ON THE EXISTENCE OF A NON-ZERO LOWER BOUND FOR THE NUMBER OF GOLDBACH PARTITIONS OF AN EVEN INTEGER

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Abstract. The Goldbach partitions of an even number greater than 2, given by the sums of two prime addends, form the non-empty set for all integers $2n$ with $2 \le n \le 2 \times 10^{14}$. It will be shown how to determine by the method of induction the existence of a non-zero lower bound for the number of Goldbach partitions of all even integers greater than or equal to 4. The proof depends on contour arguments for complex functions in the unit disk.

The order estimates of the number of partitions of an even number into a sum of primes typically have an error term. However, when the partitions are restricted to the set of sums of two primes, the form of the error term implies that the estimate does not exclude the possibility of zero being obtained. If the constant c in the error term is specified, then the integer typically should be greater than some exponential function of c for the bound to be non-zero.

For example, an estimate of the number of partitions into the sum of two primes has been obtained by using an integral representation of the density of prime powers in a certain interval [1].

$$G(2s) \;=\; \int_2^{2s-2} \frac{dy}{log\ y \cdot log(2s-y)} + \mathcal{O}\left(\frac{2s}{log^2 2s}\right) + \mathcal{O}\left(\frac{2s}{log^q 2s}\right)$$

A non-zero lower bound for G(2s) exists if

$$\frac{2s-4}{log^3 2s} > c \cdot \frac{2s}{log^3 2s} \frac{1}{\left(1+\frac{log\ 2}{log\ s}\right)^3}$$

$$s > \frac{1}{8} e^{c-4}$$

This problem does not arise in the order estimates in the lower bound for the number of partitions of odd integers into the sum of three primes, because the error term is typically less than the leading estimate by a factor of $\frac{1}{(log\ N)^A}$, where A can be chosen to be arbitrarily large [2].

It is conventional to obtain an estimate of the number of partitions of an integer into a sum of primes by considering the following exponential sum [3]

$$R(\alpha) = \sum_p e^{2\pi i p\alpha}$$

$$S(\alpha) = \sum_p (log\ p)\ e^{2\pi i p\alpha}$$

and the functions

$$g(\alpha) = R(\alpha)^2 e^{-2\pi i n\alpha}$$

$$h(\alpha) = S(\alpha)^2 e^{-2\pi i n\alpha}$$

with the integrals

$$G(n) = \int_0^1 R(\alpha)^2 e^{-2\pi i n\alpha} = \sum_{p_1,p_2} \int_0^1 e^{2\pi i(p_1+p_2-n)\alpha} d\alpha$$

$$H(n) = \int_0^1 S(\alpha)^2 e^{-2\pi i n\alpha} = \sum_{p_1,p_2} (log\ p_1)(log\ p_2) \int_0^1 e^{2\pi i(p_1+p_2-n)\alpha} d\alpha$$

Since the integral vanishes when $p_1 + p_2 \neq n$, $G(n)$ equals the number of Goldbach partitions of n with the order of summands relevant and the two functions $G(n)$ and $H(n)$ are simultaneuously greater than zero.

The method of induction shall be used to determine the existence of a non-zero lower bound for the number of Goldbach partitions of an even integer.

Suppose that the even integer $n-2$ can be written as the sum of two prime numbers. Then

$$H(n-2) = \frac{1}{2\pi}\int_0^{2\pi} S(\theta)^2 e^{-i(n-2)\theta} d\theta > 0 \qquad \theta = 2\pi\alpha$$

This integral can be viewed as a contour integral over the unit circle

$$\frac{1}{2\pi}\oint \vec{v}\cdot \vec{d\ell}$$

where

$$\vec{v}^{(n-2)} = (v_r^{(n-2)}, v_\theta^{(n-2)}) \qquad\qquad \vec{d\ell} = (0, d\theta)$$

and

$$v_\theta^{(n-2)}(r=1) = S(\theta)^2 e^{-i(n-2)\theta}$$

Let $\vec{v}^{(n)}(r=1) = e^{-2i\theta}\vec{v}^{(n-2)}(r=1)$. The extrapolation of the function into the disk is either $z^{-2}\vec{v}^{(n-2)}(r,\theta)$ or $\bar{z}^2\vec{v}^{(n-2)}(r,\theta)$. By Stokes' theorem, the non-vanishing of the line integral implies that $\vec{v}^{(n-2)}(r,\theta) \neq \vec{\nabla}\chi$ where χ is a nonsingular function of r,θ.

Consider $z^{-2}\vec{\nabla}\chi = r^{-2}e^{-2i\theta}\vec{\nabla}\chi(r,\theta)$ and suppose that there is a singularity in χ at $\theta = \theta_s$. If $\vec{\nabla}\psi_1(r,\theta) = z^{-2}\vec{\nabla}\chi$,

$$\psi_1(r,\theta_s) = e^{-2i\theta_s}\int drr^{-2}\frac{\partial\chi(r,\theta)}{\partial r}\bigg|_{\theta_s}$$

so that if $\chi(r,\theta_s)$ is a singular function of the function r, $\psi_1(r,\theta_s)$ shall also be a singular function of r, giving rise to a non-zero surface integral

$$\int \vec{\nabla}\times\vec{v}^{(n)}(r,\theta)\cdot \vec{dS}$$

and

$$H(n) = \frac{1}{2\pi}\int_0^{2\pi} S(\theta)^2 e^{-in\theta}d\theta = \frac{1}{2\pi}\int \vec{v}^{(n)}\cdot\vec{d\ell} \neq 0$$

Next consider the function defined by $\vec{\nabla}\psi_2(r,\theta) = \bar{z}^2\vec{\nabla}\chi$. Then

$$\psi_2(r,\theta_0) = e^{-2i\theta_0}\int drr^2\frac{\partial\chi(r,\theta)}{\partial r}\bigg|_{\theta=\theta_0}$$

and when θ_0 coincides with θ_s, the singularity in $\chi(r,\theta)$ may be removed the integral. However, by Stokes's theorem,

$$\oint f(z,\bar{z})dz = \int\int_D \frac{\partial f(z,\bar{z})}{\partial z} d\bar{z} \wedge dz$$

If $f(z,\bar{z}) = -\frac{i}{r}\frac{\partial \psi_2(z,\bar{z})}{\partial \theta}$. the surface integral receives a contribution from the integral

$$-\frac{i}{\pi}\int\int_D \bar{z}\frac{e^{-i\theta}}{r}\frac{\partial \chi(r,\theta)}{\partial \theta} d\bar{z} \wedge dz$$

Combining the two integrals gives

$$-\frac{i}{\pi}\int\int_D \bar{z}\frac{e^{-i\theta}}{r}\frac{\partial \chi(r,\theta)}{\partial \theta} d\bar{z} \wedge dz - \frac{i}{2\pi}\int\int_D \bar{z}^2 \frac{\partial}{\partial \bar{z}}\left(\frac{e^{-i\theta}}{r}\frac{\partial \chi(r,\theta)}{\partial \theta}\right) d\bar{z} \wedge dz$$

and using

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}e^{i\theta}\frac{\partial}{\partial r} + \frac{i}{2}\frac{e^{i\theta}}{r}\frac{\partial}{\partial \theta}$$

giving rise to the following expression

$$\frac{2}{\pi}\int\int_D r^{-2i\theta}\frac{\partial \chi(r,\theta)}{\partial \theta} dr d\theta + \frac{1}{2\pi}\int\int_D r^2 e^{-2i\theta}\frac{\partial^2 \chi(r,\theta)}{\partial \theta \partial r} dr d\theta$$
$$+ \frac{i}{2\pi}\int\int_D r e^{-2i\theta}\frac{\partial^2 \chi(r,\theta)}{\partial \theta^2} dr d\theta$$

Recalling that

$$v_\theta^{(n-2)}(r=1) = S(\theta)^2 e^{-i(n-2)\theta} = \sum_{p_1,p_2} (log\ p_1)(log\ p_2) e^{i(p_1+p_2-n)\theta}$$
$$\frac{1}{r}\frac{\partial \chi}{\partial \theta}\bigg|_{r=1} = \frac{d}{d\theta}\chi(r=1,\theta)$$

it follows that

$$\chi(r=1,\theta) = \sum_{\substack{p_1,p_2 \\ p_1+p_2 \neq n=2}} \frac{(log\ p_1)(log\ p_2)}{i(p_1+p_2-(n-2))} e^{i(p_1+p_2-(n-2))\theta}$$
$$+ \sum_{p_1+p_2=n-2} (log\ p_1)(log\ p_2)\theta$$

If this function on the unit circle is extended to the entire disk using the variables z, $\bar{z}$, it is

$$\chi(z,\bar{z}) = i\sum_{p_1+p_2<n-2} \frac{(log\ p_1)(log\ p_2)}{(n-2-(p_1+p_2)} \bar{z}^{(n-2)-(p_1+p_2)}$$
$$-\frac{i}{2}\sum_{p_1+p_2} (log\ p_1)(log\ p_2) ln\left(\frac{z}{\bar{z}}\right)$$
$$-i\sum_{p_1+p_2>n-2} \frac{(log\ p_1)(log\ p_2)}{p_1+p_2-(n-2)} z^{p_1+p_2-(n-2)}$$

This function is nonsingular, and except for the second term, it is well-defined everywhere throughout the disk besides the origin. Its value at the origin is undefined, as it depends on the direction in which the limit $r \to 0$ is taken. However, the gradient would be well-defined, and furthermore, it is nonsingular.* Setting $\vec{v}^{(n-2)}$ equal to a nonsingular gradient term is not consistent with a nonvanishing contour integral $\oint \vec{v}^{(n-2)} \cdot \vec{d\ell}$.

However, if this form is chosen for $\chi(z, \bar{z})$, the expression for $\oint \vec{v}^{(n)} \cdot \vec{d\ell}$, based on the function $\psi_2(r, \theta)$ can be computed. In terms of r, θ,

$$\chi(r,\theta) = i \sum_{\substack{p_1, p_2 \\ p_1+p_2<n-2}} \frac{(log\ p_1)(log\ p_2)}{n-2-(p_1+p-2)} r^{n-2-(p_1+p_2)} e^{-i(n-2-(p_1+p_2))\theta}$$
$$+ \sum_{p_1+p_2=n-2} (log\ p_1)(log\ p_2)\theta$$
$$- i \sum_{\substack{p_1, p_2 \\ p_1+p_2>n-2}} \frac{(log\ p_1)(log\ p_2)}{p_1+p_2-(n-2)} r^{p_1+p_2-(n-2)} e^{i(p_1+p_2-(n-2))\theta}$$

and

$$4\frac{\partial \chi(r,\theta)}{\partial \theta} + r^2 \frac{\partial^2 \chi(r,\theta)}{\partial r \partial \theta} + i\frac{\partial^2 \chi}{\partial \theta^2}$$
$$= 4 \sum_{p_1+p_2 \neq n-2} (log\ p_1)(log\ p_2) r^{|p_1+p_2-(n-2)|} e^{i(p_1+p_2-(n-2))\theta}$$
$$+ 4 \sum_{p_1+p_2=n-2} (log\ p_1)(log\ p_2)$$

and since $\int_0^{2\pi} e^{i(p_1+p_2-(n-2))\theta} d\theta = 0$ when $p_1 + p_2 \neq n$, the integral becomes

$$\frac{2}{\pi} \sum_{p_1+p_2=n} (log\ p_1)(log\ p_2) \int_0^1 r^3 dr \cdot \int_0^{2\pi} d\theta = \sum_{p_1+p_2=n} (log\ p_1)(log\ p_2) = H(n)$$

If instead, the sum $R(\theta)$ is used, the result is $\sum_{p_1+p_2=n} 1 = G(n)$.

The integral expression in $(z, \bar{z})$, coordinate is

$$-\frac{i}{\pi} \int \int_D \Bigg[\sum_{p_1+p_2=n-2} \frac{(log\ p_1)(log\ p_2)}{z} + \sum_{p_1+p_2>n-2} (log\ p_1)(log\ p_2) z^{p_1+p_2-(n-1)} \bar{z}$$
$$+ \frac{1}{z} \sum_{p_1+p_2<n-2} (log\ p_1)(log\ p_2) \bar{z}^{(n-2)-(p_1+p_2)} \Bigg] d\bar{z} \wedge dz$$
$$+ \frac{1}{2\pi} \int \int_D \frac{1}{z} \sum_{p_1+p_2<n-2} (log\ p_1)(log\ p_2)((n-2)-(p_1+p_2)) \bar{z}^{(n-1)-(p_1+p_2)} d\bar{z} \wedge dz$$

Using the complex Green formula, it follows that each of the integrals except

$$-\frac{i}{\pi}\int\int_D \sum_{p_1+p_2>n-2} z^{p_1+p_2-(n-1)}\bar{z}d\bar{z}\wedge dz$$

vanishes. Using the Cauchy formula for nonanalytic $f(z,\bar{z})$ [4]

$$f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}+\frac{1}{2\pi i}\int\int_D \frac{\frac{\partial f}{\partial z}}{z-z_0}dz\wedge d\bar{z}$$

and the vanishing of $\bar{z}z^{p_1+p_2-(n-1)}$, for $p_1+p_2>n-2$, at the origin, it follows that

$$\begin{aligned}
-\frac{i}{\pi}\int\int_D \bar{z}z^{p_1+p_2-(n-2)}d\bar{z}\wedge dz &= \frac{i}{\pi}\int\int_D \bar{z}\frac{z^{p_1+p-2-(n-2)}}{z}dz\wedge d\bar{z}\\
&= -\frac{i}{\pi}\int_D \frac{\bar{z}^2}{2}\frac{z^{p_1+p_2-(n-2)}}{z}dz\\
&= -\frac{i}{2\pi}\int_0^{2\pi} e^{i(p_1+p-2-(n-1))\theta}e^{-2i\theta}\cdot ie^{i\theta}d\theta\\
&= -\frac{i}{2\pi}\int_0^{2\pi} ie^{i(p_1+p_2-n)\theta}d\theta=\delta_{p_1+p_2,n}
\end{aligned}$$

so that the sum above is

$$\begin{aligned}
\sum_{p_1+p_2>n-2}(log\ p_1)(log\ p_2)\delta_{p_1+p_2,n} &= \sum_{p_1+p_2>n-2}(log\ p_1)(log\ p_2)\delta_{p_1+p_2,n}\\
&= \sum_{p_1+p_2=n}(log\ p_1)(log\ p_2)=H(n)
\end{aligned}$$

This result confirms the equivalence of $\chi(z)$ based on the use of the singular function

$$\begin{aligned}
\chi(z,\bar{z}) = i&\sum_{p_1+p_2<n-2}\frac{(log\ p_1)(log\ p_2)}{(n-2)-(p_1+p_2)}z^{-(n-2)-(p_1+p_2)}\\
&-\frac{i}{2}\sum_{p_1+p_2=n-2}(log\ p_1)(log\ p_2)ln\left(\frac{z}{\bar{z}}\right)\\
&-i\sum_{p_1+p_2>n-2}\frac{(log\ p_1)(log\ p_2)}{p_1+p_2-(n-2)}z^{p_1+p_2-(n-2)}
\end{aligned}$$

In terms of r,θ, this expression is

$$\begin{aligned}
\chi(r,\theta) = i&\sum_{p_1+p_2<n-2}\frac{(log\ p_1)(log\ p_2)}{(n-2)-(p_1+p_2)}r^{-((n-2)-(p_1+p_2))}e^{-i((n-2)-(p_1+p_2))\theta}\\
&+\sum_{p_1+p_2=n-2}(log\ p_1)(log\ p_2)\theta\\
&-i\sum_{p_1+p_2>n-2}\frac{(log\ p_1)(log\ p_2)}{p_1+p_2-(n-2)}r^{p_1+p_2-(n-2)}e^{i(p_1+p_2-(n-2))\theta}
\end{aligned}$$

and

$$
\begin{aligned}
&4\frac{\partial\chi}{\partial\theta}+r\frac{\partial^2\chi}{\partial\theta\partial r}+i\frac{\partial^2\chi}{\partial\theta^2}\\
&\quad=4\sum_{p_1+p_2<n-2}(log\ p_1)(log\ p_2)r^{-((n-2)-(p_1+p_2))}e^{-i((n-2)-(p_1+p_2))\theta}\\
&\quad+4\sum_{p_1+p_2=n-2}(log\ p_1)(log\ p_2)\\
&\quad+4\sum_{p_1+p_2>n-2}(log\ p_1)(log\ p_2)r^{p_1+p_2-(n-2)}e^{i(p_1+p_2-(n-2))\theta}\\
&\quad+\sum_{p_1+p_2<n-2}(log\ p_1)(log\ p_2)(p_1+p_2-(n-2))r^{-((n-2)-(p_1+p_2))}e^{-i((n-2)-(p_1+p_2))\theta}\\
&\quad+\sum_{p_1+p_2>n-2}(log\ p_1)(log\ p_2)(p_1+p_2-(n-2))r^{p_1+p_2-(n-2)}e^{i(p_1+p_2-(n-2))\theta}\\
&\quad+\sum_{p_1+p_2<n-2}(log\ p_1)(log\ p_2)(n-2-(p_1+p_2))r^{-((n-2)-(p_1+p_2))}e^{-i((n-2)-(p_1+p_2))\theta}\\
&\quad+\sum_{p_1+p_2>n-2}(log\ p_1)(log\ p_2)(n-2-(p_1+p_2))r^{p-1+p_2-(n-2)}e^{i(p_1+p_2-(n-2))\theta}
\end{aligned}
$$

The integral

$$
\frac{1}{2\pi}\int\int\ e^{-2i\theta}\left(\frac{\partial\chi}{\partial\theta}+r\frac{\partial^2\chi}{\partial\theta\partial r}+i\frac{\partial^2\chi}{\partial\theta^2}\right)rdrd\theta
$$

equals

$$
\begin{aligned}
&\frac{2}{\pi}\sum_{p_1+p_2>n-2}(log\ p_1)(log\ p_2)\int_0 2\pi e^{i(p_1+p_2-n)\theta}d\theta\int_0^1 r^{p_1+p_2-(n-3)}dr\\
&\quad+\frac{1}{2\pi}\sum_{p_1+p-2>n-2}(log\ p_1)(log\ p_2)\cdot(p_1+p_2-(n-2))\cdot\int_0^{2\pi}e^{i(p_1+p_2-(n-2))\theta}e^{-2i\theta}d\theta\cdot\\
&\qquad\qquad\int_0^1 r^{p_1+p_2-(n-3)}dr\\
&\quad-\frac{1}{2\pi}\sum_{p_1+p-2>n-2}(log\ p_1)(log\ p_2)\cdot(p_1+p_2-(n-2))\cdot\int_0^{2\pi}e^{i(p_1+p_2-(n-2))\theta}e^{-2i\theta}d\theta\cdot\\
&\qquad\qquad\int_0^1 r^{p_1+p_2-(n-3)}dr
\end{aligned}
$$

When $p_1+p_2=n$, this sum equals

$$
\begin{aligned}
&\frac{2}{\pi}\sum_{p_1+p_2=n}(log\ p_1)(log\ p_2)\cdot\frac{1}{4}\cdot 2\pi+\frac{1}{2\pi}\cdot 2\cdot 2\pi\cdot\sum_{p_1+p_2=n}(log\ p_1)(log\ p_2)\\
&\qquad-\frac{1}{2\pi}\cdot 2\cdot 2\pi\cdot\frac{1}{4}\sum_{p_1+p_2=n}(log\ p_1)(log\ p_2)\\
&\qquad=H(n)
\end{aligned}
$$

Now consider the integral in terms of the coordinates $z, \bar{z}$. Since

$$\frac{\partial \chi}{\partial \bar{z}} = \frac{i}{2} \sum_{p_1+p_2=n-2} \frac{(log\ p_1)(log\ p_2)}{\bar{z}}$$

$$\frac{\partial \chi}{\partial z} = -i \sum_{p_1+p_2<n-2} (log\ p_1)(log\ p_2) z^{-((n-1)-(p_1+p_2))} - \frac{i}{2} \sum_{p_1+p_2=n-2} \frac{(log\ p_1)(log\ p_2)}{z}$$
$$- i \sum_{p_1+p_2>n-2} (log\ p_1)(log\ p_2) z^{p_1+p_2-(n-1)}$$

the following identities are obtained

$$e^{-2i\theta} \frac{\partial \chi}{\partial \theta} = i\bar{z}\frac{\partial \chi}{\partial z} - i\frac{\bar{z}^2}{z}\frac{\partial \chi}{\partial \bar{z}} = \sum_{p_1+p_2<n-2} (log\ p_1)(log\ p_2)\bar{z} z^{-((n-1)-(p_1+p-2))}$$
$$+ \sum_{p_1+p_2=n-2} (log\ p_1)(log\ p_2)\frac{\bar{z}}{z}$$
$$+ \sum_{p_1+p_2>n-2} (log\ p_1)(log\ p_2)\bar{z} z^{p_1+p_2-(n-1)}$$

$$\frac{e^{-i\theta}}{r}\frac{\partial \chi}{\partial \theta} = \sum_{p_1+p_2=n-2} (log\ p_1)((log\ p_2) z^{-((n-1)-(p_1+p-2))}$$
$$+ \sum_{p_1+p-2=n-2} \frac{(log\ p_1)(log\ p_2)}{z}$$
$$+ \sum_{p_1+p_2>n-2} (log\ p_1)(log\ p_2) z^{p_1+p_2-(n-1)}$$

and $\bar{z}^2 \frac{\partial}{\partial \bar{z}} \left(\frac{e^{-i\theta}}{r} \frac{\partial \chi}{\partial \theta} \right) = 0$ so that the integral in $(z, \bar{z})$ coordinates is

$$-\frac{i}{\pi} \int\int_D \Bigg[\sum_{p_1+p_2<n-2} (log\ p_1)(log\ p_2)\bar{z} z^{-((n-1)-(p_1+p-2))} + \sum_{p_1+p_2=n-2} (log\ p_1)(log\ p_2)\frac{\bar{z}}{z}$$
$$+ \sum_{p_1+p_2>n-2} (log\ p_1)(log\ p_2)\bar{z} z^{p_1+p-2-(n-1)} \Bigg] d\bar{z} \wedge dz$$

This first term in this expression is the integral of a singular function. Using the identity **

$$\int\int_D \frac{\frac{\partial f}{\partial \bar{z}}}{(z-z_0)^k} d\bar{z} \wedge dz = \int_C \frac{f(z,\bar{z})}{(z-z_0)^k} dz - \frac{1}{(k-1)!} \cdot 2\pi i \frac{\partial^{(k-1)} f}{\partial z^{(k-1)}}(z_0)$$

it follows that this term vanishes.

However, this result suggests that there would be a non-zero contribution from the singular terms once an additional positive-definite function is included in the integral. An example of such a function is $e^{\delta(1-z\bar{z})} = e^{\delta(1-r^2)}$. Multiplication by this factor in the integral over r and θ gives

$$\frac{2}{\pi}\int_0^1 r^3 e^{\delta(1-r^2)}dr \cdot 2\pi \sum_{p_1+p_2=n} (log\ p_1)(log\ p_2) = 4I_3(\delta) \sum_{p_1+p_2=n} (log\ p_1)(log\ p_2)$$
$$= 4I_3(\delta)H(n)$$

With a support function in the integral over $z,\ \bar{z}$ and use of the identity for higher powers of $(z-z_0)^k$ in the denominator, linear combinations of $H(m),\ m \leq n$, are obtained. As δ is varied, or different support functions are chosen, these linear combinations will change and given the non-vanishing of $H(m)$ for $m \leq n-2$, equivalence of the integrals in the r, θ coordinates implies a nontrivial result for the magnitude of $H(n)$. The same conclusions hold for $G(n)$.

Furthermore, by considering the different extrapolations of $\chi(z,\bar{z})$ to the interior of the disk, it can be seen that the use of the nonsingular expression for $\chi(z,\bar{z})$ does not yield any constraints on $H(n)$ and therefore it would be acceptable to set this value to zero. Conversely, the feasibility of setting $H(n)$ equal to zero implies that $\chi(z,\bar{z})$ must be nonsingular, which would, in turn, imply that $H(n-2)$ vanishes, contrary to the original assumption that it is non-zero. Thus, the contour integral argument provides a method for deducing a non-zero value for $H(n)$, given that $H(m) \neq 0$ for $4 \leq m \leq n-2$.

By induction, it would then would follow that $H(n)$ is non-zero for all values of n. The non-vanishing of $H(n)$ is sufficient for a proof of a lower bound for the number of Goldbach partitions of any even number, since the induction argument also holds for $G(n) = \sum_{p_1+p_2=n} 1$.

The special role of the primes in the integrals of the summed expressions of exponential functions of θ can be made clear by noting that any exponential with exponent given by $2\pi i$ multiplied by a rational number, $e^{2\pi i \frac{m}{n}\alpha}$, $\alpha = 1$, can be obtained as a power

$$e^{2\pi i \frac{m}{n}} = \left(e^{2\pi i \frac{p_1}{n}}\right)^{k_1} = \left(e^{2\pi i \frac{p_2}{n}}\right)^{k_2}, \qquad m < \frac{n}{2}, \quad n\ even$$

so that for every even number $2m$,

$$e^{2\pi i \frac{2m}{n}} = e^{2\pi i \frac{p_1 k_1 + p_2 k_2}{n}}$$

for some p_1, p_2. Since the number of Goldbach partitions is assumed to be positive $2m \leq n-2$, there is a pair of prime such that $2m = p_1 + p_2$. Consequently, the above exponential can be equated with $e^{2\pi i \frac{p_1+p_2}{n}}$, revealing the effect of identification of the exponential of a sum of prime fractions with a sum of appropriate powers.

The property can be extended to $n+2$ by noting that

$$e^{2\pi i \frac{2m'}{n+2}} = e^{2\pi i \frac{(p'_1 k'_1 + p'_2 k'_2)}{n+2}}$$

for any pair of primes p'_1, p'_2.

It has been shown that for any two primes p_1, p_2, there exists integers k_1, k_2 such that $p_1(k_1-1)+p_2(k_2-1) \equiv 0 \ (mod\ n)$. This property can be extended to a congruence relation modulo $n+2$.

Since the number of incongruent solutions to the equations

$$a_1 x_1 + ... + a_\ell x_\ell + b \equiv 0 \quad (mod\ n)$$

where $(a_1, ..., a_\ell, n)|b$ is $n^{\ell-1}(a_1, ..., a_\ell, n)$ [5], the number of solutions to the congruence $p_1 k_1 + p_2 k_2 \equiv 2m \ (mod\ n)$ for fixed k_1, k_2 is equal to $n(k_1, k_2, n)$, and, by assumption, the intersection of this set with the solutions to $a_1(k_1-1)+a_2(k_2-1) \equiv 0 \ (mod\ n)$ includes the prime pair (p_1, p_2). For fixed k'_1, k'_2, properties of the intersection of solutions sets of $a'_1 k'_1 + a'_2 k'_2 \equiv 2m' \ (mod\ n+2)$ and $a'_1(k'_1-1)+a'_2(k'_2-1) \equiv 0 \ (mod\ n+2)$ will be determined by the linearity of the congruence relations. Extending the solution set D_n to D_{n+2}, the existence of a prime pair (p'_1, p'_2) in the set D_{n+2} can be deduced from the density of prime pairs in the planar domain represented by the congruence relation modulo $n+2$.

It follows that the use of the exponentials with exponents containing fractions with prime numerators can be used as an appropriate basis for all exponentials of type $e^{2\pi i \frac{m}{n}}$. This basis can extrapolated to describe $e^{2\pi i \alpha}$ by means of the functional analytic technique of extending the domain of functions from the set of rational numbers forming a dense subset of the continuous interval $[0,1]$ to the entire interval.

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* The gradient in the radial direction is well defined throughout the disk.
**This identity originally appeared with a factor of $\frac{1}{k!}$, which is valid when $k = 1$.

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