Elliptic Theory on Manifolds with Nonisolated Singularities III. The Spectral Flow of Families of Conormal Symbols

Vladimir Nazaikinskii*

Institute for Problems in Mechanics, Russian Academy of Sciences
e-mail: nazaik@math.uni-potsdam.de

Anton Savin*

Independent University of Moscow,
e-mail: antonsavin@mtu-net.ru

Bert-Wolfgang Schulze*

Institut für Mathematik, Universität Potsdam
e-mail: schulze@math.uni-potsdam.de

Boris Sternin*

Independent University of Moscow
e-mail: sternine@mtu-net.ru

*Supported by the DFG via a project with the Arbeitsgruppe "Partielle Differentialgleichungen und Komplexe Analysis," Institut für Mathematik, Universität Potsdam, by the DAAD via the International Quality Network "Kopplungsprozesse und ihre Strukturen in der Geo- und Biosphäre" in the framework of partnership between Moscow State University and Universität Potsdam, by RFBR grants Nos. 02-01-00118, 02-01-06515 and 00-01-00161, and by the European Training Network Program "Geometric Analysis."

Abstract

When studying elliptic operators on manifolds with nonisolated singularities one naturally encounters families of conormal symbols (i.e. operators elliptic with parameter $p \in \mathbb{R}$ in the sense of Agranovich–Vishik) parametrized by the set of singular points. For homotopies of such families we define the notion of spectral flow, which in this case is an element of the K-group of the parameter space. We prove that the spectral flow is equal to the index of some family of operators on the infinite cone.

Keywords: elliptic family, conormal symbol, spectral flow, relative index.

2000 AMS classification: Primary 58J20, Secondary 58J30, 58J40, 19K56

Contents

Introduction

The notion of spectral flow $\{A_t\}$, $t \in [0,1]$, of a family of elliptic self-adjoint operators is well known and widely used in elliptic theory for a long time (e.g., see [3]). The present paper develops a generalization motivated by the recent development of index theory on manifolds with singularities. The spectral flow of a homotopy of *conormal symbols* (that is, operators elliptic with parameter in the sense of Agranovich–Vishik) was introduced in [14] (close results can be found in [10]): if $\mathbf{D}_t(p)$, $t \in [0,1]$, is a continuous family of conormal symbols, then, intuitively speaking, the spectral flow $\operatorname{sf} \mathbf{D}_t$ is the net number (counting multiplicities) of singular points $p_j = p_j(t)$ of the family $\mathbf{D}_t(p)$ crossing the real axis in the p-plane upwards as the homotopy parameter t varies from 0 to 1. An example is

$$\mathbf{D}_t(p) = p - A_t, \tag{0.1}$$

where A_t is a family of self-adjoint elliptic (or normally elliptic, as in [16]) operators. In this special case, the precise definition of spectral flow is based on the notion of *spectral sections*, which is not defined for general conormal symbols that do not have the form (0.1). However, to give a rigorous meaning to counting the number of singular points crossing the real axis in the general case, we need another idea, see Fig. 1.

Instead of counting singular points crossing the real axis, one draws a staircase in the space $[0,1] \times \mathbb{C}_p$ (the figure shows the projection of the staircase on the $(t, \operatorname{Im} p)$-plane) such that the stairs do not touch singular points. The passage from stair to stair occurs at some time instants t_j, and one counts the singular points occurring at these instants between the stairs, taking the number of points (more precisely, the sum of their multiplicities) with the minus sign when ascending and the plus sign when descending. The advantage of this method is that the sum of multiplicities of singular points of the conormal symbol in a strip is given by simple closed-form expressions (see [18]). We can also interpret this idea as follows: It is hard to count singular points crossing the real axis, since their trajectories are in general not transversal to the real axis. Hence we deform the real axis to a broken line (formed by the horizontal stairs and the vertical segments joining the stairs) such that the trajectories of singular points meet only vertical segments, to which they are necessarily transversal; then the computation becomes easier.

One of the main theorems on index formulas on manifolds with conical singularities says that index formulas with homotopy invariant terms exist for the class of elliptic operators with principal symbols from a given set satisfying certain natural conditions if and only if the spectral flow of every periodic family of conormal symbols associated with such principal symbols is zero [12]. (This condition can be verified effectively, since the spectral flow of a periodic family depends only on the principal symbols of the operators in the family.) A similar result for closely related [13] "spectral boundary value problems" (i.e., problems in which the boundary conditions are determined by some pseudodifferential projections) was obtained in [16]. The vanishing condition for the spectral flow is not satisfied in the class of all elliptic principal symbols, and one example of a symbol class

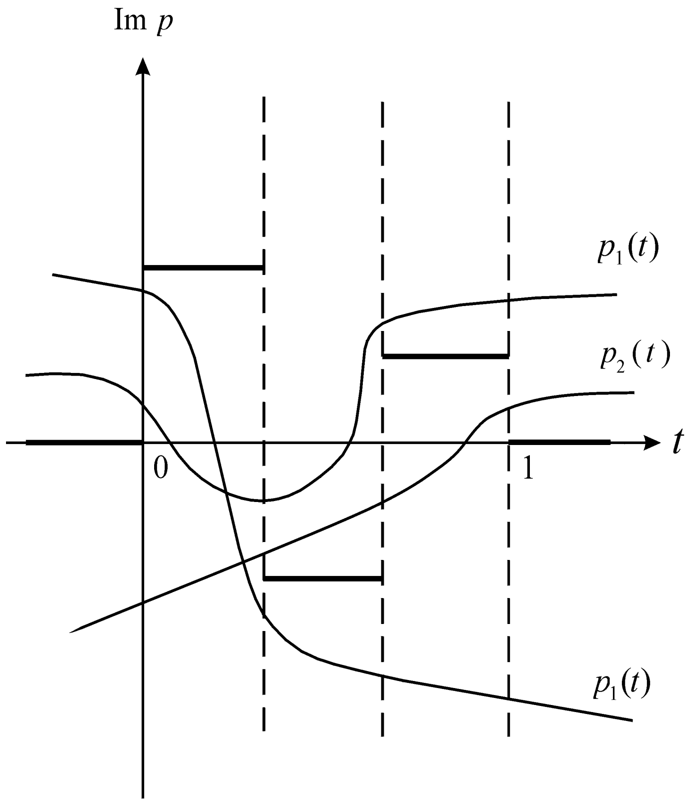

Figure 1. The definition of spectral flow

for which it is valid is the class of principal symbols whose restriction to the boundary of the cotangent bundle of the stretched manifold is symmetric with respect to the change of sign of the conormal variable:

$$D(r,\omega,p,q)\big|_{\partial T^*M^\wedge} = D(r,\omega,-p,q). \tag{0.2}$$

(Other examples can be found in [12].) The index formula for operators whose principal symbols satisfy this symmetry condition has the form [14]

$$\operatorname{ind}_a(\widehat{D}) + \operatorname{sf}\mathbf{D}_\tau = \operatorname{ind}_t(2D), \tag{0.3}$$

where the family $\mathbf{D}_\tau$ is determined by the conormal symbol and has the form of a linear homotopy

$$\mathbf{D}_\tau = (1-\tau)\mathbf{D}(p) + \tau\mathbf{D}(-p). \tag{0.4}$$

This formula generalizes the formula in [18], which uses only the multiplicities of singular points under a symmetry condition imposed on the full conormal symbol.

These constructions show that the notion of spectral flow plays an important role in elliptic theory on manifolds with singularities. However, the definition given in [14] is adequate only for the case of simplest singular manifolds, namely, manifolds with isolated singularities. The situation becomes more complicated for manifolds with nonisolated singularities. For example, consider an elliptic operator D on a manifold M with edge X. Its conormal symbol is defined at each point $x \in X$ and hence is a family that depends not only on the conormal variable p, but also on the point of the edge:

$$\sigma_c(D) = \mathbf{D}_x(p), \quad x \in X.$$

Now if we consider a homotopy $\mathbf{D}_t \equiv \mathbf{D}_{x,t}(p)$ of such parameter-dependent conormal symbols, then the spectral flow should be an element of the Grothendieck K-group $K(X)$ of the parameter space. For the case of families of self-adjoint elliptic operators, the definition was given in [7, 8] (again in terms of spectral sections). We are however interested in the case of general conormal symbols.

In the present paper, we define the spectral flow $\operatorname{sf} \mathbf{D}_t$ of a homotopy

$$\mathbf{D}_t \equiv \mathbf{D}_{x,t}(p), \quad t \in [0,1],$$

of families of conormal symbols with compact parameter space X and prove its main properties. The main novelties are as follows.

1. We consider general conormal symbols rather than linear ones of the form (0.1);

2. We give a local analytic definition that essentially treats the spectral flow as an intersection number;

3. We prove a K-theoretic formula for the spectral flow using Bott periodicity.

The passage from the case $X = \{pt\}$ to a nontrivial parameter space X dramatically complicates the technicalities, and so in the remaining part of the introduction we explain the main essential points of the construction.

1. In our situation the spectral flow is not a number, but rather an element of the K-group, that is, a formal difference $[E]$-$[F]$ of equivalence classes of two vector bundles over X. Consequently, we should count not the singular points crossing the real line (or passing between stairs), but the corresponding bundles. What are these bundles? If the family has the form

$$\mathbf{D}_{x,t}(p) = p - A(x,t) \qquad (0.5)$$

with a normally elliptic $A(x,t)$, then it is natural to define the bundle corresponding to a singular point $p_j = p_j(x,t)$, which in this case is just an eigenvalue of the operator $A(x,t)$, as the bundle formed by the corresponding root subspaces (so far we ignore the additional difficulties encountered if the eigenvalues have variable multiplicities). However, it is not obvious at first glance how to extend this scheme to arbitrary families $\mathbf{D}_{x,t}(p)$ that do not have the form (0.5). The hint is given by an analogy in the theory of elliptic boundary value problems. When analyzing an mth-order differential equation near the boundary, one freezes the coefficients, passes to the Fourier transform with respect to the tangent variables, and considers decaying solutions of the resulting ordinary differential equation with constant coefficients on the half-line $\mathbb{R}_+ \ni \tau$. The space of such solutions is isomorphic to the space of their m-jets at $\tau = 0$, and the boundary conditions are imposed on these jets. If the original equation is pseudodifferential rather than differential, then it is hard a priori to say anything about the value of m for which such an isomorphism holds, and it is more convenient to consider the space of decaying solutions themselves. In our case we proceed as follows. Let us start from the simplest family (0.5). Let h be an eigenvector of the operator $A = A(x,t)$ with eigenvalue p:

$$Ah = ph.$$

Instead of h, we consider the solution

$$f(\tau) = e^{-ip\tau}h$$

of the equation

$$(\widehat{p} - A)f(\tau) \equiv \left(\frac{\partial}{\partial\tau} - A\right)f(\tau) = 0 \tag{0.6}$$

with the initial condition $f|_{t=0} = h$. Next, let h_1 be an associated vector:

$$Ah_1 = ph_1 + h.$$

Instead of h_1, we consider the solution

$$f_1(\tau) = e^{-ip\tau}h_1 + ite^{-ip\tau}$$

of Eq. (0.6) with the initial condition $f_1|_{t=0} = h_1$, and so on (see [2]). The passage from root vectors with given λ to solutions of the nonautonomous equation is an isomorphism, and even for the simplest family (0.5) the advantages of this method are obvious: one need not distinguish between eigenvectors and associated vectors, since they all are represented by solutions of the same nonautonomous equation. Next, in the construction of the spectral flow we shall need all solutions corresponding to roots p contained in some open strip $a < \operatorname{Im} p < b$ in the complex plane. It is easy to single out such solutions: one must consider solutions that grow slower than $e^{b\tau}$ at $+\infty$ and slower than $e^{a\tau}$ at $-\infty$. (Here we continue the analogy with elliptic boundary value problems but use a strip instead of the lower half-plane.) Restating this condition, we say that the solutions should be sought in the weighted Sobolev space $H^{s,a,b}(\mathbb{R}_\tau \times \Omega)$ on the infinite cylinder (e.g., see [20], [18]). Now it is clear what to do for general conormal symbols $\mathbf{D}(p)$: the bundle corresponding to the passage between the stairs $\operatorname{Im} p = a$ and $\operatorname{Im} p = b$, $a < b$, at time t_j is formed by the kernels of the operators

$$\widehat{\mathbf{D}}_{x,t_j} \equiv \mathbf{D}_{x,t_j}\left(-i\frac{\partial}{\partial\tau}\right)$$

in the space $H^{s,a,b}$. (The smoothness parameter s does not affect anything by virtue of the smoothness theorems for solutions of elliptic equations.) The bundles E and F defining the spectral flow can be constructed as follows in the simplest case: one chooses some staircase, and the direct sum of all bundles corresponding to ascending steps is taken as F, while the direct sum of all bundles corresponding to descending steps is taken as E. Needless to say, one has to prove that the class $[E]-[F]\in K(X)$ is independent of the choice of the staircase.

2. If the parameter space X is nontrivial, then there is an additional technical difficulty: in general, there exists no staircase suitable for all $x\in X$ simultaneously. The construction can be carried out only locally; in other words, one covers the parameter space by finitely many open charts U_j, chooses a staircase S_j in each of these charts, and constructs the corresponding bundles E_j and F_j. However, this is not sufficient for defining a class in $K(X)$, even if the necessary conditions $[E_j]-[F_j]=[E_k]-[F_k]\in K(U_j\cap U_k)$ hold. Thus we in fact construct some bundles W_j and isomorphisms

$$\varphi_j k : E_j \oplus W_j \longrightarrow E_k \oplus W_k,$$
$$\psi_j k : F_j \oplus W_j \longrightarrow F_k \oplus W_k$$

on the intersections $U_j\cap U_k$ such that the cocycle conditions hold. These data already determine vector bundles E and F over entire X and hence the class $[E]-[F]\in K(X)$. A major part of this paper just deals with the construction of these bundles and with the proof of the fact that the construction is invariant (in particular, independent of the choice of staircases in the charts).

The main body of the paper consists of three sections. The first section contains some necessary preliminaries, the second section contains the construction of the spectral flow, and in the third section we prove the main properties of the spectral flow, namely, the homotopy invariance and the index theorem.

We note that the sign in the definition of spectral flow is opposite to that adopted in [14]. This has been done to avoid the unaesthetical minus sign in the statement of the index theorem.

1 Preliminaries

In this section we recall some known material in order to make the exposition as much as possible self-contained.

1.1 Conormal symbols

Let X be a compact parameter space (for simplicity assumed to be a smooth manifold), and let

$$\widetilde{\Omega} \xrightarrow{\pi} X \tag{1.1}$$

be a locally trivial bundle with fiber a smooth compact manifold Ω without boundary. In what follows the fiber over a point $x \in X$ will be denoted by Ω_x whenever we need to indicate the base point explicitly. Next, let $\widetilde{E}$ be a vector bundle over $\widetilde{\Omega}$. We denote the restriction $\widetilde{E}\,\big|_{\Omega_x}$ by E_x; this is a locally trivial vector bundle over Ω_x.

Definition 1.1. A *conormal symbol* of order m (with parameter space X) is a family, continuously depending on $x \in X$, of mth-order pseudodifferential operators Agranovich–Vishik elliptic with parameter $p \in \mathbb{R}$ [1] in the Sobolev scales $\{H^s(\Omega_x, E_x)\}_{s\in\mathbb{R}}$.

We denote a conormal symbol by $\mathbf{D} = \{\mathbf{D}_x(p)\}$, where

$$\mathbf{D}_x(p) : H^s(\Omega_x, E_x) \longrightarrow H^{s-m}(\Omega_x, E_x). \tag{1.2}$$

In what follows we omit the subscript x and the variable p unless this might lead to confusion and also write $H^s(\Omega_x)$ instead of $H^s(\Omega_x, E_x)$. Furthermore, we consider only conormal symbols of order $m = 0$. (The general case can be reduced to this by order reduction.) Unless otherwise explicitly specified, we assume throughout the following that for some $h_0 > 0$ all conormal symbols in question satisfy the following condition.

Condition 1.2. The operator function (1.2) is holomorphic in the strip $|\operatorname{Im} p| < h_0$ and is a family of pseudodifferential operators Agranovich–Vishik elliptic with parameter $\operatorname{Re} p \in \mathbb{R}$ continuously depending on $x \in X$ and $\operatorname{Im} p \in (-h_0, h_0)$.

We first give the definition of spectral flow for families of conormal symbols satisfying Condition 1.2. Later on we show how to extend the definition to the case in which this condition is violated.

It follows from Condition 1.2 that for each $x \in X$ the operator function (1.2) is finitely meromorphically invertible in the strip $|\operatorname{Im} p| < h_0$, i.e. invertible everywhere but a discrete set of *singular points*, where the inverse operator has poles with finite rank principal parts of the Laurent series. Moreover, there are only finitely many singular points in any proper substrip $|\operatorname{Im} p| < h$, $h < h_0$.

1.2 Weighted Sobolev spaces and ψDO on the infinite cone

Weighted Sobolev spaces. To a given operator function (1.2) we shall assign some families of finite rank projection operators defined on open subsets of the parameter space X. First, let us describe the spaces where these projections act. Let

$$K_\Omega = \big\{\Omega \times \mathbb{R}_+\big\}/\big\{\Omega \times \{0\}\big\} \tag{1.3}$$

be the infinite cone with base Ω. We denote points of the smooth part of the cone by (ω, r), $\omega \in \Omega$, $r \in \mathbb{R}_+$. The projections act in the spaces $H^{s,\gamma_1,\gamma_2}(K_\Omega)$ defined as follows. The cone K_Ω can be treated as a compact manifold with two conical singularity points (a "spindle"), where one point corresponds to the value $r = 0$ and the other to $r = \infty$.

(The radial variable r' in a neighborhood of the second point is related to r by the change of variables $r' = 1/r$.) Then $H^{s,\gamma_1,\gamma_2}(K_\Omega)$ is the weighted Sobolev space of order s with weight exponents γ_1 at the point $r=0$ and $-\gamma_2$ at the point $r'=0$. For $\gamma_1 \le \gamma \le \gamma_2$ there is a continuous embedding

$$H^{s,\gamma}(K_\Omega) \subset H^{s,\gamma_1,\gamma_2}(K_\Omega), \tag{1.4}$$

where $H^{s,\gamma}(K_\Omega)$ is the "standard" weighted Sobolev space on the infinite cone with the norm

$$\|u\|^2_{s,\gamma} = \int\limits_{K_\Omega} \left| \left(1 - \left(r\frac{\partial}{\partial r}\right)^2 - \triangle_\Omega\right)^{s/2} [r^{-\gamma}u] \right|^2 \frac{dr}{r}\, d\omega.$$

Moreover, if $\gamma_1 \le \gamma_3, \gamma_4 \le \gamma_2$, then there is a more general embedding

$$H^{s,\gamma_3,\gamma_4}(K_\Omega) \subset H^{s,\gamma_1,\gamma_2}(K_\Omega),$$

which turns into the previous one for $\gamma_3 = \gamma_4 = \gamma$, since

$$H^{s,\gamma,\gamma}(K_\Omega) = H^{s,\gamma}(K_\Omega).$$

For the reader's convenience, we also note that the space $H^{s,\gamma_1,\gamma_2}(K_\Omega)$ can be described as follows. Let

$$\begin{gathered} 1 = \varphi_1(r) + \varphi_2(r) \\ \varphi_1(r) = 0 \text{ for } r > 1,\ \varphi_2(r) = 0 \text{ for } r < 1/2 \end{gathered} \tag{1.5}$$

be a smooth partition of unity on $\mathbb{R}_+$. From now on we assume that this partition is chosen and fixed. Next, let $\overset{\circ}{K}_\Omega$ be the cone K_Ω without the vertex. Then

$$\begin{aligned} H^{s,\gamma_1,\gamma_2}(K_\Omega) &= \left\{ \begin{matrix} \text{the completion of } C_0^\infty(\overset{\circ}{K}_\Omega) \\ \text{with respect to the norm } \|u\|_{s,\gamma_1,\gamma_2} = \|\varphi_1 u\|_{s,\gamma_1} + \|\varphi_2 u\|_{s,\gamma_2} \end{matrix} \right\} \\ &= \left\{ \begin{matrix} \text{the set of distributions } u \in \mathcal{D}'(\overset{\circ}{K}_\Omega), \\ \text{such that } \varphi_1 u \in H^{s,\gamma_1}(K_\Omega),\ \varphi_2 u \in H^{s,\gamma_2}(K_\Omega) \end{matrix} \right\}. \end{aligned} \tag{1.6}$$

In the following, we consider only spaces $H^{s,\gamma_1,\gamma_2}(K_\Omega)$ with $\gamma_1 \le \gamma_2$.

Pseudodifferential operators with constant coefficients on the infinite cone. Let

$$\mathbf{A} \equiv \mathbf{A}(p) : H^s(\Omega) \longrightarrow H^s(\Omega), \quad s \in \mathbb{R}$$

be a pseudodifferential operator with parameter $\operatorname{Re} p \in \mathbb{R}$ in the sense of Agranovich–Vishik of zero order on Ω, defined on the weight line

$$\mathcal{L}_\gamma = \{\operatorname{Im} p = \gamma\}.$$

We define an operator

$$\widehat{\mathbf{A}}_\gamma \equiv \mathbf{A}\left(ir\frac{\partial}{\partial r}\right) : C_0^\infty(\overset{\circ}{K}_\Omega) \longrightarrow \mathcal{D}'(\overset{\circ}{K}_\Omega) \tag{1.7}$$

by setting

$$\widehat{\mathbf{A}}_\gamma = \mathfrak{M}_\gamma^{-1} \circ \mathbf{A}(p) \circ \mathfrak{M}_\gamma, \tag{1.8}$$

where

$$[\mathfrak{M}_\gamma u](p) = \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{ip} u(r)\,\frac{dr}{r}, \quad p \in \mathcal{L}_\gamma, \qquad [\mathfrak{M}_\gamma^{-1} v](r) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{L}_\gamma} r^{-ip} v(p)\,dp \tag{1.9}$$

is the Mellin transform with respect to the variable r with weight line $\mathcal{L}_\gamma$ and the inverse transform. The operator $\widehat{\mathbf{A}}_\gamma$ extends by closure to a continuous operator (denoted by the same letter)

$$\widehat{\mathbf{A}}_\gamma : H^{s,\gamma}(K_\Omega) \longrightarrow H^{s,\gamma}(K_\Omega). \tag{1.10}$$

Now suppose that the operator function $\mathbf{A}(p)$ is holomorphic in the strip $|\operatorname{Im} p| < h_0$ and is a family of Agranovich–Vishik pseudodifferential operators with parameter $\operatorname{Re} p \in \mathbb{R}$ continuously depending on $x \in X$ and $\operatorname{Im} p \in (-h_0, h_0)$. Then the operator (1.7) (i.e., the operator $\widehat{\mathbf{A}}_\gamma$ on compactly supported functions) is independent of the choice of $\gamma \in (-h_0, h_0)$ by the Cauchy theorem and can be extended not only to the continuous operators (1.10), but also to the continuous operator

$$\widehat{\mathbf{A}}_{\gamma_1,\gamma_2} : H^{s,\gamma_1,\gamma_2}(K_\Omega) \longrightarrow H^{s,\gamma_1,\gamma_2}(K_\Omega) \tag{1.11}$$

for $\gamma_1, \gamma_2 \in (-h_0, h_0)$. (This assertion is not valid if $\mathbf{A}(p)$ has singular points in the strip.)

1.3 Projections associated with conormal symbols

Let $\mathbf{D}$ be a conormal symbol satisfying Condition 1.2. We take some positive $h < h_0$ and carry out all subsequent considerations in the Hilbert spaces[1]

$$\mathfrak{H}_x = H^{s,-h,h}(K_{\Omega_x}). \tag{1.12}$$

Accordingly, all weight exponents γ will be taken from the interval $[-h, h]$, which is not explicitly mentioned every time. Consider the operator family $\mathbf{R} = \mathbf{D}^{-1}$; more precisely,

$$\mathbf{R}_x(p) = \mathbf{D}_x^{-1}(p).$$

[1]The specific value of s is irrelevant; the results are independent of the choice of s by virtue of the theorems on the smoothness of solutions of elliptic equations, which is valid also for cone-degenerate operators.

By the preceding, this is a meromorphic operator family in the strip $|\operatorname{Im} p| < h_0$; it depends on x continuously and for each x has at most finitely many poles in the strip $|\operatorname{Im} p| \leq h$. Moreover, it is a zero-order pseudodifferential operator with parameter $\operatorname{Re} p$ on every weight line $\mathcal{L}_\gamma$ that does not contain poles for a given x.

Let $U \subset X$. Until the end of this subsection we fix U and consider only parameter values $x \in U$. Let $A = [\gamma_1, \gamma_2] \subset [-h, h]$, $\gamma_1 \leq \gamma_2$, be an interval (possibly degenerating to a point) such that the weight lines $\mathcal{L}_{\gamma_j}$, $j = 1, 2$, corresponding to its endpoints do not contain poles of $\mathbf{R}$ for $x \in U$. Such intervals will be called *admissible* (with respect to U). We define an operator family

$$\widehat{\mathbf{P}}_A \equiv \widehat{\mathbf{P}}_A(x) : C_0^\infty(\mathring{K}_{\Omega_x}) \longrightarrow \mathfrak{H}_x, \quad x \in U, \tag{1.13}$$

by the formula (we omit the parameter x here and in what follows and sometimes speak of operators rather then operator families for brevity)

$$\widehat{\mathbf{P}}_A = \big(\widehat{\mathbf{R}}_{\gamma_1} - \widehat{\mathbf{R}}_{\gamma_2}\big)[\varphi_2, \widehat{\mathbf{D}}], \tag{1.14}$$

where $\widehat{\mathbf{D}} = \widehat{\mathbf{D}}_\gamma$, $\gamma \in [-h, h]$ is chose arbitrarily, φ_2 is the second element of the partition of unity (1.5), and the brackets $[\cdot, \cdot]$ stand for the commutator of operators.

Remark 1.3. If the interval A is degenerate, then $\mathbf{P}_A = 0$.

One can readily see that the operator (1.14) is well defined: it follows from the above discussion that the operator $\widehat{\mathbf{D}}_\gamma$ (and hence the commutator $[\varphi_2, \widehat{\mathbf{D}}_\gamma]$) is independent of γ, and by applying the commutator to a function from the space $C_0^\infty(\mathring{K}_\Omega)$ one obtains a function that lies in each of the spaces $H^{s,\gamma_j}(K_\Omega)$, so that the operators $\widehat{\mathbf{R}}_{\gamma_1}$ and $\widehat{\mathbf{R}}_{\gamma_2}$ can be applied. It remains to use appropriate embeddings of the form (1.4). Moreover, the operator (1.14) extends to a continuous operator in the space $H^{s,\gamma_1,\gamma_2}(K_\Omega)$: it is the projection on the finite-dimensional kernel of the operator $\widehat{\mathbf{D}}_{\gamma_1,\gamma_2}$. Thus, for each admissible interval the above-mentioned projections act in their own weighted Sobolev spaces, which complicates the comparison of their ranges. However, one can readily overcome the difficulty. Let us prove some properties of the operators $\mathbf{P}_A$ in the wider space $\mathfrak{H} \equiv \mathfrak{H}_x$.

Lemma 1.4. *The projections* (1.14) *extend by closure to continuous projections* (*denoted by the same letter*)

$$\widehat{\mathbf{P}}_A : \mathfrak{H} \longrightarrow \mathfrak{H}. \tag{1.15}$$

(*The ranges of these projections are finite-dimensional and hence are not affected by the closure.*)

Proof. The operator of multiplication by φ_2 is continuous in the spaces

$$\varphi_2 : \mathfrak{H} \equiv H^{s,-h,h}(K_\Omega) \longrightarrow H^{s,\gamma,h}(K_\Omega),$$

where γ is arbitrary. (This follows from the fact that $\varphi_2 \equiv 0$ for small r). Of course, we are interested only in the values $\gamma \in [-h, h]$. For these γ, the operator $\widehat{\mathbf{D}}$ is continuous in $H^{s,\gamma,h}(K_\Omega)$, so that the commutator is continuous in the spaces

$$[\varphi_2, \widehat{\mathbf{D}}] : \mathfrak{H} \longrightarrow H^{s,\gamma,h}(K_\Omega). \tag{1.16}$$

Now we note that

$$[\varphi_2, \widehat{\mathbf{D}}] = -[\varphi_1, \widehat{\mathbf{D}}], \tag{1.17}$$

and moreover, $\varphi_1 \equiv 0$ for large r. A similar argument shows that the commutator is continuous in the spaces

$$[\varphi_2, \widehat{\mathbf{D}}] : \mathfrak{H} \longrightarrow H^{s,-h,\gamma}(K_\Omega). \tag{1.18}$$

Combining (1.16) with (1.18), we find that the commutator is continuous in the spaces

$$[\varphi_2, \widehat{\mathbf{D}}] : \mathfrak{H} \longrightarrow H^{s,\gamma,h}(K_\Omega) \cap H^{s,-h,\gamma}(K_\Omega) \equiv H^{s,\gamma}(K_\Omega). \tag{1.19}$$

It remains to take $\gamma = \gamma_j$, $j = 1, 2$, use the continuity of the operators $\widehat{\mathbf{R}}_{\gamma_j}$ in the corresponding spaces, and apply the corresponding embeddings (1.4). $\square$

The projections thus constructed continuously depend on x. Hence for each interval $A \subset [-h, h]$ admissible with respect to U we have a finite-dimensional subbundle $\mathcal{E}_A = \mathcal{E}_A(U)$ over U of the Hilbert bundle $\widetilde{\mathfrak{H}} \xrightarrow{\pi} X$ with fiber $\mathfrak{H}_x$; namely, the subbundle is formed by the ranges of the family of projections $\widehat{\mathbf{P}}_A$.

These projection families (and the corresponding subbundles) prove to have useful properties.

Theorem 1.5. *The following assertions hold.*

1) *If A and B are admissible intervals, then*

$$\widehat{\mathbf{P}}_A \widehat{\mathbf{P}}_B = \widehat{\mathbf{P}}_B \widehat{\mathbf{P}}_A = \widehat{\mathbf{P}}_{A\cap B}, \quad \mathcal{E}_A \cap \mathcal{E}_B = \mathcal{E}_{A\cap B}. \tag{1.20}$$

Here $\widehat{\mathbf{P}}_\varnothing = 0$ by definition, and, as noted before, $\widehat{\mathbf{P}}_C = 0$ if C is a degenerate admissible interval (i.e., consists of a single point).

2) *If A and B are admissible intervals with a single common point, then*

$$\widehat{\mathbf{P}}_A + \widehat{\mathbf{P}}_B = \widehat{\mathbf{P}}_{A\cup B}, \quad \mathcal{E}_A \oplus \mathcal{E}_B = \mathcal{E}_{A\cup B} \tag{1.21}$$

Remark 1.6. The intersection of admissible intervals, as well as the union of admissible intervals with a single common point, is always an admissible interval, so the objects in the statement of the theorem are well defined.

Proof. First, let us prove item 2). Let $A = [\gamma_1, \gamma_2]$, $B = [\gamma_2, \gamma_3]$, so that $A \cup B = [\gamma_1, \gamma_3]$. We have

$$\widehat{\mathbf{P}}_A + \widehat{\mathbf{P}}_B = \bigl(\widehat{\mathbf{R}}_{\gamma_1} - \widehat{\mathbf{R}}_{\gamma_2}\bigr)[\varphi_2, \widehat{\mathbf{D}}] + \bigl(\widehat{\mathbf{R}}_{\gamma_2} - \widehat{\mathbf{R}}_{\gamma_3}\bigr)[\varphi_2, \widehat{\mathbf{D}}] = \bigl(\widehat{\mathbf{R}}_{\gamma_1} - \widehat{\mathbf{R}}_{\gamma_3}\bigr)[\varphi_2, \widehat{\mathbf{D}}] = \widehat{\mathbf{P}}_{A\cup B},$$

as desired.

Now let us pass to item 1). With regard to item 2), it suffices to consider the case in which A and B have at most one point in common. Let $A = [\gamma_1, \gamma_2]$ and $B = [\gamma_3, \gamma_4]$, where $\gamma_2 \le \gamma_3$. Then

$$\begin{aligned}\widehat{\mathbf{P}}_A\widehat{\mathbf{P}}_B &= \bigl(\widehat{\mathbf{R}}_{\gamma_1} - \widehat{\mathbf{R}}_{\gamma_2}\bigr)[\widehat{\mathbf{D}}, \varphi_1]\widehat{\mathbf{P}}_B \\ &= \bigl(\widehat{\mathbf{R}}_{\gamma_1} - \widehat{\mathbf{R}}_{\gamma_2}\bigr)\widehat{\mathbf{D}}\varphi_1\widehat{\mathbf{P}}_B - \bigl(\widehat{\mathbf{R}}_{\gamma_1} - \widehat{\mathbf{R}}_{\gamma_2}\bigr)\varphi_1\widehat{\mathbf{D}}\widehat{\mathbf{P}}_B \\ &= \bigl(\widehat{\mathbf{R}}_{\gamma_1} - \widehat{\mathbf{R}}_{\gamma_2}\bigr)\widehat{\mathbf{D}}\varphi_1\widehat{\mathbf{P}}_B,\end{aligned}$$

since

1. $\widehat{\mathbf{D}}\widehat{\mathbf{P}}_B = 0$ (the range of $\mathbf{P}_B$ is contained in the kernel of $\widehat{\mathbf{D}}$);
2. by virtue of the inequalities relating the weight exponents, the operator of multiplication by φ_1 continuously acts from the space $H^{s,\gamma_3,\gamma_4}(K_\Omega) \supset \operatorname{Im} \mathbf{P}_B$ into the space $H^{s,\gamma_1}(K_\Omega) \cap H^{s,\gamma_2}(K_\Omega)$, so that all operator products in this computation are well defined continuous operators.[2]

It remains to note that $\bigl(\widehat{\mathbf{R}}_{\gamma_1} - \widehat{\mathbf{R}}_{\gamma_2}\bigr)\widehat{\mathbf{D}} = 0$ on $H^{s,\gamma_1}(K_\Omega) \cap H^{s,\gamma_2}(K_\Omega)$.

In a similar way, on can consider the product $\widehat{\mathbf{P}}_B\widehat{\mathbf{P}}_A$. The proof is complete. □

1.4 Homotopies of projections and bundle isomorphisms

We intend to study how the above-introduced projection families (and the corresponding bundles) behave under homotopies of conormal families. Here we give some facts concerning homotopies of projection families and the corresponding bundles in the abstract case. All our constructions are "pointwise" with respect to the parameter space, and so instead of homotopies of projection families we speak for brevity of homotopies of projections and so on. The continuous (or appropriately differentiable) dependence of our constructions on the parameters is trivial.

[2]Compare with the computation for the case $A \supset B$ (i.e., $\gamma_1 \le \gamma_3 \le \gamma_4 \le \gamma_2$):

$$\widehat{\mathbf{P}}_A\widehat{\mathbf{P}}_B = \bigl(\widehat{\mathbf{R}}_{\gamma_1}[\widehat{\mathbf{D}}, \varphi_1] + \widehat{\mathbf{R}}_{\gamma_2}[\widehat{\mathbf{D}}, \varphi_2]\bigr)\widehat{\mathbf{P}}_B = \cdots = \widehat{\mathbf{R}}_{\gamma_1}\widehat{\mathbf{D}}\varphi_1\widehat{\mathbf{P}}_B + \widehat{\mathbf{R}}_{\gamma_2}\widehat{\mathbf{D}}\varphi_2\widehat{\mathbf{P}}_B = (\varphi_1 + \varphi_2)\widehat{\mathbf{P}}_B = \widehat{\mathbf{P}}_B.$$

Statement of the problem. Let $P(t)$, $t \in [0,1]$, be a continuous piecewise continuously differentiable (in the uniform operator topology) homotopy of continuous projections in a Hilbert space $\mathcal{H}$. (Instead of $[0,1]$ one can take an arbitrary interval $[a,b]$ of the real line.) To this homotopy we assign the family of subspaces

$$\mathcal{E}(t) = \operatorname{Im} P(t) \subset \mathcal{H}, \quad t \in [0,1]. \tag{1.22}$$

We shall study continuous piecewise continuously differentiable families of isomorphisms

$$W(t) = \mathcal{E}(0) \longrightarrow \mathcal{E}(t), \quad t \in [0,1], \qquad W(0) = 1. \tag{1.23}$$

More precisely, we answer the following questions:

- How to describe all such families of isomorphisms in terms of the family of projections $P(t)$?
- How to construct them?[3]

Reduction to the similarity of projections. To solve these problems, we first note that from isomorphisms of the ranges of projections one can always proceed to the similarity of the projections themselves. Namely, the following lemma holds.

Lemma 1.7. *Each family* (1.23) *can be extended to a continuous piecewise continuously differentiable family of isomorphisms* $U(t) : \mathcal{H} \longrightarrow \mathcal{H}$ *satisfying the conditions*

$$U(t)P(0)U^{-1}(t) = P(t), \quad t \in [0,1]; \qquad U(0) = 1. \tag{1.24}$$

Conversely, the restriction to $\mathcal{E}(0)$ *of a continuous piecewise continuously differentiable family of isomorphisms* $U(t)$ *satisfying conditions* (1.24) *is a family of isomorphisms of the form* (1.23).

Proof. The second part of the assertion is trivial; let us prove the first one. Given $W(t)$, we consider the family of subspaces

$$\widetilde{\mathcal{E}}(t) = \operatorname{Im}(1 - P(t))$$

and some continuous piecewise continuously differentiable family of isomorphisms

$$\widetilde{W}(t) : \widetilde{\mathcal{E}}(0) \longrightarrow \widetilde{\mathcal{E}}(t), \quad \widetilde{W}(0) = 1.$$

(The existence of such a family, in any case well-known, follows also from the results given in this subsection below.) We set

$$U(t) = W(t)P(0) + \widetilde{W}(t)(1 - P(0)).$$

[3] The answer to the second question will not be complete. We only describe the construction of a fairly large set of such families.

Then one can readily see that

$$U(t)P(0) = W(t)P(0) = P(t)W(t)P(0) = P(t)[W(t)P(0) + \widetilde{W}(t)(1 - P(0))] = P(t)U(t),$$

as desired. $\square$

From now on we sometimes omit the argument t. The t-derivative will be denoted by a dot.

A description of the set of families (1.23)**.** Now we are in a position to describe all possible families of isomorphisms (1.23). The following assertion holds.

Theorem 1.8. *Every continuous piecewise continuously differentiable family of isomorphisms* (1.23) *has the form*

$$W(t) = U(t)\big|_{\mathcal{E}(0)} \tag{1.25}$$

for an invertible operator family

$$U(t) : \mathcal{H} \longrightarrow \mathcal{H}$$

satisfying the Cauchy problem

$$\dot{U} = AU, \quad U(0) = 1 \tag{1.26}$$

with some bounded piecewise continuous operator function $A = A(t)$ such that the original projection family P satisfies the equation

$$\dot{P} = [A, P]. \tag{1.27}$$

Two such operators A and $\widetilde{A}$ generate the same isomorphism W if and only if

$$(A - \widetilde{A})P \equiv 0. \tag{1.28}$$

Proof. By Lemma 1.7, every isomorphism W has the form (1.25), where U satisfies the relation (1.24). Thus, we must show that the relation

$$P = UP(0)U^{-1}$$

holds if and only if for some operator A the family U satisfies the Cauchy problem[4] (1.26) and the projection P simultaneously satisfies Eq. (1.27). This assertion is nothing else than a variation on the theme of the well-known relationship between the Heisenberg and

[4]The initial condition follows from (1.24).

Schrödinger pictures in quantum mechanics. For completeness, we give the proof here: the operator A is uniquely determined by U as $A = \dot{U}U^{-1}$; now if $P = UP(0)U^{-1}$, then

$$\dot{P} = (UP(0)U^{-1})^{\cdot} = \dot{U}P(0)U^{-1} - UP(0)U^{-1}\dot{U}U^{-1}$$
$$= (\dot{U}U^{-1})(UP(0)U^{-1}) - (UP(0)U^{-1})(\dot{U}U^{-1}) = [A, P].$$

Conversely, if $\dot{P} = [A, P]$, then $P = UP(0)U^{-1}$ by virtue of the unique solvability of the Cauchy problem.

Finally, two isomorphisms U and $\widetilde{U}$ with property (1.24) specify the same isomorphism W if and only if

$$V \stackrel{\text{def}}{=} P(U - \widetilde{U}) \equiv 0.$$

The family V satisfies the Cauchy problem

$$V(0) = 0, \quad \dot{V} = \widetilde{A}V + (A - \widetilde{A})PU$$

and vanishes identically if and only if condition (1.28) holds. □

A construction of operators A satisfying (1.27)**.** Thus, to construct the desired isomorphisms W, it suffices to find operators $A(t)$ such that $P(t)$ satisfies the Heisenberg equation (1.27). Here we give a construction that provides an ample supply of such operators. Let $\mathcal{P} = \{P_1(t), \dots, P_N(t)\}$ be a finite set of continuous piecewise continuously differentiable disjoint projection-valued functions of the parameter $t \in [0, 1]$. This means that for each t all operators $P_j(t)$ are projections (not necessarily orthogonal), and moreover,

$$P_j(t)P_k(t) = 0 \quad \text{for } j \neq k. \tag{1.29}$$

We set

$$P_0(t) = 1 - \sum_{j=1}^{N} P_j(t), \tag{1.30}$$

so that the operators $P_0, \dots, P_N$ form a resolution of identity:

$$1 = \sum_{j=0}^{N} P_j. \tag{1.31}$$

(If $\mathcal{P}$ is a resolution of identity, then $P_0 = 0$.) Consider the piecewise continuous operator function of t given by

$$A \equiv A[\mathcal{P}] \equiv A[\mathcal{P}](t) \stackrel{\text{def}}{=} \sum_{j=0}^{N} \dot{P}_j P_j. \tag{1.32}$$

Lemma 1.9. *Let the projection P be decomposable in the system $\mathcal{P}$ in the sense that*

$$P = \sum_{j\in I} P_j,$$

where $I \in \{0, \dots, N\}$ is some subset. Then $\dot{P} = [A, P]$.

Proof. In view of linearity, it suffices to prove that $\dot{P}_j = [A, P_j]$ for all $j = 0, \dots, N$. Indeed,

$$\begin{aligned} AP_j - P_j A &= \dot{P}_j P_j - P_j \sum_{l=0}^{N} \dot{P}_l P_l = \dot{P}_j P_j - \sum_{l=0}^{N} (P_j P_l)^{\cdot} P_l + \dot{P}_j \sum_{l=0}^{N} P_l P_l \\ &= \dot{P}_j P_j - \dot{P}_j P_j + \dot{P}_j \sum_{l=0}^{N} P_l = \dot{P}_j. \quad \square \end{aligned}$$

Remark 1.10. If the $P_j(t)$ are rank one orthogonal projections on the eigenvectors of some time-dependent self-adjoint operator $B(t)$, then the corresponding unitary operator $U(t)$ specifies the evolution in the adiabatic approximation (see Berry [5] and Simon [19]).

Remark 1.11. The construction given here is a generalization of the well-known construction in which the operator U is defined via the Cauchy problem

$$\begin{cases} \dot{U} = [\dot{P}, P]U, \\ U|_{t=0} = 1. \end{cases}$$

Indeed, if we take $N = 1$ and $P_1 = P$, then $P_0 = 1 - P$ and

$$A = \dot{P}_1 P_1 + \dot{P}_0 P_0 = \dot{P}P - \dot{P}(1 - P) = 2\dot{P}P - \dot{P} = 2\dot{P}P - (PP)^{\cdot} = [\dot{P}, P].$$

Our construction can readily be generalized to the case of several sets of disjoint projections.

Lemma 1.12. *Let $\mathcal{P}_1, \dots, \mathcal{P}_s$ be continuous piecewise continuously differentiable (in the parameter $t \in [0, 1]$) finite sets of disjoint projections such that the projection P is decomposable in each of these sets. Then P satisfies Eq. (1.27) with an arbitrary operator A of the form*

$$A = \sum_{j=1}^{s} \lambda_j A[\mathcal{P}_j], \tag{1.33}$$

where the $\lambda_j = \lambda_j(t)$ are piecewise continuous functions such that

$$\sum_j \lambda_j(t) = 1.$$

The proof is trivial.

2 The Spectral Flow

Let $\mathbf{D}_t = \mathbf{D}_{x,t}(p)$ be a family of conormal symbols over a compact parameter space X continuously depending on the additional parameter $t \in [0,1]$. Suppose that the following condition holds.

Condition 2.1. For $t = 0$ and $t = 1$ the family $\mathbf{D}_t$ is invertible for all $p \in \mathbb{R}$.

Under this condition we define the notion of spectral flow of our family on the interval $[0,1]$ as an element of the K-group of the parameter space:

$$\operatorname{sf}\{\mathbf{D}_t\} = \operatorname{sf}_{t\in[0,1]}\{\mathbf{D}_t\} \in K(X).$$

Remark 2.2. If X is a singleton, then the invertibility condition is not necessary, since one can always guarantee the invertibility by moving by a small ε down from the real axis. In the general case this cannot be done, and the invertibility condition is essential.

Throughout this section we additionally assume that the family $\mathbf{D}_t$ smoothly depends on t, Condition 1.2 holds for all $t \in [0,1]$, and a positive number $h < h_0$ is chosen and fixed, so that the constructions of Subsection 1.3 in the Hilbert spaces (1.12) are well defined. Next, we equip X with some metric; for arbitrary $\varepsilon \geq 0$ and an arbitrary subset $K \subset X$, by $U_\varepsilon(K)$ we denote the ε-neighborhood of K in this metric; for a singleton $K = \{x\}$, we write $U_\varepsilon(x)$ instead of $U_\varepsilon(\{x\})$. Thus, $U_\varepsilon(x)$ is an open ball centered at x. We recall that $U_\varepsilon(A\cup B) = U_\varepsilon(A)\cup U_\varepsilon(B)$, but $U_\varepsilon(A\cap B) \subset U_\varepsilon(A)\cap U_\varepsilon(B)$.

2.1 Admissible weights and staircases

We define a *weight system* as a finite subset of $[-h,h]$ containing the point 0. Let $\Gamma \subset [-h,h]$ be a given weight system. We say that $\gamma \in \Gamma$ is an *admissible weight* at a point $x \in X$ for given $t \in [0,1]$ if the family $\mathbf{D}_{x,t}(p)$ is invertible everywhere on the weight line $\mathcal{L}_\gamma$ for these x and t. (For example, Condition 2.1 implies that the weight $\gamma = 0$ is admissible for $t = 0,1$ for any $x \in X$.) The set of all admissible weights at a point $x \in X$ for given $t \in [0,1]$ will be denoted by $\Gamma(x,t)$. We note that by definition $\Gamma(x,t) \subset \Gamma$; that is, one takes weights from a given weight system rather than arbitrary weights. Next, we set

$$\Gamma(K,t) = \bigcap_{x\in K}\Gamma(x,t), \qquad K\subset X, \quad t\in[0,1],$$
$$\Gamma(x,A) = \bigcap_{t\in A}\Gamma(x,t), \qquad x\in X, \quad A\subset[0,1],$$
$$\Gamma(K,A) = \bigcap_{x\in K, t\in A}\Gamma(x,t), \qquad K\subset X, \quad A\subset[0,1]$$

(for example, $\Gamma(x, A)$ is the set of weights admissible at x for all $t \in A$) and

$$\Gamma_\varepsilon(x,t) = \Gamma(U_\varepsilon(x),t), \qquad \Gamma_\varepsilon(K,t) = \Gamma(U_\varepsilon(K),t),$$
$$\Gamma_\varepsilon(x,A) = \Gamma(U_\varepsilon(x),A), \qquad \Gamma_\varepsilon(K,A) = \Gamma(U_\varepsilon(K),A).$$

An element $\gamma \in \Gamma_\varepsilon(x,t)$ (respectively, $\gamma \in \Gamma_\varepsilon(K,t)$) will be called an *$\varepsilon$-admissible weight* at a point x (respectively, on a set K) for given t. Furthermore, an element $\gamma \in \Gamma_\varepsilon(x,[t,\tau])$ (respectively, $\gamma \in \Gamma_\varepsilon(K,[t,\tau])$) will be called an *$\varepsilon$-admissible stair* at x (respectively, on K) on the interval $[t,\tau]$.

Let a weight system $\Gamma \subset [-h,h]$ and a finite partition $T = \{t_0, t_1, \dots, t_r\}$, $0 = t_0 < t_1 < \dots < t_r = 1$, of the interval $[0,1]$ (in what follows, simply a *partition*) be given.

Definition 2.3. Let $U \subset X$ be a subset. An *ε-admissible staircase* (for U) is a sequence $S = \{\gamma_1, \gamma_2 \dots, \gamma_r\}$ of ε-admissible stairs for U on the intervals $[t_0,t_1]$, $[t_1,t_2]$, …, $[t_{r-1},t_r]$ (see Fig. 2).

For technical reasons, we supplement each staircase by zero initial and final stairs $\gamma_0 = \gamma_{r+1} = 0$ (also shown in Fig. 2), so that for each point of the partition T there is an incoming (left) and an outgoing (right) stair.

We note that admissible staircases in general do not exist unless U is sufficiently small, T is sufficiently fine, and the weight system Γ is sufficiently ample.

The following lemma is a basis for the local construction of the spectral flow.

Lemma 2.4. *There exists an $\varepsilon > 0$, a finite open cover $\bigcup_j U_j \supset X$, a partition T of the interval $[0,1]$, and a weight system $\Gamma \subset [-h,h]$, such that for each element U_j of the cover there exists an ε-admissible staircase.*

Proof. Take some $x \in X$. Then there is a partition $T_x = \{t_0, t_1, \dots, t_r\}$, $r = r(x)$, of $[0,1]$ and a weight system $\gamma_1, \dots \gamma_r \in [-h,h]$ such that the family $\mathbf{D}_{x,t}(p)$ is invertible on the weight line $\mathcal{L}_{\gamma_j}$ for $t \in [t_{j-1},t_j]$, $j = 1,\dots,r$. By continuity, the invertibility holds in some open ball B_x centered at x. We set $\Gamma_x = \{\gamma_1, \dots \gamma_r\}$. Let $\widetilde{B}_x$ be the open ball centered at x with radius half that of B_x. Since X is compact, there exists a finite cover $X = \bigcup_j \widetilde{B}_{x_j}$. Now we can set $U_j = \widetilde{B}_{x_j}$, $T = \bigcup_j T_{x_j}$, and $\Gamma = \bigcup_j \Gamma_{x_j} \cup \{0\}$ and take the minimum radius of the balls $\widetilde{B}_{x_j}$ as ε. ∎

2.2 Auxiliary constructions

In the following subsection, we construct the bundles specifying the spectral flow using a set of ε-admissible staircases in elements of some open cover of the parameter space X. Here we only describe the relevant auxiliary constructions that depend only on the weight system Γ, the partition $T = \{t_0, t_1, \dots, t_r\}$, and the number $\varepsilon > 0$, which are assumed to be chosen in such a way that the conclusion of Lemma 2.4 holds for some open finite cover of X. (Note, however, that the cover itself does not occur in the constructions of this subsection.)

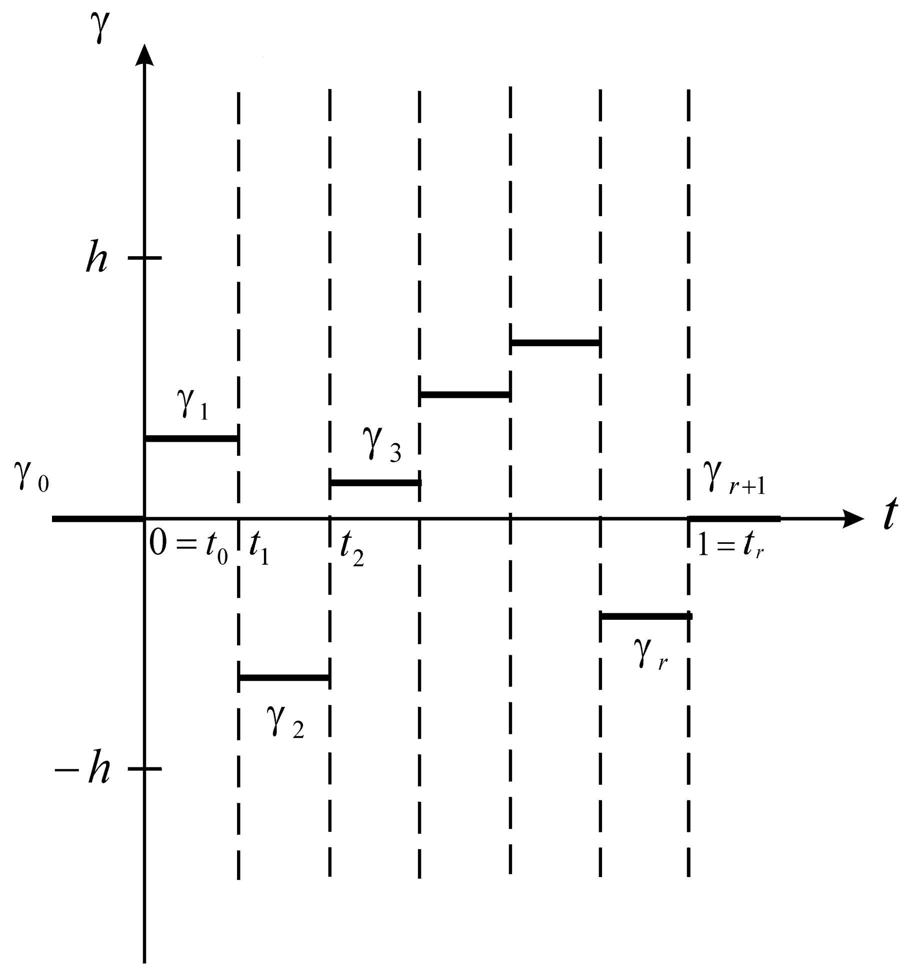

Figure 2. An admissible staircase

Morphisms relating the range of admissible projections. Let $U \subset X$ be an open subset, and let $\gamma_1, \gamma_2 \in \Gamma(U, t_j)$. According to Subsection 1.3, the subbundle $\mathcal{E}_{[\gamma_1,\gamma_2]}(U, t_j) \subset \widetilde{\mathfrak{H}}\big|_U$ is well defined as the range of the projection $\widehat{\mathbf{P}}_{[\gamma_1,\gamma_2]}(x, t_j)$, $x \in U$. The fiber of this subbundle at a point $x \in U$ will be denoted by $\mathcal{E}_{[\gamma_1,\gamma_2]}(x, t_j) \subset \mathfrak{H}_x$.

We shall introduce bundle isomorphisms

$$\widehat{\mathbf{U}}_j : \widetilde{\mathfrak{H}} \longrightarrow \widetilde{\mathfrak{H}}, \quad j = 1, \ldots, r, \tag{2.1}$$

such that the following properties hold for an arbitrary open subset $U \subset X$:

1. If $\gamma_1, \gamma_2 \in \Gamma_\varepsilon(U, [t_{j-1}, t_j])$ are ε-admissible stairs for U on $[t_{j-1}, t_j]$, then $\widehat{\mathbf{U}}_j$ restricts to an isomorphism of the bundles $\mathcal{E}_{[\gamma_1,\gamma_2]}(U, t_{j-1})$ and $\mathcal{E}_{[\gamma_1,\gamma_2]}(U, t_j)$:

$$\widehat{\mathbf{W}}_j([\gamma_1, \gamma_2], U) \equiv \widehat{\mathbf{U}}_j\Big|_{\mathcal{E}_{[\gamma_1,\gamma_2]}(U,t_{j-1})} : \mathcal{E}_{[\gamma_1,\gamma_2]}(U, t_{j-1}) \longrightarrow \mathcal{E}_{[\gamma_1,\gamma_2]}(U, t_j). \tag{2.2}$$

2. If $\gamma_1, \gamma_2, \gamma_3 \in \Gamma_\varepsilon(U, [t_{j-1}, t_j])$, $\gamma_1 < \gamma_2 < \gamma_3$, then

$$\widehat{\mathbf{W}}_j([\gamma_1, \gamma_3], U) = \widehat{\mathbf{W}}_j([\gamma_1, \gamma_2], U) \oplus \widehat{\mathbf{W}}_j([\gamma_2, \gamma_3], U). \tag{2.3}$$

In fact, property 2 holds automatically, since the operators $\widehat{\mathbf{W}}_j(\cdots)$ occurring in (2.3) are the restrictions of the same operator $\widehat{\mathbf{U}}_j$ to the subspaces

$$\mathcal{E}_{[\gamma_1,\gamma_3]}(U, t) = \mathcal{E}_{[\gamma_1,\gamma_2]}(U, t) \oplus \mathcal{E}_{[\gamma_2,\gamma_3]}(U, t).$$

Hence it suffices to guarantee property 1.

We construct the isomorphisms $\widehat{\mathbf{U}}_j$ by the method explained in Subsection 1.4. That is, we take some piecewise continuous point family $\widehat{\mathbf{A}}(x, t)$, $t \in [0, 1]$, $x \in X$, such that for any weights $\gamma_1, \gamma_2 \in \Gamma_\varepsilon(x, [t_{j-1}, t_j])$, $\gamma_1 < \gamma_2$, one has

$$\dot{\widehat{\mathbf{P}}}_{[\gamma_1,\gamma_2]}(x, t) = [\widehat{\mathbf{A}}(x, t), \widehat{\mathbf{P}}_{[\gamma_1,\gamma_2]}(x, t)]. \tag{2.4}$$

Next, we solve the Cauchy problem

$$\dot{\widehat{\mathbf{U}}} = \widehat{\mathbf{A}}\widehat{\mathbf{U}}, \quad t \in [0, 1], \qquad \widehat{\mathbf{U}}\big|_{t=t_{j-1}} = 1 \tag{2.5}$$

and set

$$\widehat{\mathbf{U}}_j = \widehat{\mathbf{U}}(t_j)\widehat{\mathbf{U}}(t_{j-1})^{-1}. \tag{2.6}$$

By Theorem 1.8, this isomorphism has the desired properties.

Lemma 2.5. *The set of piecewise continuous operator families* $\mathbf{A}(x, t)$ *with property* (2.4) *is nonempty.*

Proof. Let $X = \bigcup_s V_s$ be a finite cover of X by open balls of radius $\varepsilon/2$. Consider an arbitrary V_s; let $\Gamma(V_s, [t_{j-1}, t_j]) = \{\gamma_1, \dots, \gamma_l\}$, where the numbers $\gamma_1, \dots \gamma_l$ are arranged in ascending order. Consider the system of disjoint projections

$$\mathcal{P}_s = \bigl\{\widehat{\mathbf{P}}_{[\gamma_1,\gamma_2]}(x, t), \widehat{\mathbf{P}}_{[\gamma_2,\gamma_3]}(x, t), \dots, \widehat{\mathbf{P}}_{[\gamma_{l-1},\gamma_l]}(x, t)\bigr\}, \quad t \in [t_{j-1}, t_j]$$

and set

$$\widehat{\mathbf{A}}_s = A[\mathcal{P}_s]$$

(see (1.32)). Next, let $\{\varphi_s(x)\}$ be a partition of unity on X subordinate to the cover $\{V_s\}$. We set

$$\widehat{\mathbf{A}} = \sum_s \varphi_s \widehat{\mathbf{A}}_s, \quad t \in [t_{j-1}, t_j].$$

The operator this constructed has the desired property. Indeed, let

$$\gamma_1, \gamma_2 \in \Gamma_\varepsilon(U, [t_{j-1}, t_j]).$$

Then $\gamma_1, \gamma_2 \in \Gamma(V_s, [t_{j-1}, t_j])$ for any V_s that has a nonempty intersection with U. (Since V_s is a ball of radius $\varepsilon/2$). It follows that the projection $\widehat{\mathbf{P}}_{[\gamma_1,\gamma_2]}(x, t)$, $x \in U$, $t \in [t_{j-1}, t_j]$ is decomposable in each of the systems $\mathcal{P}_s$ for which $x \in V_s$. It remains to apply Lemma 1.12. □

Remark 2.6. The isomorphisms $\widehat{\mathbf{W}}_j(\cdots)$ are not uniquely determined; they depend on the choice of the family $\widehat{\mathbf{A}}(x,t)$. The set of such families with the desired properties is an affine space, and so any two systems of isomorphisms $\widehat{\mathbf{W}}_j(\cdots)$ given by the construction (2.2), (2.4), (2.5), (2.6) are homotopic in the class of such systems of isomorphisms.

Finite-dimensional approximations. Although the bundles $\mathcal{E}_{[\gamma_1,\gamma_2]}(U,t_j)$ are finite-dimensional, they form a system of subbundles (defined on certain open subsets) of the *infinite-dimensional* bundle $\widetilde{\mathfrak{H}}$. Now we pass (for each $j=0,\dots,r$) to an isomorphic system of subbundles of some *finite-dimensional* bundle F_j. Consider some $j\in\{0,\dots,r\}$.

Lemma 2.7. *For each $\delta>0$ there exists a bundle homomorphism*

$$f_j : \widetilde{\mathfrak{H}} \longrightarrow F_j, \tag{2.7}$$

where F_j is a finite-dimensional bundle over X, such that for any $x\in X$ and $\gamma_1,\gamma_2\in\Gamma_\delta(x,t_j)$, $\gamma_1<\gamma_2$, the restriction

$$f_{jx}\Big|_{\mathcal{E}_{[\gamma_1,\gamma_2]}(x,t_j)} : \mathcal{E}_{[\gamma_1,\gamma_2]}(x,t_j) \longrightarrow F_{jx} \tag{2.8}$$

is a monomorphism.

Remark 2.8. It follows from the lemma that for admissible weights $\gamma_1,\gamma_2\in\Gamma_\delta(U,t_j)$ the subspaces

$$E(\gamma_1,\gamma_2,x,t_j) = f_{jx}(\mathcal{E}_{[\gamma_1,\gamma_2]}(x,t_j)) \subset F_{jx}, \quad x\in U$$

form a subbundle $E(\gamma_1,\gamma_2,U,t_j)\subset F_j\big|_U$ isomorphic to $\mathcal{E}_{[\gamma_1,\gamma_2]}(U,t_j)$. Moreover, natural properties like

$$E(\gamma_1,\gamma_3,U,t_j) = E(\gamma_1,\gamma_2,U,t_j)\oplus E(\gamma_2,\gamma_3,U,t_j) \tag{2.9}$$

are valid.

For brevity, these subbundles (respectively, subspaces) will be denoted by[5]

$$\begin{aligned} [\gamma_1,\gamma_2]_j(U) &\stackrel{\text{def}}{=} E(\gamma_1,\gamma_2,U,t), \\ [\gamma_1,\gamma_2]_j(x) &\stackrel{\text{def}}{=} E(\gamma_1,\gamma_2,x,t), \end{aligned}$$

and the argument U or x (as well as the index j) will be omitted whenever it is clear from the context or irrelevant. Property (2.9) acquires then the mnemonically natural form

$$[\gamma_1,\gamma_2]\oplus[\gamma_2,\gamma_3]=[\gamma_1,\gamma_3], \quad \gamma_1\le\gamma_2\le\gamma_3;$$

likewise, for $\gamma_1\le\gamma_{2,3}\le\gamma_4$ we have

$$[\gamma_1,\gamma_3]\cap[\gamma_2,\gamma_4] = \begin{cases} [\gamma_2,\gamma_3], & \gamma_2\le\gamma_3, \\ \{0\}, & \gamma_2\ge\gamma_3. \end{cases}$$

[5]The brackets are typeset in bold to distinguish subbundles from the corresponding closed intervals.

Proof of Lemma 2.7. By the Kuiper–Jänich theorem (see [4]), the bundle $\widetilde{\mathfrak{H}}$ is trivial: $\widetilde{\mathfrak{H}} \simeq X \times \mathfrak{H}$. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis in $\mathfrak{H}$. We claim that f_j can be taken in the form of the orthogonal projection $\widehat{\mathbf{P}}_N$ on the subbundle $F_j \simeq \mathbb{C}^N$ spanned by the first N vectors of this basis for sufficiently large N. First, note that if for given x, γ_1, γ_2 the mapping (2.8), where $f_j = \widehat{\mathbf{P}}_N$, is a monomorphism for some $N = N_0$, then it is a monomorphism for $N > N_0$ as well, since $\widehat{\mathbf{P}}_{N_0} = \widehat{\mathbf{P}}_{N_0}\widehat{\mathbf{P}}_N$ in this case. Now we take arbitrary $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 < \gamma_2$, and set

$$X_\delta(\gamma_1, \gamma_2) = \overline{\bigl\{x \in X \mid \gamma_1, \gamma_2 \in \Gamma_\delta(x, t_j)\bigr\}}.$$

The bundle $\mathcal{E}_{[\gamma_1,\gamma_2]}(U, t_j)$ is defined in some neighborhood U of the set $X_\delta(\gamma_1, \gamma_2)$. Since $\widehat{\mathbf{P}} \xrightarrow{\text{strongly}} 1$ as $N \to \infty$, it follows that for each point $x_0 \in X_\delta(\gamma_1, \gamma_2)$ there exists an N such that the mapping (2.8), where $f_j = \widehat{\mathbf{P}}_N$, is a monomorphism for $x = x_0$. By continuity, this mapping is monomorphic also in a sufficiently small neighborhood of the point x_0. Considering a finite cover of the closed set $X_\delta(\gamma_1, \gamma_2)$ by such neighborhoods and choosing the maximum of the corresponding N, we provide the validity of the assertion of the lemma for the given pair $\gamma_1, \gamma_2 \in \Gamma$. Since there are finitely many such pairs, we can proceed to the maximum of N over all pairs, thus completing the proof. □

The isomorphisms $\widehat{\mathbf{W}}_j(\cdots)$ construct, via f_j, the corresponding isomorphisms

$$W_j(\gamma_1, \gamma_2) : [\gamma_1, \gamma_2]_{j-1} \longrightarrow [\gamma_1, \gamma_2]_j, \quad \gamma_1 \le \gamma_2, \tag{2.10}$$

which inherit the direct additivity from $\widehat{\mathbf{W}}_j$:

$$W_j(\gamma_1, \gamma_2) \oplus W_j(\gamma_2, \gamma_3) = W_j(\gamma_1, \gamma_3), \quad \gamma_1 \le \gamma_2 \le \gamma_3. \tag{2.11}$$

Remark 2.6 pertains to these morphisms as well.

We write $W_j(\gamma_1, \gamma_2, U)$ instead of $W_j(\gamma_1, \gamma_2)$ whenever we wish to indicate the domain U over which the subbundles related by this morphism are defined.

In the following, we assume that $\delta < \varepsilon$ (say, we choose $\delta = \varepsilon/2$).

Orthogonalization. Let some homomorphisms f_j with the properties indicated in Lemma 2.7 be given. We shall equip each of the bundles F_j with an inner product, which permits us to consider orthogonal complements of bundles. The subbundles $[\gamma_1, \gamma_2]_j$ will be orthogonal with respect to this inner product. More precisely, the following assertion holds.

Lemma 2.9. *In the bundle F_j there exists an inner product $(\,\cdot\,,\,\cdot\,)$ such that the subspaces $[a, b]_j(x)$ and $[c, d]_j(x)$ are orthogonal for any $x \in X$ and any two intervals $[a, b], [c, d]$ with disjoint interiors such that $a, b, c, d \in \Gamma_\varepsilon(x, t_j)$.*

Proof. Let $\langle \cdot\,, \cdot \rangle$ be some inner product on F_j. Consider a finite open cover $X = \bigcup_s V_s$ by balls of diameter $\le \varepsilon - \delta$. We take some ball V_s; let $\Gamma_\delta(V_s, t_j) = \{\gamma_1, \dots, \gamma_l\}$, where the numbers $\gamma_1, \dots \gamma_l$ are arranged in ascending order. We have the direct sum expansion

$$F_j\big|_{V_s} = [\gamma_1, \gamma_2] \oplus [\gamma_2, \gamma_2] \oplus \cdots \oplus [\gamma_{l-1}, \gamma_l] \oplus \widetilde{E}, \tag{2.12}$$

where $\widetilde{E}$ is the orthogonal complement to $[\gamma_1, \gamma_2] \oplus \cdots \oplus [\gamma_{l-1}, \gamma_l]$ with respect to $\langle \cdot\,, \cdot \rangle$. Now we define an inner product on $F_j\big|_{V_s}$ by setting

$$(u, v)_s = \sum_{j=1}^{l} \langle \pi_l u, \pi_l v \rangle, \tag{2.13}$$

where π_l is the projection on the lth component in the expansion (2.12). The expansion (2.12) is orthogonal with respect to the product (2.13). We set

$$(u, v) = \sum_s \varphi_s(u, v)_s \tag{2.14}$$

where $\{\varphi_s(x)\}$ is a partition of unity on X subordinate to the cover $\{V_s\}$. This inner product has the desired property, since if $\gamma \in \Gamma_\varepsilon(x, t_j)$, then $\gamma \in \Gamma_\delta(V_s, t_j)$ for each ball V_s containing the point x. □

Remark 2.10. It follows from the construction given in the proof with regard for the convexity of the set of inner products satisfying the conditions of the lemma that for a continuous deformation of the family $\mathbf{D}_t$ and/or finite-dimensional approximations f_j, the inner products $(\cdot\,, \cdot)$ can be chosen continuously depending on the deformation parameters.

In this subsection, for given weight system Γ, partition T, and number $\varepsilon > 0$, we have defined a construction (nonunique) of morphisms, finite-dimensional approximations, and inner products satisfying the above-mentioned conditions. For brevity, such morphisms, finite-dimensional approximations, and inner products will be referred to as *suitable* (for given $\Gamma, T\varepsilon$) in what follows.

2.3 Bundles determining the spectral flow

The spectral flow will be defined in the next subsection as an element of the group $K(X)$, i.e., the difference $[E] - [F]$ of equivalence classes of two bundles E and F over X. In this subsection we construct E and F, assuming that the above-mentioned objects are fixed (more details will be given below), and in the next subsection we show that the element $[E] - [F]$ is well defined, i.e., independent of the choice of these objects.

Thus, we assume that the following objects are chosen and fixed:

1. the partition of unity $1 = \varphi_1(r) + \varphi_2(r)$ occurring in the definition of the projections $\widehat{\mathbf{P}}_A$ (see Subsection 1.3);

2. a partition $T = \{0 = t_0 < t_1 < \ldots < t_r = 1\}$ of the interval $[0, 1]$;

3. a finite weight system $\Gamma \subset [-h, h]$;

4. a number $\varepsilon > 0$;

5. a cover $\bigcup_s U_s \supset X$ for which the conclusion of Lemma 2.4 holds (in the following, such covers are said to be *admissible*);

6. suitable finite-dimensional approximations f_j;

7. suitable inner products $(\,\cdot\,,\,\cdot\,)$ on F_j;

8. suitable morphisms $\widehat{\mathbf{W}}_j(\cdots)$ and the corresponding morphisms $W_j(\cdots)$.

The bundle F. We define F globally on X by the formula

$$F = \bigoplus_{j=0}^{r} F_j. \tag{2.15}$$

The bundle E: a local description. The description of E is more complicated. In this item, we give a local description, that is, describe the restrictions $E_s \equiv E|_{U_s}$ of E to the charts U_s of the cover, and in the next item we describe the transition functions. Let us take some s. Suppose that in U_s we have chosen some ε-admissible staircase $S_s = \{\gamma_{s1}, \ldots, \gamma_{sr}\}$. (The independence of the construction on the choice of staircases will be shown in the next subsection.) It proves convenient to describe E_s not only over all U_s, but also over an arbitrary open subset $U \subseteq U_s$. The smaller the subset, the larger the freedom in the description; of course, all descriptions will be isomorphic with explicitly given isomorphisms.

We define $E_s|_U$ as the direct sum of $3(r+1)$ bundles

$$E_s\,|_U = \bigoplus_{j=0}^{r} E_{sj}^{(-)} \oplus \bigoplus_{j=0}^{r} E_{sj}^{(0)} \oplus \bigoplus_{j=0}^{r} E_{sj}^{(+)} \equiv \bigoplus_{j=0}^{r} \bigl(E_{sj}^{(-)} \oplus E_j^{(0)} \oplus E_{sj}^{(+)}\bigr), \tag{2.16}$$

so that to each point $t_j \in T$ there correspond three summands with superscripts $(-), (0)$, and $(+)$. Let us describe these summands. To simplify the notation, we omit the subscript s, i.e., write $E_j^{(\pm)}, E_j^{(0)}$ instead of $E_{sj}^{(\pm)}, E_{sj}^{(0)}$, γ_j instead of γ_{sj}, etc. At the point t_j there is an incoming stair γ_j and an outgoing stair γ_{j+1} of the staircase S. Furthermore,

$$\gamma_j, \gamma_{j+1} \in \Gamma_\varepsilon(U_s, t_j) \subset \Gamma_\varepsilon(U, t_j).$$

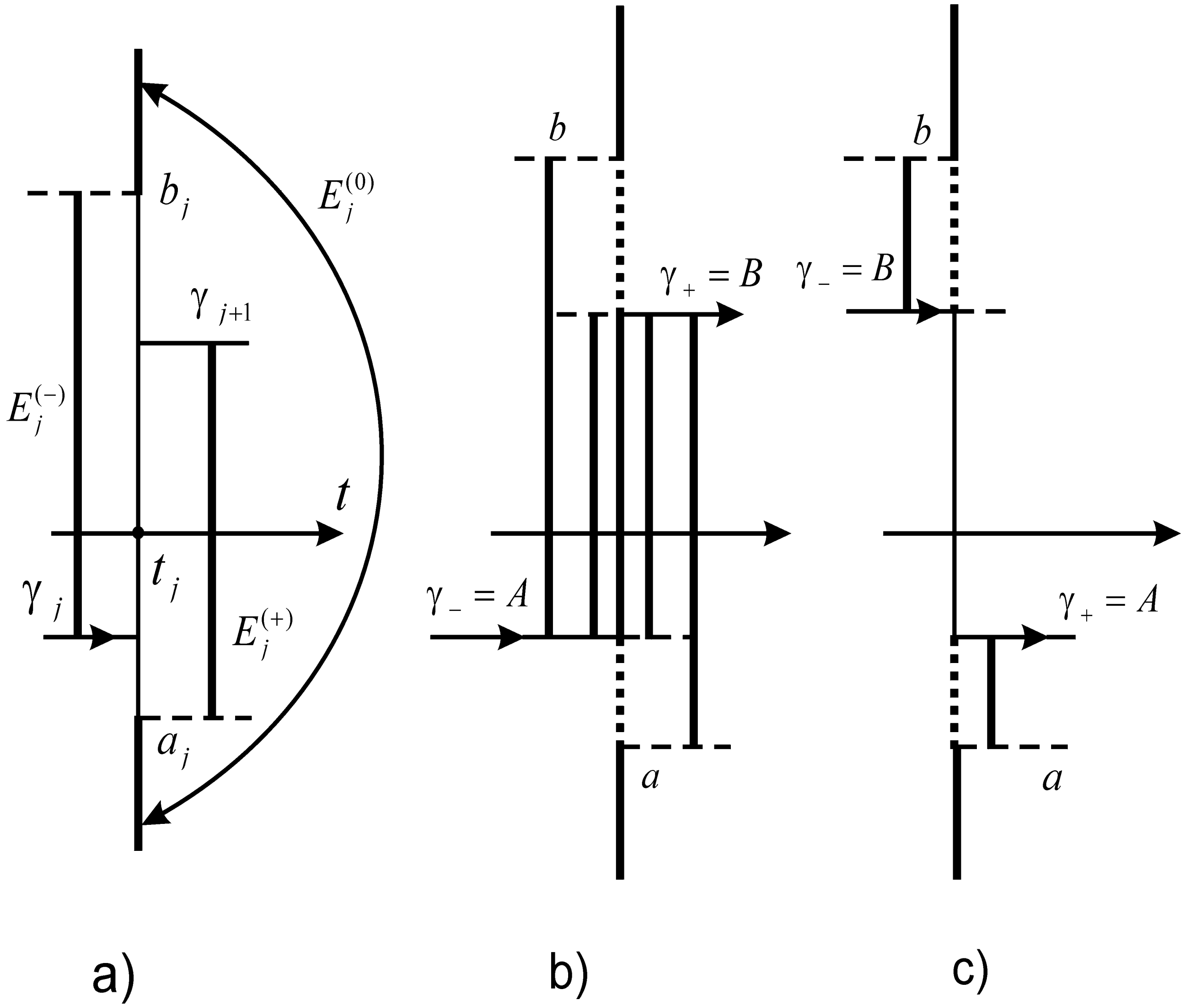

Figure 3. The construction of the bundle E for an admissible staircase

We take arbitrary admissible weights $a_j, b_j \in \Gamma_\varepsilon(U, t_j)$ such that

$$a_j \leq \gamma_j, \gamma_{j+1} \leq b_j \tag{2.17}$$

(say, one can take $a_j = \min\{\gamma_j, \gamma_{j+1}\}$, $b_j = \max\{\gamma_j, \gamma_{j+1}\}$) and set

$$E_j^{(-)} = [\gamma_j, b_j]_j(U), \quad E_j^{(0)} = [a_j, b_j]_j^\perp(U), \quad E_j^{(+)} = [a_j, \gamma_{j+1}]_j(U) \tag{2.18}$$

(see Fig. 3a, where the construction is visualized; the vertical segments stand for the bundle given by the ranges of the corresponding projections). Here $E^\perp$ is the orthogonal complement of E.

Our construction contains additional ambiguity in the choice of the weights $a_j, b_j \in \Gamma_\varepsilon(U, t_j)$ satisfying condition (2.17). Let us show that the bundle E_s is independent of this choice up to natural isomorphisms. These isomorphisms are in fact direct sums of isomorphisms corresponding to separate points of T, and so we consider the components corresponding to a given $j \in \{0, \dots, r\}$.

Lemma 2.11. *Suppose that two pairs a_j, b_j and $\widetilde{a}_j, \widetilde{b}_j$ of admissible weights $\Gamma_\varepsilon(U, t_j)$ satisfy condition* (2.17). *Then there is a natural isomorphism*

$$\begin{aligned} E_j^{(-)} \oplus E_j^{(0)} \oplus E_j^{(+)} &\equiv [\gamma_j, b_j] \oplus [a_j, b_j]^\perp \oplus [a_j, \gamma_{j+1}] \\ &\longrightarrow \widetilde{E}_j^{(-)} \oplus \widetilde{E}_j^{(0)} \oplus \widetilde{E}_j^{(+)} \equiv [\gamma_j, \widetilde{b}_j] \oplus [\widetilde{a}_j, \widetilde{b}_j]^\perp \oplus [\widetilde{a}_j, \gamma_{j+1}]. \end{aligned} \tag{2.19}$$

Moreover, if $\widetilde{\widetilde{a}}_j, \widetilde{\widetilde{b}}_j$ is a third such pair, then the triangle

$$\begin{array}{ccc} E_j^{(-)} \oplus E_j^{(0)} \oplus E_j^{(+)} & \longrightarrow & \widetilde{E}_j^{(-)} \oplus \widetilde{E}_j^{(0)} \oplus \widetilde{E}_j^{(+)} \\ \Big\| & & \Big\downarrow \\ E_j^{(-)} \oplus E_j^{(0)} \oplus E_j^{(+)} & \longleftarrow & \widetilde{\widetilde{E}}_j^{(-)} \oplus \widetilde{\widetilde{E}}_j^{(0)} \oplus \widetilde{\widetilde{E}}_j^{(+)} \end{array} \tag{2.20}$$

of the corresponding natural isomorphisms commutes.

Proof. We write out this natural isomorphism explicitly. To reduce the computations, we omit the subscript j and the arguments U and t_j and write $\gamma_- = \gamma_j$ and $\gamma_+ = \gamma_{j+1}$. Let

$$A = \min\{\gamma_-, \gamma_+\}, \quad B = \max\{\gamma_-, \gamma_+\}$$

(See Fig. 3b, c, where two possible case of arrangements of the incoming and outgoing stairs are shown.) Then

$$a \le A \le \gamma_-, \gamma_+ \le B \le b,$$

and regardless of a and b we have the isomorphism

$$\psi_{ab} : [\gamma_-, b] \oplus [a, b]^\perp \oplus [a, \gamma_+] \longrightarrow [\gamma_-, B] \oplus [A, B]^\perp \oplus [A, \gamma_+] \tag{2.21}$$

obtained as the composition (from top to bottom) of the direct sum decompositions

$$\begin{array}{ccccc} [\gamma_-, b] & \oplus & [a, b]^\perp & \oplus & [a, \gamma_+] \\ \Big\| & & \Big\| & & \Big\| \\ \overbrace{[\gamma_-, B] \oplus [B, b]} & \oplus & [a, b]^\perp & \oplus & \overbrace{[a, A] \oplus [A, \gamma_+]} \\ & & \underbrace{[B, b] \oplus [a, b]^\perp \oplus [a, A]} & & \\ \Big\| & & \Big\| & & \Big\| \\ [\gamma_-, B] & \oplus & [A, B]^\perp & \oplus & [A, \gamma_+] \end{array}$$

(the validity of these decompositions follows from the fact that the inner product satisfies the condition stated in Lemma 2.9). Now the isomorphism (2.19) can be defined as $\psi^{-1}_{\widetilde{ab}}\psi_{ab}$, and the commutativity of the diagram (2.20) becomes the trivial identity

$$\psi^{-1}_{\widetilde{\widetilde{ab}}}\psi_{\widetilde{ab}} \circ \psi^{-1}_{\widetilde{ab}}\psi_{ab} = \psi^{-1}_{\widetilde{\widetilde{ab}}}\psi_{ab}.$$

□

The bundle E: transition functions and the cocycle condition. In the preceding item we have described local representatives E_s of the bundle E in the charts U_s on X. Here we describe the transition functions. Let U_s and U_l be two charts with a nonempty intersection $U_{sl} = U_s \cap U_l$. We set

$$E_{sl} = E_s\big|_{U_{sl}}, \quad E_{ls} = E_l\big|_{U_{sl}},$$

so that, according to the construction given in the preceding item,

$$E_{sl} = \bigoplus_{j=0}^{r}\bigl(E_{sj}^{(-)} \oplus E_{sj}^{(0)} \oplus E_{sj}^{(+)}\bigr), \quad E_{ls} = \bigoplus_{j=0}^{r}\bigl(E_{lj}^{(-)} \oplus E_{lj}^{(0)} \oplus E_{lj}^{(+)}\bigr), \tag{2.22}$$

where the bundles on the right-hand side are restricted to U_{sl}. For these restrictions, in the definitions of the bundles for each $j = 0, 1, \ldots, r$, we can use the same weights $a_j, b_j \in \Gamma_\varepsilon(U_{sl}, t_j)$ both for E_{sl} and E_{ls}, so that

$$a_j \le \gamma_{sj}, \gamma_{s,j+1}, \gamma_{lj}, \gamma_{l,j+1} \le b_j. \tag{2.23}$$

(Here γ_{sj} and γ_{lj} are stairs of the staircases corresponding to the jth and lth chart, respectively). Let us define the transition function

$$g_{sl} : E_{ls} \longrightarrow E_{sl}. \tag{2.24}$$

For the above-mentioned choice of a_j and b_j, this homomorphism is given by the direct sum

$$g_{sl} = \mathrm{id} \oplus \bigoplus_{j=1}^{r} g_{sl}^{[j]} \tag{2.25}$$

of the homomorphisms

$$g_{sl}^{[j]} : E_{l,j-1}^{(+)} \oplus E_{lj}^{(-)} \longrightarrow E_{s,j-1}^{(+)} \oplus E_{sj}^{(-)} \tag{2.26}$$

corresponding to the replacement of the jth stair, $j = 1, \ldots, r$, and the identity homomorphism

$$\mathrm{id} : E_{l0}^{(-)} \oplus \bigoplus_{j=0}^{r} E_{lj}^{(0)} \oplus E_{lr}^{(+)} \longrightarrow E_{s0}^{(-)} \oplus \bigoplus_{j=0}^{r} E_{sj}^{(0)} \oplus E_{sr}^{(+)}$$

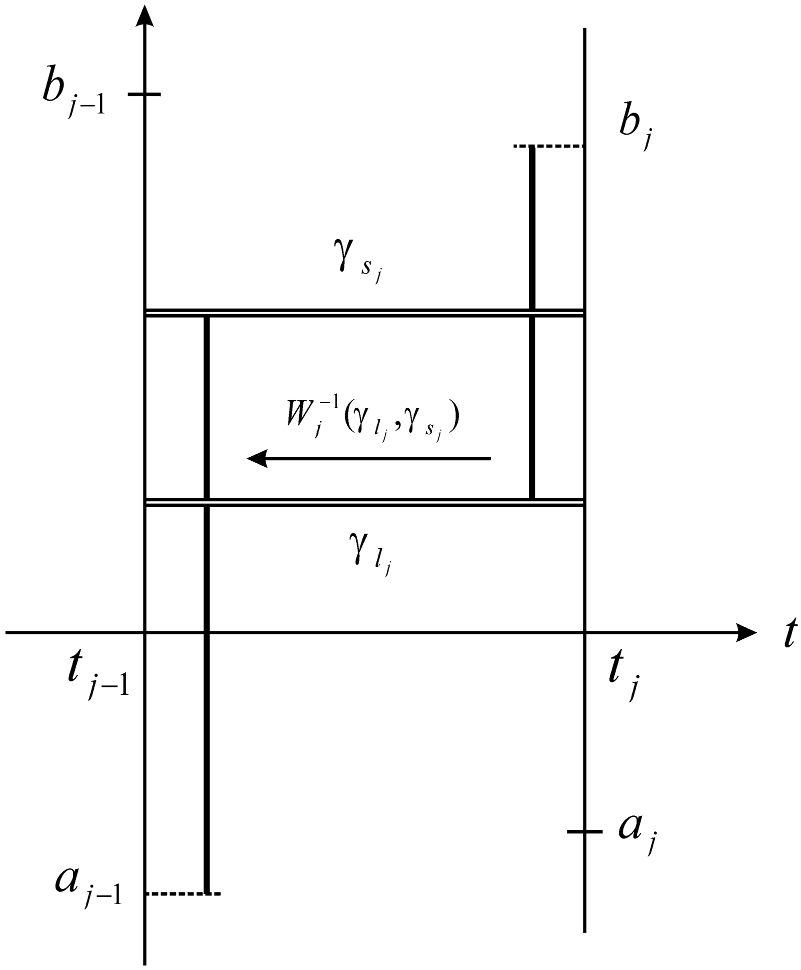

Figure 4. The stair change homomorphism

on the components lacking in (2.26). It remains to describe the homomorphism $g_{sl}^{[j]}$ (see Fig. 4).

By the definition of the spaces occurring in (2.26), we have

$$g_{sl}^{[j]} : [a_{j-1}, \gamma_{lj}]_{j-1} \oplus [\gamma_{lj}, b_j]_j \longrightarrow [a_{j-1}, \gamma_{sj}]_{j-1} \oplus [\gamma_{sj}, b_j]_j. \tag{2.27}$$

Let $\gamma_{sj} \geq \gamma_{lj}$. (In the opposite case, the formulas are similar.) Then the mapping (2.27) is obtained as the composition (from top to bottom) of the mappings

$$
\begin{array}{ccccc}
[a_{j-1},\gamma_{lj}]_{j-1} & \oplus & & [\gamma_{lj},b_j]_j & \\
\Big\| & & & \Big\| & \\
[a_{j-1},\gamma_{lj}]_{j-1} & \oplus & \overbrace{[\gamma_{lj},\gamma_{sj}]_j} & \overbrace{\oplus} & \overbrace{[\gamma_{sj},b_j]_j} \\
\Big\| & & \Big\downarrow \left(W_j(\gamma_{lj},\gamma_{sj})\right)^{-1} & & \Big\| \\
\underbrace{[a_{j-1},\gamma_{lj}]_{j-1}} & \underbrace{\oplus} & \underbrace{[\gamma_{lj},\gamma_{sj}]_{j-1}} & \oplus & [\gamma_{sj},b_j]_j \\
& \Big\| & & & \Big\| \\
& [a_{j-1},\gamma_{sj}]_{j-1} & & \oplus & [\gamma_{sj},b_j]_j .
\end{array} \tag{2.28}
$$

One can readily see (we omit the corresponding awkward commutative diagram) that the definition of the transition function (2.24) is independent of the choice of the constants a_j, b_j satisfying conditions (2.23).

Lemma 2.12. *The transition functions thus defined satisfy the cocycle condition.*

Proof. Let the intersection $U_{slm} = U_s \cap U_l \cap U_m$ be nonempty. On this intersection, in the description of our bundles we can use weights

$$a_j, b_j \in \Gamma_\varepsilon(U_{slm}, t_j) \supset \Gamma_\varepsilon(U_s, t_j), \Gamma_\varepsilon(U_l, t_j), \Gamma_\varepsilon(U_m, t_j)$$

such that

$$a_j \le \gamma_{sj}, \gamma_{s,j+1}, \gamma_{lj}, \gamma_{l,j+1}, \gamma_{mj}, \gamma_{m,j+1} \le b_j. \tag{2.29}$$

With regard for (2.25), it suffices to verify the cocycle condition separately for each of the components corresponding to changes of stairs; for these components, however, the desired property follows from (2.28) and the direct additivity of the morphisms $W_j(\cdots)$. For example, if $\gamma_{sj} \le \gamma_{lj} \le \gamma_{mj}$, then the cocycle condition follows from the relation

$$W_j(\gamma_{sj}, \gamma_{mj}) = W_j(\gamma_{sj}, \gamma_{lj}) \oplus W_j(\gamma_{lj}, \gamma_{mj}).$$

We leave details to the reader. □

Corollary 2.13. *We have simultaneously shown that the bundle E is independent of the choice of ε-admissible staircases S_s in the charts U_s.*

Indeed, one can assume that each chart in the cover is repeated several times and each copy is equipped by an ε-admissible staircases, so that all ε-admissible staircases are exhausted. Then the passage from one staircase to another is just a special case of the passage from one chart to another.

2.4 The spectral flow as an element of $K(X)$

The definition of spectral flow. In the preceding subsection, for a family $\mathbf{D}_t$, $t \in [0,1]$, of conormal symbols with compact parameter space X satisfying Conditions 2.1 and 1.2, we have constructed the bundles E and F using some arbitrary additional data.

Definition 2.14. The *spectral flow* of the family $\mathbf{D}_t$ is the element

$$\operatorname{sf}_{t\in[0,1]} \mathbf{D}_t \equiv \operatorname{sf} \mathbf{D}_t = [E] - [F] \in K(X) \tag{2.30}$$

of the K-group of the parameter space X.

Theorem 2.15. *The element* 2.14 *is well defined, that is, independent of the ambiguity in the construction of the bundles E and F.*

Proof. As was mentioned in Subsection 2.3, the construction of E and F depends on the (eight) objects listed there. Accordingly, we split the proof into several parts.

a) $[E]-[F]$ **is independent of the choice of objects 8, 1.** According to Remark 2.6, any suitable morphisms $W_j(\cdots)$ can be connected by a homotopy of suitable morphisms. In the construction of E, this results in a homotopy of the transition functions, so that E is replaced by an isomorphic bundle. The change of the morphisms $W_j(\cdots)$ does not affect other elements of the construction; in particular, F remains unchanged. Thus, we have shown that $[E]-[F]$ is independent of the choice of objects 8 for given projections $\widehat{\mathbf{P}}_A$. Now if we vary object 1, i.e., the partition of unity occurring in the definition of the projections, then the bundles defined by the projections remain the same, while the set of suitable morphisms varies. However, one partition of unity $1 = \varphi_1(r) + \varphi_2(r)$ can be connected with another by a linear homotopy, which provides homotopies of projections $\widehat{\mathbf{P}}_A$ in Eq. (2.4), so that one can obtain the corresponding homotopy of the families $\widehat{\mathbf{A}}(x,t)$ by using the construction of Lemma 2.5 with a given partition of unity $\{\varphi_s(x)\}$. Thus, we again arrive at a homotopy of suitable morphisms $W_j(\cdots)$ (where the notion suitability now depends on the homotopy parameter), and the preceding argument shows that $[E]-[F]$ is independent of the choice of the object 1.

b) $[E]-[F]$ **is independent of the choice of the objects 7 and 6.** First, we deal with inner products. Suitable inner products form a convex cone; in particular, any two of them can be connected by a linear homotopy. Such homotopies generate the corresponding homotopies of E (the definition of the local representatives E_s contains orthogonal complements with respect to a given inner product), so that the bundle E is taken to an isomorphic bundle and F remains unchanged. Hence the desired assertion follows. Now let us prove that $[E]-[F]$ is independent of the choice of suitable finite-dimensional approximations

$$f_j : \widetilde{\mathfrak{H}} \longrightarrow F_j. \tag{2.31}$$

First, we consider the case in which the bundles F_j remain unchanged and there is a continuous family $f_j(\tau) : \widetilde{\mathfrak{H}} \longrightarrow F_j$, $\tau \in [0,1]$, of suitable finite-dimensional approximations. For this family we can construct a continuous family of suitable inner products (see Remark 2.10), so that as the result we obtain a continuous homotopy of E, while F remains unchanged. Next, consider an embedding $F_j \subset \widetilde{F}_j$ and the associated passage from the approximation (2.31) to the approximation

$$\widetilde{f}_j : \widetilde{\mathfrak{H}} \longrightarrow \widetilde{F}_j, \tag{2.32}$$

where $\widetilde{f}_j$ is the composition of f_j with the embedding. Then $\widetilde{F}_j$ can be equipped with a suitable inner product whose restriction to F_j coincides with the original one; the bundles E and F will be accordingly replaced by

$$\widetilde{E} = E \oplus \bigoplus_{j=0}^{r} F_j^{\perp} \text{ and } \widetilde{F} = \bigoplus_{j=0}^{r} \widetilde{F}_j \equiv F \oplus \bigoplus_{j=0}^{r} F_j^{\perp},$$

where $F_j^{\perp}$ is the orthogonal complement to F_j in $\widetilde{F}_j$. Obviously, $[\widetilde{E}] - [\widetilde{F}] = [E] - [F]$.

Finally, from the approximation (2.32) one can pass to any other approximation

$$g_j : \widetilde{\mathfrak{H}} \longrightarrow G_j \tag{2.33}$$

by embedding F_j and G_j into $F_j \oplus G_j$ and then by joining $f_j \oplus 0$ with $0 \oplus g_j$ by the linear homotopy

$$f_j(\tau) = \begin{pmatrix} (1-\tau) f_j & 0 \\ 0 & \tau g_j \end{pmatrix}, \quad \tau \in [0,1]. \tag{2.34}$$

The suitability of $f_j(\tau)$, that is, the fact that it is monomorphic on certain subbundles for each τ, follows from the suitability of f_j and g_j.

c) $[E] - [F]$ is independent of the choice of objects 2–5. Now let us show that our definition is independent of the choice of the partition, the weight system, the number $\varepsilon > 0$, and the admissible cover $\{U_s\}$. Standard reasoning implies that it suffices to show that $[E] - [F]$ remains unchanged if we refine the cover, add a new point to the partition or the weight system, or diminish ε. The only nontrivial case is adding a new point to the partition T. In the other cases, finite-dimensional approximations, inner products, and morphisms $W_j(\cdots)$ suitable for the "new" objects are also suitable for "old" ones, and we obtain the same class $[E] - [F]$ as before by using "old" admissible staircases. (Here we have used the already known independence of $[E] - [F]$ on the choice of objects 1,6–8.)

Now let us add a new point to the partition T between t_j and t_{j+1}. We denote the new point by $t_{j+1/2}$ to avoid renumbering. Obviously, $\Gamma_\varepsilon(U, [t_j, t_{j+1}]) = \Gamma_\varepsilon(U, [t_j, t_{j+1/2}]) \cap \Gamma_\varepsilon(U, [t_{j+1/2}, t_{j+1}])$ for any subset $U \subset X$. We denote by E and F the bundles constructed for T and by $\widetilde{E}$ and $\widetilde{F}$ the bundles constructed for the partition $\widetilde{T} = T \cup \{t_{j+1/2}\}$. (We shall use tilde to indicate objects corresponding to the latter pair of bundles.) We assume that all other objects (except for morphisms on the interval $[t_j, t_{j+1}]$ and objects pertaining to the point $t_{j+1/2}$) are the same for both pairs. In particular, $F_i = \widetilde{F}_i$, $i = 0, 1, \ldots, r$, $i \neq j + 1/2$. Obviously, in the construction of $\widetilde{E}$ it suffices to use only ε-admissible staircases such that the stairs on the intervals $[t_j, t_{j+1/2}]$ and $[t_{j+1/2}, t_{j+1}]$ are the same (and hence lie in $\Gamma_\varepsilon(U_s, [t_j, t_{j+1}])$ for the corresponding element U_s of the cover). Next, using the construction (2.2)–(2.6) for the morphisms, we can assume that

$$W_{j-1}(\gamma_1, \gamma_2, U) = \widetilde{W}_{j-1/2}(\gamma_1, \gamma_2, U) \circ \widetilde{W}_{j-1}(\gamma_1, \gamma_2, U) \tag{2.35}$$

for any pair of admissible stairs $\gamma_1, \gamma_2 \in \Gamma_\varepsilon(U, [t_j, t_{j+1}])$ and any open subset $U \subset X$.

For $\widetilde{F}$ we thus obtain

$$\widetilde{F} = F \oplus \widetilde{F}_{j+1/2}. \tag{2.36}$$

The bundle $\widetilde{E}$ is thus locally (i.e., in the charts U_s) of the form (see Fig. 5)

$$\widetilde{E}_s = E_s \oplus [\gamma_{js}, b] \oplus [a, b]^\perp \oplus [a, \gamma_{js}], \tag{2.37}$$

where $a = a_{j+1/2}$ and $b = b_{j+1/2}$ are chosen in U_s arbitrarily under the same conditions as the other a_j, b_j, and we have omitted the subscript $j + 1/2$ in the notation of spaces. Since

$$[\gamma_{js}, b] \oplus [a, b]^\perp \oplus [a, \gamma_{js}] = \widetilde{F}_{j+1/2}\big|_{U_s},$$

we have

$$\widetilde{E}_s = E_s \oplus \widetilde{F}_{j+1/2}\big|_{U_s}. \tag{2.38}$$

We claim that globally one also has

$$\widetilde{E} = E \oplus \widetilde{F}_{j+1/2}, \tag{2.39}$$

so that $[E] - [F] = [\widetilde{E}] - [\widetilde{F}]$. Thus, to complete the proof of Theorem 2.15, it remains to prove the following lemma.

Lemma 2.16. *Relation* (2.39) *holds.*

Proof of the lemma. Let us analyze the transition functions of the bundles on the right- and left-hand sides of the identity to be proved. The transition functions of E and $\widetilde{E}$ are direct sums of transition functions on pairs of components (see (2.25) and (2.26)), and moreover, they coincide for the corresponding pairs in E and $\widetilde{E}$ with the only exception for the pairs that do not have counterparts, namely, the pairs

$$\widetilde{E}_j^{(+)} \oplus \widetilde{E}_{j+1/2}^{(-)} \equiv [a_j, \gamma_{sj}]_j \oplus [\gamma_{sj}, b]_{j+1/2}$$
$$\widetilde{E}_{j+1/2}^{(+)} \oplus \widetilde{E}_{j+1}^{(-)} \equiv [a, \gamma_{sj}]_{j+1/2} \oplus [\gamma_{sj}, b_{j+1}]_{j+1}$$
$$E_j^{(+)} \oplus E_{j+1}^{(-)} \equiv [a_j, \gamma_{sj}]_j \oplus [\gamma_{sj}, b_{j+1}]_{j+1}$$

corresponding to the intervals $[t_j, t_{j+1/2}]$, $[t_{j+1/2}, t_{j+1}]$, and $[t_j, t_{j+1}]$ of the partition. Consider the intersection $U_{sl} = U_s \cap U_l$. Let us write out the critical components[6] of the

[6] The components where there may be differences.

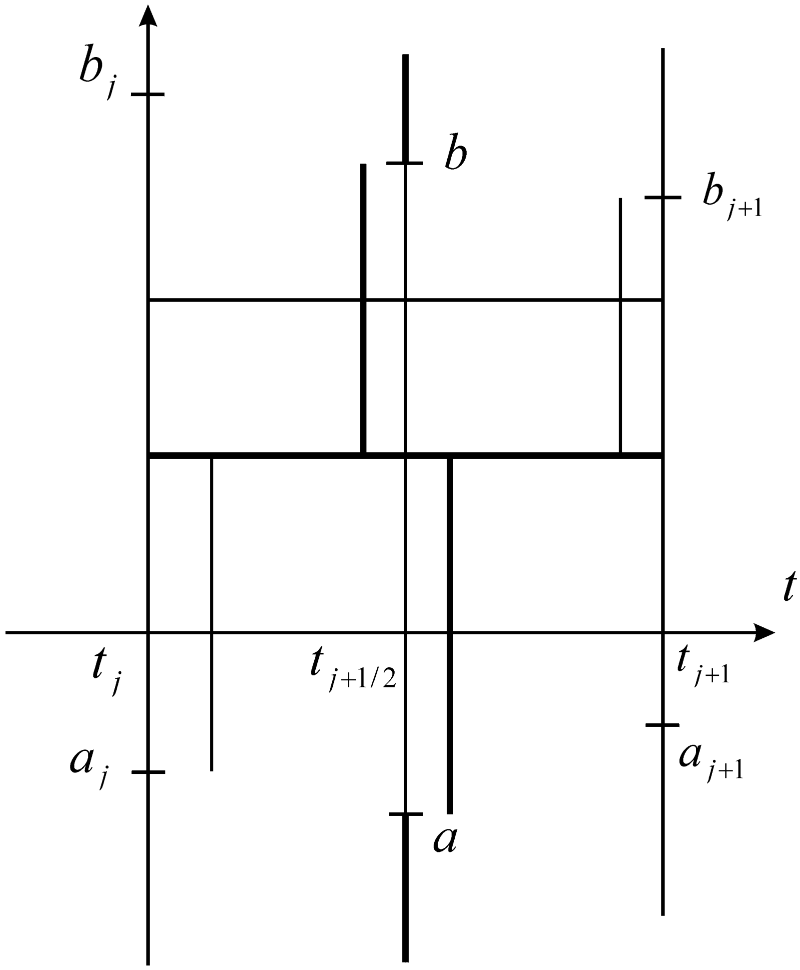

Figure 5. Adding the point $t_{j+1/2}$ to the partition

transition functions $\widetilde{g}_{sl}$ of the bundle $\widetilde{E}$ and $g_{sl} \oplus \mathrm{id}$ of the bundle $E \oplus \widetilde{F}_{j+1/2}$. All critical components of the transition function $g_{sl} \oplus \mathrm{id}$ are necessarily contained in the diagram (where we write $\gamma = \gamma_{sj}$ and $\widetilde{\gamma} = \gamma_{lj}$, assume without loss of generality that $\gamma < \widetilde{\gamma}$ and omit the subscript $j+1/2$ on spaces, that is, write $[v,w]_{j+1/2} = [v,w]$)

$$\begin{array}{ccccc}
[a_j,\widetilde{\gamma}]_j & \oplus & [\widetilde{\gamma}, b_{j+1}]_{j+1} & \oplus & \widetilde{F}_{j+1/2} \\
\| & & \| & & \| \\
\overbrace{[a_j,\gamma]_j \quad \oplus \quad [\gamma,\widetilde{\gamma}]_j} & \oplus & [\widetilde{\gamma}, b_{j+1}]_{j+1} & \oplus & \overbrace{[\gamma,\widetilde{\gamma}] \oplus [\widetilde{\gamma},b] \oplus [a,\gamma] \oplus [a,b]^{\perp}} \\
\| \qquad\qquad \big\downarrow W_{j-1}(\gamma,\widetilde{\gamma}) & & \| & & \| \quad\; \| \quad\; \| \quad\; \| \\
[a_j,\gamma]_j \quad \oplus \quad [\gamma,\widetilde{\gamma}]_{j+1} & \oplus & [\widetilde{\gamma}, b_{j+1}]_{j+1} & \oplus & [\gamma,\widetilde{\gamma}] \oplus [\widetilde{\gamma},b] \oplus [a,\gamma] \oplus [a,b]^{\perp} \\
\| & & \| & & \| \\
[a_j,\gamma]_j & \oplus & [\gamma, b_{j+1}]_{j+1} & \oplus & \widetilde{F}_{j+1/2}
\end{array} \tag{2.40}$$

(in the following, we shall see that this diagram still contains some noncritical element). The critical components of the transition function $\widetilde{g}_{sl}$ are contained in the diagram

$$\begin{array}{ccccccccc}
[a_j,\widetilde{\gamma}]_j & \oplus & [\widetilde{\gamma},b] & \oplus & [a,b]^{\perp} & \oplus & [a,\widetilde{\gamma}] & \oplus & [\widetilde{\gamma}, b_{j+1}]_{j+1} \\
\| & & \| & & \| & & \| & & \| \\
\overbrace{[a_j,\gamma]_j \quad \oplus \quad [\gamma,\widetilde{\gamma}]_j} & \oplus & [\widetilde{\gamma},b] & \oplus & [a,b]^{\perp} & \oplus & \overbrace{[a,\gamma] \quad \oplus \quad [\gamma,\widetilde{\gamma}]} & \oplus & [\widetilde{\gamma}, b_{j+1}]_{j+1} \\
\| \qquad \big\downarrow \widetilde{W}_{j-1}(\gamma,\widetilde{\gamma}) & & \| & & \| & & \| \qquad \big\downarrow \widetilde{W}_{j-1/2}(\gamma,\widetilde{\gamma}) & & \| \\
[a_j,\gamma]_j \quad \oplus \quad \underbrace{[\gamma,\widetilde{\gamma}]_j} & \oplus & \underbrace{[\widetilde{\gamma},b]} & \oplus & [a,b]^{\perp} & \oplus & [a,\gamma] \quad \oplus \quad \underbrace{[\gamma,\widetilde{\gamma}]} & \oplus & \underbrace{[\widetilde{\gamma}, b_{j+1}]_{j+1}} \\
\| & & \| & & \| & & \| & & \| \\
[a_j,\gamma]_j & \oplus & [\gamma,b] & \oplus & [a,b]^{\perp} & \oplus & [a,\gamma] & \oplus & [\gamma, b_{j+1}]_{j+1}
\end{array} \tag{2.41}$$

Dropping coinciding morphisms in (2.40) and (2.41), we see that

$$g_{sl} \oplus \mathrm{id} = G_{sl} \oplus \psi_{sl}, \quad \widetilde{g}_{sl} = G_{sl} \oplus \widetilde{\psi}_{sl}, \tag{2.42}$$

where G_{sl} is the direct sum of all common components of the mappings $g_{sl} \oplus \mathrm{id}$ and $\widetilde{g}_{sl}$, and

$$\psi_{sl} = \begin{pmatrix} W_{j-1}(\gamma,\widetilde{\gamma}) & 0 \\ 0 & \mathrm{id} \end{pmatrix} : \begin{array}{c} [\gamma,\widetilde{\gamma}]_j \\ \oplus \\ [\gamma,\widetilde{\gamma}] \end{array} \longrightarrow \begin{array}{c} [\gamma,\widetilde{\gamma}]_{j+1} \\ \oplus \\ [\gamma,\widetilde{\gamma}] \end{array} \tag{2.43}$$

$$\widetilde{\psi}_{sl} = \begin{pmatrix} 0 & \widetilde{W}_{j-1/2}(\gamma,\widetilde{\gamma}) \\ \widetilde{W}_{j-1}(\gamma,\widetilde{\gamma}) & 0 \end{pmatrix} : \begin{array}{c} [\gamma,\widetilde{\gamma}]_j \\ \oplus \\ [\gamma,\widetilde{\gamma}] \end{array} \longrightarrow \begin{array}{c} [\gamma,\widetilde{\gamma}]_{j+1} \\ \oplus \\ [\gamma,\widetilde{\gamma}] \end{array} \tag{2.44}$$

The isomorphisms (2.43) and (2.44) are homotopic; namely, the homotopy has the form

$$\psi_{sl}(\tau) = \begin{pmatrix} \tau W_{j-1} & [1-\tau+i\tau(1-\tau)]\widetilde{W}_{j-1/2} \\ [1-\tau+i\tau(1-\tau)]\widetilde{W}_{j-1} & \tau\,\mathrm{id} \end{pmatrix}, \quad \tau \in [0,1], \qquad (2.45)$$

where the arguments $\gamma, \widetilde{\gamma}$ are omitted for brevity. Indeed, $\psi_{sl}(0) = \widetilde{\psi}_{sl}$, $\psi_{sl}(1) = \psi_{sl}$; let us verify that the morphism (2.45) is invertible for all $\tau \in [0,1]$. Let

$$\begin{pmatrix} \tau W_{j-1} & [1-\tau+i\tau(1-\tau)]\widetilde{W}_{j-1/2} \\ [1-\tau+i\tau(1-\tau)]\widetilde{W}_{j-1} & \tau\,\mathrm{id} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0,$$

that is,

$$\tau W_{j-1}u + [1-\tau+i\tau(1-\tau)]\widetilde{W}_{j-1/2}v = 0,$$
$$[1-\tau+i\tau(1-\tau)]\widetilde{W}_{j-1}u + \tau v = 0.$$

We multiply the first equation by $1-\tau+i\tau(1-\tau)$ and subtract it from the second equation multiplied by $\tau\widetilde{W}_{j-1/2}$. With regard to (2.35), we obtain

$$\left\{\tau^2 - [1-\tau+i\tau(1-\tau)]^2\right\} W_{j-1/2}v = 0,$$

whence it follows that $v = 0$, since $W_{j-1/2}$ is an isomorphism and $\tau^2 - [1-\tau+i\tau(1-\tau)]^2$ does not vanish on $[0,1]$. By substituting $v = 0$ in our equations, we find that $u = 0$, as desired.

Thus, we have obtained a homotopy $g_{sl}(\tau) = G_{sl} \oplus \psi_{sl}(\tau)$ of the transition functions $\widetilde{g}_{sl}$ of the bundle $\widetilde{E}$ to the transition functions $g_{sl} \oplus \mathrm{id}$ of the bundle $E \oplus \widetilde{F}_{j+1/2}$. Note that for any value of the homotopy parameter τ the transition functions $g_{sl}(\tau)$ satisfy the cocycle condition. Indeed, consider a nonempty intersection $U_{slm} = U_s \cap U_l \cap U_m$ and assume that, say $\gamma_s \equiv \gamma < \gamma_l \equiv \widetilde{\gamma} < \gamma_m \equiv \mathring{\gamma}$. Then for all $\tau \in [0,1]$ one has the identity

$$\psi_{sm}(\tau) = \psi_{sl}(\tau) \oplus \psi_{lm}(\tau)$$

following from the direct additivity of the morphisms $W(\cdots)$ and $\widetilde{W}(\cdots)$. Hence the desired cocycle condition follows. (We omit computations, similar to the preceding ones, in which critical terms are singled out.) Thus, the bundle $\widetilde{E}$ is homotopic and hence isomorphic to the bundle $E \oplus \widetilde{F}_{j+1/2}$. The proof of Lemma 2.16 and Theorem 2.15 is complete. □

3 Homotopy Invariance and the Main Theorem

In this section we show that the spectral flow of a family of conormal symbols with parameter space X is a stable homotopy invariant of such families and establish a relationship with other homotopy invariants, namely, the index of the same family treated as a family of pseudodifferential operators with the (noncompact) parameter space $X \times \mathbb{R}^2$ and the index of the corresponding family of pseudodifferential operators with parameter space X on the infinite cone.

3.1 The spectral flow is a stable homotopy invariant

Theorem 3.1. *Let $\mathbf{D}_t(\tau)$, $t \in [0,1]$ be a family of conormal symbols with compact parameter space X continuously depending on the additional parameter $\tau \in [0,1]$. Suppose that for each τ condition* 2.1 *holds. Then the spectral flow $\operatorname{sf}_{t\in[0,1]} \mathbf{D}_t(\tau)$ is independent of τ.*

Proof. Each point $\tau_0 \in [0,1]$ has a neighborhood such that for all τ from that neighborhood in the construction of the spectral flow $\operatorname{sf}_{t\in[0,1]} \mathbf{D}_t(\tau)$ one can take the same partition, weight system, number ε, and admissible cover and the same ε-admissible staircases in elements of the cover. In a possibly smaller neighborhood, one can take fixed suitable finite-dimensional approximations for all points of the partition and then equip them with suitable inner products continuously depending on τ. Then the corresponding bundles E and F determining the spectral flow continuously depend on τ in that neighborhood, so that the desired assertion holds in that neighborhood. It remains to use the compactness of the interval $[0,1]$. ☐

Next, it is obvious that the spectral flow remains unchanged if we add a direct summand identically equal to the identity operator to the family. In conjunction with the theorem we have just proved, this shows that the spectral flow is a stable homotopy invariant.

The spectral flow of nonanalytic conormal symbols. Using the stable homotopy invariance, we can define the notion of spectral flow for families defined only for $p \in \mathbb{R}$ (nonanalytic in p) and depending on the parameter t only continuously. Namely, we apply the standard smoothing with respect to t (convolution with a δ-like sequence of compactly supported smooth functions) and analytic smoothing with respect to p (convolution with the δ-like scaling of the Fourier transform of a compactly supported smooth function) [17], thus obtaining a sequence of operator families smoothly depending on t and analytic in p. For sufficiently small values of the smoothing parameters, these families are families of conormal symbols analytic in a strip, so that the above definition applies. By virtue of the homotopy invariance, the spectral flow of the smoothed family is independent of the choice of smoothing, and we define the spectral flow of the original family as the spectral flow of the smoothed family.

3.2 The spectral flow and the index of families on the infinite cone

We introduce a smooth monotone nondecreasing function

$$\xi : \mathbb{R}_+ \longrightarrow [0,1]$$

equal to 0 for $r \le 1$ and 1 for sufficiently large r.

Theorem 3.2. *Let $\mathbf{D}_t \equiv \mathbf{D}_{x,t}$, $t \in [0,1]$, be a homotopy of families of conormal symbols with parameter space X satisfying Condition 2.1. Then the following elements of the K-group $K(X)$ coincide:*

1) *the spectral flow $\operatorname{sf}_{t\in[0,1]} \mathbf{D}_t$ given by Definition 2.14;*

2) *the index of the operator family*[7]

$$\mathbf{D}_x\left(\overset{2}{r}, ir\overset{1}{\frac{\partial}{\partial r}}\right) : H^{s,0,0}(K_\Omega) \longrightarrow H^{s,0,0}(K_\Omega)$$

   in weighted Sobolev spaces on the infinite cone, where $\mathbf{D}_x(r,p) = \mathbf{D}_{x,\xi(r)}(p)$;

3) *the element $\beta \operatorname{ind} \mathbf{D}_{x,t}(p) \in K(X)$, where $\operatorname{ind} \mathbf{D}_{x,t}(p)$ is understood as an element of the K-group $K_c(X \times \mathbb{R} \times (0,1))$ with compact supports (note that the family $\mathbf{D}_{x,t}(p)$ is invertible outside a compact set in $X \times \mathbb{R} \times (0,1)$) and*

$$\beta : K_c(X \times \mathbb{R} \times (0,1)) \longrightarrow K(X)$$

   is the Bott periodicity map.

The proof of the theorem is based on the following two lemmas.

Lemma 3.3. *There exists a number N and a family*

$$\widetilde{\mathbf{D}}_{x,t}(p) : C^\infty(\Omega_x) \oplus C^\infty(\Omega_x, \mathbb{C}^N) \to C^\infty(\Omega_x) \oplus C^\infty(\Omega_x, \mathbb{C}^N)$$

of elliptic operators with parameter p such that the operators of the family are invertible on the real line and the principal symbol of that family is obtained from the principal symbol of the original family by the direct summand equal to the unit symbol:

$$\sigma(\widetilde{\mathbf{D}}_{x,t}(p)) = \sigma(\mathbf{D}_{x,t}(p)) \oplus 1_N. \tag{3.1}$$

[7] We use Feynman indices showing the order of action of operator arguments; see [9], [15].

Proof. Consider the family $\mathbf{D}_{x,t}(p)$. By assumption, it is invertible outside some compact set in the parameter space $X \times [0,1] \times \mathbb{R}$ and is Fredholm everywhere on that space.

Just as in the theory of families index, it follows (e.g., see [4]) that for some N there exists a family

$$K(x,t,p) : C^\infty(\Omega_x) \oplus C^\infty(\Omega_x, \mathbb{C}^N) \to C^\infty(\Omega_x) \oplus C^\infty(\Omega_x, \mathbb{C}^N)$$

of compact operators vanishing outside a compact set in the space $X \times [0,1] \times \mathbb{R}$ such that the family

$$\mathbf{D}_{x,t}(p) \oplus 1_N + \sigma'_K(x,t,p)$$

is everywhere invertible. To complete the proof, it suffices to apply analytic smoothing with respect to the variable p to the family K (e.g., see [17]). □

The following lemma pertains to the homotopy classification of conormal symbols with unit principal symbol (cf. [12], where the classification was obtained for the case $X = \{pt\}$).

By $\mathcal{C}$ we denote the abelian group of stable homotopy classes of families, parametrized by points $x \in X$, of conormal symbols with unit principal symbol invertible on the real line. Let $C_0(X \times \mathbb{R}, \mathcal{K}(L^2(\Omega)))$ be the C^*-algebra of sections, decaying at infinity on the space $X \times \mathbb{R}$, of the bundle of C^*-algebras $\mathcal{K}(L^2(\Omega_x))$ of compact operators in the spaces $L^2(\Omega_x)$ of functions on the fibers Ω_x. The natural mapping

$$\mathcal{C} \longrightarrow K_1(C_0(X \times \mathbb{R}, \mathcal{K}(L^2(\Omega)))),$$

where K_1 is the odd K-group of an algebra [6] and the operator is mapped to its condition symbol at infinity, is an isomorphism. Indeed, the algebra of conormal symbols with zero principal symbol is a subalgebra in $C_0(X \times \mathbb{R}, \mathcal{K}(L^2(\Omega)))$ closed with respect to holomorphic functional calculus.

On the other hand, the latter K-group admits the explicit description

$$K_1(C_0(X \times \mathbb{R}, \mathcal{K}(L^2(\Omega)))) \simeq K_1(C_0(X \times \mathbb{R})) \simeq K^1(X \times \mathbb{R}) \simeq K(X).$$

The composite homomorphism

$$\alpha : K(X) \longrightarrow \mathcal{C}$$

from the topological K-group can be expressed as follows: to a vector bundle $V \in \mathrm{Vect}(X)$ we assign the conormal symbol

$$\alpha(V)(x,p) = 1 - Q(x) + Q(x) f(p), \tag{3.2}$$

where

$$Q(x) : L^2(\Omega_x) \to L^2(\Omega_x)$$

is a family of orthogonal projections on the finite-dimensional subbundle consisting of smooth functions and isomorphic to V. One can obtain such a family by realizing V as a subbundle in a trivial finite-dimensional bundle and then embedding the trivial bundle as a subbundle in the bundle of Hilbert spaces $L^2(\Omega)$. This is possible, since $L^2(\Omega)$ is infinite-dimensional. Finally, by f we denote a function analytic in a neighborhood of the real axis, tending to 1 as $p \to \pm\infty$ and such that the degree $\deg f$ of the corresponding map $\mathbb{R} \to \mathbb{S}^1$ is equal to 1. For example, one can take

$$f(p) = \frac{p-i}{p+i}.$$

The argument carried above shows that the following lemma holds.

Lemma 3.4. *An arbitrary family $\mathbf{D}_x(p)$, $x \in X$, of conormal symbols that has a unit principal symbol and is invertible on the real line is stably homotopic to a conormal symbol of the form* (3.2)

Proof of the theorem. It follows by simple reasoning from Lemmas 3.3 and 3.4 that it suffices to carry out the proof for the case in which

$$\mathbf{D}_{x,t}(p) = 1 + tQ(x)(f(p) - 1) \tag{3.3}$$

(the notation is the same as in Lemma 3.4. For such families one can always construct an admissible ε-staircase globally on X, and the proof of the fact that the elements 1) and 2) in the statement of the theorem coincide is carried out in the same way as in [14] with regard for the relative index theorem in [11].

To prove that the elements 2) and 3) in the statement of the theorem coincide, we note that for families of the form (3.3) one has

$$\operatorname{ind}\mathbf{D}_x\left(\overset{2}{r}, ir\overset{1}{\frac{\partial}{\partial r}}\right) = [V]\deg f = [V] \in K(X)$$

(we omit the corresponding surgery-based computation). On the other hand, the index of the corresponding family $\mathbf{D}_{t,x}(p)$ is equal to the product

$$b[V] \in K(S^*X \times (0,1) \times \mathbb{R}),$$

where $b \in K((0,1) \times \mathbb{R})$ is the Bott element.

This completes the proof. $\square$

References

[1] M. Agranovich and M. Vishik. Elliptic problems with parameter and parabolic problems of general type. *Uspekhi Mat. Nauk*, **19**, No. 3, 1964, 53–161. English transl.: Russ. Math. Surv. **19** (1964), N 3, p. 53–157.

[2] V. I. Arnold. *Ordinary Differential Equations.* MIT Press, Cambridge–London, 1978.

[3] M. Atiyah, V. Patodi, and I. Singer. Spectral asymmetry and Riemannian geometry III. *Math. Proc. Cambridge Philos. Soc.*, **79**, 1976, 71–99.

[4] M.F. Atiyah. *K-Theory.* The Advanced Book Program. Addison–Wesley, Inc., second edition, 1989.

[5] M. V. Berry. Classical adiabatic angles and quantal adiabatic phase. *J. Phys. A*, **18**, No. 1, 1985, 15–27.

[6] B Blackadar. *K-Theory for Operator Algebras.* Springer, New York–Berlin, 1986.

[7] X. Dai and W. Zhang. Higher spectral flow. *Math. Research Letters*, **3**, 1996, 93–102.

[8] X. Dai and W. Zhang. Higher spectral flow. *J. Funct. Anal.*, **157**, No. 2, 1998, 432–469.

[9] V.P. Maslov. *Operator Methods.* Nauka, Moscow, 1973. English transl.: *Operational Methods*, Mir, Moscow, 1976.

[10] R. Melrose. The eta invariant of pseudodifferential operators and families. *Math. Research Letters*, **2**, No. 5, 1995, 541–561.

[11] V. Nazaikinskii, A. Savin, B.-W. Schulze, and B. Sternin. *Elliptic Theory on Manifolds with Nonisolated Singularities. I. The Index of Families of Cone-Degenerate Operators.* Univ. Potsdam, Institut für Mathematik, Potsdam, August 1998. Preprint N 98/16.

[12] V. Nazaikinskii, B.-W. Schulze, and B. Sternin. *On the Homotopy Classification of Elliptic Operators on Manifolds with Singularities.* Univ. Potsdam, Institut für Mathematik, Potsdam, Oktober 1999. Preprint N 99/21.

[13] V. Nazaikinskii, B.-W. Schulze, B. Sternin, and V. Shatalov. Spectral boundary value problems and elliptic equations on singular manifolds. *Differents. Uravnenija*, **34**, No. 5, 1998, 695–708. English trans.: Differential Equations, **34**, N 5 (1998), pp 696–710.

[14] V. Nazaikinskii and B. Sternin. *Surgery and the Relative Index in Elliptic Theory.* Univ. Potsdam, Institut für Mathematik, Potsdam, Juli 1999. Preprint N 99/17.

[15] V. Nazaikinskii, B. Sternin, and V. Shatalov. *Methods of Noncommutative Analysis. Theory and Applications.* Mathematical Studies. Walter de Gruyter Publishers, Berlin–New York, 1995.

[16] A. Savin, B.-W. Schulze, and B. Sternin. On the invariant index formulas for spectral boundary value problems. *Differentsial'nye uravnenija*, **35**, No. 5, 1999, 705–714. [Russian].

[17] B.-W. Schulze. *Pseudodifferential Operators on Manifolds with Singularities.* North–Holland, Amsterdam, 1991.

[18] B.-W. Schulze, B. Sternin, and V. Shatalov. On the index of differential operators on manifolds with conical singularities. *Annals of Global Analysis and Geometry*, **16**, No. 2, 1998, 141–172.

[19] B. Simon. Holonomy, the quantum adiabatic theorem, and Berry's phase. *Phys. Rev. Lett.*, **51**, No. 24, 1983, 2167–2170.

[20] B. Sternin. *Quasielliptic Equations on an Infinite Cylinder.* Moscow Institute of Electronic Engineering, Moscow, 1972.

Potsdam