GLOBAL SINGULARITY STRUCTURE OF WEAK SOLUTIONS TO 3-D SEMILINEAR DISPERSIVE WAVE EQUATIONS WITH DISCONTINUOUS INITIAL DATA

Huicheng Yin — Nanjing University, P.R. China

Ingo Witt — University of Potsdam, Germany

This note is devoted to the global singularity structure of solutions to the following three-dimensional semilinear wave equation with discontinuous initial data:

$$\begin{cases} \Box u + g(u) = 0, \\ u(0,x) = 0, \\ \partial_t u(0,x) = u_1(x), \end{cases} \tag{1}$$

where $\Box u = \partial_t^2 u - \sum_{k=1}^3 \partial_k^2 u$, $u_1(x) = \varphi(x)$ when $|x| \le 1$, $u_1(x) = 0$ when $|x| > 1$, $\varphi(x) \in C^\infty(\overline{B(0,1)})$, and $x = (x_1, x_2, x_3)$. The nonlinearity $g(u) \in C^\infty(\mathbb{R})$ is assumed to satisfy the following assumptions:

$$g(0) = 0, \quad G(u) = \int_0^u g(s)\,ds \ge 0, \tag{2}$$

$$\left|g^{(j)}(u)\right| \le C_j(1+|u|)^{p-j}, \quad 1 < p \le 5. \tag{3}$$

Under the conditions (2) and (3), the global existence of weak and smooth solutions to (1) has been intensively studied in literature, [1–4] and the references therein. For instance, if $u_1(x) \in C^\infty(\mathbb{R}^3)$, the authors in [1] and [2] proved the global existence of smooth solutions to (1) when $p < 5$ and $p = 5$, respectively. Besides, if $u_1(x) \in L^2(\mathbb{R}^3)$, the authors in [3] proved the global existence and uniqueness of weak solutions to (1) in the space $u \in C([0,\infty), H^1(\mathbb{R}^3)) \cap C^1([0,\infty), L^2(\mathbb{R}^3))$ when $1 < p < 5$. J. Shatah and M. Struwe in [4] then obtained the global existence and uniqueness in the space $u \in C([0,\infty), H^1(\mathbb{R}^3)) \cap C^1([0,\infty), L^2(\mathbb{R}^3)) \cap L^5([0,\infty), L^{10}(\mathbb{R}^3))$ when $p = 5$.

Our main purpose is to study the influence of the nonlinear term $g(u)$ on the properties of the weak solution u to (1), in particular, to give a precise description of the global singularity structure of u when the initial data $u_1(x)$ are discontinuous.

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The main result is:

Theorem. *Under the conditions* (2) *and* (3), *the global weak solution* $u(t,x)$ *to* (1) *belongs to* $C^\infty\big((\mathbb{R}^+ \times \mathbb{R}^3) \setminus (\Sigma_1 \cup \Sigma_2)\big)$, *where* $\Sigma_{1,2} = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \colon (t \pm 1)^2 = \sum_{k=1}^3 x_k^2\}$.

Notice that the surface Σ_2 has a singular point $(1,0,0,0)$. In general, the solution u will be singular at this point. In [5], M. Beals showed that singularities of solutions to nonlinear wave equations at such a point can cause a very complicated singularity structure of the solutions for later times. More precisely, he proved that there are a C^∞ function $\beta(t,x)$ and a solution $u \in H^s_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^3)$, $s > \frac{3}{2}$, to the equation $\Box u + \beta(t,x)u^3 = 0$ such that the initial data $u(0,x)$, $\partial_t u(0,x)$ have compact support and are C^∞ with the exception of the point at $x = 0$, while the singularities of u fill the whole solid light cone $\{(t,x) \colon |x| \le t\}$. For the problem (1), however, thanks to the special property of the discontinuous initial data to be conormal with respect to the sphere $|x| = 1$ (for the definition of being conormal, see [6] or [7]), with the help of Strichartz' inequality we can prove that the weak solution u to (1) is also globally conormal with respect to both the surfaces Σ_1 and Σ_2. Consequently, $u \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^3) \setminus (\Sigma_1 \cup \Sigma_2))$, and the nonlinear term $g(u)$ does not bring about the nonlinear influence on the singularity structure of u.

To prove the Theorem, we will make use of the commutator technique introduced in [7]. To do so, we are required to know a basis for the C^∞ vector fields which are tangent to both Σ_1 and Σ_2.

Lemma 1. *A basis for the* C^∞ *vector fields simultaneously tangent to* Σ_1 *and* Σ_2 *is given by*

$$
\begin{aligned}
V_1 &= \big(t^2 + x_1^2 + x_2^2 + x_3^2 - 1\big)\,\partial_t + 2tx_1\partial_1 + 2tx_2\partial_2 + 2tx_3\partial_3,\\
V_2 &= 2tx_1\partial_t + \big(t^2 + x_1^2 - 1\big)\,\partial_1 + x_1x_2\partial_2 + x_1x_3\partial_3,\\
V_3 &= 2tx_2\partial_t + x_1x_2\partial_1 + \big(t^2 + x_2^2 - 1\big)\,\partial_2 + x_2x_3\partial_3,\\
V_4 &= 2tx_3\partial_t + x_1x_3\partial_1 + x_2x_3\partial_2 + \big(t^2 + x_3^2 - 1\big)\,\partial_3,\\
V_5 &= x_1\partial_2 - x_2\partial_1,\\
V_6 &= x_1\partial_3 - x_3\partial_1,\\
V_7 &= x_2\partial_3 - x_3\partial_2.
\end{aligned}
$$

Proof. As is well known, a basis for the C^∞ vector fields tangent to Σ_1 is

$$
\begin{aligned}
M_0 &= (t+1)\partial_t + x_1\partial_1 + x_2\partial_2 + x_3\partial_3,\\
M_1 &= (t+1)\partial_1 + x_1\partial_t,\\
M_2 &= (t+1)\partial_2 + x_2\partial_t,\\
M_3 &= (t+1)\partial_3 + x_3\partial_t,\\
M_4 &= x_1\partial_2 - x_2\partial_1,\\
M_5 &= x_1\partial_3 - x_3\partial_1,\\
M_6 &= x_2\partial_3 - x_3\partial_2.
\end{aligned}
$$

Obviously, M_4, M_5 and M_6 are also tangent to Σ_2.

Suppose that the C^∞ vector field V is tangent to Σ_1 and Σ_2. Then

$$\begin{aligned} V &= \sum_{i=0}^{6} a_i(t,x) M_i \\ &= \left(a_0(t+1) + a_1x_1 + a_2x_2 + a_3x_3\right)\partial_t + \left(a_0x_1 + a_1(t+1)\right)\partial_1 \\ &\qquad + \left(a_0x_2 + a_2(t+1)\right)\partial_2 + \left(a_0x_3 + a_3(t+1)\right)\partial_3 + \sum_{i=4}^{6} a_i M_i, \end{aligned}$$

where $a_i(t,x)$ $(i = 0,1,2,3)$ are appropriate C^∞ functions to be determined.

For V is tangent to Σ_2, from the Malgrange preparation theorem we infer that there is a C^∞ function $d(t,x)$ such that

$$V\left((t-1)^2 - \sum_{i=1}^{3} x_i^2\right) = 2d\left((t-1)^2 - \sum_{i=1}^{3} x_i^2\right).$$

We get

$$\left(a_0(t+1) + \sum_{i=1}^{3} a_i x_i\right)(t-1) - \sum_{i=1}^{3}\left(a_0x_i + a_i(t+1)\right)x_i = d\left((t-1)^2 - \sum_{i=1}^{3} x_i^2\right). \quad (4)$$

We now start to determine the functions a_i $(i = 0,1,2,3)$.

Setting $x_1 = x_2 = x_3 = 0$ in (4), we get

$$a_0(t,0,0,0)(t+1) = d(t,0,0,0)(t-1).$$

Then there are C^∞ functions $b_i(t,x)$ $(i = 1,2,3)$ such that

$$a_0(t+1) = d(t-1) + \sum_{i=1}^{3} b_i x_i. \quad (5)$$

Substituting (5) into (4) yields

$$\left(\sum_{i=1}^{3}(a_i + b_i)x_i\right)(t-1) - \sum_{i=1}^{3}\left(a_0x_i + a_i(t+1)\right)x_i = -d\sum_{i=1}^{3} x_i^2. \quad (6)$$

Setting $t = 1$, $x_1 = x_2 = 0$ in (6), on the analogy of the analysis made above we conclude that there are C^∞ functions $b_j(t,x)$ $(j = 4,5,6)$ such that

$$a_0x_3 + a_3(t+1) = dx_3 + b_4(t-1) + b_5x_1 + b_6x_2. \quad (7)$$

Similarly, there are C^∞ functions $b_k(t,x)$ $(k = 7,8,9)$ and $b_m(t,x)$ $(m = 10,11,12)$ such that

$$a_0x_2 + a_2(t+1) = dx_2 + b_7(t-1) + b_8x_1 + b_9x_3, \quad (8)$$

$$a_0x_1 + a_1(t+1) = dx_1 + b_{10}(t-1) + b_{11}x_2 + b_{12}x_3. \quad (9)$$

Then Eq. (6) assumes the form

$$\left(\sum_{i=1}^{3}(a_i+b_i)\,x_i\right)(t-1)-\left(b_{10}(t-1)+b_{11}x_2+b_{12}x_3\right)x_1 \\ -\left(b_7(t-1)+b_8x_2+b_9x_3\right)x_2-\left(b_4(t-1)+b_5x_1+b_6x_2\right)x_3=0. \tag{10}$$

Setting $t=1$ and $x_3=0$ in (10), we get

$$b_{11}(1,x_1,x_2,0)+b_8(1,x_1,x_2,0)=0.$$

Thus, there are C^∞ functions $c_i(t,x)$ $(i=1,2)$ such that

$$b_{11}=-b_8+c_1(t-1)+c_2x_3. \tag{11}$$

Similarly, there are C^∞ functions $c_j(t,x)$ $(j=3,\dots,8)$ such that

$$b_{12}=-b_5+c_3(t-1)+c_4x_2, \tag{12}$$

$$b_9=-b_6+c_5(t-1)+c_6x_1, \tag{13}$$

$$a_3+b_3=b_4+c_7x_1+c_8x_2. \tag{14}$$

Eq. (10) becomes

$$\left(\sum_{i=1}^{2}(a_i+b_i)x_i+(c_7x_1+c_8x_2)x_3\right)(t-1)-\Big(b_{10}(t-1)+\left(c_1(t-1)+c_2x_3\right)x_2 \\ +\left(c_3(t-1)+c_4x_2\right)x_3\Big)x_1-\left(b_7(t-1)+\left(c_5(t-1)+c_6x_1\right)x_3\right)x_2=0. \tag{15}$$

Setting $t=1$ and $x_1=0$ in (15), we infer that there are C^∞ functions $c_k(t,x)$ $(k=9,10)$ such that

$$c_2+c_4=-c_6+c_9(t-1), \tag{16}$$

$$b_7=a_2+b_2+c_8x_3-c_5x_3-c_{10}x_1. \tag{17}$$

Thus, Eq. (15) simplifies to

$$a_1+b_1-b_{10}=-c_{10}x_2-c_7x_3+c_1x_2+c_3x_3+c_9x_2x_3 \tag{18}$$

Below, we shall derive expressions for a_0,a_1,a_2,a_3 in terms of $b_1,b_2,b_3,b_5,b_6,b_8,c_6,c_7,c_8,c_{10}$, and d. From (5), (7), (8) and (9), we gather

$$a_0=\frac{d(t-1)+\sum_{i=1}^3 b_ix_i}{t+1}, \tag{19}$$

$$a_1=\frac{2dx_1}{(t+1)^2}+\frac{b_{10}(t-1)+b_{11}x_2+b_{12}x_3}{t+1}-\frac{\left(\sum_{i=1}^3 b_ix_i\right)x_1}{(t+1)^2}, \tag{20}$$

$$a_2=\frac{2dx_2}{(t+1)^2}+\frac{b_7(t-1)+b_8x_1+b_9x_3}{t+1}-\frac{\left(\sum_{i=1}^3 b_ix_i\right)x_2}{(t+1)^2}, \tag{21}$$

$$a_3=\frac{2dx_3}{(t+1)^2}+\frac{b_4(t-1)+b_5x_1+b_6x_2}{t+1}-\frac{\left(\sum_{i=1}^3 b_ix_i\right)x_3}{(t+1)^2}. \tag{22}$$

Substituting (11), (12) and (13) into (20) and (21), we arrive at new expressions for a_1 and a_2. Inserting these new expressions for a_1, a_2 and (22) into (14), (17), and (18), and taking advantage of relation (16), we obtain

$$\begin{cases} \dfrac{2b_4}{t+1} = \dfrac{2dx_3}{(t+1)^2} + \dfrac{b_5x_1+b_6x_2}{t+1} - \dfrac{\left(\sum_{i=1}^3 b_ix_i\right)x_3}{(t+1)^2} - c_7x_1 - c_8x_2 + b_3, \\ \dfrac{2b_7}{t+1} = \dfrac{2dx_2}{(t+1)^2} + \dfrac{b_8x_1 + (-b_6 + c_5(t-1) + c_6x_1)\,x_3}{t+1} - \dfrac{\left(\sum_{i=1}^3 b_ix_i\right)x_2}{(t+1)^2} \\ \qquad + b_2 + c_8x_3 - c_5x_3 - c_{10}x_1, \\ \dfrac{2b_{10}}{t+1} = \dfrac{2dx_1}{(t+1)^2} + \dfrac{(-b_8 + c_1(t-1) + c_2x_3)\,x_2 + (-b_5 + c_3(t-1) + c_4x_2)\,x_3}{t+1} \\ \qquad - \dfrac{\left(\sum_{i=1}^3 b_ix_i\right)x_1}{(t+1)^2} + b_1 + (c_{10}-c_1)\,x_2 + (c_7-c_3)\,x_3 - c_9x_2x_3. \end{cases} \tag{23}$$

By virtue of (23), we then find the sougth expressions for a_1, a_2, and a_3:

$$\begin{cases} a_1 = \dfrac{dx_1}{t+1} - \dfrac{1}{2}\left(b_8x_2 + c_6x_2x_3 + b_5x_3 + \dfrac{\left(\sum_{i=1}^3 b_ix_i\right)x_1}{t+1}\right) \\ \qquad + \dfrac{t-1}{2}\left(b_1 + c_{10}x_2 + c_7x_3\right), \\ a_2 = \dfrac{dx_2}{t+1} + \dfrac{1}{2}\left(b_8x_1 + (c_6x_1 - b_6)x_3 - \dfrac{\left(\sum_{i=1}^3 b_ix_i\right)x_2}{t+1}\right) \\ \qquad + \dfrac{t-1}{2}\left(b_2 + c_8x_3 - c_{10}x_1\right), \\ a_3 = \dfrac{dx_3}{t+1} + \dfrac{1}{2}\left(b_5x_1 + b_6x_2 - \dfrac{\left(\sum_{i=1}^3 b_ix_i\right)x_3}{t+1}\right) + \dfrac{t-1}{2}\left(b_3 - c_7x_1 - c_8x_2\right). \end{cases} \tag{24}$$

Substituting (19) and (24) into the expression for V, we get

$$\begin{aligned} V = {} & \frac{d}{t+1}V_1 + \frac{b_1}{2(t+1)}\Big\{x_1\left((t+1)^2 - x_1^2 - x_2^2 - x_3^2\right)\partial_t + (t-1)\left((t+1)^2 - x_1^2\right)\partial_1 \\ & + (1-t)x_1x_2\partial_2 + (1-t)x_1x_3\partial_3\Big\} + \frac{b_2}{2(t+1)}\Big\{x_2\left((t+1)^2 - x_1^2 - x_2^2 - x_3^2\right)\partial_t \\ & + (1-t)x_1x_2\partial_1 + (t-1)\left((t+1)^2 - x_2^2\right)\partial_2 + (1-t)x_1x_3\partial_3\Big\} \\ & + \frac{b_3}{2(t+1)}\Big\{x_3\left((t+1)^2 - x_1^2 - x_2^2 - x_3^2\right)\partial_t + (1-t)x_1x_3\partial_1 + (1-t)x_2x_3\partial_2 \end{aligned}$$

$$+ (t-1)\left((t+1)^2 - x_3^2\right)\partial_3\Big\} + \left(a_4 + \frac{b_8(t+1)}{2} + \frac{c_6 x_3(t+1)}{2} - \frac{c_{10}(t^2-1)}{2}\right)V_5$$
$$+ \left(a_5 + \frac{b_5(t+1)}{2} - \frac{c_7(t^2-1)}{2}\right)V_6 + \left(a_6 + \frac{b_6(t+1)}{2} - \frac{c_8(t^2-1)}{2}\right)V_7.$$

Finally, upon noting

$$\begin{aligned}
&x_1\left((t+1)^2 - x_1^2 - x_2^2 - x_3^2\right)\partial_t + (t-1)\left((t+1)^2 - x_1^2\right)\partial_1 + (1-t)x_1x_2\partial_2 + (1-t)x_1x_3\partial_3 \\
&\qquad = (t+1)V_2 - x_1V_1, \\
&x_2\left((t+1)^2 - x_1^2 - x_2^2 - x_3^2\right)\partial_t + (1-t)x_1x_2\partial_1 + (t-1)\left((t+1)^2 - x_2^2\right)\partial_2 + (1-t)x_2x_3\partial_3 \\
&\qquad = (t+1)V_3 - x_2V_1, \\
&x_3\left((t+1)^2 - x_1^2 - x_2^2 - x_3^2\right)\partial_t + (1-t)x_1x_3\partial_1 + (1-t)x_2x_3\partial_2 + (t-1)\left((t+1)^2 - x_3^2\right)\partial_3 \\
&\qquad = (t+1)V_4 - x_3V_1,
\end{aligned}$$

Lemma 1 is completely proved. □

Let $[A, B] = AB - BA$ denote the commutator. By direct computation, we find:

Lemma 2. *We have*

$$\begin{aligned}
&[V_1, V_2] = -2x_2tV_5 - 2x_3tV_6, \\
&[V_1, V_3] = 2x_1tV_5 - 2x_3tV_7, \\
&[V_1, V_4] = 2x_1tV_6 + 2x_2tV_7, \\
&[V_1, V_5] = [V_1, V_6] = [V_1, V_7] = 0, \\
&[V_2, V_3] = -3(t^2+1)V_5, \;\; [V_2, V_4] = 3(t^2+1)V_6, \\
&[V_2, V_5] = V_3, \;\; [V_2, V_6] = V_4, \;\; [V_2, V_7] = 0, \\
&[V_3, V_4] = -3(t^2+1)V_7, \;\; [V_3, V_5] = -V_2, \\
&[V_3, V_6] = 0, \;\; [V_3, V_7] = V_4, \\
&[\Box, V_1] = 4t\Box - 4\partial_t, \\
&[\Box, V_2] = 4x_1\Box + 2\partial_2V_5 + 2\partial_3V_6 + 4\partial_1, \\
&[\Box, V_3] = 4x_2\Box - 2\partial_1V_5 + 2\partial_3V_7 + 4\partial_2, \\
&[\Box, V_4] = 4x_3\Box - 2\partial_1V_6 - 2\partial_2V_7 + 4\partial_3, \\
&[\Box, V_5] = [\Box, V_6] = [\Box, V_7] = 0.
\end{aligned}$$

From Lemma 2, one easily derives the following result:

Corollary. *Let* $\{V^k u\} = \left\{V_{l_1}^{k_1}V_{l_2}^{k_2}\dots V_{l_j}^{k_j}u\colon k_1 + \dots + k_j = |k|,\, k_1, \dots, k_j \in \mathbb{N}_0\right\}$*, where* $V_{l_1}, \dots, V_{l_j} \in \{V_1, \dots, V_7\}$*. If* u *is a solution to (1), then* $\{V^k u\}$ *satisfies the following system of equations:*

$$\Box V^k u + \sum_{j=1}^{3} c_{kj}(t,x)\partial_j V^k u + f_k(t,x,u)V^k u$$
$$= \sum_{|l_1|+\dots+|l_j| \le |k|-1} a_{kl_1\dots l_j}(x,t,u)V^{l_1}u\dots V^{l_j}u + \sum_{\substack{1\le j\le 3\\ |l|\le|k|-1}} b_{klj}\partial_j V^l u, \quad (25)$$

where $c_{kj}(t,x)$, $f_k(t,x,u)$, *and* $a_{kl_1\dots l_j}(x,t,u)$ *are* C^∞ *functions,* b_{klj} *are constants.*

Next, we determine the initial data for the $V^k u$.

Lemma 3. *For each* k, *we have*

$$\begin{cases} V^k u\big|_{t=0} = g_{1k}(x)\left(x_1^2+x_2^2+x_3^2-1\right)H(x_1^2+x_2^2+x_3^2-1)+g_{2k}(x), \\ \partial_t V^k u\big|_{t=0} = q_{1k}(x)H(x_1^2+x_2^2+x_3^2-1)+q_{2k}(x), \end{cases} \tag{26}$$

where $H(s)$ *is the Heaviside function, and* $g_{1k}(x)$, $g_{2k}(x)$, $q_{1k}(x)$, $q_{2k}(x)$ *are* C^∞ *on* $\mathbb{R}^3$.

Proof. For $|k|=0$, (26) obviously holds.

We now assume that (26) holds for some $|k|$. Then we want to show that (26) also holds when $|k|$ is replaced with $|k|+1$.

Let $N_1 = (x_1^2-1)\partial_1 + x_1x_2\partial_2 + x_1x_3\partial_3$, $N_2 = x_1x_2\partial_1 + (x_2^2-1)\partial_2 + x_2x_3\partial_3$, and $N_3 = x_1x_3\partial_1 + x_2x_3\partial_2 + (x_3^2-1)\partial_3$. Then $\{N_1, N_2, N_3, V_5, V_6, V_7\}$ constitutes a basis for the C^∞ vector fields tangent to the circle $x_1^2+x_2^2+x_3^2=1$. A direct computation gives

$$\begin{aligned}
V_1V^k u\big|_{t=0} &= \left(x_1^2+x_2^2+x_3^2-1\right)\partial_t V^k u\big|_{t=0} \\
&= q_{1k}(x)\left(x_1^2+x_2^2+x_3^2-1\right)H(x_1^2+x_2^2+x_3^2-1) \\
&\qquad + q_{2k}(x)(x_1^2+x_2^2+x_3^2-1), \\
V_{i+1}V^k u\big|_{t=0} &= (N_i g_{1k}(x) + 2x_i g_{1k}(x))(x_1^2+x_2^2+x_3^2-1)H(x_1^2+x_2^2+x_3^2-1) \\
&\qquad + N_i g_{2k}(x), \qquad i=1,2,3, \\
V_{j+4}V^k u\big|_{t=0} &= V_{j+4}g_{1k}(x)(x_1^2+x_2^2+x_3^2-1)H(x_1^2+x_2^2+x_3^2-1) \\
&\qquad + V_{j+4}g_{2k}(x), \qquad j=1,2,3, \\
V_1\partial_t V^k u\big|_{t=0} &= (x_1^2+x_2^2+x_3^2-1)\partial_t^2 V^k u\big|_{t=0} \\
&= (x_1^2+x_2^2+x_3^2-1)\Bigg(\sum_{k=1}^{3}\partial_k^2 V^k u - \sum_{j=1}^{3} c_{kj}(t,x)\partial_j V^k u \\
&\qquad - f_k(t,x,u)V^k u + \sum_{|l_1|+\cdots+|l_j|\le|k|-1} a_{kl_1\dots l_j}(x,t,u)V^{l_1}u\dots V^{l_j}u \\
&\qquad + \sum_{\substack{1\le j\le 3\\ |m|\le|k|-1}} b_{kmj}\partial_j V^m u\Bigg)\Bigg|_{t=0} \\
&= g_{1,k+1}(x)(x_1^2+x_2^2+x_3^2-1)H(x_1^2+x_2^2+x_3^2-1)+g_{2,k+1}(x), \\
V_{i+1}\partial_t V^k u\big|_{t=0} &= N_i q_{1k}(x)H(x_1^2+x_2^2+x_3^2-1)+N_i q_{2k}(x), \qquad i=1,2,3, \\
V_{j+1}\partial_t V^k u\big|_{t=0} &= V_{j+4}q_{1k}(x)H(x_1^2+x_2^2+x_3^2-1)+V_{j+4}q_{2k}(x), \qquad j=1,2,3.
\end{aligned}$$

By the principle of induction and a straightforward computation, Lemma 3 is then proved. □

To solve (25) with the initial data (26), we have to show that the weak solution $u(t,x)$ to (1) belongs to $L^\infty_{\text{loc}}([0,\infty)\times\mathbb{R}^3)$. Let us recall Strichartz' inequality from [3].

Lemma 4. *There is a constant $C > 0$ such that, for each $T > 0$,*

$$\|u\|_{L^q(0,T;L^{3r}(\mathbb{R}^3))} \le C\left(\|\Box u\|_{L^1(0,T;L^2(\mathbb{R}^3))} + \|\nabla_x u(0,x)\|_{L^2(\mathbb{R}^3)} + \|\partial_t u(0,x)\|_{L^2(\mathbb{R}^3)}\right), \quad (27)$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, $r \in [2, \infty)$.

By virtue of the estimate (27), we arrive at (see [3, Lemma 3.3]):

Lemma 5. *For all $T > 0$, $r \in [2,\infty)$, $M > 0$, there is a constant $C(T,r,M) > 0$ such that the solution u to (1) with $p < 5$ satisfies*

$$\|u\|_{L^q(0,T;L^{3r}(\mathbb{R}^3))} \le C(T,r,M), \quad (28)$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, $r \in [2,\infty)$, *and* $\|u_1\|_{L^2(\mathbb{R}^3)} \le M$.

Based on Lemmas 4 and 5, we can now establish an L^∞_{loc} bound for solutions u to (1) when $p < 5$.

Lemma 6. *Let $1 < p < 5$. Then the weak solution u to (1) satisfies*

$$u(t,x) \in L^\infty_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^3).$$

Proof. Suppose that $v(t,x)$ is the solution of the following linear wave equation:

$$\begin{cases} \Box v = 0, \\ v(0,x) = 0, \\ \partial_t v(0,x) = u_1(x). \end{cases}$$

By Kirchhoff's formula,

$$v(t,x) = \frac{1}{4\pi t}\int_{|x-y|=t} u_1(y)\,dS_y = \frac{t}{4\pi}\int_{S^2} u_1(x+t\omega)\,d\omega.$$

Hence, $v(t,x) \in L^\infty_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^3)$. By virtue of the energy inequality, we addionally have

$$v(t,x) \in C([0,\infty), H^{s+1}_{\mathrm{comp}}(\mathbb{R}^3)) \cap C^1([0,\infty), H^s_{\mathrm{comp}}(\mathbb{R}^3)) \qquad \text{for any } s < \frac{1}{2}.$$

Set $w = u - v$. Then w satisfies

$$\begin{cases} \Box w + g(w+v) = 0, \\ w(0,x) = 0, \\ \partial_t w(0,x) = 0. \end{cases} \quad (29)$$

To show $u \in L^\infty_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^3)$, it suffices to show that $w \in L^\infty_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^3)$.

From (29), one deduces

$$\begin{cases} \Box \partial_x w + g'(w+v)\partial_x(w+v) = 0, \\ \partial_x w(0,x) = 0, \\ \partial_t \partial_x w(0,x) = 0. \end{cases} \quad (30)$$

What we want to show is $g'(w+v)\partial_x(w+v) \in L^1_{\rm loc}([0,\infty); L^2(\mathbb{R}^3))$. If this holds, then $\partial_x^2 w \in L^\infty_{\rm loc}([0,\infty); L^2(\mathbb{R}^3))$ and $w \in L^\infty_{\rm loc}([0,\infty)\times\mathbb{R}^3)$ by Sobolev's embedding theorem.

In fact, using (28) and choosing $q=5$, $r=\frac{10}{3}$ in Lemma 4, we have

$$\begin{aligned} \|\partial_x w\|_{L^5(0,T;L^{10})} &\le C\,\|g'(w+v)\,\partial_x(w+v)\|_{L^1(0,T;L^2)} \\ &\le C_{T,M}\Big(\|\partial_x w\|_{L^\infty(0,T;L^2)} + \|\partial_x v\|_{L^\infty(0,T;L^2)} \\ &\qquad + \big\||w|^{p-1}\partial_x w\big\|_{L^1(0,T;L^2)} + \big\||w|^{p-1}\partial_x v\big\|_{L^1(0,T;L^2)}\Big) \\ &\le C_{T,M}\Big(1 + \|w\|^{p-1}_{L^{\frac{5(p-1)}{4+\theta}}\left(0,T;L^{\frac{5(p-1)}{2(1-\theta)}}\right)} \|\partial_x w\|^{\theta}_{L^\infty(0,T;L^2)} \|\partial_x w\|^{1-\theta}_{L^5(0,T;L^{10})} \\ &\qquad + \big\||w|^{p-1}\partial_x v\big\|_{L^1(0,T;L^2)}\Big), \end{aligned}$$

where $\theta = \frac{5-p}{8}$, and $C_{T,M} > 0$ is a generic constant depending only on $T>0$ and $M>0$, where $\|u_1\|_{L^2(\mathbb{R}^3)} \le M$. Taking into account that $\frac{5(p-1)}{4+\theta} < 5$, $\frac{5(p-1)}{2(1-\theta)} < 10$, and $\|w\|_{L^5(0,T;L^{10})} \le C_{T,M}$ in view of Lemma 5, we further obtain

$$\|\partial_x w\|_{L^5(0,T;L^{10})} \le C_{T,M}\left(1 + \|\partial_x w\|^{1-\theta}_{L^5(0,T;L^{10})} + \big\||w|^{p-1}\partial_x v\big\|_{L^1(0,T;L^2)}\right). \tag{31}$$

To estimate the remaining term $\big\||w|^{p-1}\partial_x v\big\|_{L^1(0,T;L^2)}$, we now distinguish three cases:

Case 1: $1<p<2$. By Sobolev's embedding theorem, $\partial_x v \in L^\infty(0,T;L^{3-\varepsilon_0})$ for any $\varepsilon_0>0$. Since $|w|^{p-1} \in L^\infty(0,T;L^{\frac{6}{p-1}})$, upon choosing $\varepsilon_0 = \frac{6-3p}{4-p} < 1$ we obtain

$$\big\||w|^{p-1}\partial_x v\big\|_{L^1(0,T;L^2)} \le C_{T,M}\,\|\partial_x v\|_{L^\infty(0,T;L^{3-\varepsilon_0})}\,\big\||w|^{p-1}\big\|_{L^\infty(0,T;L^{\frac{6}{p-1}})} \le C_{T,M}$$

by Hölder's inequality. Hence, for $1<p<2$, (31) implies

$$\|\partial_x w\|_{L^5(0,T;L^{10})} \le C_{T,M}\left(1 + \|\partial_x w\|^{1-\theta}_{L^5(0,T;L^{10})}\right). \tag{32}$$

We deduce $\|\partial_x w\|_{L^5(0,T;L^{10})} \le C_{T,M}$ from (32), i.e., $g'(w+v)\partial_x(w+v) \in L^1(0,T;L^2)$ and $w \in L^\infty((0,T)\times\mathbb{R}^3)$.

Case 2: $2\le p<4$. By Hölder's inequality, we have

$$\begin{aligned} &\big\||w|^{p-1}\partial_x v\big\|_{L^1(0,T;L^2)} \\ &\qquad \le \|1\|_{L^{p_1}(0,T;L^{q_1}(B(0,1+T)))}\,\|w\|^{p-1}_{L^{(p-1)p_2}(0,T;L^{(p-1)q_2})}\,\|\partial_x v\|_{L^\infty(0,T;L^{3-\varepsilon_0})}, \end{aligned}$$

where $p_1, p_2, q_1, q_2 \ge 1$ satisfy

$$\frac{1}{p_1}+\frac{1}{p_2}=1,\ \ \frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{3-\varepsilon_0}=\frac12,\ \ \frac{1}{(p-1)p_2}+\frac{3}{(p-1)q_2}=\frac12,\ \ (p-1)q_2\ge 6.$$

Choose $0<\varepsilon_0<\min\{1,4-p\}$ and set $p_1=\frac{2}{4-p-\varepsilon_0}$, $q_1=\frac{6(3-\varepsilon_0)}{\varepsilon_0(1-\varepsilon_0)}$, $p_2=\frac{2}{p-2+\varepsilon_0}$, and $q_2=\frac{6}{1-\varepsilon_0}$ to satisfy the above requirements. Hence, in view of Lemma 5, we obtain $\||w|^{p-1}\partial_x v\|_{L^1(0,T;L^2)} \le C_{T,M}$. Similar to Case 1, we then deduce $w \in L^\infty((0,T)\times\mathbb{R}^3)$.

Case 3: $4 \le p < 5$. Choose $\varepsilon_0 > 0$ sufficiently small so that $p - 2 + \varepsilon_0 < 3$ and $\frac{(p-2+\varepsilon_0)(3-\varepsilon_0)}{1-\varepsilon_0} < 9$. By Hölder's inequality and Sobolev's embedding theorem,

$$\begin{aligned}\big\||w|^{p-1}\partial_x v\big\|_{L^1(0,T;L^2)} &= \big\||w|^{p-2+\varepsilon_0}|w|^{1-\varepsilon_0}\partial_x v\big\|_{L^1(0,T;L^2)} \\ &\le \|w\|^{p-2+\varepsilon_0}_{L^{p-2+\varepsilon_0}\left(0,T;L^{\frac{2(p-2+\varepsilon_0)(3-\varepsilon_0)}{1-\varepsilon_0}}\right)} \|\partial_x^2 w\|^{1-\varepsilon_0}_{L^\infty(0,T;L^2)}\|\partial_x v\|_{L^\infty(0,T;L^{3-\varepsilon_0})}.\end{aligned}$$

By Lemma 5 (with $q = 3$, $r = 6$), we get

$$\big\||w|^{p-1}\partial_x v\big\|_{L^1(0,T;L^2)} \le C_{T,M}\|\partial_x^2 w\|^{1-\varepsilon_0}_{L^\infty(0,T;L^2)}. \tag{34}$$

Substituting (34) into (31) gives

$$\|\partial_x w\|_{L^5(0,T;L^{10})} \le C_{T,M}\left(1 + \|\partial_x w\|^\theta_{L^5(0,T;L^{10})} + \|\partial_x^2 w\|^{1-\varepsilon_0}_{L^\infty(0,T;L^2)}\right).$$

Consider the function $h(s) = s - C_0 - C_{T,M}s^\theta$, where $C_0 = C_{T,M}\left(1 + \|\partial_x^2 w\|^{1-\varepsilon_0}_{L^\infty(0,T;L^2)}\right)$. Thereby, we can assume that $C_{T,M} > 1$. Obviously, $h(0) = -C_0 < 0$ and

$$\begin{aligned}h(C_0 + C_0C^m_{T,M}) &= C_0^\theta C_{T,M}\left(C_0^{1-\theta}C^{m-1}_{T,M} - (1 + C^m_{T,M})^\theta\right) \\ &> C_0^\theta C_{T,M}\left(C^{m-1}_{T,M} - (1 + C^m_{T,M})^\theta\right) > 0\end{aligned}$$

for $m > 0$ large enough. Hence,

$$\|\partial_x w\|_{L^5(0,T;L^{10})} \le C_0(1 + C^m_{T,M}) \le C_{T,M}(1 + \|\partial_x^2 w\|^{1-\varepsilon_0}_{L^\infty(0,T;L^2)}).$$

Upon applying the energy estimate to (30), we infer

$$\begin{aligned}\|\partial_x^2 w\|_{L^\infty(0,T;L^2)} &\le C_{T,M}\,\|g'(w+v)\partial_x(w+v)\|_{L^1(0,T;L^2)} \\ &\le C_{T,M}\left(1 + \|\partial_x^2 w\|^{1-\varepsilon_0}_{L^\infty(0,T;L^2)} + \|\partial_x^2 w\|^{(1-\varepsilon_0)\theta}_{L^\infty(0,T;L^2)}\right).\end{aligned}$$

Hence, $\|\partial_x^2 w\|_{L^\infty(0,T;L^2)} \le C_{T,M}$, and we find $w \in L^\infty(0,T;H^2(\mathbb{R}^3)) \subset L^\infty((0,T)\times\mathbb{R}^3)$.

Because $T > 0$ is arbitrary, Lemma 6 is proved. □

Now, we establish the same result when $p = 5$.

Lemma 7. *For $p = 5$, the weak solution $u \in C([0,\infty), H^1(\mathbb{R}^3)) \cap C^1([0,\infty), L^2(\mathbb{R}^3)) \cap L^5([0,\infty), L^{10}(\mathbb{R}^3))$ to (1) satisfies*

$$u(t,x) \in L^\infty_{\text{loc}}([0,\infty)\times\mathbb{R}^3).$$

Remark. In case $p = 5$, (1) has a unique global weak solution $u \in C([0,\infty), H^1(\mathbb{R}^3)) \cap C^1([0,\infty), L^2(\mathbb{R}^3)) \cap L^5_{\text{loc}}([0,\infty), L^{10}(\mathbb{R}^3))$ by [4]. Moreover, $u \in L^5((0,\infty), L^{10}(\mathbb{R}^3))$ by [8].

Proof. One easily gets that there is a time $T^*>0$ such that Eq. (29) has a local solution $w(t,x)\in C([0,T^*),H^{s+2}_{\text{comp}}(\mathbb{R}^3))\cap C^1([0,T^*),H^{s+1}_{\text{comp}}(\mathbb{R}^3))$ for any $0<s<\frac{1}{2}$. Thus, $w(t,x)\in L^5_{\text{loc}}([0,T^*),L^{10}(\mathbb{R}^3))$. By the preceding remark, $w(t,x)\in L^5((0,T^*),L^{10}(\mathbb{R}^3))$.

We want to show that $w(t,x)\in L^\infty((0,T^*)\times\mathbb{R}^3)$.

Applying the pseudodifferential operator $\Lambda^{s+\frac12}$ with symbol $(1+|\xi|^2)^{\frac12(s+\frac12)}$ to both sides of (29), we find

$$\begin{aligned}\Box\Lambda^{s+\frac12}w+\Lambda^{s+\frac12}g(w+v)&=0,\\ \Lambda^{s+\frac12}w(0,x)&=0,\\ \partial_t\Lambda^{s+\frac12}w(0,x)&=0.\end{aligned}\tag{35}$$

Choosing $t_0<s_0<T^*$, by virtue of Lemma 4 we have

$$\begin{aligned}\|\Lambda^{s+\frac12}w\|_{L^5(t_0,s_0;L^{10})}&\le C\Big(\|\partial_x\Lambda^{s+\frac12}w(t_0,.)\|_{L^2}+\|\partial_t\Lambda^{s+\frac12}w(t_0,.)\|_{L^2}\\ &\quad+\|\Lambda^{s+\frac12}g(w+v)\|_{L^1(t_0,s_0;L^2)}\Big)\le C(t_0)+C\|(w+v)^5\|_{L^1(t_0,s_0;H^{s+\frac12})}.\end{aligned}$$

For $\mu>0, 1<p<\infty$, the following inequality holds (see [9]):

$$\|\Lambda^\mu(f_1f_2)\|_{L^p(\mathbb{R}^n)}\le C\left(\|f_1\|_{L^{q_1}(\mathbb{R}^n)}\|\Lambda^\mu f_2\|_{L^{q_2}(\mathbb{R}^n)}+\|f_2\|_{L^{r_1}(\mathbb{R}^n)}\|\Lambda^\mu f_1\|_{L^{r_2}(\mathbb{R}^n)}\right)$$

provided that $\frac1p=\frac1{q_1}+\frac1{q_2}=\frac1{r_1}+\frac1{r_2}$, q_1, $r_1\in(1,\infty]$, q_2, $r_2\in(0,\infty)$.

Thus, we obtain, for $\|u_1\|_{L^2}\le M$,

$$\begin{aligned}\|w^5(t,.)\|_{H^{s+\frac12}}&\le C\left(\|w(t,.)\|_{L^{10}}\|\Lambda^{s+\frac12}(w^4)(t,.)\|_{L^{\frac52}}+\|w^4(t,.)\|_{L^{\frac52}}\|\Lambda^{s+\frac12}w(t,.)\|_{L^{10}}\right)\\ &\le C\|w(t,.)\|^4_{L^{10}}\|\Lambda^{s+\frac12}w(t,.)\|_{L^{10}},\\ \|w^4v(t,.)\|_{H^{s+\frac12}}&\le C\left(\|w^4(t,.)\|_{L^6}\|\Lambda^{s+\frac12}v(t,.)\|_{L^3}+\|v(t,.)\|_{L^\infty}\|\Lambda^{s+\frac12}(w^4)(t,.)\|_{L^2}\right)\\ &\le C_{T^*,M}\left(\|w(t,.)\|^3_{L^{18}}\|\Lambda^{s+\frac32}w(t,.)\|_{L^2}+\|\Lambda^{s+\frac12}(w^4)(t,.)\|_{L^3}\right)\\ &\le C_{T^*,M}\|w(t,.)\|^3_{L^{18}}\|\Lambda^{s+\frac32}w(t,.)\|_{L^2},\end{aligned}$$

$$\begin{aligned}\|w^3v^2(t,.)\|_{H^{s+\frac12}}&\le C\left(\|w^3(t,.)\|_{L^6}\|\Lambda^{s+\frac12}(v^2)(t,.)\|_{L^3}+\|v^2(t,.)\|_{L^\infty}\|\Lambda^{s+\frac12}(w^3)(t,.)\|_{L^2}\right)\\ &\le C_{T^*,M}\Big(\|w(t,.)\|^3_{L^{18}}\|v^2(t,.)\|_{H^{s+1}}\\ &\qquad+\|v(t,.)\|^2_{L^\infty}\|w(t,.)\|^2_{L^{10}}\|\Lambda^{s+\frac12}w(t,.)\|_{L^{10}}\Big),\\ \|w^2v^3(t,.)\|_{H^{s+\frac12}}&\le C\left(\|w^2(t,.)\|_{L^6}\|\Lambda^{s+\frac12}(v^3)(t,.)\|_{L^3}+\|v^3(t,.)\|_{L^\infty}\|\Lambda^{s+\frac12}(w^2)(t,.)\|_{L^2}\right)\\ &\le C_{T^*,M}\Big(\|w(t,.)\|^2_{L^{12}}\|v^3(t,.)\|_{H^{s+1}}\\ &\qquad+\|v(t,.)\|^3_{L^\infty}\|w(t,.)\|_{L^{10}}\|\Lambda^{s+\frac12}w(t,.)\|_{L^{10}}\Big),\\ \|wv^4(t,.)\|_{H^{s+\frac12}}&\le C\left(\|w(t,.)\|_{L^6}\|v^4(t,.)\|_{H^{s+1}}+\|v(t,.)\|^4_{L^\infty}\|\Lambda^{s+\frac12}w(t,.)\|_{L^2}\right).\end{aligned}$$

Hence, we have

$$
\begin{aligned}
\|\Lambda^{s+\frac{1}{2}} w\|_{L^5(t_0,s_0;L^{10})} \leq C_{T^*,M}\Big(1 &+ \|w\|^4_{L^5(t_0,s_0;L^{10})}\|\Lambda^{s+\frac{1}{2}} w\|_{L^5(t_0,s_0;L^{10})} \\
&+ \|w\|^3_{L^3(t_0,s_0;L^{18})}\|\Lambda^{s+\frac{3}{2}} w\|_{L^\infty(t_0,s_0;L^2)} + \|w\|^2_{L^5(t_0,s_0;L^{10})}\|\Lambda^{s+\frac{1}{2}} w\|_{L^5(t_0,s_0;L^{10})} \\
&+ \|w\|_{L^5(t_0,s_0;L^{10})}\|\Lambda^{s+\frac{1}{2}} w\|_{L^5(t_0,s_0;L^{10})} + (s_0-t_0)^{\frac{4}{5}}\|\Lambda^{s+\frac{1}{2}} w\|_{L^5(t_0,s_0;L^{10})}\Big).
\end{aligned} \tag{36}
$$

Since $w \in L^5((0,T^*),L^{10})$, we have $w \in L^3([0,T^*),L^{18})$ by virtue of Lemma 4. When t_0 is sufficiently close to T^*, we infer from (36)

$$
\|\Lambda^{s+\frac{1}{2}} w\|_{L^5(t_0,s_0;L^{10})} \leq C_{T^*,M}\Big(1 + \|w\|^3_{L^3(t_0,s_0;L^{18})}\|\Lambda^{s+\frac{3}{2}} w\|_{L^\infty(t_0,s_0;L^2)}\Big).
$$

Applying the energy estimate to (35), we get

$$
\begin{aligned}
&\|\partial_t\Lambda^{s+\frac{1}{2}} w\|_{L^\infty(t_0,s_0;L^2)} + \|\Lambda^{s+\frac{3}{2}} w\|_{L^\infty(t_0,s_0;L^2)} \\
&\qquad\qquad \leq C_{T^*,M}\Big(1 + \|w\|^3_{L^3(t_0,s_0;L^{18})}\|\Lambda^{s+\frac{3}{2}} w\|_{L^\infty(t_0,s_0;L^2)}\Big).
\end{aligned}
$$

If t_0 is sufficiently close to T^* so that $C_{T^*,M}\|w\|^3_{L^3(t_0,s_0;L^{18})} < \frac{1}{2}$, then we have

$$
\|\partial_t\Lambda^{s+\frac{1}{2}} w\|_{L^\infty(t_0,s_0;L^2)} \leq C_{T^*,M}, \quad \|\Lambda^{s+\frac{3}{2}} w\|_{L^\infty(t_0,s_0;L^2)} \leq C_{T^*,M}.
$$

More precisely,

$$
\|\partial_t\Lambda^{s+\frac{1}{2}} w\|_{C([t_0,T^*),L^2)} \leq C_{T^*,M}, \quad \|\Lambda^{s+\frac{3}{2}} w\|_{C([t_0,T^*),L^2)} \leq C_{T^*,M}.
$$

Using Eq. (35) once again, we find

$$
\|\partial_t\Lambda^{s+1} w\|_{C([t_0,T^*),L^2)} \leq C_{T^*,M}, \quad \|\Lambda^{s+2} w\|_{C([t_0,T^*),L^2)} \leq C_{T^*,M}.
$$

We obtain $w \in C([0,T^*],H^{s+2}(\mathbb{R}^3)) \cap C^1([0,T^*],H^{s+1}(\mathbb{R}^3))$ by Arzela-Ascoli's Theorem. Furthermore, for some $\delta > 0$, we can extend this solution to the time interval $[0,T^*+\delta]$ utilizing the local existence of solutions to (35).

Continuing this way, the proof of Lemma 7 is completed. □

Proof of the Theorem. We solve Eq. (25) with the initial data (26). To prove the Theorem, we need to show that

$$
V^k u \in C([0,\infty),L^2(\mathbb{R}^3)) \tag{37}
$$

for all k.

For $|k| = 0$, (37) obviously holds. Now suppose that (37) holds when $|l| \leq |k|-1$. We shall then prove this result for k.

Since $u \in L^\infty([0,T]\times\mathbb{R}^3)$ for $T > 0$ according to Lemmas 6 and 7, respectively, the Gagliardo-Nirenberg inequality (see [10]) when applied to (25) gives

$$
\sum_{|l_1|+\cdots+|l_j|\leq|k|-1} a_{kl_1\ldots l_j}(x,t,u)V^{l_1}u\ldots V^{l_j}u \in C([0,T],L^2(\mathbb{R}^3)).
$$

Hence, by the standard energy estimate and the regularity of initial data as stated in Lemma 3, we get

$$
V^k u \in C([0,T],L^2(\mathbb{R}^3)) \quad \text{for any } T > 0.
$$

Therefore, (37) holds and the Theorem is proved. □

References

1. K. Jörgens, *Das Anfangswertproblem im Großen für eine Klasse nichtlinearer Wellengleichungen*, Math. Z. **77** (1961), 295–307.
2. G. Grillakis, *Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity*, Ann. of Math. **132** (1990), 485–509.
3. J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Klein-Gordan equation*, Math. Z. **189** (1985), 487–505.
4. J. Shatah and M. Struwe, *Well-posedness in the energy space for semilinear wave equations with critical growth*, Int. Math. Res. Not. **7** (1994), 303–309.
5. M. Beals, *Self-spreading and strength of singularities for solutions to semilinear wave equations*, Ann. of Math. **118** (1983), 187–214.
6. J.-M. Bony, *Interaction de singularitiés pour les équations aux dérivées partielles non linéaries*, 11 p., Sém. Goulaouic-Meyer-Schwartz, 1981–1982, exp. no. 2.
7. J.-M. Bony, *Second microlocalization and propagation of singularities for semilinear hyperbolic equations*, Taniguchi Symp., Katata 1984, pp. 11–49.
8. H. Bahouri and J. Shatah, *Decay estimates for the critical semilinear wave equation*, Ann. Inst. Henri Poincare **15** (1998), 783-789.
9. M. Christ and M. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. **100** (1991), 87-109.
10. R. Melrose and N. Ritter, *Interaction of progressing waves for semilinear wave equation I*, Ann. of Math. **121** (1985), 187-213.

Department of Mathematics, Nanjing University, Nanjing 210093, P.R.China
E-mail address: huicheng@nju.edu.cn

Institute of Mathematics, University of Potsdam, P.O. Box 60 15 53, D-14469 Potsdam, Germany
E-mail address: ingo@math.uni-potsdam.de