Elliptic Theory on Manifolds with Nonisolated Singularities

IV. Obstructions to Elliptic Problems on Manifolds with Edges

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Abstract

The obstruction to the existence of Fredholm problems for elliptic differential operators on manifolds with edges is a topological invariant of the operator. We give an explicit general formula for this invariant. As an application we compute this obstruction for geometric operators.

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Introduction

This paper is a continuation of the series of papers [1], [2], [3] on elliptic operators on manifolds with edges.

The program of the present part is to compute the obstruction to the existence of well-posed (Fredholm) problems for elliptic operators on such manifolds. This question is of similar relevance as an analogous question for boundary value problems, since there exist elliptic operators that have no (co)boundary conditions with the Fredholm property in Sobolev spaces. Some examples of such operators (together with an explicit computation of the obstruction for them) were presented in our previous paper [2].

In the general case, the obstruction is of topological nature; namely, it is determined by the principal symbol of the operator. The main result of this paper is a general formula expressing the obstruction in topological form. It represents the obstruction as an element of the group $K^1(T^*X)$ with compact supports, where X is the edge, in terms of the principal symbol. If the edge consists of a single point, i.e. the original manifold has an isolated conical singularity, then the obstruction vanishes. On the contrary, if the edge has codimension 1 (so that the manifold near an edge point looks like an open book with finitely many sheets), the obstruction coincides with that in the theory of nonlocal elliptic boundary value problems in the sense of [4]. In particular, if the edge manifold is actually a smooth manifold with boundary and the edge is the boundary, then the obstruction coincides with the Atiyah–Bott obstruction [5] to the existence of classical boundary value problems for elliptic operators on manifolds with boundary.

Note that the obstruction for an edge of an arbitrary dimension coincides with that computed in [6], which refers to the existence of Fredholm problems on manifolds with fibered boundary in the class of boundary conditions with discontinuous symbols.

Let us briefly describe the scheme used in the computation of the obstruction. Technically, the problem of finding the obstruction can be restated as an index problem for some operator family on an infinite cone. We prove that this index is equal to the index of a family of elliptic operators on a compact manifold but with a noncompact parameter space. It turns out that the latter family is actually a parameter-elliptic family in the sense of Agranovich–Vishik [7], and the computation of the obstruction is reduced to an index problem for parameter-elliptic families. The desired formula for the index of such families is obtained by a reduction to the Atiyah–Singer index theorem for families.

An index theorem (in a slightly different form) for parameter-elliptic families was proved earlier by Nistor [8]. Note that elliptic edge operators with non-vanishing obstruction can be regarded as elements in a Toeplitz analogue of the edge algebra from [9] with global projection conditions on the edge, cf. [10].

Let us describe the contents of the paper. To make the paper self-contained, the first section contains all necessary notions from elliptic theory on manifolds with edges. Then we state the main theorem of the paper, which gives a topological formula for the obstruction to the existence of elliptic edge problems. The next section contains examples. For geometric operators, we find conditions ensuring that the obstruction vanishes. The remaining part of the paper deals with the proof of the main theorem. To this end, we first represent the obstruction as the index of an operator family on a closed manifold with noncompact parameter space. Then we reduce this family to a parameter-elliptic family in the sense of Agranovich–Vishik and compute the index of the latter family.

1 Statement of the problem

Presently, there are several approaches to elliptic theory on manifolds with edges (e.g., see [11, 12, 9, 13]). We follow the approach developed in [9].

1. Manifolds with edges. Let M be a compact smooth manifold with boundary ∂M such that the boundary is the total space of a locally-trivial bundle

$$\pi : \partial M \to X$$

with fiber $\Omega_x \simeq \Omega$ over $x \in X$.

We consider the following equivalence relation on M:

$$x \sim y \Longleftrightarrow x = y \text{ or } \{x, y \in \partial M \text{ and } \pi(x) = \pi(y)\}.$$

This relation identifies boundary points lying in the same fiber.

Definition 1.1. The topological space $\mathcal{M} = \overline{M}^{\pi}$ of equivalence classes of points $x \in M$ is called a *manifold with edge* $X = \overline{\partial M}^{\pi} \subset \overline{M}^{\pi}$, or simply an *edge manifold.*

One can readily see that the space $\overline{M}^{\pi}$ in a neighborhood of $x \in X$ looks like a neighborhood of $(0, \alpha)$ in the product $\mathbb{R}^n \times K_\Omega$, where K_Ω is the infinite cone

$$K_\Omega = \left(\overline{\mathbb{R}}_+ \times \Omega\right) / \left(\{0\} \times \Omega\right)$$

with base Ω and $\alpha = (\{0\} \times \Omega)/(\{0\} \times \Omega)$ is the cone vertex.

Let us recall the definitions of the tangent and cotangent bundles of a manifold $\mathcal{M}$ with edge X (e.g., see [11]). Let $\mathcal{D}(M, \pi)$ be a linear space of C^∞ vector fields on M everywhere tangent to the fibers of π on ∂M. This space is a (locally free) $C^\infty(M)$-module and hence is the space of sections of a vector bundle over M:

$$\mathcal{D}(M, \pi) = C^\infty(M, E).$$

The bundle E is denoted by $T\mathcal{M}$ and called the (*compressed*) *tangent bundle* of the manifold $\mathcal{M}$ (even though it is a bundle over M).

Let $\overset{\circ}{M} = M \setminus \partial M = \mathcal{M} \setminus X$ be the smooth part of $\mathcal{M}$. An arbitrary point $x \in \overset{\circ}{M}$ has a neighborhood such that an arbitrary vector field defined in this neighborhood extends to an element of $\mathcal{D}(M, \pi)$. Therefore,

$$T\mathcal{M}|_{\overset{\circ}{M}} = T\overset{\circ}{M}.$$

The dual bundle for $T\mathcal{M}$ is denoted by $T^*\mathcal{M}$ and called the *compressed cotangent bundle* of the manifold $\mathcal{M}$ with edge X. Because of

$$T^*\mathcal{M}|_{\overset{\circ}{M}} = T^*\overset{\circ}{M}.$$

we obtain an embedding

$$T^*\overset{\circ}{M} \subset T^*\mathcal{M}$$

of total spaces, and

$$T^*\mathcal{M} \setminus T^*\overset{\circ}{M} = \partial T^*\mathcal{M}.$$

Let us describe this embedding in local coordinates. To this end, we choose a collar neighborhood $U_{\partial M} \subset M$ of ∂M and a diffeomorphism $U_{\partial M} \simeq \partial M \times [0,1)$. The normal coordinate to the boundary is denoted by r and local tangential coordinates (i.e., coordinates on ∂M) by (x, ω), where x is a coordinate on the base and ω is a coordinate along the fiber. The dual coordinates to (x, r, ω) on T^*M are denoted by $(\xi, \tilde{p}, \tilde{q})$ and those on $T^*\mathcal{M}$ by (η, p, q). Then the embedding $T^*\overset{\circ}{M} \subset T^*\mathcal{M}$ is given by the formulas

$$\eta = r\xi, \quad p = r\tilde{p}, \quad q = \tilde{q}. \tag{1.1}$$

For what follows, we choose some diffeomorphism $U_{\partial M} \simeq \partial M \times [0,1)$. Then for each $\varepsilon \in (0,1)$ we obtain a vector bundle isomorphism

$$\partial T^*M \overset{\varphi_\varepsilon}{\simeq} \partial T^*\mathcal{M}$$

over the boundary ∂M. In local coordinates in the fiber over a point $(x, 0, \omega)$, this diffeomorphism is given by (1.1) with r replaced by ε. All these isomorphisms for various ε are homotopic to each other (and to isomorphisms associated with a different choice of the collar of the boundary and the trivialization). Thus, there is a well-defined isomorphism

$$K(\partial T^*M) \simeq K(\partial T^*\mathcal{M})$$

of K-groups with compact supports.

2. Operators on manifolds with edges. We consider differential operators

$$D : C_0^\infty(\overset{\circ}{M}) \longrightarrow C_0^\infty(\overset{\circ}{M})$$

with smooth coefficients that have edge degeneration in the sense of the following definition.

Definition 1.2. A differential operator D of order m is said to be *edge degenerate* if in some neighborhood $U_{\partial M}$ of the boundary it has the form

$$D|_{U_{\partial M}} = r^{-m} \sum_{k+|l| \le m} a_{kl}\left(x, r, \omega, -i\frac{\partial}{\partial\omega}\right)\left(ir\frac{\partial}{\partial r}\right)^k \left(-ir\frac{\partial}{\partial x}\right)^l,$$

where the multiindex $l = (l_1, l_2, \ldots, l_n)$ has $n = \dim X$ nonnegative components and the

$$a_{kl}\left(x, r, \omega, -i\frac{\partial}{\partial\omega}\right) \in \mathrm{Diff}^{m-k-|l|}(\Omega_x)$$

are families of differential operators on the bases of the cones K_{Ω_x} smoothly depending on $(x, r) \in X \times [0,1)$, i.e., remaining smooth up to $r = 0$. We denote the principal symbol of $a_{kl}(x, r, \omega, -i\partial/\partial\omega)$ by $a_{kl}(x, r, \omega, q)$.

One can show that the principal symbol of an edge-degenerate operator of order m is r^{-m} times a homogeneous function of order m on the space $T^*\mathcal{M}\setminus\{0\}$ smooth up to the boundary. Indeed, let $\widetilde{\sigma}(D)$ be the principal symbol of D on $T^*\overset{\circ}{M}$. Then

$$\begin{aligned} r^m\widetilde{\sigma}(D)(x,r,\omega,\xi,\tilde{p},q) &= \sum_{k+|l|\le m} a_{kl}(x,r,\omega,q)(r\tilde{p})^k(r\xi)^l \\ &= \sum_{k+|l|\le m} a_{kl}(x,r,\omega,q)p^k\eta^l. \end{aligned}$$

Here we use the embedding (1.1).

We define the principal symbol $\sigma(D)$ on $T^*\mathcal{M}$ by the formula

$$\sigma(D)=\psi(r)\tilde{\sigma}(D),$$

where $\psi(r)$ is a smooth function positive for $r>0$ and such that

$$\psi(r)=\begin{cases} r^m & \text{for small } r,\\ 1 & \text{for large } r.\end{cases}$$

(Thus, $\psi(r)$ is well-defined on $\mathcal{M}$.)

Remark 1.3. The definition of edge-degenerate operators acting on sections of vector bundles over M is an obvious generalization.

3. The ellipticity condition on manifolds with edges. We assume that the operator D satisfies the following *interior ellipticity condition*:

the principal symbol $\sigma(D)$ is invertible on the compressed cotangent bundle $T^*\mathcal{M}$ without the zero section.

In addition to the principal symbol, edge-degenerate operators have a certain operator-valued symbol.

Definition 1.4. The *edge symbol* $\sigma_\Lambda(D)$ of operator D is the operator family

$$\sigma_\Lambda(D)(x,\xi)=r^{-m}\sum_{k+|l|\le m} a_{kl}\left(x,0,\omega,-i\frac{\partial}{\partial\omega}\right)\left(ir\frac{\partial}{\partial r}\right)^k(r\xi)^l, \quad (x,\xi)\in T^*X,$$

parametrized by the cotangent bundle T^*X of the edge X.

For given $(x,\xi)\in T^*X$, the operator $\sigma_\Lambda(D)(x,\xi)$ is defined on the infinite cone K_{Ω_x}. Obviously, $\sigma_\Lambda(D)(x,\xi)$ is cone degenerate for $r=0$. For $\xi\neq 0$, it is treated as a bounded operator in special weighted spaces:

$$\sigma_\Lambda(D)(x,\xi):\mathcal{K}^{s,\gamma}(K_{\Omega_x})\longrightarrow\mathcal{K}^{s-m,\gamma-m}(K_{\Omega_x}),$$

where s and γ are some real numbers. The definition of these spaces can be found in [9, 14] (see also [1]).

It is well known (e.g., see [9]) that $\sigma_\Lambda(D)$ is a family of Fredholm operators over $T^*X\setminus\{0\}$ provided that the following two conditions are satisfied.

1. The principal symbol $\sigma(D)$ satisfies the interior ellipticity condition.

2. The conormal symbol

$$\sigma_c\left(\sigma_\Lambda(D)\right)(x,p) = \sum_{0\le k\le m} a_{k0}\left(x,0,\omega,-i\frac{\partial}{\partial\omega}\right)p^k \qquad (1.2)$$

of the edge symbol $\sigma_\Lambda(D)(x,\xi)$ at the conical point $r=0$ is invertible for all $x\in X$ and all p on the weight line

$$L_\gamma = \left\{p\in\mathbb{C} \mid \operatorname{Im} p = \gamma\right\}.$$

4. The obstruction to the existence of edge elliptic problems. Consider an operator D satisfying conditions 1) and 2). In this setting, it was proved in [9] that *the obstruction to the existence of an elliptic problem for the operator D with boundary and coboundary conditions along the edge* is equal to the index $\operatorname{ind}\sigma_\Lambda(D)$ of the edge symbol viewed as an element of the following quotient group:

$$[\operatorname{ind}\sigma_\Lambda(D)] \in K(S^*X)/\pi_0^*K(X). \qquad (1.3)$$

(Here π_0 is the natural projection $S^*X \longrightarrow X$). In other words, an elliptic operator D satisfying conditions 1) and 2) has an elliptic edge problem in the sense of [9] if and only if the element (1.3) is zero.

Remark 1.5. We shall show later on that the index of the edge symbol in

$$K(S^*X)/\pi_0^*K(X),$$

unlike the original index in $K(S^*X)$, is determined by the principal symbol of D and is a homotopy invariant of $\sigma(D)$. In particular, it is independent of the choice of the weights.

2 Main results

2.1 A formula for the obstruction

As announced in the beginning, the main result of this paper is a formula expressing the obstruction $[\operatorname{ind}\sigma_\Lambda(D)]$ in topological terms, i.e. via the principal symbol of D.

By virtue of the isomorphisms

$$\partial T^*\mathcal{M} \simeq \partial T^*M \simeq T^*\partial M\times\mathbb{R},$$

the restriction

$$\sigma(D)|_{\partial M} \overset{def}{=} \sigma(D)|_{\partial T^*M}$$

of the principal symbol of D gives the well-defined difference element

$$[\sigma(D)|_{\partial M}] \in K(T^*\partial M\times\mathbb{R})$$

in the K-group with compact supports.

Theorem 2.1. *One has*

$$\alpha\left[\operatorname{ind}\sigma_\Lambda\left(D\right)\right]=\pi_!\left[\sigma\left(D\right)|_{\partial M}\right]\in K\left(T^*X\times\mathbb{R}\right), \tag{2.1}$$

where

$$\pi_!:K\left(T^*\partial M\times\mathbb{R}\right)\longrightarrow K\left(T^*X\times\mathbb{R}\right)$$

is the direct image mapping induced by the projection π *and* α *is the Atiyah–Patodi–Singer isomorphism* [15]

$$K(S^*X)/\pi_0^*K(X)\overset{\alpha}{\simeq}K\left(T^*X\times\mathbb{R}\right).$$

The proof will be given in Section 3.5, and in this section we consider some examples.

Remark 2.2. If the edge consists of a single point, i.e.,

$$\pi:\partial M\longrightarrow\{pt\},$$

then we arrive at cone-degenerate operators, where it is well known that the obstruction is zero.

On the contrary, for the identity projection π

$$\pi=Id:\partial M\longrightarrow\partial M,$$

the obstruction (2.1) coincides with the Atiyah–Bott obstruction to the existence of classical boundary value problems for elliptic operators on manifolds with boundary (see [5]).

There is an interesting special case in which the fiber bundle is a covering:

$$\pi:\partial M\longrightarrow X,\quad \dim\pi^{-1}(x)=0.$$

Here the obstruction coincides with the obstruction to the existence of nonlocal boundary value problems in the sense of [4].

Finally, for a general projection π the obstruction coincides with the obstruction to the existence of boundary value problems on manifolds with fibered boundary in the sense of [6].

2.2 Examples

Let ${}^vT(\partial M)\in\operatorname{Vect}(\partial M)$ be the tangent bundle along the fibers of π, and let ${}^vT^*\partial M\in\operatorname{Vect}(\partial M)$ be the dual bundle. Then the cotangent bundle of the total space ∂M has the direct sum decomposition

$$T^*\partial M\simeq\pi^*(T^*X)\oplus{}^vT^*\partial M\in\operatorname{Vect}(\partial M),$$

which enables one to define a product

$$K^i(T^*X)\times K^j\left({}^vT^*\partial M\right)\longrightarrow K^{i+j}(T^*\partial M) \tag{2.2}$$

(e.g., see [16]) on K-groups as the composition

$$K^i(T^*X) \times K^j\left({}^vT^*\partial M\right) \to K^i(\pi^*T^*X) \times K^j\left({}^vT^*\partial M\right) \to K^{i+j}(T^*\partial M).$$

Proposition 2.3. *Suppose that the element* $[\sigma(D)|_{\partial M}] \in K(T^*\partial M \times \mathbb{R}) = K^1(T^*\partial M)$ *has the decomposition*

$$[\sigma(D)|_{\partial M}] = xy, \quad x \in K^i(T^*X), \;\; y \in K^{i+1}\left({}^vT^*\partial M\right). \tag{2.3}$$

Then

$$\pi_![\sigma(D)|_{\partial M}] = x\pi'_!(y),$$

where the direct image mapping $\pi'_!$ *on the right-hand side is the group homomorphism*

$$\pi'_! : K^{i+1}\left({}^vT^*\partial M\right) \longrightarrow K^{i+1}(X)$$

induced by the family of projections $\pi'_x : \Omega_x \to \{x\} \subset X$, $x \in X$.

Proof. This is a restatement of the well-known property (e.g., see [17])

$$\pi_!(xy) = x\pi'_!(y)$$

of direct images. □

Example 2.4 (a trivial bundle). Consider the Cartesian product

$$\pi : \partial M = X \times \Omega \longrightarrow X.$$

The Künneth formula in K-theory gives the equality

$$K^1(T^*\partial M) = K^1(T^*X) \otimes K(T^*\Omega) \oplus K(T^*X) \otimes K^1(T^*\Omega) \,(\mathrm{mod}\;\; \mathrm{Tors}),$$

modulo torsion. Let us represent $[\sigma(D)|_{\partial M}]$ as a sum $ab + cd$ in accordance with the above decomposition. Then the obstruction is

$$\pi_![\sigma(D)|_{\partial M}] = a\pi'_!(b) + c\pi'_!(d) = (\mathrm{ind}\, b)a,$$

modulo torsion, where $\mathrm{ind}\, b \in \mathbb{Z}$ is the index of an elliptic operator with difference element $b \in K(T^*\Omega)$.

Let us apply these results to geometric operators.

Example 2.5 (the Euler operator). Let M be an oriented even-dimensional Riemannian manifold with metric having the form

$$\pi^*\left(dx^2\right) + dr^2 + r^2 d\omega^2(x)$$

near the boundary, where dx^2 is a smooth metric on the edge X and $d\omega^2(x)$ is a smooth family of metrics on the fibers of π. One can show that the Euler operator (e.g., see[18])

$$d+\delta : \Lambda^{ev}(\overset{\circ}{M}) \longrightarrow \Lambda^{odd}(\overset{\circ}{M}),$$

acting from even to odd forms is an edge-degenerate operator. On the other hand, even the Atiyah–Bott obstruction

$$[\sigma(d+\delta)|_{\partial M}] \in K^1(T^*\partial M)$$

is zero. So much the more, the Euler operator has the trivial obstruction (2.1).

Example 2.6 (the Hirzebruch (signature) operator). In addition to the assumptions of the previous example, we require that X be oriented. The bundle π must also be oriented, i.e. each fiber should be equipped with an orientation continuously depending on the point of the base. Moreover, the orientations must be compatible in the sense that the orientation of the boundary must be consistent with the orientations of the base and the fiber. In this case, the signature operator (e.g., see [18])

$$D_M = d+\delta : \Lambda^+(\overset{\circ}{M}) \longrightarrow \Lambda^-(\overset{\circ}{M}),$$

acting from self-dual to antiself-dual forms is also edge degenerate. [1] The Atiyah–Bott obstruction for the signature operator

$$[\sigma(D_M)|_{\partial M}] \in K^1(T^*\partial M)$$

is *never* zero. However, the obstruction (2.1) sometimes vanishes. Let us study this possibility.

Since the obstruction is independent of the metric chosen, we consider a product metric on the cylinder $[0,1)\times\partial M$, where the metric on ∂M is given by

$$\pi^*\left(dx^2\right) + d\omega^2(x).$$

It is well known (e.g., see [19]) that the signature operator on the Cartesian product $[0,1)\times\partial M$ defines a K-theory element as in (2.3):

$$[\sigma(D_M)|_{\partial M}] = [\sigma(D_X)][\sigma(D_{\Omega\times[0,1)})|_{r=\varepsilon}],$$

where the factors are the difference elements

$$[\sigma(D_X)] \in K^{\dim X}(T^*X), \quad [\sigma(D_{\Omega\times[0,1)})|_{r=\varepsilon}] \in K^{\dim\Omega+1}\left({}^vT^*\partial M\right),$$

[1] To show this, one has to extend the bundles $\Lambda^{\pm}(\overset{\circ}{M})$ to vector bundles on M.

of the signature operator D_X on the base and the family $D_{\Omega\times[0,1)}$ of signature operators on the fibers $\Omega_x \times [0,1)$ of the bundle

$$\partial M \times [0,1) \longrightarrow X.$$

Here we assume that the edge X is even-dimensional. For odd-dimensional edges, we consider the different product

$$[\sigma(D_{X\times[0,1)})|_{r=\varepsilon}][\sigma(D_\Omega)],$$
$$[\sigma(D_{X\times[0,1)})|_{r=\varepsilon}] \in K^{\dim X+1}(T^*X), \quad [\sigma(D_\Omega)] \in K^{\dim\Omega}({}^vT^*\partial M)$$

of the signature operators $D_{X\times[0,1)}$ on the cylinder $X\times[0,1]$ by the family of signature operators D_Ω on the fibers Ω_x. By Proposition 2.3, this gives

$$\pi_![\sigma(D_M)|_{\partial M}] = [\sigma(D_X)]\pi'_![\sigma(D_{\Omega\times[0,1)})|_{r=\varepsilon}].$$

The signature operator D_X is a generator of the free $K^*(X)\otimes\mathbb{Q}$-module $K^{\dim X+*}(T^*X)\otimes\mathbb{Q}$ (see [18]). This leads to the following proposition.

Proposition 2.7. *For the signature operator, the obstruction* (2.1) *rationally vanishes if and only if the index*

$$\operatorname{ind} D_\Omega \in K^{\dim\Omega}(X)\otimes\mathbb{Q} \tag{2.4}$$

of the family of signature operators in the fibers is zero. Here D_{Ω_x} in the case of odd-dimensional manifolds is the so-called odd signature operator; e.g., see [19]*. In particular, the condition is always satisfied if the fibers are odd-dimensional, while for even-dimensional fibers of dimension $2d$ it is satisfied under the additional assumption that the cohomology of the fiber in dimension d is trivial.*

Proof. It suffices to apply Proposition 2.2. Indeed, by the Atiyah–Singer index theorem for families, $\pi'_![\sigma]$ is equal to the index of the corresponding operator family

$$\pi'_![\sigma(D_\Omega)] = \operatorname{ind} D_\Omega.$$

Let us prove the last part of the statement dealing with the vanishing of the index for families of signature operators.

1. *Even-dimensional fibers.* It was shown in [20] that the signature operator, modulo invertible operators, is equivalent to the elliptic operator

$$d\delta - \delta d : \Lambda^+(\Omega)\cap\Lambda^d(\Omega) \longrightarrow \Lambda^-(\Omega)\cap\Lambda^d(\Omega).$$

The kernel and cokernel of the latter operator are trivial as subspaces of the space of harmonic forms of degree d, since by the assumption the fiber has no dimension d cohomology. Thus, we arrive at the desired equality for the index: $\operatorname{ind} D_\Omega = 0$.

2. *Odd-dimensional fibers.* The desired vanishing follows from the fact that the kernel of the self-adjoint signature operator on an odd-dimensional manifold has a constant dimension (since it coincides with the space of harmonic forms), so that the index of this family of elliptic self-adjoint operators is trivial as an element of the group $K^1(X)$ corresponding to the parameter space (see [21]).

□

Example 2.8 (the Dirac operator). In addition to the assumptions of the previous example, let as assume that M and X are equipped with *spin* structures (see [22]). We also assume that the fibers of π are equipped with a *spin* structure. The induced *spin* structure on the boundary must coincide with the structure induced by the spin structures of the base and the fibers. In this case, the Dirac operator

$$D : \mathbb{S}^+(\overset{\circ}{M}) \longrightarrow \mathbb{S}^-(\overset{\circ}{M})$$

on M (e.g., see [22]) acting from even spinors to odd spinors is edge degenerate, and its difference element in K-theory is decomposed as before:

$$[\sigma(D_M)|_{\partial M}] = [\sigma(D_X)][\sigma(D_{\Omega\times[0,1)})|_{r=\varepsilon}],$$

where

$$[\sigma(D_X)] \in K^{\dim X}(T^*X), \quad [\sigma(D_{\Omega\times[0,1)})|_{r=\varepsilon}] \in K^{\dim\Omega+1}\left({}^vT^*\partial M\right)$$

are the difference elements of the Dirac operator on the base and the family of Dirac operators in the fibers of π. Therefore, the obstruction is also the product

$$\pi_![\sigma(D_M)|_{\partial M}] = [\sigma(D_X)]\pi_![\sigma(D_{\Omega\times[0,1)})|_{r=\varepsilon}].$$

Since D_X is a generator of the free $K^*(X)$-module $K^{\dim X+*}(T^*X)$, we arrive at the following proposition.

Proposition 2.9. *For the Dirac operator, the obstruction* (2.1) *is trivial if and only if the index of the Dirac operators*

$$\operatorname{ind} D_\Omega \in K^{\dim\Omega}(X)$$

in the fibers is zero. For example, this condition is satisfied if the fibers of π are equipped with a metric of positive scalar curvature continuously depending on the fiber.

Proof. The latter assertion follows from the fact that the Dirac operator on a manifold of positive scalar curvature metric is invertible (e.g., see [22]). Therefore, the index is trivial independent of the dimension. □

Remark 2.10. The problem of finding the weight γ such that the edge symbols of the geometric operators in question are families of Fredholm operators will be considered elsewhere.

3 Computation of the obstruction

3.1 Reduction of the index to the base of the cone

Consider an elliptic edge-degenerate operator D of order m on an edge manifold $\overline{M}^{\pi}$ satisfying conditions 1) and 2) in Section 1.

We define the operator family

$$\sigma'_\Lambda(D)\left(x, r\eta, \omega, -i\frac{\partial}{\partial\omega}, p\right) = \sum_{k+|l|\leq m} a_{kl}\left(x, 0, \omega, -i\frac{\partial}{\partial\omega}\right) p^k \left(r\eta\right)^l, \tag{3.1}$$
$$\sigma'_\Lambda(D)\left(x, \eta, r, p\right) : C^\infty(\Omega_x) \to C^\infty(\Omega_x),$$

obtained from the edge symbol by replacing $ir\partial/\partial r$ by a new variable p. The family (3.1) is parametrized by the points

$$(x, \eta, r, p) \in S^*X \times \mathbb{R}_+ \times L_\gamma.$$

The pair η, r can be represented as a single point $r\eta \in T^*X \setminus \{0\}$ of the cotangent bundle of the edge minus the zero section; then the family $\sigma'_\Lambda(D)$ extends to a continuous family parametrized by the product $T^*X \times L_\gamma$.

Lemma 3.1. *The family $\sigma'_\Lambda(D)$ is a family of Fredholm operators parametrized by $T^*X \times L_\gamma$. Moreover, the family is invertible in a neighborhood of the zero section $X \times L_\gamma \subset T^*X \times L_\gamma$ and outside some compact subset.*

Proof. Indeed, (3.1) is a parameter-elliptic family with parameters (ξ, p), where $\xi = r\eta$. Therefore for $|\xi| + |p| > R$ (for a sufficiently large R) the family is invertible (e.g., see [23]). On the other hand, on the zero section $X \times L_\gamma$, i.e. for $\xi = 0$, the family coincides with the conormal symbol of the edge symbol and hence is invertible by virtue of 2). By continuity, the family is invertible in a neighborhood of the zero section. □

By virtue of this lemma, $\sigma'_\Lambda(D)$ is invertible outside a compact set in $S^*X \times \mathbb{R}_+ \times L_\gamma$ and has an index

$$\operatorname{ind} \sigma'_\Lambda(D) \in K\left(S^*X \times \mathbb{R}_+ \times L_\gamma\right).$$

Theorem 3.2. *One has*

$$\beta\left(\operatorname{ind} \sigma_\Lambda\left(D\right)\right) = \operatorname{ind} \sigma'_\Lambda(D) \in K\left(S^*X \times \mathbb{R}_+ \times L_\gamma\right), \tag{3.2}$$

where β is the Bott periodicity isomorphism

$$\beta : K\left(S^*X\right) \xrightarrow{\simeq} K\left(S^*X \times \mathbb{R}_+ \times L_\gamma\right).$$

Proof. **1. Reduction to the conical spaces $H^{s,\gamma}$.** Let us choose R large enough to ensure that for all r on the closure of $(R,+\infty)$ the family $\sigma'_\Lambda(D)$ is invertible on the weight line L_γ for all $(x,\eta)\in S^*X$. After a small deformation of $\sigma'_\Lambda(D)$, we can assume that in a neighborhood of the point $r=R$ this family is independent of r. By

$$\widetilde{\sigma}_\Lambda(D)(x,\eta,r,p): C_0^\infty(\mathbb{R}_+\times\Omega_x)\longrightarrow C_0^\infty(\mathbb{R}_+\times\Omega_x)$$

we denote a differential operator on K_{Ω_x} that coincides with $\sigma_\Lambda(D)$ for $r<R$ and is the restriction of

$$r^{-m}\sigma'_\Lambda(D)\left(x,\eta,R,ir\frac{\partial}{\partial r}\right)$$

for $r>R$. This operator family is continuous in the weighted Sobolev spaces

$$\widetilde{\sigma}_\Lambda(D): H^{s,\gamma,\gamma}(K_\Omega)\longrightarrow H^{s-m,\gamma-m,\gamma-m}(K_\Omega)$$

on the infinite cone and is Fredholm. To make the paper self-contained, we recall the definition of the weighted Sobolev spaces H^{s,γ_1,γ_2} on the cone (see [24]). The cone K_Ω is viewed as a manifold with two conical points, at zero and infinity. The radial variable near infinity is $r'=1/r$. Then the space $H^{s,\gamma_1,\gamma_2}(K_\Omega)$ is the weighted Sobolev space of order s, weight γ_1 at $r=0$, and weight $(-\gamma_2)$ at $r'=0$. We proved in [2] that the following assertion holds.

Lemma 3.3. *One has*

$$\operatorname{ind}\sigma_\Lambda(D)=\operatorname{ind}\widetilde{\sigma}_\Lambda(D).$$

2. Computation of the index on an infinite cone in the weighted Sobolev spaces $H^{s,\gamma}$. It is well known that the index in weighted spaces can be expressed in terms of the spectral flow. In a special case, this was used in computations with geometric operators in [2]. In the general case, we shall use the results of [3].

By virtue of Lemma 3.3, it suffices to prove that

$$\beta(\operatorname{ind}\widetilde{\sigma})=\operatorname{ind}\sigma'\in K(S^*X\times\mathbb{R}_+\times L_\gamma), \tag{3.3}$$

where on the left-hand side there is an arbitrary family

$$\widetilde{\sigma}=\widetilde{\sigma}(x,\eta): H^{s,\gamma,\gamma}(K_{\Omega_x})\longrightarrow H^{s-m,\gamma-m,\gamma-m}(K_{\Omega_x})$$

of elliptic operators on the bundle K_Ω of infinite cones. The operators are cone degenerate at $r=0$ and $r=\infty$. Just this relation was proved in [3] (Theorem 3.2).

This completes the proof of Theorem 3.2. ☐

Remark 3.4. The simpler formula

$$\beta\left(\operatorname{ind}P\right)=\operatorname*{ind}_{p\in\mathbb{R},r\in\mathbb{R}_+}\left(P\left(r,\omega,-i\frac{\partial}{\partial\omega},p\right)\right)\in K\left(\mathbb{R}\times\mathbb{R}_+\right)\simeq\mathbb{Z}$$

holds for a cone-degenerate operator P on an infinite cone. This formula is obtained from the above formula by forgetting the parameter ξ. This formula gives an equality (up to the periodicity isomorphism) of the index of an operator P on a noncompact manifold and the index of an operator family on a smooth closed manifold.

Remark 3.5. In terms of the family $\sigma'_\Lambda(D)$, one can give an alternative form of the ellipticity condition. For the ellipticity, it is necessary that this family have a well-defined index as an element of the group $K(S^*X \times \mathbb{R} \times \mathbb{R}_+)$; i.e., *$\sigma'_\Lambda(D)$ must be Fredholm everywhere and invertible outside a compact set.*

Remark 3.6. The family (3.1) is formed by operators on a closed manifold, but the parameter space $S^*X \times L_\gamma \times \mathbb{R}_+$ is noncompact. Thus, the Atiyah–Singer formula [25] for *families with a compact parameter space* cannot be applied in this case. On the other hand, one can show that the formula nevertheless remains valid if we consider families of elliptic operators $\{P_x\}_{x\in X}$ with noncompact parameter space X such that the operators are vector bundle isomorphisms for x outside a compact set $K \subset X$. In our case, this condition is not satisfied. Thus, the index must be computed by a different method.

3.2 Reduction to the cotangent bundle

Let us now compute the obstruction

$$[\operatorname{ind}\sigma_\Lambda(D)] \in K(S^*X)/\pi_0^*K(X).$$

Quite remarkably, the projection to the quotient group $K(S^*X)/\pi_0^*K(X)$ admits an alternative description. Namely, we realize $S^*X \times \mathbb{R}_+$ as the subspace

$$S^*X \times \mathbb{R}_+ \simeq T^*X\backslash\{0\} \overset{i}{\subset} T^*X$$

in T^*X. Here $\mathbb{R}_+$ corresponds to the radius. This embedding enables us to define the diagram

$$\begin{array}{ccc} K(S^*X) & \overset{\beta}{\longrightarrow} & K(S^*X \times \mathbb{R} \times \mathbb{R}_+) \\ \downarrow & & \downarrow i_! \\ K(S^*X)/\pi_0^*K(X) & \overset{\simeq}{\longrightarrow} & K(T^*X \times \mathbb{R}). \end{array} \tag{3.4}$$

Here $i_!$ is induced by the embedding, and the bottom map is the Atiyah–Patodi–Singer isomorphism [15]. Note that this isomorphism is induced by the composition $i_!\beta$; i.e., the diagram is commutative.

Diagram (3.4) shows that the index $[\operatorname{ind}\sigma_\Lambda(D)]$ in the quotient group is represented as

$$\alpha[\operatorname{ind}\sigma_\Lambda(D)] = \operatorname{ind}\sigma'_\Lambda(D) \in K(T^*X \times \mathbb{R}). \tag{3.5}$$

Let us consider the family $\sigma'_\Lambda(D)$ in more detail. For given x, we obtain a family of operators on a closed manifold Ω_x that is parameter-elliptic with parameters $(\eta, p) \in$

$T_x^*X \times \mathbb{R}$ [7]. Thus, to compute the obstruction it suffices to obtain an index formula for families of parameter-elliptic operators. We obtain this formula in the next section (in a slightly more general setting).

3.3 Families of parameter-elliptic operators

1. Parameter-elliptic symbols. Consider a real vector bundle V and a locally trivial bundle $\pi : E \to X$ with fiber Ω on some manifold X. By $TE \in \mathrm{Vect}(E)$ we denote the tangent vector bundle along the fibers. The dual bundle is denoted by T^*E. Let us also consider two vector bundles $E_1, E_2 \in \mathrm{Vect}(E)$.

Definition 3.7. *A family of symbols of order m with parameters in a vector bundle V* is a smooth bundle homomorphism

$$\sigma = \sigma(x, v, \omega, q) : \pi_1^* E_1 \longrightarrow \pi_1^* E_2,$$

homogeneous of degree m in the variables

$$(q, v) \in \{T^*_{x,\omega}E \oplus V_x\} \setminus \{0\},$$

where $\pi_1 : \{\pi^* V \oplus T^*E\} \setminus \{0\} \longrightarrow E$ is the natural projection on the base.

Remark 3.8. For the special case in which X is a point, we obtain the usual definition of a homogeneous symbol with parameters; e.g., see [23].

The linear space of symbols of order m in bundles E_1, E_2 is denoted by $\mathrm{Smbl}^m(E_1, E_2)$.

2. Families of parameter-dependent operators. The *quantization mapping* taking each family of symbols to a family of operators with these symbols is constructed as usual in the theory of families (see [25]).

To this end, we take some quantization of symbols on the fiber Ω. Consider a cover $\{U_i\}_{i\in I}$ of the base X of both bundles E and V such that both bundles are trivial on the elements of the cover and the bundles $E_{1,2}$ are constant along the base. Thus, we choose some trivializations

$$q_i : E_{U_i} \simeq U_i \times \Omega, \quad r_i : V_{U_i} \simeq U_i \times \mathbb{R}^n, \quad e_{1,2} : (E_{1,2})_{U_i} \simeq q_i^*(E'_{1,2})_i, \;\; (E'_{1,2})_i \in \mathrm{Vect}(\Omega).$$

We denote the induced mapping for q_i by $q'_i : (T^*E)_{U_i} \simeq U_i \times T^*\Omega$. On U_i the symbol $\sigma(x, v, \omega, q)$ can be equivalently represented as a family $\sigma_i(x, v_0, \omega_0, q_0)$ of parameter-elliptic symbols on Ω with parameter $v_0 \in \mathbb{R}^n$, which also depend on an additional variable $x \in U_i$. More precisely, the family

$$\sigma_i(x, v_0) : \pi_0^* E'_{1,i} \longrightarrow \pi_0^* E'_{2,i}, \quad \pi_0 : U_i \times \mathbb{R}^n \times T^*\Omega \to \Omega,$$

on U_i is represented via the original symbol as

$$\sigma_i = I_{2,i} \sigma I_{1,i}^{-1},$$

for the isomorphisms $I_{j,i} : \pi_0^* E'_{2,i} \to \pi_1^*(E_j)_{U_i}$ defined by the composition

$$\pi_1^* E_j \simeq \pi_1^* q_i^* E'_{j,i} \simeq U_i \times \mathbb{R}^n \times \pi_0^* E'_{j,i}.$$

Then the symbol σ_i in U_i can be quantized as usual. The corresponding operator is denoted by $\widehat{\sigma}_i$. A global family can be obtained by means of a partition of unity χ_i subordinate to the cover $\{U_i\}_{i\in I}$ of the base by the formula

$$Q(\sigma) = \sum_i I_{2,i}^{-1}\,(\widehat{\chi_i\sigma_i})\,I_{1,i}.$$

Thus, we have a quantization mapping

$$Q : \mathrm{Smbl}^m(E_1, E_2) \longrightarrow \Gamma(V, \pi_2^*\mathcal{L}(H^s(E_1), H^{s-m}(E_2))), \quad \pi_2 : V \to X,$$

where the $H^s(E_i)$ are the Hilbert space bundles over X with fiber over a point x being the Sobolev space $H^s(\Omega_x, (E_i)_{\Omega_x})$ on the corresponding fiber of π (see [25]) and the fiber of the Hilbert space bundle $\mathcal{L}(H^s(E_1), H^s(E_2))$ over a point $(x, v) \in V$ is the space of bounded operators acting between the corresponding Sobolev spaces; finally, Γ stands for the functor of global sections.

3. Parameter-elliptic symbols and operators.

Definition 3.9. A family of parameter-dependent symbols $\sigma \in \mathrm{Smbl}^m(E_1, E_2)$ is said to be *parameter-elliptic* if the corresponding bundle homomorphism is invertible for $|v|+|q| > 0$.

The operator family $Q(\sigma)$ corresponding to an elliptic family of symbols consists of Fredholm operators invertible outside some compact set in V. This follows from the compactness of the base X and the fact that locally we deal with a family of usual parameter-elliptic operators for which the statement is well known. Thus, the family has a well-defined index

$$\mathrm{ind} Q(\sigma) \in K(V)$$

that is an element of the K-group with compact supports. One can readily show that the index is determined by the principal symbol σ, is homotopy invariant (within the class of elliptic families), and is independent of the choice of the quantization mapping Q.

3.4 Index theorem for parameter-elliptic families

Theorem 3.10. *Let σ be a parameter-elliptic symbol with parameters in a vector bundle V. Then*

$$\mathrm{ind} Q(\sigma) = (\pi_3)_![\sigma], \tag{3.6}$$

where $[\sigma] \in K(T^*E \oplus \pi^*V)$ *is the difference element of the elliptic symbol and*

$$(\pi_3)_! : K(T^*E \oplus \pi^*V) \longrightarrow K(V)$$

is the direct image mapping in K*-theory induced by the projection*

$$\pi_3 : T^*E \oplus \pi^*V \to V.$$

Proof. 1. First, note that it suffices to prove the formula only for zero-order families. Indeed, consider the invertible family

$$\left(1 + |v|^2 - \Delta_\Omega\right)^{-m/2}$$

of order $-m$ in the fibers of π. Here Δ_Ω stands for a family of nonpositive Laplacians. Obviously, the difference element of the symbol of this family is zero in the K-group. Thus, taking the product of a family of order m by this family changes neither the analytical index (3.6) nor the topological index.

2. By virtue of the parameter ellipticity, the family $Q(\sigma)(x, v)$ is invertible for all v of sufficiently large absolute value. Let $V_R \subset V$ be the subbundle of balls of radius R. Let us show that for R sufficiently large the restriction of $Q(\sigma)(x, v)$ to the boundary of V_R is homotopic in the class of invertible families to a family of vector bundle isomorphisms.

Let $S(V)$ be the unit sphere bundle in V with respect to some metric in V.

For each point $(x, v_1) \in S(V)$, consider the bundle

$$S^+(T^*\Omega_x \oplus V_x) = \{(\omega, q, v) \in S(T^*\Omega_x \oplus V_x) \ \mid (v_1, v) \geq 0\},$$

over Ω_x. Its fibers are hemispheres consisting of unit vectors having an acute angle with the fixed vector $(\omega, 0, v_1)$. Obviously, this space retracts to the subspace consisting of the points $(\omega, 0, v_1)$. A retraction can be given by the expression

$$\gamma_t(x, v_1) : S^+(T^*\Omega_x \oplus V_x) \longrightarrow S^+(T^*\Omega_x \oplus V_x), \quad t \in [0, 1],$$

$$\gamma_t(x, v_1)(\omega, q, v) = \frac{1}{((1-t)v + tv_1)^2 + ((1-t)q)^2} \left((1-t)q, (1-t)v + tv_1\right).$$

For given (x, ω, v_1), this family of mappings is a smooth deformation of the identity mapping (for $t = 0$) to the mapping to the point $(0, v_1)$ (for $t = 1$).

Now for a family of parameter-elliptic symbols $\sigma = \sigma(x, v, \omega, q)$ with parameter v we define the following family of symbols:

$$\sigma(\gamma_t(x, v_1)(\omega, q, v)). \tag{3.7}$$

This family of symbols with parameter v is elliptic in a conical neighborhood of the ray $v = \alpha v_1, \alpha > 0$ for all parameter values $t \in [0, 1]$ and $(x, v_1) \in S(V)$. Thus, for sufficiently large R the corresponding family

$$Q(\sigma(\gamma_t(x, v_1, v)))$$

of parameter-elliptic operators with parameter v is invertible on the ray

$$\{v = \alpha v_1, \alpha \geq R\}$$

for all $(t, x, v_1) \in [0,1] \times S(V)$.

Now the desired homotopy of the family $Q(\sigma)(x, v)$ on the spheres of radius R is defined as

$$Q_t(x, v) = Q\left(\sigma\left(\gamma_t\left(x, \frac{v}{|v|}, v\right)\right)\right), \quad t \in [0,1].$$

For $t = 1$, this family is induced by a vector bundle isomorphism.

3. The original family together with the homotopy thus constructed can be viewed as a new family defined on the ball bundle of radius $R + 1$:

$$Q'(x, v) = \begin{cases} Q(\sigma)(x, v), & \text{if } |v| \leq R, \\ Q\left(\sigma\left(\gamma_{|v|-R}\left(x, \frac{v}{|v|}, v\right)\right)\right), & \text{if } |v| > R. \end{cases}$$

To the remaining part of V, this family is extended constantly in the radial directions. The obtained family Q' is a continuous family of elliptic pseudodifferential operators which are invertible outside the compact subset V_R. On the other hand, on the subspace V_R the new family coincides with the original one. Therefore, by the definition of the index for families we obtain

$$\text{ind} Q(\sigma) = \text{ind} Q' \in K(V). \tag{3.8}$$

Thus, to prove the theorem it suffices to compute the latter index.

4. The family Q' consists of elliptic operators induced by a vector bundle isomorphism for parameter values outside a compact subset. The index of such families is computed by the Atiyah–Singer index formula for families. This formula gives

$$\text{ind} Q' = (\pi_3)_! [\text{smbl} Q'],$$

where $[\text{smbl} Q'] \in K(T^*E \oplus \pi^* V)$ is the difference element determined by the principal symbol of the family Q'.

The proof of the index formula is finished once we note the equality

$$[\text{smbl} Q'] = [\sigma]$$

of the difference elements of the family Q' and of the family of parameter-elliptic symbols σ. □

3.5 Proof of the main theorem

Proof of Theorem 2.1. Let us replace the index of the parameter-elliptic family in Eq. (3.5) by its topological expression given by Theorem 3.10. As a result we obtain the desired equality

$$\alpha[\text{ind}\, \sigma_\Lambda(D)] = \text{ind}\, \sigma'_\Lambda(D) = \pi_! [\sigma(D)|_{\partial M}].$$

□

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