An Algebra of Meromorphic Corner Symbols

L. Maniccia* and B.-W. Schulze

Dipartimento di Matematica
Università di Bologna
P.zza di Porta S. Donato 5,
40126 Bologna, ITALY
maniccia@dm.unibo.it

Institut für Mathematik
Univerität Potsdam
Postfach 601553,
14415 Potsdam, GERMANY
schulze@math.uni-potsdam.de

Abstract

Operators on manifolds with corners that have base configurations with geometric singularities can be analysed in the frame of a conormal symbolic structure which is in spirit similar to the one for conical singularities of Kondrat'ev's work. Solvability of elliptic equations and asymptotics of solutions are determined by meromorphic conormal symbols. We study the case when the base has edge singularities which is a natural assumption in a number of applications. There are new phenomena, caused by a specific kind of higher degeneracy of the underlying symbols. We introduce an algebra of meromorphic edge operators that depend on complex parameters and investigate meromorphic inverses in the parameter-dependent elliptic case. Among the examples are resolvents of elliptic differential operators on manifolds with edges.

Contents

*Supported by the EU Research and Training Network "Geometric Analysis"

Introduction

Meromorphic families of differential (and pseudo-differential) operators on a compact C^∞ manifold X belong to the crucial elements in the description of asymptotics of solutions to elliptic equations on a manifold B with conical singularities. Here, X plays the role of the base of a cone that locally models B near a conical point.

Basic observations in this context go back to Kondrat'ev [12] who studied elliptic boundary value problems in a domain with conical singularities and characterised asymptotics in terms of the poles, multiplicities and Laurent coefficients of the inverse of a parameter-dependent elliptic holomorphic family of boundary value problems on the base. Asymptotics can be interpreted as a kind of elliptic regularity of solutions in weighted Sobolev spaces. Phenomena become particularly transparent, if we embed the given operators in an algebra of pseudo-differential operators that are of Fuchs type in the distance variable $r \in \mathbb{R}_+$ to the conical singularity [20].

To answer similar questions for higher singularities, i.e., when X itself has conical or edge singularities, it seems, in fact, indispensable to employ a sufficiently developed (parameter-dependent) calculus of pseudo-differential operators on X.

The case when X has conical points has been treated by Schulze [21] in the framework of an operator algebra with iterated asymptotics and with a hierarchy of principal symbols.

Applications in connection with Fedosov's index formulas have been given in [5]. Meromorphic operator functions on X also occur in the context of Euler solutions of equations on an infinite cylinder with cross-section X, cf. [23] for the case of smooth X, [10] for the case of a manifold X with conical singularities. Another application concerns long-time asymptotics of solutions to parabolic operators on spatial configurations that have conical singularities [14]. Note that parameter-dependent theories for differential equations have been studied in other situations before, in particular, by Agranovich and Vishik in [1] in connection with parabolic operators. Meromorphic Fredholm functions in a more general functional analytic set-up are studied by Gohberg and Segal in [8]. Other aspects (factorisations into holomorphic invertible and "smoothing" meromorphic factors) are investigated by Gramsh and Kaballo in [9], and Witt in [25].

The present paper is aimed at developing the analysis of parameter-dependent meromorphic operator families for the case when X is a manifold with edges. The values of operator functions belong to the edge algebra, i.e., a block matrix calculus of operators on a stretched manifold $\mathbb{W}$ belonging to a manifold W with edges Y, cf. [4] or [20] . The operators in the upper left corners are edge-degenerate; other entries represent trace and potential conditions on the edge. We study here a parameter-dependent theory, where the parameters play the role of covariables in a higher floor of the hierarchy of operator algebras on stratified spaces, in this case on manifolds with corner singularities, where the base manifolds are of type W, cf. also [24].

Chapter 1 presents the necessary elements of this edge operator theory. Weight and asymptotic data are controlled in a such a way that the edge algebra can be written as a union of Fréchet subspaces. Their topologies are given by symbolic structures and smoothing operators that refer to spaces with asymptotics. We mainly consider spaces and operators with continuous asymptotics, formulated in terms of vector or operator-valued analytic functionals in the complex z-plane of the Mellin covariable, with the Mellin transform operating on the half-axis of the distance $r \in \mathbb{R}_+$ to the edge. The motivation is that conormal symbols of operators on a manifold with edges depend on edge variables, and the "spectral" points z (that determine asymptotic data via inversion of meromorphic operator functions) are variable and, in general, of changing multiplicity. A description of such phenomena by families of analytic functionals was originally introduced in [15], see also [19] or [20]. Here we employ structures of that kind as a part of asymptotic information associated with the model cone of W.

In Chapter 2 we construct a new algebra of holomorphic and meromorphic operator functions, globally operating on the corner base W. Spectral points $w \in \mathbb{C}$ determine another contribution to asymptotics of solutions in an associated corner operator calculus, though we focus here on meromorphic families themselves. In the case of polynomial dependence on w they are related to the resolvent structure of differential operators on a manifold with edges. For the simpler case of conical singularities such relations are studied in [21]; applications to heat trace asymptotics are given by Gil [6].

We consider parameter-dependent ellipticity in our algebra and show that there are meromorphic inverses of elliptic elements. A crucial point is that kernel cut-off constructions (introduced for a simpler situation in [19]) can be applied again in the corner covariable, and we show that for an arbitrary parameter-dependent element with real parameter λ there is a holomorphic representative in $w \in \mathbb{C}$ for $\lambda = \mathrm{Re}\ w$, modulo a family of order $-\infty$.

Let us finally note that the scenario of [21] (for corners based on manifolds with conical singularities), shows how our algebra of corner symbols can be applied again in a calculus of Mellin operators along the half-axis $t \in \mathbb{R}_+$ for a corner singularity of base W and that iterated edge-corner asymptotics for elliptic equations are encoded by the poles and Laurent expansions of our Mellin symbols.

1 Elements of the edge calculus

1.1 Manifolds with edges and associated operators

Let X be a closed C^∞ manifold, and form the quotient space $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ that is a cone with base X, where $\{0\} \times X$ is shrunk to a point v, the tip of the cone. Then $X^\Delta \setminus \{v\} \cong \mathbb{R}_+ \times X$ is a C^∞ manifold. Two splittings of variables (r, x), $(\tilde{r}, \tilde{x})$ on $\mathbb{R}_+ \times X$ are said to define the same cone structure on $X^\wedge := \mathbb{R}_+ \times X$, if $(r, x) \to (\tilde{r}, \tilde{x})$ is induced by a diffeomorphism $\overline{\mathbb{R}}_+ \times X \to \overline{\mathbb{R}}_+ \times X$.

Given an open set $\Omega \subseteq \mathbb{R}^q$ we can pass to a wedge $X^\Delta \times \Omega$ or to the (open) stretched wedge $X^\wedge \times \Omega$. Two splittings of variables (r, x, y) and $(\tilde{r}, \tilde{x}, \tilde{y})$ on $X^\wedge \times \Omega$ are said to define the same wedge structure on $X^\wedge \times \Omega$, if $(r, x, y) \to (\tilde{r}, \tilde{x}, \tilde{y})$ is induced by a diffeomorphism $\overline{\mathbb{R}}_+ \times X \times \Omega \to \overline{\mathbb{R}}_+ \times X \times \Omega$, where $\tilde{r}(r, x, y)|_{r=0} = 0$, $x \to \tilde{x}(r, x, y)|_{r=0}$ represents a diffeomorphism $X \to X$ for every $y \in \Omega$, and $y \to \tilde{y}(r, x, y)|_{r=0}$ is independent of x and represents a diffeomorphism $\Omega \to \Omega$.

A topological space W (locally compact and paracompact) is said to be a manifold with edges $Y \subset W$, if $W \setminus Y$ and Y are C^∞ manifolds of dimension $1 + n + q$ and q, respectively, and every $y \in Y$ has a neighbourhood V in W that is homeomorphic to a wedge $X^\Delta \times \Omega$ with a fixed wedge structure on $X^\wedge \times \Omega$, where X is a certain closed C^∞ manifold, $n = \dim X$. In addition, we require that such so-called singular charts $V \to X^\Delta \times \Omega$ induce diffeomorphisms $V \setminus Y \to X^\wedge \times \Omega$ and $V \cap Y \to \Omega$, and that the transition diffeomorphisms to different singular charts preserve the local wedge structure on $X^\wedge \times \Omega$.

With W we can associate a stretched manifold $\mathbb{W}$ that is a C^∞ manifold with boundary such that $\mathbb{W} \setminus \partial\mathbb{W} \cong W \setminus Y$ and $\mathbb{W}$ is locally near $\partial\mathbb{W}$ modelled by $\overline{\mathbb{R}}_+ \times X \times \Omega$. This is an invariant definition, and $\partial\mathbb{W}$ is an X-bundle on Y.

For simplicity, in this paper we assume that our manifolds W with edges Y satisfy the following condition. There exists a neighbourhood V of Y in W and a homeomorphism $\chi : V \to X^\Delta \times Y$ that restricts to diffeomorphisms $V \setminus Y \to X^\wedge \times Y$ and $V \cap Y \to Y$, such that the local wedge structures are defined by a global splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X \times Y$. In particular, we assume $\partial\mathbb{W}$ to be a trivial X-bundle. If this is not the case, we can fix an atlas on $\mathbb{W}$, where the transition maps near $\partial\mathbb{W}$ (i.e., for small r) are independent of r; this is always possible. The essential results of our calculus extend to this situation but this would require extra invariance discussions for our operators that we wish to avoid.

We may admit $q = \dim Y = 0$; then we recover the definition of a manifold B with conical singularities including its associated stretched manifold $\mathbb{B}$. In particular, X^Δ is a manifold with conical singularity and $\overline{\mathbb{R}}_+ \times X$ the stretched manifold. If B has conical singularities S, then $W := B \times Y$ for any C^∞ manifold Y is a manifold with edge $S \times Y$, and $\mathbb{W} = \mathbb{B} \times Y$ is the corresponding stretched manifold.

Differential and pseudo-differential operators on a manifold W with edges Y will be expressed as operators on int $\mathbb{W}$ with a special behaviour of symbols near $\partial\mathbb{W}$. Given a symbol

$$\tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}_{\tilde{\rho}, \xi, \tilde{\eta}}) \ ,$$

where $\Sigma \subseteq \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^q$ are open sets and $\mu \in \mathbb{R}$, we form operators on $\mathbb{R}_+ \times \Sigma \times \Omega$ in terms of $r^{-\mu} p(r, x, y, \rho, \xi, \eta)$, where

$$p(r, x, y, \rho, \xi, \eta) := \tilde{p}(r, x, y, r\rho, \xi, r\eta). \tag{1.1.1}$$

Notation in connection with classical symbols of Hörmander's type, $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}_{\tilde{\rho},\xi,\tilde{\eta}})$ as well as non-classical (denoted by $S^{\mu}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}_{\tilde{\rho},\xi,\tilde{\eta}})$), will be explained in this section below for an operator-valued variant that contains scalar symbols as a special case. Most of our results (but not all) are true both for classical and non-classical symbols. If a relation is valid in both cases we often write "(cl)" as subscript.

Symbols of the form (1.1.1) are called edge-degenerate (cf. [18]) . We will be interested, in fact, in a parameter-dependent variant with a parameter $\lambda \in \mathbb{R}^l$ and $(\eta, \lambda) \in \mathbb{R}^{q+l}$ in place of $\eta \in \mathbb{R}^q$. In other words, we talk about symbols

$$p(r, x, y, \rho, \xi, \eta, \lambda) = \tilde{p}(r, x, y, r\rho, \xi, r\eta, r\lambda)$$

for $\tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda}) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q+l}_{\tilde{\rho},\xi,\tilde{\eta},\tilde{\lambda}})$.

For the moment we shall omit λ, but later on we return to the parameter-dependent case.

Special examples are edge-degenerate differential operators that are locally near $\partial\mathbb{W}$, in variables $(r, x, y) \in X^{\wedge} \times \Omega$, of the form

$$A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y)(-r\frac{\partial}{\partial r})^j (rD_y)^{\alpha} \tag{1.1.2}$$

with coefficients $a_{j\alpha}(r, y) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$.

Here $\mathrm{Diff}^{\nu}(\cdot)$ denotes the space of all differential operators of order ν with smooth coefficients on the space in the brackets. In particular, the Laplace-Beltrami operator to a wedge metric on $X^{\wedge} \times \Omega$ of the form $dr^2 + r^2 g_X + dy^2$ with a Riemannian metric g_X on X is edge-degenerate of order $\mu = 2$.

The basics of the calculus of edge-degenerate operators may be found in [20] or [22], see also the monograph Egorov and Schulze [4].

Notice that edge-degeneracy is an invariant property under transition maps that preserve the wedge structure.

Moreover, considering $W := \mathbb{R}^{n+1} \times \Omega$ as a manifold with edge $\Omega \subseteq \mathbb{R}^q$, $(\tilde{x}, y) \in W$, and model cone $\mathbb{R}^{n+1} \cong (\overline{\mathbb{R}}_+ \times S^n)/(\{0\} \times S^n)$ (with S^n being the unit sphere in $\mathbb{R}^{n+1}$), substituting of polar coordinates $\tilde{x} \to (r, x)$, $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}_+ \times S^n$, transforms every differential operator $\tilde{A}$ on W with smooth coefficients in $(\tilde{x}, y)$ into an edge degenerate one on $\mathbb{W} = (\mathbb{R}_+ \times S^n) \times \Omega$, including the weight factor $r^{-\mu}$, cf. (1.1.2) . In this sense, the space of edge-degenerate operators is much larger than that of operators with smooth coefficients.

Let $\mathrm{Diff}^{\mu}_{\mathrm{edge}}(\mathbb{W})$ denote the space of all differential operators A of order μ on $\mathrm{int}\mathbb{W}$ with smooth coefficients, such that A near $\partial\mathbb{W}$ has the form (1.1.2) (this is a Fréchet space in a canonical way). Then

$$t^{-\mu}\sum_{k=0}^{\mu} A_k(t)(-t\frac{\partial}{\partial t})^k \tag{1.1.3}$$

with coefficients $A_k(t) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-k}_{\mathrm{edge}}(\mathbb{W}))$ is a typical differential operator on a (stretched) corner configuration $\mathbb{R}_+ \times \mathbb{W}$. Operators of that kind (for $\mu = 2$) occur as Laplace-Beltrami operators for corner metrics, locally of the form $dt^2 + t^2(dr^2 + r^2 g_X + dy^2)$ where g_X is a Riemannian metric on X that may smoothly depend on r, t, y (smooth up to $r = 0$, $t = 0$).

Examples of operator functions in $w \in \mathbb{C}$ as they are studied below in Chapter 2 are

$$a(w) := \sum_{k=0}^{\mu} A_k(0)w^k.$$

They have the meaning of principal conormal symbols for corner operators (1.1.3). The main difficulty to analyse them lies in the corner degeneracy in w (which substitues $-t\frac{\partial}{\partial t}$) that causes terms of the form $rt\frac{\partial}{\partial t}$ in operators (1.1.3) themselves for r, t close to zero, apart from edge-degeneracy far from $t = 0$.

It will be convenient to reformulate edge-degenerate operators in terms of the Mellin transform with respect to the axial variable $r \in \mathbb{R}_+$, namely

$$Mu(z) = \int_0^\infty r^{z-1}u(r)dr \ .$$

For $u \in C_0^\infty(\mathbb{R}_+)$ the Mellin transform $Mu(z)$ is an entire function in $z \in \mathbb{C}$. Later on we extend M to larger distribution spaces, also vector-valued ones; then the variable z will run over a line

$$\Gamma_\beta := \{z \in \mathbb{C} : \ \mathrm{Re}\ z = \beta\}$$

for an appropriate $\beta \in \mathbb{R}$. Recall, in particular, that M extends by continuity to an isomorphism $M_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\gamma})$ for every $\gamma \in \mathbb{R}$; L^2-spaces are equipped with standard scalar products (i.e., on $\mathbb{R}_+$ induced by the Lebesgue measure on $\mathbb{R}$, and on $\Gamma_{\frac{1}{2}-\gamma}$ by $\mathbb{R}$ from the identification of $\Gamma_{\frac{1}{2}-\gamma}$ with $\mathbb{R}$ by $z \to \mathrm{Im}\ z$). We call M_γ the weighted Mellin transform with weight γ. Often we shall set $M_0 = M$. The identity $-r\frac{\partial}{\partial r} = M^{-1}zM$ (first considered on $C_0^\infty(\mathbb{R}_+)$ and then on larger spaces) motivates to formulate pseudo-differential operators, based on the (weighted) Mellin transform, namely,

$$\mathrm{op}^\gamma_M(f)u(r) := \frac{1}{2\pi i}\int\int_0^\infty \left(\frac{r}{r'}\right)^{-z} f(r, r', z)u(r')\frac{dr'}{r'}dz \ .$$

Here, $z = \frac{1}{2} - \gamma + i\rho$, and $f(r, r', z)$ is a symbol in $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$ (identified with $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R})$ via $\Gamma_{\frac{1}{2}-\gamma} \cong \mathbb{R}$). Similar notation is used for vector-valued functions, say, $u \in C_0^\infty(\mathbb{R}_+, C^\infty(X))$, and operator valued symbols

$$f(r, r', z) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^{\mu}_{(\mathrm{cl})}(X; \Gamma_{\frac{1}{2}-\gamma})) \ .$$

In general, if X is a C^∞ manifold (locally compact and paracompact), $L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^n)$ denotes the space of all parameter-dependent (with parameter $\lambda \in \mathbb{R}^l$) pseudo-differential operators on X (classical or non-classical) in its natural Fréchet topology, cf. [20] and [22]. More details on the Fréchet space aspect will be studied in Section 2.3 below. Notation on symbols and pseudo-differential operator spaces will be added later on.

Consider charts $\nu : \mathbb{R}_+ \times U \to \Gamma$ on $X^\wedge$ for a coordinate neighbourhood U on X and an open conical set $\Gamma \subset \mathbb{R}^{n+1} \setminus \{0\}$, such that

(i) $\nu(\delta r, x) = \delta \nu(r, x)$ for all $\delta \in \mathbb{R}_+$, $(r, x) \in \mathbb{R}_+ \times U$,

(ii) $\nu_1(x) := \nu(1, x)$, $x \in U$, induces a diffeomorphism $\nu_1 : U \to \Gamma \cap S^n$.

Let $H^s_{\mathrm{cone}}(X^\wedge)$, $s \in \mathbb{R}$, denote the space of all $u(r, x) \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}$, such that for every $\varphi(x) \in C_0^\infty(U)$ for any coordinate neighbourhood U on X and every excision function $\chi(r)$ (i.e., $\chi \in C^\infty(\mathbb{R}_+)$, $\chi = 0$ near $r = 0$, $\chi = 1$ near $r = \infty$) we have $\chi\varphi u \circ \nu^{-1} \in H^s(\mathbb{R}^{n+1})$; here ν is any chart on $\mathbb{R}_+ \times U$ of the above kind. On $H^s_{\mathrm{cone}}(X^\wedge)$ we fix a Hilbert space structure, such that

$$H^0_{\mathrm{cone}}(X^\wedge) = \langle r \rangle^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times X) \ ,$$

where $L^2(\mathbb{R}_+ \times X)$ refers to $drdx$ with dx being associated with a fixed Riemannian metric on X.

Moreover, let $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s, \gamma \in \mathbb{R}$ denote the completion of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} ||R^s(\mathrm{Im}\ z) Mu(z)||^2_{L^2(X)} dz \right\}^{\frac{1}{2}} \ , \quad n = \dim X \ .$$

Here, $R(\rho)$ is a parameter-dependent elliptic operator belonging to the space $L^s_{\mathrm{cl}}(X; \mathbb{R})$ that induces isomorphisms $R^s(\rho) : H^t(X) \to H^{t-s}(X)$ for all $t, s \in \mathbb{R}$. $H^s(X)$ is the standard Sobolev space on X of smoothness $s \in \mathbb{R}$, and $H^0(X)$ is identified with $L^2(X)$ with the scalar product $(u, v) = \int_X u\overline{v} dx$.

Concerning basic properties of the spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$, cf. [20]. In the present paper a cut-off function on $\mathbb{R}_+$ is any real valued function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega = 1$ near 0. We then define

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u + (1 - \omega)v : u \in \mathcal{H}^{s,\gamma}(X^\wedge) \ , v \in H^s_{\mathrm{cone}}(X^\wedge)\}$$

for any cut-off function ω (the choice of ω is unessential). We endow the spaces with Hilbert space structures in a natural way, in particular, $\mathcal{K}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times X)$.

Remark 1.1.1 *Setting* $(\kappa_\delta u)(r,x) := \delta^{\frac{n+1}{2}} u(\delta r, x)$, $\delta \in \mathbb{R}_+$, *we get a family of continuous operators*

$$\kappa_\delta : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge), \qquad s,\gamma \in \mathbb{R},$$

that is strongly continuous in δ.

Let A be an edge-degenerate differential operator of the form (1.1.2), and suppose the coefficients $a_{j\alpha}$ to be independent of r for large r. Set

$$a(y,\eta) := r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y)(-r\frac{\partial}{\partial r})^j (r\eta)^\alpha \ , \tag{1.1.4}$$

regarded as a family of operators $C_0^\infty(X^\wedge) \to C_0^\infty(X^\wedge)$, parametrised by $(y,\eta) \in \Omega \times \mathbb{R}^q$. Then (1.1.4) extends to continuous operators

$$a(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

for every $s,\gamma \in \mathbb{R}$, and this operator function may be interpreted as an operator-valued symbol in the sense specified below.

If E is a Hilbert space and $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ a fixed strongly continuous group of isomorphisms $\kappa_\delta : E \to E$ (where $\kappa_\delta\kappa_\beta = \kappa_{\delta\beta}$), we say that E is endowed with a group action $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$. For $E = \mathbb{C}^N$ we always suppose $\kappa_\delta = \mathrm{id}_E$ for all $\delta \in \mathbb{R}_+$.

Let $(E, \{\kappa_\delta\}_{\delta\in\mathbb{R}_+})$ and $(\tilde{E}, \{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+})$ be Hilbert spaces with group actions. Then, if $U \subseteq \mathbb{R}^p$ is any open set, $S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ denotes the space of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E,\tilde{E}))$ such that

$$\sup_{(y,\eta)\in K\times\mathbb{R}^q} \langle\eta\rangle^{-\mu+|\beta|} ||\tilde{\kappa}^{-1}_{\langle\eta\rangle}(D_y^\alpha D_\eta^\beta a(y,\eta))\kappa_{\langle\eta\rangle}||_{\mathcal{L}(E,\tilde{E})} \tag{1.1.5}$$

are finite for all $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, for arbitrary $K \subset\subset U$, with constants $c = c(\alpha,\beta,K) > 0$. We consider the space $S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ with its natural Fréchet topology given by the semi-norms (1.1.5).

There is also a natural notion of classical symbols, based on "twisted homogeneity" of order μ:

$$f(y,\delta\eta) = \delta^\mu \tilde{\kappa}_\delta f(y,\eta)\kappa_\delta^{-1} \qquad \text{for all } \delta \in \mathbb{R}_+ \ . \tag{1.1.6}$$

Let $S^{(\mu)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$ denote the subspace of all $f \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}); \mathcal{L}(E,\tilde{E}))$ such that (1.1.6) holds. Then, $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$ is defined to be the subspace of all $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ such that there are elements $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$, $j \in \mathbb{N}$, satisfying

$$r_N(y,\eta) := a(y,\eta) - \chi(\eta) \sum_{j=0}^N a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; E, \tilde{E}) \qquad \text{for all } N \in \mathbb{N} \ .$$

The homogeneous components $a_{(\mu-j)}(y,\eta)$ are uniquely determined by $a(y,\eta)$. Also $S^{\mu}_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\tilde{E})$ is Fréchet in a natural way. An adequate semi-norm system comes from all semi-norms of $a_{(\mu-j)}(y,\eta)$ in $C^\infty(U\times(\mathbb{R}^q\setminus\{0\});\mathcal{L}(E,\tilde{E}))$, $j\in\mathbb{N}$, together with all semi-norms of $r_N(y,\eta)\in S^{\mu-(N+1)}(U\times\mathbb{R}^q;E,\tilde{E})$, $N\in\mathbb{N}$ (the choice of χ is unessential).

Let $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q;E,\tilde{E})$ denote the subspace of all elements of $S^{\mu}_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})$ with constant coefficients (i.e., $a=a(\eta)$). The space $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q;E,\tilde{E})$ is closed in $S^{\mu}_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})$ in the induced topology, and we have $S^{\mu}_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})=C^\infty(U,S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q;E,\tilde{E}))$.

In the case $U=\Omega\times\Omega$, for an open set $\Omega\subseteq\mathbb{R}^q$, $(y,y')\in U$, we set, for every $a(y,y',\eta)\in S^{\mu}(\Omega\times\Omega\times\mathbb{R}^q;E,\tilde{E})$,

(1.1.7) $$\sigma_\wedge(a)(y,\eta):=a_{(\mu)}(y,y',\eta)|_{y'=y}$$

and

$$\mathrm{Op}(a)u(y):=\iint e^{i(y-y')\eta}a(y,y',\eta)u(y)dy'đ\eta\ ,\quad đ\eta=(2\pi)^{-q}d\eta$$

which is the associated pseudo-differential operator on Ω, based on the Fourier transform

$$Fu(\eta)=\int e^{-iy\eta}u(y)dy=\hat{u}(\eta)\quad\text{ in }\mathbb{R}^q\ .$$

Parallel to the symbol spaces we have so-called abstract edge Sobolev spaces introduced in [18]. Given a Hilbert space E with group action $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$, the space $\mathcal{W}^s(\mathbb{R}^q,E)$, $s\in\mathbb{R}$, is defined to be the completion of $\mathcal{S}(\mathbb{R}^q,E)$ (the Schwartz space of E-valued functions defined on $\mathbb{R}^q$) with respect to the norm

$$\Big\{\int\langle\eta\rangle^{2s}||\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)||^2_E d\eta\Big\}^{\frac{1}{2}}\ .$$

We then have $\mathcal{W}^s(\mathbb{R}^q,E)\subseteq\mathcal{S}'(\mathbb{R}^q,E)$ $(=\mathcal{L}(\mathcal{S}(\mathbb{R}^q),E))$.

Similarly to "comp" and "loc" versions of scalar Sobolev spaces we have $\mathcal{W}^s_{\mathrm{comp}}(\Omega,E)$ and $\mathcal{W}^s_{\mathrm{loc}}(\Omega,E)$, for any open set $\Omega\subseteq\mathbb{R}^q$. Then

$$\mathrm{Op}(a):C_0^\infty(\Omega,E)\to C^\infty(\Omega,\tilde{E})$$

for $a(y,y',\eta)\in S^{\mu}(\Omega\times\Omega\times\mathbb{R}^q;E;\tilde{E})$ induces continuous operators

(1.1.8) $$\mathrm{Op}(a):\mathcal{W}^s_{\mathrm{comp}}(\Omega,E)\to\mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega,\tilde{E})$$

for all $s\in\mathbb{R}$. Basic properties on the scale of spaces $\mathcal{W}^s(\mathbb{R}^q,E)$ may be found in [22] or [20], see also [11].

We also employ such constructions for the case of Fréchet spaces, written as projective limits of Hilbert spaces, E^j, $j\in\mathbb{N}$ with continuous embeddings $E^{j+1}\hookrightarrow E^j$ and a group action on E^0 that restricts to group actions on E^j for all j. In that case we say that the Fréchet space is endowed with a group action. We then have edge spaces $\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j)$ as well as corresponding "comp" and "loc" versions on open subsets of $\mathbb{R}^q$.

Symbol spaces $S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E})$ also make sense for Fréchet spaces E and $\tilde{E}$ with group actions, and we use again the notation $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})$ for the spaces of elements with constant coefficients. Precise definitions and further material may be found, e.g., in [22], Section 1.3.1.

Remark 1.1.2 *For the operator function* (1.1.4) *that is associated with an edge-degenerate differential operator we have*

$$a(y,\eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))$$

for all $s \in \mathbb{R}$, $\gamma \in \mathbb{R}$. If the coefficients $a_{j,\alpha}$ are independent of r, the symbol $a(y,\eta)$ is classical.

Our edge-degenerate differential operator A can be written in the form $A = \mathrm{Op}(a)$, and hence A induces continuous operators

$$A : \mathcal{W}^s_{\mathrm{comp}}(\Omega, \mathcal{K}^{s,\gamma}(X^{\wedge})) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})) \tag{1.1.9}$$

for all $s \in \mathbb{R}$ (in this case we may also write "comp" or "loc" on both sides).

To get continuous operators in (1.1.9) for an edge-degenerate pseudo-differential operator A with local symbol $r^{-\mu}p(r,x,y,\rho,\xi,\eta)$, where p has the form (1.1.1), we apply a so-called Mellin operator convention.

Let us content ourselves with classical symbols and operators, though many constructions (not all) easily extend to the non-classical case. If $U \subseteq \mathbb{C}^n$ is an open set and E a Fréchet space, $\mathcal{A}(U,E)$ denotes the space of all holomorphic functions on U with values in E in the (Fréchet) topology of uniform convergence on compact sets. We denote by $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C}_z \times \mathbb{R}^{n+q}_{\xi,\eta})$ the space of all $h(r,x,y,z,\xi,\eta) \in \mathcal{A}(\mathbb{C}_z, S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{n+q}))$ such that

$$h(r,x,y,\beta + i\rho,\xi,\eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \Gamma_{\beta} \times \mathbb{R}^{n+q})$$

for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for arbitrary $c \leq c'$. We then employ the following result (cf. [22], Section 3.2.2).

Theorem 1.1.3 *Let $\tilde{p}(r,x,y,\tilde{\rho},\xi,\tilde{\eta}) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$ and set $p(r,x,y,\rho,\xi,\eta) := \tilde{p}(r,x,y,r\rho,\xi,r\eta)$.*

Then there exists an $\tilde{h}(r,x,y,z,\xi,\tilde{\eta}) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ such that $h(r,x,y,z,\xi,\eta) := \tilde{h}(r,x,y,z,r\eta)$ satisfies the relation

$$\mathrm{op}^{\gamma}_{M}(h)(x,y,\xi,\eta) = \mathrm{op}_r(p)(x,y,\xi,\eta) \tag{1.1.10}$$

modulo $C^{\infty}(\Sigma \times \Omega, L^{-\infty}(\mathbb{R}_+; \mathbb{R}^{n+q}_{\xi,\eta}))$ for every $\gamma \in \mathbb{R}$.

Here, and in future, the variable r that occurs as factor at covariables η as well as at ρ (in p) will be treated as a multiplication from the left (in other words, symbols h and p are interpreted as left symbols in r). Operator families in the latter theorem are interpreted in the sense $C_0^\infty(\mathbb{R}_+) \longrightarrow C^\infty(\mathbb{R}_+)$. Below we pass to several extensions by continuity to weighted Sobolev spaces.

Definition 1.1.4 *Let $M_{\mathcal{O}}^\mu(X;\mathbb{R}^q)$ defined to be the space of all operator functions $h(z,\eta) \in \mathcal{A}(\mathbb{C}, L_{\mathrm{cl}}^\mu(X;\mathbb{R}^q))$ such that*

$$h(z,\eta)|_{\Gamma_\beta \times \mathbb{R}^q} \in L_{\mathrm{cl}}^\mu(X;\Gamma_\beta \times \mathbb{R}^q)$$

for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for arbitrary $c \leq c'$.

The space $M_{\mathcal{O}}^\mu(X;\mathbb{R}^q)$ is Fréchet in a natural way. As a corollary of Theorem 1.1.3 we then obtain a (y,η)-dependent Mellin operator convention as follows.

Fix an open covering $\{U_1,...,U_N\}$ of X by coordinate neighbourhoods, a subordinate partition of unity $\{\varphi_1,...,\varphi_N\}$ and a system of functions $\{\psi_1,...,\psi_N\}$, $\psi_j \in C_0^\infty(U_j)$, such that $\psi_j = 1$ on supp φ_j for all j. Let $p_j(r,x,y,\rho,\xi,\eta)$ be symbols of the form (1.1.1), where the open sets Σ and Ω are simply assumed to be the same for all $j = 1,...,N$. Form the operator family

$$p(r,y,\rho,\eta) := \sum_{j=1}^{N} \varphi_j\{(\chi_j)_*^{-1}\mathrm{op}_x(p_j)(r,y,\rho,\eta)\}\psi_j \tag{1.1.11}$$

where $\chi_j : U_j \to \Sigma$ are fixed charts and $(\chi_j)_*^{-1}$ the pseudo-differential operator push-forward under χ_j^{-1}.

We then have $\mathrm{op}_r(p)(y,\eta) \in C^\infty(\Omega, L_{\mathrm{cl}}^\mu(X^\wedge;\mathbb{R}^q))$.

Corollary 1.1.5 *Given an operator family of the form* (1.1.11) *there is an element $\tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^\mu(X;\mathbb{R}^q))$ such that $h(r,y,z,\eta) := \tilde{h}(r,y,z,r\eta)$ satisfies the relation*

$$\mathrm{op}_M^\gamma(h)(y,\eta) = \mathrm{op}_r(p)(y,\eta) \tag{1.1.12}$$

modulo $C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q))$ for every $\gamma \in \mathbb{R}$. Moreover, forming $p_0(r,y,\rho,\eta)$ by an analogous expression as (1.1.11) *with $p_{j,0}(r,x,y,\rho,\xi,\eta) = \tilde{p}_j(0,x,y,r\rho,\xi,r\eta)$ in place of p_j and $h_0(r,y,z,\eta) = \tilde{h}(0,y,z,r\eta)$, we also have*

$$\mathrm{op}_M^\gamma(h_0)(y,\eta) = \mathrm{op}_r(p_0)(y,\eta)$$

modulo $C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q))$ for every $\gamma \in \mathbb{R}$.

Remark 1.1.6 *The operator function $a(y,\eta)$, cf.* (1.1.4)*, associated with an edge-degenerate differential operator A, can be viewed as a family of the form $r^{-\mu}\mathrm{op}_r(p)(y,\eta)$; in this case we have*

$$h(r,y,z,\eta) = \sum_{j+|\alpha|\leq\mu} a_{j,\alpha}(r,y)z^j(r\eta)^\alpha ,$$

and $A = r^{-\mu}\mathrm{Op}(\mathrm{op}_M^{\gamma-\frac{n}{2}}(h))$ for every $\gamma\in\mathbb{R}$ (first, as an operator $C_0^\infty(X^\wedge\times\Omega)\to C^\infty(X^\wedge\times\Omega)$ and then extended in the sense of (1.1.9) *).*

In the considerations below we need relation (1.1.12) only in the form $\sigma(r)\mathrm{op}_M^\gamma(h)(y,\eta) = \sigma(r)\mathrm{op}_r(p)(y,\eta)$ for some cut-off function $\sigma(r)$. In other words, we may assume $\tilde{h}(r,y,z,\tilde{\eta})$ to have bounded support in r. The Mellin pseudo-differential family $\mathrm{op}_M^\gamma(\tilde{h})(y,\tilde{\eta})$ then belongs to a global calculus on $\mathbb{R}_+$, where $\tilde{h}(r,y,z,\tilde{\eta})$ plays the role of a left symbol that can be recovered from the operator action on the weight line $\Gamma_{\frac{1}{2}-\gamma}$ in a unique way. Clearly, $\tilde{h}(r,y,\frac{1}{2}-\gamma+i\rho,\tilde{\eta})$ uniquely determines the holomorphic extension for all $z\in\mathbb{C}$.

1.2 Asymptotics in spaces on the model cone

We now construct an algebra of parameter-dependent cone operators with continuous asymptotics, with parameters $(y,\eta)\in U\times\mathbb{R}^q$, where $U\subseteq\mathbb{R}^p$ is an open set. These families will appear as operator-valued symbols of the edge operator calculus below.

Let us first recall the definition of subspaces of $\mathcal{K}^{s,\gamma}(X^\wedge)$ with continuous asymptotics. $\mathcal{V}$ denotes the system of all closed subsets $V\subset\mathbb{C}$ with the following properties:

(i) $V\cap\{z : c\leq \mathrm{Re}\, z\leq c'\}$ is compact for every $c\leq c'$,

(ii) $z_0,\, z_1\in V$ and $\mathrm{Re}\, z_0 = \mathrm{Re}\, z_1$ imply $(1-\lambda)z_0+\lambda z_1\in V$ for all $0\leq\lambda\leq 1$.

The sets $V\in\mathcal{V}$ will be interpreted as carrier of continuous asymptotics in the following sense. Given a compact set $K\in\mathcal{V}$, $K\subset\{z:\ \mathrm{Re}\, z<\frac{n+1}{2}-\gamma\}$, we form the space

(1.2.1) $$\mathcal{E}_K(X^\wedge) := \{\omega(r)\langle\zeta, r^{-z}\rangle : \zeta\in\mathcal{A}'(K,C^\infty(X))\} .$$

Here, $\omega(r)$ is a fixed cut-off function, and $\mathcal{A}'(K,E)$ for a Fréchet space E, denotes the space of all E-valued analytic functionals carried by K. The functional ζ in (1.2.1) is applied to r^{-z} with respect to the complex variable z. There is a natural isomorphism

(1.2.2) $$\mathcal{E}_K(X^\wedge)\cong\mathcal{A}'(K,C^\infty(X))$$

induced by the weighted Mellin transform $M_{\gamma'} : \mathcal{E}_K(X^\wedge)\to\mathcal{A}(\mathbb{C}\setminus K, C^\infty(X))$ for any $\gamma'\in\mathbb{R}$ such that $\sup\,\{\mathrm{Re}\ z : z\in K\} < \frac{1}{2}-\gamma'$. We have in fact, $\mathcal{A}'(K,C^\infty(X))\cong\mathcal{A}(\mathbb{C}\setminus K,C^\infty(X))/\mathcal{A}(\mathbb{C},C^\infty(X))$.

Remark 1.2.1 *If $K \subset \{z \in \mathbb{C} : \text{Re}\, z < \frac{n+1}{2} - \gamma\}$ is a compact set, there is a smooth curve $L \subset \{z \in \mathbb{C} : \text{Re}\, z < \frac{n+1}{2} - \gamma\}$ surrounding K, such that for every $\zeta \in \mathcal{A}'(K, C^\infty(X))$ there is an $f \in C^\infty(L, C^\infty(X))$ with*

$$\langle \zeta, h \rangle = \frac{1}{2\pi i} \int_L h(z) f(z) dz$$

for all $h \in \mathcal{A}(\mathbb{C})$. In fact, it suffices to choose any $\tilde{f} \in \mathcal{A}(\mathbb{C} \setminus K, C^\infty(X))$ that represents ζ and to set $f := \tilde{f}|_L$.

Remark 1.2.2 *$\mathcal{E}_K(X^\wedge)$ is a subspace of $\mathcal{K}^{\infty,\gamma}(X^\wedge)$. In fact, for every fixed $z \in \{z \in \mathbb{C} : \text{Re}\, z < \frac{n+1}{2} - \gamma\}$, we have $\omega(r) r^{-z} \in \mathcal{K}^{\infty,\gamma}(X^\wedge)$, and $z \in U \to \omega(r) r^{-z} \in \mathcal{K}^{\infty,\gamma}(X^\wedge)$ defines a continuous map for any open neighbourhood U of K, $U \subset \{z \in \mathbb{C} : \text{Re}\, z < \frac{n+1}{2} - \gamma\}$. Then, according to Remark* 1.2.1 *the function $\langle \zeta, \omega(r) r^{-z} \rangle$ can be regarded as a linear superposition of elements in $\mathcal{K}^{\infty,\gamma}(X^\wedge)$ which again belongs to this space.*

Relation (1.2.2) gives us a Fréchet space structure in the space (1.2.1). Fix now weight data (γ, Θ), $\gamma \in \mathbb{R}$, where $\Theta = (\theta, 0]$, $-\infty < \theta < 0$, is interpreted as a weight interval relative to γ where we control asymptotics, and set $\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) = \bigcap_{\epsilon > 0} \mathcal{K}^{s,\gamma-\theta-\epsilon}(X^\wedge)$ with the Fréchet topology of the projective limit (for $\Theta = (-\infty, 0]$ we define $\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) = \mathcal{K}^{s,\infty}(X^\wedge)$).

Write $u \sim v$ for $u, v \in \mathcal{E}_K(X^\wedge)$ if and only if $u - v \in \mathcal{K}^{\infty,\gamma}_\Theta(X^\wedge)$. Then the quotient space $\mathcal{E}_K(X^\wedge)/\sim$ is called a continuous asymptotic type P, associated with the weight data (γ, Θ). Let $\text{As}(X, (\gamma, \Theta))$ denote the set of all such P, and write $\pi_{\mathbb{C}} P = K \cap \{z : \text{Re}\, z > \frac{n+1}{2} - \gamma + \theta\}$. For $K \subset \{z : \text{Re}\, z \leq \dfrac{n+1}{2} - \gamma + \theta\}$ we get the trivial asymptotic type $O \in \text{As}(X, (\gamma, \Theta))$, characterised by $\pi_{\mathbb{C}} O = \emptyset$. This is coherent with the fact that, in this case, $\mathcal{E}_K(X^\wedge) \subseteq \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$.

We then define

$$\mathcal{K}^{s,\gamma}_P(X^\wedge) = \mathcal{K}^{s,\gamma}_\Theta(X^\wedge) + \mathcal{E}_K(X^\wedge) \tag{1.2.3}$$

with the Fréchet topology of the non-direct sum.

Let us briefly explain what we understand by a non-direct sum. If E_0, E_1 are Fréchet spaces, embedded in a Hausdorff topological vector space H, we form $E_0 + E_1 = \{e_0 + e_1 : e_0 \in E_0, e_1 \in E_1\}$ and endow the space with the Fréchet topology from the algebraic isomorphism $E_0 + E_1 \cong E_0 \oplus E_1 / \Delta$ for $\Delta = \{(e, -e) : e \in E_0 \cap E_1\}$. Moreover, if a Fréchet space E is a (left-) module over an algebra A, $[a]E$ for any fixed $a \in A$ denotes the closure of $\{ae : e \in E\}$ in E.

Let us extend the definition of spaces with continuous asymptotics to the case $\Theta = (-\infty, 0]$, where instead of a compact set we admit an arbitrary set $V \in \mathcal{V}$, $V \subset \{z : \text{Re}\, z < \frac{n+1}{2} - \gamma\}$. We simply choose a sequence $(\theta_k)_{k \in \mathbb{N}}$, $\theta_{k+1} < \theta_k < 0$, $\theta_k \to -\infty$ as $k \to \infty$, and form the sequence of compact sets

$$V_k = V \cap \{z : \text{Re}\, z \geq \frac{n+1}{2} - \gamma + \theta_{k+1}\}\ , \quad k \in \mathbb{N}\ .$$

Then V_k induces a continuous asymptotic type P_k connected with the weight data (γ, Θ_k), $\Theta_k = (\theta_k, 0]$, and it is obvious that we get a chain of continuous embeddings $\mathcal{K}^{s,\gamma}_{P_{k+1}}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}_{P_k}(X^\wedge)$. We then set

$$\mathcal{K}^{s,\gamma}_{P}(X^\wedge) = \varprojlim_{k\in\mathbb{N}} \mathcal{K}^{s,\gamma}_{P_k}(X^\wedge) \ ,$$

and call P a continuous asymptotic type associated with the weight data $(\gamma, (-\infty, 0])$. In this case we set $V = \pi_{\mathbb{C}} P$. Analogously to the above notation we denote by $\mathrm{As}(X, (\gamma, (-\infty, 0]))$ the set of all P arising by this construction, using the sets $V \in \mathcal{V}$ (clearly, in this case we have a one-to-one correspondence $\{V \in \mathcal{V} \ : \ V \subset \{z \ : \ \mathrm{Re}\, z < \frac{n+1}{2} - \gamma\}\} \leftrightarrow \mathrm{As}(X, (\gamma, (-\infty, 0]))$).

Set

$$\mathcal{S}^{\gamma}_{P}(X^\wedge) := [\omega]\mathcal{K}^{\infty,\gamma}_{P}(X^\wedge) + [1-\omega]\mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X)) \qquad (1.2.4)$$

for a cut-off function $\omega(r)$. The space (1.2.4) is Fréchet in a natural way. It can be written in the form

$$\mathcal{S}^{\gamma}_{P}(X^\wedge) = \varprojlim_{k\in\mathbb{N}} E^k$$

for a chain of Hilbert spaces $(E^k)_{k\in\mathbb{N}}$ with continuous embeddings $E^{k+1} \hookrightarrow E^k \hookrightarrow \cdots \hookrightarrow E^0 := \mathcal{K}^{s,\gamma}(X^\wedge)$, such that $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ from the space E^0 induces strongly continuous groups of isomorphisms on E^k for every k.

Remark 1.2.3 *For every $P \in \mathrm{As}(\mathrm{X}, (\gamma, \Theta))$ we can pass to the "complex conjugate" $\overline{P} \in \mathrm{As}(\mathrm{X}, (\gamma, \Theta))$ by replacing the set K in relation* (1.2.3) *by $\overline{K} = \{\overline{z} \ : \ z \in K\}$ when Θ is finite; for infinite Θ we simply replace $V \in \mathcal{V}$ by $\overline{V}$, using the one-to-one correspondence of $\mathrm{As}(\mathrm{X}, (\gamma, \Theta))$ with a corresponding subset of $\mathcal{V}$. Then the map $u \to \overline{u}$ gives us antilinear maps $\mathcal{K}^{s,\gamma}_{P}(X^\wedge) \to \mathcal{K}^{s,\gamma}_{\overline{P}}(X^\wedge)$ and $\mathcal{S}^{\gamma}_{P}(X^\wedge) \to \mathcal{S}^{\gamma}_{\overline{P}}(X^\wedge)$.*

In the sequel we will have symbol spaces referring to direct sums $E = H \oplus \mathbb{C}^j$ where H is one of the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ or $\mathcal{S}^{\gamma}_{P}(X^\wedge)$ with the above mentioned group action. On $H \oplus \mathbb{C}^j$ we then take the group action $\mathrm{diag}\{\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}, \mathrm{id}_{\mathbb{C}^j}\}$.

In the following definition we set $\boldsymbol{g} = (\gamma, \sigma, \Theta)$ for $\gamma, \sigma \in \mathbb{R}$, $\Theta = (\theta, 0]$, $-\infty \le \theta < 0$, and $\boldsymbol{w} = (e, f; j_-, j_+)$, where e, f and j_-, j_+ play the role of fibre dimensions of vector bundles below.

Definition 1.2.4 *$R^{\mu}_{G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$, $\mu \in \mathbb{R}$, denotes the set of all*

$$g(y,\eta) \in \bigcap_{s\in\mathbb{R}} C^\infty(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge, \mathbb{C}^e) \oplus \mathbb{C}^{j_-}, \mathcal{K}^{\infty,\sigma}(X^\wedge, \mathbb{C}^f) \oplus \mathbb{C}^{j_+}))$$

such that

$$g_0(y,\eta) = \mathrm{diag}(\mathrm{id}, \langle\eta\rangle^{-\frac{n+1}{2}}) g(y,\eta) \mathrm{diag}(\mathrm{id}, \langle\eta\rangle^{\frac{n+1}{2}})$$

satisfy the relations

$$g_0(y,\eta) \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \mathcal{S}) \ , \quad g_0^*(y,\eta) \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^q; \tilde{E}, \tilde{\mathcal{S}})$$

for

$$E := \mathcal{K}^{s,\gamma}(X^\wedge, \mathbb{C}^e) \oplus \mathbb{C}^{j_-} \ , \quad \mathcal{S} := \mathcal{S}^{\sigma}_P(X^\wedge, \mathbb{C}^f) \oplus \mathbb{C}^{j_+} \ , \tag{1.2.5}$$

$$\tilde{E} := \mathcal{K}^{s,-\sigma}(X^\wedge, \mathbb{C}^f) \oplus \mathbb{C}^{j_+} \ , \quad \tilde{\mathcal{S}} := \mathcal{S}^{-\gamma}_Q(X^\wedge, \mathbb{C}^e) \oplus \mathbb{C}^{j_-} \tag{1.2.6}$$

for all $s \in \mathbb{R}$, *where* $P \in \mathrm{As}(X, (\sigma, \Theta))$ *and* $Q \in \mathrm{As}(X, (-\gamma, \Theta))$ *are asymptotic types; here* $*$ *denotes the* (y,η)*-wise formal adjoint in the sense*

$$(g_0 u, v)_{\mathcal{K}^{0,0}(X^\wedge, \mathbb{C}^f) \oplus \mathbb{C}^{j_+}} = (u, g_0^* v)_{\mathcal{K}^{0,0}(X^\wedge, \mathbb{C}^e) \oplus \mathbb{C}^{j_-}}$$

for all $u \in C_0^\infty(X^\wedge, \mathbb{C}^e) \oplus \mathbb{C}^{j_-}$, $v \in C_0^\infty(X^\wedge, \mathbb{C}^f) \oplus \mathbb{C}^{j_+}$.

The elements of $R^{\mu}_G(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ will be called Green symbols of order μ. In particular, if we write $g = (g_{ij})_{i,j=1,2}$, the element g_{21} will also be called a trace symbol and g_{12} a potential symbol. Note that the lower right corner g_{22} is a classical symbol of order μ on U with covariables $\eta \in \mathbb{R}^q$ in the standard sense.

The (order reduced) elements $g_0(y,\eta)$ in Definition 1.2.4 as classical operator-valued symbols of order μ have a homogeneous principal (so-called edge-) symbol of $g_0(y,\eta)$ of order μ, denoted by $\sigma^{\mu}_{\wedge}(g_0)(y,\eta)$ for $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$. We then define

$$\sigma^{\mu}_{\wedge}(g)(y,\eta) := \begin{pmatrix} 1 & 0 \\ 0 & |\eta|^{\frac{n+1}{2}} \end{pmatrix} \sigma^{\mu}_{\wedge}(g_0)(y,\eta) \begin{pmatrix} 1 & 0 \\ 0 & |\eta|^{-\frac{n+1}{2}} \end{pmatrix}, \tag{1.2.7}$$

called the homogeneous principal edge symbol of $g(y,\eta)$ of order μ (in the sense of DN-orders; "DN" stands for Douglis-Nirenberg). We then have

$$\sigma^{\mu}_{\wedge}(g)(y,\delta\eta) = \delta^{\mu} \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \delta^{\frac{n+1}{2}} \end{pmatrix} \sigma^{\mu}_{\wedge}(g)(y,\eta) \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \delta^{\frac{n+1}{2}} \end{pmatrix}^{-1}$$

for all $\delta \in \mathbb{R}_+$, $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$.

Set $R^{\mu}_G(U \times \mathbb{R}^q, \boldsymbol{g}; (e,f)) = \{g_{11} : g \in R^{\mu}_G(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})\}$. For $e = f = 1$ we simply omit (e,f) in the notation. For $\boldsymbol{w} = (1,1;0,0)$ we drop it at all.

Let $R^{\mu}_G(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_{P,Q}$ denote the subspace of all elements of $R^{\mu}_G(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ with fixed P and Q in formulas (1.2.6) and (1.2.5).

Remark 1.2.5 *The space* $R^{\mu}_G(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_{P,Q}$ *is Fréchet in a natural way.*

To illustrate this, for simplicity, assume $j_- = j_+ = 0$. Then we have two systems of linear maps

$$R^\mu_G(U\times\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_{P,Q} \longrightarrow S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge,\mathbb{C}^e), \mathcal{S}^\sigma_P(X^\wedge,\mathbb{C}^f)) \tag{1.2.8}$$

and

$$R^\mu_G(U\times\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_{P,Q} \longrightarrow S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q; \mathcal{K}^{s,-\sigma}(X^\wedge,\mathbb{C}^f), \mathcal{S}^{-\gamma}_{\overline{Q}}(X^\wedge,\mathbb{C}^e)) \tag{1.2.9}$$

with s running over $\mathbb{N}$. While (1.2.8) is immediate from Definition 1.2.4, we define (1.2.9) by the composition $g \to g_0 \to g_0^* \to l$ where l is given by $lv := \overline{g_0^*\overline{v}}$.

The space $R^\mu_G(U\times\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_{P,Q}$ is then endowed with the topology of the projective limit with respect to the mappings (1.2.8), (1.2.9) for all $s\in\mathbb{N}$.

Notice that we have a continuous embedding

$$R^\mu_G(U\times\mathbb{R}^q, \boldsymbol{g}, \boldsymbol{w})_{P,Q} \subseteq R^\mu_G(U\times\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_{\tilde{P},\tilde{Q}} \text{ whenever } \pi_{\mathbb{C}}P\subseteq\pi_{\mathbb{C}}\tilde{P},\ \pi_{\mathbb{C}}Q\subseteq\pi_{\mathbb{C}}\tilde{Q}\ . \tag{1.2.10}$$

Let us set

$$R^\mu_G(U\times\mathbb{R}^q, \boldsymbol{g}, \boldsymbol{w})_{\mathcal{O}} := R^\mu_G(U\times\mathbb{R}^q, \boldsymbol{g}, \boldsymbol{w})_{O,O} \tag{1.2.11}$$

for $\boldsymbol{g} = (\gamma,\sigma,\Theta)$, where $O\in \mathrm{As}(X,(\delta,\Theta))$ denotes the trivial asymptotic types for $\delta=\gamma$ and $\delta=\sigma$, respectively. Notice that for the special case $\Theta=(-\infty,0]$ the weight data $\boldsymbol{g}$ in (1.2.11) become irrelevant insofar the spaces are isomorphic to analogous spaces with arbitrary other weights. Parallel to the spaces with (continuous) asymptotics we now define spaces of Mellin symbols that also reflect asymptotic information.

Let $V\in\mathcal{V}$, fix finite reals $c_1<c_2$ and set $K := V\cap S(c_1-\epsilon, c_2+\epsilon)$ for any $\epsilon>0$, $S(B)=\cup\{\Gamma_\beta : \beta\in B\}$ for any $B\subseteq\mathbb{C}$. Similarly to (1.2.1) we form the space

$$\mathcal{E}_K(L^{-\infty}(X)) := \{\omega(r)\langle\zeta, r^{-z}\rangle : \zeta\in\mathcal{A}'(K, L^{-\infty}(X))\} \tag{1.2.12}$$

which is again a Fréchet space and isomorphic to $\mathcal{A}'(K,L^{-\infty}(X))$. If $\gamma'\in\mathbb{R}$ is chosen in such a way that $\{\mathrm{Re}\,z : z\in K\} < 1/2-\gamma'$, the weighted Mellin transform $M_{\gamma'}$, induces an isomorphism of $\mathcal{E}_K(L^{-\infty}(X))$ to a closed subspace of $\mathcal{A}(\mathbb{C}\backslash K, L^{-\infty}(X))$. Let $\mathcal{A}^{-\infty}(S(c_1,c_2), L^{-\infty}(X))$ denote the set of all $h\in\mathcal{A}(S(c_1,c_2),L^{-\infty}(X))$ such that $h|_{\Gamma_\beta}\in\mathcal{S}(\Gamma_\beta, L^{-\infty}(X))$ for every $c_1<\beta<c_2$ unifomly in compact intervals. The latter space is Fréchet, and we can form the space

$$\mathcal{A}^{-\infty}(S(c_1,c_2), L^{-\infty}(X)) + M_{\gamma'}(\mathcal{E}_K(L^{-\infty}(X))) \tag{1.2.13}$$

as a subspace of $\mathcal{A}(S(c_1,c_2)\setminus K, L^{-\infty}(X))$ in the topology of the non-direct sum. The space (1.2.13) is independent of ϵ that is involved in the choice of K. Notice that for $\tilde{c_1}<c_1$, $\tilde{c_2}>c_2$ the spaces (1.2.13) associated with $(\tilde{c_1},\tilde{c_2})$ are continuously embedded into the ones associated with (c_1,c_2). We now define $\mathcal{M}^{-\infty}_V(X)$ to be the projective limit of the spaces (1.2.13) for $c_1\to-\infty$, $c_2\to+\infty$. The space $\mathcal{M}^{-\infty}_V(X)$ is Fréchet (it can be proved that its Fréchet topology is nuclear).

We identify the set $V \in \mathcal{V}$ with a so-called continuous asymptotic type (with coefficients in $L^{-\infty}(X)$) R of Mellin symbols and set $\mathcal{M}_R^{-\infty}(X) := \mathcal{M}_V^{-\infty}(X)$. The set of all such R, will be denoted by $\boldsymbol{As}(X)$. In this notation we keep in mind the $L^{-\infty}(X)$-coefficients arising in the spaces of analytic functionals (1.2.13). If R is associated with V we also write $\pi_{\mathbb{C}}(R) = V$.

Let $M_{\mathcal{O}}^{\mu}(X)$, $\mu \in \mathbb{R}$, defined to be the subspace of all $h(z) \in \mathcal{A}(\mathbb{C}, L_{\mathrm{cl}}^{\mu}(X))$ such that $h(z)|_{\Gamma_\beta} \in L_{\mathrm{cl}}^{\mu}(X; \Gamma_\beta)$ for each $\beta \in \mathbb{R}$ and uniformly in compact β-intervals. We consider $M_{\mathcal{O}}^{\mu}(X)$ with its canonical Fréchet topology and set

$$M_R^{\mu}(X) := M_{\mathcal{O}}^{\mu}(X) + M_R^{-\infty}(X) \ ,$$

endowed with the Fréchet topology of the non-direct sum.

Remark 1.2.6 *Given two sets V_1, $V_2 \in \mathcal{V}$ we can form $V_1 + V_2 := (V_1 \cup V_2)^I$, where*

$$V^I := \{(1-\lambda)z_0 + \lambda z_1 \ : \ z_0, z_1 \in V, \ \mathrm{Re}\ z_0 = \ \mathrm{Re}\ z_1, 0 \le \lambda \le 1\} \ .$$

For the elements R_1, R_2 and R in $\boldsymbol{As}(X)$ associated with V_1, V_2 and $V_1 + V_2$, respectively, we then write $R = R_1 + R_2$. In this case we have

$$M_{R_1+R_2}^{\mu}(X) = M_{R_1}^{\mu}(X) + M_{R_2}^{\mu}(X)$$

as a non-direct sum of Fréchet spaces.

A standard functional analytic argument then gives us

$$C^\infty(U, M_{R_1+R_2}^{\mu}(X)) = C^\infty(U, M_{R_1}^{\mu}(X)) + C^\infty(U, M_{R_2}^{\mu}(X))$$

for an open set $U \subseteq \mathbb{R}^p$. With elements $f(z) \in M_R^{\mu}(X)$ we can associate Mellin pseudo-differential operators

$$\mathrm{op}_M^{\beta}(f) : C_0^\infty(X^\wedge) \to C^\infty(X^\wedge)$$

whenever $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{1}{2}-\beta} = \emptyset$. In that case, setting $\beta = \gamma - \frac{n}{2}$ for $n = \dim X$, we get continuous operators

$$\omega \mathrm{op}_M^{\gamma - \frac{n}{2}}(f)\tilde{\omega} : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma}(X^\wedge)$$

as well as

$$\omega \mathrm{op}_M^{\gamma - \frac{n}{2}}(f)\tilde{\omega} : \mathcal{K}_P^{s,\gamma}(X^\wedge) \to \mathcal{K}_Q^{s-\mu,\gamma}(X^\wedge)$$

for every $P \in \mathrm{As}(X, (\gamma, \Theta))$ with some resulting $Q \in \mathrm{As}(X, (\gamma, \Theta))$ for any choice of the weight interval Θ and cut-off funtions ω, $\tilde{\omega}$.

1.3 The edge symbolic algebra

Consider the operator families $\mathrm{op}_r(p)(y,\eta)$ and $\mathrm{op}_M^\beta(h)(y,\eta)$ from Corollary 1.1.5, fix cut-off functions $\omega_i(r)$, $i=1,2,3$, such that $\omega_2=1$ in a neighbourhood of supp ω_1, and $\omega_1=1$ in a neighbourhood of supp ω_3.

Choose any strictly positive function $\eta\to[\eta]$ in $C^\infty(\mathbb{R}^q)$ such that $[\eta]=|\eta|$ for $|\eta|\geq c$ for some constant $c>0$. Set

$$a_M(y,\eta)=\omega_1(r[\eta])r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\omega_2(r[\eta])\ , \tag{1.3.1}$$

$$a_F(y,\eta)=(1-\omega_1(r[\eta]))r^{-\mu}\mathrm{op}_r(p)(y,\eta)(1-\omega_3(r[\eta]))\ . \tag{1.3.2}$$

By using Corollary 1.1.5, it is then easy to verify that

$$a_M(y,\eta)+a_F(y,\eta)=r^{-\mu}\mathrm{op}_r(p)(y,\eta)\qquad \mathrm{mod}\ C^\infty(\Omega,L^{-\infty}(X^\wedge;\mathbb{R}^q))\ .$$

For arbitrary cut-off functions σ_1, σ_2, with σ_2 equals 1 in a neighbourhood of the support of σ_1, set

$$a_\psi(y,\eta)=\sigma_1(r)\{a_M(y,\eta)+a_F(y,\eta)\}\sigma_2(r). \tag{1.3.3}$$

In [4], Section 9.2.3 it is shown that the use of parameter-dependent cut-off functions in the expressions (1.3.1) and (1.3.2) gives

$$a_\psi(y,\eta)\in S^\mu(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))\qquad\text{for all } s\in\mathbb{R}.$$

The following remark establishes an alternative representation for $a_\psi(y,\eta)$, the proof can be found in [7], Propositions A.4 and A.8.

Remark 1.3.1 *Given $a_\psi(y,\eta)$ in the form* (1.3.3) *there exists an*

$$\tilde{f}(r,y,z,\tilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times\Omega,M_{\mathcal{O}}^\mu(X;\mathbb{R}^q_{\tilde{\eta}}))$$

such that for $f(r,y,z,\eta)=\tilde{f}(r,y,z,r\eta)$ we have

$$a_\psi(y,\eta)=\sigma_1(r)r^{-\mu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(y,\eta)\sigma_2(r)+g_0(y,\eta) \tag{1.3.4}$$

for a certain $g_0(y,\eta)\in R_G^\mu(\Omega\times\mathbb{R}^q,\boldsymbol{g}_\infty)_{\mathcal{O}}$, $\boldsymbol{g}_\infty:=(\gamma,\gamma-\mu,(-\infty,0])$ (recall that the latter class of flat Green symbols is independent of the weight in $\boldsymbol{g}_\infty$). Conversely, any operator family (1.3.4) *can be rewritten as* (1.3.3) mod $R_G^\mu(\Omega\times\mathbb{R}^q,\boldsymbol{g}_\infty)_{\mathcal{O}}$.

In [7], Proposition A.13, there is proved the independence of relation (1.3.4) of the cut-off functions σ_i, $i = 1, 2$, and the equality

$$\sigma_1(r) r^{-\mu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(f)(y, \eta) \sigma_2(r) = r^{-\mu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(f_1)(y, \eta) + g_1(y, \eta) \tag{1.3.5}$$

for $f_1(r, y, z, \eta) = \sigma_1(r) f(r, y, z, \eta)$ and $g_1(y, \eta) = \sigma_1(r) r^{-\mu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(f)(y, \eta)(\sigma_2(r) - 1) \in R_G^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}_\infty)_{\mathcal{O}}$.

Remark 1.3.2 *The operator*

$$\mathrm{Op}_y(a_\psi) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge)) \to \mathcal{W}^s_{\mathrm{loc}}(\Omega, \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

belongs to $L^\mu_{\mathrm{cl}}(X^\wedge \times \Omega)$ *and has (up to the cut-off functions* σ_1, σ_2*) the above-mentioned system* $\{r^{-\mu} p_j(r, x, y, \rho, \xi, \eta)\}_{j=1,\cdots,N}$ *as local amplitude functions (where we assume that suitable compatibility conditions in the intersections of coordinate neighbourhoods of* X *are satisfied, see [20], Definition 4 of Section 3.3).*

With the notation of Corollary 1.1.5, we set, for $(y, \eta) \in U \times (\mathbb{R}^q \setminus \{0\})$,

$$\sigma^\mu_\wedge(a_M)(y, \eta) = \omega_1(r|\eta|) r^{-\mu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(h_0)(y, \eta) \omega_2(r|\eta|), \tag{1.3.6}$$

$$\sigma^\mu_\wedge(a_F)(y, \eta) = (1 - \omega_1(r|\eta|)) r^{-\mu} \mathrm{op}_r(p_0)(y, \eta)(1 - \omega_2(r|\eta|)). \tag{1.3.7}$$

The edge calculus contains another kind of operator-valued symbols, namely the smoothing Mellin symbols. In this case we set

$$\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta) \quad \text{for } \Theta = (-(k+1), 0]$$

with reals γ, $\mu \in \mathbb{R}$ and $k \in \mathbb{N}$.

Given any

$$f(y, z) \in C^\infty(U, M_R^{-\infty}(X)) \ ,$$

$R \in \boldsymbol{As}(X)$ and a weight $\delta \in \mathbb{R}$ where

$$\gamma - j \le \delta \le \gamma \text{ for some } j \in \mathbb{N} \ , 0 \le j \le k \ , \pi_{\mathbb{C}} R \cap \Gamma_{\frac{n+1}{2} - \delta} = \emptyset \ , \tag{1.3.8}$$

we form

$$m(y, \eta) := \omega_1(r[\eta]) r^{-\mu + j} \mathrm{op}_M^{\delta - \frac{n}{2}}(f)(y) \eta^\alpha \omega_2(r[\eta]) \ . \tag{1.3.9}$$

Remark 1.3.3 *Operator families of the form (1.3.9) belong to* $S^{\mu - j + |\alpha|}_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$ *for* $E = \mathcal{K}^{s,\gamma}(X^\wedge)$, $\tilde{E} = \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)$ *as well as for* $E = \mathcal{K}^{s,\gamma}_P(X^\wedge)$, $\tilde{E} = \mathcal{K}^{\infty,\gamma-\mu}_Q(X^\wedge)$ *for all* $s \in \mathbb{R}$ *and every* $P \in \mathrm{As}(X, (\gamma, \Theta))$ *with some resulting* $Q \in \mathrm{As}(X, (\gamma - \mu, \Theta))$ *that depends on* P, f *and on the weight* δ.

Operator valued symbols of the form (1.3.9) have been studied in [20], Section 3.3.3., and it is known that when we form

$$\tilde{m}(y,\eta)=\tilde{\omega}_1(r[\eta])r^{-\mu+j}\mathrm{op}_M^{\tilde{\delta}-\frac{n}{2}}(f)(y)\eta^{\alpha}\tilde{\omega}_2(r[\eta])$$

with the same $f(y,z)$ as before but other $\tilde{\omega}_i$, $i=1,2$, and $\tilde{\delta}$, satisfying analogous conditions as (1.3.8), we have $m(y,\eta)-\tilde{m}(y,\eta)\in R_G^{\mu-j+|\alpha|}(U\times\mathbb{R}^q,\boldsymbol{g})$. In the latter notation the dimension data $\boldsymbol{w}$ are $(1,1;0,0)$.

Note that, by Remark 1.2.6, we can find, for every $R\in\boldsymbol{As}(X)$ and $f(y,z)\in C^\infty(U,M_R^{-\infty}(X))$, and every decomposition $R=R^1+R^2$ for $R^1,R^2\in\boldsymbol{As}(X)$, elements $l_i(y,z)\in C^\infty(U,M_{R^i}^{-\infty}(X))$, $i=1,2$, such that $f=l_1+l_2$. We apply this to f in (1.3.9) for the case $j>0$ and choose an arbitrary decomposition $R=R_\beta+R_{\tilde{\beta}}\in\boldsymbol{As}(X)$ satisfying $\pi_{\mathbb{C}}R_\beta\cap\Gamma_{\frac{n+1}{2}-\beta}=\pi_{\mathbb{C}}R_{\tilde{\beta}}\cap\Gamma_{\frac{n+1}{2}-\tilde{\beta}}=\emptyset$ for reals $\beta\neq\tilde{\beta}$, $\gamma-j\leq\beta,\tilde{\beta}\leq\gamma$, and write $f(y,z)=l_\beta(y,z)+l_{\tilde{\beta}}(y,z)$ for corresponding l_β and $l_{\tilde{\beta}}$. Then, if we set

$$n(y,\eta)=\omega_1(r[\eta])r^{-\mu+j}\{\mathrm{op}_M^{\beta-\frac{n}{2}}(l_\beta)(y)+\mathrm{op}_M^{\tilde{\beta}-\frac{n}{2}}(l_{\tilde{\beta}})(y)\}\eta^{\alpha}\omega_2(r[\eta])\ ,$$

we have

$$m(y,\eta)=n(y,\eta)\mod R_G^{\mu-j+|\alpha|}(U\times\mathbb{R}^q,\boldsymbol{g})_{P,Q}\tag{1.3.10}$$

for certain $P\in\mathrm{As}(X,(\gamma-\mu,\Theta))$, $Q\in\mathrm{As}(X,(-\gamma,\Theta))$, depending on R, β, $\tilde{\beta}$. Because of the semi-ordering of the spaces of Green symbols in the sense of relation (1.2.10) we can choose P and Q in (1.3.10) in a suitable way, such that (1.3.10) holds for every decomposition of f into $l_\beta+l_{\tilde{\beta}}$ for arbitrary β, $\tilde{\beta}$ satisfying the above-mentioned relations. Previous considerations also say that one can fix P and Q in (1.3.10) not depending on $0\leq j\leq k$. This is also true when, instead of f, in (1.3.9) appears a finite sum over $|\alpha|\leq j$ of Mellin symbols $f_{j\alpha}(y,z)\in C^\infty(U,M_{R_{j\alpha}}^{-\infty}(X))$ $(\pi_{\mathbb{C}}R_{j\alpha}\cap\Gamma_{\frac{n+1}{2}-\delta}=\emptyset)$.

Let $R_{M+G}^{\mu}(U\times\mathbb{R}^q,\boldsymbol{g})$ denote the space of all operator functions of the form $m(y,\eta)+g(y,\eta)$, for arbitrary $g(y,\eta)\in R_G^{\mu}(U\times\mathbb{R}^q,\boldsymbol{g})$, where

$$m(y,\eta)=\omega_1(r[\eta])r^{-\mu}\sum_{j=0}^{k}r^j\sum_{|\alpha|\leq j}\{\mathrm{op}_M^{\beta_{j\alpha}-\frac{n}{2}}(l_{\beta_{j\alpha}})(y)+\mathrm{op}_M^{\tilde{\beta}_{j\alpha}-\frac{n}{2}}(l_{\tilde{\beta}_{j\alpha}})(y)\}\eta^{\alpha}\omega_2(r[\eta])\tag{1.3.11}$$

with given Mellin symbols

$$l_{\beta_{j\alpha}}(y,z)\in C^\infty(U,M_{R_{j\alpha}}^{-\infty}(X))\ ,\quad l_{\tilde{\beta}_{j\alpha}}(y,z)\in C^\infty(U,M_{\tilde{R}_{j\alpha}}^{-\infty}(X))\tag{1.3.12}$$

for $R_{j\alpha}$, $\tilde{R}_{j\alpha}\in\boldsymbol{As}(X)$, $\gamma-j\leq\beta_{j\alpha}$, $\tilde{\beta}_{j\alpha}\leq\gamma$, $\beta_{j\alpha}\neq\tilde{\beta}_{j\alpha}$ for $j>0$, and

$$\pi_{\mathbb{C}}R_{j\alpha}\cap\Gamma_{\frac{n+1}{2}-\beta_{j\alpha}}=\pi_{\mathbb{C}}\tilde{R}_{j\alpha}\cap\Gamma_{\frac{n+1}{2}-\tilde{\beta}_{j\alpha}}=\emptyset\quad\text{for all } j,\alpha\ .\tag{1.3.13}$$

By virtue of Remark 1.3.3, operator families of the form (1.3.11) represent classical symbols of order μ, with spaces E and $\tilde{E}$ as in Remark 1.3.3.

Let us set

$$\sigma_M^{\mu-j}(m+g)(y,\eta,z) = \sum_{|\alpha|\leq j} f_{j\alpha}(y,z)\eta^\alpha \ ,$$

called the conormal symbol of $(m+g)(y,\eta)$ of order $\mu - j$, $j = 0, ..., k$. We then have

$$\sigma_M^{\mu-j}(m+g)(y,\eta,z) = 0 \text{ for } 0 \leq j \leq k \Leftrightarrow (m+g)(y,\eta) \in R_G^\mu(U \times \mathbb{R}^q, \boldsymbol{g}) \ .$$

Using the fact that elements $(m+g)(y,\eta) \in R^\mu_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g})$ are classical symbols of order μ, define the homogeneous principal (so-called edge) symbol of order μ by

$$\sigma_\wedge^\mu(m+g)(y,\eta) := \sigma_\wedge^\mu(m)(y,\eta) + \sigma_\wedge^\mu(g)(y,\eta), \tag{1.3.14}$$

where $\sigma_\wedge^\mu(g)(y,\eta)$ is given by (1.2.7) and (in the notation of formula (1.3.11))

$$\sigma_\wedge^\mu(m)(y,\eta) = \omega_1(r|\eta|)r^{-\mu}\sum_{j=0}^k r^j \sum_{|\alpha|=j}\{\mathrm{op}_M^{\beta_{j\alpha}-\frac{n}{2}}(l_{\beta_{j\alpha}})(y) + \mathrm{op}_M^{\tilde{\beta}_{j\alpha}-\frac{n}{2}}(l_{\tilde{\beta}_{j\alpha}})(y)\}\eta^\alpha\omega_2(r|\eta|).$$

We now want to define a system of Fréchet subspaces of $R^\mu_{M+G}(U\times\mathbb{R}^q,\boldsymbol{g})$. For this purpose let us consider a fixed sequence $R = (R_{j\alpha}, \tilde{R}_{j\alpha})_{0\leq|\alpha|\leq j, 0\leq j\leq k}$ with $R_{j\alpha}, \tilde{R}_{j\alpha} \in \boldsymbol{As}(X)$ such that $R_{00} = \tilde{R}_{00}$ and satisfying conditions (1.3.13) with

$$\beta_{j\alpha} = \tilde{\beta}_{j\alpha} = \gamma \quad \text{for } j = 0, \qquad \beta_{j\alpha} = \gamma - \frac{1}{3}\ , \ \tilde{\beta}_{j\alpha} = \gamma - \frac{2}{3} \quad \text{for } j > 0 \text{ for all } |\alpha| \leq j\ . \tag{1.3.15}$$

Choosing any $l'_{\beta_{j\alpha}}(y,z) \in C^\infty(U, M^{-\infty}_{R_{j\alpha}}(X))$ and $l'_{\tilde{\beta}_{j\alpha}}(y,z) \in C^\infty(U, M^{-\infty}_{\tilde{R}_{j\alpha}}(X))$ satisfying

$$f_{j\alpha}(y,z) = l_{\beta_{j\alpha}}(y,z) + l_{\tilde{\beta}_{j\alpha}}(y,z) = l'_{\beta_{j\alpha}}(y,z) + l'_{\tilde{\beta}_{j\alpha}}(y,z) \tag{1.3.16}$$

in the space $C^\infty(U, M^{-\infty}_{R_{j\alpha}+\tilde{R}_{j\alpha}}(X))$, and forming a family of operators $m'(y,\eta)$ as in (1.3.11) with $l'_{\beta_{j\alpha}}$ and $l'_{\tilde{\beta}_{j\alpha}}$ instead of $l_{\beta_{j\alpha}}$ and $l_{\tilde{\beta}_{j\alpha}}$, respectively, we get, for suitable asymptotic types $P \in \mathrm{As}(X,(\gamma,\Theta))$ and $Q \in \mathrm{As}(X,(-\gamma+\mu,\Theta))$

$$m(y,\eta) - m'(y,\eta) \in R^\mu_G(U\times\mathbb{R}^q,\boldsymbol{g})_{P,Q}. \tag{1.3.17}$$

We call P and Q compatible to R if their carriers are chosen so large (which is always possible) that (1.3.17) holds for every $l_{\beta_{j\alpha}}$, $l_{\tilde{\beta}_{j\alpha}}$, $l'_{\beta_{j\alpha}}$, $l'_{\tilde{\beta}_{j\alpha}}$ satisfying relation (1.3.16).

Let $\boldsymbol{As}_{M+G}(X,\boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$ denote the set of all sequences $S = (R; P, Q)$, for $R = (R_{j\alpha}, \tilde{R}_{j\alpha})_{0\leq|\alpha|\leq j, 0\leq j\leq k}$ with $R_{j\alpha}$, $\tilde{R}_{j\alpha} \in \boldsymbol{As}(X)$ being as before and $P \in \mathrm{As}(X,(\gamma,\Theta))$, $Q \in \mathrm{As}(X,(-\gamma+\mu,\Theta))$ compatible to R.

We then denote by $R^\mu_{M+G}(U\times\mathbb{R}^q,\boldsymbol{g})_S$, for $S \in \boldsymbol{As}_{M+G}(X,\boldsymbol{g})$, the subspace of all elements of $R^\mu_{M+G}(U\times\mathbb{R}^q,\boldsymbol{g})$ of the form $m(y,\eta)+g(y,\eta)$ with $m(y,\eta)$ being given of the form (1.3.11) with the weights defined by (1.3.15).

Remark 1.3.4 *For every fixed $S \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g})$, the space $R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g})_S$ is a Fréchet space in a natural way.*

In fact, let T denote the subspace of all $a(y, \eta) \in R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g})_S$ such that the coefficients $f_{j\alpha}(y, z)$ of $\sigma^{\mu-j}_M(a)(y, \eta, z)$ belong to $C^\infty(U, M^{-\infty}_{R_{j\alpha}}(X))$ for all $|\alpha| \leq j$, $j = 0, ..., k$. In a similar manner we define $\tilde{T}$ by requiring $f_{j\alpha}(y, z) \in C^\infty(U, M^{-\infty}_{\tilde{R}_{j\alpha}}(X))$ for all j, α (here, by definition $R_{00} = \tilde{R}_{00}$). We introduce Fréchet topologies in T and $\tilde{T}$ and then set $R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g})_S = T + \tilde{T}$ in the topology of the non-direct sum. To apply the general notion of a non-direct sum of Fréchet spaces, we have to choose a Hausdorff topological vector space H containing the summands as subspaces; in our case we can set $H = S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge))$ for any real s. Let us consider T, for $\tilde{T}$ we can proceed in an analogous manner. The space T is isomorphic to

$$\left\{ \bigoplus_{j=0}^{k} \bigoplus_{|\alpha| \leq j} C^\infty(U, M^{-\infty}_{R_{j\alpha}}(X)) \right\} \oplus R^{\mu}_{G}(U \times \mathbb{R}^q, \boldsymbol{g})_{P,Q}, \tag{1.3.18}$$

since $\sigma_M : a \longrightarrow \sigma_m(a) := \{f_{j\alpha}\}_{|\alpha| \leq j, j=0,...,k}$ defines a surjective map of T to the space in the brackets $\{...\}$ in (1.3.18), where $\ker \sigma_M = R^{\mu}_{G}(U \times \mathbb{R}^q, \boldsymbol{g})_{P,Q}$. Moreover, σ_M has a right inverse, namely

$$\sigma_M(a) \longrightarrow \omega_1(r[\eta]) r^{-\mu} \sum_{j=0}^{k} r^j \sum_{|\alpha| \leq j} \mathrm{op}_M^{\beta_{j\alpha} - \frac{n}{2}}(f_{j\alpha})(y) \eta^\alpha \omega_2(r[\eta])$$

for any fixed choice of cut-off functions ω_1, ω_2 and weights $\beta_{j\alpha}$ as in (1.3.15). The summands in (1.3.18) are Fréchet spaces, hence T itself becomes a Fréchet space.

It can be proved (by using a Cousin problem argument) that the space $R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g})_S$ with its Fréchet topology from $T + \tilde{T}$ only depends on the sequence

$$S = ((R_{j\alpha} + \tilde{R}_{j\alpha})_{|\alpha| \leq j, j=0,...,k}; P, Q) \tag{1.3.19}$$

but not on the specific decomposition of $S_{j\alpha} = R_{j\alpha} + \tilde{R}_{j\alpha}$ which justifies the notation in the previous formula.

Similarly to Definition 1.2.4 we can introduce the space $R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ as the set of all $\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}(y, \eta) + g(y, \eta)$ for arbitrary $g(y, \eta) \in R^{\mu}_{G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ and an $f \times e$-matrix $m(y, \eta)$ of elements in $R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g})$. Let $R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_S$ for S as in (1.3.19) denote the subspace of all $a(y, \eta) \in R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ such that the entries of the $(f \times e)$-matrix valued upper left corner belong to $R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g})_S$ while the other entries are Green of asymptotic types P, Q. The Fréchet topology of $R^{\mu}_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_S$ is immediate from that of the $(f \times e)$-upper left corners and the one of the Green families in the other entries.

Definition 1.3.5 *$R^\mu(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta = (-(k+1), 0]$, $k \in \mathbb{N}$, and $\boldsymbol{w} = (e, f; j_-, j_+)$ is defined to be the space of all operator families*

$$a(y,\eta) = \begin{pmatrix} \sigma(a_M + a_F)\tilde{\sigma} & 0 \\ 0 & 0 \end{pmatrix}(y,\eta) + r(y,\eta), \tag{1.3.20}$$

where $(a_M + a_F)(y,\eta)$ is an $f \times e$-matrix of elements (1.3.3) with arbitrary cut-off functions $\sigma(r)$, $\tilde{\sigma}(r)$, with $\tilde{\sigma} = 1$ in a neighbourhood of $\operatorname{supp}\sigma$ and $r(y,\eta) \in R^\mu_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$.

Remark 1.3.6 *Set $E = \mathcal{K}^{s,\gamma}(X^\wedge, \mathbb{C}^e) \oplus \mathbb{C}^{j_-}$ and $\tilde{E} = \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge, \mathbb{C}^f) \oplus \mathbb{C}^{j_+}$ endowed with the group actions $\operatorname{diag}(\kappa_\delta, \delta^{\frac{n+1}{2}})_{\delta \in \mathbb{R}_+}$. We have $R^\mu(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w}) \subset S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ for every $s \in \mathbb{R}$. Moreover, for every $a(y\eta) \in R^\mu(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ and every $P \in \operatorname{As}(X, (\gamma, \Theta))$ there is a $Q \in \operatorname{As}(X, (\gamma - \mu, \Theta))$ such that $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E_P, \tilde{E}_Q)$ when we set $E_P = \mathcal{K}^{s,\gamma}_P(X^\wedge, \mathbb{C}^e) \oplus \mathbb{C}^{j_-}$, $\tilde{E}_Q = \mathcal{K}^{s-\mu,\gamma-\mu}_Q(X^\wedge, \mathbb{C}^f) \oplus \mathbb{C}^{j_+}$, for all $s \in \mathbb{R}$.*

If $a \in R^\mu(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$, let us set, for $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$,

$$\sigma^\mu_\wedge(a)(y,\eta) = \begin{pmatrix} \sigma^\mu_\wedge(a_M + a_F)(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + \sigma^\mu_\wedge(r)(y,\eta),$$

where $\sigma^\mu_\wedge(a_M + a_F)(y,\eta)$ is the matrix with elemts given by (1.3.6) and (1.3.7) and $\sigma^\mu_\wedge(r)(y,\eta)$ as in (1.3.14).

In an obvious way we can define subspaces $R^\mu(U \times \mathbb{R}^q; \boldsymbol{g}; \boldsymbol{w})_S$ of $R^\mu(U \times \mathbb{R}^q; \boldsymbol{g}; \boldsymbol{w})$ (for fixed asymptotic data (1.3.19)) by considering in (1.3.20) $r(y,\eta) \in R^\mu_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_S$. We conclude this section by introducing a Fréchet topology in these subspaces. Since $R^\mu_{M+G}(U \times \mathbb{R}^q; \boldsymbol{g}; \boldsymbol{w})_S$ is already treated, it remains the space of upper left corners.

Equality (1.3.5) tells us that the space to topologyse is $\Psi^\mu([0,c)_0)$ defined to be the space of all operator families of the form $r^{-\mu}\operatorname{op}_M^{\gamma-\frac{n}{2}}(f)(y,\eta)$ with $f(r,y,z,\eta) := \tilde{f}(r,y,z,r\eta)$ for arbitrary $\tilde{f}(r,y,z,\tilde{\eta}) \in C^\infty([0,c)_0 \times U, M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}}))$.

To introduce a Fréchet topology in $\Psi^\mu([0,c)_0)$ it suffices to establish a canonical isomorphism

$$\Psi^\mu([0,c)_0) \longrightarrow C^\infty([0,c)_0 \times U, M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}}))$$

and carry over the Fréchet topology from the space on the right to $\Psi^\mu([0,c)_0)$. This isomorphism can be defined to be the composition of well-defined maps

$$b(y,\eta) \in \Psi^\mu([0,c)_0) \to r^\mu b(y,\eta) \to r^\mu b(y, \frac{\tilde{\eta}}{r}) \to \operatorname{symb}(r^\mu b(y, \frac{\tilde{\eta}}{r})),$$

where "symb" means the symbolic map

$$\operatorname{op}_M^{\gamma-\frac{n}{2}}(f)(y, \frac{\tilde{\eta}}{r}) \to f(r,y,z,\tilde{\eta})$$

that is well-defined as a map to a parameter-dependent left symbol (with respect to r) that first gives the values on Re $z = \frac{n+1}{2} - \gamma$ but then extends in a unique way to a holomorphic function in $z \in \mathbb{C}$.

Summing up, the space $\Psi^\mu([0,c)_0)$ is now equipped with a natural Fréchet topology, and we can pass to the non-direct sum of Fréchet spaces

$$R^\mu(U \times \mathbb{R}^q), \boldsymbol{g}; \boldsymbol{w})_{c,S} := \Big(\Psi^\mu([0,c)_0) \oplus \{0\}\Big) + R^\mu_{M+G}(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})_S \tag{1.3.21}$$

for fixed $S \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g})$, $c > 0$. The space $R^\mu(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ itself is the union of the spaces of the form (1.3.21) over all $c > 0$ and $S \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g})$.

1.4 Edge operators

Let W be a compact manifold with edge Y and $\mathbb{W}$ the associated stretched manifold. By the assumptions of Section 1.1 we have a global splitting of variables near $\partial\mathbb{W}$, i.e., there is a neighbourhood of $\partial\mathbb{W}$ of the form

$$\mathbb{V} = [0,1) \times X \times Y, \quad (r,x,y) \in \mathbb{V}.$$

Let $\{V_1, ..., V_L\}$ be an open covering of Y by coordinate neighbourhoods and $\chi_l : V_l \to \Omega$, $l = 1, ..., L$, charts for some open $\Omega \subseteq \mathbb{R}^q$. Then the sets $[0,1) \times X \times V_l$ form a open covering of $\mathbb{V}$, and we have diffeomorphisms

$$\chi_l : [0,1) \times X \times V_l \to [0,1) \times X \times \Omega\ , \quad l = 1, ..., L\ .$$

Taking the sets $[0,1) \times X \times \Omega$ as local models of $\mathbb{W}$ near $\partial\mathbb{W}$, the transition maps between different such "singular charts" are independent of r and x. Let Vect($\mathbb{W}$) denote the set of all smooth complex vector bundles E on $\mathbb{W}$, where we assume that the transition maps between realisations of $E|_{[0,1)\times X \times V_l}$ on $[0,1) \times X \times \Omega$ are independent of r.

For local considerations it will be convenient to work with $\overline{\mathbb{R}}_+ \times X \times \Omega$; contributions for large r will be unessential after applying corresponding cut-offs. Let e be the fibre dimension of E. Then we form the spaces

$$\mathcal{K}^{s,\gamma}(X^\wedge, \mathbb{C}^e) := \mathcal{K}^{s,\gamma}(X^\wedge) \otimes \mathbb{C}^e$$

and, similarly, $\mathcal{K}^{s,\gamma}_P(X^\wedge, \mathbb{C}^e)$ for some $P \in \mathrm{As}(X, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \Theta)$. This gives rise to spaces

$$\mathcal{W}^s_{\mathrm{comp}}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge, \mathbb{C}^e)),\ \mathcal{W}^s_{\mathrm{loc}}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge, \mathbb{C}^e))$$

as well as

$$\mathcal{W}^s_{\mathrm{comp}}(\Omega, \mathcal{K}^{s,\gamma}_P(X^\wedge, \mathbb{C}^e)),\ \mathcal{W}^s_{\mathrm{loc}}(\Omega, \mathcal{K}^{s,\gamma}_P(X^\wedge, \mathbb{C}^e))\ .$$

They are invariant with respect to transition maps such that we have the spaces

$$\mathcal{W}^s_{\text{comp}}(V, \mathcal{K}^{s,\gamma}(X^\wedge, E|_{X^\wedge \times V})) \ ,$$

(where V denotes one of the sets in the covering of Y) as well as those with asymptotic types P and with subscript "loc". Now if $\sigma(r)$ is a cut-off function supported in a neighbourhood of $r = 0$, and if $\{\varphi_1, ..., \varphi_L\}$ is a partition of unity subordinate to $\{V_1, ..., V_L\}$, we set

$$\mathcal{W}^{s,\gamma}(\mathbb{W}, E) := \{\sigma \sum_{l=1}^{L} \varphi_l u_l + (1-\sigma) u_{\text{int}}\}$$

where $u_l \in \mathcal{W}^s_{\text{loc}}(V_l, \mathcal{K}^{s,\gamma}(X^\wedge, E|_{X^\wedge \times V_l}))$ and $u_{\text{int}} \in H^s_{\text{loc}}(\text{int}\mathbb{W}, E)$.

Similarly, we define the spaces $\mathcal{W}^{s,\gamma}_P(\mathbb{W}, E)$ by inserting $\mathcal{K}^{s,\gamma}_P(X^\wedge, E|_{X^\wedge \times V_l})$ in place of $\mathcal{K}^{s,\gamma}(X^\wedge, E|_{X^\wedge \times V_l})$.

Remark 1.4.1 *The space $\mathcal{W}^{s,\gamma}_P(\mathbb{W}, E)$ for every fixed asymptotic type P can be written as a projective limit of Hilbert spaces; as such it is a Fréchet space with a countable system of norms.*

Bundles $E \in \text{Vect}(\mathbb{W})$ are assumed to be equipped with Hermitian metrics that are independent of r for small r (i.e., lifted Hermitian metrics from $E|_{\partial\mathbb{W}}$ in a collar neighbourhood $\cong [0,1) \times X \times Y$ of $\partial\mathbb{W}$). We then have $\mathcal{W}^{0,0}(\mathbb{W}, E) \cong h^{-\frac{n}{2}} L^2(\mathbb{W}, E)$ where $L^2(\mathbb{W}, E)$ is the space of square integrable sections in E (with a measure that treats $\mathbb{W}$ as a C^∞ manifold with boundary) and h^ρ, $\rho \in \mathbb{R}$, is a strictly positive function in $C^\infty(\text{int}\mathbb{W})$ such that $h^\rho = r^\rho$ in a neighbourhood of 0.

The $\mathcal{W}^{0,0}$- scalar product then induces a non-degenerate sesquilinear pairing

$$\mathcal{W}^{s,\gamma}(\mathbb{W}, E) \times \mathcal{W}^{-s,-\gamma}(\mathbb{W}, E) \to \mathbb{C} \quad \text{for every } s, \gamma \in \mathbb{R}.$$

Definition 1.4.2 *An operator $G : \mathcal{W}^{0,\gamma}(\mathbb{W}, E) \to \mathcal{W}^{0,\delta}(\mathbb{W}, F)$, for $\gamma, \delta \in \mathbb{R}$ and $E, F \in$* Vect$(\mathbb{W})$, *is said to be a smoothing Green operator with asymptotics of types $P \in \text{As}(X, (\delta, \Theta))$, $Q \in \text{As}(X, (-\gamma, \Theta))$ (for some weight interval $\Theta = (\theta, 0]$) if, for all $s \in \mathbb{R}$, G and G^* induce continuous operators*

$$G : \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \to \mathcal{W}^{\infty,\delta}_P(\mathbb{W}, F) \text{ and } G^* : \mathcal{W}^{s,-\delta}(\mathbb{W}, F) \to \mathcal{W}^{\infty,-\gamma}_Q(\mathbb{W}, E). \tag{1.4.1}$$

Here G^* denotes the formal adjoint of G in the sense

$$(Gu, v)_{\mathcal{W}^{0,0}(\mathbb{W},F)} = (u, G^* v)_{\mathcal{W}^{0,0}(\mathbb{W},E)} \quad \text{for all } u \in C_0^\infty(\text{int}\mathbb{W}, E), v \in C_0^\infty(\text{int}\mathbb{W}, F).$$

It suffices to require conditions (1.4.1) for all $s \in \mathbb{Z}$; then the operators G are continuous for all $s \in \mathbb{R}$. Set $\boldsymbol{g} = (\gamma, \delta, \Theta)$ and let $\mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; E, F)_{P,Q}$ denote the space of all smoothing Green operators with asymptotic types P and Q. The definition gives us linear maps

$$\mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; E, F)_{P,Q} \longrightarrow \mathcal{L}(\mathcal{W}^{s,\gamma}(\mathbb{W}, E), \mathcal{W}^{\infty,\gamma}_P(\mathbb{W}, F)), \tag{1.4.2}$$

$$(1.4.3) \qquad \mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};E,F)_{P,Q} \longrightarrow \mathcal{L}(\mathcal{W}^{s,-\delta}(\mathbb{W},F),\mathcal{W}^{\infty,-\gamma}_{\overline{Q}}(\mathbb{W},E))$$

for all $s \in \mathbb{N}$, where (1.4.3) is defined as $G \to L$, obtained as a composition $G \to G^* \to L$ where $Lv := \overline{G^*\overline{v}}$, $v \in \mathcal{W}^{s,-\delta}(\mathbb{W},F)$. For every fixed s the spaces on the right hand side of (1.4.2) and (1.4.3) are Fréchet spaces with a countable system of operator norms, obtained in terms of the contable systems of norms in the respective spaces with asymptotics, cf. Remark 1.4.1.

In the space $\mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};E,F)_{P,Q}$ we introduce the Fréchet topology of projective limit with respect to the mappings (1.4.2), (1.4.3), $s \in \mathbb{N}$. This enables us to define parameter-dependent Green operators with parameter $\lambda \in \mathbb{R}^l$. We set

$$\mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};E,F;\mathbb{R}^l)_{P,Q} = \mathcal{S}(\mathbb{R}^l,\mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};E,F)_{P,Q}).$$

In an analogous manner, given elements $E,F \in \mathrm{Vect}(\mathbb{W})$, J_-, $J_+ \in \mathrm{Vect}(\mathrm{Y})$, we define the spaces

$$\mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};\boldsymbol{v})_{P,Q} \text{ and } \mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^l)_{P,Q}$$

for $\boldsymbol{v} = (E,F;J_-,J_+)$, where $G \in \mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};\boldsymbol{v})_{P,Q}$ is defined to be the space of all operator block matrices

$$G = (G_{ij})_{i,j=1,2} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W},E) \\ \oplus \\ H^s(Y,J_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{\infty,\delta}_P(\mathbb{W},F) \\ \oplus \\ C^\infty(Y,J_+) \end{array}$$

such that the formal adjoint with respect to the $\mathcal{W}^{0,0}(\mathbb{W},\cdot) \oplus H^0(Y,\cdot)$-scalar products have an analogous mapping property with opposite weights and the asymptotic type Q in the image, for all $s \in \mathbb{R}$. We endow the space $\mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};\boldsymbol{v})_{P,Q}$ with a natural Fréchet topology that is defined in an analogous manner as that for the corresponding space of upper left corners. Let $\mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^l)$ denote the union of all the spaces $\mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^l)_{P,Q}$ over all P,Q.

Next we pass to pseudo-differential operators on Ω with amplitude functions

$$a(y,y',\eta,\lambda) \in R^\mu(\Omega \times \Omega \times \mathbb{R}^{q+l}_{\eta,\lambda},\boldsymbol{g};\boldsymbol{w})$$

for $\boldsymbol{g} = (\gamma,\gamma-\mu,\Theta)$, $\Theta = (-(k+1),0]$, $k \in \mathbb{N}$, and $\boldsymbol{w} = (e,f;j_-,j_+)$. If we form operator families $\mathrm{Op}(a)(\lambda)$ where, first for $u(y) \in C_0^\infty(\Omega,\mathcal{K}^{s,\gamma}(X^\wedge,\mathbb{C}^e) \oplus \mathbb{C}^{j_-})$,

$$\mathrm{Op}(a)(\lambda)u(y) = \iint e^{i(y-y')\eta}a(y,y',\eta,\lambda)u(y')dy'đ\eta\ ,$$

we get, for all λ, continuous operators

$$\mathrm{Op}(a)(\lambda) : \begin{array}{c} \mathcal{W}^s_{\mathrm{comp}}(\Omega,\mathcal{K}^{s,\gamma}(X^\wedge,\mathbb{C}^e)) \\ \oplus \\ H^{s-\frac{n+1}{2}}_{\mathrm{comp}}(\Omega,\mathbb{C}^{j_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega,\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge,\mathbb{C}^f)) \\ \oplus \\ H^{s-\mu-\frac{n+1}{2}}_{\mathrm{loc}}(\Omega,\mathbb{C}^{j_+}). \end{array}$$

There is then invariance under substituting transition maps from $E, F \in \mathrm{Vect}(\mathbb{W})$, $J_-, J_+ \in \mathrm{Vect}(Y)$, between trivialisations of the respective bundles belonging to "charts" $\chi_l : \mathbb{R}_+ \times X \times V_l \to X^\wedge \times \Omega$ and $\chi'_l : V_l \to \Omega$, respectively (see also the notation at the beginning of this section). This gives us operators

$$(1.4.4) \qquad \mathcal{A}_l(\lambda) : \begin{array}{c} \mathcal{W}^s_{\mathrm{comp}}(V_l, \mathcal{K}^{s,\gamma}(X^\wedge, E)) \\ \oplus \\ H^{s-\frac{n+1}{2}}_{\mathrm{comp}}(V_l, J_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu}_{\mathrm{loc}}(V_l, \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge, F)) \\ \oplus \\ H^{s-\mu-\frac{n+1}{2}}_{\mathrm{loc}}(V_l, J_+). \end{array}$$

Here, for brevity, in the spaces we wrote the bundles themselves rather than their restrictions to $X^\wedge \times V_l$ and V_l, respectively.

Let us now fix cut-off functions $\sigma(r)$, $\tilde{\sigma}(r)$, $\tilde{\tilde{\sigma}}(r)$, supported in $[0, 1)$, such that $\sigma\tilde{\sigma} = \sigma$, $\sigma\tilde{\tilde{\sigma}} = \tilde{\tilde{\sigma}}$, choose a partition of unity $\{\varphi_1, ..., \varphi_L\}$ subordinate to $\{V_1, ..., V_L\}$, and let $\{\psi_1, ..., \psi_L\}$ be functions $\psi_j \in C^\infty_0(V_j)$ such that $\varphi_j\psi_j = \varphi_j$ for all j. Moreover, let $L^\mu_{\mathrm{cl}}(\mathrm{int}\mathbb{W}; E, F; \mathbb{R}^l)$ denote the space of all classical parameter-dependent pseudo-differential operators on $\mathrm{int}\mathbb{W}$, operating between spaces of distributional sections of bundles E, F. In the following definition we set $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ and $\boldsymbol{v} = (E, F; J_-, J_+)$ for $\gamma, \mu \in \mathbb{R}$, $\Theta = (\theta, 0]$, $-\infty \le \theta < 0$, $E, F \in \mathrm{Vect}(\mathbb{W})$, $J_-, J_+ \in \mathrm{Vect}(Y)$.

Definition 1.4.3 *$\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ is defined to be the space of all operator families*

$$\begin{aligned} \mathcal{A}(\lambda) &= \sum_{j=1}^{L} \mathrm{diag}(\sigma\varphi_j, \varphi_j)\mathcal{A}_j(\lambda)\mathrm{diag}(\tilde{\sigma}\psi_j, \psi_j) \\ &+ \mathrm{diag}(1-\sigma, 0)A_{\mathrm{int}}(\lambda)\mathrm{diag}(1-\tilde{\tilde{\sigma}}, 0) + \mathcal{G}(\lambda) \end{aligned}$$

for arbitrary operators of the form (1.4.4)*, an operator $A_{\mathrm{int}}(\lambda) \in L^\mu_{\mathrm{cl}}(\mathrm{int}\mathbb{W}; E, F; \mathbb{R}^l)$ and $\mathcal{G}(\lambda) \in \mathcal{Y}^{-\infty}_G(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)_{P,Q}$ and $\mathcal{A}_j(\lambda)$ of the form* (1.4.4)*.*

The case $l = 0$ in Definition 1.4.3 is also admitted. In this case we write $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for the corresponding space of operators. Note that $\mathcal{A}(\lambda, \tilde{\lambda}) \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+\tilde{l}}_{\lambda,\tilde{\lambda}})$ implies $\mathcal{A}(\lambda, \tilde{\lambda}_0) \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l_\lambda)$ for every fixed $\tilde{\lambda}_0 \in \mathbb{R}^{\tilde{l}}$.

Let us set $\mathcal{W}^{s,\gamma}(\mathbb{W}; \boldsymbol{m}) = \mathcal{W}^{s,\gamma}(\mathbb{W}; E) \oplus H^{s-\frac{n+1}{2}}(Y, J)$, $s \in \mathbb{R}$, for any pair $\boldsymbol{m} \in \mathrm{Vect}(\mathbb{W}) \times \mathrm{Vect}(Y)$. Similarly, we define $\mathcal{W}^{s,\gamma}_P(\mathbb{W}; \boldsymbol{m}) = \mathcal{W}^{s,\gamma}_P(\mathbb{W}; E) \oplus H^{s-\frac{n+1}{2}}(Y, J)$ for any asymptotic type $P \in \mathrm{As}\ (X, (\gamma, \Theta))$.

Theorem 1.4.4 *The elements $\mathcal{A}(\lambda) \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ for $\boldsymbol{v} = (E, F; J_-, J_+)$ induce families of continuous operators*

$$\mathcal{A}(\lambda) : \mathcal{W}^{s,\gamma}(\mathbb{W}; \boldsymbol{m}) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}; \boldsymbol{n})$$

for $\boldsymbol{m} = (E, J_-)$, $\boldsymbol{n} = (F, J_+)$, *as well as*

$$\mathcal{A}(\lambda) : \mathcal{W}_P^{s,\gamma}(\mathbb{W}; \boldsymbol{m}) \longrightarrow \mathcal{W}_Q^{s-\mu,\gamma-\mu}(\mathbb{W}; \boldsymbol{n})$$

for every $P \in \mathrm{As}\ (X, (\gamma, \Theta))$ *with some resulting* $Q \in \mathrm{As}\ (X, (\gamma - \mu, \Theta))$.

Theorem 1.4.4 is a consequence of corresponding local continuity results, cf. relation (1.1.8) and Remark 1.3.6, together with (1.4.1) for global smoothing operators. Let us now establish the (parameter-dependent) principal symbolic structure of operator families $\mathcal{A}(\lambda)$ as in the previous definition (for convenience, we often write $\mathcal{A}$ instead of $\mathcal{A}(\lambda)$). Incidentally, we write $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2}$ and u.l.c. $\mathcal{A}$ (upper left corner of $\mathcal{A}$) in place of $\mathcal{A}_{11}$. We then have a space of upper left corners

$$\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; E, F; \mathbb{R}^l) = \text{u.l.c. } \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)\ ,$$

where

$$\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; E, F; \mathbb{R}^l) \subset L^\mu(\mathrm{int}\mathbb{W}; E, F; \mathbb{R}^l)\ .$$

Here, analogously to notation in Section 1.1, $L^\mu_{\mathrm{cl}}(\cdot; E, F; \mathbb{R}^l)$ is the space of all classical parameter-dependent pseudo-differential operators on a manifold (indicated by the dot), operating between spaces of sections in the respective vector bundles E and F. In our case the manifold is $\mathrm{int}\mathbb{W}$ and we have standard parameter-dependent homogeneus principal symbols

$$\sigma_\psi^\mu(\mathcal{A}) = \sigma_\psi^\mu(\text{u.l.c. } \mathcal{A})$$

that are bundle homomorphisms

$$\sigma_\psi^\mu(\mathcal{A}) : \pi^*_{\mathrm{int}\mathbb{W}} E \to \pi^*_{\mathrm{int}\mathbb{W}} F,$$

where $\pi_{\mathrm{int}\mathbb{W}} : (T^*(\mathrm{int}\mathbb{W}) \times \mathbb{R}^l_\lambda) \setminus 0 \to \mathrm{int}\mathbb{W}$ is the canonical projection. In our case, because of the edge-degenerate nature of operators, we have in the splitting of variables $(r, x, y) \in [0, 1) \times X \times \Omega$ near $\partial\mathbb{W}$ with covariables (plus parameter) $(\rho, \xi, \eta, \lambda)$ a representation

$$\sigma_\psi^\mu(\mathcal{A})(r, x, y, \rho, \xi, \eta, \lambda) = \tilde{\sigma}_\psi^\mu(\mathcal{A})(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta, \tilde{\lambda}=r\lambda}\ ,$$

where $\tilde{\sigma}_\psi^\mu(\mathcal{A})$ is smooth in r up to 0 and has an invariant meaning as a bundle homomorphism

$$\tilde{\sigma}_\psi^\mu(\mathcal{A}) : \tilde{\pi}^*_{\mathbb{W}} E \to \tilde{\pi}^*_{\mathbb{W}} F$$

between pull-backs to a so-called stretched cotangent bundle (+parameter) $\tilde{T}^*(\mathbb{W} \times \mathbb{R}^l) \setminus 0$ (0 means $(\tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda}) = 0$) with the canonical projection $\tilde{\pi}_{\mathbb{W}} : \tilde{T}^*(\mathbb{W} \times \mathbb{R}^l) \setminus 0 \to \mathbb{W}$. In addition, the homogeneus principal edge symbols of local operator-valued amplitude functions that are involved in $\mathcal{A}_l(\lambda)$, cf. the previous definition and Section 1.3, gives us a bundle homomorphism

$$\sigma_\wedge^\mu(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \longrightarrow \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F' \\ \oplus \\ J_+ \end{pmatrix}$$

between pull-backs of corresponding bundles with infinite-dimensional fibres to $(T^*Y \times \mathbb{R}^l_\lambda) \setminus 0$ with respect to the canonical projection $\pi_Y : (T^*Y \times \mathbb{R}^l) \setminus 0 \to Y$, $E' := E|_{\partial \mathbb{W}}$, $F' := F|_{\partial \mathbb{W}}$. We now set

$$\sigma(\mathcal{A}) = (\sigma^\mu_\psi(\mathcal{A}), \sigma^\mu_\wedge(\mathcal{A})) \ ,$$

called the principal symbol of $\mathcal{A}$ of order μ.

In the above definitions of symbol and operator spaces the order is the same as μ in the weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$. We can also define symbol spaces $R^\nu(U \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ with the same $\boldsymbol{g}$ but any other order $\nu \in \mathbb{R}$ such that $\mu - \nu \in \mathbb{N}$.
The generalisation for Green symbols is straightforward; concerning smoothing Mellin symbols it suffices to replace $r^{-\mu}$ in formula (1.3.11) by $r^{-\nu}$, while for the non-smoothing parts $a_M(y,\eta)$ and $a_F(y,\eta)$ we simply take symbols of order ν instead of μ. We then get associated operator spaces $\mathcal{Y}^\nu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ for $\mu - \nu \in \mathbb{N}$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, with associated principal symbols $\sigma^\nu(\mathcal{A}) = (\sigma^\nu_\psi(\mathcal{A}), \sigma^\nu_\wedge(\mathcal{A}))$. Note that then

$$\mathcal{Y}^{\nu-1}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l) = \{\mathcal{A} \in \mathcal{Y}^\nu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l) \ : \ \sigma^\nu(\mathcal{A}) = 0\}.$$

Remark 1.4.5 $\mathcal{A} \in \mathcal{Y}^{\mu-1}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w}; \mathbb{R}^l)$ *implies that*

$$\mathcal{A} : \mathcal{W}^{s,\gamma}(\mathbb{W}; \boldsymbol{m}) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}; \boldsymbol{n})$$

is a compact operator for every $s \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}^l$*.*

Theorem 1.4.6 *Let* $\mathcal{A}_j(\lambda) \in \mathcal{Y}^{\nu-j}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$*,* $j \in \mathbb{N}$*, be an arbitrary sequence and suppose that the asymptotic types in the Green symbols (of the local representations) are independent of* j*. Then there is an* $\mathcal{A}(\lambda) \in \mathcal{Y}^\nu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ *such that, for every* $N \in \mathbb{N}$*,*

$$\mathcal{A}(\lambda) - \sum_{j=0}^{N} \mathcal{A}_j(\lambda) \in \mathcal{Y}^{\nu-(N+1)}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l),$$

and $\mathcal{A}(\lambda)$ *is unique* mod $\mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$*.*

This result is a consequence of the fact that local amplitude functions belonging to $\mathcal{A}_j(\lambda)$ can be summed up asymptotically, uniquely with remainders of order $-\infty$. In the following theorem we consider operators $\mathcal{B}$ and $\mathcal{A}$ with weight and bundle data

$$\boldsymbol{f} = (\gamma, \gamma - \nu, \Theta) \ , \quad \boldsymbol{w} = (E, G; J_-, J)$$

and

$$\boldsymbol{g} = (\gamma - \nu, \gamma - (\nu + \mu), \Theta) \ , \quad \boldsymbol{v} = (G, F; J, J_+)$$

respectively, and set

$$\boldsymbol{f} \circ \boldsymbol{g} = (\gamma, \gamma - (\nu + \mu), \Theta) \ , \quad \boldsymbol{w} \circ \boldsymbol{v} = (E, F; J_-, J_+) \ .$$

The next Theorem is proved in [22], Section 3.4.4.

Theorem 1.4.7 *Let $\mathcal{A}$ be in $\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ and $\mathcal{B}$ be in $\mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{f}; \boldsymbol{w}; \mathbb{R}^l)$. We then have for the composition $\mathcal{AB} \in \mathcal{Y}^{\mu+\nu}(\mathbb{W}, \boldsymbol{g} \circ \boldsymbol{f}; \boldsymbol{w} \circ \boldsymbol{v}; \mathbb{R}^l)$, and $\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ (with componentwise composition).*

1.5 Ellipticity and parametrices

We now turn to (parameter-dependent) ellipticity of elements in the space

$$\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l), \tag{1.5.1}$$

where $\mathbb{W}$ is a (not necessarily compact) stretched manifold with edge Y, and

$$\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta), \quad \boldsymbol{v} = (E, F; J_-, J_+) \tag{1.5.2}$$

for $\Theta = (-(k+1), 0]$, $k \in \mathbb{N} \cup \{\infty\}$, $\gamma, \mu \in \mathbb{R}$, and $E, F \in \mathrm{Vect}(\mathbb{W})$, $J_-, J_+ \in \mathrm{Vect}(Y)$.

Definition 1.5.1 *An element $\mathcal{A}(\lambda)$ in* (1.5.1) *is said to be (parameter-dependent) elliptic (of order μ) if*

(i)

$$\tilde{\sigma}^{\mu}_{\psi}(\mathcal{A}) : \tilde{\pi}^*_{\mathbb{W}} E \longrightarrow \tilde{\pi}^*_{\mathbb{W}} F$$

is an isomorphism, $\tilde{\pi}_{\mathbb{W}} : (\tilde{T}^*\mathbb{W} \times \mathbb{R}^l) \setminus 0 \longrightarrow \mathbb{W}$;

(ii)

$$\sigma^{\mu}_{\wedge}(\mathcal{A}) : \pi^*_Y \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \longrightarrow \pi^*_Y \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F' \\ \oplus \\ J_+ \end{pmatrix}$$

is an isomorphism for some $s \in \mathbb{R}$, $\pi_Y : (T^*Y \times \mathbb{R}^l) \setminus 0 \longrightarrow Y$.

The case $l = 0$ is also admitted; we then talk about ellipticity (without parameters). Notice that when $\mathcal{A}(\lambda) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ is parameter-dependent elliptic, $\mathcal{A}(\lambda_0) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ is elliptic without parameters for every $\lambda_0 \in \mathbb{R}^l$.

If (ii) is satisfied for some $s = s_0 \in \mathbb{R}$, then it is true for all $s \in \mathbb{R}$. This is a consequence of the fact that the upper left corners of edge symbols belong to the cone algebra on $X^\wedge$ and are elliptic for $(\eta, \lambda) \neq 0$ both with respect to the tip of the cone and to $r \to \infty$, the exit of $X^\wedge$ to infinity. Moreover, as is well-known, kernel and cokernel are independent of s.

Theorem 1.5.2 *Let $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ be elliptic. Then there is a parametrix $\mathcal{P}$ of $\mathcal{A}$ with $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1}; \mathbb{R}^l)$, $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$, $\boldsymbol{v}^{-1} = (F, E; J_+, J_-)$.*

This means

$$\mathcal{I} - \mathcal{PA} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}}; \mathbb{R}^l), \quad \mathcal{I} - \mathcal{AP} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \tilde{\tilde{\boldsymbol{g}}}; \tilde{\tilde{\boldsymbol{v}}}; \mathbb{R}^l),$$

where $\tilde{\boldsymbol{g}} = (\gamma, \gamma, \Theta)$, $\tilde{\boldsymbol{v}} = (E, E; J_-, J_-)$, $\tilde{\tilde{\boldsymbol{g}}} = (\gamma - \mu, \gamma - \mu, \Theta)$, $\tilde{\tilde{\boldsymbol{v}}} = (F, F; J_+, J_+)$.

For the proof see [22], Section 3.5.2.

Corollary 1.5.3 *Let $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ be elliptic and assume*

$$u \in \mathcal{W}^{-\infty,\gamma}(\mathbb{W}; \boldsymbol{m}), \qquad \mathcal{A}u = f \in \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}; \boldsymbol{n}),$$

for some $s \in \mathbb{R}$. Then we have $u \in \mathcal{W}^{s,\gamma}(\mathbb{W}; \boldsymbol{m})$. Moreover, $\mathcal{A}u = f \in \mathcal{W}_Q^{s-\mu,\gamma-\mu}(\mathbb{W}; \boldsymbol{n})$ for a $Q \in \mathrm{As}(X, (\gamma-\mu, \Theta))$ implies $u \in \mathcal{W}_P^{s,\gamma}(\mathbb{W}; \boldsymbol{m})$ for a resulting asymptotic type $P \in \mathrm{As}(X, (\gamma, \Theta))$.

Corollary 1.5.4 *Let $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ be elliptic. Then*

$$\mathcal{A}(\lambda) : \mathcal{W}^{s,\gamma}(\mathbb{W}; \boldsymbol{m}) \to \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}; \boldsymbol{n}) \tag{1.5.3}$$

is a family of Fredholm operators for all $s \in \mathbb{R}$. In the case $l > 0$ the operators are of index zero, and there is a $C > 0$ such that (1.5.3) are isomorphisms for all $|\lambda| > c$ and all $s \in \mathbb{R}$.

We conclude the section with a remark that has been proved in [3].

Remark 1.5.5 *For every $\mu, \gamma \in \mathbb{R}$, $l \in \mathbb{N}$ and $E \in \mathrm{Vect}(\mathbb{W})$ there exists a parameter-dependent elliptic element $R(\lambda) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; E, E; \mathbb{R}^l)$ that induces isomorphisms*

$$R(\lambda) : \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, E) \quad \text{for all } s \in \mathbb{R},\ \lambda \in \mathbb{R}^l,$$

where $R^{-1}(\lambda) \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; E, E; \mathbb{R}^l)$, $R^{-1}(\lambda) = (R(\lambda))^{-1}$.

2 Meromorphic corner symbols

2.1 General kernel cut-off

In this Section we prepare some general constructions on operator-valued symbols that depend holomorphically on a complex covariable. Symbols of that kind may be constructed in terms of so-called kernel cut-off operators. Let us first introduce spaces of parameter-dependent operator-valued symbols with holomorphic dependence on parameters. In Section 1.1 we have defined the symbol spaces $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^m; E, \tilde{E})$ (with constant coefficients), where E and $\tilde{E}$ are spaces (either Hilbert spaces of Fréchet spaces written as projective limits of Hilbert spaces) with group actions $\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ and $\{\tilde{\kappa}_\delta\}_{\delta \in \mathbb{R}_+}$, respectively.

The following considerations will be formulated for the Hilbert space case. The generalisation for Fréchet spaces is straightforward and will tacitly be used below.

Definition 2.1.1 *Let $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E})$ denote the space of all $h(\eta, z) \in \mathcal{A}(\mathbb{C}^l_z, S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E}))$ such that $h(\eta, \lambda + i\tau) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+l}_{\eta,\lambda}; E, \tilde{E})$ for every $\tau \in \mathbb{R}^l$ and uniformly in $\tau \in K$ for every $K \subset\subset \mathbb{R}^l$.*

Note that $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E})$ are Fréchet spaces in a canonical way (an adequate semi-norm system immediately follows from the definition).

Let us set

$$K(a)(\eta, \theta) = (2\pi)^{-l} \int_{\mathbb{R}^l} e^{i\lambda\theta} a(\eta, \lambda) d\lambda \ ;$$

this exists as an element of $\mathcal{S}'(\mathbb{R}^l_\theta, \mathcal{L}(E, \tilde{E}))$ for every fixed η, and we have $\chi(\theta)K(a)(\eta, \theta) \in \mathcal{S}(\mathbb{R}^l_\theta, \mathcal{L}(E, \tilde{E}))$ for each excision function χ (i.e., vanishing in a neighbourhood of 0 and being 1 outside another neighbourhood of 0). We have

$$\varphi(\theta)K(a)(\eta, \theta) \in \mathcal{S}'(\mathbb{R}^l, \mathcal{L}(E, \tilde{E}))$$

for every $\varphi \in C_0^\infty(\mathbb{R}^l)$, and we set

$$H(\varphi)a(\eta, \lambda) = \int e^{-i\lambda\theta} \varphi(\theta) K(a)(\eta, \theta) d\theta \ .$$

We call $H(\varphi)$ a kernel cut-off operator.

The following results and observations on kernel cut-off constructions may be found in different versions in [19] or in [20], Section 3.2.2, [3], Section 1.5.2.

Theorem 2.1.2 *Let $\varphi(\theta) \in C_0^\infty(\mathbb{R}^l)$ and $a(\eta, \lambda) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+l}; E, \tilde{E})$. Then $H(\varphi)a(\eta, \lambda) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+l}; E, \tilde{E})$, and there is an $h(\eta, z) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E})$ such that*

$$h(\eta, z)|_{\mathrm{Im}\, z = 0} = H(\varphi)a(\eta, \lambda) \ .$$

Clearly, $h(\eta, z)$ is uniquely determined by φ and a, and we will also set $h(\eta, z) = H(\varphi)a(\eta, z)$.

Remark 2.1.3 *For every fixed $\varphi \in C_0^\infty(\mathbb{R}^l)$ the map $a(\eta, \lambda) \to H(\varphi)a(\eta, z)$ defines a continuous operator*

$$H(\varphi) : S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+l}; E, \tilde{E}) \to S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E}) \ .$$

Moreover, if $a(\eta, \lambda) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+l}; E, \tilde{E})$ is fixed, the map $\varphi \to H(\varphi)a$ defines a continuous operator

$$C_0^\infty(\mathbb{R}^l) \to S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E}).$$

Remark 2.1.4 *Let $a(\eta, \lambda) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+l}; E, \tilde{E})$, and define $h(\eta, z) = (H(\varphi)a)(\eta, z)$ as before. Then, for every fixed $\tau \in \mathbb{R}^l$ the restriction $a_\tau(\eta, \lambda) = h(\eta, \lambda + i\tau)$ admits an asymptotic expansion*

$$a_\tau(\eta, \lambda) \sim \sum_{\alpha \in \mathbb{N}^l} c_\alpha D^\alpha_\lambda a(\eta, \lambda)$$

in the space $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+l}; E, \tilde{E})$, with constants $c_\alpha = c_\alpha(\varphi, \tau)$, $\alpha \in \mathbb{N}^q$. If $\varphi \in C_0^\infty(\mathbb{R}^l)$ equals 1 in a neighbourhood of $\theta = 0$, we have $c_0 = (\varphi, \tau) = 1$.

Remark 2.1.5 *The kernel cut-off construction can also be started from elements* $b(\eta,\lambda) = h(\eta,\lambda+i\tau)$ *for any* $h(\eta,z) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E})$ *and any fixed* $\tau \in \mathbb{R}^l$.

For every $\varphi \in C_0^\infty(\mathbb{R}^l)$ *we then obtain a map*

$$H_\tau(\varphi) : S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E}) \to S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E})$$

when we set $H_\tau(\varphi)h(\eta,z) = H(\varphi)b(\eta,z)$.

Theorem 2.1.6 *Let* $\psi \in C_0^\infty(\mathbb{R}^l)$ *be a function such that* $\psi(\theta) = 1$ *in a neighbourhood of* $\theta = 0$, *and let* $a(\eta,\lambda) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{q+l}; E, \tilde{E})$. *Then* $h(\eta,z) = H(\psi)a(\eta,z) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E})$ *satisfies the relation*

$$h(\eta,z)|_{\mathrm{Im} z=0} = a(\eta,\lambda) \mod S^{-\infty}(\mathbb{R}^{q+l}; E, \tilde{E}) \ .$$

Remark 2.1.7 *From Remark* 1.1.51 *of* [22] *we get the following property. Let* a *and* ψ *as in Theorem* 2.1.6, *and set* $\psi_r(\theta) := \psi(r\theta)$ *for* $r > 0$. *Then* $H(\psi_r)a(\eta,z)|_{\mathrm{Im}\ z=0} - a(\eta,\lambda)$ *tends to zero in* $S^{-\infty}(\mathbb{R}^{q+l}; E, \tilde{E})$ *as* $r \to 0$.

Corollary 2.1.8 *Let* $h(\eta,z) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E})$, *and let* $\tau \in \mathbb{R}^l$ *be fixed.*

Then $h(\eta,\lambda+i\tau) \in S^{\mu-1}_{(\mathrm{cl})}(\mathbb{R}^{q+l}_{\eta,\lambda}; E, \tilde{E})$ *implies* $h(\eta,z) \in S^{\mu-1}_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E})$. *In particular,* $h(\eta,\lambda+i\tau) \in S^{-\infty}(\mathbb{R}^{q+l}_{\eta,\lambda}; E, \tilde{E})$ *entails* $h(\eta,z) \in S^{-\infty}(\mathbb{R}^q \times \mathbb{C}^l; E, \tilde{E})$.

Remark 2.1.9 *The above kernel cut-off constructions directly extend to the spaces of symbols* $S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^{q+l}; E, \tilde{E})$ *with "non-constant" coefficients dependent on* $y \in U$, *cf. the notation in Section* 1.1. *The kernel cut-off operators only act on covariables; in the sequel we tacitly use results in the evident generalisation to the* y*-dependent case.*

2.2 Further results on the edge-operator algebra

To carry out our program on edge operator-valued meromorphic functions we now study the structure of edge pseudo-differential operators in more detail.

First, if $\Omega \subseteq \mathbb{R}^q$ is an open set, E and $\tilde{E}$ Hilbert spaces with group actions $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+}$, respectively, we have our spaces of symbols $a(y,y',\eta,\lambda) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{q+l}; E, \tilde{E})$ and associated parameter-dependent pseudo-differential operators

$$L^{\mu}_{(\mathrm{cl})}(\Omega; E, \tilde{E}; \mathbb{R}^l) := \big\{\mathrm{Op}(a)(\lambda) : \ a(y,y',\eta,\lambda) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{q+l}; E, \tilde{E})\big\}.$$

The following considerations are valid with obvious modifications also for the case of Fréchet spaces E or $\tilde{E}$ with group actions, and we then employ that without further comment. Moreover, let us mainly discuss the classical case; the considerations for non-classical symbols and operators are completely analogous. We want to endow the space $L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)$ with a canonical Fréchet topology. To this end we first observe that $L^{-\infty}(\Omega; E, \tilde{E}) := \bigcap_{\mu\in\mathbb{R}} L^{\mu}(\Omega; E, \tilde{E})$ is isomorphic to $C^{\infty}(\Omega \times \Omega; \mathcal{L}(E, \tilde{E}))$ such that $L^{-\infty}(\Omega; E, \tilde{E})$ becomes Fréchet, and then we set $L^{-\infty}(\Omega; E, \tilde{E}; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(\Omega; E, \tilde{E}))$.

Moreover, let $K \subset \Omega \times \Omega$ be any proper relatively closed set containing $\text{diag}(\Omega \times \Omega)$ in its interior, and let $\omega(y, y') \in C^{\infty}(\Omega \times \Omega)$ be any element supported by K where $\omega(y, y') = 1$ in a neighbourhood of $\text{diag}(\Omega \times \Omega)$. A well-known construction for scalar pseudo-differential operators then also applies to the vector-valued case. Every $A(\lambda) \in L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)$ admits a decomposition

$$A(\lambda) = A_0(\lambda) + C(\lambda), \tag{2.2.1}$$

where the $\mathcal{L}(E, \tilde{E})$-valued distributional kernel of $A_0(\lambda)$ is supported by K for all $\lambda \in \mathbb{R}^l$, while $C(\lambda) \in L^{-\infty}(\Omega; E, \tilde{E}; \mathbb{R}^l)$.

Given $A(\lambda) = \text{Op}(a)(\lambda)$ as above, it suffices to set $A_0(\lambda) = \text{Op}(\omega a)(\lambda)$. Let $L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)_K$ denote the subspace of all $A(\lambda) \in L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)$ with distributional kernel supported in K. Elements in $L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)_K$ induce families of maps $C_0^{\infty}(\Omega, E) \to C_0^{\infty}(\Omega, \tilde{E})$ as well as $C^{\infty}(\Omega, E) \to C^{\infty}(\Omega, \tilde{E})$. In particular, setting $e_{\eta}(y) := e^{iy\eta}$ we can form $A(\lambda)e_{\eta}u \in C^{\infty}(\Omega, \tilde{E})$ for every $u \in E$.

Lemma 2.2.1 *Given $A_0(\lambda) \in L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)_K$ we have*

$$a(y, \eta, \lambda) := e_{-\eta}(y) A_0(\lambda) e_{\eta}(\cdot) \in S^{\mu}_{\rm cl}(\Omega \times \mathbb{R}^{q+l}; E, \tilde{E})_K$$

and $A_0(\lambda) = \text{Op}(a)(\lambda)$. Moreover,

$$S^{\mu}_{\rm cl}(\Omega \times \mathbb{R}^{q+l}; E, \tilde{E})_K := \{e_{-\eta}(y) A_0(\lambda) e_{\eta}(\cdot) \ : \ A_0(\lambda) \in L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)_K\}$$

is a closed subspace of $S^{\mu}_{\rm cl}(\Omega \times \mathbb{R}^{q+l}; E, \tilde{E})$ (concerning the Fréchet space structure of classical symbol spaces, cf. Section 1.1*).*

The proof is similar to the corresponding case of scalar operators.

Remark 2.2.2 *The correspondence $A_0(\lambda) \to a(y, \eta, \lambda)$ from Lemma* 2.2.1 *yields an isomorphism*

$$L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)_K \longrightarrow S^{\mu}_{\rm cl}(\Omega \times \mathbb{R}^{q+l}; E, \tilde{E})_K. \tag{2.2.2}$$

We now get a Fréchet topology in the space $L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)_K$ by using (2.2.2), i.e., carrying over the Fréchet topology from the space $S^{\mu}_{\rm cl}(\Omega \times \mathbb{R}^{q+l}; E, \tilde{E})_K$. Relation (2.2.1) gives us

$$L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l) = L^{\mu}_{\rm cl}(\Omega; E, \tilde{E}; \mathbb{R}^l)_K + L^{-\infty}(\Omega; E, \tilde{E}; \mathbb{R}^l) \tag{2.2.3}$$

as vector spaces. Both summands on the right are Fréchet (and contained in the space of linear and continuous operators $\mathcal{W}^s_{\text{comp}}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\text{loc}}(\Omega, \tilde{E})$, cf. relation (1.1.8)). We then endow (2.2.3) with the Fréchet topology of non-direct sum; an easy consideration shows that it is independent of the specific K.

We now specify these constructions for symbols of the classes $R^\mu(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g}; \boldsymbol{w})$ and for the subspaces $R^\mu_{M+G}(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g}; \boldsymbol{w})$ and $R^\mu_G(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g}; \boldsymbol{w})$.

Let us start from Green symbols, cf. Definition 1.2.4. For simplicity, we take upper left corners and assume $e = f = 1$, cf. (1.2.5) and (1.2.6); the considerations for block matrices in general are completely analogous and left to the reader. In other words, we have the Fréchet spaces $R^\mu_G(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q}$ for $\boldsymbol{g} = (\gamma, \sigma, \Theta)$, with asymptotic types $P \in \text{As}(X, (\sigma, \Theta))$, $Q \in \text{As}(X, (-\gamma, \Theta))$, cf. Remark 1.2.5.

Lemma 2.2.3 *If $g_j \in R^{\mu-j}_G(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q}$, $j \in \mathbb{N}$, is any sequence, there is a $g \in R^\mu_G(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q}$ that is the asymptotic sum, i.e.,*

$$g - \sum_{j=0}^{N} g_j \in R^{\mu-(N+1)}_G(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q}$$

for every $N \in \mathbb{N}$, and g is unique mod $R^{-\infty}_G(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q}$.

Lemma 2.2.3 is a direct consequence of Definition 1.2.4 and of the fact that symbols in the context of twisted homogeneity can be summed up asymptotically within the classes.

Let $P \in \text{As}(X, (\sigma, \Theta))$, $Q \in \text{As}(X, (-\gamma, \Theta))$, $\boldsymbol{g} = (\gamma, \sigma, \Theta)$, and set

$$\text{(2.2.4)} \qquad \mathcal{Y}^\mu_G(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q} := \{\text{Op}(g)(\lambda) \ : \ g(y, y', \eta, \lambda) \in R^\mu_G(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q}\}.$$

(2.2.4) is just the space of parameter-dependent edge-operators of Green type on the (stretched) wedge $X^\wedge \times \Omega$. From the definition we have

(2.2.5)

$$\mathcal{Y}^\mu_G(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q} \subset \{\text{Op}(g)(\lambda) \ : \ g(y, y', \eta, \lambda) \in S^\mu_{\text{cl}}(\Omega \times \Omega \times \mathbb{R}^{q+l}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}^\sigma_P(X^\wedge))\}$$

for all $s \in \mathbb{R}$; an analogous relation is true for the space of adjoints; both inclusions then characterise the space $\mathcal{Y}^\mu_G(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q}$. In particular,

$$C \in \mathcal{Y}^{-\infty}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q} := \bigcap_{\mu \in \mathbb{R}} \mathcal{Y}^\mu_G(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q}$$

is characterised by the relations

$$C \in \mathcal{S}(\mathbb{R}^l, C^\infty(\Omega \times \Omega, \mathcal{S}^\sigma_P(X^\wedge) \hat{\otimes}_\pi \mathcal{K}^{-s,-\gamma}(X^\wedge))),$$

$$C^* \in \mathcal{S}(\mathbb{R}^l, C^\infty(\Omega \times \Omega, \mathcal{S}^{-\gamma}_Q(X^\wedge) \hat{\otimes}_\pi \mathcal{K}^{s,\sigma}(X^\wedge)))$$

for all $s \in \mathbb{Z}$. This employs the fact that

$$\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}_P^\sigma(X^\wedge)) \cong \mathcal{S}_P^\sigma(X^\wedge)\hat{\otimes}_\pi \mathcal{K}^{-s,-\gamma}(X^\wedge),$$

and, similarly, for dual maps.

The above constructions for "abstract" operator-valued symbols and associated pseudo-differential operators may be applied to (2.2.5) and (2.2.4), and we have to observe that general results specialise to the specific case of Green symbols and operators in the right way. As noted before, the above Hilbert space $\tilde{E}$ may be replaced by a Fréchet space with group action, and in the present case we have $\mathcal{S}_P^\sigma(X^\wedge)$. To define a semi-norm system for the Fréchet topology in the space $\mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q}$ we consider the relations for all $s \in \mathbb{Z}$ (which suffices) together with analogous relations for adjoints. The discussion for adjoints will be easy as well and left to the reader.

Let $K \subset \Omega \times \Omega$ be as before, and assume for simplicity that $(y, y') \in K \Leftrightarrow (y', y) \in K$. Then $\mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q;K}$ denotes the subspace of all $G(\lambda) \in \mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q}$ whose (operator-valued) distributional kernel (with respect to (y, y')-variables) is supported by K. Then every $G(\lambda) \in \mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q}$ admits a decomposition

$$G(\lambda) = G_0(\lambda) + C(\lambda)$$

for $G_0(\lambda) \in \mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q;K}$, $C(\lambda) \in \mathcal{Y}^{-\infty}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q}$.

In other words, we have

$$\mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q} = \mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q;K} + \mathcal{Y}^{-\infty}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q} \tag{2.2.6}$$

as vector spaces. $\mathcal{Y}^{-\infty}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q}$ is Fréchet in a canonical way. So we have to Fréchet topologise the space $\mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q;K}$. This can be done as in the general situation.

Lemma 2.2.4 *Given $G_0(\lambda) \in \mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q;K}$ we have*

$$g(y, \eta, \lambda) := e_{-\eta}(y) G_0(\lambda) e_\eta(\cdot) \in R_G^\mu(\Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q;K} \tag{2.2.7}$$

where $G_0(\lambda) = \mathrm{Op}(g)(\lambda)$, and the space $R_G^\mu(\Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q;K}$, defined to be the set of all $g(y, \eta, \lambda)$ when $G_0(\lambda)$ runs over $\mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q;K}$, is a closed subspace of $R_G^\mu(\Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q}$.

Proof. The arguments are practically the same as in the set-up with abstract operator valued-symbols, cf. the beginning of this section, because Green symbols are, up to conditions for pointwise adjoints, operator-valued symbols where the second space $\tilde{E}$ is Fréchet. □

Now the bijection between $\mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q;K}$ and $R_G^\mu(\Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q;K}$ gives us a Fréchet topology also in $\mathcal{Y}_G^\mu(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q;K}$, and (2.2.6) can be equipped with the Fréchet topology of the non-direct sum.

The next step is the space of smoothing Mellin plus Green symbols $R_{M+G}^\mu(\Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_S$ that we consider for the weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, (-(k+1), 0])$, $k \in \mathbb{N}$, and a given sequence S

of asymptotic types, cf. formula (1.3.19). Because of the representation of the Mellin part of a symbol as a sum (1.3.11) with symbols (1.3.12) satisfying condition (1.3.13), we may concentrate on the space consisting of first or second parts of such sums; the full space is then a corresponding non-direct sum. In addition, we observed that weights may be normalized as (1.3.15). For that reason, it suffices to study that case.

Let us take smoothing Mellin sums with $\beta_{j\alpha}$ as in (1.3.15). In other words, we content ourselves with the space of Mellin plus Green symbols $R^{\mu}_{M+G}(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{S_1}$ that is defined to be the set of all

$$a(y, y', \eta, \lambda) = m(y, y', \eta, \lambda) + g(y, y', \eta, \lambda)$$

where

$$g(y, y', \eta, \lambda) \in R^{\mu}_{G}(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{P,Q}$$

and

$$m(y, y', \eta, \lambda) = \omega_1(r[\eta, \lambda]) \sum_{j=0}^{k} r^j \sum_{|\alpha| \le j} \{\mathrm{op}_M^{\beta_{j\alpha} - \frac{n}{2}}(l_{j\alpha})(y, y')\}(\eta, \lambda)^{\alpha} \omega_2(r[\eta, \lambda]) \tag{2.2.8}$$

with $l_{j\alpha}(y, y', z)$ in $C^{\infty}(\Omega \times \Omega, M^{-\infty}_{R_{j\alpha}}(X))$, $\pi_{\mathbb{C}} R_{j\alpha} \cap \Gamma_{\frac{n+1}{2} - \beta_{j\alpha}} = \emptyset$. In this case the fixed asymptotic data are given by $S_1 := ((R_{j\alpha})_{|\alpha| \le j, j=0,\dots,k}; P, Q)$. If we consider $\tilde{\beta}_{j,\alpha}$ as in (1.3.15) and set $S_2 := ((\tilde{R}_{j\alpha})_{|\alpha| \le j, j=0,\dots,k}; P, Q)$, $\pi_{\mathbb{C}} \tilde{R}_{j\alpha} \cap \Gamma_{\frac{n+1}{2} - \tilde{\beta}_{j\alpha}} = \emptyset$, we can define exactly in the same way the space $R^{\mu}_{M+G}(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{S_2}$. In what follows we carry on the discussion by considering $R^{\mu}_{M+G}(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{S_1}$ but same results hold, of course, for $R^{\mu}_{M+G}(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{S_2}$.

Proposition 2.2.5 *Let $\omega(y, y') \in C^{\infty}(\Omega \times \Omega)$ be as before, then if one defines $g(y, y', \eta, \lambda) := (1 - \omega(y, y'))m(y, y', \eta, \lambda)$ where m is given by (2.2.8), we have*

$$\mathrm{Op}(g)(\lambda) \in \mathcal{Y}^{-\infty}(X^{\wedge} \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q}$$

for suitable asymptotic types P and Q.

Proof. Analogously to the standard pseudo-differential calculus, from symbols $g(y, y', \eta, \lambda)$ we can pass to left symbols $g_L(y, y', \eta, \lambda)$, such that $\mathrm{Op}(g - g_L)(\lambda)$ is of order $-\infty$ (here, with parameters). $g_L(y, \eta, \lambda)$ is determined by the asymptotic formula

$$g_L(y, \eta, \lambda) \sim \sum_{\alpha} \frac{1}{\alpha!} D^{\alpha}_{\eta} \partial^{\alpha}_{y'} g(y, y', \eta, \lambda)_{|_{y'=y}}.$$

Moreover, because of the factor $1 - \omega(y, y')$ all summands in the previous formula vanish, i.e., we may set $g_L \equiv 0$, and hence $\mathrm{Op}(g)(\lambda)$ itself is of order $-\infty$. □

Let $\mathcal{Y}^{\mu}_{M+G}(X^{\wedge} \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_1}$ denote the space of all $\mathrm{Op}(a)(\lambda)$, where $a(y, y', \eta, \lambda)$ belongs to $R^{\mu}_{M+G}(\Omega \times \Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{S_1}$, and let $\mathcal{Y}^{\mu}_{M+G}(X^{\wedge} \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_1;K}$ be the subspace of all elements whose distributional kernel (with respect to (y, y')-variables) is supported by K. Then every $A(\lambda) \in \mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_1}$ admits a decomposition $A(\lambda) = A_0(\lambda) + C(\lambda)$ for $A_0(\lambda) \in \mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_1;K}$ and $C(\lambda) \in \mathcal{Y}^{-\infty}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q}$. This gives us a decomposition

$$\mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_1} = \mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_1;K} + \mathcal{Y}^{-\infty}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{P,Q} \tag{2.2.9}$$

as vector spaces. To get a Fréchet topology we have to Fréchet topologize the first summand on the right of (2.2.9).

The bijection between $\mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_1;K}$ and $R^{\mu}_{M+G}(\Omega \times \mathbb{R}^{q+l}, \boldsymbol{g})_{S_1;K}$ gives us a Fréchet topology in the operator space and then the Fréchet topology of (2.2.9) as non-direct sum. In a similar manner we can define $\mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_2}$ and $\mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^2)_{S_1;K}$ and topologize the former space by means of the latter. We finally set

$$\mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S} := \mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_1} + \mathcal{Y}^{\mu}_{M+G}(X^\wedge \times \Omega, \boldsymbol{g}; \mathbb{R}^l)_{S_2}$$

as a non-direct Fréchet sum.

A standard procedure now gives us a Fréchet topology also in the spaces $\mathcal{Y}^{\mu}_{M+G}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}, \mathbb{R}^l)_S$, $S \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g}; \boldsymbol{v})$, namely, as non-direct sum of spaces like

$$[\varphi_j]\mathcal{Y}^{\mu}_{M+G}(\mathbb{W}_j, \boldsymbol{g}; \boldsymbol{v}, \mathbb{R}^l)_S[\psi_j] + \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}, \mathbb{R}^l)_{P,Q}$$

where $\{\varphi_j\}$, $\{\psi_j\}$ are C^∞ functions in a collar neighbourhood of $\partial\mathbb{W}$, written as a union of neighbourhoods $\mathbb{W}_j$, and $\sum \varphi_j = 1$ near $\partial\mathbb{W}$, $\psi_j = 1$ on supp φ_j for all j.

2.3 Holomorphic families of edge operators

Similarly to the previous section one could show that the space $\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m})$ of parameter-dependent edge operators with parameters $(\lambda, \tau) \in \mathbb{R}^{l+m}$ can be viewed as a union of Fréchet subspaces of the kind $\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m})_R$, $R \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g}; \boldsymbol{v})$; recall that R contains asymptotic information that is contained in Green and Mellin operators. However, we want to avoid here the corresponding lenghty discussion and proceed in a more direct way.

Let $w = \tau + i\beta$, τ, $\beta \in \mathbb{R}^m$, and set $\Gamma_\beta := \{w \in \mathbb{C}^m : \text{Im}\, w = \beta\}$. If the parameter $\tau =$ Re w varies on Γ_β, we also write $\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \Gamma_\beta)_R$.

We want to get an analogue of Section 2.1 to operator families $\mathcal{A}(\lambda, \tau) \in \mathcal{Y}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m})_R$ with respect to the parameter $\tau \in \mathbb{R}$. This may be done for the summands in the representation

$$\begin{aligned} \mathcal{A}(\lambda, \tau) &= \sum_{j=1}^{L} \text{diag}(\sigma\varphi_j, \varphi_j)\mathcal{A}_j(\lambda, \tau)\text{diag}(\tilde{\sigma}\psi_j, \psi_j) \\ &+ \text{diag}(1 - \sigma, 0)A_{\text{int}}(\lambda, \tau)\text{diag}(1 - \tilde{\tilde{\sigma}}, 0) + \mathcal{G}(\lambda, \tau) \end{aligned}$$

separately, cf. the notation of Definition 1.4.3.

Let us first define $\mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m} \times \mathbb{C}^m)_{P,Q}$ as the space of all operator functions

$$g(\lambda, w) \in \mathcal{A}(\mathbb{C}^m, \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)_{P,Q}) \tag{2.3.1}$$

such that

$$g(\lambda, \tau + i\beta) \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m}_{\lambda,\tau})_{P,Q}$$

for every $\beta \in \mathbb{R}^m$, uniformly in $\beta \in K$ for every $K \subset\subset \mathbb{R}^m$. Recall that $\mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m}_{\lambda,\tau})_{P,Q}$ is a Fréchet space in a natural way, cf. Section 1.4. Moreover, the space $L^\mu_{\mathrm{cl}}(\mathrm{int}\mathbb{W}; E, F; \mathbb{R}^l \times \mathbb{C}^m)$ is defined to be the set of all

$$p(\lambda, w) \in \mathcal{A}(\mathbb{C}^m, L^\mu_{\mathrm{cl}}(\mathrm{int}\mathbb{W}; E, F; \mathbb{R}^l), \tag{2.3.2}$$

such that

$$p(\lambda, \tau + i\beta) \in L^\mu_{\mathrm{cl}}(\mathrm{int}\mathbb{W}; E, F; \mathbb{R}^{l+m}_{\lambda,\tau})$$

for every $\beta \in \mathbb{R}^m$, uniformly in $\beta \in K$ for every $K \subset\subset \mathbb{R}^m$.

Holomorphic dependence of summands $\mathcal{A}_j(\lambda, w)$ on $w \in \mathbb{C}^m$ will be introduced on the level of local amplitude functions in $R^\mu(\Omega \times \Omega \times \mathbb{R}^{q+l+m}_{\eta,\lambda,\tau}, \boldsymbol{g}; \boldsymbol{w})$. We take the representation of Definition 1.3.5 for $U = \Omega \times \Omega$ and $\mathbb{R}^{q+l+m}$ in place of $\mathbb{R}^q$ and consider the summands of $a(y, y', \eta, \lambda, \tau)$ separately, that means

$$\big(\sigma(a_M + a_F)\tilde{\sigma}\big)(y, y', \eta, \lambda, \tau)$$

and

$$r(y, y', \eta, \lambda, \tau) \in R^\mu_{M+G}(\Omega \times \Omega \times \mathbb{R}^{q+l+m}, \boldsymbol{g}; \boldsymbol{w}).$$

Definition 2.3.1 *Let $U \subseteq \mathbb{R}^p$ open, $S \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g})$. Then $R^\mu_{M+G}(U \times \mathbb{R}^{q+l} \times \mathbb{C}^m, \boldsymbol{g}; \boldsymbol{w})_S$ is defined to be the space of all*

$$a(y, \eta, \lambda, w) \in \mathcal{A}(\mathbb{C}^m_w, R^\mu_{M+G}(U \times \mathbb{R}^{q+l}, \boldsymbol{g}; \boldsymbol{w})_S)$$

(cf. Remark 1.3.4) such that

$$a(y, \eta, \lambda, \tau + i\beta) \in R^\mu_{M+G}(U \times \mathbb{R}^{q+l+m}, \boldsymbol{g}; \boldsymbol{w})_S$$

for every $\beta \in \mathbb{R}^n$, uniformly in $\beta \in K$ for every $K \subset\subset \mathbb{R}^m$.

Let us now turn to $\big(\sigma(a_M + a_F)\tilde{\sigma}\big)(y, y', \eta, \lambda, \tau)$. According to Remark 1.3.1 we have

$$\big(\sigma(a_M + a_F)\tilde{\sigma}\big)(y, y', \eta, \lambda, \tau) = \sigma r^{-\mu} \mathrm{op}_M^{\gamma - n/2}(f)(y, y', \eta, \lambda, \tau)\tilde{\sigma} + g_0(y, y', \eta, \lambda, \tau) \tag{2.3.3}$$

with

$$f(r, y, y', z, \eta, \lambda, \tau) = \tilde{f}(r, y, y', z, r\eta, r\lambda, r\tau) \tag{2.3.4}$$

for $\tilde{f}(r,y,y',z,\tilde{\eta},\tilde{\lambda},\tilde{\tau}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega \times \Omega, M_{\mathcal{O}}^\mu(X;\mathbb{R}^{q+l+m}_{\tilde{\eta},\tilde{\lambda},\tilde{\tau}})$ and a certain $g_0(y,y',\eta,\lambda,\tau) \in R_G^\mu(\Omega \times \Omega \times \mathbb{R}^{q+l+m}, \boldsymbol{g}_\infty)_{\mathcal{O}}$. The flat Green symbol can be subsumed under $r(y,y',\eta,\lambda,\tau)$. Holomorphy of Green operator families is covered by Definition 2.3.1. Thus it remains to look at the first summand on the right of (2.3.3).

Given any family $f(r,y,y',z,\eta,\lambda,\tau)$ of the form (2.3.4) for a given $\tilde{f}$, we set $\sigma_M(f)(y,y',z) := \tilde{f}(0,y,y',z,0,0,0)$ which is, by notation, the conormal symbol of (2.3.3) from the calculus for conical singularities.

In the following, assume for a moment that operator-valued functions are independent of y and y'. An $f(r,z,\eta,\lambda,w)$ for $(r,z,\eta,\lambda,w) \in \mathbb{R}_+ \times \mathbb{C}_z \times \mathbb{R}^{q+l}_{\eta,\lambda} \times \mathbb{C}^m_w$ with values in $L^\mu_{\mathrm{cl}}(X)$ is said to be edge-degenerate and holomorphic in $w \in \mathbb{C}^m$, if there is an $L^\mu_{\mathrm{cl}}(X)$-valued function

$$\tilde{f}(r,z,\tilde{\eta},\tilde{\lambda},\tilde{w}) \in \mathcal{A}(\mathbb{C}^m_{\tilde{w}}, C^\infty(\mathbb{R}_+, M_{\mathcal{O}}^\mu(X;\mathbb{R}^{q+l}_{\tilde{\eta},\tilde{\lambda}}))$$

such that $f(r,z,\eta,\lambda,w) := \tilde{f}(r,z,r\eta,r\lambda,rw)$ belongs to the space $\mathcal{A}(\mathbb{C}^m_w, C^\infty(\mathbb{R}_+, M_{\mathcal{O}}^\mu(X;\mathbb{R}^{q+l}_{\eta,\lambda}))$ and $\tilde{f}(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\tau}+ir\beta)$ to in $C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^\mu(X;\mathbb{R}^{q+l+m}_{\tilde{\eta},\tilde{\lambda},\tilde{\tau}}))$ for every $\beta \in \mathbb{R}$ uniformly in $c \le \beta \le c'$ for every $c \le c'$. Let

$$M_{\mathcal{O}}^\mu(X;\overline{\mathbb{R}}_+ \times \mathbb{R}^{q+l}_{\eta,\lambda} \times \mathbb{C}^m) \tag{2.3.5}$$

denote the space af all edge-degenerate holomorphic operator families in that sense. Similar notation makes sense for the case of vector bundles E', F' on X; we then write, for the corresponding spaces, $M_{\mathcal{O}}^\mu(X;E',F';\overline{\mathbb{R}}_+ \times \mathbb{R}^{q+l}_{\eta,\lambda} \times \mathbb{C}^m)$.

Let us now consider an element $h(r,z,\eta,\lambda,\tau) = \tilde{h}(r,z,r\eta,r\lambda,r\tau)$ for any $\tilde{h}(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\tau})$ in $C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^\mu(X;\mathbb{R}^{q+l+m}_{\tilde{\eta},\tilde{\lambda},\tilde{\tau}}))$, and

$$K(h)(r,z,\eta,\lambda,\theta) = \int e^{i\tau\theta} h(r,z,\eta,\lambda,\tau) đ\tau. \tag{2.3.6}$$

Theorem 2.3.2 *For every $\psi(\theta) \in C_0^\infty(\mathbb{R}^m)$ the function*

$$H(\psi)h(r,z,\eta,\lambda,w) = \int e^{-i\tau w} \psi(\theta) K(h)(r,z,\eta,\lambda,\theta) d\theta \tag{2.3.7}$$

belongs to the space $M_{\mathcal{O}}^\mu(X;\overline{\mathbb{R}}_+ \times \mathbb{R}^{q+l} \times \mathbb{C}^m)$; if $\psi(\theta)$ is a cut-off function, we have

$$H(\psi)h(r,z,\eta,\lambda,w)|_{\mathrm{Im}\ w=0} = h(r,z,\eta,\lambda,\tau)$$

modulo elements of the form $c(r,z,\eta,\lambda,\tau) = \tilde{c}(r,z,r\eta,r\lambda,r\tau)$ for $\tilde{c}(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\tau})$ in $C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{-\infty}(X;\mathbb{R}^{q+l+M}_{\tilde{\eta},\tilde{\lambda},\tilde{\tau}}))$. Moreover, we have $\sigma_M(H(\psi)h)(z) = \sigma_M(h)(z)$.

Proof. For convenience, we concentrate on the case of r-independent $\tilde{h}$, i.e., $\tilde{h}(z,\tilde{\eta},\tilde{\lambda},\tilde{\tau}) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}^{q+l+m})$. The extra r-dependence in the general case, smooth up to $r=0$, does not cause any difficulties and may be ignored. Let us set

$$K(h)(r,z,\eta,\lambda,\theta) = \int e^{i\tau\theta} h(r,z,\eta,\lambda,\tau) đ\tau$$

and

$$\tilde{K}(\tilde{h})(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\theta}) = \int e^{i\tilde{\tau}\tilde{\theta}} \tilde{h}(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\tau}) đ\tilde{\tau}$$

where $h(r,z,\eta,\tau) = \tilde{h}(z,r\eta,r\tau)$. For arbitrary cut-off functions $\psi(\theta)$ and $\tilde{\psi}(\tilde{\theta})$ we then have

$$(H(\psi)h)(r,z,\eta,\lambda,w) = \int e^{-iw\theta}\psi(\theta)K(h)(r,z,\eta,\lambda,\theta)d\theta$$

and

$$(H(\tilde{\psi})\tilde{h})(r,z,\tilde{\eta},\tilde{\lambda},\tilde{w}) := \int e^{-i\tilde{w}\tilde{\theta}}\tilde{\psi}(\tilde{\theta})\tilde{K}(\tilde{h})(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\theta})d\tilde{\theta}.$$

Let us now show that

$$(H(\psi)h)(r,z,\eta,\lambda,w) = (H(\psi_r)\tilde{h})(r,z,r\eta,r\lambda,rw)$$

for $\psi_r(\theta) = \psi(r\theta)$. In fact, we have

$$\begin{aligned} K(h)(r,z,\eta,\lambda,\theta) &= \int e^{i\tau\theta} h(r,z,\eta,\lambda,\tau) đ\tau \\ &= \int e^{i\tau\theta}\tilde{h}(r,z,r\eta,r\lambda,r\tau)đ\tau \\ &= r^{-m}\int e^{ir^{-1}\tilde{\tau}\theta}\tilde{h}(r,z,r\eta,r\lambda,\tilde{\tau})đ\tilde{\tau} \\ &= r^{-m}\tilde{K}(\tilde{h})(r,z,\tilde{\eta},\tilde{\lambda},r^{-1}\theta)|_{\tilde{\eta}=r\eta,\tilde{\lambda}=r\lambda}. \end{aligned}$$

Now the properties that are required for the space (2.3.5) are satisfied for (2.3.7); all steps to verify that are evident by the general properties of kernel cut-off operations (see Section 2.1). The only more subtle point is perhaps the smoothness in r up to zero. However, this is a consequence of Remark 2.1.7.□

Remark 2.3.3 *There is an immediate analogue of Theorem* 2.3.2 *for the case of y-dependent functions $\tilde{h}$ in $C^\infty(U, C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{\mu}(X;\mathbb{R}^{q+l+m}_{\tilde{\eta},\tilde{\lambda},\tilde{\tau}})))$, $U \subseteq \mathbb{R}^p$ open, as well as in the context of pairs of vector bundles E', F' on X.*

Notation in the following definition are analogous to those in Definition 1.4.3.

Definition 2.3.4 *$\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)_R$ for $R \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g}; \boldsymbol{v})$ is defined to be the space of all operator families*

$$\mathcal{A}(\lambda, w) = \sum_{j=1}^{L} \mathrm{diag}(\sigma\varphi_j, \varphi_j)\mathcal{A}_j(\lambda, w)\mathrm{diag}(\tilde{\sigma}\psi_j, \psi_j) \tag{2.3.8}$$

$$+ \ \mathrm{diag}(1-\sigma, 0)A_{\mathrm{int}}(\lambda, w)(1-\tilde{\tilde{\sigma}}, 0) + g(\lambda, w) \tag{2.3.9}$$

for arbitrary families of operators $\mathcal{A}_j(\lambda, w)$ with local amplitude functions being block matrices of the form

$$a_j(y, \eta, \lambda, w) = \begin{pmatrix} b_j(y, \eta, \lambda, w) & 0 \\ 0 & 0 \end{pmatrix} + r_j(y, \eta, \lambda, w)$$

for $r_j(y, \eta, \lambda, w) \in R^\mu_{M+G}(\Omega \times \mathbb{R}^{q+l} \times \mathbb{C}^m, \boldsymbol{g}; \boldsymbol{v})_R$, cf. Definition 2.3.1, and

$$b_j(y, \eta, \lambda, w) = \sigma_1 r^{-\mu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(f_j)(y, \eta, \lambda, w)\tilde{\sigma}_1,$$

$f_j(r, y, z, \eta, \lambda, w) \in C^\infty(\Omega, M^\mu_{\mathcal{O}}(X; E', F'; \overline{\mathbb{R}}_+ \times \mathbb{R}^{q+l}_{\eta,\lambda} \times \mathbb{C}^m))$ with cut-off functions $\sigma_1(r)$ and $\tilde{\sigma_1}(r)$ such that $\sigma_1 = 1$ on $\mathrm{supp}\,\sigma$ and $\tilde{\sigma}_1 = 1$ on $\mathrm{supp}\,\tilde{\sigma}$. Moreover, we assume $A_{\mathrm{int}}(\lambda, w) \in L^\mu_{\mathrm{cl}}(\mathrm{int}\mathbb{W}; E, F; \mathbb{R}^l \times \mathbb{C}^m)$, and $g(\lambda, w) \in \mathcal{A}(\mathbb{C}^m, \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)_{P,Q})$ (with P, Q being contained in R, cf. notation of Section 1.3) such that

$$g(\lambda, w)|_{\mathbb{R}^l \times \Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)_{P,Q})$$

for every $\beta \in \mathbb{R}^m$, uniformly in $\beta \in K$ for every $K \subset\subset \mathbb{R}^m$.

Set $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m) = \bigcup_R \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)_R$.

The space $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)$ may be regarded as an analogue of $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)$ with holomorphy in a extra complex parameter $w \in \mathbb{C}^m$. To introduce holomorphy combined with parameter-dependence we also could refer to a Fréchet topology in $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l)_R$ for every R. Our point of view makes it necessary to explain differentiations of $\mathcal{A}(\lambda, w)$ with respect to parameters on the level of amplitude functions that are involved in Definition 2.3.4, but this not a problem. In particular, we can form

$$D^\alpha_\lambda D^\beta_w \mathcal{A}(\lambda, w) \qquad \text{for} \quad \mathcal{A}(\lambda, w) \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)$$

for every $\alpha \in \mathbb{N}^l$, $\beta \in \mathbb{N}^m$, and get again elements in $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)$.

There is, in fact, a decrease in the orders when we differentiate. First, let us formulate the following observation.

Remark 2.3.5 *Let* $\mathcal{A}(\lambda,\tau) \in \mathcal{Y}^{\mu}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^{l}\times\mathbb{C}^{m})$*, and set*

$$\mathcal{A}_{\beta}(\lambda,\tau) := \mathcal{A}(\lambda,w)|_{\mathrm{Im}\ w=\beta} \in \mathcal{Y}^{\mu}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^{l}\times\Gamma_{\beta}).$$

Then

$$\sigma^{\mu}_{\psi}(\mathcal{A}) := \sigma^{\mu}_{\psi}(\mathcal{A}_{\beta}), \quad \sigma^{\mu}_{\wedge}(\mathcal{A}) := \sigma^{\mu}_{\wedge}(\mathcal{A}_{\beta}) \tag{2.3.10}$$

are independent of the choice of β*.*

This is a consequence of the properties of the kernel cut-off construction, applied to the involved amplitude functions in expression (2.3.8). As usual, we set

$$\sigma^{\mu}(\mathcal{A}) := \big(\sigma^{\mu}_{\psi}(\mathcal{A}), \sigma^{\mu}_{\wedge}(\mathcal{A})\big).$$

Recall that we assumed $\boldsymbol{g} = (\gamma, \gamma-\mu, \theta)$. Write, for abbrevation,

$$\mathcal{Y}^{\mu} := \mathcal{Y}^{\mu}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^{l}\times\mathbb{C}^{m}),$$

and the same with the subscript R; set $\mathcal{Y}^{\mu-1} := \{\mathcal{A}\in\mathcal{Y}^{\mu} : \sigma^{\mu}(\mathcal{A}) = 0\}$. This space still refers to $\boldsymbol{g}$, and we have a pair of principal symbols $\sigma^{\mu-1}(\mathcal{A})$ for the elements of $\mathcal{Y}^{\mu-1}$. We now define inductively, for all $j = 1, 2, ...,$

$$\mathcal{Y}^{\mu-j} := \{\mathcal{A}\in\mathcal{Y}^{\mu-(j-1)} : \sigma^{\mu-(j-1)}(\mathcal{A}) = 0\}.$$

Remark 2.3.6 *For* $\mathcal{A}(\lambda,w) \in \mathcal{Y}^{\mu}$ *we have* $D^{\alpha}_{\lambda}D^{\beta}_{w}\mathcal{A}(\lambda,w) \in \mathcal{Y}^{\mu-(|\alpha|+|\beta|)}$ *for every* $\alpha\in\mathbb{N}^{l}$*,* $\beta\in\mathbb{N}^{m}$*.*

Theorem 2.3.7 *For every sequence* $\mathcal{A}_j \in \mathcal{Y}^{\mu-j}_{R}$*,* $R \in \boldsymbol{As}_{M+G}(X,\boldsymbol{g};\boldsymbol{v})$*,* $j\in\mathbb{N}$*, there exists an element* $\mathcal{A}\in\mathcal{Y}^{\mu}_{R}$ *such that*

$$\mathcal{A} - \sum_{j=0}^{N}\mathcal{A}_j \in \mathcal{Y}^{\mu-(N+1)}_{R}$$

for all $N\in\mathbb{N}$*, moreover* $\mathcal{A}$ *is unique* mod $\mathcal{Y}^{-\infty}_{P,Q}$ *(with asymptotic types* P*,* Q *defined by* R*).*

Proof. It is a direct consequence of corresponding results on asymptotic summation of underlying local amplitude functions. □

Given any $\mathcal{A}(\lambda,\tau) \in \mathcal{Y}^{\mu}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^{l+m})_{R}$, we set $K(\mathcal{A})(\lambda,\zeta) = \int e^{i\tau\zeta}\mathcal{A}(\lambda,\tau)đ\tau$ and

$$H(\varphi)\mathcal{A}(\lambda,w) = \int e^{-iw\zeta}\varphi(\zeta)K(\mathcal{A})(\lambda,\zeta)d\zeta\ ,$$

for any $\varphi\in C^{\infty}_{0}(\mathbb{R}^{m})$ and $w = \tau + i\beta \in \mathbb{C}^{m}$.

Theorem 2.3.8 *For every $\varphi(\zeta) \in C_0^\infty(\mathbb{R}^m)$ the operator $H(\varphi)$ induces a linear map*

$$H(\varphi) : \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m})_R \longrightarrow \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)_R$$

for every $\mu \in \mathbb{R}$. For every $\beta \in \mathbb{R}^m$ there are constants $c_\alpha(\varphi; \beta)$ such that

$$H(\varphi)\mathcal{A}(\lambda, \tau + i\beta) \sim \sum_{\alpha \in \mathbb{N}^m} c_\alpha(\varphi; \beta) D_\tau^\alpha \mathcal{A}(\lambda, \tau) \tag{2.3.11}$$

as an asymptotic sum in $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m})_R$.

Proof. The kernel cut-off operation $H(\varphi)$ can be applied to all the local amplitude functions that are involved in the summands of the representation $\mathcal{A}(\lambda, \tau) \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m})$. For all those amplitude functions we have an analogue of Remark 2.1.4, here with τ as a parameter. This gives us formula (2.3.11).□

From Theorem 2.1.6 and its analogues for specific amplitude functions involved in the representation of $\mathcal{A}(\lambda, \tau)$ we get the following result.

Theorem 2.3.9 *Let $\mathcal{A}(\lambda, \tau) \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m})_R$, and let $\psi(\zeta) \in C_0^\infty(\mathbb{R}^m)$, $\psi(\zeta) = 1$ in a neighbourhood of $\zeta = 0$. Then we have*

$$\mathcal{A}(\lambda, \tau) - H(\psi)\mathcal{A}(\lambda, \tau) \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m})_{P,Q} \tag{2.3.12}$$

for P and Q defined by R. Moreover, we have in formula (2.3.11) for ψ instead of φ, $c_0(\psi; 0) = 1$.

Remark 2.3.10 (i) *Let ψ, $\tilde{\psi} \in C_0^\infty(\mathbb{R}^m)$ be functions as in Theorem 2.3.9, apply $H(\tilde{\psi})$ to $H(\psi)a(\lambda, \tau)$ and let $H(\tilde{\psi})H(\psi)a(\lambda, w)$ denote the corresponding extension as an element in the space $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)_R$. Then we have*

$$H(\psi)a(\lambda, w) - H(\tilde{\psi})H(\psi)a(\lambda, w) \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)_{P,Q} \ .$$

(ii) *$\mathcal{A}(\lambda, w) \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)_R$ and $\mathcal{A}(\lambda, \tau + i\beta) \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)_R$ for any fixed $\beta \in \mathbb{R}$ implies $\mathcal{A}(\lambda, w) \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)_{P,Q}$.*

Remark 2.3.11 *Let $\mathcal{A}(\lambda, \tau) \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m})_R$ be parameter-dependent elliptic.*

Then $H(\psi)\mathcal{A}(\lambda, \tau + i\beta)$ is parameter-dependent elliptic for every $\beta \in \mathbb{R}^m$, uniformly for β varying in any compact set of $\mathbb{R}^m$.

2.4 Ellipticity of holomorphic families

Definition 2.4.1 *An element* $\mathcal{A}(\lambda, w) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)$ *is called elliptic, if there is a* $\beta \in \mathbb{R}^m$ *such that* $\mathcal{A}(\lambda, \tau + i\beta) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m}_{\lambda,\tau})$ *is parameter-dependent elliptic with parameters* λ, τ *in the sense of Definition* 1.5.1.

Remark 2.4.2 *The ellipticity of* $\mathcal{A}(\lambda, w) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)$ *is independent of the choice of* β *in Definition* 2.4.1. *This a direct consequence of Remark* 2.3.5.

An element $\mathcal{P}(\lambda, w) \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1}; \mathbb{R}^l \times \mathbb{C}^m)$ is called a parametrix of the element $\mathcal{A}(\lambda, w)$ in $\mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)$, if

$$\mathcal{C}_l(\lambda, w) = \mathcal{I} - \mathcal{P}(\lambda, w)\mathcal{A}(\lambda, w) \in \mathcal{Y}^{-\infty}(\mathbb{W}, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}}; \mathbb{R}^l \times \mathbb{C}^m)\ , \tag{2.4.1}$$

$$\mathcal{C}_r(\lambda, w) = \mathcal{I} - \mathcal{A}(\lambda, w)\mathcal{P}(\lambda, w) \in \mathcal{Y}^{-\infty}(\mathbb{W}, \tilde{\tilde{\boldsymbol{g}}}; \tilde{\tilde{\boldsymbol{v}}}; \mathbb{R}^l \times \mathbb{C}^m)\ , \tag{2.4.2}$$

cf. notation of Theorem 1.5.2.

Remark 2.4.3 *Assume an element* $\mathcal{P}(\lambda, w) \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1}; \mathbb{R}^l \times \mathbb{C}^m)$ *satisfies the following conditions*

$$\mathcal{C}_l(\lambda, \tau + i\beta) = \mathcal{I} - \mathcal{P}(\lambda, \tau + i\beta)\mathcal{A}(\lambda, \tau + i\beta) \in \mathcal{Y}^{-\infty}(\mathbb{W}, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}}; \mathbb{R}^{l+m}_{\lambda,\tau}),$$
$$\mathcal{C}_r(\lambda, \tau + i\beta) = \mathcal{I} - \mathcal{A}(\lambda, \tau + i\beta)\mathcal{P}(\lambda, \tau + i\beta) \in \mathcal{Y}^{-\infty}(\mathbb{W}, \tilde{\tilde{\boldsymbol{g}}}; \tilde{\tilde{\boldsymbol{v}}}; \mathbb{R}^{l+m}_{\lambda,\tau})$$

for one $\beta \in \mathbb{R}^m$. *Then, by virtue of Remark* 2.3.10, $\mathcal{P}(\lambda, w)$ *fulfills relations* (2.4.1) *and* (2.4.2).

Theorem 2.4.4 *Let* $\mathcal{A}(\lambda, w) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)$ *be elliptic. Then there is a parametrix* $\mathcal{P}(\lambda, w) \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1}; \mathbb{R}^l \times \mathbb{C}^m)$.

Proof. Applying Theorem 1.5.2 to $\mathcal{A}(\lambda, \tau) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^{l+m}_{\lambda,\tau})$ we get a parametrix $\mathcal{F}(\lambda, \tau) \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1}; \mathbb{R}^{l+m})$. Choose any $\psi(\zeta) \in C_0^{\infty}(\mathbb{R}^m)$ that equals 1 in a neighbourhood of $\zeta = 0$ and form $\mathcal{P}(\lambda, w) = H(\psi)\mathcal{F}(\lambda, w) \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1}; \mathbb{R}^l \times \mathbb{C}^m)$. Then relation (2.4.1) shows that $\mathcal{P}(\lambda, \tau) \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1}; \mathbb{R}^{l+m}_{\lambda,\tau})$ is also a parametrix of $\mathcal{A}(\lambda, \tau)$. The assertion then follows from Remark 2.4.3.□

Remark 2.4.5 *Let* $\mathcal{A}(\lambda, w) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}^l \times \mathbb{C}^m)$ *be elliptic. Then, for every* $s \in \mathbb{R}$

$$\mathcal{A}(\lambda, w) : \mathcal{W}^{s,\gamma}(\mathbb{W}, \boldsymbol{m}) \to \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, \boldsymbol{n}) \tag{2.4.3}$$

is a family of Fredholm operators holomorphic in $w \in \mathbb{C}^m$, *and there exists a* $c > 0$ *such that* (2.4.3) *is invertible for* $|(\lambda, w)| > 0$.

Theorem 2.4.6 *Let $\mathcal{A}(\lambda,\tau) \in \mathcal{Y}^{\mu}(\mathbb{W},\boldsymbol{g};\mathbb{R}^{l+m})_R$ be a parameter dependent elliptic element, and let $\psi(\zeta) \in C_0^{\infty}(\mathbb{R}^m)$ be a cut-off function (that equals 1 in a neighbourhood of $\zeta = 0$). Then $\mathcal{A}_0(\lambda,w) := H(\psi)\mathcal{A}(\lambda,w) \in \mathcal{Y}^{\mu}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^l \times \mathbb{C}^m)_R$ is elliptic in the sense of Definition* 2.4.1.

Proof. By virtue of Theorem 2.3.9 we see that $\mathcal{A}_0(\lambda,w) \in \mathcal{Y}^{\mu}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^{l+m})_R$ is parameter-dependent elliptic, with parameters $(\lambda,\tau) \in \mathbb{R}^{l+m}$. Thus, $\mathcal{A}_0(\lambda,w)$ satisfies conditions of Definition 2.4.1 for $\beta = 0$, and hence, by Remark 2.4.2 for all $\beta \in \mathbb{R}$. □

2.5 The algebra of corner symbols

We now specify our holomorphic operator spaces to the case $m = 1$ and introduce notation in analogy to cone Mellin symbols (cf. Section 1.2). Let

$$\mathcal{M}^{\mu}_{R,\mathcal{O}}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^l) = \{\mathcal{A}(\lambda,w) : \mathcal{A}(\lambda,iw) \in \mathcal{Y}^{\mu}(\mathbb{W},\boldsymbol{g};\boldsymbol{v};\mathbb{R}^l \times \mathbb{C}_w)_R\} .$$

We call $\mathcal{A}(\lambda,w)$ elliptic if $\mathcal{A}(\lambda,iw)$ is elliptic in the sense of Definition 2.4.1. For the case $l = 0$ we simply omit $\mathbb{R}^l$ in the notation.

Remark 2.5.1 *Let $\mathcal{A}(w) \in \mathcal{M}^{\mu}_{R,\mathcal{O}}(\mathbb{W},\boldsymbol{g};\boldsymbol{v})$ be elliptic. Then there is a countable set $D \subset \mathbb{C}$, where $D \cap \{w \in \mathbb{C} : \alpha \le \operatorname{Re} w \le \alpha'\}$ is finite for every $\alpha \le \alpha'$, such that the operators*

$$\mathcal{A}(w) : \mathcal{W}^{s,\gamma}(\mathbb{W};\boldsymbol{m}) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W};\boldsymbol{n}) \tag{2.5.1}$$

are isomorphisms for all $w \in \mathbb{C} \setminus D$ and all $s \in \mathbb{R}$, and there is an $S \in \boldsymbol{As}_{M+G}(X,\boldsymbol{g}^{-1};\boldsymbol{v}^{-1})$ such that $\mathcal{A}^{-1}(w) \in \mathcal{A}(\mathbb{C} \setminus D, \mathcal{Y}^{-\mu}(\boldsymbol{g}^{-1};\boldsymbol{v}^{-1})_S)$.

It is known, from "abstract" holomorphic Fredholm families operating between Hilbert spaces that $\mathcal{A}^{-1}(w)$ extends to $\mathbb{C}$ as a meromorphic operator function with poles at points $d_j \in D$ of certain multiplicities $n_j + 1$, where the Laurent coefficients at $(w - d_j)^{-(k+1)}$, $0 \le k \le n_j$, are operators of finite rank. In the present situation we know more, and the corresponding result below, can be formulated in terms of discrete asymptotic types.

Let us fix $S \in \boldsymbol{As}_{M+G}(X,\boldsymbol{g};\boldsymbol{v})$ and let $\boldsymbol{As}^{\bullet}(X;\boldsymbol{v})_S$ denote the set of all sequences $T = \{(d_j,n_j,L_j)\}_{j\in\mathbb{Z}}$, so-called discrete corner asymptotic types (associated with S), where the intersection of the set $\pi_{\mathbb{C}}T = \{d_j\}_{j\in\mathbb{Z}} \subset \mathbb{C}$ with $\{w \in \mathbb{C} : \alpha \le \operatorname{Re} w \le \alpha'\}$ is finite for every $\alpha \le \alpha'$, $n_j \in \mathbb{N}$, and $L_j \subset \mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};\boldsymbol{v})_{P,Q}$ (with P and Q defined by S) is a finite-dimensional subspace of operators of finite rank for all j.

Definition 2.5.2 *The space $\mathcal{M}^{-\infty}_{S,T}(\mathbb{W},\boldsymbol{g};\boldsymbol{v})$ for $S \in \boldsymbol{As}_{M+G}(X,\boldsymbol{g};\boldsymbol{v})$ and $T \in \boldsymbol{As}^{\bullet}(\mathbb{W},\boldsymbol{v})_S$ is defined to be the set of all $\mathcal{F}(w) \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}T, \mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};\boldsymbol{v})_{P,Q})$ that are meromorphic with poles at the points $d_j \in \pi_{\mathbb{C}}T$ of multiplicities $n_j + 1$ and Laurent coefficients at $(w - d_j)^{-(k+1)}$ belonging to L_j for all $0 \le k \le n_j$, $j \in \mathbb{Z}$, such that for any $\pi_{\mathbb{C}}T$-excision function $\chi(w)$ we have*

$\chi(w)\mathcal{F}(w)|_{\Gamma_\beta} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \Gamma_\beta)_{P,Q}$ *for every* $\beta \in \mathbb{R}$, *uniformly in* $c \leq \beta \leq c'$ *for arbitrary* c, c' (*the latter condition means the corresponding property on the level of local amplitude functions, see Definition* 2.3.4).

Here, by a $\pi_{\mathbb{C}}T$-excision function we mean any $\chi \in C^\infty(\mathbb{C})$ that vanishes in a neighbourhood of $\pi_{\mathbb{C}}T$ and equals 1 on $\{w \in \mathbb{C} : \text{dist } (w, \pi_{\mathbb{C}}T) > c\}$ for some $c > 0$. Let us set

$$\mathcal{M}^\mu_{S,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}) = \mathcal{M}^\mu_{S,\mathcal{O}}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}) + \mathcal{M}^{-\infty}_{S,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}) \tag{2.5.2}$$

as a non-direct sum.

Theorem 2.5.3 *Let* $\mathcal{A}(w) \in \mathcal{M}^\mu_{R,V}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, $\mathcal{B}(w) \in \mathcal{M}^\nu_{\tilde{R},\tilde{V}}(\mathbb{W}, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}})$ *for pairs of asymptotic types*

$$(R, V) \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g}; \boldsymbol{v}) \times \boldsymbol{As}^\bullet(\mathbb{W}; \boldsymbol{v})_R \ ,$$
$$(\tilde{R}, \tilde{V}) \in \boldsymbol{As}_{M+G}(X, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}}) \times \boldsymbol{As}^\bullet(\mathbb{W}; \tilde{\boldsymbol{v}})_{\tilde{R}},$$

where

$$\boldsymbol{g} = (\gamma - \nu, \gamma - (\mu + \nu); \Theta), \quad \boldsymbol{v} = (G, F; J, J_+),$$
$$\tilde{\boldsymbol{g}} = (\gamma, \gamma - \nu; \Theta), \quad \tilde{\boldsymbol{v}} = (E, G; J_-, J) \ .$$

Then we have $(\mathcal{A}\mathcal{B})(w) \in \mathcal{M}^{\mu+\nu}_{S,T}(\mathbb{W}, \boldsymbol{h}; \boldsymbol{w})$ *for a resulting pair* $(S, T) \in \boldsymbol{As}_{M+G}(X, \boldsymbol{h}; \boldsymbol{w}) \times \boldsymbol{As}^\bullet(\mathbb{W}, \boldsymbol{w})_S$, *where* $\boldsymbol{h}$ *and* $\boldsymbol{w}$ *are given by*

$$\boldsymbol{h} = (\gamma, \gamma - (\mu + \nu); \Theta), \quad \boldsymbol{w} = (E, F; J_-, J_+) \ .$$

Proof. From Theorem 1.4.7 we have a corresponding composition result for

$$\mathcal{A}_\beta(\tau) := \mathcal{A}(w)|_{\text{Re } w = \beta} \quad \text{and} \quad \mathcal{B}_\beta(\tau) := \mathcal{B}(w)|_{\text{Re } w = \beta}$$

for every $\beta \in \mathbb{R}$ such that $(\pi_{\mathbb{C}}V \cup \pi_{\mathbb{C}}\tilde{V}) \cap \Gamma_\beta = \emptyset$. In other words, it follows that

$$\mathcal{A}_\beta(\tau)\mathcal{B}_\beta(\tau) \in \mathcal{Y}^{\mu+\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \Gamma_\beta)_S$$

for a suitable $S \in \text{As}_{M+G}(X, \boldsymbol{h}; \boldsymbol{w})$. For the case $\pi_{\mathbb{C}}V = \pi_{\mathbb{C}}\tilde{V} = \emptyset$ this holds for all $\beta \in \mathbb{R}$, and we immediatelly obtain $\mathcal{A}(w)\mathcal{B}(w) \in \mathcal{M}^{\mu+\nu}_{S,\mathcal{O}}(\mathbb{W}, \boldsymbol{h}; \boldsymbol{w})$. In the general case there is an additive decomposition $\mathcal{A} = \mathcal{A}_{\mathcal{O}} + \mathcal{C}$, $\mathcal{B} = \mathcal{B}_{\mathcal{O}} + \mathcal{D}$ for $\mathcal{A}_{\mathcal{O}} \in \mathcal{M}^\mu_{R,\mathcal{O}}(X, \boldsymbol{g}; \boldsymbol{v})$, $\mathcal{C} \in \mathcal{M}^{-\infty}_{R,V}(X, \boldsymbol{g}; \boldsymbol{v})$, $\mathcal{B}_{\mathcal{O}} \in \mathcal{M}^\nu_{\tilde{R},\mathcal{O}}(X, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}})$, $\mathcal{D} \in \mathcal{M}^{-\infty}_{\tilde{R},\tilde{V}}(X, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}})$.

It is an easy consequence of the definitions that

$$\mathcal{A}_{\mathcal{O}}\mathcal{D}, \ \mathcal{C}\mathcal{B}_{\mathcal{O}}, \ \ \mathcal{C}\mathcal{D} \in \mathcal{M}^{-\infty}_{S,T}(X, \boldsymbol{h}; \boldsymbol{w})$$

for suitable (S,T) as asserted.$\square$

An element $\mathcal{A}(w) \in \mathcal{M}^{\mu+\nu}_{S,T}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ is called elliptic, if $\mathcal{A}(-\beta + i\tau) \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}; \mathbb{R}_{\tau})$ is elliptic for some $\beta \in \mathbb{R}$ where $\pi_{\mathbb{C}} T \cap \Gamma_{-\beta} = \emptyset$.

Clearly, this definition is independent of the choice of β.

Theorem 2.5.4 *Let* $\mathcal{A} \in \mathcal{M}^{\mu}_{R,V}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *for* $R \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g}; \boldsymbol{v})$, $V \in \boldsymbol{As}^{\bullet}(\mathbb{W}; \boldsymbol{v})_R$ *be elliptic. Then there is an element* $\mathcal{A}^{-1} \in \mathcal{M}^{-\mu}_{S,T}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ *for certain* $S \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$, $T \in \boldsymbol{As}^{\bullet}(\mathbb{W}, \boldsymbol{v}^{-1})_S$ *such that* $\mathcal{A}^{-1}$ *is the inverse in the sense of Theorem* 2.5.3.

Proof. By construction, $\mathcal{A}$ can be written as $\mathcal{A} = \mathcal{H} + \mathcal{G}$ for certain $\mathcal{H} \in \mathcal{M}^{\mu}_{R,\mathcal{O}}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ and $\mathcal{G} \in \mathcal{M}^{-\infty}_{R,V}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$. Clearly, $\mathcal{H}$ is also elliptic, and by Theorem 2.4.4 we can find a parametrix $\mathcal{P} \in \mathcal{M}^{-\mu}_{S_1,\mathcal{O}}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ of $\mathcal{H}$ for a certain $S_1 \in \boldsymbol{As}_{M+G}(X, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$. This gives us $\mathcal{P}\mathcal{A} = \mathcal{P}\mathcal{H} + \mathcal{P}\mathcal{G} = 1 + \mathcal{L} + \mathcal{P}\mathcal{G}$ for certain $\mathcal{L} \in \mathcal{M}^{-\infty}_{S_2,\mathcal{O}}(\mathbb{W}, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}})$ (cf. notation of Definition 1.5.1), where by Theorem 2.5.3 $\mathcal{P}\mathcal{G} \in \mathcal{M}^{-\infty}_{S_3,V_1}(\mathbb{W}, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}})$ for certain asymptotic data S_2, S_3, V_1. We now apply Lemma 2.5.5 below that gives us $(1 + \mathcal{L} + \mathcal{P}\mathcal{G})^{-1} = 1 + \mathcal{M}$ for a certain $\mathcal{M} \in \mathcal{M}^{-\infty}_{S_3,V_2}(\mathbb{W}, \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{v}})$. Applying Theorem 2.5.3 we obtain $\mathcal{A}^{-1} = (1+\mathcal{M})\mathcal{P} \in \mathcal{M}^{-\mu}_{S,T}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ for certain asymptotic types S, T.$\square$

Lemma 2.5.5 *Let* $\mathcal{L} \in \mathcal{M}^{-\infty}_{R,V}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *be an arbitrary element. Then there is an element* $\mathcal{M} \in \mathcal{M}^{-\infty}_{S,T}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ *such that* $(1 + \mathcal{L})^{-1} = (1 + \mathcal{M})$.

Proof. The proof is formally similar to a corresponding result of [16], Lemma 4.3.13, and can easily be adapted to the present situation.$\square$

References

[1] M.S. Agranovich and M.I. Vishik: *Elliptic problems with parameter and parabolic problems of general type.* Uspekhi Mat. Nauk **19**, 3 (1964), 53-161.

[2] De Donno and B.-W. Schulze: *Meromorphic symbolic structures for boundary value problems on manifolds with edges.* In preparation (2002).

[3] C. Dorschfeldt: *Algebras of pseudo-differential operators near edge and corner singularities.* Math. Research **102** Akademie Verlag Berlin (1998) .

[4] J.V. Egorov and B.-W. Schulze: *Pseudo-differential operators, singularities, applications.* Birkhäuser Verlag Basel (1997).

[5] B.V. Fedosov, B.-W. Schulze, and N.N. Tarkhanov: *Analytic index formulas for elliptic corner operators.* Ann. Inst. Fourier Grenoble **52**, 3 (2002), 899-982.

[6] J.B. Gil: *Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators.* Preprint math. 0004161, Temple University, Dep. Math., Philadelphia, April 2000.

[7] J.B. Gil, B.-W. Schulze, and J. Seiler: *Cone pseudodifferential operators in the edge symbolic calculus.* Osaka J. Math. **37** (2000), 219-258.

[8] I.C. Gohberg and E.I. Sigal: *An operator generalization of the logarithmic residue theorem and the theorem of Rouché.* Math. USSR Sbornik **13**, 4 (1971), 603-625.

[9] B. Gramsch and W. Kaballo: *Multiplicative decompositions of holomorphic Fredholm functions and Ψ^*-algebras.* Math. Nachr. **204** (1999), 83-100.

[10] G. Harutjunjan and B.-W. Schulze: *Euler solutions on a cylinder with conical cross section.* Preprint Institut für Mathematik (to appear), Potsdam (2002).

[11] T. Hirschmann: *Functional analysis in cone and edge Sobolev spaces.* Ann. Global Anal. Geom. **8**, 2 (1990), 167-192.

[12] V.A. Kondrat'ev: *Boundary value problems for elliptic equations in domains with conical points.* Trudy Mosk. Mat. Obshch. **16** (1967), 209-292.

[13] D. Kapanadze and B.-W. Schulze: *Symbolic calculus for boundary value problems on manifolds with edges.* Preprint 2001/21, Institut für Mathematik, Potsdam (2001). Integral Equ. and Operator Theory (to appear).

[14] T. Krainer and B.-W. Schulze: *Long-time asymptotics with geometric singularities in the spatial variables.* Preprint 2000/17, Institut für Mathematik, Potsdam (2000).

[15] S. Rempel and B.-W. Schulze: *Complete Mellin symbols and the conormal asymptotics in boundary value problems.* Proc. Journées "Equ. aux Dériv. Part.", St.-Jean de Monts, Conf. No. V, 1984.

[16] E. Schrohe and B.-W. Schulze: *Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities I.* Advances in Partial Differential Equations, "Pseudo-Differential Calculus and Mathematical Physics", Akademie Verlag Berlin (1994), 97-209.

[17] E. Schrohe and B.-W. Schulze: *A symbol algebra for pseudodifferential boundary value problems on manifolds with edges.* Differential Equations, Asymptotic Analysis, and Mathematical Physics, Math. Research **100**, Akademie Verlag Berlin (1997), 292-324.

[18] B.-W. Schulze: *Pseudo-differential operators on manifolds with edges.* Symposium: "Partial Differential Equations", Holzhau 1988. Teubner-Texte zur Mathematik, vol.112, Teubner, Leipzig, 1989, 259-287.

[19] B.-W. Schulze: *Corner Mellin operators and reduction of orders with parameters.* Ann. Sc. Norm. Sup. Pisa, Cl. Sci., **16**,1 (1989), 1-81.

[20] B.-W. Schulze: *Pseudo-differential operators on manifolds with singularities.* North-Holland, Amsterdam (1991).

[21] B.-W. Schulze: *The Mellin pseudo-differential calculus on manifolds with corners.* Symposium: "Analysis in Domains and on Manifolds with Singularities", Breitenbrunn 1990. Teubner-Texte zur Mathematik, vol.131, Teubner, Leipzig, 1992, 208-289.

[22] B.-W. Schulze: *Boundary value problems and singular pseudo-differential operators.* J. Wiley, Chichester (1998).

[23] B.-W. Schulze and N. Tarkhanov: *Euler solutions of pseudodifferential equations.* Integral Equ. and Operator Theory **33** (1999), 98-123.

[24] B.-W. Schulze: *Operators with symbol hierarchies and iterated asymptotics.* Publ. of RIMS, Kyoto University **38**, 4 (2002), 735-802.

[25] I. Witt: *On the factorization of meromorphic Mellin symbols.* Advances in Partial Differential Equations, "Parabolicity, Volterra Calculus, and Conical Singularities"(S. Albeverio, M. Demuth, E. Schrohe, B.-W. Schulze Eds.), Oper. Theory Adv. Appl., **138**, Birkhäuser Basel (2002), 279-306.