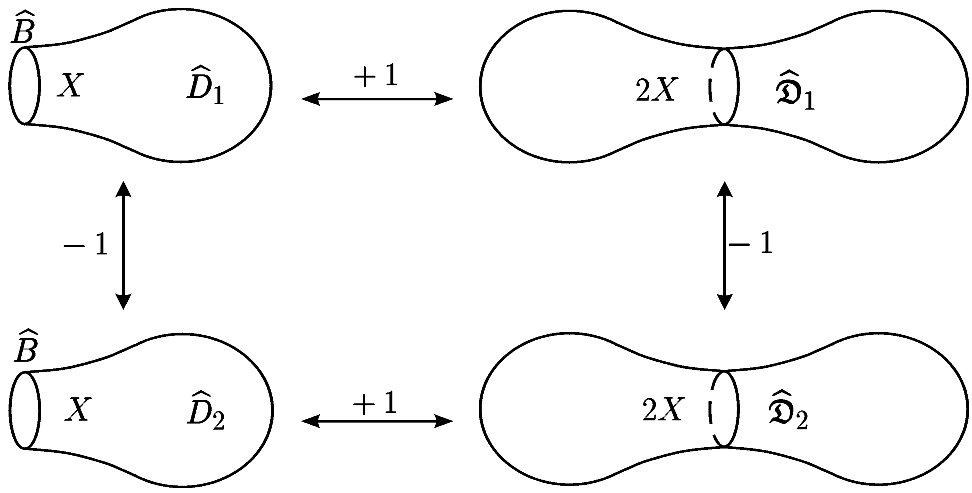

Figure 1.3. Extension to the double.

By the locality principle for the relative index, we obtain

$$\operatorname{ind}(\widehat{D}_1, \widehat{B}) - \operatorname{ind}(\widehat{D}_2, \widehat{B}) = \operatorname{ind}(\widehat{\mathfrak{D}}_1) - \operatorname{ind}(\widehat{\mathfrak{D}}_2) = \operatorname{ind}(\widehat{\mathfrak{D}_1 \mathfrak{D}_2^{-1}}).$$

But it is obvious that

$$\operatorname{ind}(\widehat{\mathfrak{D}_1\mathfrak{D}_2^{-1}}) = \operatorname{ind}(\widehat{D_1 D_2^{-1}}),$$

since the symbol $\mathfrak{D}_1\mathfrak{D}_2^{-1}$ of the operator $\widehat{\mathfrak{D}_1\mathfrak{D}_2^{-1}}$ is equal to unity on the second copy of X and in a neighborhood of Y, so that this operator can be homotopied to an operator acting as the identity operator on functions supported on the second copy of X or in a neighborhood of Y. □

1.3.6. Bojarski's theorem and its generalizations

In the middle 1970s, Bojarski put forward the following *cutting conjecture* in the framework of a surgery proof of the Atiyah–Singer index theorem, which he was developing at the time. Consider a Dirac operator $\widehat{D}$ on a closed connected manifold M. We cut M by a two-sided hypersurface S into two parts M_+ and M_-, $\partial M_+ = \partial M_- = S$, and equip the resulting Dirac operators on M_+ and M_- with the Atiyah–Patodi–Singer conditions $\widehat{P}_+ u_+ = 0$, $\widehat{P}_- u_- = 0$. Then the index of the Dirac operator on M is equal to the relative index of the Fredholm pair of subspaces

$$(\widehat{L}_+ = \operatorname{Im} \widehat{P}_+,\ \widehat{L}_- = \operatorname{Im} \widehat{P}_-).$$

Recall that the relative index of the pair $(\widehat{L}_+, \widehat{L}_-)$ is defined as the index of the Fredholm operator

$$\widehat{P}_+ : \widehat{L}_- \longrightarrow \widehat{L}_+.$$

Later this conjecture was proved (Bojarski's theorem); see the book (Booß-Bavnbek and Wojciechowski 1993) for details. Here we shall prove a theorem on cutting an arbitrary elliptic operator into boundary value problems.

Let M be a closed C^∞ manifold, $\widehat{D}$ an elliptic differential operator on M, and $S \subset M$ a smooth two-sided hypersurface. We cut M along S into two manifolds M_+ and M_- with boundary $\partial M_+ = \partial M_- = S$ and consider general elliptic boundary value problems on M_+ and M_-:

$$\left\{ \begin{array}{rcll} \widehat{D} u_+ &=& f_+, & \text{on} \quad M_+, \\ \widehat{B}_+ j_S^{m-1} u_+ &=& g_+ \in \mathcal{L}_+, & \end{array} \right. \tag{1.40}$$

$$\left\{ \begin{array}{rcll} \widehat{D} u_+ &=& f_-, & \text{on} \quad M_-, \\ \widehat{B}_- j_S^{m-1} u_- &=& g_- \in \mathcal{L}_-, & \end{array} \right. \tag{1.41}$$

where

$$\widehat{B}_+ : \mathcal{H}^s_m(S) \to \mathcal{L}_+ \tag{1.42}$$

$$\widehat{B}_- : \mathcal{H}^s_m(S) \to \mathcal{L}_-, \tag{1.43}$$

are some operators of boundary conditions such that problems (1.40) and (1.41) are Fredholm and $\mathcal{L}_+$ and $\mathcal{L}_-$ are some Hilbert spaces. The restrictions of $\widehat{B}_+$ to $\widehat{L}_+$ and $\widehat{B}_-$ to $\widehat{L}_-$, where $\widehat{L}_\pm = \operatorname{Im} \widehat{P}_\pm$ and the projections $\widehat{P}_\pm$ correspond to the conormal symbol of the operator $\widehat{D}$, will be denoted by the same letters.

THEOREM 1.16.

$$\operatorname{ind}\widehat{D} = \operatorname{ind}(\widehat{D}_+, \widehat{B}_+) + \operatorname{ind}(\widehat{D}_-, \widehat{B}_-) - \operatorname{ind}\begin{pmatrix} \widehat{B}_+ & \widehat{L}_+ & & \mathcal{L}_+ \\ \oplus & : \oplus & \longrightarrow & \oplus \\ \widehat{B}_- & \widehat{L}_- & & \mathcal{L}_- \end{pmatrix} \tag{1.44}$$

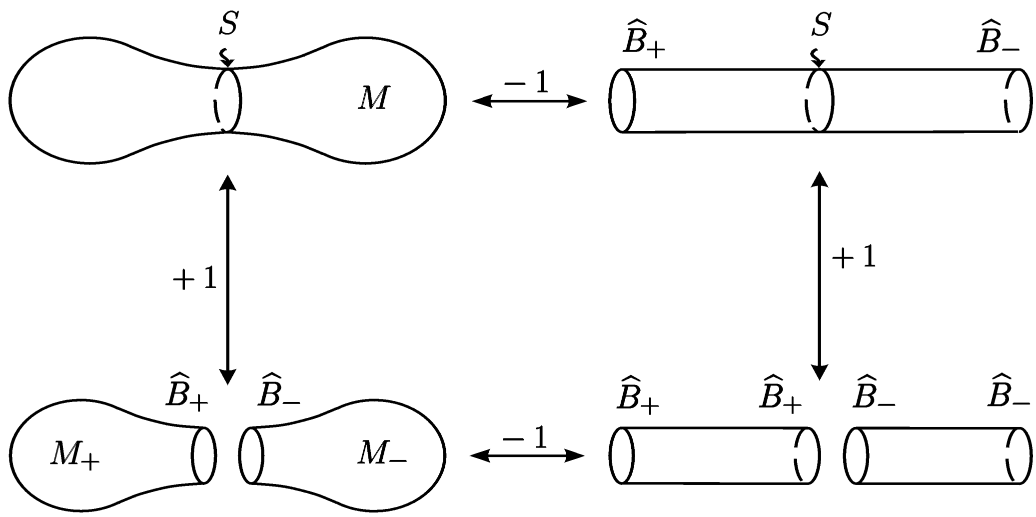

Figure 1.4. Cutting into boundary value problems.

Proof. Consider the surgery diagram shown in Fig. 1.4. In the left column, the main elliptic operator is the operator $\widehat{D}$ on M and its restrictions to M_+ and M_-. In the right column, the main operator is given by the extension to the finite cylinder C of the operator $\widehat{D}$ with coefficients freezed on S. Needless to say, just as before, we assume that the coefficients of $\widehat{D}$ are independent of the collar variable t. By the relative index theorem, we have

$$\operatorname{ind}\widehat{D} - \operatorname{ind}(\widehat{D}_+, \widehat{B}_+) - \operatorname{ind}(\widehat{D}_-, \widehat{B}_-) = \operatorname{ind}\widehat{B}_+ \oplus \widehat{B}_-$$

(the remaining two terms on the right-hand side are zero). The proof is complete. □

1.4. (Micro)localization in Lefschetz theory

Lefschetz theory also enjoys a localization principle, which essentially states that the Lefschetz number of a geometric endomorphism is equal to the sum of contributions of fixed points of the corresponding mapping. This principle also has a microlocal version pertaining to the case in which the endomorphism is associated with a mapping of the phase space, i.e., is a Fourier integral operator. In this section we explain the corresponding results for the case of smooth manifolds.

1.4.1. The Lefschetz number

Let

$$0 \longrightarrow E_0 \xrightarrow{D_0} E_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{m-1}} E_m \longrightarrow 0 \tag{1.45}$$

be a complex of vector spaces over $\mathbb{C}$ with finite-dimensional cohomology, and let

$$T = \{T_j : E_j \to E_j\}\big|_{j=0,\dots,m} \tag{1.46}$$

be an endomorphism of the complex (1.45), that is, a collection of linear mappings such that the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & E_0 & \xrightarrow{D_0} & E_1 & \xrightarrow{D_1} & \cdots & \xrightarrow{D_{m-1}} & E_m & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T_0} & & \downarrow{\scriptstyle T_1} & & & & \downarrow{\scriptstyle T_m} & & \\ 0 & \longrightarrow & E_0 & \xrightarrow{D_0} & E_1 & \xrightarrow{D_1} & \cdots & \xrightarrow{D_{m-1}} & E_m & \longrightarrow & 0 \end{array} \tag{1.47}$$

commutes. It follows from the commutativity of this diagram that

$$T_j(\operatorname{Ker} D_j) \subset \operatorname{Ker} D_j, \quad T_j(\operatorname{Im} D_{j-1}) \subset \operatorname{Im} D_{j-1}, \quad j = 0, \dots, m,$$

where $\operatorname{Ker} A$ and $\operatorname{Im} A$ are the kernel and range of the operator A, respectively, and by definition D_{-1} and D_m are zero maps. Hence the endomorphism (1.46) induces well-defined operators

$$\widetilde{T}_j : H^j(E) \to H^j(E), \quad j = 0, \dots, m \tag{1.48}$$

on the cohomology spaces $H^j(E) = \operatorname{Ker} D_j / \operatorname{Im} D_{j-1}$ of the complex (1.45), and the Lefschetz number of the endomorphism (1.46) is defined as the alternating sum

$$\mathcal{L} \equiv \mathcal{L}(D, T) = \sum_{j=0}^{m} (-1)^j \operatorname{Trace} \widetilde{T}_j \tag{1.49}$$

of traces of the finite-dimensional operators (1.48).

1.4.2. Localization and the contributions of singular points

Suppose that (1.45) is an elliptic complex of differential operators on a smooth compact manifold M without boundary and (1.46) is a geometric endomorphism associated with a smooth mapping $f : M \to M$:

$$E_j = C^\infty(M, F_j), \quad \text{where } F_j \text{ is a vector bundle over } M,$$
$$T_j\varphi(x) = A_j(x)\varphi(g(x)), \quad \text{where } A_j(x) : F_{j f(x)} \to F_{jx} \text{ is a homomorphism.} \tag{1.50}$$

Here F_{jx} is the fiber of F_j over a point x. In the following, we assume for simplicity that $m = 1$ (i.e., the complex has length 1) and omit the index 0 on the operator D. To distinguish this case from the general case of abstract operators, we equip the operators D and T_j with hats.

In this subsection, we show that the Lefschetz number $\mathcal{L}(\widehat{D}, \widehat{T})$ can be expressed as the sum of contributions corresponding to connected components of the set fix(g) of fixed points of g. Although these contributions are defined here as integrals over some neighborhoods of these components and there is an ambiguity in the definition of the corresponding integrands, the values of these integrals are independent of the cited ambiguity as well as on the structure of the operators $\widehat{D}, \widehat{T}_1$, and $\widehat{T}_2$ outside an arbitrarily small neighborhood of the set $\mathrm{fix}(g)$.

Our starting point is the following trace formula for the Lefschetz number (e.g., see (Fedosov 1993)):

$$L = \text{Trace } \widehat{T}_1(1 - \widehat{R}\widehat{D}) - \text{Trace } \widehat{T}_2(1 - \widehat{D}\widehat{R}). \tag{1.51}$$

Here $\widehat{R}$ is an arbitrary almost inverse of $\widehat{D}$ modulo trace class operators. (That is, the operators $1 - \widehat{R}\widehat{D}$ and $1 - \widehat{D}\widehat{R}$ are trace class.)

◀ Recall how this formula can be established. Let N be the orthogonal complement of the kernel of $\widehat{D}$ and Z the orthogonal complement of the range $\mathcal{R}(\widehat{D})$. (Thus, Z is naturally isomorphic to the cokernel of $\widehat{D}$). Consider the operator $\widehat{R}$ that vanishes on Z and is equal to the inverse of $\widehat{D}|_N : N \longrightarrow \mathcal{R}(\widehat{D})$ on the range $\mathcal{R}(\widehat{D})$. Then $1 - \widehat{R}\widehat{D}$ is the projection on the kernel of $\widehat{D}$, and $1 - \widehat{D}\widehat{R}$ is the projection on the cokernel of $\widehat{D}$, so that formula (1.51) for this case is reduced to the definition of the Lefschetz number. Now if $\widehat{R}_1$ is another almost inverse of $\widehat{D}$ modulo trace class operators, then after replacing $\widehat{R}$ by $\widehat{R}_1$ the right-hand side changes by

$$\text{Trace } \widehat{T}_1(\widehat{R} - \widehat{R}_1)\widehat{D} - \text{Trace } \widehat{T}_2\widehat{D}(\widehat{R} - \widehat{R}_1).$$

Using the cyclic invariance of the trace and the relation $\widehat{D}\widehat{T}_1 = \widehat{T}_2\widehat{D}$, we find that this difference of traces is zero, so that formula (1.51) is valid for an *arbitrary* almost inverse modulo trace class operators. ▶

It turns out that for a special choice of regularizers the right-hand side of (1.51) splits into a sum of integrals over arbitrarily small neighborhoods of components of the fixed point set. For the sake of our argument, it is convenient to equip M with a Riemannian metric $d\rho^2$ and a smooth measure $d\mu(x)$. Using this measure, we can treat integral kernels of PDO as (generalized) functions (or sections of the corresponding bundles) on $M \times M$.

Let $\mathrm{fix}(g)$ be the fixed point set of g, and let

$$\mathrm{fix}(g) = K_1 \cup \ldots \cup K_k,$$

where $K_j,\ j = 1, \ldots, k$, are disjoint compact sets.

Next, let $V_1, \ldots, V_k$ be sufficiently small neighborhoods of these sets. On the compact set $M \setminus \bigl(V_1 \cup \ldots \cup V_k\bigr)$, the function $\rho(x, g(x))$ does not vanish and is continuous and hence has a nonzero minimum ε. Since $\widehat{D}$ is an elliptic differential operator on M, it follows that for an arbitrary $\varepsilon > 0$ there exists an almost inverse $\widehat{R}$ of $\widehat{D}$ modulo trace class operators such that its integral kernel $R(x, y)$ has the property

$$R(x, y) = 0 \text{ for } \rho(x, y) > \varepsilon/2. \tag{1.52}$$

(An operator $\widehat{R}$ with this property is said to be ε-narrow.) Then the kernels $K_1(x, y)$ and $K_2(x, y)$ of the operators $1 - \widehat{R}\widehat{D}$ and $1 - \widehat{D}\widehat{R}$ have the same property, since the differential operator $\widehat{D}$ does not enlarge supports. In terms of these kernels, formula (1.51) becomes

$$L = \int_M (\text{Trace } A_1(x)K_1(g(x), x) - \text{Trace } A_2(x)K_2(g(x), x))\ d\mu(x), \tag{1.53}$$

where Trace in the integrand stands for the matrix trace. With regard to property (1.52), this expression can be rewritten in the form

$$L = \sum_{j=1}^{k} \int_{V_j} (\operatorname{Trace} A_1(x) K_1(g(x), x) - \operatorname{Trace} A_2(x) K_2(g(x), x)) \; d\mu(x), \tag{1.54}$$

since the integrand vanishes identically outside the union of V_j.

It turns out that not only the whole sum but also each individual term is independent of the choice of a ε-narrow almost inverse (at least provided that the V_j are sufficiently small).

◀ This assertion follows from the *local nature* of the construction of an almost inverse operator: for two ε-narrow almost inverses, there always exists a third ε-narrow almost inverse coinciding with the first operator in some neighborhood V_{j_0} and with the second operator in all V_j for $j \neq j_0$. ▶

Thus, the following definition is meaningful.

DEFINITION 1.17. The number

$$L_j = \int_{V_j} \Big(\operatorname{Trace} A_1(x) K_1 \big(g(x), x\big) - \operatorname{Trace} K_2 \big(g(x), x\big) \Big) \, d\mu(x) \tag{1.55}$$

is called the *contribution of the component K_j of the fixed point set* to the Lefschetz number. (In particular, if K_j is a singleton, this is the contribution of the corresponding fixed point.)

1.4.3. The semiclassical method and microlocalization

The class of geometric endomorphisms is not a natural framework for the problem on the Lefschetz number if one does not restrict oneself to complexes of differential operators but has in mind also pseudodifferential operators. As differential operators form a subclass of the more general class of pseudodifferential operators, so geometric endomorphisms form a subclass of the class of Fourier integral operators, and hence one can naturally try to obtain a Lefschetz type formula for the case in which (1.45) is an elliptic complex of *pseudodifferential* operators and the endomorphism (1.46) is given by a set of Fourier integral operators. It turns out that once we pass to endomorphisms associated with mappings of the phase space, it is more convenient to deal with *asymptotic* theory. To obtain meaningful formulas, one should introduce a small parameter $h \in (0, 1]$ and consider semiclassical pseudodifferential operators (or $1/h$-pseudodifferential operators; e.g., see (Maslov 1972, Maslov 1973, Mishchenko, Shatalov and Sternin 1990)) and Fourier–Maslov integral operators associated with a canonical transformation

$$g \,:\, T^*M \,\to\, T^*M.$$

Thus, it is *symplectic* rather than contact geometry that underlies the Lefschetz formula.) Then the Lefschetz number depends on h, and under appropriate assumptions about the fixed points of g one obtains an expression for the asymptotics of the Lefschetz number as $h \to 0$ by applying the stationary phase method to the trace integrals representing this number. Namely, the Lefschetz number of the endomorphism (1.46) (with $m = 1$ for simplicity) is given by the formula (1.51). Now if D and R are pseudodifferential operators and T_0 and T_1 are Fourier–Maslov integral operators, then $T_0(1 - RD)$ and

$T_1(1-DR)$ are also Fourier–Maslov integral operators associated with the same canonical transformation as T_0 and T_1. Hence, the problem is reduced to the evaluation of *traces of Fourier–Maslov integral operators*. This is carried out with the help of the stationary phase method; only fixed points of g give a nonzero contribution to the asymptotics of these traces as $h \to 0$. As usual in the stationary phase method, the contribution of each isolated component of the set of fixed points can be treated separately; in other words, the *microlocalization principle* holds in this case for the Lefschetz number.

We see that the semiclassical method provides a straightforward computation of the Lefschetz number. Let us now state one of the main results for semiclassical endomorphisms of elliptic complexes on smooth manifolds.

Let M be a smooth closed manifold of dimension n. We suppose that M is oriented and is equipped with a positive volume form dx. For a canonical transformation

$$g : T^*M \to T^*M \tag{1.56}$$

and a smooth function φ on T^*M satisfying appropriate conditions at infinity in the fibers, by $T(g,\varphi)$ we denote the semiclassical Fourier–Maslov integral operator (i.e., a Fourier–Maslov integral operator with small parameter $h \in (0,1]$) with amplitude φ associated with the graph $\operatorname{graph} g$ of the transformation g. We assume that the graph of g is a quantized Lagrangian submanifold in $T^*M \times T^*M$. (A detailed definition of Fourier–Maslov integral operators can be found in (Mishchenko, Shatalov and Sternin 1990), and precise conditions ensuring that $T(g,\varphi)$ is well defined are given in (Sternin and Shatalov 1998).)

Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^\infty(M,F_1) & \xrightarrow{\widehat{D}} & C^\infty(M,F_2) & \longrightarrow & 0 \\ & & \big\downarrow \widehat{T}_1 & & \big\downarrow \widehat{T}_2 & & \\ 0 & \longrightarrow & C^\infty(M,F_1) & \xrightarrow{\widehat{D}} & C^\infty(M,F_2) & \longrightarrow & 0, \end{array} \tag{1.57}$$

where F_1 and F_2 are vector bundles over M, $\widehat{T_j} = T(g,\varphi_j)$, $j=1,2$, are Fourier–Maslov integral operators associated with some canonical transformation (1.56), and $\widehat{D}$ is an elliptic[5]

THEOREM 1.18. *Let the transformation g have only isolated nondegenerate fixed points $\alpha_1,\dots,\alpha_N$. (The nondegeneracy condition means that $\det(1-g_*(\alpha_k)) \neq 0, \quad k=1,\dots,N$, where $g_*(\alpha_k)$ is the derivative mapping of g at the point α_k.) Then the Lefschetz number of the diagram* (1.57) *has the asymptotics*

$$\mathcal{L} = \sum_{k=1}^{N} \exp\left(\frac{i}{h} S_k\right) \frac{\operatorname{Trace}\varphi_1(\alpha_k) - \operatorname{Trace}\varphi_2(\alpha_k)}{\sqrt{\det\bigl(1-g_*(\alpha_k)\bigr)}} + O(h), \tag{1.58}$$

where φ_1 and φ_2 are the amplitudes of $\widehat{T}_1$ and $\widehat{T}_2$, S_k is the value of the generating function of g (the choice of which is fixed in the definition of the Fourier–Maslov integral operators $\widehat{T}_1$ and $\widehat{T}_2$) at the point α_k, and the branch of the square root is chosen according to the stationary phase method.[6]

[5]By definition, this means that there exists a $1/h$-pseudodifferential operator $\widehat{R}$ on M such that the symbols of the operators

$$1-\widehat{D}\widehat{R} \quad \text{and} \quad 1-\widehat{R}\widehat{D}$$

belong to the Hörmander class $S^{-\infty}(T^*M)$ uniformly with respect to $h \in (0,1]$.

[6]See the explicit formulas in (Sternin and Shatalov 1998).

1.4.4. The classical Atiyah–Bott–Lefschetz theorem

This theorem (Atiyah and Bott 1967) deals with the case in which (1.45) is an elliptic complex of differential operators on a smooth compact manifold M without boundary and (1.46) is a geometric endomorphism associated with a smooth mapping $f : M \longrightarrow M$ (see (1.50)). Suppose that the fixed points of f are nondegenerate in the sense that

$$\det\left(1 - \frac{\partial f}{\partial x}(x)\right) \neq 0, \quad x \in \operatorname{fix}(f). \tag{1.59}$$

(Here $\operatorname{fix}(f)$ is the set of fixed points of f.) Then they are isolated, and the Atiyah–Bott–Lefschetz theorem states that the Lefschetz number $\mathcal{L}$ can be expressed by the formula

$$\mathcal{L} = \sum_{x \in \operatorname{fix}(f)} \frac{\sum_{j=0}^{m}(-1)^j \operatorname{Trace} A_j(x)}{|\det\bigl(1 - f'(x)\bigr)|}. \tag{1.60}$$

Thus, the Lefschetz number of the endomorphism (1.46) is expressed in classical terms. Note also that the operators D_j themselves do not occur in (1.60); only the *existence* of a complex (1.45) that makes the diagram (1.47) commute is relevant to the theorem.

The *Lefschetz fixed point theorem* is the special case of (1.60) in which (1.45) is the de Rham complex on a smooth compact oriented m-dimensional manifold M and T is the endomorphism induced on differential forms by a smooth mapping $f : M \to M$. In other words,

$$E_k = \Lambda^k(M)$$

is the space of differential k-forms on M,

$$D_k = d : \Lambda^k(M) \to \Lambda^{k+1}(M)$$

is the exterior differential, and

$$T_k = f^* : \Lambda^k(M) \to \Lambda^k(M)$$

is the induced mapping of differential forms.

The Lefschetz number $\mathcal{L} \equiv \mathcal{L}(f)$ is given in this case by the formula

$$\mathcal{L}(f) = \sum_{x \in \operatorname{fix}(f)} \operatorname{sgn}\det\left(\frac{\partial f}{\partial x}(x) - 1\right)$$

under the assumption that all fixed points are nondegenerate.

Now let us show how the classical result by Atiyah–Bott (Atiyah and Bott 1967) follows from Theorem 1.18. In doing so, we suppose that the map $f : M \to M$ determining the corresponding geometric endomorphism is a local diffeomorphism and has only nondegenerate fixed points. We again assume for simplicity that $m = 1$. Let operators $\widehat{T}_1$ and $\widehat{T}_2$ determine a geometric endomorphism of the elliptic differential complex

$$0 \longrightarrow C^\infty(M, E_1) \xrightarrow{\widehat{D}} C^\infty(M, E_2) \longrightarrow 0, \tag{1.61}$$

corresponding to the mapping f.

Multiplying the operator $\widehat{D}$ by h^m, where m is the order of this operator, we can view it as a semiclassical differential operator. It is also easy to see that the principal symbol of the operator $h^m\widehat{D}$ treated as a $1/h$-PDO coincides with the principal symbol of the differential operator $\widehat{D}$. In particular, the operator $h^m\widehat{D}$ is elliptic. Furthermore, after multiplying by h^m the spaces $\operatorname{Ker}\widehat{D}$ and $\operatorname{Coker}\widehat{D}$ remain unchanged, and so the Lefschetz number $\mathcal{L}(h^m\widehat{a}, \widehat{T}_1, \widehat{T}_2)$ does not depend on h.

The operators $\widehat{T}_1$ and $\widehat{T}_2$ can be rewritten in the form

$$\widehat{T}_j u(x) = \left(\frac{1}{2\pi h}\right)^n \int \exp\left\{\frac{i}{h} q(f(x) - y)\right\} A_j(x) u(y)\, dq\, dy, \qquad j = 1, 2, \tag{1.62}$$

and can be viewed as semiclassical Fourier integral operators associated with the graph L of the symplectic transformation corresponding to the mapping f.

More precisely, the form (1.62) corresponds to a nonsingular chart, and the generating function of the symplectic transformation g determined by the map f has the form

$$s(x, q) = qf(x).$$

Accordingly, the equations of the canonical transformation read

$$p = q\frac{\partial f}{\partial x}(x), \qquad y = f(x),$$

or

$$x = f^{-1}(y), \qquad p = q\frac{\partial f}{\partial x}(f^{-1}(y)).$$

Let $x = f(x)$ be a nondegenerate fixed point of f. Then $(x, 0)$ is a fixed point of the map g, which is unique in the fiber of the bundle T^*M over the point x. At this point, we have

$$\det(1 - g_*) = \det\begin{pmatrix} 1 - (\frac{\partial f}{\partial x})^{-1} & 0 \\ * & 1 - \frac{\partial f}{\partial x} \end{pmatrix} = (-1)^n \det\left(\frac{\partial f}{\partial x}\right)^{-1} \det\left(1 - \frac{\partial f}{\partial x}\right)^2 \neq 0,$$

so that it is nondegenerate. Computations by the stationary phase method give

$$\mathcal{L} = \sum_{x = f(x)} \frac{\operatorname{Trace} A_1(x) - \operatorname{Trace} A_2(x)}{|\det\left(1 - \frac{\partial f}{\partial y}\right)|},$$

which coincides with the classical result due to Atiyah–Bott (Atiyah and Bott 1967).

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