LOCAL ASYMPTOTIC TYPES

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Abstract. The local theory of asymptotic types is elaborated. It appears as coordinate-free version of part of Gohberg-Sigal's theory of the inversion of finitely meromorphic, operator-valued functions at a point.

1. Introduction

Asymptotic types arise in singular analysis. They provide a link of formal asymptotic analysis to functional analysis. The basic idea consists in assigning certain types to conormal asymptotic expansions of solutions to elliptic P.D.E. and, more generally, to elliptic pseudodifferential equations near singularities of the underlying geometric configuration. This allows to set up a functional-analytic frame by incorporating such asymptotic types into function spaces. The idea goes back to Rempel–Schulze [8] in the one-dimensional case and to Schulze [9] in the higher-dimensional one. Meanwhile, there is a great variety of different notions of asymptotic types used in different situations and emphasizing the one or other aspect of this concept, see Schulze [10].

The needs of a treatment of non-elliptic equations, and of certain nonlinear elliptic equation as well, however, require still more refined notions of asymptotic types. Already in the simplest case of a conical point the answer to the question what is the "best," i.e., most refined, notion of an asymptotic type is quite involved, at least in its formulation, see Liu-Witt [5]. To be able to provide an answer in even more complicated situations, like egdes, corners,

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and so on, a general concept of a (pre-)asymptotic algebra (that being a situation in which *a priori* the notion of asymptotic types makes sense) has been introduced by Witt [12].

The local theory of asymptotic types, i.e., asymptotic types "at only one singular exponent," as presented in this paper, is fundamental in all further developments for two reasons: First, the answer in that case is part of the answer in all succeeding cases; secondly, various techniques particularly developed here can be generalized to more intricate situations. (This concerns, e.g., the factorization of symbols, see (3.20), and generalizations of Corollary 1.3 below.) In fact, the local theory of asymptotic types appears in a well-defined sense, see Remark 3.5, as coordinate-free version of part of the theory of the inversion of finitely meromorphic functions at a point. For the latter, we refer to Gohberg-Sigal [4].

Let us describe the content of this paper in more detail: For E being a Banach space, $p \in \mathbb{C}$, let $\mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$ be the space of all germs of finitely meromorphic functions $F(z)$ at $z = p$ taking values in $\mathcal{L}(E)$, see Definition 3.1 (a). The algebra $\mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$ is regarded as (canonically) acting on the space $\mathcal{M}_p(E)$ of all germs of meromorphic functions $u(z)$ at $z = p$ taking values in E. As distinguished subspace in the sense of asymptotic algebras, we choose the space $\mathcal{A}_p(E)$ of all germs of holomorphic functions in $\mathcal{M}_p(E)$. Further, we identify the quotient space $\mathcal{M}_p(E)/\mathcal{A}_p(E)$ with the space E^∞ of all finite sequences in E, see (3.4). Under this identification, a linear subspace $J \subset E^\infty$ is called an asymptotic type (local asymptotic type, at the singular exponent p) if $\dim J < \infty$ and $TJ \subseteq J$, where T is the right shift operator on E^∞, see (3.2).

Theorem 1.1. $\big(\mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E)), \mathcal{M}_p(E), \mathcal{A}_p(E), \mathcal{J}(E)\big)$ *constitutes an asymptotic symbol algebra in the sense of* Definition 2.2.

There are two fundamental characteristics associated with an asymptotic type $J \in \mathcal{J}(E)$: The number and the sizes of Jordan blocks, respectively, with respect to T, the latter acting as nilpotent operator on J. Let $\ell(J)$ be the number of Jordan blocks; then we always have $\ell(J) \leq \dim E$. (This condition is void if $\dim E = \infty$.) For $F \in \mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$, $J \subseteq E^\infty$ being a linear subspace, there is the notion of push-forward J^F of J under F, see (2.1) and Lemma 3.3 (b). $J^F \subseteq E^\infty$ is then a linear subspace and, additionally, $J^F \in \mathcal{J}(E)$ when $J \in \mathcal{J}(E)$. Let $\mathcal{O}^F$ be the amount of asymptotics "produced" by F, i.e., the push-forward of the empty asymptotic type $\mathcal{O}$ under F, and L_F be the amount of asymptotics "annihilated" by F, i.e., the largest linear subspace of E^∞ such that $(L_F)^F = \mathcal{O}^F$, see (2.2) and Lemma 3.3 (a). If F is normally meromorphic, i.e., it belongs to the group $\mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$ of invertible elements of the algebra $\mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$, see Definition 3.1 (b), then, in fact, $L_F \in \mathcal{J}(E)$. Moreover, $\mathcal{O}^F = \mathcal{O}$ (no asymptotics is "produced" by F) if and only if $F \in \mathcal{A}_p(\mathcal{L}(E))$.

The second main result of this paper is the following:

Theorem 1.2. *Let J, $K \in \mathcal{J}(E)$. Then there is an $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$ such that $L_F = J$ and $\mathcal{O}^F = K$ if and only if*

$$\ell(J) + \ell(K) \leq \dim E. \tag{1.1}$$

Note that Proposition 2.4 below shows that L_F and $\mathcal{O}^F$ are, in fact, the two most important quantities for controlling asymptotics of solutions to elliptic equations. In our case, these elliptic equations are of the form

$$F(z)u(z) = v(z), \quad F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)),$$

where, for a given $v \in \mathcal{M}_p(E)$, $u \in \mathcal{M}_p(E)$ is sought. (The general case $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$ corresponds to an elliptic pseudodifferential equation, while in case $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$ we have a good analogue of an elliptic *differential* equation.) There are two extreme cases to discuss for (1.1): $\dim E = 1$, i.e., $E = \mathbb{C}$, in which case either $F(z)$ or $F^{-1}(z)$ is holomorphic (or both), and $\dim E = \infty$ in which case this condition is void.

As an immediate consequence of Theorem 1.2 we obtain:

Corollary 1.3. We have

$$\mathcal{J}(E) = \big\{L_F;\, F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))\big\}. \tag{1.2}$$

The paper is organized as follows: In Section 2, for the reader's convenience the notion of an asymptotic algebra is recalled. In particular, we recall some rules for manipulating asymptotic types then used in the following computations in Section 3. In Section 3.1, we introduce some basic notion and prepare for the application of the general concept of an asymptotic algebra. In the situation under consideration, the most refined notion of asymptotic type comes up canonically, this is exposed in Section 3.2. Then, in Section 3.3, the information gained so far is used to rewrite (basically) known results about the singularity structure of inverses to normally meromorphic functions in these new terms. In particular, this enables us to prove Theorem 1.2 that concerns the emergence of asymptotic types, more precisely, the "production"/"annihilation" of asymptotics in terms of asymptotic types. This latter result is prototypical for the construction of function spaces with asymptotics in various applications, see Remark 2.3 below.

2. Asymptotic algebras

Here, we introduce a simplified, but sufficient for the application we have in mind, version of asymptotic algebra. Comments on more general concepts of asymptotic algebras are given in the remarks. For a detailed exposition of the latter, see Witt [12].

2.1. **Generalities.** Let $\mathfrak{M}$ be a unital algebra, $\mathfrak{F}$ be a linear space, $\mathfrak{F}_0$ be its linear subspace, and ϱ_0 be a faithful representation of $\mathfrak{M}$ on $\mathfrak{F}$. The quadruple $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0)$ is called a pre-asymptotic algebra. (We also write $\mathfrak{M}$ if the triple $(\varrho_0, \mathfrak{F}, \mathfrak{F}_0)$ resp. $(\mathfrak{M}, \mathfrak{F}, \mathfrak{F}_0)$ if the representation ϱ_0 is understood from the context.) It is important to note that on this level no topologies are involved. The linear subspace $\mathfrak{F}_0$ is, in general, not invariant under the action by elements of $\mathfrak{M}$. This leads to the notion of asymptotic type: In a sense, an asymptotic type measures the deviation of $P\mathfrak{F}_0$ for $P \in \mathfrak{M}$ (P is $\varrho_0(P)$) from the distinguished subspace $\mathfrak{F}_0$. Therefore, asymptotic types are linear subspaces of the quotient space $\mathfrak{F}/\mathfrak{F}_0$. These linear subspaces have, of course, to satisfy certain further properties.

Before we proceed, we introduce some notation associated with the quadruple $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0)$: Let $\pi\colon \mathfrak{F} \to \mathfrak{F}/\mathfrak{F}_0$ be the canonical projection. For a linear subspace $J \subseteq \mathfrak{F}/\mathfrak{F}_0$ and $P \in \mathfrak{M}$, we define $\mathfrak{F}_J = \pi^{-1}(J)$ and

$$J^P = (P\mathfrak{F}_J + \mathfrak{F}_0)/\mathfrak{F}_0 \tag{2.1}$$

as the push-forward of J with respect to P. $\mathcal{O} = \mathfrak{F}_0/\mathfrak{F}_0$ is the empty asymptotic type. In particular, $\mathfrak{F}_{\mathcal{O}} = \mathfrak{F}_0$, and $\mathcal{O}^P$ is the amount of asymptotics "produced" by $P \in \mathfrak{M}$. The elements of $\mathfrak{F}_0$ are interpreted as the elements having no asymptotics, i.e., the "flat" elements. For $P \in \mathfrak{M}$, L_P denotes the largest linear subspace of $\mathfrak{F}/\mathfrak{F}_0$ such that $(L_P)^P = \mathcal{O}^P$, i.e.,

$$L_P = (P^{-1}\mathfrak{F}_0 + \mathfrak{F}_0)/\mathfrak{F}_0. \tag{2.2}$$

L_P is the amount of asymptotics "annihilated" by $P \in \mathfrak{M}$.

Definition 2.1. A quintuple $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$, where $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0)$ is a pre-asymptotic algebra as above and $\mathfrak{J}$ is an l.a.t. (lattice of asymptotic types) on it, is called an asymptotic algebra. Thereby, $\mathfrak{J}$ is an l.a.t. on $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0)$ if it is a sub-lattice of the complete lattice of all linear subspaces of $\mathfrak{F}/\mathfrak{F}_0$ such that

(i) $\mathcal{O} \in \mathfrak{J}$;
(ii) for all $J \in \mathfrak{J}$ and $P \in \mathfrak{M}$, there is a $K \in \mathfrak{J}$ such that $J^P \subseteq K$;
(iii) $\mathfrak{J}$ is closed under arbitrary non-empty intersections.

The elements of $\mathfrak{J}$ are called asymptotic types; $\mathfrak{F}_J$ for $J \in \mathfrak{J}$ is the space of the elements of $\mathfrak{F}$ having asymptotics of type J.

Requirement (ii) is flexible enough to encompass different notions of asymptotic type (depending on the respective context) on a given pre-asymptotic algebra $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0)$. There is always a distinguished l.a.t., denoted by

$$\mathfrak{J}_0 = \mathfrak{J}_{0,\mathfrak{M}}, \tag{2.3}$$

namely the least l.a.t. on $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0)$ having the property that $J^P \in \mathfrak{J}$ whenever $J \in \mathfrak{J}$, $P \in \mathfrak{M}$, i.e., equality holds in (ii). Since one wants asymptotic types to reflect a certain "internal structure" of the problem under consideration, the l.a.t. $\mathfrak{J}_0$ is, in general, too large in order to be useful. (Exceptions are asymptotic symbol algebras, see Section 2.2.) L.a.t.'s arising in applications are typically generated (as lattices) by the L_P's for elliptic $P \in \mathfrak{M}$ admitting a "distinguished" parametrix.

Remark 2.1. (a) In the above context, elements of $\mathfrak{M}$ are interpreted as "operators." Typically, however, asymptotic types are determined on the level of "symbols of operators." In general, such a symbol does not act on $\mathfrak{F}$, but it acts only on "$\mathfrak{F}$ modulo $\mathfrak{F}_0$ in the image." Then, a pre-asymptotic algebra is a quadruple $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0)$, where $\mathfrak{M}$, $\mathfrak{F}$, $\mathfrak{F}_0$ are as above and ϱ is an (injective) linear map $\varrho\colon \mathfrak{M} \to L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ sending the unit of $\mathfrak{M}$ to the identity such that, for all P, $Q \in \mathfrak{M}$, the diagram

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{Q \circ P} & \mathfrak{F}/\mathfrak{F}_0 \\ {\scriptstyle P}\downarrow & & \downarrow \\ \mathfrak{F}/\mathfrak{F}_0 & \xrightarrow{Q} & (\mathfrak{F}/\mathfrak{F}_0)/Q\mathfrak{F}_0 \end{array}$$

commutes. (The second horizontal line is induced by $Q\colon \mathfrak{F} \to \mathfrak{F}/\mathfrak{F}_0$, the second vertical line is the quotient map.) An example for ϱ is the composite $\pi \circ \varrho_0$, where ϱ_0 is a genuine (faithful) representation of $\mathfrak{M}$ on $\mathfrak{F}$. L.a.t.'s $\mathfrak{J}$ on $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0)$ and asymptotic algebras $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ are introduced as before, where now $J^P = P\mathfrak{F}_J$.

(b) Property (iii) of Definition 2.1 implies that every non-empty subset $\mathcal{S} \subseteq \mathfrak{J}$ possesses a greatest lower bound, $\bigwedge \mathcal{S} = \bigcap_{J \in \mathcal{S}} J$, and every bounded subset $\mathcal{T} \subseteq \mathfrak{J}$ possesses a least upper bound,

$$\bigvee \mathcal{T} = \bigwedge \{K;\ K \supseteq J \text{ for all } J \in \mathcal{T}\}.$$

In particular, $\bigwedge \mathfrak{J} = \mathcal{O}$.

There is a number of useful rules which permit a manipulation of asymptotic types. Let $\mathfrak{M}^{-1}$ denote the group of invertible elements of $\mathfrak{M}$.

Lemma 2.2. *Let* $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ *be an asymptotic algebra as in* Definition 2.1.

(a) *If* $P \in \mathfrak{M}^{-1}$, *then* $L_P = \mathcal{O}^{P^{-1}}$.

(b) *For* P, $Q \in \mathfrak{M}$, *and* $J \in \mathfrak{J}$, $\left(J^P\right)^Q = J^{QP} \vee \mathcal{O}^Q$.

(c) *For* $P \in \mathfrak{M}$, $Q \in \mathfrak{M}^{-1}$, $(L_P)^Q = L_{PQ^{-1}} \vee L_{Q^{-1}}$.

Proof. The proofs of (a) to (c) are straightforward. □

2.2. **Asymptotic symbol algebras.** In pseudodifferential analysis, ellipticity is the same as invertibility on the level of principal symbols. This motivates the next definition.

Definition 2.3. An asymptotic algebra $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ is called a symbol algebra if it is reduced, i.e., $S \in \mathfrak{M}$ and $S\mathfrak{F} \subseteq \mathfrak{F}_J$ for some $J \in \mathfrak{J}$ implies $S = 0$ and if, in addition, $\mathfrak{J} = \mathfrak{J}_0$.

Remark 2.2. (a) For an asymptotic algebra $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ as in Remark 2.1, we have the ideal

$$\mathfrak{S}_{\mathfrak{J}} = \{S \in \mathfrak{M};\ S\mathfrak{F} \subseteq J \text{ for some } J \in \mathfrak{J}\}$$

of the residual elements and the multiplicatively closed set

$$\mathfrak{E}_{\mathfrak{J}} = \{P \in \mathfrak{M};\ \ PQ - 1,\ QP - 1 \in \mathfrak{S}_{\mathfrak{J}} \text{ for some } Q \in \mathfrak{M}\}$$

of the elliptic elements. Reducibility for an asymptotic algebra means that $\mathfrak{E}_{\mathfrak{J}}$ consists entirely of invertible elements.

(b) In general, a symbol algebra $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ for a pre-asymptotic algebra $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0)$ is an asymptotic algebra, where $\mathfrak{J} = \mathfrak{J}_{0,\mathfrak{N}}$, together with a homomorphism $\Theta \colon \mathfrak{M} \to \mathfrak{N}$ of unital algebras such that

$$(\rho(P) - \sigma(\Theta P))\, \mathfrak{F} \subseteq J$$

for all $P \in \mathfrak{M}$ and some $J \in \mathfrak{J}_{0,\mathfrak{N}}$ (where J may depend on P). In that case, $\mathfrak{J}_{0,\mathfrak{N}}$ is also an l.a.t. for $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0)$. Whenever a symbol algebra

$(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J}_{0,\mathfrak{N}})$ for the pre-asymptotic algebra $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0)$ is known, $\mathfrak{J}_{0,\mathfrak{N}}$ is a natural candidate for an l.a.t. to equip $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0)$ with it. The advantage of $\mathfrak{J}_{0,\mathfrak{N}}$ lies in its computability. Asymptotic algebras occuring in singular analysis can often be reduced to symbol algebras, see LIU–WITT [5], WITT [12].

For the rest of this section, we shall assume that $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ is a symbol algebra as in Definition 2.3. The major result in this context admitting in its consequence to operate on asymptotic types is stated next.

Proposition 2.4. *Let $(\mathfrak{M}, \varrho, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ be a symbol algebra. Then, for $P \in \mathfrak{M}^{-1}$, L_P is an asymptotic type. Furthermore, there is an order-preserving bijection*

$$\{J \in \mathfrak{J};\ J \supseteq L_P\} \to \{K \in \mathfrak{J};\ K \supseteq L_{P^{-1}}\},\quad J \mapsto J^P, \tag{2.4}$$

with its inverse given by $K \mapsto K^{P^{-1}}$.

Proof. By Lemma 2.2 (a), $L_P = \mathcal{O}^{P^{-1}} \in \mathfrak{J}$. Moreover, Lemma 2.2 (b) implies $\left(J^P\right)^{P^{-1}} = J \vee L_P$ and $\left(K^{P^{-1}}\right)^P = K \vee L_{P^{-1}}$ for any $J, K \in \mathfrak{J}$. This yields the order isomorphism in (2.4). □

As an example, we state the following useful consequence of Lemma 2.2 and Proposition 2.4. Let $\mathfrak{A} = \{P \in \mathfrak{M};\ P\mathfrak{F}_0 \subseteq \mathfrak{F}_0\}$ be the sub-algebra of all elements $P \in \mathfrak{M}$ leaving the distinguished subspace $\mathfrak{F}_0$ invariant. Note that $P \in \mathfrak{A}$ if and only if $\mathcal{O}^P = \mathcal{O}$.

Proposition 2.5. *For $P, Q \in \mathfrak{M}^{-1} \cap \mathfrak{A}$, $PQ^{-1}, QP^{-1} \in \mathfrak{A}$ if and only if $L_P = L_Q$.*

Proof. We show that, for $P \in \mathfrak{M}$, $Q \in \mathfrak{M}^{-1} \cap \mathfrak{A}$, $L_P \subseteq L_Q$ if and only if $L_{PQ^{-1}} = \mathcal{O}$. If, in addition, $P \in \mathfrak{M}^{-1}$, then this latter condition is equivalent to $QP^{-1} \in \mathfrak{A}$.

In fact, by Lemma 2.2 (c),

$$L_P \vee L_Q = \left(L_{PQ^{-1}}\right)^{Q^{-1}}.$$

Thus $L_P \subseteq L_Q$ if and only if $L_Q = \left(L_{PQ^{-1}}\right)^{Q^{-1}}$ and this holds, by the foregoing proposition, if and only if $L_{PQ^{-1}} \subseteq L_{Q^{-1}} = \mathcal{O}$. □

Remark 2.3. For $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ being an asymptotic algebra, we have

$$F\colon \mathfrak{F}_J \to \mathfrak{F}_K$$

for any $F \in \mathfrak{M}$, where the asymptotic types $J, K \in \mathfrak{J}$ are so that $J^F \subseteq K$. That is, elements of $\mathfrak{F}$ having asymptotics of type J are mapped by F to elements of $\mathfrak{F}$ having asymptotics of type K. If $(\mathfrak{M}, \varrho_0, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ is a symbol algebra, then we can take J^F for K. Moreover, if $u \in \mathfrak{F}$, $Fu \in \mathfrak{F}_K$ for some $K \in \mathfrak{J}$, where now $F \in \mathfrak{M}^{-1}$ (and $\mathfrak{M}$ continues to be a symbol algebra), then $u \in \mathfrak{F}_{K^{F^{-1}}}$, and Proposition 2.4 states that F realizes an isomorphism from $\mathfrak{F}_J$ onto $\mathfrak{F}_{J^F}$ if $J \supseteq L_F$.

Furthermore, if $J \in \mathfrak{J}$ is such that $J = \bigvee_{\iota\in\mathcal{I}} L_{G_\iota}$ for some bounded family $\{G_\iota\}_{\iota\in\mathcal{I}} \subset \mathfrak{M}^{-1} \cap \mathfrak{A}$, then $\mathfrak{F}_J = \sum_{\iota\in\mathcal{I}}\{u \in \mathfrak{F};\, G_\iota u \in \mathfrak{F}_0\}$, and from Proposition 2.5 we recover that this characterization is actually independent of the choice of the family $\{G_\iota\}_{\iota\in\mathcal{I}}$. Via the construction of suitable G_ι, this is the way employed in the definition of function spaces with asymptotics, see, e.g., Liu–Witt [5].

3. Local theory of asymptotic types

The local theory of asymptotic types is the theory of asymptotic types at a fixed singular exponent, but allowing all the other characteristic to vary, e.g., its algebraic multiplicity (regarding the singular exponent as an eigenvalue).

3.1. **Finitely meromorphic functions.**

Let E be a Banach space. We shall consider the m-fold product E^m for $m \in \mathbb{N}_0$, where $E^0 = \{0\}$. We identify E^m with a subspace of E^{m+1} via the map

$$E^m \to E^{m+1}, \quad (\phi_0, \ldots, \phi_{m-1}) \mapsto (0, \phi_0, \ldots, \phi_{m-1}). \tag{3.1}$$

Further, we set

$$E^\infty = \bigcup_{m\in\mathbb{N}_0} E^m.$$

Thus, E^∞ is the linear space of all finite sequences in E, where the sequences $(\phi_0, \ldots, \phi_{m-1})$ and $(\underbrace{0, \ldots, 0}_{h \text{ times}}, \phi_0, \ldots, \phi_{m-1})$ for $h \in \mathbb{N}_0$ are identified.

On E^∞, we define the right shift operator T by

$$T\colon E^\infty \to E^\infty, \quad (\phi_0, \dots, \phi_{m-2}, \phi_{m-1}) \mapsto (\phi_0, \dots, \phi_{m-2}). \tag{3.2}$$

By $\mathcal{M}_p(E)$ for $p \in \mathbb{C}$ we shall denote the space of all germs of E-valued meromorphic functions at p. Moreover, $\mathcal{A}_p(E)$ is the space of all germs of E-valued analytic functions at p.

Definition 3.1. (a) $\mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$ is the space of all germs of $\mathcal{L}(E)$-valued, finitely meromorphic functions at p, i.e., the space of all $F \in \mathcal{M}_p(\mathcal{L}(E))$ such that

$$F(z) = \frac{F_0}{(z-p)^\nu} + \frac{F_1}{(z-p)^{\nu-1}} + \dots + \frac{F_{\nu-1}}{z-p} + F_\nu + \sum_{j \geq 1} F_{\nu+j}\,(z-p)^j \tag{3.3}$$

with finite-rank operators F_0, $F_1, \dots,$ $F_{\nu-1} \in \mathcal{L}(E)$.

(b) $\mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$ is the space of all germs of $\mathcal{L}(E)$-valued, normally meromorphic functions at p, i.e., the space of all $F \in \mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$ such that $F(z) \in \mathcal{L}(E)$ is invertible for z close to p, $z \neq p$, and, moreover, $F_\nu \in \mathcal{L}(E)$ in the representation (3.3) is a Fredholm operator (then necessarily of index 0).

For the next result, see Bleher [2].

Proposition 3.2. *$\mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$ is an algebra and $\mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$ is its group of invertible elements.*

3.2. **Local asymptotic types.** The idea is to choose $\mathfrak{M} = \mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$, $\mathfrak{F} = \mathcal{M}_p(E)$, and $\mathfrak{F}_0 = \mathcal{A}_p(E)$ in Definitions 2.1, 2.3.

The quadruple $\big(\mathcal{M}_p^{\text{fin}}(\mathcal{L}(E)), \varrho_0, \mathcal{M}_p(E), \mathcal{A}_p(E)\big)$, where ϱ_0 is the canonical action of $\mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$ on $\mathcal{M}_p(E)$, constitutes a pre-asymptotic algebra. Equipped with the l.a.t. $\mathfrak{J}_0$, see (2.3), this pre-asymptotic algebra turns out to be an asymptotic symbol algebra. We shall uncover this l.a.t. $\mathfrak{J}_0$.

We identify the quotient space $\mathcal{M}_p(E)/\mathcal{A}_p(E)$ with E^∞ via the map

$$\frac{\phi_0}{(z-p)^m} + \frac{\phi_1}{(z-p)^{m-1}} + \dots + \frac{\phi_{m-1}}{z-p} \mapsto (\phi_0, \phi_1, \dots, \phi_{m-1}). \tag{3.4}$$

This identification is compatible with (3.1). Moreover, multiplication by $z-p$ corresponds to the action by T.

Lemma 3.3. (a) *For $F \in \mathcal{M}_p^{\mathrm{fin}}(E)$, the space L_F consists of all sequences $(\phi_0, \phi_1, \ldots, \phi_{m-1}) \in E^\infty$ for which there is a $\tilde{\phi}(z) \in \mathcal{A}_p(E)$ such that*

$$F(z)\left(\frac{\phi_0}{(z-p)^m} + \frac{\phi_1}{(z-p)^{m-1}} + \cdots + \frac{\phi_{m-1}}{z-p} + \tilde{\phi}(z)\right) \in \mathcal{A}_p(E),$$

see (2.2).

(b) *For $F \in \mathcal{M}_p^{\mathrm{fin}}(E)$ and $J \subseteq E^\infty$ being a linear subspace,*

$$J^F = \big\{\big(F_0\phi_0, F_1\phi_0 + F_0\phi_1, \ldots, F_{m+\nu-1}\phi_0 + F_{m+\nu-2}\phi_1 + \cdots + F_0\phi_{m+\nu-1}\big);$$
$$(\phi_0, \ldots, \phi_{m-1}) \in J,\ \phi_m, \phi_{m+1}, \ldots, \phi_{m+\nu-1} \in E\big\}$$

where F is given in the form (3.3), *see* (2.1).

Remark 3.1. For $F \in \mathcal{A}_p(\mathcal{L}(E))$, i.e., $\nu = 0$ in (3.3), the operation $J \mapsto J^F$ is given directly on the level of the space E^∞,

$$(\,)^F\colon E^\infty \to E^\infty, (\phi_0, \ldots, \phi_{m-1})$$
$$\mapsto \big(F_0\phi_0, F_1\phi_0 + F_0\phi_1, \ldots, F_{m-1}\phi_0 + \cdots + F_0\phi_{m-1}\big). \quad (3.5)$$

Then $J^F = \{\Phi^F;\, \Phi \in J\}$. Moreover, L_F is the kernel of the map $(\,)^F$. The map $(\,)^F$ restricted to E^m only depends on $F_0, F_1, \ldots, F_{m-1}$. Therefore, we will occasionally write $(\,)^F = (\,)^{(F_0, F_1, \ldots, F_{m-1})}$ on E^m.

Lemma 3.4. *Let $F \in \mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$. Then:*

(a) *$L_F \subseteq E^\infty$ is a linear subspace that is left invariant under the action by the right shift operator T, i.e., $TL_F \subseteq L_F$;*

(b) *If $J \subseteq E^\infty$ is a linear subspace that is left invariant under the action by T, then $J^F \subseteq E^\infty$ is a linear subspace that is also left invariant under the action by T;*

(c) $\dim J < \infty$ *implies* $\dim J^F < \infty$.

Hence, we introduce local asymptotic types as follows:

Definition 3.5. An asymptotic type, J, on E is a finite-dimensional subspace of E^∞ such that $TJ \subseteq J$. The set of all asymptotic types on E is denoted by $\mathcal{J}(E)$.

Remark 3.2. By virtue of Lemma 3.4 (b), (c), $\mathcal{J}(E)$ already supplies an l.a.t. for the pre-asymptotic algebra $\big(\mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E)), \varrho_0, \mathcal{M}_p(E), \mathcal{A}_p(E)\big)$. The asymptotic algebra $\big(\mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E)), \varrho_0, \mathcal{M}_p(E), \mathcal{A}_p(E), \mathcal{J}(E)\big)$ is readily seen to be reduced.

Notice that, for $J \in \mathcal{J}(E)$, $J \subseteq E^m$ for some $m \in \mathbb{N}_0$ and, therefore, the right shift operator T is nilpotent on J, since $T^m = 0$ on E^m. We will need the following fact from linear algebra.

Lemma 3.6. *Let J be a finite-dimensional linear space and $T\colon J \to J$ be a nilpotent linear operator. Then there are $\Phi_1, \dots, \Phi_e \in J$ such that*

$$\Phi_1, T\Phi_1, \dots, T^{m_1-1}\Phi_1, \dots, \Phi_e, T\Phi_e, \dots, T^{m_e-1}\Phi_e, \tag{3.6}$$

where $m_j \in \mathbb{N}_0$, $m_j \geq 1$, is a linear basis of J, while $T^{m_j}\Phi_j = 0$ for $1 \leq j \leq e$. Furthermore, the numbers $m_1, \dots, m_e$ are uniquely determined up to permutation.

Proof. Choose a linear basis for J for which the associated matrix to T is in Jordan form. □

Hence, the numbers $m_1, \dots, m_e$ appear as the sizes of Jordan blocks; $\dim J = m_1 + \dots + m_e$. The tuple $(m_1, \dots, m_e)$ is called the characteristic of J (with respect to the nilpotent operator T), e is called the length of its characteristic, and $\Phi_1, \dots, \Phi_e$ is said to be a characteristic basis of J, of characteristic $(m_1, \dots, m_e)$, or simply an $(m_1, \dots, m_e)$-characteristic basis of J. Note that the space $\{0\}$ has empty characteristic with length $e = 0$.

The next lemma hints at an effective method for finding the characteristic and a characteristic basis upon constructing a suitable basis of $\ker T$. A proof will appear elsewhere.

Lemma 3.7. *Let J be a finite-dimensional linear space and $T\colon J \to J$ be a nilpotent linear operator. Suppose that the characteristic of J is $(m_1, \dots, m_e)$. Then $\Phi_1, \dots, \Phi_e \in J$ is an $(m_1, \dots, m_e)$-characteristic basis of J if and only if $T^{m_1-1}\Phi_1, \dots, T^{m_e-1}\Phi_e$ constitute a linear basis of* $\ker T$.

In particular, the length e of the characteristic of J equals the number $\dim\ker T$.

We add a general remark concerning the appearance of asymptotic types.

Remark 3.3. Let $J \in \mathcal{J}(E)$ have characteristic $(m_1, m_2, \dots, m_e)$. Let

$$\Phi_1 = \big(\phi_0^{(1)}, \dots, \phi_{m_1-1}^{(1)}\big), \dots, \Phi_e = \big(\phi_0^{(e)}, \dots, \phi_{m_e-1}^{(e)}\big) \tag{3.7}$$

be an $(m_1, \dots, m_e)$-characteristic basis of J. The vectors $\phi_0^{(1)}, \dots, \phi_0^{(e)}$ are linearly independent, since $T^{m_j-1}\Phi_j = \big(\phi_0^{(j)}\big)$ for $1 \le j \le e$. We set

$$L_l = \operatorname{span}\{\phi_k^{(j)};\ m_j - k \ge \bar{m} - l + 1\},\ \ 1 \le l \le \bar{m},$$

where $\bar{m} = \max_{1 \le j \le e} m_j$. The spaces L_l are actually independent of the choice of $\Phi_1, \dots, \Phi_e$, since L_l is the projection of J on the lth component of $E^{\bar{m}}$. In particular, $J \subseteq L_1 \times \cdots \times L_{\bar{m}} \subseteq E^{\bar{m}}$. In this relation, however, equality in general fails to hold.

Equality holds, i.e.,

$$J = L_1 \times L_2 \times \cdots \times L_{\bar{m}} \tag{3.8}$$

if and only if $\phi_k^{(j)} \in \operatorname{span}\{\phi_0^{(h)};\ m_h \ge m_j - k\}$ for $1 \le j \le e$, $1 \le k \le m_j - 1$. Again, this is a condition that is independent of the choice of $\Phi_1, \dots, \Phi_e$. This condition, in turn, is fulfilled if and only if

$$L_l = \operatorname{span}\{\phi_0^{(j)};\ m_j \ge \bar{m} - l + 1\},\ \ 1 \le l \le \bar{m}, \tag{3.9}$$

see also (3.18) and Remark 3.7 (a).

For $J \in \mathcal{J}(E)$, let $\ell(J)$ denote the length of its characteristic. Note that the linear independence of $\phi_0^{(1)}, \dots, \phi_0^{(e)}$ implies that $\ell(J) = e \le \dim E$.

Example 3.4. For $\dim E = 1$, an asymptotic type is uniquely determined by a number $m \in \mathbb{N}_0$. Indeed, $E = \mathbb{C}$ in this case and, if the asymptotic type J has characteristic (m), then $J = \mathbb{C}^m$. Moreover, for $F \in \mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(\mathbb{C})) = \mathcal{M}_p(\mathbb{C})$, $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(\mathbb{C}))$ exactly means that $F(z) \not\equiv 0$. Then $L_F = \mathbb{C}^m$ if and only if $F(z)$ has a zero of order m at $z = p$.

Lemma 3.8. (a) *For each tuple* $(m_1, \dots, m_e)$ *with* $e \le \dim E$, *there is a* $J \in \mathcal{J}(E)$ *having characteristic* $(m_1, \dots, m_e)$. *Moreover,* $J \in \mathcal{J}(E)$ *can be chosen, via* (3.9), *in the particular form* (3.8).

(b) *For* $J, K \in \mathcal{J}(E)$,

$$\ell(J \wedge K) \ge (\ell(J) + \ell(K) - \dim E)^+ . \tag{3.10}$$

Furthermore, if $(m_1, \dots, m_e)$, $(n_1, \dots, n_f)$ *are tuples with* $e \le \dim E$, $f \le \dim E$, *then there are* $J, K \in \mathcal{J}(E)$ *having* $(m_1, \dots, m_e)$ *and* $(n_1, \dots, n_f)$, *respectively, as characteristics such that in* (3.10) *equality holds.*

Proof. (a) is follows from the description given in Remark 3.3. (b) is concluded from the fact that $\ell(J)$ equals the dimension of $\ker T$, where the operator T is considered as acting on J, see Lemma 3.7. □

Moreover, for $J, K \in \mathcal{J}(E)$, it is seen in the same way that $\ell(J \wedge K) + \ell(J \vee K) = \ell(J) + \ell(K)$. Thus

$$\ell(J \vee K) \le \min\{\ell(J) + \ell(K), \dim E\},$$

and equality holds if and only if equality holds in (3.10). Note that if these equalities take place, then $J, K \in \mathcal{J}(E)$ are understood to be in general position (with respect to each other).

Remark 3.5. In Gohberg-Sigal [4], $p \in \mathbb{C}$ was called a *characteristic value* for $F \in \mathcal{M}_p^{\text{fin}}(\mathcal{L}(E))$ if $\dim L_F > 0$. If, additionally, $F \in \mathcal{M}_p^{\text{nor}}(\mathcal{L}(E))$, then one finds an $(m_1, \dots, m_e)$-characteristic basis $\Phi_1, \dots, \Phi_e$ of L_F as in (3.6). If $\Phi_1, \dots, \Phi_e$ are given by (3.7), then

$$\phi_0^{(1)}, \phi_1^{(1)}, \dots, \phi_{m_1-1}^{(1)}, \dots, \phi_0^{(e)}, \phi_1^{(e)}, \dots, \phi_{m_e-1}^{(e)}, \tag{3.11}$$

was called a *canonical system of eigenvectors and associated vectors* for $F(z)$ at $z = p$. Vica versa, if a canonical system of eigenvectors and associated vectors (3.11) is given, then the vectors $\Phi_1, \dots, \Phi_e$ formed in (3.7) constitute an $(m_1, \dots, m_e)$-characteristic basis $\Phi_1, \dots, \Phi_e$ of L_F. In this sense, we talk about a coordinate-free version of part of Gohberg-Sigal's theory. The numbers

m_j for $1 \le j \le e$ were called *partial null multiplicities* and $m_1+m_2+\cdots+m_e$ ($= \dim L_F$) was called the *null multiplicity* of the characteristic value p of $F(z)$.

3.3. **Singularity structure of inverses.** Let E' be the topological dual to E. For $\Phi \in E^\infty$, $\Psi \in (E')^\infty$, where we have $\Phi = (\phi_0, \phi_1, \dots, \phi_{m-1})$, $\Psi = (\psi_0, \psi_1, \dots, \psi_{m-1})$, we define

$$(\Phi \otimes \Psi)[z-p] = \frac{\phi_0 \otimes \psi_0}{(z-p)^m} + \frac{\phi_1 \otimes \psi_0 + \phi_0 \otimes \psi_1}{(z-p)^{m-1}} + \cdots + \frac{\phi_{m-1} \otimes \psi_0 + \cdots + \phi_0 \otimes \psi_{m-1}}{z-p},$$

where, for $\phi \in E$, $\psi \in E'$, $\phi \otimes \psi \in \mathcal{L}(E)$ is the rank-one operator $h \mapsto \langle \psi, h \rangle \phi$, $h \in E$, with $\langle\ ,\ \rangle$ denoting the dual pairing between E, E'.

Proposition 3.9. *Let $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, $J \in \mathcal{J}(E)$. Let $\Phi_1, \dots, \Phi_e \in J$ be an $(m_1, \dots, m_e)$-characteristic basis of J. Then $L_F \subseteq J$ if and only if there are $\Psi_1, \dots, \Psi_e \in (E')^\infty$ such that $T^{m_j}\Psi_j = 0$ for $1 \le j \le e$ and the principal part of the Laurent expansion of $F^{-1}(z)$ at $z = p$ equals*

$$\sum_{j=1}^{e} (\Phi_j \otimes \Psi_j)[z-p]. \tag{3.12}$$

In that case, $\Psi_1, \dots, \Psi_e \in (E')^\infty$ are uniquely determined. Furthermore, $L_F = J$ if and only if, in addition,

$$\Psi_1, T\Psi_1, \dots, T^{m_1-1}\Psi_1, \dots, \Psi_e, T\Psi_e, \dots, T^{m_e-1}\Psi_e$$

are linearly independent.

Proof. Let $\Phi_1 = \big(\phi_0^{(1)}, \dots, \phi_{m_1-1}^{(1)}\big), \dots, \Phi_e = \big(\phi_0^{(e)}, \dots, \phi_{m_e-1}^{(e)}\big)$ be as in (3.7). Recall that the eigenvectors $\phi_0^{(1)}, \dots, \phi_0^{(e)}$ are linearly independent. Choose $\phi_j' \in E'$ for $1 \le j \le e$ such that $\langle \phi_j', \phi_0^{(j')} \rangle = \delta_{jj'}$ for $1 \le j, j' \le e$. Furthermore, let $G = F^{-1}$,

$$G(z) = \frac{G_0}{(z-p)^\sigma} + \frac{G_1}{(z-p)^{\sigma-1}} + \cdots + \frac{G_{\sigma-1}}{z-p} + G_\sigma + \sum_{k \ge 1} G_{\sigma+k}(z-p)^k,$$

with finite-rank operators $G_0, G_1, \dots, G_{\sigma-1} \in \mathcal{L}(E)$. Under the assumption $G_0 \neq 0$,

$$\sigma \leq \bar{m} = \max_{1\leq j\leq e} m_j$$

is a necessary condition for both $L_F = \mathcal{O}^G \subseteq J$ and (3.12) to hold. Thus, by possibly enlarging σ, we may assume that $\sigma = \bar{m}$.

Condition (3.12) is equivalent to

$$G_l = \sum_{m_j\geq \bar{m}-l} \sum_{r+s=m_j-\bar{m}+l} \phi_r^{(j)} \otimes \psi_s^{(j)}, \quad 0 \leq l \leq \bar{m}-1, \tag{3.13}$$

for suitable $\psi_s^{(j)}$, $1 \leq s \leq e$, $0 \leq j \leq m_j - 1$, where the first sum is extended over all j such that $m_j \geq \bar{m} - l$. The relation with the $\Psi_1, \dots, \Psi_e \in (E')^\infty$ is $\Psi_1 = \big(\psi_0^{(1)}, \dots, \psi_{m_1-1}^{(1)}\big), \dots, \Psi_e = \big(\psi_0^{(e)}, \dots, \psi_{m_e-1}^{(e)}\big)$.

Suppose that (3.13) holds. Then, for $h(z) = \sum_{j\geq 0} h_j (z-p)^j \in \mathcal{A}_p(E)$, where $h_j \in E$,

$$\begin{aligned}\big(G_0h_0, G_1h_0 + G_0h_1, \dots, G_{\bar{m}-1}h_0 + \dots + G_0h_{\bar{m}-1}\big)\\ = \sum_{j=1}^{e}\sum_{s=1}^{m_j} \Big(\langle\psi_{s-1}^{(j)}, h_0\rangle + \dots + \langle\psi_0^{(j)}, h_{s-1}\rangle\Big) T^{s-1}\Phi_j\end{aligned}$$

in E^∞, i.e., $L_F = \mathcal{O}^G \subseteq J$. Moreover, equality holds if and only if Ψ_1, $T\Psi_1, \dots, T^{m_1-1}\Psi_1, \dots, \Psi_e, T\Psi_e, \dots, T^{m_e-1}\Psi_e$ are linearly independent.

Now, suppose that $L_F = \mathcal{O}^G \subseteq J$. Then, for $1 \leq l \leq \bar{m}$, we have

$$G_0E + \dots + G_{l-1}E \subseteq \operatorname{span}\{\phi_s^{(j)};\, 1 \leq j \leq e,\, 0 \leq s \leq m_j - \bar{m} + l - 1\}$$

such that the hypothetical relation (3.13) determines $\psi_s^{(j)}$ uniquely. To see this, we proceed by induction on $\bar{m} - m_j + s$. For $\bar{m} - m_j + s = 0$, i.e., $m_j = \bar{m}$, $s = 0$, we obtain $\langle\psi_0^{(j)}, h\rangle = \langle\phi_j', G_0h\rangle$, $h \in E$, for $m_j = \bar{m}$, i.e., $\psi_0^{(j)} = G_0'\phi_j'$. This leads to $G_0 = \sum_{m_j=\bar{m}} \phi_0^{(j)} \otimes \psi_0^{(j)}$. Furthermore, if $\psi_s^{(j)}$ for $\bar{m} - m_j + s < l$ for some $l \geq 1$ have already been found, then

$$\langle\psi_s^{(j)}, h\rangle = \langle\phi_j', G_lh\rangle - \sum_{m_j\geq\bar{m}-l} \sum_{\substack{r+s'=m_j-\bar{m}+l,\\ \bar{m}-m_j+s'<l}} \langle\psi_{s'}^{(j)}, h\rangle\phi_r^{(j)}, \quad h \in E,$$

for $\bar{m} - m_j + s = l$, i.e.,

$$\psi_s^{(j)} = G_l' \phi_j' - \sum_{m_j \geq \bar{m} - l} \sum_{\substack{r+s' = m_j - \bar{m} + l, \\ \bar{m} - m_j + s' < l}} \phi_r^{(j)} \otimes \psi_{s'}^{(j)},$$

which gives us (3.13). This furnishes the proof. □

Corollary 3.10. Under the assumptions of the previous proposition, if $\Phi_1, \ldots, \Phi_e$ is an $(m_1, \ldots, m_e)$-characteristic basis of L_F, i.e., we have $J = L_F$, then $\Psi_1, \ldots, \Psi_e \in (E')^\infty$ is an $(m_1, \ldots, m_e)$-characteristic basis of $L_{F'}$. In particular, L_F and $L_{F'}$ have the same characteristic.

Remark 3.6. In Gohberg-Sigal [4, Theorem 7.1], it was shown that, for each $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, there exist an $(m_1, \ldots, m_e)$-characteristic basis $\Phi_1, \ldots, \Phi_e$ of L_F and an $(m_1, \ldots, m_e)$-characteristic basis $\Psi_1, \ldots, \Psi_e$ of $L_{F'}$ (this stated in the language expounded in Remark 3.5) such that the principal part of the Laurent expansion of $F^{-1}(z)$ at $z = p$ has the form (3.12). In that respect, Corollary 3.10 is more general.

3.4. **Proofs of the main theorems.** In view of Lemma 2.2 (a) and Remark 3.2, Theorem 1.1 follows after Theorem 1.2 is proved. Thus, we immediately enter the proof of our second main theorem.

Lemma 3.11. *Let J, K, $L \in \mathcal{J}(E)$, $J \wedge K = \mathcal{O}$. Then there is an $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$ satisfying $L_F = J$ and $K^F = L$ if and only if K and L have the same characteristic.*

Proof. For $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$, $L_F = J$, and $K^F = L$, the operator $(\ )^F$ from (3.5) induces an isomorphism $(\ )^F \colon K \to L$ which commutes with T. Thus, the condition is necessary.

Suppose that K and L have the same characteristic. First, we treat the case $J = \mathcal{O}$. Let K and L have characteristic $(n_1, \ldots, n_f)$. Further, let $(\Phi_1, \ldots, \Phi_f)$ be an $(n_1, \ldots, n_f)$-characteristic basis of K and $(\Psi_1, \ldots, \Psi_f)$ be an $(n_1, \ldots, n_f)$-characteristic basis of L. We are going to construct operators

$F_0, F_1, \ldots, F_{\bar{n}-1} \in \mathcal{L}(E)$, where $\bar{n} = \max_{1\le k\le f} n_k$, such that F_0 is invertible and

$$\Phi_k^{(F_0,\ldots,F_{\bar{n}-1})} = \Psi_k, \quad 1 \le k \le f.$$

These relations mean that $F_0, F_1, \ldots, F_{\bar{n}-1}$ satisfy

$$\begin{aligned} F_0\phi_0^{(k)} &= \psi_0^{(k)}, \\ F_1\phi_0^{(k)} + F_0\phi_1^{(k)} &= \psi_1^{(k)}, \quad n_k \ge 2, \\ &\cdots\cdots\cdots \\ F_{\bar{n}-2}\phi_0^{(k)} + \cdots + F_0\phi_{\bar{n}-2}^{(k)} &= \psi_{\bar{n}-2}^{(k)}, \quad n_k \ge \bar{n}-1, \\ F_{\bar{n}-1}\phi_0^{(k)} + F_{\bar{n}-2}\phi_1^{(k)} + \cdots + F_0\phi_{\bar{n}-1}^{(k)} &= \psi_{\bar{n}-1}^{(k)}, \quad n_k = \bar{n}, \end{aligned} \tag{3.14}$$

for $1 \le k \le f$, where $\Phi_k = (\phi_0^{(k)}, \ldots, \phi_{n_k-1}^{(k)})$, $\Psi_k = (\psi_0^{(k)}, \ldots, \psi_{n_k-1}^{(k)})$ for $1 \le k \le f$.

We proceed as follows: For $1 \le l \le \bar{n}$, $0 \le k \le l-1$, let

$$K_l^{(k)} = \operatorname{span}\{\phi_r^{(h)};\, r \le l-k-1,\, n_h \ge r+k+1\}.$$

Note that, for $1 \le l < \bar{n}$, $0 \le k \le l-1$,

$$K_{l+1}^{(k)} = K_l^{(k)} + \operatorname{span}\{\phi_{l-k}^{(h)};\, n_h \ge l+1\}.$$

By induction on $l = 1, \ldots, \bar{n}$, we shall determine F_0 on $K_l^{(0)}$, F_1 on $K_l^{(1)}$, ..., F_{l-1} on $K_l^{(l-1)}$ such that F_0 is bijective from the space $K_l^{(0)}$ onto its image and the system (3.14) of linear equations is fulfilled up to the lth equation.

The beginning of induction, $l = 1$, is clear: $F_0\phi_0^{(k)} = \psi_0^{(k)}$ for $1 \le k \le f$ determines F_0 uniquely on the space $K_1^{(0)}$. Furthermore, F_0 is a bijection from $K_1^{(0)}$ onto $\operatorname{span}\{\psi_0^{(k)};\, 1 \le k \le f\}$, since both spaces are spanned by linearly independent sets of f eigenvectors. Now, suppose that, for some $1 \le l \le \bar{n}-1$, the construction has already been performed and consider the equation

$$F_l\phi_0^{(k)} + F_{l-1}\phi_1^{(k)} + \cdots + F_0\phi_l^{(k)} = \psi_l^{(k)} \tag{3.15}$$

which should hold for all $1 \le k \le f$ such that $n_k \ge l+1$. Let a be the number of h such that $n_h \ge l+1$. Assuming $n_1 \ge n_2 \ge \cdots \ge n_f$, we, therefore, have $n_h \ge l+1$ if and only if $h \le a$. Now, by induction on $k = 1, \ldots, a$, we extend the operator F_0 to $K_l^{(0)} + \operatorname{span}\{\phi_l^{(h)};\ 1 \le h \le k\}$, the operator F_1 to $K_l^{(1)} + \operatorname{span}\{\phi_{l-1}^{(h)};\ 1 \le h \le k\}$, ..., the operator F_{l-1} to $K_l^{(l-1)} + \operatorname{span}\{\phi_1^{(h)};\ 1 \le h \le k\}$, and the operator F_l to $\operatorname{span}\{\phi_0^{(h)};\ 1 \le h \le k\}$.

The beginning of induction, $k = 0$, is empty. In the inductive step, from $k-1$ to k for some $1 \le k \le a$, the value for $F_l\phi_0^{(k)}$ and the values for $F_{l-r}\phi_r^{(k)}$ for all these $1 \le r \le l$ satisfying

$$\phi_r^{(k)} \notin K_l^{(l-r)} + \operatorname{span}\{\phi_r^{(h)};\ 1 \le h < k\} \tag{3.16}$$

are to be chosen in such a way that (3.15) is fulfilled. If l is not among the r satisfying (3.16), then after the corresponding choice we still have

$$F_0 \text{ is injective on the space } K_l^{(0)} + \operatorname{span}\{\phi_l^{(h)};\ 1 \le h \le k\}; \tag{3.17}$$

while if l is among these r, then there is at least one other $0 \le r \le l-1$ such that $F_{l-r}\phi_r^{(k)}$ is to be chosen, e.g., $r = 0$, and this choice can always be made in a manner such that (3.17) is fulfilled.

Having defined, for $0 \le l \le \bar{n} - 1$, the operators F_l on $K_{\bar{n}}^{(l)}$, it remains to extend these operators to bounded operators F_l on E such that F_0 is invertible. We set

$$F(z) = F_0 + F_1(z-p) + \cdots + F_{\bar{n}-1}(z-p)^{(\bar{n}-1)}$$

and obtain $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$ such that $F^{-1} \in \mathcal{A}_p(\mathcal{L}(E))$, i.e., $L_F = \mathcal{O}$, and, moreover, $K^F = L$ by construction.

Now, we switch to the general case. Let the asymptotic type J have characteristic $(m_1, \ldots, m_e)$ and let $J_0 = L_1 \times L_2 \times \cdots \times L_{\bar{m}}$ be as in (3.8) also having characteristic $(m_1, \ldots, m_e)$. Consider the operator

$$F_0(z) = \Pi_0 + \sum_{j=1}^{e} (z-p)^{m_j} \Pi_j, \tag{3.18}$$

where $\Pi_j \in \mathcal{L}(E)$ for $1 \le j \le e$ is a rank-one projection onto the space spanned by the jth eigenvector of T on J_0, orthogonal to all other projections Π_h for $1 \le h \le e$, $h \ne j$, and $\Pi_0 = 1 - \sum_{j=1}^{e} \Pi_j$. Then $F_0 \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$, and $L_{F_0} = J_0$. By the first part of the proof, there are G, $H \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$ such that $G^{-1}, H^{-1} \in \mathcal{A}_p(\mathcal{L}(E))$ and $J^G = J_0$, $L_0^H = L$, where

$K^G = K_0$ and $K_0^{F_0} = L_0$. Note that $J_0 \wedge K_0 = (J \wedge K)^{\,G} = \mathcal{O}^G = \mathcal{O}$ such that K, K_0, L_0, and L all have the same characteristic. We set

$$F(z) = H(z)F_0(z)G(z) \tag{3.19}$$

and conclude that $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$, $L_F = J$, and $K^F = L$, since

$$\begin{aligned} L_F &= L_{HF_0G} = \mathcal{O}^{(F_0G)^{-1}H^{-1}} = \mathcal{O}^{(F_0G)^{-1}} = L_{F_0G} \\ &= (L_{F_0})^{\,G^{-1}} = J_0^{G^{-1}} = J, \\ K^F &= K^{HF_0G} = \left(K^G\right)^{\,HF_0} = K_0^{HF_0} = \left(K_0^{F_0}\right)^{\,H} = L_0^H = L \end{aligned}$$

by the help of Lemma 2.2. □

Remark 3.7. (a) In the proof of Lemma 3.11, normal factorization enters via the construction in (3.18), (3.19). This technique played a predominant role in Gohberg-Sigal's paper, see, e.g., [4, Theorem 3.2].

(b) The previous lemma remains valid if the condition $J \wedge K = \mathcal{O}$ is skipped. In that case, there is an $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$ satisfying $L_F = J$ and $K^F = L$ if and only if $K/\,(J \wedge K)$ and L have the same characteristic. Here, the right shift operator induces a nilpotent operator on $K/\,(J \wedge K)$, since both K and $J \wedge K$ are invariant under the action by T.

Proof of Theorem 1.2. Assume that $L_F = J$, $\mathcal{O}^F = K$ for a certain $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$. Let $\phi_0^{(1)}, \dots, \phi_0^{(e)}$ be a canonical system of eigenvectors for $F(z)$ at $z = p$ and $\bar{\psi}_0^{(1)}, \dots, \bar{\psi}_0^{(f)}$ be a canonical system of eigenvectors for $(F')^{-1}(z)$ at $z = p$. Since the representation (3.12) is valid for both F, $F^{-1} \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, the relation $F(z)F^{-1}(z) = 1$ immediately implies that $\langle \bar{\psi}_0^{(k)}, \phi_0^{(j)} \rangle = 0$ for $1 \le j \le e$, $1 \le k \le f$. This yields $\ell(J) + \ell(K) = e + f \le \dim E$.

Now, let $\ell(J) + \ell(K) \le \dim E$. We choose J_1, $K_1 \in \mathcal{J}(E)$ such that J_1 has the same characteristic as J, K_1 has the same characteristic as K, and $J_1 \wedge K_1 = \mathcal{O}$, i.e., J_1, $K_1 \in \mathcal{J}(E)$ are in general position. (This is possible in view of Lemma 3.8 (c).) By virtue of Lemma 3.11, there are F, $G \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E)) \cap \mathcal{A}_p(\mathcal{L}(E))$ such that $L_F = J_1$, ${K_1}^F = K$, $L_G = K_1$, and ${J_1}^G = J$. By

Lemma 2.2 (c), we obtain

$$L_{FG^{-1}} = (L_F)^{\,G} = J_1{}^G = J, \quad L_{GF^{-1}} = (L_G)^{\,F} = K_1{}^F = K, \tag{3.20}$$

showing that $FG^{-1} \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$ is as required. □

The statement that L_F characterizes the amount of asymptotics annihilated by $F \in \mathcal{M}_p^{\mathrm{fin}}(\mathcal{L}(E))$, while $\mathcal{O}^F$ contains the asymptotics produced by it, has to be read with some care. In fact, Theorem 1.2 shows that, already for $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, $L_F \wedge \mathcal{O}^F \neq \mathcal{O}$ is possible provided that $\dim E > 1$.

We conclude with a simple example.

Example 3.8. Let

$$F(z) = 1 + \frac{A}{z-p},$$

where $A \in \mathcal{L}(E)$ is a finite-rank operator and $A^2 = 0$ ($\dim E > 1$ if $A \neq 0$). Then $F \in \mathcal{M}_p^{\mathrm{nor}}(\mathcal{L}(E))$, with its inverse being given by $F^{-1}(z) = 1 - A(z-p)^{-1}$. We get

$$L_F = \mathcal{O}^F = AE.$$

Here, asymptotics of type AE are annihilated, while at the same time in a complementary direction exactly this sort of asymptotics is produced.

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